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THE EFFECT OF RADIATIVE EMISSION AND SELF-ABSORPTION ON THE
FLOW FIELD AND HEAT TRANSFER BEHIND A REFLECTED SHOCK
WAVE IN AIR

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the
Degree Doctor of Philosophy in the Graduate School of The
Ohio State University

By
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* * * * * * *

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### SYMBOLS

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<tr>
<td>( B_\nu )</td>
<td>Black body specific radiative intensity, ( \frac{2\hbar \nu^3}{c^2(e^{\hbar \nu/kT}-1)} )</td>
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<td>( B )</td>
<td>Black body radiative intensity, ( \int_0^\infty B_\nu , d\nu )</td>
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<td>( B_i )</td>
<td>( \int_{\nu_i}^{\nu_f} B_\nu , d\nu )</td>
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<tr>
<td>( c )</td>
<td>Speed of light in a vacuum</td>
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<tr>
<td>( C_p )</td>
<td>Specific heat at constant pressure</td>
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<tr>
<td>( E )</td>
<td>Radiative energy emitted per unit volume and time, ( \frac{4\pi J}{J} )</td>
</tr>
<tr>
<td>( E_n(z) )</td>
<td>Integro-exponential function of order, ( n ), ( \int_0^\infty \omega^{n-2} e^{-3i\omega} , d\omega )</td>
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<td>( \overrightarrow{F_i} )</td>
<td>Force exerted on a particle of species, ( i ), due to gravity, electric fields, magnetic fields, etc.</td>
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<tr>
<td>( h )</td>
<td>Static enthalpy; also Planck's constant</td>
</tr>
<tr>
<td>( I_\nu )</td>
<td>Specific radiative intensity, i.e., energy emitted per unit area, per unit time, per unit frequency, per unit solid angle</td>
</tr>
<tr>
<td>( J_\nu )</td>
<td>Energy emitted per unit volume, per unit time, per unit frequency, per unit solid angle</td>
</tr>
<tr>
<td>( J )</td>
<td>( \int_0^\infty J_\nu , d\nu )</td>
</tr>
<tr>
<td>( k )</td>
<td>Total thermal conductivity, including effects of diffusion; also, Boltzmann constant</td>
</tr>
<tr>
<td>( L )</td>
<td>Characteristic length, arbitrarily chosen to be one inch</td>
</tr>
<tr>
<td>( N_i )</td>
<td>Number density of species, ( i )</td>
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Pressure

Initial pressure ahead of incident shock

Prandtl number, \( \frac{\mu C_p}{\kappa} \)

Radiative flux of frequency, \( v \), i.e., energy emitted per unit area, per unit time, per unit frequency

\[ \int_0^\infty q_v \, dv \]

Radiative heat transfer at the end wall

Conductive heat transfer at the end wall

Radiative heat transfer from a flow field which has no radiative cooling, i.e., one which is adiabatic with respect to radiation

Transformed variable, \( s = t \)

Temperature

Time after reflection of shock

Flow velocity relative to end wall

Diffusion velocity for species, \( i \)

Incident shock velocity relative to end wall

Reflected shock velocity relative to end wall

Reflected shock velocity at \( t = 1 \mu\text{sec} \)

Distance measured from end wall

Location of the reflected shock

Constant corresponding to the \( i \)-th level in the non-gray absorption coefficient
Γ
Radiation loss parameter defined by equation (41)

Γ_{CH}
Characteristic radiation loss parameter,

\[ \Gamma_{CH} = \frac{4\pi k_{Si} c T_{Si}}{\beta_{Si} W_{Rz} h_{Si}} \]

η
Transformed variable,

\[ \eta = \left[ \frac{(\rho/\mu)_{e}}{2\rho_{e}} \right]^{1/2} \int_{0}^{x} \rho \, dx \]

K_v
Spectral absorption coefficient with dimension of (l/length)

K
A mean absorption coefficient

K_P
Planck mean absorption coefficient

K_R
Rosseland mean absorption coefficient

μ
Coefficient of viscosity

ν
Frequency

ρ
Density

σ
Stephan-Boltzmann constant

τ_v
Optical length corresponding to radiation of frequency, ν

τ
Optical length corresponding to a mean absorption coefficient

τ_o
Characteristic optical thickness based on length x_o

Subscripts
AD
Conditions for a flow field without radiative cooling

e
Conditions at the edge of the thermal boundary layer

s
Conditions behind the reflected shock

si
Conditions behind the reflected shock at t=1 μsec (reference condition), also, the same as conditions behind an ideal reflected shock
Conditions at the end wall

Incident shock conditions
CHAPTER I
INTRODUCTION

Flow field variables in high temperature gas dynamics are usually influenced by the physical effect of chemically reacting gases. If the temperature of the gas is sufficiently high, certain elements of the flow will noticeably emit radiation, and the physical effect of this radiative energy transfer also may influence and may even dominate the gas dynamic problem. Although astrophysicists have been interested in radiative transfer for the past sixty years\textsuperscript{1,2} in conjunction with studies of the atmospheres of planets, stars, nebulae and galaxies, gas dynamicists have only recently been concerned with the transfer of radiation in conjunction with such problems as nuclear fireballs, high temperature plasma facilities, and in particular, high velocity atmospheric re-entry of lunar and planetary space vehicles. This recent engineering interest in radiation has given birth to a relatively new fluid dynamic speciality called radiation gas dynamics, which constitutes the basic area for the particular investigation reported in this paper.

A specific engineering motivation for studies in radiation gas dynamics, in general, and for the present investigation, in particular, can be obtained from Figure 1, which shows the convective and equilibrium radiative stagnation point heat transfer to a re-entry vehicle with a two foot nose radius. The convective heat transfer is calculated from the results of Lees\textsuperscript{3} and the equilibrium radiative heat transfer is obtained
Fig. 1.--Equilibrium radiative and convective Stagnation point heat transfer for nose radius = 2.0 ft and altitude = 160,000 ft
from the constant property, semi-infinite slab approximation. (This aspect of radiative heat transfer is discussed in a subsequent section.) Figure 1 demonstrates the important qualitative fact that equilibrium radiative heat transfer can equal and exceed the convective heat transfer for practical re-entry velocities. This fact by itself is a strong motivation for the intensive study of radiative transfer in connection with aerodynamic flow fields, and provides the incentive to delve into the basic aspects of radiation gas dynamics.

As a qualitative introduction to the general concept of radiation gas dynamics, and as a means to provide some physical insight for the quantitative discussions to follow, the effects of radiative transfer on gas dynamic problems can be reduced to two general phenomena. First, each fluid element in a high temperature flow field loses energy due to radiative emission, thus tending to locally cool the flow and contribute to the radiation field. Second, each fluid element in the presence of a radiation field tends to absorb some of the radiative energy, thus tending to locally warm the flow and attenuate the radiation field. One net result of the above two phenomena, namely, radiative emission and self-absorption, is a local non-adiabatic effect at each point in the flow. A second net result is that radiation obviously adds an additional mode to the energy transfer across the boundary of a gaseous system, thus making radiative heat transfer an important quantity along with the more usual mode of conductive heat transfer. In addition, an important effect of the self-absorption is that each fluid element is coupled to other regions of the flow field by means of the radiation field, and this physical coupling has strong ramifications in the mathematical formulation.
of the energy equation as will be discussed later. The radiative emission and self-absorption phenomena discussed above have a dominant influence on the particular problem investigated in this paper.

The above physical discussion of the effects of radiative transfer on gas dynamic flow fields can be summarized by defining the concept of a radiation--gas dynamic coupling as follows. The radiation field (radiative intensity) is dependent upon the gas dynamic properties of the flow field such as temperature and density, and conversely, the gas dynamic properties of the flow field are dependent upon the radiation. Throughout the remainder of this paper, the term radiation--gas dynamic coupling will denote the physical interaction described above.

In order to place the present investigation in its proper perspective, a qualitative review will now be made of the existing physical formulations of problems in radiation gas dynamics. In almost any physical problem, simplifying assumptions must be made to make the problem more tractable for analysis, and radiation gas dynamics is no exception. Figure 2 is a block diagram which helps to provide a general idea of the avenues that lead to the treatment of flow problems involving some aspect of thermal radiation. The simplest but sometimes least realistic avenue is shown on the left of Figure 2. Here, each fluid element is assumed to be at a sufficiently high temperature such that it noticeably emits radiation, but the amount of energy loss in comparison to its total internal energy is so small that for all practical purposes the flow is adiabatic. However, the radiative emission from all the fluid elements accumulates through the flow field, and radiative heat transfer across the boundary of the system certainly becomes an item of interest
Flow Fields Involving Some Aspect of Thermal Radiation

Adiabatic Flow

Non-Adiabatic Flow

- Transparent
- Self-Absorbing

Gray Gas

Non-Gray Gas

Items of interest concerning effects of radiation:
1. Radiative heat transfer
2. Convective heat transfer
3. Detailed flow field structure

Fig. 2. -- Block diagram for various approaches to radiation gas dynamic analysis
even though the flow is locally assumed to be adiabatic. The simplification imposed by this adiabatic assumption is obvious. The radiation-gas dynamic coupling is not present, and the flow field can be determined quite independently of the radiation field. Then the radiative heat transfer can be obtained from the known temperature and density distributions throughout the flow. This approach certainly has some merit for the analysis of re-entry radiative heat transfer in the velocity range from 30,000 to 37,000 ft/sec, where the above assumptions are somewhat valid. Yoshikawa et al., Kivel, and Wick employed the above approach in order to obtain some predictions for the stagnation point radiative heat transfer.

A more realistic but considerably more complex avenue is shown on the right of Figure 2. Here, the local non-adiabatic effect on the flow field is taken into account. Each fluid element is assumed to emit radiation in an amount which is sizable in comparison to its total internal energy, thus cooling the flow and noticeably influencing the flow variables. This is the important case of the radiation-gas dynamic coupling, where the flow field and radiation field must be simultaneously determined. Re-entry radiative and convective heat transfer calculations for velocities above 37,000 ft/sec (escape velocity from the Earth) are affected by the above phenomena. Compounding the non-adiabatic nature of the problem, an additional concern enters the physical picture, namely, the tendency for the fluid elements to absorb as well as emit radiation. For conditions where the self-absorption is very weak, it is suitable to ignore the effect completely and assume the gas is transparent to radiation. Several non-adiabatic transparent flow
problems have been analyzed by Hoshizaki and Wilson,7,11 Nerem,8 and by Burggraf.13 On the other hand, if the temperature, density and characteristic dimension of the flow field are large enough, self-absorption cannot be ignored, and its presence complicates the analysis of the problem. However, when including self-absorption, there are two general alternatives as shown in Figure 2. One choice is to assume a gray gas where the absorption processes are described in terms of average quantities independent of the radiation frequency, whereas the more realistic choice is to consider a non-gray gas where the frequency dependent absorption processes are taken into account. Several non-adiabatic gray gas problems have been treated by Yoshikawa and Chapman,9 Howe and Viegas,10 Goulard,14 and Olstad,12 whereas some non-adiabatic non-gray problems have been investigated by Hoshizaki and Wilson13 and by Olstad.12 All of the above investigations have been oriented toward the re-entry heat transfer problem, and have treated flow fields around blunt bodies, over blunted cones, and at stagnation points. One exception is Reference 9 where the steady state flow behind a normal shock wave is investigated.

At this point, it is appropriate to mention that if gray gas self-absorption becomes extreme (the optically thick case), the radiation field approaches that of a black body. Under this condition, most of the radiative energy is contained within the bulk of the flow field, and this effect tends to reduce the complexity of the problem. However, even though this condition is of practical interest to the astrophysicists, it does not usually apply to engineering problems in radiation gas dynamics. On the other hand, for a non-gray gas, certain wave length
regions may be more self-absorbing than other wave length regions. As a result, a high temperature radiating gas may be optically thick for some wave lengths and optically thin for others. This is indeed the case for high temperature air, which is strongly self-absorbing in the vacuum-ultraviolet. The variation of the self-absorption phenomenon with wave length for a non-gray gas is definitely of practical interest in engineering radiation gas dynamics, as shown by the results of the present investigation.

Also, it is appropriate to mention that whereas radiative heat transfer is the only item of interest concerning effects of radiation for adiabatic flows, the non-adiabatic flow problem introduces conductive heat transfer and detailed flow field structure in addition to radiative heat transfer as phenomena affected by the presence of radiation. This latter flow problem is the area considered in the present investigation.

The intent of the above discussion has been to establish a qualitative background and appreciation for the basic problems in radiation gas dynamics. As a conclusion to this introductory chapter, the specific problem analyzed in the present investigation will now be described, and its motivations and contributions set forth.

The present investigation is concerned with the analysis of the radiation-gas dynamic coupling occurring in the flow field behind a reflected shock wave in air. Ideal shock theory predicts a stagnant, constant property region behind a reflected shock, the velocity of which is constant and uniquely determined by the initial conditions and the incident shock velocity. By the inclusion of the radiation-gas dynamic
coupling in the reflected shock region, all of the simple aspects mentioned above are destroyed. Flow properties are no longer constant, but vary both with time and location. Mass motion is induced behind the reflected shock, and the wave velocity itself becomes a function of time. These aspects are discussed in detail both qualitatively and quantitatively in Chapter III. However, they are briefly mentioned here in order to emphasize that the straightforward ideal reflected shock problem is transformed into a very interesting and complex gas dynamic study by the effects of radiative transfer. The quantities of interest that are determined in the course of the present analysis are (1) detailed flow field structure, (2) end wall radiative heat transfer, and (3) end wall conductive heat transfer, all of which are influenced by the presence of radiation. In reference to the earlier discussion, the analysis is carried out for both a non-adiabatic, transparent flow and a non-adiabatic, self-absorbing flow. In all cases, the transport mechanisms of conduction and diffusion are included.

A strong motivation for the present analysis can be seen in light of some of the previous investigations described above. In many cases, the radiation gas dynamic problems which are treated in the literature deal with flows such as blunt body flows, which are complex even without radiative transfer effects. Therefore, it is reasonable to choose a high temperature gas dynamic problem which is simple in the ideal case, and analyze it with respect to the radiation—gas dynamic coupling. In this way, it is possible to emphasize the fundamental influence of radiation without clouding the picture with additional fluid dynamic complexities. The radiating reflected shock problem has not been treated
in the literature, and this by itself tends to generate interest in its investigation. However, more importantly, the results of the reflected shock analysis provide information on the physical parameters that govern the radiation—gas dynamic coupling, in general, thus making the problem more utilitarian than its specialized nature might initially indicate. Finally, a practical engineering contribution is obtained from the reflected shock problem through its analog in several respects with the steady state stagnation point problem. This analogy is formulated and demonstrated in detail in Chapter V.

An engineering motivation for specifically treating the self-absorption case can be seen from Figure 3, which shows several re-entry trajectories in reference to the velocity-altitude regions where non-adiabatic and gray gas self-absorption effects are important. These regions are obtained from Reference 9, whereas the trajectories are from Reference 10. For the more realistic case of a non-gray gas, the self-absorption region will be displaced to the left in Figure 3. The non-adiabatic region, in Figure 3, is defined as that region where, for a body with shock detachment distance equal to 0.1 ft., the ratio of radiative energy lost by a fluid element to its initial energy is greater than 0.1. The gray gas self-absorption region is defined as that region where the shock detachment distance of 0.1 ft. is greater than one tenth of the gas thickness which would be required to produce black body radiation at the same temperature and density. It can be seen from Figure 3 that particularly high velocity re-entries dip into the self-absorption regime. This is certainly the case for meteoric trajectories, which are currently gaining more research emphasis. However, an additional
Fig. 3.—Velocity-altitude map with some typical trajectories, showing the non-adiabatic and gray gas self-absorbing regions based on a shock detachment distance of 0.1 ft
motivation for including self-absorption is that it tends to remove certain peculiarities analogous to the anomaly observed by Goulard\textsuperscript{14} and Hoshizaki\textsuperscript{7} which appears in an inviscid, non-adiabatic, transparent blunt body flow. In brief, this anomaly is a consequence of the infinite time theoretically taken by a fluid particle to approach the body surface at the stagnation point. If the flow is both non-conducting (inviscid) and transparent, the fluid elements near the stagnation point continue to emit radiation until they lose all their energy. This produces a radiation-cooled layer near the surface and a zero enthalpy condition at the stagnation point and along the body surface streamline. Of course, this anomaly is physically unrealistic, and is a consequence of the inviscid and transparent assumptions. The anomaly can be removed by the more realistic approach of including self-absorption in the flow. Thus, the cool fluid elements near the surface absorb radiation, and the enthalpy is finite at the surface. Thomas\textsuperscript{15} has investigated the magnitude of this effect. On the other hand, the anomaly can also be removed by including thermal conduction in the flow, which is certainly appropriate due to the large temperature gradients in the highly cooled surface layer. Nerem\textsuperscript{8} and Burggraf\textsuperscript{43} have treated the case of a transparent and conducting gas. Even though the anomaly discussed above would not be present in the reflected shock problem due to the finite times involved, it does serve as a motivation to include both condition and self-absorption in the analysis in order to gain as much physical realism as possible.

In summary, this introduction has presented some reasons for engineering interest in radiation gas dynamics. It has discussed in a
very general manner the physical aspects of radiative effects on flow fields, and has defined the radiation--gas dynamic coupling in order to provide some physical insight into the more detailed discussions to come. In addition, it has reviewed and generally classified the existing approaches towards the analysis of radiative problems in gas dynamics, and has partially reviewed some of the existing literature. Finally, the specific problem analyzed in this investigation, namely, the nonadiabatic, self absorbing reflected shock problem, has been described and its motivations and engineering applications have been discussed. Now it is appropriate to discuss in some detail the physical description and theoretical formulation of the radiation--gas dynamic coupling.
CHAPTER II
FORMULATION OF THE GENERAL RADIATION--GAS DYNAMIC COUPLING

The physical description and general theoretical formulation of the radiation--gas dynamic coupling combines the principles of radiative transfer in gases with the gas dynamic equations of change. In order to lay the foundation for the reflected shock problem, it is necessary to develop and present some basic aspects of both disciplines. However, for the sake of brevity and clarity, only those aspects of radiative transfer and gas dynamics which are directly pertinent to the present investigation will be discussed. First, the relevant concepts from radiative transfer theory will be presented, followed by a brief description of physical radiative properties. Finally, these concepts will be combined with the gas dynamic equations.

A. Macroscopic radiative transfer in gases

The macroscopic theory of the transfer of radiative energy through matter has been thoroughly developed over a period of years by the astrophysicists. References 1 and 2 are authoritative presentations of this theory. Several concepts from this classical theory have been adopted for use in radiation gas dynamics, and these will now be discussed.

The two basic quantities which describe a radiation field are the specific radiative intensity and the radiative flux, which are two different descriptions of the amount of energy crossing an arbitrary surface.
The distinction can simply be made by considering a small surface element arbitrarily oriented in a field of radiation as shown in Figure 4. The direction normal to the surface element is associated with the unit vector \( \hat{L} \). The specific radiative intensity, \( I_\nu \), along the direction of \( \hat{L} \) is defined as the amount of radiative energy of a given frequency, \( \nu \), crossing this surface element per unit area, per unit time, and per unit solid angle around the direction of \( \hat{L} \). For example, the units of \( I_\nu \) in the c.g.s. system are \( \frac{\text{erg/sec}}{(\text{cm}^2)(1/\text{sec})(\text{steradian})} \). The above definition of \( I_\nu \) specifies the direction of energy transfer to be along the normal to the surface element. This is different from the standard presentations, \(^1,^2\) which first specify the energy transfer to be along an arbitrary direction of unit vector \( \hat{s} \), and then later imply that the surface area normal to \( \hat{s} \) is really the fundamental area. By shrinking the surface element until it is infinitesimally small, \( I_\nu \) can be seen to be a point property which, in general, varies with location in the field.

It is important to note that intensity is directional in nature. When specifying the magnitude of specific intensity at any point in a radiation field, one must also specify whatever direction is being considered. On the other hand, the quantity which specifies the amount of radiative energy of frequency, \( \nu \), crossing the surface element per unit area, per unit time for all directions is defined as the net radiative flux, \( q_\nu \). The units of flux in the c.g.s. system are \( \frac{\text{erg/sec}}{(\text{cm}^2)(1/\text{sec})} \). The relation between \( q_\nu \) and \( I_\nu \) can be obtained directly from Figure 4 by first considering the energy crossing the surface element per unit area, per unit...
Fig. 4.—Elemental surface area and directions for the definition of radiative intensity and net flux
time, contributed by the intensity in an arbitrary direction along the unit vector \( \mathbf{s} \) within an elemental solid angle, \( dw \),

\[
dq_r = I_r(\mathbf{s}) \cos \theta \, dw
\]

and summing over all directions,

\[
q_r = \int I_r(\mathbf{s}) \cos \theta \, dw
\]

In terms of spherical coordinates,

\[
q_r = \int_0^{2\pi} \int_0^\pi I_r(\theta, \phi) \cos \theta \sin \theta \, d\theta \, d\phi
\]

The above discussion implies that both intensity and flux vary with position in the radiation field. The equation which describes this variation is the **radiative transfer equation**, which can be obtained by considering an element of matter in the field of radiation. The element alters the intensity by both emitting radiation which originates from its constituents, and by absorbing a fraction of the incident energy. The intensity is also altered by the influence of scattering, and this effect can be included in the general emission and absorption terms in the transfer equation. However, scattering is neglected in the present investigation, and only true emission and absorption will be considered in the following discussion. Let \( J_v \) denote the energy of frequency, \( v \), emitted by the element of matter per unit volume, per unit time, and per unit solid angle along a direction \( \mathbf{s} \) in the field. In addition, if \( I_v \) is the incident specific intensity along the direction
\( s \), let \( K_v \), denote the energy of frequency \( \nu \) absorbed by the element of matter per unit volume, per unit time, and per unit solid angle, where \( K_v \) is the spectral absorption coefficient. Then, due to the element of matter, the change in intensity along the path \( s \) is

\[
\frac{dI_\nu}{ds} = J_\nu - K_\nu I_\nu
\]  

(2)

This equation physically describes the local change in intensity due to emission and absorption, and is called the radiative transfer equation. Note that both \( J_\nu \) and \( K_\nu \) are defined quantities in the above equation, and, for a gas dynamic flow field, both are functions of the density, temperature and chemical composition.

Even though the above equation describes the physical variation of intensity in a radiating flow field, it is a differential equation which must be solved in order to obtain practical expressions for intensity and flux. Towards this end, the assumption of local thermodynamic equilibrium is made, which implies that the gas particles are distributed over their various energy levels according to the Boltzmann distribution. In turn, this implies that the density, consequently the collision frequency, is sufficiently high to maintain the Boltzmann distribution in the face of radiative transitions. It also implies that each energy mode, namely, translation, rotation, vibration and electronic is in equilibrium with each other, thus causing the temperatures associated with the different modes to be equal. Therefore, for this case, the emission term is
\[ J_\nu = K_\nu B_\nu \]  

where \( B_\nu \) is the black body specific radiative intensity given by

\[
B_\nu = \frac{2\beta \nu^3}{c^2 (e^{\beta \nu kT} - 1)}
\]

By including equation (3), and by dividing by \( K_\nu \), the transfer equation becomes

\[
\frac{dI_\nu}{K_\nu ds} = B_\nu - I_\nu
\]

The expression \( K_\nu ds \) introduces a new parameter of particular physical significance in radiating flow fields, namely, the optical length, \( \tau_\nu \), defined by

\[
d\tau_\nu = K_\nu ds
\]

The optical length is an index which describes the relative influence of self-absorption; as \( \tau_\nu \) increases, the effect of self-absorption becomes stronger. This will be markedly demonstrated in subsequent sections.

The solution of equation (4) which is pertinent to the present investigation assumes a one-dimensional radiation field. In order to interpret the form of the solution, consider a semi-infinite slab of radiating gas of thickness \( X_s \), where all properties depend only on one
preferred coordinate, \( X \). This preferred coordinate can just as well be described in terms of optical length, \( \tau_v \), corresponding to radiation of frequency, \( \nu \), where

\[
\tau_v = \int_0^X k_v \, dx
\]  

(6)

and

\[
\tau_{sv} = \int_0^{x_0} k_v \, dv
\]  

(7)

Since the solution of equation (4) for the one-dimensional case is carried out in some detail in the literature, only the result is given below. Considering only the radiation which originates within the gas itself, solving for the specific intensity, and employing equation (1), the expression for net flux of radiative energy of frequency, \( \nu \), at position, \( \tau_v \), in the gas is

\[
q_v = 2\pi \int_{\tau_v}^{\tau_{sv}} B_v(\tau_v) E_2(\nu) \, d\tau_v
\]

\[
- 2\pi \int_0^{\tau_v} B_v(\tau_v) E_2(\tau_v - \nu) \, d\tau_v
\]  

(8)

where, by astrophysical convention, a positive value of \( q_v \) denotes flux of energy in the negative \( X \) direction. The function \( E_2 \) belongs to a special class of integro-exponential functions of order \( n \), defined by

\[
E_n(\tau_v) = \int_0^\nu \omega^{n-2} e^{-\tau_v/\nu} \, d\nu
\]
Note that z is a dummy variable of integration. This expression for net radiative flux will subsequently be used for generating the radiation term in the gas dynamic energy equation and for calculating the end wall radiative heat transfer.

At this point, it is appropriate to express the above results in terms of a gray gas, in which by definition the absorption coefficient, \( K \), is a mean value, independent of frequency. For a gray gas, the optical length is independent of \( \nu \), and by equation (6), becomes

\[
\tau = \int_{0}^{\infty} K \, dx
\]

As a result, the net radiative flux including all frequencies,

\[
\varphi = \int_{0}^{\infty} \varphi_{r} \, dr
\]

becomes, with the aid of equation (8)

\[
\varphi(\tau) = 2\alpha \int_{\lambda}^{\infty} \frac{\nu}{T^{\frac{3}{2}}(\lambda)} E_{2}(\lambda-\tau) \, d\lambda - 2\alpha \int_{0}^{\infty} \frac{\nu}{T^{\frac{3}{2}}(\lambda)} E_{2}(\tau-\lambda) \, d\lambda
\]

where the integrated black body flux has been inserted:

\[
\varphi_{s} = \int_{0}^{\infty} B_{\nu} \, d\nu = \frac{\sigma T^{\frac{3}{2}}}{\pi}
\]

In the above, \( \sigma \) is the Stefan-Boltzmann constant. These expressions will also be useful in subsequent sections.
B. The absorption coefficient

The above discussion has dealt with some aspects of radiative transfer from the point of view of macroscopic classical theory, which does not concern itself with the details of the actual microscopic physical processes that generate the radiative energy. The present study of the radiating reflected shock problem is primarily concerned with the gas dynamic consequences of radiation, and is concerned with the detailed microscopic radiative mechanisms only in obtaining somewhat accurate radiative properties for quantitative analysis of the flow field. The radiative properties for the emissivity of air, as relevant to the present investigation, are discussed in some detail in Chapter IV. However, it is appropriate to emphasize at this point that the absorption coefficient can be considered to be a bridge between microscopic and macroscopic radiative theory. The classical theory as described above buries all consideration of radiative properties in the concept of the absorption coefficient.

Since the concept of a gray gas is pertinent to the present study, the mean absorption coefficient should be mentioned. Two different types of mean absorption coefficients have been useful in previous radiative studies; the Planck mean absorption coefficient, defined as

$$K_P = \frac{\int_0^\infty k_r B_r(T) dv}{B(T)}$$  \hspace{1cm} (11)

and the Rosseland mean absorption coefficient, defined as

$$K_R^{-1} = \int_0^\infty k_r \frac{d B_r(T)}{d T} dv$$
The theory of radiative transfer can be used to show that $K_p$ is appropriate for the optically thin case, whereas $K_R$ is useful for the optically thick case. Sampson\textsuperscript{17} has recommended use of a mean absorption coefficient which is a function of $\tau$ for intermediate cases. Since the present investigation is partly concerned with a gray gas self-absorbing region which is reasonably optically thin ($\tau \leq 0.3$), the Planck mean absorption coefficient will be used when treating the gray gas case.

C. Gas dynamic equations

The previous sections have been concerned with the radiation aspects of the radiation--gas dynamic coupling. This section will briefly describe the inclusion of the radiation effects in the gas dynamic equations. Goulard\textsuperscript{18} has discussed these effects in detail, so only those aspects which apply to the present problem will be mentioned.

In general, the influence of radiation is felt in both the momentum and energy equations through the radiation pressure, radiation energy density, and the divergence of the radiation flux vector. Whereas all the above quantities are of concern in astrophysical problems, only the latter influence is important for most radiation gas dynamic problems, the radiation pressure and energy density being insignificant. Therefore, the ramification of radiation which pertains to this investigation is the inclusion of the radiative flux term in the energy equation. As a result, the general equations of change which describe the radiation gas dynamic problem are

\begin{equation}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0
\end{equation}
Species continuity equation  
\[ \rho \frac{Dc_i}{Dt} + \nabla \cdot (\rho_c \vec{U}_i) = \dot{w}_i \]  
(13)

Momentum equation  
\[ \rho \frac{D\vec{U}_i}{Dt} + \nabla \cdot [P] - \sum_{\xi} N_{\xi} \vec{X}_i = 0 \]  
(14)

Energy equation  
\[ \rho \frac{DH_i}{Dt} = \nabla \cdot (k \nabla T) - \nabla \cdot \sum_{\xi} p_{\xi} \vec{U}_i h_i \]  
\[ - \nabla \cdot \vec{q} + \frac{Dp}{Dt} + \Phi \]  
(15)

where \( c_i \) is the mass fraction of chemical species, \( i \), \( \dot{w}_i \) is the species diffusion velocity, \( \dot{w}_i \) is the chemical source function due to reactions, \([ P ]\) is the ordinary pressure tensor, \( V \) is the flow velocity, \( N_{\xi} \) is the number density of species, \( i \), \( \vec{X}_i \) is the external force due to gravity, electromagnetic fields, etc. on a particle of species, \( i \), \( k \) is thermal conductivity, \( \Phi \) is the dissipation function,\(^{19}\) and \( \nabla \cdot \vec{q} \) is the divergence of the radiative heat flux vector.

The quantitative evaluation of \( \nabla \cdot \vec{q} \) can be made by considering its physical meaning. It represents the net energy change of a fluid element due to radiative emission and absorption per unit volume, per unit time. This is precisely the physical significance of the right hand side of equation (2) when integrated over all solid angles and all frequencies. Therefore

\[ \nabla \cdot \vec{q} = \iint_{\omega} J_{\omega} dw dv - \iint_{\omega} k_{\omega} J_{\omega} dw dv \]

Since the local radiative emission usually is the same in all directions
(isotropic emission), the first integral can be partially integrated, and the above expression becomes

$$\nabla \cdot \vec{q}_R = 4\pi J - \int_\omega \int_\nu \kappa_\nu I_\nu d\nu d\nu$$  \hspace{1cm} (16)$$

where

$$J = \int_\nu J_\nu d\nu$$ \hspace{1cm} (17)$$

Further evaluation of equation (16) depends on the geometry of the particular problem being considered.

Even though the only appearance of radiation in the above equations is through the $\nabla \cdot \vec{q}$ term, this is sufficient to couple the two disciplines of radiative transfer and gas dynamics. In Chapter III the above equations will be reduced to the forms appropriate for analyzing the reflected shock problem.

**D. Summary of the radiation—gas dynamic coupling**

This chapter has discussed the basic aspects of classical radiative transfer theory which pertain to the present investigation, and has shown how this theory enters the mathematical formulations of gas dynamics. In brief, the mathematical statement of the physical radiation—gas dynamic coupling is the inclusion of the $\nabla \cdot \vec{q}$ term in the energy equation, where the magnitude of $\vec{q}$ for the one dimensional problem considered here is given by equation (8) integrated over all frequencies. The intent of this chapter has been to discuss in a general fashion the
quantitative aspects of radiation gas dynamics in order to provide the necessary background for the present study. Now it is appropriate to formulate the particular physical and mathematical analysis for the radiating reflected shock problem.
CHAPTER III

FORMULATION OF THE RADIATING REFLECTED SHOCK PROBLEM

The preceding chapters contain some fundamental concepts from radiation gas dynamics which are necessary for generally understanding and interpreting the radiating reflected shock problem. This chapter will now describe the detailed physical and theoretical formulation of the problem itself.

A. Qualitative formulation and assumptions

In order to form a physical picture of the present analysis, consider the radiating flow field extending from the end wall to the reflected shock. The extent of this flow field is continually increasing as the shock moves away from the wall. As mentioned in Chapter I, the ideal picture of a stagnant, constant property gas behind a reflected shock traveling at constant velocity is totally destroyed by the physical effects of radiation and end wall conductive heat transfer. These effects constitute the dominant characteristics of the present investigation, and will now be discussed in detail.

Probably the most obvious characteristic is the non-adiabatic nature of the flow caused by the local emission and self-absorption of each fluid element. For the present study, the overall magnitude of radiative emission is greater than the magnitude of absorption, and the resulting non-adiabatic effect is that of cooling the flow.
Therefore, for conciseness, the non-adiabatic effect will hereafter be denoted as the **radiative cooling** effect.

Several other characteristics of the flow field are consequences of the radiative cooling described above. For instance, flow properties such as temperature, density, and enthalpy will vary with both time and location between the end wall and the reflected shock. This unsteady nature of the flow is accompanied by an attenuation of the shock velocity due to density changes resulting from the local cooling. Thus, the shock velocity itself becomes an unknown function of time. Finally, the non-uniform and the time-dependent radiative cooling will induce some **mass motion** between the end wall and the shock. These four main characteristics, namely, (1) radiative cooling, (2) time dependent conditions, (3) induced mass flow, and (4) shock wave attenuation, are of primary concern in the present investigation, and are fully taken into account by the analysis.

In order to concentrate on the main characteristics above, several simplifying assumptions are made concerning other aspects of the problem. First, the flow is assumed to be one-dimensional, thus making **time** and **distance** the two independent variables. This assumption seems reasonable for geometries where the shock distance from the wall is less than one-quarter of the wall diameter. It also seems reasonable in light of the three-dimensional versus one-dimensional comparison made by Kennet and Strack during their analysis of radiative heat transfer from a spherical gas cap. In addition, the gas between the wall and the shock is assumed to be in **thermodynamic and chemical equilibrium**. This assumption is quite valid when taken in the light of relaxation measurements.
behind reflected shock reported as auxiliary information in Reference 20. This information indicates that, for the density and temperature levels considered in the present investigation, less than one per cent of the entire region of a typical flow field will be in a non-equilibrium condition. (Typical flow field thickness can be considered about 8 millimeters or more.) Also, in order to isolate and study the effect of radiation which originates in the reflected shock region only, extraneous sources of radiation from the incident shock region and driver gas are assumed to be weak and are neglected, the shock tube walls are assumed to be cold black bodies, and the shock front itself is assumed transparent to radiation. These assumptions seem reasonable in light of practical testing conditions in some shock tubes. Finally, the reflected shock is assumed to move into a region described by ideal incident shock conditions, and no side wall boundary layer or contact region interactions occur.

With the above dominant characteristics and simplifying assumptions in mind, it is now appropriate to discuss the mathematical equations and boundary conditions which describe the physical problem.

B. Quantitative formulation

The mathematical simplification derived from ideal shock conditions is that the governing differential equations reduce to algebraic equations. However, for the present analysis, where properties in the flow vary with distance and time, the partial differential equations discussed in Chapter II must be solved.

Toward this end, consider the model of the reflected shock region shown in Figure 5. Here, the reflected shock is moving away from the
RADIATING FLOW FIELD

The diagram illustrates a schematic of the radiating reflected shock flow field.

Fig. 5.—Schematic of the radiating reflected shock flow field
end wall at a velocity relative to the wall, \( W \), which is a function of time, \( t \). The location of the end wall is \( x = 0 \), and the position of the shock is \( x = x_\text{s} \), where \( x_\text{s} = x_\text{s}(t) \). The region defined by \( 0 \leq x \leq x_\text{s} \) contains a high temperature radiating flow field in which all properties are functions of \( x \) and \( t \). The influence of viscosity is neglected in this region because the velocity gradients in the radiating flow field, though finite, are very small, as verified by the numerical results to be discussed later. In addition, the species continuity equation is superfluous for flows in chemical equilibrium. Therefore, the equations of continuity, momentum and energy which describe flow properties for the one-dimensional radiating flow field, and which are obtained from the more general form shown by equations (12) through (15), are as follows:

Continuity

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0 \tag{18}
\]

Momentum

\[
\frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} = - \frac{\partial p}{\partial x} \tag{19}
\]

Energy

\[
\frac{\partial h}{\partial t} + \rho u \frac{\partial h}{\partial x} = \frac{1}{\gamma-1} \left( k \frac{\partial T}{\partial x} \right) - \frac{\sum_{i} \rho_i u_i h_i}{\rho} - \frac{\partial q_R}{\partial x} + \frac{\partial p}{\partial x} + u \frac{\partial p}{\partial x} \tag{20}
\]

The radiation term in equation (20), \( - \frac{\partial q_R}{\partial x} \), is obtained by differentiating the expression for net flux as formulated in Chapter II, equation (8). This is a rather involved differentiation, and, for this
reason, it is appropriate to carry out the steps in Appendix I. The
result from Appendix I, equation (I-8), for the one-dimensional case,
is

\[- \frac{3\partial R}{2x} = -4\pi J + 2\pi \int_0^\infty B_r(3,t) E_1(1r-3) \, dq \, dr \]  

(21)

Note that equation (21) should follow directly from equation (16) with
suitable reduction for the one-dimensional case. On a physical basis,
the first term on the right hand side of equation (21) is the local
energy emitted per unit volume and time, and the second term is the
local energy absorbed per unit volume and time. The integral term con­
tains the effect of radiative emission from all regions of the flow field,
and the integro-exponential function, \( E_1 \), is the attenuating factor that
accounts for absorption along the paths leading from the emissive sources
in the flow field to the point in question. It is this integral term
that physically couples each fluid element of the flow field to all
other elements of the flow, thus constituting an example of "action at
a distance." It is also this integral term that increases the math­
ematical complexity of the problem by introducing an integro-differential
equation, namely, equation (20). As usual, increased physical com­
plicity leads to increased mathematical complexity. As a final note
concerning the radiation term in the energy equation, equation (21) can
be specialized for the case of a gray gas, and the result is

\[- \frac{3\partial R}{2x} = -4\pi J + \kappa \int_0^\infty 2\sigma T(3,t) E_1(1r-3) \, dq \]  

(22)
In addition to the above discussion of the radiation term, a very important point is the fact that the pressure gradient, \( \frac{\partial p}{\partial x} \), is very small and can be neglected. This is verified by numerical results obtained from the momentum equation as discussed in Chapter IV. However, it is sufficient to state here that the pressure gradient term, \( u \frac{\partial p}{\partial x} \), in the energy equation is a full order of magnitude less than the next largest term, \( \frac{\partial p}{\partial t} \), which in turn is two orders of magnitude less than the radiation term. Therefore, there is no particular loss in accuracy by ignoring \( \frac{\partial p}{\partial x} \), which will be the case throughout the remainder of the analysis. However, by ignoring \( \frac{\partial p}{\partial x} \), and, therefore, considering pressure to be only a function of time throughout the remainder of the analysis, two considerable simplifications are obtained. First, the numerical analysis, described in Chapter IV, requires fewer calculations at each point in the flow field. However, the most important simplification occurs with the diffusion term. By the combined effect of chemical equilibrium and constant pressure, with respect to \( x \), the concentration gradients can be expressed in terms of the temperature gradients. Consequently, following the results of Hirschfelder,\(^{22}\) and Butler and Brokaw,\(^{23}\) the effects of diffusion can be obtained in terms of a "reaction conductivity" which is added directly to the ordinary thermal conductivity. Hanson\(^{24}\) has calculated values for the total thermal conductivity including diffusion for air, and, as will be discussed in Chapter IV, these values are employed for the present investigation. As a result of the above simplifications, it is straightforward to write the energy equation as
where the Prandtl number, Pr, contains the effect of diffusion. Equation (23) will be the form of the energy equation used throughout the remainder of the present investigation.

The partial differential equations which describe the radiating reflected shock flow field have been formulated above. Now, the appropriate boundary conditions will be stated. They are,

at the end wall \(x = 0\),
\[
T = T_w, \quad u = 0
\]

and immediately behind the reflected shock \(x = x_s\),
\[
\begin{align*}
\rho &= \rho_s, \quad h = h_s \\
T &= T_s, \quad u = u_s \\
p &= p_s
\end{align*}
\]

where \(\rho_s, T_s, p_s, h_s, \) and \(u_s\) are directly obtained from the usual normal shock relations for flow in chemical equilibrium once the reflected wave velocity, \(W_R\), is specified. However, in the present problem, \(W_R\) is initially unknown, and is obtained as part of the solution.

The fact that \(W_R\) is a function of time which must be obtained during the course of the solution implies that the problem is not completely defined by equations (18), (19), and (23) along with the boundary conditions stated above. An additional relation is necessary which links the flow field variations to the change in reflected shock velocity. This
additional relation can be obtained from an overall mass balance between the flow field and the flux of mass through the wave, as follows. At any given time, \( t \), the total mass of gas between the end wall and the shock is \( \int_{0}^{x_s(t)} \rho A dx \) where \( A \) is the flow field cross-sectional area parallel to the shock. On the other hand, the mass flow through the shock is \( \rho^2 (W_R + u_2) A \) where \( \rho, u_2 \) are density and mass motion in the incident shocked region just ahead of the reflected wave. This mass flow must instantaneously equal the time rate of change of the total mass of the flow field, thus

\[
P^2 (W_R + u_2) = \frac{d}{dt} \int_{0}^{x_s(t)} \rho(x, t) \, dx
\]  

(24)

From Leibniz' Rule (states as equations (I-2) and (I-3) in Appendix I), the above differentiation is

\[
\frac{d}{dt} \int_{0}^{x_s(t)} \rho(x, t) \, dx = \rho(x_s, t) \frac{dx_s}{dt} + \int_{0}^{x_s(t)} \frac{\partial \rho(x, t)}{\partial t} \, dx
\]  

(25)

However, by identity, \( \frac{dx_s}{dt} = W_R \) and \( \rho(x_s, t) = \rho_s \), and equation (25) becomes

\[
\frac{d}{dt} \int_{0}^{x_s(t)} \rho(x, t) \, dx = \rho_s W_R + \int_{0}^{x_s(t)} \frac{\partial \rho(x, t)}{\partial t} \, dx
\]  

(26)

Therefore, equation (26) can be substituted into equation (24), resulting in

\[
P^2 (W_R + u_2) = \rho_s W_R + \int_{0}^{x_s(t)} \frac{\partial \rho(x, t)}{\partial t} \, dx
\]  

(27)
It is interesting to note that the above equation also follows directly from an integral of equation (18) together with the conservation of mass across the shock discontinuity. Solving for $W_R$ in equation (27),

$$W_R(t) = \int_0^{\xi(t)} \frac{\partial p}{\partial t} \, dx - \int_{\rho_2}^{\rho_3} u_2 \, dz$$

Equation (28) will be referred to as the continuity condition throughout the remainder of this paper. For the ideal shock case, where $\frac{\partial \rho}{\partial t} = 0$, the continuity condition reduces to

$$\rho_2 (W_R + u_2) = W_R \rho s$$

which is the familiar continuity equation for the ideal reflected shock. On a physical basis, the continuity condition relates the time variation of $W_R$ with the rate of change of density integrated throughout the unsteady flow field.

C. Summary of the formulation

The radiating reflected shock problem has now been completely formulated. It consists of solving the partial differential equations (18), (19), and (23) in conjunction with the boundary conditions stated above and with the continuity condition, equation (28). The important radiative effects on the problem have been qualitatively emphasized, and the simplifying assumptions have been discussed. The above formulation leads to a solution for the detailed flow field properties, which is one of the primary items of interest in the present investigation. In
turn, detailed knowledge of the flow field structure gives the resulting end wall radiative heat transfer, obtained from

\[ \dot{q}_{r_w} = 2m \int_0^\infty B_r(z) E_2(z) \, dz \, dv \]  \hfill (29)

and the end wall conductive heat transfer, obtained from

\[ \dot{q}_{c_w} = \left( \frac{\mu}{A_1} \right) \omega \left( \frac{\partial h}{\partial z} \right) \]  \hfill (30)

Now that the problem has been formulated, it is appropriate to describe the method of solution employed in the present investigation.
CHAPTER IV

A SOLUTION FOR THE RADIATING REFLECTED SHOCK PROBLEM

The radiating reflected shock problem, as formulated in Chapter III, does not lend itself to a straightforward solution due to the integro-differential equations involved and to the time dependent boundary conditions behind the shock, whose velocity variation constitutes one of the unknowns. The radiation term in the energy equation seems to preclude any transformation of the partial differential equations into ordinary equations, as is common in many self-similar fluid dynamic analysis, and there is certainly no existing general solution for the system as formulated earlier. These considerations point to a numerical method as a feasible solution. Therefore, this chapter describes a finite-difference technique for obtaining the structure of the radiating flow field, and discusses several auxiliary matters concerning the quantitative analysis. First, the evaluation of some physical properties of air is mentioned. Then the finite difference solution for the flow field is described, followed by a discussion of the self-similar, non-radiating solution which is necessary to start the finite difference method, and which provides a convenient comparison for the radiating case. In addition, some conduction effects on the formation and initial development of the reflected shock trajectory are discussed. Finally, the actual numerical evaluation of the radiation integrals is presented.
A. Physical properties of air

Throughout the course of the following solution, values for the thermodynamic, transport, and radiative properties of high temperature air are required. In addition, the necessity arises to compute the equilibrium properties behind a reflected shock as it moves into the dissociated and ionized incident shocked region. Evaluation of these properties will now be discussed.

1. Thermodynamic Properties.—The thermodynamic properties of high temperature air as calculated by Hansen, and correlated by Viegas and Howe, are employed in the present analysis. The correlations, which are extremely accurate representations of Hansen’s data, are given by the general form,

\[ a + by + cxy + d \gamma^2 + e_1x + e_2x^2 + e_3x^3 + \ldots + e_nx^n = 0 \]  

where the independent variable, \( x \), is the enthalpy, and the dependent variable, \( y \), is either temperature or density. The coefficients, \( a, b, c, e, e_1, e_2, \ldots, e_n \), are functions of the pressure, and depend upon whether density or temperature is being calculated. In brief, the input to the correlations is enthalpy and pressure, and the output is temperature and density.

Since the original calculations of Hansen were based on the results of statistical thermodynamics, and even though a simplified gas model was used, the accuracy of the thermodynamic properties used in this analysis is considered sufficient (less than two per cent error).

2. Transport Properties.—Viegas and Howe, in the same reference, have also correlated Hansen’s transport property calculations for high
These correlations are of the same form as equation (31), and are used for the present analysis. For this case, the input to the correlations is again enthalpy and pressure, and the output is the product, $\rho u/Pr$, from which the ratio, $\frac{\mu}{Pr}$, pertinent to the present analysis, can be obtained.

The accurate evaluation of transport properties for high temperature air is in a rather tenuous state. Hansen's calculations were some of the earliest, and employed a hard sphere model for molecular collisions. More recent values obtained by Peng and Pindroh, and by Yos, also using a hard sphere model but more recent values for collision cross sections, have shown $\mu/Pr$ to be as much as an order of magnitude smaller than Hansen's calculations. However, all of the above calculations employed certain approximations to simplify the computations. In his recent critical evaluation of existing methods for calculating transport coefficients, Ahtye has compared the above results of both Yos, and Peng and Pindroh, to calculations by the rigorous second-order Chapman-Enskog formulation, and has found that the former values of thermal conductivity (thus $\mu/Pr$) were underestimated by up to 60 percent. Therefore, in light of this present status, Hansen's original calculations are considered to be as valid for the present investigation as some of the more recent work.

3. Radiation Properties.--The present analysis employs the radiative emissive properties of high temperature air over the wavelength range of 0.05 $\mu$ to 10 $\mu$, as calculated by Nardone et al. These radiative properties give values of $J$ as a function of temperature and density, and include contributions from the molecular bands of $N_2$, $O_2$, $O_3$, $N_2^+$, and $O_2^+$.  


NO, and N$^+$, the deionization continuum of O$^+$ and N$^+$, and the Bremsstrahlung continuum of N, N$^+$, N$^{++}$, O, O$^+$, and O$^{++}$. The contribution of atomic lines is not included. The present author has correlated the radiative properties as calculated by Reference 29 for the limited conditions encountered in the present analysis. These correlations are valid for densities approximately equal to one tenth of standard atmospheric density, and are,

for $5000^\circ R < T < 24,300^\circ R$,

$$J = 9160 \left( \frac{P}{\bar{P}} \right)^{0.89} \left[ e^{\left( \frac{2.12}{1050} \right)} \right]^{2.2}$$

and for $24,300^\circ R < T < 29,200^\circ R$

$$J = 19800 \left( \frac{P}{\bar{P}} \right)^{0.89} \left[ (42.1) \left( \frac{T}{1.8} \right) - (100.2 \times 10^{-4}) \left( \frac{T}{1.8} \right)^2 
+ (5.8 \times 10^{-8}) \left( \frac{T}{1.8} \right)^3 - (6.82 \times 10^{-12}) \left( \frac{T}{1.8} \right)^4 \right]$$

where the units of J are $(ft-lb) / (sec)(ft)^3(ster)$ and T is in degrees Rankine. The errors in the correlations range from 30% at the lowest temperature (where radiation is relatively insignificant) to less than 1% over the range from $22,000^\circ R$ to $29,000^\circ R$.

The present knowledge of radiative properties of air occupies about the same tenuous status as the transport properties. A detailed review of this status is beyond the scope of the present paper. However, it is relevant to mention that extensive experimental programs are being
conducted by Nerem,\textsuperscript{20} and by Gruszczynski and Warren,\textsuperscript{30} in order to measure the radiative properties of air, and to put the existing theoretical calculations in proper perspective. Nerem has shown reasonable agreement with Nardone \textit{et al.} for the wave length range from 0.17\(\mu\) to 6\(\mu\). On the other hand, Reference 30 has obtained measurements extending to 880 \(\AA\) in the vacuum-ultra-violet, and has found agreement with the theoretical calculations of Biberman \textit{et al.}\textsuperscript{31} which include the effect of atomic lines. The radiation properties used in the present investigation fall about half way between the measurements of References 20 and 30, and are considered sufficient for the purposes of the present study.

4. \textbf{Normal Shock Properties}.—Since there are no existing general tabulations for equilibrium properties behind normal shocks moving into initially dissociated and ionized regions, the properties immediately behind the time dependent reflected shock in the present analysis have to be obtained from the general normal shock equations. These equations are

\textbf{Continuity:} \[ f_2 (W_R + u_2) = f_s (W_R + u_s) \]

\textbf{Momentum:} \[ p_2 + f_2 (W_R + u_2) = p_s + f_s (W_R + u_s)^2 \]

\textbf{Energy:} \[ h_2 + \frac{1}{2} (W_R + u_2)^2 = h_s + \frac{1}{2} (W_R + u_s)^2 \]

where the subscripts 2 and s denote conditions ahead of and behind the reflected shock respectively, and where \(W_R\) and \(u\) denote reflected wave velocity and mass motion respectively, both measured relative to the end
wall. Note that, corresponding to the actual physical case, the direction of the mass motion, $u_2$ and $u_s$, is toward the end wall in the above normal shock equations, and, therefore, the value of $u_s$ must be interpreted as a negative number when used in the reference frame indicated in Figure 5, where $x$ is increasing away from the wall. A solution of these equations, employing the thermodynamic properties of high temperature air mentioned above, has been obtained by the present author through the use of an iterative procedure, and has been programed as a computer subroutine for use in the subsequent numerical analysis. For the given initial conditions, $p_2$, $\rho_2$, $u_2$, and $h_2$, and for a specified wave velocity, $W_R$, the solution produces equilibrium values of $p_s$, $h_s$, $\rho_s$, $u_s$, and $T_s$ immediately behind the shock. The accuracy of this solution is as good as the thermodynamic properties which it employs.

B. Finite difference solution for the radiating case

As mentioned previously, the inherent complexity that characterizes the radiating reflected shock problem and its mathematical formulation strongly suggests a numerical solution of the relevant partial differential equations. Such a numerical solution, namely, a finite difference technique, will now be discussed and applied to the present problem.

The general idea of the finite difference approach is to replace partial derivatives by algebraic difference quotients. This results in a system of algebraic equations which replaces the partial differential equations, and allows for the step-by-step simultaneous calculation of the dependent variables. The accuracy and convergence of such finite difference solutions are influenced by the choice of step size, and the solution is, in principle, exact in the limit of zero increments. The
obvious advantage of a finite difference solution is that it provides an evaluation of the problem in cases where general analytical solutions are difficult, if not impossible, to find. In turn, a disadvantage is that it does not directly provide concise formulas which describe the governing physical parameters, even though the numerical results may sometimes lead to useful physical correlations. In addition, the execution of a finite difference technique is usually tedious, and is impractical if done by hand. However, the recent wide-spread use of high speed computers has made the application of finite difference methods quite feasible.

It is beyond the scope of this paper to discuss the many aspects of finite difference techniques. However, several authoritative presentations can be found in the literature. The particular numerical solution for the present study will now be described.

The partial differential equations which describe the present physical problem are given by equations (18), (19), and (23). If the following dimensionless variables are defined, where the subscript $si$ denotes conditions immediately behind an ideal reflected shock, and where $L$ is a characteristic length,

\[
\begin{align*}
\rho^* &= \frac{\rho}{\rho_{si}} \\
K^* &= \frac{K}{K_{si}} \\
x^* &= \frac{x}{L} \\
T^* &= \frac{T}{T_{si}} \\
h^* &= \frac{h}{h_{si}} \\
u^* &= \frac{u}{w_{Rsi}} \\
t^* &= \frac{t}{\frac{W_{Rsi}}{L}} \\
(\mu/Pr)^* &= \frac{\mu/Pr}{(\mu/Pr)_{si}}
\end{align*}
\]
then the above equations are transformed into the following non-dimensional relations:

$$\frac{\partial P^*}{\partial x^*} + \frac{\partial (P^* \dot{u}^*)}{\partial x^*} = 0$$

(32)

$$\rho^* \frac{\partial \dot{u}^*}{\partial x^*} + \rho^* \dot{u}^* \frac{\partial \dot{u}^*}{\partial x^*} = - \left[ \frac{\rho_{si}}{\rho_{si} (W_{rsi})^2} \right] \frac{\partial P^*}{\partial x^*}$$

(33)

$$\rho^* \frac{\partial h^*}{\partial x^*} + \rho^* \dot{u}^* \frac{\partial h^*}{\partial x^*} = \left[ \frac{(\mu/P_{si}) \dot{T}_{si}}{P_{si} W_{rsi} L} \right] \frac{\partial}{\partial x^*} \left[ (\mu/P_{si}) \frac{\partial h^*}{\partial x^*} \right]$$

$$+ \Gamma_{CH} \left[ -\kappa^* T^* \dot{T}^* + \frac{\kappa^*}{2} \int_0^{T_{si}} T^* \dot{T}^* \left[ 1 - (1 - \tau_A) \right] d\tau \right]$$

$$+ \left( \frac{P_{si}}{P_{si} h_{si}} \right) \left( \frac{\partial P^*}{\partial x^*} + \dot{u}^* \frac{\partial P^*}{\partial x^*} \right)$$

(34)

For purposes of illustration, the radiation terms in equation (34) are written for a gray gas. Also, $\Gamma_{CH}$ is a characteristic radiation loss parameter, defined as

$$\Gamma_{CH} = \frac{4 \kappa_{si} \sigma T_{si}^4 L}{\rho_{si} W_{rsi} h_{si}}$$

(35)

The above expression for $\Gamma_{CH}$ is one of the radiation gas dynamic similarity parameters discussed by Goulard. If the optical thicknesses, $\tau$ and $z$, in equation (34) would have been based on a reference value, $\tau_{ref} = K_{ref} L$, then $\tau_{ref}$ would have explicitly appeared as a second similarity parameter. These similarity parameters will be discussed in detail in Chapter V.
If the x-t plane is divided into a network of horizontal and vertical lines of spacing $\Delta t$ and $\Delta x$ respectively as shown in Figure 6, the partial derivatives considered above can be replaced by finite difference expressions such as $\frac{\partial h}{\partial x} \approx \frac{h(x + \Delta x, t) - h(x, t)}{\Delta x}$ and $\frac{\partial^2 h}{\partial x^2} \approx \frac{h(x + \Delta x, t) - 2h(x, t) + h(x - \Delta x, t)}{(\Delta x)^2}$, with similar expressions for the time derivatives. Applying this technique to equation (34), and neglecting the term $u^* \frac{\partial p^*}{\partial x^*}$ for reasons stated earlier, the non-dimensional finite difference energy equation results, after some rearrangement, in the following form:

$$
\begin{align*}
    h_d^* &= h_b^* + \frac{A_t^*}{\Delta x^*} u_b^* (h_c^* - h_c^*) + \\
    &\quad \frac{A_t^*}{(\Delta x^*)^2} \left( \frac{1}{f_b^*} \right) \left[ \left( \frac{\mu/\rho_t}{s_i} \right) \right] \left( \frac{\mu}{\rho_t} \right)_b (h_c^* - 2h_b^* + h_d^*) + \\
    &\quad (h_c^* - h_b^*) \left[ \left( \frac{\mu}{\rho_t} \right)_c^* - \left( \frac{\mu}{\rho_t} \right)_b \right] + \\
    &\quad \frac{1}{\Delta t} \left( \frac{\Delta t^*}{f_b^*} \right) \left( -\kappa_b^* T_b^* \right) + \frac{K_b^*}{2} \int_0^{T_s} T^* E_i (1T - g) \, dz + \\
    &\quad \left( \frac{\rho_i}{\rho_t} \right) \left( \frac{\rho_c^* - \rho_b^*}{\rho_c^*} \right)
\end{align*}
$$

(36)

For simplicity, the subscripts a,b,c, and d refer to the lattice points of the grid shown in Figure 6. The finite difference continuity equation is obtained from equation (32), and results in
Fig. 6.--Elementary finite difference network
The assumption, $\frac{\partial p}{\partial x} = 0$, which was discussed in Chapter III, essentially takes the place of the momentum equation in the numerical solution, and, therefore, a finite difference expression for the momentum equation is not absolutely necessary. However, the momentum equation is used in conjunction with the velocities obtained from equation (37) to calculate representative values of $\frac{\partial p}{\partial x}$. The results of this calculation indicate that $\frac{\partial p}{\partial x}$ is indeed negligible, thus, serving to justify the original assumption, $\frac{\partial p}{\partial x} = 0$.

Now that the finite difference equations have been stated, the steps required for the numerical solution will be discussed. These steps are influenced by two considerations. First, the finite difference solution must begin with a known flow field at some initial value of time, $t$, hereafter denoted as the starting conditions. These starting conditions are obtained from a self-similar analysis for the non-radiating flow field as discussed in a subsequent section. In addition, the fact that $\mathcal{W}_R$ and pressure are unknown functions of time requires an iterative procedure throughout the course of the solution.

With these considerations in mind, the radiating flow field behind the reflected shock can be constructed from one value of time, $t_1$, to the next value, $t_1 + \Delta t$, by means of equations (36) and (37). This step-by-step procedure, illustrated by the grid shown on the x-t diagram in Figure 7, is briefly outlined below.
Fig. 7.—A representative x-t diagram for the radiating reflected shock analysis showing the grid and starting conditions
1. Assume, by some means, that all the flow field variables are known as a function of $x$ at some value of time, $t_1$. Then, all terms on the right hand side of equation (36) can be evaluated except $p_d^*$, which is the pressure at time, $t_1 + \Delta t$.

2. The unknown value of $p_d^*$ can be initially estimated by assuming the value of $W_R$ at time, $t_1 + \Delta t$, and calculating the pressure behind the reflected shock according to the normal shock method discussed previously.

3. With the above estimated value of $p_d^*$, and with the known flow field properties at time, $t_1$, the values of $h_d^*$ as a function of $x$ at time, $t_1 + \Delta t$, are directly obtained from equation (36).

4. From the values of $p_d^*$ and $h_d^*$ obtained above, values for temperature and density at the corresponding points are calculated from the thermodynamic correlations discussed earlier.

5. With values of density now estimated as functions of $x$ for the time, $t_1 + \Delta t$, as well as for time, $t_1$, the continuity condition expressed as equation (28) can be evaluated numerically, resulting in a value of $W_R$ at time, $t_1 + \Delta t$. This value for $W_R$ will in general be different than the value initially assumed in step (2).

6. Using this new value for $W_R^*$, again calculate a new value of $p_d^*$ in step (2), and repeat the procedure described above to obtain new values of enthalpy, density and temperature as functions of $x$ for time, $t_1 + \Delta t$.

7. Repeat the above iteration, steps (2) through (6), until agreement is obtained for $W_R$ between the final iteration and the one immediately preceding it. The values obtained for $W_R^*$, $h$, $T$, $\rho$ and $p$ during this final iteration are the final values at time, $t_1 + \Delta t$. 

8. With the final results obtained from the previous step, the right hand side of equation (37) is completely determined and yields the velocity, $u_d^*$, as a function of $x$ for time, $t_1 + \Delta t$. This step completes the solution of the flow field for time, $t_1 + \Delta t$, and provides the initial values to be used in step (1) in order to repeat the entire procedure for the subsequent increment in time.

The above eight steps describe the numerical method developed for solving the entire radiating reflected shock flow field as a function of $x$ and $t$. However, it is obvious that the technique must start from some initially known flow field at some initial value of time. This starting condition will now be discussed, not only because of its relevance to the above technique, but also because of its importance as a solution to a somewhat independent problem, namely, the non-radiating and conducting reflected shock problem.

C. Self-similar solution for the non-radiating case

During the initial portion of the reflected shock wave trajectory, where the distance between the end wall and the shock is very small, the influence of conduction should overshadow that of radiation. This suggests that the starting conditions at some early initial time can be obtained by solving the reflected shock problem with conduction only, neglecting the effect of radiation. Fortunately, the equations for the non-radiating case lend themselves to a self-similar solution, as shown by Fay and Kemp.\textsuperscript{37} This case will now be described.

The physical picture of the non-radiating reflected shock flow field is essentially that of a constant property inviscid region in contact with a thermal boundary layer adjacent to the end wall. The
inviscid region is determined by ideal reflected shock properties. The equations which describe this flow field, assuming constant pressure with respect to distance and time, are:

\[
\begin{align*}
\text{Continuity} & \quad \frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0 \\
\text{Energy} & \quad \rho \frac{\partial h}{\partial t} + \rho u \frac{\partial h}{\partial x} = \frac{\partial}{\partial x} \left[ \left( \frac{\mu}{\rho c_s} \right) \frac{\partial h}{\partial x} \right]
\end{align*}
\]

This system of partial differential equations, where \( x \) and \( t \) are independent variables, can be transformed into one ordinary differential equation by means of the similarity variable \( \eta \), where

\[
\eta = \eta(x, t) = \left[ \frac{(P_e/\mu) e^{-x/2}}{2P_e t} \right]^{1/2} \int_0^x \rho \, dx
\]

The subscript, \( e \), denotes conditions at the edge of the boundary layer. The transformation is rather long and complex, and because it is not derived (only the results presented) in the literature, the derivation is presented in Appendix II.

By means of \( \eta \), defined above, the transformation yields an expression for the convective operator,

\[
\rho \frac{\partial}{\partial t} + \rho u \frac{\partial}{\partial x} = - \frac{\rho \eta}{2t} \frac{\partial}{\partial \eta}
\]
which in turn reduces the energy equation to

\[ \eta \frac{dh}{d\eta} + \frac{d}{d\eta} \left[ \left( \frac{\mu}{\rho_e} \right) \rho_e \frac{dh}{d\eta} \right] = 0 \]  \tag{37}  

Equation (37) is an ordinary differential equation for enthalpy as a function of \( \eta \), which itself is a function of time and distance. This second order equation is solved by reducing it to two first order equations, which are then simultaneously solved by the very accurate Runge-Kutta numerical method.\(^{36}\)

The results of the self-similar non-radiating solution described above are pertinent to the present investigation for two reasons. First, they provide the initial flow field profiles for the necessary starting condition described earlier. These initial profiles are evaluated for time, \( t = 10^6 \) sec, which is considered early enough such that radiation effects are quite small. In addition, the self-similar results provide an expression for end wall conductive heat transfer as a function of time for the non-radiating case. This heat transfer can be obtained from

\[ \dot{Q}_{cw}^{AD} = \left( \frac{\mu}{\rho_e} \right)_{e} \left[ \frac{(P_e/\mu)_{e}}{2 \rho_e T} \right]^{1/2} \rho_e \left( \frac{\partial h}{\partial \eta} \right)_{e} \]  \tag{38}  

where the subscripts AD and w imply conditions for non-radiating flow (flow which is adiabatic with respect to radiation) and conditions at the wall, respectively. The results obtained from equation (38) are used to assess the integrity of the finite difference solution, as will be described in Chapter V.
Fay and Kemp\textsuperscript{37} have solved the end wall conductive heat transfer problem for an ionized monatomic gas, and have presented results for argon. The self-similar results included in the present analysis are the first reported for high temperature dissociated and partially ionized air.

In conjunction with the starting condition, it is relevant to mention that several investigators have studied the conduction effect on the formation and early trajectory of a reflected shock wave. Theoretical investigations by Goldsworthy,\textsuperscript{38} and experimental measurements by Sturtevant et al.,\textsuperscript{39} and by Baganoff,\textsuperscript{40} have indicated a strong perturbation on the reflected shock velocity due to conduction to the end wall. This perturbation is essentially the result of the negative displacement thickness caused by the large increase in density in the end wall thermal boundary layer. However, this strong effect diminishes rapidly with time, and is consequently limited to very early times after reflection. Calculation of this perturbation effect on the starting condition for the present investigation shows that, for $t = 1\mu$ sec, the only noticeable influence is a slight displacement in shock location compared to that obtained from ideal shock tables. This displacement is taken into account for the present numerical analysis.

D. Evaluation of the radiation integrals and heat transfer

The previous sections have discussed the important aspects of the main numerical solution for the radiating reflected shock problem. However, there are several auxiliary considerations which support and are consequences of the main solution. These considerations will now be discussed.
The first consideration deals with the evaluation of the radiation term in the energy equation, as expressed by equation (21) for a general non-gray gas, and by equation (22) for a gray gas. These expressions involve the integro-exponential function of order one, $E_1(z)$, which has a singularity at $z = 0$. This singularity complicates the numerical evaluation of the radiation integral, and, therefore, prompts some modification of the expression. Such a modification is discussed in Reference 10, and when applied to the radiation term in the finite difference energy equation, equation (36), the result is, for a gray gas,

$$
\Gamma_{CH} \left( \frac{A_f^*}{\rho_s^*} \right) \left( \frac{k_{s}^*}{2} \right) \left\{ -T_s^{\#} E_2 (\tau_e - \tau) \right. \\
- \sum_{i=0}^{I} \left( \frac{T_s^{\#} - T_i^{\#}}{\Delta \tau} \right) \left[ E_3 (1 \tau - \bar{3}_{i+1}) - E_3 (1 \tau - \bar{3}_i) \right] \right\}
$$

The above expression is the complete radiation term, including both emission and absorption for a gray gas. It is basically derived by replacing the curve of $T_s^\#$ with a series of $I + 1$ straight line segments, and integrating over each segment individually, employing the identity

$$
E_m (\bar{3}) \Delta \bar{3} = -E_{m+1} (\bar{3})
$$

No derivation of the modification mentioned above is given in Reference 10. As a result, the present author has carried out the lengthy derivation in detail, and an outline of the necessary steps is given in Appendix III. Thus, the finite series expression for the radiation term given above has certainly been verified, and is a reasonable evaluation.
of the radiation term as long as the segmented temperature distribution mentioned above is an accurate representation of the actual distribution. For the case of a non-gray gas, a single step model for absorption coefficient is assumed, and is shown schematically in Figure 15 (Chapter V). This model is a crude representation of the wave length variation of absorption coefficient for high temperature air, and is similar to that employed by Olstad$^{12}$ except for the magnitude and wave length location of the step. The step at 1100 Å is chosen to correspond with data presented by Hoshizaki and Wilson.$^{13}$ The magnitude of the step is chosen such that the integrated local radiative emission corresponds to the total emissive properties of Nardone et al. used for the transparent and gray gas cases. The variation in absorption coefficient is given quantitatively by $\alpha_1 K_p$ from 0 to 1100 Å and by $\alpha_2 K_p$ from 1100 Å and above, where $\alpha_1$ and $\alpha_2$ are 20.9 and 0.26, respectively. In a manner similar to that of the gray gas described above, the radiation term in the finite difference energy equation, equation 36, is obtained, for a non-gray gas, as

$$
-2\pi \frac{\Delta \tau}{P_0} \left[ \frac{L}{P_0 \cdot W_{0,i} \cdot h_{0,i}} \right] K_P \sum_{i} \Delta \lambda_i \left\{ B_i \left( \tau_{0,i} \right) E_2 \left( \tau_{0,i} - \tau_{i} \right) \right\} \\
+ \sum_{j=0}^{j=I} \left[ \frac{B_{i,j+1} - B_{i,j}}{\Delta \lambda} \right] \left[ E_3 \left( \tau_{i} - \tau_{j+1} \right) - E_3 \left( \tau_{i} - \tau_{j} \right) \right]
$$

where

$$
B_i = \int_{\lambda_i}^{\lambda_{i+1}} B_\lambda \, d\lambda
$$
and $\tau_i$ is the optical depth based on the absorption coefficient, $\alpha_{iP}$. Additional description of this type of step model for the absorption coefficient can be obtained from Reference 12.

In addition to the radiation term in the energy equation discussed above, the end wall radiative heat transfer expression, equation (29), must be evaluated. However, this evaluation is straightforward since no singularities appear in the $E_2$ function, and the integral is calculated numerically by employing Simpson's Rule.

As a final consideration, the end wall conductive heat transfer is evaluated from equation (30), where the enthalpy gradient at the wall is obtained from the solution for flow field structure, as discussed earlier. However, in order to obtain an accurate value for the enthalpy slope at the wall, the region adjacent to the end wall, but larger than the thermal boundary layer, is subdivided into a finite difference net which is twenty-five times finer than the grid employed for the bulk of the flow field. This region is coupled to the main network, and its properties are calculated simultaneously with the rest of the flow field in a manner identical to the finite difference technique previously described.

E. Summary and additional comments

The previous sections have described in some detail the method of solution for the high temperature radiating and conducting flow field behind a reflected shock, which was formulated in Chapter III. The complexity of the mathematical formulation leads to a numerical solution which is based on the finite difference technique, and which requires a formidable iteration procedure. In addition, a self-similar solution
for the non-radiating reflected shock problem is described. This solution provides necessary information in several respects for the general radiating reflected shock case.

At this point, it is relevant to note that the present author has coded the numerical analysis described in this chapter for solution on an IBM 7094 computer, using the SCATRAN algebraic computer language. The resulting program is quite long in respect to both number of statements (approximately 600 for the non-gray case), and to execution time on the computer. As an example, representative execution times to carry the flow field thickness out to 15.7 mm for several different cases involving one incident shock condition are listed as follows:

1. Self-similar solution - 0.3 min.
2. Transparent radiating case - 18.5 min.
3. Self-absorbing gray gas case - 52.3 min.

For the non-gray case, the program efficiency was considerably improved by utilizing more of the computer memory storage, and as a result, the non-gray execution time was 50 minutes. A similar improvement could be employed to reduce the execution time for the gray case below the figure listed above.

Now that the radiating reflected shock problem has been described and formulated, and the method of solution outlined, it is appropriate to present and discuss the results.
CHAPTER V

DISCUSSION OF RESULTS

The results for the present study of the radiative emission and self-absorption effects on the high temperature flow field behind a reflected shock are discussed in the following sections. In most instances, they are obtained for the conditions created by an incident shock which has a velocity of 29,000 ft/sec and which is moving into a region where the initial pressure is 1mm of mercury. In addition, some data are also presented for the conditions due to a 25,000 ft/sec incident shock velocity. However, before discussing these results, it is appropriate to mention that the integrity and accuracy of the numerical solution for the radiating reflected shock problem can be checked and verified by applying it to a known solution. The known solution is for an ideal case where the flow field experiences no radiative cooling or self-absorption. For this application, the radiation term in the finite difference energy equation is deleted, but all other aspects of the method, such as variable $W_R$ and the iteration procedure, are maintained. Therefore, flow field structure as well as end wall radiative and conductive heat transfer are calculated by the finite difference program without the physical influence of radiative cooling or self-absorption. Some of the above results are given in Figures 8 and 9, which compare the finite difference solution with the results obtained from ideal calculations. Figure 8 shows the
RADIATIVE HEAT TRANSFERS \(\text{watt/cm} \times 10^{-5}\)

\[W_1 = 29,000 \text{ ft/sec}\]
\[P_1 = 1 \text{ mm Hg}\]

IDEAL \(\dot{Q}_R = 2.065 \times 10^4 t\)

FINITE DIFFERENCE SOLUTION

IDEAL CONSTANT PROPERTY SOLUTION

Fig. 8.--Comparison of the finite difference solution with the semi-infinite slab approximation for end wall radiative heat transfer without radiative cooling or self-absorption
Fig. 9.—Comparison of the finite difference solution with the self-similar results for end wall conductive heat transfer without radiative cooling or self-absorption.
end wall radiative heat transfer, where the ideal calculation is that of a constant property, transparent, radiating gas behind the reflected shock. For such an ideal model, the radiative heat transfer is given by the semi-infinite slab approximation \(4,10\) and has the form,

\[
\hat{Q}_w = \frac{E X_s}{2}
\]  

(39)

Here, \(E = 4\pi J\), which is the energy radiated per unit volume and time, and \(X_s\) is the distance between the end wall and the reflected shock.

For the condition of incident shock velocity, \(W_1 = 29,000\) ft/sec, equation (39) becomes

\[
\hat{Q}_w = 2.065 \times 10^4 \hat{t}
\]

where \(\hat{Q}_w\) is in watt/cm\(^2\), and \(\hat{t}\) is in \(\mu\)sec. From Figure 8, it can be seen that the finite difference solution accurately reproduces the ideal case. In addition, Figure 9 shows the end wall conductive heat transfer, where the ideal calculation is the self-similar solution for the non-radiating thermal boundary layer, as discussed in Chapter IV.

For \(W_1 = 29,000\) ft/sec, the self-similar result for conductive heat transfer is

\[
\hat{Q}_w = 6.08 \times 10^4 \hat{t}^{-\frac{1}{2}}
\]

where, again, \(\hat{Q}_w\) is in watt/cm\(^2\), and \(\hat{t}\) is in \(\mu\)sec. As seen from Figure 9, the finite difference solution again accurately reproduces the ideal case. In fact, the non-radiating solution obtained from the present finite difference technique reproduces the ideal case in every
respect, such as enthalpy, density and temperature distributions, and wave velocity. The above comparisons provide a very convenient check on the validity and accuracy of the present numerical solution.

The following results for the radiating reflected shock problem are organized into three main categories. First, the results pertaining to detailed flow field structure are discussed, followed by those for radiative and conductive end wall heat transfer. The cases of a transparent, gray and non-gray gas are treated for each of the above categories. Finally, an analogy is drawn between the present unsteady reflected shock problem and the steady shock layer near the stagnation point on a blunt body.

A. Flow field structure

Figure 10 shows several enthalpy profiles corresponding to different values of time after the shock reflects from the end wall for a transparent gas and $W_1 = 29,000$ ft/sec. These profiles markedly demonstrate the strong effect of radiative cooling, especially for larger values of time. For this case, the temperature immediately behind the reflected shock is on the order of $16,200^\circ$K, producing a local energy loss by radiation of $0.336 \times 10^5$ watt/cm$^3$. As a result, the radiation--gas dynamic coupling is quite predominant. On the other hand, the enthalpy profiles shown in Figure 11 for the case of $W_1 = 25,000$ ft/sec are only slightly influenced by radiative cooling. For this shock velocity, the temperature immediately behind the reflected shock is on the order of $13,500^\circ$K, producing a local energy loss by radiation of $0.49 \times 10^4$ watt/cm$^3$. The enthalpy profiles in Figure 11 are shown for the sake of comparison with those in Figure 10 in order to demonstrate
Fig. 10.—Static enthalpy profiles for several reflected shock positions in a transparent gas, \( W_1 = 29,000 \text{ ft/sec} \).
Fig. 11.—Static enthalpy profiles for several reflected shock positions in a transparent gas, \( W_1 = 25,000 \) ft/sec

- \( W_1 = 25,000 \) ft/sec
- \( P_1 = 1 \) mm Hg
- \( h_{si} = 0.674 \times 10^{-9} \) ft lb/slug

**Transparent Gas**

Distance from end wall, \( X \) in mm

Dimensionless enthalpy, \( h/h_{si} \)
the "onset" nature of radiation effects. That is, since the radiative emission varies approximately as the fourth power of temperature, the influence of radiation on a gas dynamic flow field grows rapidly with temperature above a certain onset level.

The similarity parameter which governs the above profiles for a transparent gas is the radiation loss parameter, $\Gamma$. The physical significance of $\Gamma$ can easily be seen by considering the ratio of radiative energy lost per unit volume by a fluid element over a period of time to its initial enthalpy content at the onset of radiation. In terms of the reflected shock problem, this ratio is

$$
\frac{\int \nabla \cdot \overline{Q_R} \, dt}{\rho^2 \ h_5}
$$

A very useful characteristic value of this ratio for a transparent gas is obtained by assuming that the fluid element loses energy by radiation at a constant rate. This characteristic ratio is the radiation loss parameter, $\Gamma$ where

$$
\Gamma = \frac{E \ x}{\rho^3 \ h_5} \quad (40)
$$

$E$ is the constant rate of radiative emission per unit volume and time. Goulard, among others, has shown that $\Gamma$, evaluated at a suitable reference condition for the particular problem involved, is a similarity parameter in radiation gas dynamics. This is easily shown by casting the energy equation in the suitable non-dimensional form. Using the initial conditions immediately behind the reflected shock, at $t = 1$
\( \mu \text{sec} \) as a reference conditions for the present reflected shock problem, the radiation loss parameter becomes

\[
\Gamma = \frac{E_s t}{k_s h_s} \tag{41}
\]

where \( t \) is the time after reflection. Note that the flow fields for different positions of the reflected shock are identified with different values of \( \Gamma \) because of the variable time, \( t \). For example, for the case of \( W_1 = 29,000 \, \text{ft/sec} \), \( \Gamma = 0.0213 \, t \), where \( t \) is in \( \mu \text{sec} \). Thus, for the flow field at \( t = 4.22 \, \mu \text{sec} \), \( \Gamma = 0.09 \); for \( t = 8.42 \, \mu \text{sec} \), \( \Gamma = 0.18 \); and for \( t = 12.62 \, \mu \text{sec} \), \( \Gamma = 0.27 \). These values of \( \Gamma \) apply to the three profiles shown in Figure 10. Obviously, \( \Gamma \) is a convenient index for the radiative cooling effect; as \( \Gamma \) increases, more energy is removed from the flow field by radiation, as is graphically illustrated in Figure 10. At this point, it is relevant to mention that an alternate physical interpretation of \( \Gamma \) can be made in terms of overall radiative flux out of the flow field versus the energy flux convected by mass motion into the flow field. However, the earlier physical interpretation seems slightly more basic to the present reflected shock problem due to the nature of its unsteady wave motion.

With the above description of \( \Gamma \) in mind, and with the knowledge that \( \Gamma \) is a similarity parameter for the flow, it is expected that all the enthalpy profiles given in Figures 10 and 11 can be correlated by the radiation loss parameter given by equation (41). The present author has made such a correlation, which is shown in Figure 12. Here, all the enthalpy profiles from Figures 10 and 11 are folded into one
RADIATION LOSS PARAMETER

\[ \Gamma = \frac{E_{sl}}{P_{sl} h_{sl}} \]

Fig. 12.—Correlation of enthalpy profiles for a transparent gas
curve which gives \( \frac{\Gamma}{(1-h^*)^{1.15}} \) as a function of relative location, \( x/x_s \), behind the shock. In the above expression, \( h^* \) is the same dimensionless enthalpy defined in Chapter IV, namely, \( h^* = h/h_{si} \). The exponent in the denominator of the above expression accounts for the non-uniform rate of radiative energy loss as the temperature of a fluid element decreases.

The correlation shown in Figure 12 has two purposes. First, it demonstrates the validity and usefulness of \( \Gamma \) as a similarity parameter. However, in addition, it provides a curve from which the flow field enthalpy profiles can be calculated for various reflected shock positions and for various incident shock velocities.

One intention of the present investigation is to study the influence of self-absorption as well as radiative emission. First, the gray gas case will be discussed, followed by the results for the non-gray gas. In conjunction with the present discussion of flow field structure, Figure 13 shows the enthalpy profiles for a self absorbing gray gas in comparison to those for a transparent gas shown earlier. The effect of a gray gas on the enthalpy profiles is somewhat mild but accumulative for the present conditions, and its influence is certainly noticeable at larger times. Of course, the physical consequence of self-absorption, whether the gas is gray or non-gray, is the containment of some of the radiative energy which would otherwise be lost from a transparent flow. This trend is obvious from Figure 13 which shows the gray gas enthalpy distributions higher than those for the transparent case.
DIMENSIONLESS ENTHALPY, \( h / h_s \).

\[ t = 8.42 \mu \text{sec} \quad t = 12.62 \mu \text{sec} \]

TRANSPARENT GAS
GRAY GAS

\( W_1 = 29,000 \, \text{ft/sec} \)
\( P_1 = 1 \, \text{mm Hg} \)
\( h_s = 0.902 \times 10^9 \, \text{ft lb/slug} \)

Fig. 13.—Comparison between gray gas and transparent enthalpy profiles for several reflected shock positions, \( W_1 = 29,000 \, \text{ft/sec} \).
The similarity parameter that governs self absorption effects on the flow field is the characteristic optical thickness, which is nothing more than the optical length defined in Chapter II, based on some characteristic dimension. This characteristic optical thickness is the second similarity parameter in radiation gas dynamics, and is a consequence of the absorption integral term in the energy equation. It is basically an index for the magnitude of the self-absorption influence on a gas dynamic flow field; as the characteristic optical thickness increases, the effect of self-absorption increases. For the present reflected shock analysis, this similarity parameter is denoted by $\tau^o$, and is defined as

$$\tau^o = K_s \cdot X_s$$  \hspace{1cm} (42)$$

where $K_s$ is the Planck mean absorption coefficient based on the initial starting conditions. Note that, analogous to $\Gamma$, flow fields for different positions of the reflected shock are identified with different values of $\tau^o$ because of the variable characteristic distance, $X_s$. For example, for the case of $W_1 = 29,000$ ft/sec, $\tau^o = 0.212 \cdot X_s$, where $X_s$ is in centimeters. Thus, for the flow field at $t = 4.22$ usec, $\tau^o = 0.109$; for $t = 8.43$ usec, $\tau^o = 0.216$; and for $t = 12.61$ usec, $\tau^o = 0.322$. These values of $\tau^o$ apply to the three gray gas enthalpy profiles shown in Figure 13.

Since both $\Gamma$ and $\tau^o$ are the governing similarity parameters for a flow field which is both emitting and absorbing radiation, it is expected that the gray gas enthalpy profiles in Figure 13 can be correlated by these parameters, defined by equations (41) and (42).
Such a correlation has been made by the present author, and is presented in Figure 14. Here, all the gray gas enthalpy profiles from Figure 13 are folded into one curve which gives \( \frac{e^{(-0.518\tau_0)\Gamma}}{(1-\tau)1.18} \) as a function of \( x/X_g \). This correlation demonstrates the validity and usefulness of both \( \Gamma \) and \( \tau_0 \) as similarity parameters for radiating and self-absorbing flow fields.

Comparison of the correlation curves in Figures 12 and 14 brings out an interesting point. These two curves are essentially the same thus, indicating for a gray gas that the expression, \( e^{(-0.518\tau_0)\Gamma} \) acts like an "effective gamma" for radiative cooling. That is, the radiative cooling effect on the enthalpy profiles for a gray gas is governed by an effective radiation loss parameter which is smaller than the transparent case by the factor, \( e^{(-0.518\tau_0)\Gamma} \). Of course, this is compatible with the physical fact that, for the same conditions of temperature and density, a gray gas will experience a smaller rate of loss of energy by radiation than its transparent counterpart. Note that the "effective gamma," \( e^{(-0.518\tau_0)\Gamma} \) approaches \( \Gamma \) in the limit of transparent gases (\( \tau_0 \to 0 \)).

Even though considerable discussion has been devoted above to the self-absorption effects of a gray gas, the more important and realistic case is that of a non-gray gas. The non-gray self-absorption effect on the enthalpy profile for a given flow field thickness is shown in Figure 15 for convenient reference. In addition, the gray and transparent results are shown for the sake of comparison. The striking result of this comparison is that the non-gray profile is considerably above the corresponding profiles for the gray and
Fig. 14.—Correlation of enthalpy profiles for a gray gas

\[ \gamma_0 = k_{sl} X_s \]

\[ e^{(-0.518 X_h)} \frac{\Gamma}{(1-h^+)^{1.18}} \]

\[ x/x_s \]
Fig. 15.—Comparison of the non-gray, gray and transparent enthalpy profiles for a reflected shock location of approximately 15.7 millimeters from the end wall.
transparent cases. Obviously, the non-gray gas physically traps within the flow field a large portion of the radiative energy which would otherwise be lost from a transparent or even a gray gas. This result is a consequence of the very strong self-absorption in the vacuum-ultraviolet wave length region. At this point, it is emphasized that both the non-gray and gray gas models are employed in an effort to account for the effect of self-absorption on the flow field variables in contradistinction to the purely transparent case. It is rather apparent that the gray gas model is unsuccessful in accounting for the effect of self-absorption on the enthalpy profile, and that the strong absorption in the vacuum ultra-violet demands the use of some type of non-gray model. Further evidence supporting this conclusion will be shown in relation to the subsequent end wall heat transfer results.

As an additional aspect of flow field structure, it is interesting to note that, whereas the enthalpy profiles are strongly influenced by radiative cooling, the reflected shock velocity is relatively insensitive to radiation effects. This result is shown in Figure 16, which is a distance-time plot of the reflected shock trajectory for a transparent gas. It can be seen that the shock velocity is only slightly attenuated, amounting to only a 6.7% reduction for the transparent case at the most extreme condition treated by the present analysis. As expected from physical considerations, the attenuation for the self-absorption cases is even less, amounting to maximum reductions of 4.9% for the gray gas and 1.25% for the non-gray gas.
$W_l = 29000 \text{ ft/sec}$

$W_{rl} = 4013 \text{ ft/sec}$

*Fig. 16.* An $x$-$t$ diagram for the reflected shock in a transparent gas
B. End wall heat transfer

The total heat transfer to the end wall from the high temperature reflected shock flow field is the sum of radiative and conductive contributions. First, the results for the radiative contribution will be discussed, followed by those for the conductive contribution.

Figure 17 shows the results for end wall radiative heat transfer for a transparent gas as influenced by radiative cooling of the flow field. For the sake of comparison, the radiative heat transfer without radiative cooling (obtained from equation (39)) is also shown. The strong reduction in radiative heat transfer due to the cooling effect on the flow field is striking. This reduction takes place primarily because the total radiative emission from each fluid element decreases rapidly as the element becomes cooler. In addition, some reduction is caused by the decreased flow field thickness due to the attenuation of the reflected shock velocity. However, the shock wave attenuation is very slight, as discussed earlier, and this latter effect is minor in comparison to the reduced emission.

Figure 18 shows the relationship between the reduction in radiative heat transfer discussed above and the radiation loss parameter, $\Gamma$. Here, $q_{RW}$ is the radiative heat transfer with radiative cooling, and $(q_{RW})_{AD}$ is the corresponding quantity without radiative cooling. Figure 18 will subsequently be useful in an analogy with the blunt body stagnation point case.

The influence of a gray gas on end wall radiative heat transfer is shown in Figure 19, which compares results for the three cases of a constant property transparent gas, a transparent gas with radiative
Fig. 17—End wall radiative heat transfer for a transparent gas

---TRANSPARENT GAS---

\[ W_1 = 29,000 \text{ ft./sec.} \]

\[ P_1 = 1 \text{ mm. Hg.} \]
Fig. 18.—Ratio of radiative heat transfers with and without radiative cooling as a function of the radiation loss parameter
Fig. 19.--The effect of a gray gas on end wall radiative heat transfer

$W_i = 29,000 \text{ ft/sec}$

$P_i = 1 \text{ mmHg}$
cooling, and a gray gas with radiative cooling. This figure graphically demonstrates the strong attenuating effect of a gray gas on \( \dot{q}_{R_w} \). On a physical basis, even though the gray gas enthalpy levels are higher than for a transparent gas, thus, causing higher local emission, the attenuation of the net radiative flux as it traverses the absorbing flow field is dominant and results in a value of \( \dot{q}_{R_w} \) which is lower than the transparent case. For all practical purposes, the dual effects of radiative cooling and self-absorption both work to reduce the end wall radiative heat transfer below the levels predicted by an ideal, constant property, transparent solution. Figure 19 shows that this combined effect amounts to as much as 65% for the condition of \( W_1 = 29,000 \) ft/sec. Note also that the influence of self-absorption increases with time, corresponding to increasing values of \( \tau_0 \). Of course, in the limit of very large values for \( \tau_0 \), the radiative flux will approach that of a black body. In addition, it is interesting to observe that the additive effects of radiative cooling and gray gas self-absorption cause \( \dot{q}_{R_w} \) to vary approximately as \( t^{3/4} \) rather than \( t \) as in the ideal transparent, constant property case.

In connection with the above discussion, it is appropriate to mention that self-absorption is usually neglected in problems characterized by small optical thicknesses. However, the above results indicate that caution must be taken to establish exactly what constitutes a small optical thickness. For example, the reduction in \( \dot{q}_{R_w} \) due only to gray gas self-absorption amounts to 16% at the maximum value of optical thickness, \( \tau_c = 0.322 \), encountered in the present
analysis. On the other hand, the reduction in $\dot{q}^{*}_{R_{w}}$ for $\tau_{0}$ as low as 0.045 is still quite noticeable, amounting to 9%. In mathematical terms, this is a consequence of the attenuating factor, $E_{2}(z)$, in the radiation flux integral. $E_{2}(z)$ decreases rapidly as $z$ increases for small values of $z$, but this trend becomes less severe for larger values of $z$. Therefore, the point emphasized here is that the self-absorption effect on radiative heat transfer should be accounted for when $\tau_{0}$ is at least as small as 0.045.

As emphasized during the discussion on flow field structure, the case of a non-gray gas is more important and more realistic than the gray gas case. This is particularly true when dealing with the end wall radiative heat transfer, which seems to be more sensitive to the influence of radiative cooling and self-absorption than all other phenomena considered in the present investigation. The strong self-absorption which takes place in the vacuum ultraviolet wavelength region considerably reduces the end wall radiative heat transfer. This reduction is markedly evident from Figure 20, which compares the non-gray end wall radiative heat transfer to the cases for a transparent and gray gas. It is obvious from Figure 20 that the combined effects of radiative cooling and non-gray self-absorption are much stronger than the other cases. In fact, two important points are demonstrated by the results shown in Figure 20. First, the assumption of an ideal constant property, transparent gas, which has been employed in a number of previous estimates of stagnation point radiative heat transfer, gives results for end wall radiative heat transfer under the conditions of the present investigation which are
Fig. 20.—End wall radiative heat transfer for the cases of non-gray, gray and transparent gases with radiative cooling, and for the ideal transparent constant property case (no radiative cooling).
almost an order of magnitude higher than the more realistic non-gray case with radiative cooling. Second, the assumption of a gray gas is a rather feeble attempt to account for self-absorption effects on radiative heat transfer in comparison to the non-gray case. This lack of success on the part of a gray gas substantiates the trend which was first observed during the investigation of enthalpy profiles. In addition, it is interesting to observe that the additive effects of radiative cooling and non-gray self-absorption cause $q_{Rw}$ to vary as $0.51 \sqrt{t}$ (approximately as the square root of $t$) rather than $t$ as in the ideal transparent, constant property case. This makes a rather striking comparison with the stagnation point results reported by Hoshizaki and Wilson, who have indicated that the additive effects of radiative cooling and non-gray self-absorption cause stagnation point radiative heat transfer to vary as the square root of nose radius, $R$, rather than directly as $R$ as in the ideal constant property, transparent case.

In conjunction with the present discussion of end wall radiative heat transfer, it is pertinent to present some results for the portion of $q_{Rw}$ in the wavelength region from 0 to 1100Å. These results are shown in Figure 21 along with the corresponding black body values for two different temperatures. The results shown in Figure 21 indicate that the strong self-absorption in the vacuum ultraviolet causes the flow field to be optically thick in this wavelength region. In fact, it is apparent that this portion of $q_{Rw}$ initially approaches a black body limit corresponding to some characteristic temperature which is slightly less than the maximum flow
field temperature (which occurs immediately behind the shock). The fact that this characteristic temperature is slightly less than the maximum temperature is certainly reasonable on a physical basis in light of the temperature variation throughout the flow field plus the fact that the higher temperature region is farthest from the end wall, and the lower temperature region is closer to the end wall. It is also interesting to note from Figure 21 that the heat transfer appears to reach this characteristic black body limit at about 3 μsec, after which the actual magnitude decreases due to the general cooling of the flow field at later times, which favors the longer wave length radiation.

As additional information concerning the portion of $\dot{q}_{Rw}$ between 0 and 1100 Å, the ratio of this quantity with the total radiative heat transfer is shown in Figure 22. It can be seen that this ratio decreases as the flow field thickness increases (i.e., as the time increases).

As mentioned earlier, the second contribution to end wall heat transfer is conduction. For a transparent gas, Figure 23 shows the influence of radiative cooling of the flow field on conductive heat transfer, $\dot{q}_{cw}$. The results without radiative cooling, obtained from the self-similar solution discussed in Chapter IV, are also presented for the sake of comparison. As in the case for $\dot{q}_{Rw}$, there is also a marked decrease in conductive heat transfer due to radiative cooling. This trend is only slightly modified by the assumption of a gray gas, as seen in Figure 24. Here, the influence of a gray gas is to increase $\dot{q}_{cw}$, but certainly not enough to regain or even approach the non-radiating level. On the other hand, the influence of a non-gray gas on $\dot{q}_{cw}$ is quite noticeable, as seen in Figure 25. Here, the combined
Fig. 21.--End wall radiative heat transfer for the wave length region between 0 and 1100 A--non-gray gas case.
Fig. 22.—Fraction of the end wall radiative heat transfer between 0 and 1100 A—non-gray gas case

\[ \frac{\Phi_R}{\Phi_R^{0-1100A}} \]

TIME AFTER REFLECTION, \( \mu \) sec

\[ W_1 = 29,000 \text{ ft/sec} \]

\[ P = 1 \text{ mm Hg} \]
CONDUCTIVE HEAT TRANSFER, $\dot{Q}_C$ watt/cm$^2 \times 10^{-4}$

Fig. 23.--End wall conductive heat transfer for a transparent gas

- TRANSPARENT GAS -

$W_l = 29,000$ ft/sec
$P_l = 1$ mm Hg

WITHOUT RADIATIVE COOLING

WITH RADIATIVE COOLING

TIME AFTER REFLECTION, $t_n$ µsec
Fig. 24.—End wall conductive heat transfer for a gray gas
Fig. 25.--End wall conductive heat transfer for the non-gray, gray and transparent cases with radiative cooling, and for the ideal self-similar solution (no radiative cooling)

\[ W_1 = 29,000 \text{ ft/sec} \]
\[ P_1 = 1 \text{ mm Hg} \]
effects of radiative cooling and non-gray gas self-absorption still reduce \( q_{cw} \) below the values for the ideal non-radiating case, but this reduction is certainly not as much as the transparent or gray gas cases. In other words, the realistic assumption of non-gray self-absorption considerably weakens the influence of radiative transfer on \( q_{cw} \) as compared to the effect predicted by the transparent or gray gas cases. Again, these results demonstrate the failure of the gray gas assumption as a means to properly account for self-absorption.

Lengthy attempts to correlate the above results for \( q_{cw} \) in terms of the enthalpy potential across the boundary layer and the \( \mu a \) product at the edge, as suggested by ordinary boundary layer theory, have not been successful. However, this lack of success is not uncommon. Howe and Viegas first reported failure in this respect during their stagnation point analysis. Thomas also noted a similar discrepancy. Koskizaki and Neren, in their analyses dealing with radiation effects on \( q_{cw} \), do not mention a correlation as attempted above. The common conclusion from the above discussion is that ordinary boundary layer theory may not be applicable because it requires a zero enthalpy gradient at the boundary layer edge. Obviously, as shown in Figures 10 and 13, radiative cooling establishes enthalpy gradients throughout the flow field, thus, destroying the zero gradient condition necessary for classical theory. In fact, the definition of the "edge" of the boundary layer becomes rather arbitrary. However, much of the above distress has been cleared away by Burggraf, who, by means of matched asymptotic expansions, has shown for the radiating stagnation point case that the effective outer conditions for the boundary layer are properties...
at some suitable point in the inviscid shock layer. Unfortunately, as mentioned earlier, attempts to find such a point which correlates the present numerical results for the reflected shock problem have not been fruitful.

As a partial summary of some of the present results for end wall heat transfer, Figure 26 gives radiative, conductive and total heat transfer values for the transparent case. There are two trends which are worth noting from Figure 26. First, because radiative cooling of the flow field reduces both the radiative and conductive contributions, its effect on total end wall heat transfer is certainly additive, amounting to as much as a 40% reduction for the transparent case. Second, for the conditions of the present investigation, the radiative heat transfer considerably exceeds the conductive heat transfer over most of the time interval considered, thus, demonstrating the importance of radiation as a mode of energy transfer.

An additional summary of the present end wall heat transfer results, Figure 27 shows that total end wall heat transfer for the transparent, gray and non-gray cases. The striking trend indicated by Figure 27 is that the combined effects of radiative cooling and non-gray self-absorption reduce the total end wall heat transfer considerably below the values predicted by the ideal transparent solution without radiative cooling. It is also interesting to note that these combined effects tend to result in a relatively constant value for the total end wall heat transfer.

Now, the results for flow field structure and end wall heat transfer have been presented, this chapter will conclude with a discussion of the analogy between the radiating reflected shock flow field and the radiation stagnation point shock layer.
Fig. 26.—Summary of end wall heat transfer for a transparent gas

- TRANSPARENT GAS -

RADIATIVE HEAT TRANSFER

TOTAL HEAT TRANSFER

CONDUCTIVE HEAT TRANSFER

WITH RADIATIVE COOLING

- - - - WITHOUT RADIATIVE COOLING

\[ W_i = 29,000 \text{ ft./sec.} \]

\[ P_i = 1 \text{ mm. Hg.} \]
Fig. 27.—Total end wall heat transfer for the non-gray gray and transparent cases with radiative cooling, and for the ideal case without radiative cooling.

$I_{0} = 29,000$ ft/sec
$P_{i} = 1$ mm Hg
C. Stagnation point analogy

In many respects, the hypersonic shock layer near the stagnation point of a blunt body is dissimilar to the high temperature flow field behind a reflected shock. For instance, the former is usually treated as a steady flow phenomena, whereas the latter is obviously unsteady. This has an important consequence on the enthalpy profiles as influenced by radiative cooling. In the steady flow stagnation point case, a fluid element has essentially an infinite time to radiate as it traverses the distance from the bow shock to the body surface. As a result, there is a tendency for a strongly radiation cooled layer to form near the body surface. (This has been discussed in more detail in Chapter I.) On the other hand, the maximum time scale for a radiating fluid element in the reflected shock flow field is dictated by the instantaneous position of the shock, and is certainly a finite quantity for finite values of $X_s$. Consequently, the enthalpy profiles for the reflected shock case maintain a concave appearance as can be seen in Figure 10, whereas those for the radiating stagnation point case are characterized by convex curvature for moderate values of $\Gamma$. The obvious conclusion from the above discussion is that enthalpy profiles are not similar between the two cases.

However, when such overall quantities as heat transfer and flow field thickness are considered, there is a strong incentive to look for an analogy between the reflected shock and stagnation point cases. A strong hint of such an analogy comes from the heat transfer variations for both problems without radiative cooling, as follows. For the stagnation point case, $\dot{q}_{cw}$ (transparent) varies as $R$, and $\dot{q}_{cw}$ varies as
where \( t \) is time after reflection. Obviously, through \( R \) and \( t \), there is some analogy between the two problems for the case of no radiative cooling. This prompts the question whether there exists some analogy in respect to radiative cooling effects on heat transfer, thus, adding engineering significance to the present end wall heat transfer results. However, before describing such an analogy, it is worthy to note that Yoshikawa and Chapman have analyzed the variation of radiative flux with position behind a standing normal shock wave, and have simply assumed that stagnation point radiative heat transfer can be obtained by evaluating the normal shock radiative flux at a downstream location equivalent to the stagnation point shock detachment distance. On the other hand, the analogy which is made in the following discussion is supported by a comparison of the present reflected shock results with results from an independent investigation of the stagnation point case. The bridge connecting these two cases is the radiation loss parameter, \( \Gamma \). This analogy is now described.

Figure 17 has already indicated the fact that the ratio of end wall radiative heat transfers with the without radiative cooling of the flow field is a smooth function of the radiation loss parameter, \( \Gamma \). For the reflected shock case, \( \Gamma \) has been defined as \( \Gamma = E_{si}t/\rho_{si}h_{si} \). However, assuming a constant reflected shock velocity of \( W_R \), the time, \( t \), can be expressed as,

\[
t = \frac{X_S}{W_R}
\]
Thus, an alternate form for $\Gamma$ is,

$$\Gamma = \frac{E_s \cdot x_s}{\rho_s \cdot W_R \cdot h_{s'}}$$  \hspace{1cm} (43)

Equation (43) for the reflected shock case has its counterpart in the stagnation point case, which utilizes an identical radiation loss parameter defined as

$$\Gamma = \frac{E_s \cdot \delta}{\rho_s \cdot \omega_s \cdot h_s}$$  \hspace{1cm} (44)

where $\delta$ is shock detachment distance, and the subscript $s$ denotes conditions immediately behind the bow shock. As a result, values of $\dot{q}_{R_w}/(\dot{q}_{R_w})_{AD}$ for the stagnation point case can be plotted versus the radiation loss parameter defined in equation (44), and such a curve is given by Wilson and Hoshizaki\textsuperscript{7} for a transparent gas. If these points are then superimposed on the results given by Figure 17, a striking comparison is made, as shown by Figure 28. Here, the solid curve applies to the reflected shock case, and the data points apply to the stagnation point results of Wilson and Hoshizaki. Figure 28 demonstrates that the above two sets of data are practically identical, thus, indicating that the radiative cooling effects on $\dot{q}_{R_w}$ are the same for both the reflected shock and stagnation point problems. To be explicit, the ratio $\dot{q}_{R_w}/(\dot{q}_{R_w})_{AD}$ as a function of $\Gamma$ for the reflected shock case is directly analogous to the corresponding variation for the stagnation point case.

The analogy between these two cases can possibly be extended to cover shock displacement. Previous investigations\textsuperscript{7,10,11} have shown
Fig. 28.—Radiative heat transfer analogy between the reflected shock and stagnation point cases
that radiative cooling of the stagnation point shock layer decreases the bow shock detachment distance, a consequence of the increased shock layer density. On the other hand, the present investigation has shown that radiative cooling of the reflected shock flow field attenuates the shock velocity, thus, causing the flow field thickness to be smaller than the corresponding adiabatic, constant property case. This displacement effect on reflected shock position is shown by the solid curve of Figure 29, which gives the ratio of shock position with radiative cooling to that without radiative cooling versus the radiation loss parameter. In addition, a point obtained for the similar quantities from the stagnation point case\textsuperscript{10} is shown on Figure 29. This point falls reasonable close to the reflected shock curve. Unfortunately, the data presented in Reference 10 cannot be read with reasonable accuracy for small values of $\Gamma$, and the above comparison is consequently limited to one point. Even though this is certainly not enough to establish a general trend, it does provide evidence toward an analogy between the reflected shock and stagnation point cases with respect to radiative cooling effects on shock detachment distance.

At this point, it is appropriate to mention that previous investigations\textsuperscript{8,10,11} of the stagnation point problem have found that radiative cooling of a transparent or gray gas reduces $\dot{q}_c$. The present investigation has found a similar trend for the reflected shock problem. However, attempts to strike a quantitative analogy in this respect have not been successful. None-the-less, the present investigation has demonstrated in some respects that a direct analogy exists between the radiative cooling effects on the reflected shock
Fig. 29.—Shock displacement analogy between the reflected shock and stagnation point cases.
problem and those on the stagnation point problem. The extent of the analogy which has been shown above is sufficient to warrant closer engineering investigation of the radiating reflected shock phenomena.

Now that the results from the present investigation have been presented and discussed, the pertinent conclusions will now be summarized.
CHAPTER VI
CONCLUSIONS

The following general conclusions can be summarized from the present study of the effects of the radiation–gas dynamic coupling on the high temperature flow field and heat transfer behind a reflected shock wave in air.

(1) The combined effects of radiative cooling and self-absorption on enthalpy profiles, radiative heat transfer and conductive heat transfer are so pronounced for the conditions of the present investigation that the corresponding results predicted for the ideal transparent, constant property case are totally unacceptable. In other words, the radiation–gas dynamic coupling exerts a strong influence on the flow field and heat transfer behind the reflected shock, and this influence cannot be neglected.

(2) The assumption of a gray gas does not successfully account for the influence of self-absorption on enthalpy profiles and heat transfer. The results of the present investigation indicate that the influence of self-absorption, due to the more realistic assumption of a non-gray gas model, is considerably more pronounced than the relatively weak influence of a gray gas. Consequently, if self-absorption is to be realistically included in an analysis, a non-gray model for the absorption coefficient should be employed.

In addition to the above general conclusions, the following remarks can be made in regard to some of the more specific results from the present analysis.
(1) The flow field enthalpy profiles are reduced by radiative cooling. The profiles are the lowest for a transparent gas, are only moderately increased by the additional effect of a gray gas, but are considerably increased for a non-gray gas.

(2) The overall influence of radiation on the flow field is an "onset" effect, being strong for $W_1 = 29,000$ ft/sec, but barely noticeable for $W_1 = 25,000$ ft/sec.

(3) The enthalpy profiles can be correlated by $\Gamma$ for a transparent gas, and by $\Gamma$ and $\tau_0$ for a gray gas. This provides evidence of the validity and usefulness of the radiation gas dynamic similarity parameters, and also provides a simplification for the calculation of the enthalpy profiles.

(4) The expression, $e^{-(0.518 \tau_0)} \Gamma$ acts as an effective radiation loss parameter for a gray gas. This parameter approaches $\Gamma$ as $\tau_0$ goes to zero, and approaches zero as $\tau_0$ goes to infinity, both of which are physically correct limits.

(5) The reflected shock velocity is only slightly attenuated by radiative cooling for the present range of conditions. Out of all the phenomena investigated, the shock velocity was the least affected by the radiation--gas dynamic coupling.

(6) End wall radiative heat transfer is markedly reduced by the combined effects of both radiative cooling and non-gray gas self-absorption. Out of all the phenomena investigated, the radiative heat transfer was the most affected by the radiation--gas dynamic coupling.

(7) End wall conductive heat transfer is reduced by radiative cooling. This reduction is only slightly mitigated by the influence
of gray gas self-absorption. However, self-absorption, due to the non-gray gas model, cancels a noticeable portion of the reduction initially indicated by the transparent case with radiative cooling.

(8) For the range of temperature and density considered in the present study, end wall radiative heat transfer is greater than conductive heat transfer (except for small values of time). Therefore, total end wall heat transfer is strongly reduced by both the effects of radiative cooling and non-gray self-absorption. In addition, these dual effects tend to force the total end wall heat transfer towards a relatively constant value.

(9) There exists a quantitative engineering analogy for the radiative cooling effects on radiative heat transfer and shock layer thickness between the reflected shock and stagnation point problems.

(10) In general, the overall results indicate that radiative cooling and self-absorption should be integral parts of most engineering radiation gas dynamic analyses, at least where $\Gamma > 0.05$ (for $\Gamma = 0.05$, $q_{Rw}$ is reduced 10% due just to cooling) and where $\tau_o > 0.05$ (for $\tau_o = 0.05$, $q_{Rw}$ is reduced 10% due just to gray gas self-absorption).

(11) The combined influence of radiative cooling and non-gray self-absorption causes end wall radiative heat transfer to vary approximately as $t^{1/2}$. This makes an interesting comparison with the similar results for stagnation point radiative heat transfer,\textsuperscript{13} which varies as the square root of nose radius.

In addition to the above conclusions, the following suggestions are made for improvements and extensions to the present investigation.
(1) The accuracy of the quantitative results can be improved by utilizing more accurate transport and radiation properties of air. In particular, the present radiation properties can be improved by including atomic line and more reliable vacuum-ultraviolet continuum radiation.

(2) The ranges of $\Gamma$ and $\tau_0$ covered by the present study can be extended by either running longer execution times on the computer or simplifying and shortening the analysis.

(3) The suggested analogy between the reflected shock and stagnation point problems should be further investigated in hopes of widening its scope.

(4) It would be very interesting to extend the present analysis into the non-equilibrium regime, possibly by first treating the case for a monatomic gas for sake of simplicity.
APPENDIX I

EVALUATION OF THE RADIATION TERM IN THE GAS DYNAMIC ENERGY EQUATION

In Chapter II, the net radiative flux of frequency, \( \nu \), was formulated as equation (8), copied below for convenience.

\[
\mathcal{Q}_\nu (\tilde{\tau}_r) = 2\pi \int_{\tilde{\tau}_r}^{\tau_r} B_r (z) E_2 (z - \tilde{\tau}_r) \, dz
- 2\pi \int_{0}^{\tilde{\tau}_r} B_r (z) E_2 (\tilde{\tau}_r - z) \, dz
\]  

This equation must now be differentiated to obtain the radiation term for the energy equation. Noting that \( \tilde{\tau}_r \) is a variable which appears in the limits of integration, the differentiation must be carried out according to Leibniz' Rule, which states that if

\[
I (e) = \int_{x_i (e)}^{x_2 (e)} f (x, e) \, dx
\]

then

\[
\frac{dI}{de} = f (x_2, e) \frac{dx_2}{de} - f (x_i, e) \frac{dx_i}{de} + \int_{x_i (e)}^{x_2 (e)} \frac{d}{de} f (x, e) \, dx
\]

Then, from equations (I-1) and (I-3),

\[
\frac{d\mathcal{Q}_\nu}{d\tilde{\tau}_r} = 2\pi \left\{ B_r (\tau_r) E_2 (\tau_r - \tilde{\tau}_r) \frac{d\tilde{\tau}_r}{d\tau_r} - B_r (\tilde{\tau}_r) E_2 (\tilde{\tau}_r - \tau_r) \frac{d\tau_r}{d\tilde{\tau}_r} 
+ \int_{\tilde{\tau}_r}^{\tau_r} B_r (z) \frac{dE_2 (z - \tau_r)}{d\tau_r} \, dz - \left[ B_r (\tilde{\tau}_r) E_2 (\tilde{\tau}_r - \tau_r) \frac{d\tau_r}{d\tilde{\tau}_r} 
- B_r (0) E_2 (\tau_r) (0) + \int_{0}^{\tilde{\tau}_r} B_r (z) \frac{dE_2 (\tau_r - z)}{d\tau_r} \, dz \right] \right\}
\]  

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Noting that,\[ \frac{d \tau_r}{d \tau_r} = 0 \]
\[ \frac{d \tau_r}{d \tau_r} = 1 \]

and employing the identities:\[ \frac{2 E_2 (\tau_r - \tau_r)}{2 \tau_r} = E_1 (\tau_r - \tau_r) \]
\[ \frac{2 E_2 (\tau_r - \tau_r)}{2 \tau_r} = -E_1 (\tau_r - \tau_r) \]
\[ E_2 (\tau_r - \tau_r) = E_2 (0) = 1 \]
equation (I-4) now becomes
\[ \frac{d q_r}{d \tau_r} = 2 \pi \left\{ 0 - B_r (\tau_r) + \int_{\tau_r}^{\tau_{av}} B_r (\tau_r) E_1 (\tau_r - \tau_r) d \tau_r + B_r (\tau_r) + 0 + \int_{\tau_r}^{\tau_{av}} B_r (\tau_r) E_1 (\tau_r - \tau_r) d \tau_r \right\} \]

This reduces to
\[ \frac{d q_r}{d \tau_r} = -4 \pi B_r (\tau_r) + 2 \pi \int_{\tau_r}^{\tau_{av}} B_r (\tau_r) E_1 (\tau_r - \tau_r) d \tau_r \]
(I-5)

Since \( \frac{d \tau_r}{d x} = K \frac{d x}{d x} \), equation (I-5) can be written as
\[ \frac{d q_r}{d x} = -4 \pi K_r B_r (\tau_r) + 2 \pi K_r \int_{\tau_r}^{\tau_{av}} B_r (\tau_r) E_1 (\tau_r - \tau_r) d \tau_r \]
(I-6)

The flux integrated over all frequencies is
\[ \mathcal{Q} = \int_{0}^{\infty} q_r d \tau_r \]
and equation (1-6) becomes
\[ \frac{dq}{dx} = \frac{d}{dx} \int_0^\infty q_r \, dv \]

\[ \frac{dq}{dx} = -4\pi \int_0^\infty K_r B_r \, dv + 2\pi \int_0^\infty K_r \int_0^{\tau_r} B_r(\tau) E(1\tau_r-3) \, d\tau \, dv \]

(I-7)

Recalling, that by definition,
\[ J = \int_0^\infty K_r B_r \, dv \]

the final form of the radiation term in the gas dynamic energy equation is, from equation (I-7),

\[ \frac{dq}{dx} = -4\pi J + 2\pi \int_0^\infty K_r \int_0^{\tau_r} B_r(\tau) E(1\tau_r-3) \, d\tau \, dv \]

(I-8)

Note, that because of the astrophysical convention for the sign of \( q \), namely, that \( q \) is positive in the negative \( x \) direction, the above expression for \( \frac{dq}{dx} \) corresponds to the term \( -\frac{dq}{dx} \) in the energy equation.
APPENDIX II

SIMILARITY TRANSFORMATION FOR THE END WALL THERMAL BOUNDARY LAYER WITHOUT RADIATION EFFECTS

The self-similar solution to the end wall thermal boundary layer, ignoring radiation, is described in Chapter IV. This solution requires a transformation of the independent variables, $x$ and $t$, to $\eta$ and $s$, where

$$\eta = \eta(x, t) = \left[ \frac{(P_e/\mu_e)_0}{2P_e t} \right]^{1/2} \int_0^x \rho \, d\xi$$

(II-1)

and

$$s = t$$

(II-2)

The transformation involves the following derivatives,

$$\frac{d}{dx} = \frac{2\eta}{2\eta} \left( \frac{d}{d\eta} \right) + \frac{2s}{2x} \left( \frac{d}{ds} \right)$$

(II-3)

$$\frac{d}{dt} = \frac{2\eta}{2\eta} \left( \frac{d}{d\eta} \right) + \frac{2s}{2x} \left( \frac{d}{ds} \right)$$

(II-4)
where the terms on the right hand side are, after some re-arrangement,
\[
\frac{2s}{2x} = 0, \quad \frac{2s}{2t} = 1.
\]
\[
\frac{2n}{2x} = \left[ \frac{(Pr/\mu)e}{2Pe} \right] \frac{2}{2t} \int_0^x \rho \, dx
\]
\[
\frac{2n}{2t} = -\frac{n}{2t} + \left[ \frac{(Pr/\mu)e}{2Pe} \right] \frac{2}{2t} \int_0^x \rho \, dx
\]
Substituting the above into equations (II-3) and (II-4), the following transformation results.
\[
\frac{2}{2x} = \left[ \frac{(Pr/\mu)e}{2Pe} \right] \frac{2}{2t} \int_0^x \rho \, dx \left( \frac{2}{2n} \right)
\]  
(II-5)
\[
\frac{3}{2t} = \left\{ -\frac{n}{2t} + \left[ \frac{(Pr/\mu)e}{2Pe} \right] \frac{2}{2t} \int_0^x \rho \, dx \right\} \left( \frac{2}{2n} \right) + \left( \frac{2}{2s} \right)
\]  
(II-6)

In order to apply the above transformation to the convective operator, an expression for the product \( \rho u \) must be obtained from the continuity equation, as follows.

The continuity equation is,
\[
\frac{3p}{2t} + 2 \left( \frac{\rho u}{2x} \right) = 0
\]  
(II-7)
Substituting the transformation (II-5) and (II-6) into equation (II-7),

\[
\left\{ - \frac{n}{2s} + \left[ \frac{(\rho \mu \kappa_0)}{2p_0} \right]^\frac{1}{2} \frac{2}{2^x} \int_0^x p_dx \right\} \frac{d^2}{2^s} + \frac{d^2}{2^s} + \frac{2(\rho u)}{2^s} = 0
\]

(II-8)

Assuming similarity exists, (i.e., \( \frac{d}{ds} = 0 \)), and solving for \( \rho u \) in equation (II-8),

\[
\frac{d(\rho u)}{d \eta} = \left\{ \frac{\eta}{2^x p \left[ \frac{(\rho \mu \kappa_0)}{2p_0} \right]^\frac{1}{2}} - \frac{1}{\rho} \frac{2}{2^x} \int_0^x p_dx \right\} \frac{d^2}{d \eta}
\]

Thus, by integration,

\[
\rho u = \int \left\{ \frac{\eta}{2^x p \left[ \frac{(\rho \mu \kappa_0)}{2p_0} \right]^\frac{1}{2}} - \frac{1}{\rho} \frac{2}{2^x} \int_0^x p_dx \right\} \frac{d^2}{d \eta}
\]

(II-9)

Integrating the right hand side of equation (II-9) by parts,

\[
\rho u = \left\{ \frac{\eta}{2^x p \left[ \frac{(\rho \mu \kappa_0)}{2p_0} \right]^\frac{1}{2}} - \frac{1}{\rho} \frac{2}{2^x} \int_0^x p_dx \right\} \rho
\]

\[
- \int \frac{\rho}{2^x p \left[ \frac{(\rho \mu \kappa_0)}{2p_0} \right]^\frac{1}{2}} d \eta
\]

(II-10)
The first and third terms on the right hand side cancel, and

\[ \rho u = -\frac{2}{2x} \int_0^x \rho d\chi \]  

(II-11)

Now that the desired expression for \( \rho u \) has been obtained, the convective operator,

\[ \frac{\partial}{\partial t} \frac{2}{2x} + \rho u \frac{2}{2x} \]

can be transformed by use of equations (II-5), (II-6), and (II-11) as follows.

\[ \frac{\partial}{\partial t} \frac{2}{2x} + \rho u \frac{2}{2x} = \rho \left\{ -\frac{\eta}{2x} + \left[ \frac{(\rho \eta / \mu) e}{2 Pe + \eta} \right] \frac{2}{2x} \int_0^x \rho d\chi \right\} \frac{2}{2\eta} \]

- \[ \left( \frac{2}{2x} \int_0^x \rho d\chi \right) \left[ \frac{(\rho \eta / \mu) e}{2 Pe + \eta} \right] \int_0^x \frac{2}{2\eta} \]

The above expression reduces to

\[ \frac{\partial}{\partial t} \frac{2}{2x} + \rho u \frac{2}{2x} = -\frac{\rho \eta}{2x} \frac{d}{d\eta} \]  

(II-12)

Equation (II-12) is the desired expression for the convective operator.

As a final step, the energy equation,

\[ \frac{\partial}{\partial t} \frac{2h}{2x} + \rho u \frac{2h}{2x} = \frac{2}{2x} \left[ \left( \frac{\mu}{P \zeta} \right) \frac{2h}{2x} \right] \]
is directly transformed by equations (II-5), (II-6), and (II-12), and results in

\[ \eta \frac{dh}{d\eta} + \frac{d}{d\eta} \left( \frac{(\mu/\rho)}{(\mu/\rho)_e p_e} \frac{dh}{d\eta} \right) = 0 \]

(II-13)

which must be solved numerically for the enthalpy profiles.
APPENDIX III

NUMERICAL EVALUATION OF THE RADIATION INTEGRAL

As mentioned in Chapter IV, the need arises during the numerical solution of the governing partial differential equations to evaluate the radiation integral appearing in the energy equation. This evaluation is complicated by the $E_1 \frac{1}{t}$ function which has a singularity at zero. This prompts a modification of the radiation integral, and such a modification is described below for the case of a gray gas. The evaluation of the radiation integral for the non-gray step function model is also mentioned below. Since the details of these modifications are tedious, they will only be outlined.

For a gray gas, the radiation integral which is of interest is

$$\int_0^{\tau} \frac{d}{\tau} E, (1 \tau - \eta) d\eta$$  \hspace{1cm} (III-1)

Expression (III-1) can be written as two integrals in order to remove the absolute value signs.

$$\int_0^{\tau} \frac{d}{\tau} E, (1 \eta - \tau) d\eta = \int_0^{\tau} \frac{d}{\tau} E, (\tau - \eta) d\eta +$$

$$\int_{\tau}^{\tau} \frac{d}{\tau} E, (\eta - \tau) d\eta$$  \hspace{1cm} (III-2)

If the curve for $T^4(z)$ is split into many straight line segments with projections along the z-axis of length $\Delta z$ ($\Delta z$ need not, in general, be
identical for all the segments), and if the linear variation of $T^i(z)$ over each segment is given by

$$T^i(z) = T_i^i + (T_i^i - T_i^i) \left( \frac{z - z^i}{\Delta z} \right)$$  \hspace{1cm} (III-3)

for

$$z_1 \leq z \leq z_{i+1}$$

then the integral over each segment, from $z_1$ to $z_{i+1}$ is

$$\int_{z_i}^{z_{i+1}} T^i(z) E_i(\tau - z) \, dz = T_i^i \int_{z_i}^{z_{i+1}} E_i(\tau - z) \, dz +$$

$$\left( \frac{T_i^{i+1} - T_i^i}{\Delta z} \right) \int_{z_i}^{z_{i+1}} (z - z^i) E_i(\tau - z) \, dz$$  \hspace{1cm} (III-4)

Employing the identity

$$\int E_i(\tau - z) \, dz = E_2(\tau - z)$$

integrating by parts, and combining terms, equation (II-4) becomes

$$\int_{z_i}^{z_{i+1}} T^i(z) E_i(\tau - z) \, dz = T_i^{i+1} E_2(\tau - z_{i+1}) - T_i^i E_2(\tau - z_i) - \frac{T_i^{i+1} - T_i^i}{\Delta z} \left[ E_3(\tau - z_{i+1}) - E_3(\tau - z_i) \right]$$  \hspace{1cm} (III-5)

In a similar fashion, the following integral is also obtained, where $\tau$ and $z$ are reversed.
The above result comes from the identity,

$$\int_{j_i}^{j_{i+1}} E_i (j - \tau) \, dj = T_i^u E_2 (j_{i+1} - \tau) - T_i^u E_2 (j_i - \tau) \quad (III-6)$$

The integral over the whole range of \( z \) is now obtained by combining equations (III-5) and (III-6) with (III-2) as follows

$$\int_{0}^{z_S} \frac{T}{T_i (j) \cdot E_i (j - \tau)} \, dz = \sum_{i=0}^{i=J} \left( \int_{j_i}^{j_{i+1}} \frac{T}{T_i (j) \cdot E_i (j - \tau)} \, dz \right) + \sum_{j=J+1}^{J + I} \int_{j_j}^{j_{j+1}} \frac{T}{T_i (j) \cdot E_i (j - \tau)} \, dz$$

$$= \left\{ T_i^u (\tau) - T_{i+1}^u E_2 (\tau) \right\} + \sum_{j=0}^{j=J} \left[ \frac{T_i^u - T_j^u}{\Delta j} \right] \left[ E_3 (\tau - j_{i+1}) - E_3 (\tau - j_i) \right] \right\} + \left\{ T_i^u (\tau) - T_5^u E_2 (\tau_{5} - \tau) - \sum_{j=J+1}^{J + I} \left[ \frac{T_i^u - T_j^u}{\Delta j} \right] \left[ E_3 (j_{i+1} - \tau) - E_3 (j_i - \tau) \right] \right\}$$

In the above equation, \( J \) labels the point immediately before \( \tau \), and \( I \) labels the point immediately behind the shock. If \( T_{w}^u \) is neglected in comparison to \( T_i^u \), where \( T_w \) is the cold end wall-temperature, then the above equation becomes
This is the absorption integral in the energy equation. If it is combined with the emission term, the leading term on the right hand side of equation (III-7) is cancelled. Consequently, the radiation term in the finite difference energy equation, equation (36), becomes

\[
\int_C \left( \frac{\Delta x}{f_b} \right) \left( \frac{K_{b}^*}{2} \right) \left\{ - T_s \frac{\partial}{\partial z} \right\} \left[ E_3 (1\gamma - \gamma_{i+1}) - E_3 (1\gamma - \gamma_{i+1}) \right] \right. \\
= - \sum_{k=0}^{k=L} \sum_{i=0}^{i=L} \left[ \frac{T_{k+1}^* - T_k^*}{\Delta z} \right] \left[ E_3 (1\gamma - \gamma_{i+1}) - E_3 (1\gamma - \gamma_{i+1}) \right] \\
\] (III-8)

as mentioned in Section D of Chapter IV.

For the non-gray case, if a step function model is used to represent the absorption coefficient, the radiation integral can be evaluated for each step in a similar manner as described above. For this case, \( B_1(z) \), where \( B_1 \) is defined in Chapter IV, is replaced by a segmented curve. The result for the non-gray radiation term in the finite difference energy equation is

\[
- 2 \pi \frac{\Delta x}{f_b} \left[ \frac{L}{\beta_{b} \omega_{x} W_{x} h_{x}} \right] K_P \sum_{i=0}^{i=L} \left\{ B_i (\gamma_{i+1}) E_2 (\gamma_{i+1} - \gamma_{i}) \right\} \\
+ \sum_{j=0}^{j=L} \left[ \frac{B_{i,j+1} - B_{i,j}}{\Delta x} \right] \left[ E_3 (1\gamma - \gamma_{j+1}) - E_3 (1\gamma - \gamma_{j+1}) \right] \right. \\
where all the above terms are identified in Section D of Chapter IV.


