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DISSERTATION

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By
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CHAPTER I

INTRODUCTION

A large portion of today's military resources are devoted to manned bomber and interceptor aircraft. Despite the growth of long range missile capabilities, it is generally considered likely that bombers will continue to be an important attack weapon in the future. Since interceptor aircraft appear to offer one of the best means of defending against bombers, it is also probable that interceptors will remain to represent a significant portion of the world's military resources.

Modern aircraft, both bombers and interceptors, have become very complex and therefore increasingly expensive. No nation can afford to purchase such expensive weapons without a justification of the expenditure. It is also apparent that the worth of such systems is dependent upon the weapons that will be carried in a typical mission. Any analysis of these systems must therefore consider what weapons can be included in the weapon load of the vehicle. Such
analyses must consider both the types and numbers of such weapons, both defensive and offensive.

The ability of a bomber aircraft to deliver bombs to a ground target will depend primarily upon the bomber's ability to survive the enemy defenses, both ground and air. Since the locations of ground defenses are often known, it is possible that future bombing raids will not have to be concerned with the threat of ground defenses that are not clustered around assigned targets. Unless the ground threats are extremely numerous the bomber should be able to select a flight path that will avoid all ground threats not located near assigned targets. It is also possible that future bombers will carry long range air-to-ground missiles instead of more conventional bombs or short range attack missiles. Armed with such offensive weapons the bomber will be able to attack intended targets without exposure to surrounding ground threats. Thus it is possible that future bombers will be vulnerable only to airborne threats (interceptors). For this reason the analysis of the effectiveness of a force of bombers contained in this volume will consider only the effect of the airborne threat on the bombers.

This chapter outlines the problem of determining the operating group sizes, weapon loads and firing
doctrines for both the bombers and the interceptors to allow the fighting of an air battle in an optimum (or near optimum) manner. However any solution to this problem must take into consideration the limitation on the number of weapons that a bomber can carry. Since existing air battle models do not take this factor into consideration, it will be necessary to develop a model (or more correctly a series of models) that includes a limitation on the number of weapons carried by a bomber. Since the model giving the expected outcome of an air battle subject to bomber weapon load limitations will prove to be computationally burdensome, simpler models yielding upper and lower bounds on the expected value solution will also be developed. These simpler bounding models will be used to screen the set of possible decisions of both sides and thus reduce the number of options that must be considered using the expected value model.

To determine the optimum decisions that should be made by the combatants it will be necessary to select a criterion function. This means that the analyst must decide what is to be termed a successful completion of the bomber's task. It may be that the bombers are felt to have accomplished their task only if the targets are bombed and the bombers survive. However, in the case of an all-out war, survivability would seem to be of
secondary importance (to all but the crew). The main concern must be to destroy the assigned targets. The loss of a bomber would be of secondary importance if its bombs (or air-to-ground missiles) have all been delivered. Given a preassigned set of target locations (which will be assumed generally known to both sides), the bombers should attempt to maximize target damage while the interceptors attempt to minimize this function. This is the situation that will be explored in this volume.

This problem has the structure of a game involving two persons - the offense and the defense. It will be assumed, in the analysis in this volume, that the rules of a two person, zero sum game apply, and thus, it is necessary that both participants view the situation in the same manner. That is, it will be assumed that each side knows the target distribution, force size estimates, weapon load capabilities, weapon effectiveness, etc.

There is the possibility that the decisions made by one or both sides must be made known to the opposition before the battle takes place. For example, if groups of interceptors must train together, it may be that the selected group size will become known to the offense. Similarly if it is not easy to change the weapon load carried by a bomber, the final design selected for the bomber may indicate to the defense what options have been
selected by the offense. Thus the problem may not be a true game - but a case of one side selecting its tactics and then the other side optimizing the resulting situation.

There may be a question concerning the value of an optimum (either classical or from the game theory viewpoint) solution when it is considered unlikely that both or either side will really operate in an optimum fashion. It may be observed that historically battle tactics selected have not usually resulted from optimality considerations. It is often asked if the pressures of battle allow one the ability to act in an optimum fashion or even to know what optimum is. However it is also logical to ask if the participants in modern warfare can afford to act in a non-optimum manner. It must be realized that a great portion of the wealth of today's major nations is presently spent in military endeavors. An effort must be made to ensure that these resources are used efficiently. Fortunately the wide use of computers in today's world does allow much efficiency in military planning. Possessing computational means to select an optimum or at least near optimum procedure, there should be no other acceptable course of action.

Of course a game theory solution can be of value
even if the opposition does not act in accordance with the assumptions of classical game theory (and thus in what is defined to be an optimum manner). Any solutions to a zero-sum game found assuming the opposition acts in an optimum manner will be a lower bound on the expected return. If the opposition does not act as specified by the game solution, the expected return will be even better than predicted.

The Options Available to the Offense

Many defensive weapons have been developed or proposed in order to increase the probability of survival of a bomber that may be attacked by one or more interceptors. These weapons fall into three general categories:

Type One: A weapon that attacks the interceptors and thus both increases the probability of survival of the attacked bomber and decreases the probability of future attacks on all bombers by attempting to reduce the number of interceptors.
Type Two: A weapon that attacks or decoys the attacking missile (launched by the interceptor) and thus improves the probability of survival of the attacked bomber. Use of this type of weapon does not reduce the potential frequency of future attacks.

Type Three: A weapon that decreases the rate of attack but does not destroy interceptors (such as dilution-decoys).

Weapons that destroy the interceptors would today generally be limited to some sort of attack missile although any form of gun or cannon would also fall into this category. In any case this class of weapon would attempt to destroy the threatening interceptor hopefully before any missiles could be launched at or at least directed to the bomber. Weapons in this class would not only improve the bomber's probability of surviving the present engagement, but if the interceptor could conceivably initiate additional attacks in the future, also decrease the rate of occurrence of future attacks.
upon the whole bomber force.

Although weapons of type two do not have the ability to affect the rate of future attacks upon the force of bombers, it may be easier to destroy the interceptor's missile than the interceptor itself. The threat missile could be intercepted at ranges considerably less than the separation between the aircraft when the interceptor launches its missile and still allow the bomber to survive. Thus a missile designed to attack the threat missile might be much smaller than one designed to attack the interceptor. If type two missiles were in fact smaller than type one missiles a bomber could carry many more weapons of type two than weapons of type one. Obviously, if a weapon of type two were no smaller than one of type one, it would have to be capable of achieving a higher probability of destroying the intended target to be selected over the weapon of type one. It is possible that both types of weapons should be carried and used on the same or different encounter. It may be better to launch both types of weapons when attacked, rather than limiting the response to either one alone. It also would generally be more desirable to destroy interceptors at the start of battle rather than near the end. The act of reducing the rate of future attacks has no value in itself at the
end of the battle. The bomber may choose to attack the interceptors at the start of the raid and later on rely upon weapons of type two.

The final category of defensive weapons contains those weapons that are intended to make the bomber hard to find. For example the bomber might deploy decoys which fly with the bomber (possibly in a different direction) thereby presenting additional targets to the defense. It is hoped by the bomber to in this way force the defense to utilize some of its capability to investigate and possibly attack these worthless targets. Electronic counter measures (ECM) equipments which are designed to disrupt or confuse radar searching by the defense would also fall into category three.

Thus there are several types of weapons that could be carried and utilized by the bomber in its defense. However since modern weapons have a tendency to grow in size due to increased performance demands, the bomber will be capable of carrying only a limited number of such devices. It also must be realized that space allotted to defensive weapons subtracts from the space available for carrying offensive weapons (bombs). Thus a decision must be made concerning the number and types of defensive weapons that should be carried by a bomber.

The decision concerning the defensive weapon load carried is of course a function of the manner in which
weapons will be used. This in turn is a function of the doctrine that will be followed in utilizing weapons in encounters with interceptors in addition to the anticipated frequency of such encounters. It must be decided how many weapons of each type will be used during each encounter. Since the number of weapons of type one used affects the rate of future attacks, this is not a simple problem. It must also be realized that a truly optimum policy will probably change with time. At the start of battle it is necessary to conserve ammunition for future needs. However in the last stages of the battle a better tactic may call for maximizing survival on the present encounter ignoring future needs.

The question of weapon load size should really be asked prior to the selection of a final design for the bomber. Would it be better to have a few large bombers each carrying many weapons or a large number of smaller aircraft? The answer to this question should be a major factor in determining the proper size of a new bomber. This of course would depend upon the relative costs of both options as well as the expected system effectiveness.

Finally the possible advantages and disadvantages of grouping the bombers must be considered. Assume each bomber had a supply of missiles of type one and/or type two that were capable of defending other bombers in the immediate neighborhood. In this situation the bombers
might find it advantageous to fly in groups to form a common defense. It would of course be less likely for a group to run out of weapons than a single bomber working alone.

However the advantage resulting from grouping must be weighed against several inherent disadvantages. First, grouping will require that each bomber in the group travel to each target assigned to any and all members of the group. This could greatly increase the average distance flown by the bombers during the mission. This of course increases the duration of the battle and thus the exposure of the bombers to the defenses. Also since the bombers are in groups they may be easier to find. Certainly any interceptor finding one target would then be able to find additional targets from the group. Finally since the bombers' missiles must now supply coverage for accompanying neighbors, the missile range requirements could increase. If an interceptor must be destroyed before it reaches some specified range from its intended target, and if the target is between the interceptor and the closest bomber with a remaining weapon supply, then the range requirements of the counter missile would be greater than that required for the missile when the bomber supplies its own defense. Of course as the range requirements of the missiles go
up so does the cost and size. This could result in fewer missiles carried per bomber.

Because of all of these inherent disadvantages resulting from grouping of bombers it will be assumed in this volume that the bombers are not grouped or at least cannot take advantage of the possibility of mutual support. This assumption should be acceptable in most analyses of bomber interceptor interactions since modern concepts about the operation of bomber defensive systems tend to treat each bomber as an autonomous vehicle. Thus the resulting analytical techniques should generally be applicable to present air force problems.

The Options Available to the Defense

Consider next the options open to the interceptors. Interceptors usually must be capable of flying at greater speeds than the bombers to be attacked in order to make an intercept. Therefore, they are typically smaller and carry lighter weapon loads. Since the interceptor's task is to destroy the bomber, the weapons carried are generally limited to offensive (not defensive) types. Interceptors do not normally use weapons of types two or three.

The interceptors will of course have to decide upon the number of weapons to be fired in each encounter
with a bomber. Assume that the interceptors are traveling so fast that only one firing (salvo) will be possible per target encountered (assume that a turning maneuver to reacquire the target is not practical). If the interceptors fire all of their missiles in each encounter, they will be limited to one attack before having to return to a base for possible rearming. However, if only part of the supply is used in each encounter, several encounters might be possible between rearmings. Since the probability of killing the target will normally increase as the number of weapons fired increases, a decision as to the proper salvo size must be made.

Another decision that must be made by the defense concerns selecting the bombers to be attacked. This in turn leads naturally to the thought of grouping on the part of the interceptors. Assume that the bombers carry a limited number of some very effective defensive weapon. It may be nearly impossible for an interceptor to achieve a kill as long as the bomber has such weapons available. If at some point in the battle some bombers had these weapons and others did not (due to exhaustion from previous encounters with interceptors) it would be advantageous for the interceptors to select the unarmed bombers in the next series of attacks. However the interceptors may not know who has weapons
remaining and who does not. How then can they possibly take advantage of this possible source of weakness of the bombers? One obvious answer would involve grouping of the interceptors. If the first attacks of the group were to deplete the bomber's weapon supply, then the rest of the group would be presented an unarmed target. Of course if one of the first members of the group achieved a kill, the rest would have to find a new bomber to attack (which may not be possible) or return to their base without making an attack. Thus the defense is faced with the problem of deciding what size group will maximize the interceptors' effectiveness.

When considering the grouping of interceptors two possibilities must be included. If the interceptors are able to return to a base, rearm, and initiate another attack (or series of attacks if an interceptor can carry more than one salvo of weapons) it may be possible that the surviving interceptors upon returning to their bases can be reorganized into any desirable group size (in the simplest case the defense might choose to always deploy its interceptors in the same size groups). If the interceptors could regroup after each series of attacks, the defense could attempt to determine an optimum grouping for each sequence of attacks. Thus the group size selection would be a dynamic problem that should
be optimized for each cycle of attacks.

However it may be desirable or even necessary to retain the original groupings of interceptors. This may be necessary due to the need for experience in working together on the part of the pilots as well as the need to simplify the operation of the control system. Working under the stress of battle it may be extremely difficult to reorganize the interceptors in any efficient manner. If the original groupings are to be retained, the survivors of each group will continue to work together in the next cycle of the battle. In this case selection of an optimum group size will be made only at the start of the battle.

A final question concerns the operation of the members within a group. Once a bomber is found (or selected) the interceptors might attack one at a time or they might choose to form subgroups to make simultaneous attacks. If the bomber could not respond to all of the attackers simultaneously, forming subgroups could greatly increase interceptor effectiveness. However if the bomber could respond to all attacks up to the limit of the bomber's defensive weapon load size, simultaneous attacks would be a poor tactic on the part of the interceptors. By attacking simultaneously, each attacker is subjected to the bomber's weapons and uses
up his own weapons although the first attacker might have been able to achieve an unaided kill. A proof of the claim that simultaneous attacking by interceptors is not a good tactic when the bomber can use all of its defensive weapons is given in Appendix A. In this paper it will be assumed that the bombers have this capability or that the interceptors cannot achieve the coordination required for simultaneous group attacks. The interceptors will therefore be assumed to attack in a serial fashion.

Conclusions

Modern military planning places a large emphasis on the value of air power. This will apparently continue to be true in the foreseeable future. Thus, (as in the past) military leaders will probably devote a large portion of their resources to the development of aircraft and related systems (such as weapons carried by the aircraft). If these resources are to be used wisely it is necessary that optimum or at least near optimum tactics for fighting the air battle be determined.

Modern technology has given the military the option of selecting from a large number of potentially beneficial weapons to use in fighting the air war. If the best of these options is to be selected we need to be
able to evaluate the effectiveness of each of these weapons when it is used in a reasonable manner. Thus we need to be able to determine the best or nearly best manner in which to use the weapons at our disposal, if a proper choice of weapons is to be made.

Many of the weapons that are of interest to military planners tend to be quite large. Thus aircraft can only carry a limited number of such devices. This leads to the possibility that an aircraft so armed will run out of weapons during the course of the battle. It also suggests that an intelligent enemy may be able to select his tactics so that the effectiveness of these weapons is reduced due to their limited supply. Any analysis that attempts to address this problem will almost certainly have to use models that consider the limitation in the available supply of such weapons.

Thus there is a very real need to be able to define optimum tactics to be used in fighting an air battle when the weapons that will be used are in limited supply. The next chapter will give a review of existing battle models to see how well they can handle this problem. It will be found that the present models are lacking and thus there is a need for a different approach.

The rest of this volume will then center on the development of a family of models that will be capable of
analyzing a battle when the ammunition supply is limited. In addition to an expected value model, two especially simple bounding models will be developed.
Lanchester’s Formulations

One of the earliest attempts to formulate a model of an air battle is found in the work of F. W. Lanchester (Reference 1). Lanchester proposed two possible models for air battles – the square law and the linear law.

The Square Law

Consider the following situation. Assume that the interceptors and bombers each attempt to attack the other throughout the battle and each aircraft is able to continually make attacks upon enemy aircraft. The attrition rates are thus proportional to the number of opponents. Let $E_i$ and $E_b$ represent the effectiveness or fighting value of an individual member in the interceptor and bomber forces, respectively. $E_i$ and $E_b$ have units of kills per attacker per unit time. Let $I(t)$ and $B(t)$ represent the number of interceptors and bombers in action at time $t$. The battle is thus represented by the
Following differential equations:

\[ \frac{dI(t)}{dt} = -B(t) \frac{E_b}{I(t)} \]  \hspace{1cm} (1)

\[ \frac{dB(t)}{dt} = -I(t) \frac{E_i}{I(t)} \]  \hspace{1cm} (2)

Therefore:

\[ \frac{dI(t)}{dB(t)} = \frac{B(t)}{I(t)} \left( \frac{E_b}{E_i} \right) \]  \hspace{1cm} (3)

\[ I(t) \frac{dI(t)}{dB(t)} = \left( \frac{E_b}{E_i} \right) B(t) dB(t) \]  \hspace{1cm} (4)

Integrating both sides gives:

\[ I^2(0) - I^2(t) = \left( \frac{E_b}{E_i} \right) (B^2(0) - B^2(t)) \]  \hspace{1cm} (5)

The name square law is a result of the appearance of squared terms in equation (5).

Thus we have a relationship between the size of both forces at various points in time. One can now find the forces remaining on one side given the forces remaining to the other side. Lanchester noted that the condition for equality (which is defined to be the condition that leads to both sides reaching a zero force level at the same instant of time) is:

\[ \frac{dB(t)}{dt} \bigg|_{B(t)} = \frac{dI(t)}{dt} \bigg|_{I(t)} \]  \hspace{1cm} (6)
Substituting (1) and (2) into (6) yields:

\[-I(t) \frac{E_i}{B(t)} = -B(t) \frac{E_b}{I(t)} \quad (7)\]

\[E_i I^2(t) = E_b B^2(t) \quad (8)\]

Therefore the fighting strength of the two forces are equal when the fighting value of the individual units multiplied by the square of the numerical size of the two forces are equal.

The Linear Law

The linear law can be used to represent a situation in which two forces fire randomly at an area held by the other side without being able to detect individual opponents. The area being attacked is assumed to be independent of the number of surviving opponents. In this case the attrition rate is proportional to the number of targets offered to the enemy as well as the number of attackers. The differential equations describing this situation are:

\[\frac{dB(t)}{dt} = -B(t) I(t) E_i, \quad (9)\]

\[\frac{dI(t)}{dt} = -B(t) I(t) E_b \quad (10)\]

where \(E_b\) and \(E_i\) now have units of kills per attacker per target per unit time.
Dividing (9) by (10) yields:

\[ \frac{dB(t)}{dI(t)} = \frac{E_i}{E_b} \]  

(11)

Integrating both sides:

\[ B(0) - B(t) = \frac{E_i}{E_b} (I(0) - I(t)) \]  

(12)

The linear law proposed by Lanchester would also represent the situation where the battle is in reality a series of single duels where only one member from each side can participate in a given engagement. Consider only those engagements that result in the elimination of one of the combatants (simultaneous kills, which rarely occur, are assumed impossible). Let \( E \) represent the ratio of effectiveness of the individual units from the two forces (that is \( E_b/E_i \)). This ratio is traditionally called the exchange ratio of the two forces. The expected loss per duel for both sides would be:

\[ \text{Exp(bombers lost/duel)} = \frac{1}{1 + E} \]  

(13)

\[ \text{Exp(interceptors lost/duel)} = \frac{E}{1 + E} \]  

(14)

If time is now represented as a string of duels considered to occur with a uniform temporal spacing, then the following differential equations would represent the battle:

\[ \frac{dB(t)}{dt} = -\frac{1}{1 + E} \]  

(15)

\[ \frac{dI(t)}{dt} = -\frac{E}{1 + E} \]  

(16)
therefore:

$$\frac{dB(t)}{dI(t)} = E^{-1} = E_i/E_b$$  \hspace{1cm} (17)$$

which is identical to equation (11).

If the engagements are not equally spaced in time but are distributed by some function of the number of bombers still in action, the number of interceptors still in action, and the actual time of occurrence, then the differential equations become:

$$\frac{dB(t)}{dt} = \left(-\frac{1}{1/E}\right)F(B(t), I(t), t)$$  \hspace{1cm} (18)$$

$$\frac{dI(t)}{dt} = \left(-\frac{E}{1/E}\right)F(B(t), I(t), t)$$  \hspace{1cm} (19)$$

This formulation will also lead to equation (11) if $F(B(t), I(t), t)$ does not equal zero by simply dividing equation (18) by equation (19).

**Additional Contributions to Lanchester's Formulation**

Many authors have made additional contributions to the theory of Lanchester's formulation only a few of which will be reviewed here.

Helmbold (Reference 7) noted that Lanchester's square law assumes that attrition is independent of the size of the enemy's force. He felt that some adjustment was in order if the forces were in fact greatly different in size. He suggested changing the differential
equations to:

\[
\frac{dI(t)}{dt} = -E_b B(t) h(I(t)/B(t)) \quad (20)
\]
\[
\frac{dB(t)}{dt} = -E_i I(t) g(B(t)/I(t)) \quad (21)
\]

where \( h(I(t)/B(t)) \) and \( g(B(t)/I(t)) \) are functions added to account for differences in force size. Helmbold suggested that the following restrictions on these additional terms seemed reasonable:

\[
g(1) = h(1) = 1 \quad (22)
\]
\[
g(q) = h(q) \quad (23)
\]
\[
\frac{dh(q)}{dq} \geq 0 \quad (24)
\]

One possible form for these terms would be:

\[
h = (I(t)/B(t))^c \quad (25)
\]
\[
g = (B(t)/I(t))^c \quad (26)
\]

Equations (20) and (21) then become:

\[
\frac{dI(t)}{dt} = -E_b B(t)^{1-c} I(t)^c \quad (27)
\]
\[
\frac{dB(t)}{dt} = -E_i B(t)^c I(t)^{1-c} \quad (28)
\]

If \( c=0 \) the above equations become the square law.

If \( c \neq 1 \) the linear law results.

If \( c=1 \) the equations (20) and (21) become:

\[
\frac{dI(t)}{dt} = -E_b I(t) \quad (29)
\]
\[
\frac{dB(t)}{dt} = -E_i B(t) \quad (30)
\]
which yield as a solution:
\[
\left[ \ln(I_0) - \ln(I(t)) \right] / \left[ \ln(B_0) - \ln(B(t)) \right] = E_b / E_i \tag{31}
\]
which is called the logarithmic law. Note that this represents the situation where the attrition of each force is a function of its own size and not that of the enemy.

Brackney (Reference 8) introduced the notion of the effect of search for an opponent on the attrition rates. He defined the square law as:
\[
\frac{dB(t)}{dt} = -E_i r_i I(t) \tag{32}
\]
\[
\frac{dI(t)}{dt} = -E_b r_b B(t) \tag{33}
\]
where \( r_i \) and \( r_b \) are the rates at which the combatants attack each other and are thus a function of the combatants' ability to find each other. The rates of attack were further defined by Brackney to account for the number of opponents available and the attackers' searching capability.

Weiss (Reference 9) included the effect of groups of aircraft working together as a unit. The attrition equations for the linear law were defined as:
\[
\frac{dB(t)}{dt} = -E_i I(t) B(t)/a \tag{34}
\]
\[
\frac{dI(t)}{dt} = -E_b B(t) I(t)/b \tag{35}
\]
where $a$ and $b$ are the number of units in each group of bombers and interceptors respectively. If $a$ equals $b$ equals one (implying no grouping) this is the linear law. If $a$ and $b$ equal the total forces available, this becomes the square law.

The solutions given by Lanchester do not contain time as a variable. Thus they do not convey the time at which kills occur. Morse and Kimball (Reference 2) give a time solution for the square law while time solutions for the linear and logarithmic law can be found in Clark (Reference 13).

Morse and Kimball (Reference 2) also included factors in the differential equations to account for the replacement of forces and losses due to factors other than enemy actions. Several authors (See References 10, 11, and 12) have proposed methods of evaluating the attrition rate coefficients ($E_i$ and $E_b$) when the probability of kill is a function of the results of previous encounters. Clark (Reference 13) and Snow (Reference 14) have explored the effect of heterogeneous forces upon Lanchester's formulation. Clark also considered the effect of mobile forces by allowing the parameter values to change during the battle. Gamow and Zimmerman (Reference 19) included the effect of adding a reserve force at some point in the battle.
More Exact Solutions

Many authors have concerned themselves with the difference between Lanchester's deterministic model and a stochastic model. By deterministic, it is meant that Lanchester's equations are based on average or expected values for the results of each engagement. From a probabilistic standpoint the results of a specific sequence of encounters may be quite different from those predicted by Lanchester. Also, Lanchester's equations are considered to give results that apply up to the specific point in time when one of the forces will be annihilated. If there is a possibility that one of the forces is destroyed before the predicted time of battle termination, the average outcomes defined by Lanchester's differential equations are biased. For example, solutions to equations (8) and (9) represent the average outcome of the accumulated battles up to the time \( t \), which includes the possibility that one side might achieve all kills in all encounters up to that time (which is, of course, in general highly unlikely). Now if this were to occur and if the number of encounters that occur up to time \( t \) exceeds the opposition's initial force size, then this is in reality not a possibility. Thus the average rate of attrition as specified by the differential equations is in error.
This error will occur whenever the battle lasts long enough to allow the possibility that one or the other side has been destroyed. As this possibility increases, so does the probable error in Lanchester's results.

Morse and Kimball (Reference 2) recognized this fallacy in Lanchester's models and defined probabilistic models (stochastic models) that correspond to both the linear law and the square law proposed by Lanchester. Only the results that apply to the linear law version will be reviewed here. The results of \( n \) encounters defined by equations (15) and (16) were noted by Morse and Kimball to have a multinominal distribution. On the average each encounter would result in the loss of a bomber with probability \( \frac{1}{(1+E)} \) and an interceptor with probability \( \frac{E}{(1+E)} \). Therefore the probability that there are \( x \) bombers and \( y \) interceptors lost (where \( x+y = n \)) would be:

\[
P(x,y) = \frac{n!}{(x! \cdot y!)} \frac{E^x}{(1+E)^n}
\]

where: \( n \leq B(0) \quad n \leq I(0) \)

The requirements that the number of encounters not exceed the initial force size of both sides is included to avoid the situation where force sizes might go to zero.

The following equations were given by Morse and Kimball to represent the case where one or the other
force goes to zero:

\[ P(x, I(0)) = \frac{(x+I(0)-1)!}{x! (I(0)-1)!} \frac{E^x}{(1+E)^{x+I(0)}} \]  

(37)

\[ P(B(0), y) = \frac{(y+B(0)-1)!}{y! (B(0)-1)!} \frac{E^B(0)}{(1+E)^{y+B(0)}} \]  

(38)

\[ P(B(0), I(0)) = 0 \]  

(39)

An example was presented by Morse and Kimball that showed that the average results based upon the above probability distribution when compared with the results obtained using Lanchester's equations were not in complete agreement when the number of encounters exceeded either initial force size. The authors speculated that the difference between the exact results from equations (36) through (39) and Lanchester's predictions would diminish as the force sizes increased. Snow (Reference 14) also considered this error in Lanchester's results and concluded that Lanchester's equations were good approximations to the mean value of the stochastic solution in the early stages of battle. Clark (Reference 13) also studied this problem and found that although large forces resulted in smaller percentage errors, the absolute error might remain significant.
Brown (Reference 3) developed a simplifying approximation to the final battle outcome based on the probability distribution defined by equations (36) through (39) since the probability distribution is quite difficult to use in its present form. Assume that at some point in time $B(t)$ and $I(t)$ represent the remaining combatants. The next casualty will be a bomber with some probability say $A(B(t), I(t))$. The probability that the next casualty is an interceptor is $1 - A(B(t), I(t))$. The problem posed by Brown was the determination of the probability that the bomber force will be annihilated assuming that the battle continues until one or the other side is completely destroyed. This probability as a function of $B(t)$ and $I(t)$ is denoted as $P(I(t), B(t))$ where $P(x, 0) = 1$ and $P(0, x) = 0$ for all $x$ greater than zero.

The probability of destroying the bombers as predicted at time $t$ is related to the probability as predicted at the next instant of time (that is, after the next encounter) by the following equation:

$$P(I(t), B(t)) = A(B(t), I(t)) P(B(t), I(t)-1)$$

$$+ (1-A(B(t), I(t))) P(B(t)-1, I(t))$$  \(40\)
with boundary conditions:

\[ P(x,0) = 1 \]
\[ x \text{ any positive integer} \quad (41) \]

\[ P(0,x) = 0 \]

Limiting the discussion to Lanchester's linear case, \( A(B(t), I(t)) \) simplifies to a constant \(-p\).

\[ P(x,y) = p P(x,y-1) + (1-p) P(x-1,y) \quad (42) \]

This is the same problem as that posed by Morse and Kimball, with solution:

\[ P(x,y) = \sum_{j=0}^{x-1} \left( \frac{((y+j-1):(y-1):j:)}{((y-1):j:)} \right) p^y (1-p)^j \quad (43) \]

However since this problem can be represented as a sequence of Bernoulli trials with constant probability of success \(p\) (see Reference 4), and \(P(x,y)\) is the same as the probability of at most \(x-1\) failures in \(x+y-1\) trials, an alternate form for (43) is:

\[ P(x,y) = \sum_{j=0}^{x-1} \left( \frac{((x+y-1):(j):(x+y-1-j):)}{(j):(x+y-1-j):)} \right) (1-p)^j p^{x+y-1-j} \quad (44) \]

This second form for \(P(x,y)\) does not appear to be any simpler than that originally proposed by Morse and Kimball. However Brown noted that (44) can be
approximated using the Theorem of DeMoivre and Laplace (approximating $P(x,y)$ by a normal distribution) by:

$$\frac{1}{(2\pi)^\frac{1}{2}} \int_{-\infty}^{w} \exp(-t^2/2) \, dt \quad (45)$$

for a suitably chosen $w$. Brown suggests the following form for $w$ which is claimed to yield a fairly accurate approximation:

$$w = \frac{(px-(1-p)y)}{((x+y)p(1-p))^{\frac{1}{2}}} \quad (46)$$

Although this final solution (45) is simpler than the results obtained by Morse and Kimball, it is still not in a very convenient form. Also since (45) is an approximation, it may not always be more accurate than the deterministic model developed by Lanchester. Thus although it would be preferable in some respects to work with the more exact stochastic models, they do not appear to presently be computationally convenient. It should also be recognized that Brown's solution concerns only the probability that all of the bombers are destroyed assuming a battle to the finish. Solving the problem posed in Chapter I requires a more complete solution. Not only is there interest in cases where neither side will necessarily be destroyed, but there is also a need to obtain a time history of attritions to
evaluate effectiveness as defined by targets reached.

Since the problem posed by Lanchester has not yielded a very satisfactory stochastic solution, it is highly probable that it would be very difficult to obtain a useful stochastic solution to the more complex problem posed in Chapter I. The solutions to be developed in this volume will therefore be deterministic in nature.

**Thomas' Formulation**

Clayton Thomas (Reference 15) offered a different approach to the formulation of the air battle problem, in which the battle is visualized as a series of sub-battles. Thus Thomas saw the battle as a discrete process rather than a continuous process as indicated by Lanchester's formulation. In each sub-battle a force of interceptor aircraft engaged the bombers some of which were mortally wounded in the process. All bombers were assumed to be lost between the sub-battles. Thus a 'killed' bomber was available for additional attacks during the course of the sub-battle. The bombers were not assumed to be able to destroy interceptors and thus the attrition of one side only was considered.

Thomas proposed two types of assignment of
interceptors to bombers, a uniform assignment and a random assignment. In the first case the interceptors were divided up evenly among the bombers while in the second case each interceptor randomly selected a bomber to attack with the probability of a given bomber being selected in sub-battle \( n \) being \( 1/B(n-1) \) (where \( B(n-1) \) is the number of bombers surviving sub-battle \( n-1 \)). For these two cases Thomas gives the following solutions:

**Uniform Assignment**

\[
B(n) = B(n-1) \frac{(1-k)l_{n-1}/B(n-1)}{B(n-1)}
\]  
(47)

**Random Assignment**

\[
B(n) = B(n-1) \left( 1 - \frac{k}{B(n-1)} \right) l_{n-1} \tag{48}
\]

\[= B(n-1) \exp \left( -\frac{kI(n-1)}{B(n-1)} \right) \]

where \( k \) is the probability of an interceptor killing a bomber given an encounter.

**A More Recent Approach**

A different approach to the problem of analyzing air battles is found in Reference 5 by Fawcett and Jones. Consider the case where the penetrating bombers are equipped with defensive missiles (type one or two) and/or decoys (type three weapons). The defense is assumed to be unable to distinguish between bombers and decoys. The ratio of the number of decoys to the number
of bombers is assumed to remain constant throughout the battle. This last assumption requires that both bombers and decoys are equally likely to be attacked and if the bomber has defensive missiles of type one they are used to defend decoys as well as the bombers. This of course makes it necessary that the decoys remain in the geographic vicinity of the bombers.

The interceptors are assumed to operate in a cyclic fashion. That is they take off, make a series of attacks, land, rearm, and repeat the process until all the bombers are destroyed or have left the area to be defended or all interceptors are destroyed. The time to complete one cycle is assumed to be a known constant throughout the battle.

During each cycle a number of bombers are removed from the battle by interceptor action and a number of the interceptors are removed because of bomber defensive actions. An interceptor may also be removed at the end of a cycle due to failure to recycle for one of several reasons.

The model developed in Reference 5 is based on the assumption that the event "the interceptor kills the bomber" is stochastically independent of the event "the bomber kills the interceptor". Thus it is possible to have simultaneous kills during a given engagement. The
probability that the interceptor kills the bomber and the probability that the bomber kills the interceptor during an encounter are treated as constants throughout the battle. This model will therefore represent only situations in which the bombers are initially unarmed or have a sufficient number of defensive weapons to assure never running out.

Since it is assumed that only one side (the interceptors) is aggressive, the number of encounters is proportional to the number of interceptors. The number of interceptors lost in a given cycle is represented as:

$$\triangle I(n) = -(1-(1-P_{ki})^{c}P_{c}) I(n-1)$$ (49)

where

n = the number of cycles that have occurred.
I(n) = the number of surviving interceptors at the end of the nth cycle.
c = the number of attacks an interceptor will make per interceptor cycle assuming that the interceptor is not lost previously.
P_{c} = the probability that an interceptor fails to recycle for reasons other than bomber actions.
Next consider the attrition of bombers. The joint probability that a given attack is made at a bomber rather than a decoy and that the interceptor kills the bomber is given by:

$$\beta = \frac{P_{kb}}{1 + D_b}$$

(50)

where

$P_{kb}$ = the conditional probability that the interceptor kills the bomber when the interceptor makes an attack given that the object under attack is a bomber not a decoy.

$D_b$ = Number of decoys in the air per bomber.

Now the probability that the interceptor survives an encounter is simply $1 - P_{ki}$ and so the expected number of bombers killed per cycle per interceptor is:

$$\beta + (1 - P_{ki}) \beta + (1 - P_{ki})^2 \beta + \cdots + (1 - P_{ki})^{c-1} \beta$$

(51)

$$= \beta \sum_{j=0}^{c-1} (1 - P_{ki})^j = \beta (1 - (1 - P_{ki})^c) / P_{ki}$$
In the event that $p_{ki} = 0$ (indicating that the bomber has no weapons of type one):

$$\lim_{p_{ki} \to 0} \beta (1-(1-p_{ki})c)/p_{ki} = \beta c$$  \hspace{1cm} (52)

Substituting equation (50) into equations (51) and (52) yields the following expressions for the number of bombers killed per cycle per interceptor:

$$\theta_1 = \frac{p_{kb}(1-(1-p_{ki})c)/((1+d_b)p_{ki})}{p_{ki} \neq 0} \hspace{1cm} (53)$$

$$\theta_1 = \frac{p_{kb}c/(1+d_b)}{p_{ki} = 0} \hspace{1cm} (54)$$

The number of bombers lost per cycle will then be:

$$\Delta B(n) = -\theta_1 I(n-1) \hspace{1cm} (55)$$

Letting $\theta_2 = (1-p_{ki})cpp_c$, the number of interceptors lost per cycle becomes:

$$\Delta I(n) = -(1-\theta_2) I(n-1) \hspace{1cm} (56)$$

The solutions to equations (55) and (56) are:

$$I(n) = I(0) \theta_2^n \hspace{1cm} (57)$$

$$B(n) = B(0) - I(0) \theta_1 (1-\theta_2^n)/(1-\theta_2) \hspace{1cm} p_{ki} \neq 0$$

$$= B(0) - I(0) \theta_1^n \hspace{1cm} p_{ki} = 0 \hspace{1cm} (58)$$
Continuous versions of equations (57) and (58) are obtained by letting \( n = t/\tau \) (\( \tau \) = the time to complete one cycle) which yield:

\[
I(t) = I(0) \theta_2^{t/\tau} \tag{59}
\]

\[
B(t) = B(0) - I(0) \theta_1 (1 - \theta_2^{t/\tau})/(1 - \theta_2) \quad P_{ki} = 0
\]

\[
= B(0) - I(0) \theta_1 t/\tau \quad P_{ki} = 0 \tag{60}
\]

**Limitations of Existing Models**

The Lanchester square law contains several assumptions that are inconsistent with the problem outlined in Chapter I. It should be noticed that the effectiveness of a fighting unit is considered constant throughout the battle. This is certainly not the case when it is possible that the bombers may run out of ammunition. It may be thought that this objection can be handled by allowing the bomber effectiveness to change at some point in time. However, it should be realized that different bombers will normally run out of weapons at different points in time. Thus the average effectiveness (\( E_b \)) should change with time from the point where the first bomber runs out of ammunition to the point where the last survivor's weapons are exhausted. It would seem that no simple adjustment of \( E_b \) could account for this problem.
A second problem concerns the fact that both sides are assumed to be aggressive in Lanchester's square law. In particular, note that the interceptor attrition is a function of the number of remaining bombers. In the situation outlined in Chapter I, the bombers are not assumed to hunt out interceptors, but use their defensive weapons only when under attack. The bomber interceptor problem would more reasonably consider the attrition of both forces to be proportional to the number of interceptors remaining.

Consider next the linear law. Since \( F(B(t), I(t), t) \) is any general function, it can be defined so that the attrition on both sides is a function of only the interceptor force size and not the bomber force size. This would then represent the case where only the interceptors are aggressive. However nothing in the above formulation indicates how \( F(B(t), I(t), t) \) can be defined to yield a realistic rate of attack. Since the problem posed in Chapter I treats effectiveness as a function of time (since the bomber's targets occur at different points in time), any solution of value would have to define the time solution. Also, since effectiveness \( (E_b) \) is again treated as constant throughout the battle, this solution has the same drawbacks as those cited under the square law when
investigating aircraft that may run out of ammunition.

The logarithmic law represents the situation in which the attrition of each force is a function of its own size and not that of the enemy. In cases where this law applies, the losses by both forces would not be caused by the opposition but would be due to other causes such as unreliable operation of the aircraft. The relatively high reliabilities generally assumed for modern aircraft should make such non-combat losses quite small in comparison to combat type losses. Thus this law would not seem to be appropriate for the study of the problem outlined in Chapter I.

Although Thomas' model and Fawcett and Jones' model do represent the situation in which only the interceptors are aggressive, those models also fail to consider several important factors contained in the problem outlined in Chapter I. The effectiveness of an aircraft is assumed to be constant throughout the battle in those formulations. This would not be true if the bombers are very limited in their ammunition carrying capability (which is certainly true today). Since the inability to carry large numbers of highly sophisticated weapons may greatly limit the effectiveness of such devices, it is necessary that any model used to predict the value of such weapons include this limitation.
The assumption of constant kill probabilities per encounter throughout the battle also ignores the possibility that the bombers or interceptors might alter their firing doctrine with time. The bombers may initially try to conserve ammunition but later in the battle start to increase salvo sizes. The interceptors may at first attempt to make several attacks per cycle and thus limit the number of weapons fired per encounter. However in later stages, the possible scarcity of bombers could make multiple attacks per cycle less likely. In this situation the salvo size may well increase. Notice that changes in doctrine by the interceptors might be handled by adjusting the interceptors' kill probability with time. However an adjustment to handle bomber changes in doctrine requires a knowledge of the weapon supply available at later stages of the battle. Since this factor is ignored in the models discussed in this chapter, this knowledge would not be available.

It is also apparent that all of the models in this chapter treat the aircraft as isolated independent units (with the exception of the Lanchester Linear Law as defined by Weiss in Reference 9) that are not coordinated in any way. Since grouping of interceptors may give the defense the capability of minimizing the
effectiveness of defensive weapons, an analysis based on group effectiveness would seem to be required. This of course violates the random target selection (per interceptor) assumption built into all of the above models. Since these models ignored weapon-carrying capabilities, a random selection policy would appear to be as good as any other policy. The order in which bombers are selected for attack will not affect their attrition when the probability of kill is the same for all bombers. However a study of the problem outlined in Chapter I would include the effect of group actions and would not be modeled as a random selection process (except possibly by the group as a whole).

Thus there are several factors of interest (primarily associated with the limitation on the number of weapons carried by a bomber) that are not included in the models discussed in this chapter. The only consideration found in this literature search to ammunition-limited situations concerns the outcome of one-on-one duels (see References 16 and 17) and not total force effectiveness. Therefore there is a need for the development of new models which will be considered in the following chapters.
CHAPTER III

ESTABLISHING A LOWER BOUND
ON BOMBER EFFECTIVENESS

If some of the bombers have at their disposal a defensive weapon that reduces the effectiveness of attacking interceptors a good tactic for the defense to follow would be to attack unarmed bombers rather than armed bombers whenever possible. In fact the defense may want to plan its actions so as to create (by weapon exhaustion) unarmed bombers for future attacks by the interceptors.

Consider the following simple but illustrative example. There are two penetrating bombers each armed with one missile that has 80 per cent probability of killing an attacking interceptor. These bombers are opposed by two interceptors each armed with one missile that has an 80 per cent probability of destroying an unarmed bomber. However the interceptors must survive the encounter with the bomber's missile if one is used before they can launch their missile. That is, the
bomber is capable of making a kill before the interceptor can launch its weapon. Now if a bomber has no weapon available the probability that the interceptor kills the bomber \( P_{kb/w} \) is .8. If the bomber has a weapon the probability that the interceptor kills the bomber is:

\[
P_{kb/w} = (\text{Probability that the interceptor survives the bomber's weapon}) \\
(\text{Probability that the interceptor's missile can destroy the bomber when the interceptor survives})
\]

\[
= (1 - .8) (.8) = .16 
\] (61)

Now assume that the first interceptor attacks one bomber after which the second interceptor attacks the other bomber. This plan of attack will yield the following results:

First Attack - Bomber #1 killed with Probability .16

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Interceptor #1</td>
<td></td>
<td></td>
<td>.8</td>
</tr>
</tbody>
</table>

Second Attack - Bomber #2 killed with Probability .16

<p>| | | | |</p>
<table>
<thead>
<tr>
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<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Interceptor #2</td>
<td></td>
<td></td>
<td>.8</td>
</tr>
</tbody>
</table>

Therefore the expected number of bombers killed is 0.32 and the expected number of interceptors killed is 1.60.

Next assume that the first interceptor still attacks
bomber #1, but that the second interceptor also attacks bomber #1 if it is still alive, otherwise the second interceptor attacks bomber #2. In this case the results of the first attack will be the same as before. Since the bomber survived the first attack with probability .84, the probability that the second interceptor kills bomber #1 is:

\[
\text{Bomber #1 killed with Probability } 0.84 (P_{kb/w}) = 0.672
\]

and the probability that bomber #2 is killed becomes:

\[
\text{Bomber #2 killed with Probability } 0.16 (P_{kb/w}) = 0.0256
\]

Since the second interceptor can be killed only if Bomber #2 is attacked, the probability that Interceptor #2 is killed is:

\[
\text{Interceptor #2 killed with Probability } 0.16 (P_{ki}) = 0.128
\]

where \( P_{ki} \) = Probability of kill of a single interceptor attacking an armed bomber.

Thus for this second plan of attack the expected number of bombers killed is 0.8576 and the expected number of interceptors lost is 0.928. Certainly this outcome is more desirable than the previous one to the defense.

Assuming that bombers are so numerous that an interceptor can always find another bomber to attack if
another interceptor destroys the intended bomber, it is always beneficial to the defense to gang up on particular bombers until they are destroyed and then move on to new targets. In fact if bombers are so numerous that new targets can always be found so that all interceptors use up all of their weapons without an appreciable time delay, then the optimum tactic by the defense would be to place all of the interceptors in one large group and attack selected bombers until they are destroyed. Of course if this tactic were used in real combat not only might it not be possible to find (or reach) the additional bombers but it would also tend to increase the average time between attacks by an interceptor. Some interceptors may have to fly from one bomber to another for some time before getting a chance to attack. However if it is assumed that this flying time from bomber to bomber has no appreciable effect on the total time of battle (maybe time to rearm, time to take off, etc., is much much greater than any possible time in battle) and that additional bombers can be reached to allow the use of all of the interceptors' weapons, then forming one group would maximize the defense's effectiveness.

Therefore a lower bound on the survivability of the bombers can be obtained based on the above
assumptions. Certainly the interceptors can do no better. If the above assumptions are not true then the effectiveness of the defense will be less than that predicted using these assumptions.

Interpretation of Defensive Weapon Effects

The use of the model to be developed to establish a lower bound on the bomber penetration problem will require an interpretation of the effect of the various defensive weapon types the bomber could use on the model parameters. It will be assumed in this chapter that the bomber is limited to using only one type of weapon. Limiting our consideration to one weapon type only will yield some initial insight into the problem while avoiding the additional complications (which will be considered in a later chapter) inherent in studying situations where more than one weapon type is available. It also is possible that economic considerations will not allow the development and/or production of more than one type of defensive weapon. Thus, this limitation would often be encountered in real life problems.

If the bomber has missiles that are intended to destroy the interceptor, the effectiveness of these weapons can be represented as:
\[ P_{ki} = \text{(Probability of the missile killing the interceptor if launched and guided (when necessary) to the target) (Probability of missile being launched and guided (when necessary) to the target).} \] (62)

The probability of killing the interceptor is thus made up of two components. The first term simply gives the effectiveness of an unopposed launch while the second term gives the probability of the launch. This second term is a function of the distribution of firing times of both combatants. Notice that if the counter missile requires guidance from the bomber, the bomber must survive until (approximately) the time of missile impact with the interceptor for the missile to function. If on the other hand the missile needs no guidance after launch, the bomber need only survive to the launch point to achieve a possible interceptor kill. It is assumed in this expression that an unguided missile will not be effective.

The determination of the probability of the interceptor killing the bomber given that a bomber is found will require consideration of two factors similar to those found in expression (62).
\[ P_{kb/w} = \text{(Probability of the interceptor killing the bomber given that a missile (or missiles) is launched and guided (when necessary) to the target) (Probability that a missile (or missiles) is launched and guided (when necessary) to the target).} \quad (63) \]

Again an unguided missile is assumed to be ineffective. If the bomber has no defensive missiles the second term becomes unity and:

\[ P_{kb/w}^* = \text{(Probability of the interceptor killing the bomber given that a missile (or missiles) is launched and guided to the target).} \quad (64) \]

If the bomber uses missiles that are intended to destroy interceptor missiles, the interceptor can kill the bomber only if the counter missiles fail to destroy the threatening missiles.

\[ P_{kb/w} = (P_{kb/w}^*) \quad \text{(Probability that the bomber's anti-missile missiles do not destroy or disrupt the interceptor missiles).} \quad (65) \]
In this volume it will be assumed that the information required to estimate the various kill probabilities is available.

**Derivation of the Model**

The solution to the lower bound problem will involve several approximations some of which are not unlike those found in Lanchester's solutions (Reference 1). Thus these solutions will contain some bias which could be troublesome. However using the bounds in the manner to be presented in this volume should not cause any trouble even though they are not exact. After all of the bounds are calculated, the largest of the lower bounds will be compared with the upper bounds (to be defined in Chapter IV) for purposes of rejecting some of the options. Since this procedure could never justify eliminating the option yielding the largest lower bound, this particular option will remain a candidate for the optimum solution, and so, will be evaluated using the expected value model. If the expected value solution exceeds the lower bound solution defined by the equations to be developed in this section, then the bound was legitimate and no problem results. If the lower bound should exceed the expected value solution then the bound was not legitimate. However this problem
can be easily rectified by repeating the comparison process using the expected value solution in place of the lower bound solution.

The lower bound solution will require a determination of the expected number of attacks required to kill a bomber. Assuming a nonunity probability of kill, there would be some probability of a bomber kill requiring any number of attacks from one to infinity. Considering all of these possibilities, the expected number of attacks for a kill can be determined. However since there are only a finite number of interceptors available, it is not strictly correct to include all of the possibilities out to an infinite number of attacks. Including these possibilities results in a bias in the model. Since the number of available interceptors will normally be quite large, the approximation found by considering all possibilities even when the available force size is exceeded should not be very inaccurate and will be used in the analysis.

The solution procedure will also involve the determination of the expected number of interceptors killed per bomber killed. This will require a consideration of all possible kills where each bomber could kill up to k interceptors (where k is the number
of anti-interceptor missiles carried by each bomber). However if \( k \) times the expected number of bombers killed exceeds the available number of interceptors, using the expected number of interceptors killed per bomber killed calculated as outlined above, will not be strictly correct. Thus this calculation might also be an approximation. If the number of available interceptors is quite large then this approximation should not be very inaccurate.

Assume that the bombers have \( k \) defensive weapons (or \( k \) salvos of weapons) one of which will be used against each attacking interceptor. If a bomber used a defensive weapon its probability of survival is \( (1 - P_{kb/w}) \) and if a weapon is not used the probability of survival is \( (1 - P_{kb/w}) \) where \( P_{kb/w} \ll P_{kb/w} \). Each bomber that is killed will receive at least one attack. If the bomber survives the first attack, then it will receive a second attack if a kill is to occur. When the bomber survives the \( i \)th attack, then it will take at least \( i + 1 \) attacks for a kill. Therefore if a bomber is attacked repeatedly until killed:

\[
\text{Exp. (#attacks/kill)} = \\
1 + 1 \sum_{i=1}^{\infty} (1-P_{kb/w})^{i-1} \sum_{j=0}^{k} \frac{1}{j!} (1-P_{kb/w})^{j} + \\
1 \sum_{i=1}^{\infty} (1-P_{kb/w})^{i} (1-P_{kb/w})^{i} + \\
+ \ldots + 1 \sum_{i=1}^{\infty} (1-P_{kb/w})^{i} (1-P_{kb/w})^{i}
\]
This expression can be simplified by using the following relationships (see Reference 6):

\[ \sum_{i=0}^{\infty} (1-P_{kb/w})^i = \frac{1}{P_{kb/w}} \quad (67) \]

\[ \sum_{i=0}^{k-1} (1-P_{kb/w})^i = 1 - (1-P_{kb/w})^k \quad (68) \]

\[ \therefore \text{Exp. (# attacks/bomber killed)} = \frac{1-(1-P_{kb/w})^k}{P_{kb/w}} + (1-P_{kb/w})^k \frac{1}{P_{kb/w}} = ENA \quad (69) \]

Therefore the expected number of bombers killed during the nth cycle as a function of the number of interceptors surviving the (n-1)st cycle is approximately:\(^\dagger\)

\[ \Delta B(n) = - \frac{I_{(n-1)}}{ENA} = - \theta_1 I_{(n-1)} \quad (70) \]

where \( \theta_1 = 1/ENA \)

\(^\dagger\) ENA is the expected number of attacks/kill while what we really want is the expected number of kills/attack. However the expected number of kills/attack is approximately 1/ENA.
Notice that it is assumed that all surviving interceptors make one attack per cycle.

The expression for ENA can also be arrived at by noting that the probability that a bomber is killed by the jth attacker is simply the product of the probability that the bomber survives the first (j-1) attacks and the probability that the bomber does not survive the jth attack. Therefore:

\[
\text{Exp. (# attacks/kill)} = 1 \times \frac{P_{kb/w}}{2} + \sum_{j=1}^{k} \frac{P_{kb/w}(1-P_{kb/w})^{k-j}}{j!} + \frac{P_{kb/w}(1-P_{kb/w})^{k}}{k!} + \frac{P_{kb/w}(1-P_{kb/w})^{k+1}}{(k+1)!} + \ldots
\]

This can also be reduced to equation (69).

Next consider the attrition of interceptors. The bomber will certainly counter-attack his first attacker (assuming the bomber uses anti-interceptor missiles). If the bomber survives the first engagement, a second weapon if available will be used in the next attack. Following this line of thought and letting \( P_{ki} \) be the probability of killing an attacking interceptor whenever a missile (or salvo of missiles) is used, the expected number of interceptors killed per bomber killed is:
\[ \text{Exp. (# interceptors killed/bomber killed)} = \]
\[ P_{ki} + (1-P_{kb/w})P_{ki} + \ldots + (1-P_{kb/w})^{k-1}P_{ki} \]
\[ + (1-P_{kb/w})^k 0 + (1-P_{kb/w})^k (1-P_{kb/w}) 0 \]
\[ = P_{ki} \sum_{i=0}^{k-1} (1-P_{kb/w})^i \]
\[ = P_{ki} \frac{1 - (1-P_{kb/w})^k}{P_{kb/w}} = \text{NIK} \quad (71) \]

Therefore the expected change in the number of interceptors in the nth cycle is:

\[ \Delta I(n) = \text{NIK} \quad \Delta B(n) = -\frac{\text{NIK}}{\text{ENA}} I(n-1) = -(1-\theta_2)I(n-1) \quad (72) \]

where \(1-\theta_2) = \frac{\text{NIK}}{\text{ENA}}\)

Equations (70) and (72) are the same as the difference equations discussed in Reference 5 (with the constants having different values however) and thus have as a solution equations (57) and (58).

Finally consider the case where the bomber depends upon decoys for a defense. The incorporation of this effect will require an adjustment in difference

\(^2\) Note it is assumed that the interceptor kills are equally likely to occur on any attack which is not strictly true, and thus, this is an approximation.
equation (70). First it must be determined how the interceptors are likely to react to the decoys. Will they be as likely to attack a decoy as a bomber? Will they try to destroy the decoys? If the interceptors attempt to destroy decoys, will the expected number of attacks per kill be different from the expected number required to destroy a bomber? If by chance a large number of decoys (not bombers) are attacked and destroyed during one cycle, might this not tend to reduce the ratio of decoys to bombers in future cycles? However if the decoys have relatively short ranges of flight and must be replenished from time to time (assuming the same deployment each time), then the ratio of decoys to bombers should remain approximately constant.

Obviously how the above questions are answered will determine the correct model formulation. In this volume the following assumptions will be made:

a) A decoy is as likely to be attacked as a bomber.

b) The bombers will continually replenish the decoys (due to their short flight times) maintaining the same ratio of decoys to bombers.
If the interceptors will not attempt to kill decoys, then the lower bound model will ignore the effect of decoys since the interceptors are assumed to be able to reach additional bombers to attack. If the interceptors will kill decoys then the following analysis applies:

Let $I(n-1)d$ be the number of attacks directed at decoys and $I(n-1)b$ be the number of attacks directed at bombers. Therefore:

\[ \Delta \text{Decoys} (n) = \frac{I(n-1)d}{ENA_d} \]  
(73)

\[ \Delta \text{Bombers} (n) = \frac{I(n-1)b}{ENA_b} \]  
(74)

where $I(n-1)d + I(n-1)b = I(n-1)$  
(75)

Since there are $N_d$ decoys in the air per bomber and objects are selected at random by the interceptors and attacked until destroyed before a next object is selected for attack in the lower bound model, then it follows that the expected number of decoys killed is $N_d$ times the expected number of bombers killed. Therefore:

\[ \Delta \text{Decoys} (n) = N_d \Delta \text{Bomber} (n) \]  
(76)

Substituting equations (73) and (74) into the above yields:

\[ \frac{I(n-1)d}{ENA_d} = N_d \frac{I(n-1)b}{ENA_b} \]  
(77)
Now using equation (75):

\[ \frac{I(n-1)}{ENA_d} = \frac{I(n-1)}{ENA_d} + N_d \left( \frac{I(n-1)}{ENA_b} \right) \]  

(78)

\[ I(n-1)_b = \frac{I(n-1)/ENA_d}{(1/ENA_d) + (N_d/ENA_b)} \]  

(79)

Substituting this relationship into equation (74) then yields:

\[ \text{Bombers}(n) = \frac{I(n-1)}{ENA_b + N_d ENA_d} = -\theta I(n-1) \]  

(80)

Thus a lower bound on bomber survivability can be readily obtained if the bomber uses only one of the types of defensive weapons considered. Situations where more than one type of weapon may be used will be considered in a later chapter. However initially it is probably better to focus attention on one defensive weapon candidate at a time and gain some initial insight into the problem.

An Example Problem

The Options Considered

The following example problem will be used to show how the models developed in this chapter and the following chapters can be used to analyze the bomber
effectiveness problem. Several of the questions raised in Chapter I will be considered in this example problem such as:

What defensive weapons should the bombers use?

How should the interceptors be grouped?

However, several of the options suggested in Chapter I will not be included. The following paragraph summarizes the options not analyzed.

The example problem will not consider the following:

Possible alternative weapon deployment policies for both the offense and the defense.

Weapon deployment policies that change with time.

Interceptor group sizes that change with time.

Bombers carrying more than one type of defensive weapons.

However, (as will be discussed in the following paragraphs) this does not imply that the procedures developed in this volume cannot be used to evaluate the above options. These options were not included to keep
the example problem from becoming too complicated, and thus, failing to be illustrative.

Weapon deployment could be considered by simply changing the assumed salvo size. This will change the kill probabilities and the number of available salvos which merely changes the inputs to the models developed in this volume. In fact there is no reason why the salvo size used by either the bombers or the interceptors has to be constant from cycle to cycle. Since the attrition for each cycle is calculated separately, the input values could be changed thus simulating changing weapon deployment policies.

In the example problem it will be assumed that the interceptors surviving one cycle can be regrouped for the next cycle. However it will also be assumed that the resulting groups will be of the same size as the original groups. There would seem to be at least two other possibilities that might be of interest. First it may be impossible to regroup interceptors, and so, the survivors of each group might have to be deployed as a unit. Second, if it is possible to regroup, it may be possible to find a better grouping for the next cycle than the grouping that was best in the first cycle. Either of these possibilities can be explored by changing the group size used in the models from
cycle to cycle. This would not effect the lower bound solution since the interceptor force is not divided up into groups in the lower bound model but would effect the upper bound and expected value solutions.

The models developed in Chapters III, IV, and V are based upon the assumption that only one type of defensive weapon is used by the bombers. However these models can be altered to include mixed bomber defensive weapon loads. The required adjustments are discussed in Chapter VII.

The Problem Formulation

Consider the following situation. A force of 100 bombers is to be sent to attack a certain region that is known to be defended by 200 interceptors. Each time an interceptor encounters a bomber it will be assumed that the interceptor fires one salvo of missiles, if possible (that is if not destroyed by a bomber missile). If an interceptor survives an engagement it will land, rearm, and attempt to initiate another attack. It is felt that each interceptor that is not destroyed by a bomber missile will be able to make two attacks before the bombers reach their final targets. All interceptors will be assumed to make their initial attack before any make their second attack.
The surviving bombers will be assumed to drop their bombs after the two rounds of attacks by interceptors are completed. Thus the battle scenario is one in which a region defended by interceptors must be penetrated before the targets are reached. However, once this region is penetrated it is assumed that there will be no further interceptor encounters. Thus the bombers desire to maximize the number of weapons that survive through the defended region. No consideration has to be given to the way in which the bombs will be used if we merely desire to select the best weapon mix, since maximizing the number of surviving bombs is all that is necessary to maximize bomber effectiveness. The more complicated situation in which targets are reached during the engagement with the interceptors will be considered in Chapter VIII.

The bombers are able to carry a maximum of 10 bombs each. However they must sacrifice some bomb carrying capability when defensive weapons are carried. The offense has the option of selecting decoys or one of several candidate missiles to use in bomber defense. However more effective missiles are assumed to take up more space. If anti-missile missiles are used, only one is deployed against the interceptor missile in a given encounter. If anti-interceptor missiles are used, it is assumed that only one can be fired per encounter.
Table 1 indicates the weapons under consideration for bomber defense. The decoy space trade off is that required to be able to deploy one decoy for the total duration of the battle. A decoy is assumed to be as likely to be attacked as a bomber and thus is considered a perfect decoy.\(^3\)

The interceptor missile is considered to have a 75 per cent probability of destroying an unarmed bomber. However the interceptor must survive the bomber's missile before a kill can occur. That is, the bomber always gets the first shot.

The problem is one of determining the effectiveness of bombers armed with various feasible pure loadings of each type of weapon (that is no mixed loadings of defensive weapons are considered). When defensive weapons are used, only the maximum loading in the space sacrificed need be considered. For example if a weapon replaces one fourth of a bomb, it is necessary to predict the outcome only when multiples of four of these weapons are carried. The offense should always attempt to add more defensive weapons (ignoring

\(^3\) The interceptors will attempt to kill the decoys which are as difficult to destroy as bombers.
<table>
<thead>
<tr>
<th>Weapon</th>
<th>Space Required (in bombs displaced)</th>
<th>Effectiveness</th>
</tr>
</thead>
<tbody>
<tr>
<td>Decoy</td>
<td>1.0</td>
<td>Considered perfect decoy</td>
</tr>
<tr>
<td>Missile #1</td>
<td>1.0</td>
<td>60% Prob. of Destroying Interceptor</td>
</tr>
<tr>
<td>Missile #2</td>
<td>1.5</td>
<td>75%</td>
</tr>
<tr>
<td>Missile #3</td>
<td>2.0</td>
<td>90%</td>
</tr>
<tr>
<td>Missile #4</td>
<td>0.25 per salvo</td>
<td>60%</td>
</tr>
<tr>
<td>Missile #5</td>
<td>0.375 per salvo</td>
<td>75%</td>
</tr>
<tr>
<td>Missile #6</td>
<td>0.50 per salvo</td>
<td>90%</td>
</tr>
</tbody>
</table>
the possibility of aircraft speed gains from reduced
weight carried) if the number of bombs carried is not
affected.

Further notice that some feasible combinations
are clearly better than others. Although the offense
could give up one bomb per bomber and carry two #5
missiles, it would obviously be better to carry two #6
missiles which also replace one bomb.

Table 2 gives a lower bound on bomber
survivability and effectiveness as determined by using
equations (70) and (72) for the logical weapon load
candidates not including load mixes containing more
than one type of defensive weapon.

Conclusions

A method of determining a lower bound on the
bomber penetration problem was developed for situations
in which the bomber uses either decoys, anti-interceptor
missiles, or anti-missile missiles in its defense
against a force of attacking interceptors. The number
of bombs on target was selected as the sole criterion of
effectiveness.

The resulting bounds, although possibly not very
close to the expected value solutions, do yield insight
into the problem. For example, some possible load mixes
<table>
<thead>
<tr>
<th>Defensive Weapon</th>
<th>Number/ Bomber</th>
<th>Bombs/ Bomber</th>
<th>Lower Bound on Surv. B(1)</th>
<th>Lower Bound on Bombs Surv. B(2)</th>
<th>Lower Bound on Bombs Surv. B(3)</th>
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<tbody>
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<td>Lower Bound on Bombs Surv.</td>
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may be eliminated from further consideration. In the example problem considered above load mixes were identified that resulted in a lower bound of approximately 250 bombs on target. Any load mix that devotes eight or more bomb spaces out of ten to defensive weapons can never result in more than 200 bombs on target and so can be ignored in future evaluations.

A lower bound may also be found that ensures an acceptable final result and so further search may be unnecessary. If it is felt that 250 or more bombs on target is adequate, five decoys or eight #6 missiles may be selected as an acceptable defensive loading. However since the lower bound contains several approximations and thus is biased, this might not be advisable unless the expected value solution were obtained for the particular loading selected.
CHAPTER IV

ESTABLISHING AN UPPER BOUND

ON BOMBER EFFECTIVENESS

The next step in arriving at a solution to the bomber penetration problem will be to establish an upper bound on the effectiveness of the bombers. Of course there is always one upper bound available in the initial weapon load carried by the force since, even if all bombers survive, they cannot deliver more than the initial number of bombs carried. As pointed out in Chapter III, this upper bound, although possibly not always too useful, might allow the elimination of some candidate weapon mixes when lower bounds are known. In the sample problem introduced earlier a lower bound greater than 200 was identified and thus all load mixes containing 200 or fewer bombs in the initial force can be eliminated. Our task is now to identify an additional upper bound that will hopefully allow the elimination of still more possible load mixes.

Consider the bomber penetration problem as seen
by the defense. Some or all of the attacking bombers will most likely be detected and responded to. This response will take the form of interceptors going out to meet the bombers. As discussed in Chapter III there may be a real advantage to the defense in sending out the interceptors in some useful group size. One response that the defense can and traditionally will make is to divide the attacking bombers into groups and send a group of interceptors after each group. That is, it is unlikely that any given bomber will be considered as a potential target for more than one group of interceptors. Instead the various interceptor groups will tend to divide the bombers up and respond only to their own part of the total. This philosophy of response will be assumed throughout the rest of this volume.

Now if each interceptor can attack a maximum of \( C \) bombers (due to weapon limitations) before returning to its base and there are \( N_i \) interceptors in one of these defensive groups, then it is possible that \( CN_i \) bombers could be attacked by the group. However if, as is likely, more than one attack must be made before some bombers are destroyed, not all \( CN_i \) bombers can be attacked by the group. In any event, the effectiveness of the defense will be partially determined by the number of bombers that can in fact be reached by a given
group of interceptors. Notice that as the group gets larger, the possible number of bombers that could be attacked if they could be reached will tend to increase while the number of bombers that actually can be reached based on flight range limitations will probably be unchanged. Thus forming a larger group is not always desirable.

From the preceding discussion it is apparent that any exact analysis of the problem must include a determination of the probable or expected number of bombers that can be reached by a defensive unit. In this analysis it will be assumed that the ground based radars have discovered all of the bombers, and that a group of interceptors is committed only when a bomber is discovered. Therefore it is assumed that whenever a group of interceptors take off the first target is always reached. However there will be some uncertainty about the ability of the interceptors to reach second, third, etc., bombers. This is certainly the situation if the interceptors had to initiate a random search for additional bombers. In fact, even if a ground control center was postulated to know where all targets are and tried to direct the interceptors to new targets, the ability of the interceptors to fly to the next target would be limited by range constraints. Thus there is
a question as to the interceptors' ability to find additional targets with their capability generally decreasing as the number of available bombers decreases.

Another source of potential uncertainty to the defense concerns the available defensive weapon supply of the bombers. Even if the initial loading is known through intelligence sources (which will be assumed to be true in this development), the defense's effectiveness would depend upon the number of weapons used in previous engagements. That is, if a bomber were attacked previously and survived, it would presumably have fewer defensive weapons remaining.

Establishing an Upper Bound

Since it is the intent of the present section of this volume to establish an upper bound on bomber effectiveness and not necessarily to arrive at an exact solution, several assumptions will be made that will allow an analysis of this problem without considering the ability of interceptors to reach additional bombers or the effects of past actions upon the outcome of present attacks. Since these assumptions will be defense conservative, the resulting solution will be an upper bound on bomber effectiveness.
First it will be assumed that the interceptors can attack only the initial bomber selected by the ground control system. Thus the resulting model will not allow the interceptors to attack any additional bombers. Since it was previously assumed that the interceptors can reach at least one target, the above assumption is definitely defense conservative. It should be pointed out, however, that the defense will be assumed to select a desirable group size based on a consideration of this assumption. Thus we are assuming an intelligent defense. This group size will probably tend to be small enough that the need for additional bombers is not too great. That is, it is possible that the interceptors can be grouped so that this assumption does not greatly reduce interceptor effectiveness, and thus the resulting bound may not be overly conservative.

The second assumption is that any bomber selected for attack at any point in the battle has a full load of defensive weapons. That is, no credit is given to the defense for depletion of weapon supplies in prior engagements. This too is obviously a defense conservative assumption. Again it is possible that the defense will be able to group so as to reduce the effect of this assumption on the results. If a bomber is attacked but
survives, the defense is given no credit for its action. Thus the defense may choose to group so as to give the bomber little chance of surviving the initial attack. The ability of the defense to select an optimum group size will of course depend upon the available information about the weapons carried by the bombers. The effect of the defense's knowledge of the bomber's weapon loading on the solution procedure is considered in the example problem at the end of this chapter and in Appendix B.

The following model results from the above assumptions. If the interceptors decide to attack bombers in groups of size \( N_i \), then at any point in time \( t \), no more than \( I_t(N_i) \) bombers can be attacked. Therefore the attrition of bombers in a given cycle can be represented as:

\[
\begin{align*}
\Delta B(n) &= (\text{Number of Bombers attacked}) \\
&= (\text{Prob. of a Bomber under attack not surviving}) \\
&= (\text{Min} \left( \frac{I(n-1)}{N_i}, B(n-1) \right)) \left(1-(1-P_{k_b/w})^N_i\right) \\
&= (\text{Min} \left( \frac{I(n-1)}{N_i}, B(n-1) \right)) \left(1-(1-P_{k_b/w})^N_i\right) \\
&= (\text{Min} \left( \frac{I(n-1)}{N_i}, B(n-1) \right)) \left(1-(1-P_{k_b/w})^k \left(1-P_{k_b/w}\right)^{N_i-k}\right) \\
&= (\text{Min} \left( \frac{I(n-1)}{N_i}, B(n-1) \right)) \left(1-(1-P_{k_b/w})^k \left(1-P_{k_b/w}\right)^{N_i-k}\right)
\end{align*}
\]
where \( k \) is the number of defensive missiles (or salvos) carried by a bomber.

The number of interceptors killed by each bomber attacked assuming the bomber uses a missile designed to destroy the interceptor is:

\[
\Delta I = P_{ki} \left( \frac{1-(1-P_{kb/w})^{N_i}}{P_{kb/w}} \right)
\]

\( \text{for } N_i \leq k \)

\[
= P_{ki} \left( \frac{(1-(1-P_{kb/w})^k}{P_{kb/w}} \right)
\]

\( \text{for } N_i > k \)

(82)

**Establishing an Upper Bound when Decoys are Used**

If decoys are used and are spread out away from the launching bomber (thus when a decoy is selected for attack it is not necessarily true that a bomber will be nearby), they will increase the number of possible targets available to the defense. Assuming that decoys are attacked, then the attrition rate will be reduced by the factor:

\[
\text{Attrition rate reduction} = \frac{1}{1+N_d}
\]

(83)

where \( N_d \) is the number of decoys deployed per bomber.
It should be noted that when decoys are the only defensive weapons used by the bombers, grouping of interceptors offers no advantage to the defense. In this case there is no missile supply to be depleted. Since grouping results in the possibility that some of the interceptors never make an attack, grouping is in fact undesirable. Thus the group size is best minimized by being set to one wherever possible.

Example Problem (Continued)

Consider the example problem outlined in Chapter III. Assume that the defense will know what weapon loading is ultimately chosen by the bombers and can alter its groupings accordingly.\(^1\) Thus the offense must select its loading after which the defense selects its tactics to minimize bomber effectiveness. The effect of lack of knowledge of the bomber loading on the groupings selected by the defense is considered in Appendix B. Since the 100 bombers are opposed by 200 interceptors, it would be foolish to commit less than

\(^1\) It will be assumed that the defense will redeploy the interceptors in the same size groups during each cycle of the battle. As mentioned in Chapter III, the models developed in this volume are not sensitive to this assumption. The group size can easily be changed from cycle to cycle.
two interceptors per bomber (since there are only 100 bombers it is not possible to utilize more than 100 groups of interceptors) if no decoys were deployed. If $N_i$ equals two, all 100 bombers would be attacked. The following probabilities then describe each attack upon a bomber carrying no defensive missiles.

- Prob. Bomber is killed by 1st Inter. $= 0.75$
- Prob. Bomber is killed by 2nd Inter. $= 0.25 \times 0.75 = 0.1875$
- Prob. Bomber Survives $= 0.25 \times 0.25 = 0.0625$

Thus after the first round of attacks there would be an expectation of only 6.25 surviving bombers while all 200 interceptors will survive. Even if the defense limits its response to a continuation of its initial policy (that is, fixing $N_i$ at two) the expected number of bombers surviving the second cycle would be down to 0.4. In this example $N_i$ will be considered constant throughout the battle.

If the offense deployed one or more decoy per penetrator, the defense could respond by sending out one interceptor per target (bomber or decoy) detected with a resulting expectation of destroying 150 objects (assuming all objects are eventually detected) some of which would be decoys and some of which would be bombers.
Since this is the best the defense could do under the upper bound assumptions, this solution should be used as the upper bound when an intelligent defense is assumed. The expected number of surviving bombers would then be:

\[
\text{Exp. (\# Bombers Surviving)} = 100 - \frac{150}{N_d + 1} \tag{84}
\]

Continuing the analysis for various decoy deployments yields the results given in Table 3.

Notice that if \(N_d\) is greater than two the upper bounds and the lower bounds (found in Chapter III) are identical. This follows from the fact that when three or more decoys are used it is expected that all interceptors are able to always make an attack. Since as noted previously grouping of interceptors offers no advantage in cases where missiles are not used in defense of bombers, the solution is the same as the lower bound.

**Upper Bounds Resulting when Missiles are Used**

Applying equations (81) and (82) to the sample problem introduced in Chapter III when missiles are used in bomber defense yields the results shown in Table 4. In each case \(N_i\) was selected to minimize the upper bound on bombs delivered. In calculating these results it was
### Table 3

Upper Bounds on Bomber Effectiveness When Decoys are Used

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<th>Number of Decoys/Bomber</th>
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<th>Upper Bound on Bombs Surv.</th>
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### Table 4

**Upper Bounds on Bomber Effectiveness**

When Defensive Missiles are Used

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<td>Bombs/Bomber</td>
<td>$N_i$</td>
<td>Upper Bound on Surv. B(1)</td>
<td>Upper Bound on Surv. B(2)</td>
<td>Upper Bound on Bombs Surv.</td>
</tr>
<tr>
<td>-------------------</td>
<td>---------------</td>
<td>--------------</td>
<td>------</td>
<td>-------------------------</td>
<td>-------------------------</td>
<td>---------------------------</td>
</tr>
<tr>
<td>10</td>
<td>6</td>
<td>2</td>
<td></td>
<td>66.0</td>
<td>43.6</td>
<td>261.6</td>
</tr>
<tr>
<td>13</td>
<td>5</td>
<td>2</td>
<td></td>
<td>66.0</td>
<td>43.6</td>
<td>218.0</td>
</tr>
<tr>
<td>16</td>
<td>4</td>
<td>2</td>
<td></td>
<td>66.0</td>
<td>43.6</td>
<td>174.4</td>
</tr>
<tr>
<td>18</td>
<td>3</td>
<td>2</td>
<td></td>
<td>66.0</td>
<td>43.6</td>
<td>130.8</td>
</tr>
<tr>
<td>Missile #6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>9</td>
<td>4</td>
<td></td>
<td>52.4</td>
<td>5.4</td>
<td>48.6</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td>5</td>
<td></td>
<td>67.4</td>
<td>34.8</td>
<td>278.4</td>
</tr>
<tr>
<td>6</td>
<td>7</td>
<td>7</td>
<td></td>
<td>75.9</td>
<td>51.7</td>
<td>361.9</td>
</tr>
<tr>
<td>8</td>
<td>6</td>
<td>10</td>
<td></td>
<td>80.7</td>
<td>61.4</td>
<td>368.4</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
<td>11</td>
<td></td>
<td>83.9</td>
<td>67.8</td>
<td>339.0</td>
</tr>
<tr>
<td>12</td>
<td>4</td>
<td>2</td>
<td></td>
<td>85.5</td>
<td>72.0</td>
<td>288.0</td>
</tr>
<tr>
<td>14</td>
<td>3</td>
<td>2</td>
<td></td>
<td>85.5</td>
<td>72.0</td>
<td>216.0</td>
</tr>
</tbody>
</table>
assumed that $N_i$ would be constant from cycle to cycle. Appendix D gives the results that would be obtained if the original groupings were retained from cycle to cycle thus causing $N_i$ to decrease by the rate of attrition of the interceptor force.

Notice that the value of $N_i$ that minimizes the upper bound on bomber effectiveness is generally equal to one plus the number of defensive weapons carried by each bomber or the minimum reasonable value $(I(0)/B(0) = 2)$. In the first case the defense tends to deplete the weapon supply and then achieve the kill. In the second case depletion usually does not occur (unless $k < 2$) since the weapons used in one cycle do not affect the following cycles. However if the bomber carries a great number of weapons or if the weapons are relatively ineffective, the interceptors are likely to achieve a kill before exhaustion occurs. When this is true, the additional interceptors sent to achieve exhaustion of the weapon supply are unused since it is assumed that additional bombers cannot be attacked. Thus when $k$ is large or weapon effectiveness is low, the defense is more likely to not attempt to exhaust weapons but to attack more bombers.

In some cases (for example a bomber carrying three #3 missiles) the minimum upper bound occurred when $N_i$
equaled two plus the number of offensive weapons carried by each bomber. This occurs when the bomber uses very effective (90 per cent) defensive weapons. It appears that in this case the advantage offered by an additional attack on an unarmed bomber (when compared with the possibility of attacking an armed bomber) offsets the probability of a kill occurring before the second attack on an unarmed bomber.

Conclusions

An upper bound on bomber effectiveness (as defined by weapons delivered) was determined for situations in which the bombers use only one type of defensive weapon. By comparing these upper bounds with the previously determined lower bounds it may be possible to eliminate many candidate weapon loadings for the bombers.

For example in the sample problem a load mix with a lower bound of 252 bombs delivered was found. Any mix that has an upper bound below 252 can thus be
eliminated. This then leaves the following candidates:

Table 5

<table>
<thead>
<tr>
<th>Candidate</th>
<th>Lower Bound on Bombs Delivered</th>
<th>Upper Bound on Bombs Delivered</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 #5 Missiles</td>
<td>0.0</td>
<td>316.6</td>
</tr>
<tr>
<td>8 &quot;</td>
<td>85.4</td>
<td>305.2</td>
</tr>
<tr>
<td>10 &quot;</td>
<td>102.0</td>
<td>261.6</td>
</tr>
<tr>
<td>4 #6 Missiles</td>
<td>97.6</td>
<td>278.4</td>
</tr>
<tr>
<td>6 &quot;</td>
<td>218.4</td>
<td>362.6</td>
</tr>
<tr>
<td>8 &quot;</td>
<td>252.0</td>
<td>368.4</td>
</tr>
<tr>
<td>10 &quot;</td>
<td>245.0</td>
<td>339.0</td>
</tr>
<tr>
<td>12 &quot;</td>
<td>214.4</td>
<td>289.6</td>
</tr>
</tbody>
</table>

The decision as to which of these remaining candidates to select can then be determined by using the model proposed in the next chapter or any other model such as one of the many existing Monte-Carlo battle simulations. If a simulation is used at this point it should be consistent with the assumptions made in this paper. If this is not true then the bounding models may not actually be bounds on the simulation model.
However it may be that a decision can be made based on the above bounds. It may be felt that the lower bound on a load of 8 #6 missiles is close enough to the remaining candidates' upper bounds to justify selection of this loading. On the other hand delivering 368.4 bombs on target may not be considered an acceptable outcome thereby causing the rejection of all of the possible load mixes, thus leading to the initiation of a search for new options.

Notice that if the anti-missile missiles were not being considered, loading of five decoys can be selected with no further searching. In this case a lower bound was found that exceeds all other candidate upper bounds.
CHAPTER V

AN EXPECTED VALUE SOLUTION

Chapters III and IV were devoted to establishing bounds on the expected value of the effectiveness of a force of penetrating bombers using various types of defensive weapons. The resulting solutions were then used to eliminate from further consideration some of the possible candidate weapon load mixes. However the point will usually be reached in any evaluation of weapon systems where the expected value solution is needed. In some cases more than one possible weapon load configuration may still be candidates for the best configuration after examining upper and lower bounds. Even if the bounding solutions point out one configuration as best, it still may be necessary to find the expected effectiveness value. Management usually needs more than bounds on the selected solution before a decision can be made about its acceptability. It may happen that the best solution falls short of requirements and that a search for new candidates is needed. It is
also often necessary to have expected value solutions for other planning purposes (such as selection of force size requirements).

Thus there is a need for a procedure that considers the factors omitted in Chapter IV - that is, the probability of interceptors reaching bombers and the effect of bomber weapon depletion from previous attacks. Initially it will be assumed that the probability of interceptors reaching bombers is known (one possible procedure for determining these probabilities will be discussed later in this chapter). The model to be developed will determine the survivability of the bombers and the probability a given surviving bomber has various possible numbers of defensive weapons remaining. Initially the number of weapons available will equal the original number carried. The procedure will be to predict the survivability of a given bomber given the number of available weapons and further to predict the probability distribution describing the number of weapons remaining on board surviving bombers. This procedure will then be iterated for each cycle of the battle where a cycle is as previously defined. The resulting model will have to be expanded to represent the use of decoys. This is discussed in Chapter VII.
The model to be developed will be observed to have the characteristic of a Markov process. Initially the mechanics of the computational procedure will be developed after which the Markovian nature of the model will be discussed.

Overview of the Model

The model to be developed will use as inputs the number of bombers and interceptors that start a cycle of the battle and a probability distribution function that gives the probability a given bomber starts the cycle with various possible numbers of defensive weapons remaining. The model then calculates the expected force sizes surviving the cycle and a new probability distribution function to describe the defensive weapon loading remaining on surviving bombers. Figure 1 shows the procedure that is used in arriving at a solution.

Model Description

Consider the situation as faced by the defense. Groups of interceptors will be sent out to engage the bombers. Each group will encounter the first bomber with some probability and attempt to destroy it. If the bomber survives, its defensive weapon load is reduced by the number of attacks made by the interceptors or

Divide the Interceptors into $y$ Groups of Size $N_i$ Each.

Determine the Number of Bombers ($t$) that could be Attacked by Each Group of Interceptors (Note it will be assumed that the Bomber Force Size is Greater Than or Equal to $ty$).

Determine How Many of the $t$ Bombers that could be Attacked by Each Group of Interceptors Survive. Also Determine a Probability Distribution Function Describing the Defensive Weapons Remaining on Board these Surviving Bombers.

Determine the Total Number of Interceptors Surviving.

Determine the Total Number of Bombers Surviving (Realizing some Bombers may not be Potential Targets for any Group of Interceptors).

Determine a Probability Distribution Function Describing the Defensive Weapons Remaining on Board Surviving Bombers (Realizing some Bombers may not be Potential Targets for any Group of Interceptors and thus Retain the Original Probability Distribution Function to Describe their Weapon Loading).

FIGURE 1 MODEL OVERVIEW
possibly is exhausted. If the bomber is killed and some of the interceptors have not made attacks, the armed interceptors will attempt to reach a second bomber. If a second bomber is reached it may also be destroyed. Additional bombers may thus be attacked until all of the interceptors have made attacks or until no new bombers can be found (or reached if perfect location information is available - for instance from ground control).

Notice that each additional bomber attacked by the group of interceptors has fewer interceptor attacks to survive. Thus the second bomber attacked will normally have a higher probability of surviving than the first and so on. Also, the second bomber attacked will normally need fewer defensive weapons in its defense and thus will tend to have a different expectation of weapons remaining when the bomber survives.

The calculations required to analyze the above situation are shown in Table 6. The first column contains an indicator (i) to be defined for the succeeding columns. The second column indicates the probability that the first bomber to be attacked is destroyed on the indicated attack (i). For example, in the case shown the first bomber would be killed on the fifth attack with probability .113. The table was constructed assuming that the bombers have two weapons
Table 6

Analysis of One Battle Cycle

<table>
<thead>
<tr>
<th>i</th>
<th>1st Bomber</th>
<th>2nd Bomber</th>
<th>3rd Bomber</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>PK(i)</td>
<td>PS(i)</td>
<td>P(i)</td>
</tr>
<tr>
<td>10</td>
<td>.004</td>
<td>.004</td>
<td>---</td>
</tr>
<tr>
<td>9</td>
<td>.007</td>
<td>.007</td>
<td>.050</td>
</tr>
<tr>
<td>8</td>
<td>.014</td>
<td>.014</td>
<td>.048</td>
</tr>
<tr>
<td>7</td>
<td>.028</td>
<td>.028</td>
<td>.451</td>
</tr>
<tr>
<td>6</td>
<td>.056</td>
<td>.056</td>
<td>.226</td>
</tr>
<tr>
<td>5</td>
<td>.113</td>
<td>.113</td>
<td>.113</td>
</tr>
<tr>
<td>4</td>
<td>.226</td>
<td>.226</td>
<td>.056</td>
</tr>
<tr>
<td>3</td>
<td>.451</td>
<td>.451</td>
<td>.028</td>
</tr>
<tr>
<td>2</td>
<td>.048</td>
<td>.903</td>
<td>.014</td>
</tr>
<tr>
<td>1</td>
<td>.050</td>
<td>.950</td>
<td>.007</td>
</tr>
<tr>
<td>0</td>
<td>.000</td>
<td>1.000</td>
<td>.007</td>
</tr>
</tbody>
</table>
available and each interceptor can make but one attack per cycle (a procedure for analyzing multiple-attack per cycle interceptors will be discussed later in the chapter). The probability that the first bomber is killed on the fifth attack is:

\[
\text{Prob. (1st bomber is killed on 5th attack)} = (1-P_{kb/w})^2(1-P_{kb/w})^2P_{kb/w} \quad \text{PA}(1) \quad (85)
\]

where \( PA(j) \) - probability the \( j \)th bomber is found (can be reached) - \( j = 1,2,3,--- \)

The table was constructed using the following constants:

\[
\begin{align*}
P_{kb/w} &= .05 \\
P_{kb/w} &= .50 \\
N_i &= 10 \\
PA(j) &= 1 \quad j = 1, 2, 3 \\
&= 0 \quad j > 3
\end{align*}
\]

Therefore:

\[
\text{Prob. (1st bomber is killed on 5th attack)} = (.95)^2 (.50)^2 (.50) (1) = .113
\]
The next column in the table contains the probability the first bomber can survive through the \( i \)th attack \( (PS(i)) \). This is found by summing up the probabilities that the bomber is killed on the indicated and all previous attacks (assuming \( PA(1) \) equals one) and subtracting from one. The probability the first bomber survives is given by:

\[
PS(1\text{st bomber}) = 1 - PA(1) + PA(1) \times PS(N_i) \quad (86)
\]

The rest of the table is divided into groups of four columns each. Each grouping is used to indicate the survivability of the next bomber to be attacked by the group of interceptors. The first column within each group gives the probability that exactly \( i \) interceptors reach the bomber (noted as \( P(i) \)). Note that no more than \( N_i - j + 1 \) interceptors can possibly reach the \( j \)th bomber. If \( i \) interceptors reach the \( k \)th bomber, then it must be true that the \((k-1)\)th bomber was killed on the \((N_i - i)\)th attack by the group of interceptors. The probability that the second bomber is attacked by \( i \) interceptors is thus equal to the probability that the first bomber is killed on the \((N_i - i)\)th attack times the probability that the interceptors will reach the second bomber - \( PA(2) \).
The second column within each group of four gives the probability that the bomber will survive given that \(i\) interceptor attacks will be made. Since the same probability distribution of weapons remaining is used to characterize all penetrators, the survival of the second penetrator given \(i\) attackers is identical to the probability of the first bomber surviving \(i\) attacks (column 3).

Now the probability that the bomber survives but must use \(j\) weapons can be found from the data already tabulated. The first bomber will use either \(k\) (the number available) or \(N_i\) weapons (whichever is smaller) in its defense if not killed. For example, in the case shown in the table the first bomber has but two weapons available and so must use both if it survives. This bomber has a 0.4 per cent probability of surviving with no weapons remaining after the attacks and 0 per cent probability of surviving with one or more weapons remaining after the attacks.

The second bomber has some chance of surviving without using a defensive weapon (that is, not being attacked). In this case there is a 0.7 per cent probability that this bomber survives without being attacked (found by multiplying the zero element of the column giving the probability of \(i\) attackers by the
zero element of the column giving survivability given i attackers) and thus has two weapons remaining. Of course there is the possibility that the second bomber has to absorb one or more attacks and still survives. In this case there is a 0.7 per cent probability that the second bomber is attacked by exactly one interceptor and survives. When this happens this bomber survives with one defensive weapon remaining. If the second bomber receives two or more attacks it must use the total remaining supply of defensive weapons. Thus summing the product of the probability of i attacks times the probability of surviving i attacks for i from 2 to \((N_i - 1)\) gives the probability that the second bomber survives with zero weapons remaining.

The calculations required to generate the columns used to describe the possible attacks on the third bomber that this group of interceptors might reach could proceed in the same manner as the calculations used in describing the situation of the second bomber if the distribution of the number of attackers reaching the third bomber were known. The probability that the third bomber is reached by i interceptors is dependent upon the probability that the second bomber is killed on the \((N_i - i)\)th attack by the interceptor group. Thus the
next step is to determine the probability distribution associated with the number of attacks required to kill both the first and second bombers (which is found in column four for the second bomber).

The second bomber will be killed on the ith attack if the first bomber is killed on the jth attack and the second bomber is killed after i−j attacks against it for any j less than i. Now the probabilities associated with these occurrences can be found in column two in the table.

\[
\text{Prob. (2nd bomber is killed on ith attack)} = \sum_{j=1}^{i-1} \text{Prob. (1st bomber is killed on jth attack)} \times \text{Prob. (2nd bomber is killed after (i-j) add. attacks)}
\]

For example:

\[
\text{Prob. (2nd bomber is killed on 5th attack)} = \cdot050(\cdot226) + \cdot048(\cdot451) + \cdot451(\cdot048) + \cdot226(\cdot050) = 0.066
\]

It should be stressed that these calculations do not include the probability of reaching the third bomber.
which must be used to adjust the calculations. If the probabilities in column two include a non-unity probability of reaching the first bomber, they must also be adjusted before these calculations proceed. All of these calculations are made assuming that the bombers can be reached with an adjustment at the end for the possible inability of the interceptors to reach the bomber. This procedure can be generalized to the nth bomber where we only have to consider when the n-1 th bomber might have been killed.

The calculations outlined on the preceding pages are of course burdensome. However, it is fairly easy to write a computer program to generate these results. The last section of this chapter gives the results of this type of analysis when applied to the sample problem introduced in Chapter III. The computer program used in this analysis is described in Appendix C.

At this point in the procedure development, bomber survivability and remaining weapons available to the bombers can be predicted. Consider the three bombers indicated in Table 7. This table lists the probabilities that each bomber survives with zero, one or two weapons remaining.
Table 7

<table>
<thead>
<tr>
<th>Bomber</th>
<th>PW(0)</th>
<th>PW(1)</th>
<th>PW(2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.004</td>
<td>.000</td>
<td>.000</td>
</tr>
<tr>
<td>2</td>
<td>.130</td>
<td>.007</td>
<td>.007</td>
</tr>
<tr>
<td>3</td>
<td>.310</td>
<td>.103</td>
<td>.149</td>
</tr>
<tr>
<td>Totals</td>
<td>.444</td>
<td>.110</td>
<td>.156</td>
</tr>
</tbody>
</table>

Therefore for each group of three bombers there is an expectation that .71 bombers will survive. Assume that there are forty bombers and 100 interceptors at the start of the battle. Since the interceptors form groups of size ten each capable of attacking three bombers, thirty of the forty bombers are potential targets for attack with an expectation that 7.1 of the thirty targeted bombers will survive. Thus there is an expectation that 17.1 of the forty bombers will survive. Each of the ten unattacked bombers will still have two defensive weapons available for future needs. The 7.1 surviving targeted bombers have associated with them the previously calculated distribution of possible remaining weapons (the totals in Table 7) which, of course, must be adjusted to give the probability of a
surviving bomber having \( i \) weapons remaining and not (as given in Table 7) the probability of a targeted bomber surviving with \( i \) weapons.

Since the interceptors form ten groups in this example, there is an expectation that of the 7.1 bombers that were selected for attack and survived 4.44 have zero weapons remaining, 1.10 have one weapon remaining and 1.56 have two weapons remaining. Thus the surviving bomber force has an expectation of having the following numbers of bombers remaining with the following defensive weapon loads:

<table>
<thead>
<tr>
<th>Number of Surviving Bombers</th>
<th>Surviving Bombers with ( i ) Defensive Weapons</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4.44</td>
</tr>
<tr>
<td>1</td>
<td>1.10</td>
</tr>
<tr>
<td>2</td>
<td>1.56</td>
</tr>
</tbody>
</table>

These results can now be used to define a probability distribution for the number of weapons available to each bomber attacked in the next cycle of the battle.
Markovian Nature of the Model

The process represented by the model developed in the previous section is in reality a Markov Process. Each group of interceptors is depicted as going out and attempting to reach and attack a series of bombers in a given cycle of the battle. The outcome of this action (or process) on the jth bomber that may be attacked can be represented as the outcome of a stochastic process that can have one of several values each corresponding to a possible state for the bomber. The jth bomber may be killed on the ith attack made by the group of interceptors during the cycle, or may survive with k defensive weapons available for use during other cycles, where i and k can each have one of several values. The model developed in the previous section is based upon the assumption that the probability that this jth bomber reaches one of the above states depends only on the state reached by the (j-1)th bomber, and is independent of the states reached by bombers that are attacked earlier in the sequence. Thus, this is a Markov Process (see Reference 18).

To more explicitly show the Markovian nature of the model let $X_i$ represent the state of the ith bomber that might be attacked by the group of interceptors,
after the cycle is completed where:

\[ X_i = j \]

if the bomber is killed on the jth attack of the cycle by this group of interceptors, \( j = 1, \ldots, N \)

\[ X_i = N_i + k \]

if the bomber survives the cycle with \((k-1)\) defensive weapons (or salvos) available for use during subsequent cycles

Then for the process described by the expected value model the following is true:

\[
P \left\{ X_{i+1} = q / X_0 = k_0, X_1 = k_1, \ldots, X_i = r \right\} - P \left\{ X_i+1 = q / X_i = r \right\} \]

(88)

Thus this is a Markov Process. It is also true that if the probability of reaching the jth bomber is independent of the value \( j \) then:

\[
P \left\{ X_{i+1} = q / X_i = r \right\} = P \left\{ X_2 = q / X_1 = r \right\} = P_{qr}\]

(89)

for all \( i = 1, 2, \ldots \)
and the (one-step) transition probabilities \( P_{qr} \) are stationary. However in the normal case there will be different probabilities of reaching succeeding bombers in the chain and the transition probabilities will not be stationary. Since the available supply of defensive weapons will change from cycle to cycle, the transition probabilities will also change from cycle to cycle.

The transition matrix shown in Table 9 represents the situation used in generating Table 6. These values do not include the probability of reaching the \( i \)th bomber. For the situation used in constructing Table 6, the first two transition matrices would be identical to Table 9 since the probability that the first three bombers can be reached is one. However since the probability that more than three bombers can be reached is zero, the transition matrices for \( i \) greater than three are as shown in Table 10. Notice that in all of these matrices state 13 is absorbing.

The final item needed to define the Markov Process is the set of probabilities associated with the 1st bomber in the sequence reaching various states. The probabilities associated with states 1 through \( N_i \) can be found in column one of Table 6. The other state probabilities can be found from the procedure used in generating Table 7. Thus the Markov Process is easily
Table 9

Transition Matrix for the First Two Transitions

<table>
<thead>
<tr>
<th>State of the ith Bomber (i = 2 or 3)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>---</td>
<td>.05</td>
<td>.05</td>
<td>.45</td>
<td>.23</td>
<td>.11</td>
<td>.06</td>
<td>.03</td>
<td>.01</td>
<td>.01</td>
<td>.00</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>State</td>
<td>2</td>
<td>---</td>
<td>.05</td>
<td>.05</td>
<td>.45</td>
<td>.23</td>
<td>.11</td>
<td>.06</td>
<td>.03</td>
<td>.01</td>
<td>.01</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>of the (i-1)th Bomber</td>
<td>3</td>
<td>---</td>
<td>---</td>
<td>.05</td>
<td>.05</td>
<td>.45</td>
<td>.23</td>
<td>.11</td>
<td>.06</td>
<td>.03</td>
<td>.02</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>Bomber</td>
<td>4</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>.05</td>
<td>.05</td>
<td>.45</td>
<td>.23</td>
<td>.11</td>
<td>.06</td>
<td>.05</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>5</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>.05</td>
<td>.05</td>
<td>.45</td>
<td>.23</td>
<td>.11</td>
<td>.11</td>
<td>.11</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>6</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>.05</td>
<td>.05</td>
<td>.45</td>
<td>.23</td>
<td>.22</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>7</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>.05</td>
<td>.05</td>
<td>.45</td>
<td>.45</td>
<td>---</td>
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Table 10

Transition Matrix for the Third and All Succeeding Transitions

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Probabilistic versus Deterministic Weapon Load

The description of the defensive weapon load available for succeeding cycles of the battle used in the expected value model represents the first departure in this analysis from deterministic modeling. For example there are no probability distributions associated with the number of surviving bombers and interceptors in the model, just expected values. Would it not be reasonable therefore to use an expected value to describe the weapons available for the next cycle rather than a probability distribution?

The justification for not using expected values in this case lies in the ease with which the computer model described in Appendix C can use a probabilistic description of the weapon load remaining. It is always more accurate to use the probability distribution of possible states rather than the expected value (see References 2 and 18). If a factor can easily be incorporated probabilistically in a model, then it might as well be.

To explore this question further let us compare the deterministic results with the probabilistic results that would be used in analyzing the next cycle of the
battle outlined previously. In this case the following distribution describes the weapon load available for the second cycle:

\[
\begin{align*}
PW(0) &= 0.26 \\
PW(1) &= 0.06 \\
PW(2) &= 0.68
\end{align*}
\]

These values yield an expectation of 1.42 weapons per surviving bomber. Thus an expected-value model must deal with weapon loads containing partial weapons. This of course can be done by not requiring that the number of weapons available \((k)\) in the survivability equations (for example equation (81)) take on only integer values.

Since there may be up to ten attackers reaching a given bomber in the next cycle consider the survivability of the bomber for one to ten attacks. Table 11 gives the bomber survivability using probabilistic and expected value defensive weapon load models for one to ten attacks.

Although the deterministic and stochastic results for this case are similar they are not identical. The expected value model did introduce some error. As long as the probability distribution is easily incorporated in the model, then this is the proper course of action.
Table 11

Comparison of Deterministic and Probabilistic Weapon Load Models

<table>
<thead>
<tr>
<th>Attacking Interceptor Group Size</th>
<th>Probability of Bomber Survivability</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>Deterministic</td>
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<tr>
<td></td>
<td>Model</td>
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<tr>
<td>1</td>
<td>0.95</td>
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</table>

|                                  | Probabilistic                      |
|                                  | Model                              |
| 1                                | 0.83                               |
| 2                                | 0.71                               |
| 3                                | 0.35                               |
| 4                                | 0.18                               |
| 5                                | 0.09                               |
| 6                                | 0.04                               |
| 7                                | 0.02                               |
| 8                                | 0.01                               |
| 9                                | 0.01                               |
| 10                               | 0.00                               |
Interceptor Attrition

The next step in using the model developed in this chapter involves finding the expected number of interceptors surviving each cycle. Of course this is required only when the bombers use missiles that are capable of destroying interceptors. If only decoys and/or anti-missile missiles are used to defend the bombers, the interceptors are always assumed to survive (however some fraction may not recycle due to factors such as clobber on landing, etc.).

If the number of weapons fired against the interceptor is known, then the expected number of interceptors killed can be determined. The expected number of weapons available to the bombers after each cycle is of course known from the expected number of surviving bombers and the distribution of weapons remaining per survivor determined previously. The difference between the number of weapons available before a cycle and the number available after the cycle, is the number of weapons fired at interceptors plus the number lost during battle (unused weapons remaining on destroyed bombers). Thus if the expected number of unused weapons destroyed by successful interceptor attacks were known, the expected number of weapons used (and thus the expected number of
interceptors killed) could be determined.

Consider each bomber selected for attack by each group of interceptors. There is some probability that the bomber has \( m \) weapons \((0 \leq m \leq \text{initial number available})\) available which of course is known from the previous cycle analysis. There also is some probability that \( n \) interceptors do in fact reach the bomber. If the bomber is the first target selected by the interceptors, \( n \) can equal only \( N_i \). For other bombers the probability of \( n \) equaling various values is found in the column headed \( P(i) \) in Table 6. Assume that \( m \) is greater than or equal to \( n \). The expected number of weapons lost by the bomber is given by:

\[
\begin{align*}
\text{Expected number of weapons lost} & = (m-1)P_{kb/w} + (m-2)(1-P_{kb/w})P_{kb/w} + (m-3)(1-P_{kb/w})^2P_{kb/w} \\
& + \cdots + (m-n)(1-P_{kb/w})^{n-1}P_{kb/w} \\
& + \sum_{i=1}^{n} (m-i)(1-P_{kb/w})^{i-1}P_{kb/w}
\end{align*}
\]

If \( i \) is replaced by \( j = i-1 \), this expression becomes:

\[
P_{kb/w}((m-1) \sum_{j=0}^{n-1} (1-P_{kb/w})^j - \sum_{j=0}^{n-1} j(1-P_{kb/w})^j)
\]

(90)
A closed form expression for the first summation in Equation (90) was given previously (Equation 68).

The following closed form expression for the second summation which is of the form \( \sum_{j=0}^{n-1} a^j \) is given in Reference 20:

\[
\sum_{j=0}^{n-1} a^j = \frac{(a-1)n a^n - a^{n+1} + a}{(1-a)^2}
\]

The expected number of defensive weapons lost by the bomber is thus obtained from Equation (90).

Expected number of weapons lost

\[
(m-1)(1-(1-P_{kb/w})^n) + n(1-P_{kb/w})^n + (1-P_{kb/w})^{n+1}/P_{kb/w}
\]

\[-(1-P_{kb/w})/P_{kb/w}
\]

If on the other hand \( m \) is less than \( n \):

Expected number of weapons lost =

\[
(m-1)P_{kb/w} + (m-2)(1-P_{kb/w})P_{kb/w}
\]
If equations (92) and/or (93) are weighted for all values of \( m \) and \( n \) with their corresponding probabilities for each bomber selected for possible attack, then the expected number of weapons lost by each bomber can be determined. This will then allow determination of interceptor attrition. For example consider the battle situation used in constructing Table 6. Since the results presented represent the first cycle of the battle and the initial weapon load contains two defensive weapons, \( m \) is equal to two. For the first target \( n \) equals ten with probability one and thus Equation (93) applies. Therefore:

\[
\text{Expected number of weapons lost} = 2 - 1 + (0.95)^2 + (0.95)^3/(0.05) - (0.95)/(0.05) = 0.05
\]
The expected number of weapons lost per bomber by the second and third bombers selected for attack is also .05 if \( n \) is greater than or equal to \( m \) (which equals two) since Equation (93) would again apply. If \( n \) equals one:

\[
\text{Expected number of weapons lost} = 1(1-.95) + .95 + (.95)^2/(.05) - (.95)/(.05) = .05
\]

Thus in this case the expected number of weapons lost is .05 whenever \( n \) is greater than or equal to 1. The probability that each bomber selected for attack by a given interceptor group is in fact attacked (meaning \( n \) is greater than or equal to one) is given in Table 12.

**Table 12**

<table>
<thead>
<tr>
<th>Probability of Attack of ith Bomber</th>
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</thead>
<tbody>
<tr>
<td>i</td>
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<tr>
<td>---</td>
</tr>
<tr>
<td>1</td>
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<tr>
<td>2</td>
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<tr>
<td>3</td>
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<td>&gt;3</td>
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</table>
Therefore:

Expected number of weapons lost/Group of interceptors = \(0.05(1.00 + 0.993 + 0.851) \approx 0.142\) and the expected number of weapons fired by bombers during the cycle is:

\[
\text{Expected number of weapons fired} = \text{Expected number initially available} - \text{Expected number of weapons surviving} - (94)
\]

\[
\text{Expected number of weapons lost} = 40(2.0) - 17.1(1.42) - 10(0.142) - 54.3
\]

**Multiple Attacks/Cycle Interceptors**

The model development to this point has been based on the assumption that each interceptor in a group will make at most one attack per cycle. However suppose each interceptor is armed with more than one missile. The defense may decide to use all of an interceptor's missiles in each attack and therefore achieve but one attack per cycle or the defense may choose to divide the missiles into more than one salvo, using but one salvo for each attack. In the latter case since fewer missiles are used per attack than for the first case, the probability of killing the bomber during a given attack would normally be lower. For a particular
situation it may not be obvious which policy is better for the defense and so there is a need to be able to study the case where multiple attacks per cycle per interceptor are possible.

If the bomber does not use missiles to attack the interceptors themselves, then the attrition caused by $N_i$ interceptors with $n$ salvos per interceptor would be equivalent to the attrition caused by an interceptor group of size $N_i$ times $n$ with one salvo per interceptor. Thus the model developed in this chapter (as well as the bounding models developed previously) can be used by simply adjusting the group size.

The problem is more complicated when anti-interceptor missiles are used. Since some interceptors may be destroyed before firing all weapon salvos, one cannot simply multiply the interceptor group size by the number of weapon salvos per interceptor to obtain the "effective" single-salvo group size. Not all salvos are always available to the interceptor group.

Assume that all of the interceptors in a given group will make one attack before any make the next. Then the model as developed in this chapter can be used to study the first round of firings (where a round corresponds to all interceptors firing one salvo). After the interceptor attrition is determined for this first round,
an increased interceptor group size can be defined to account for the first two rounds of attack within the cycle and the model rerun. This procedure can be repeated until all possible rounds of firings are accounted for. If the procedure should call for a non-integer group size the model will have to be run using the two closest integer group sizes and the results properly weighted.

Of course the above solution procedure is based on the assumption that all the interceptors make one attack before any make a second which may not seem desirable from the defense's standpoint. Assume each interceptor has two missiles and the first kill is made by the first bomber during the second attack. Following the assumption in the above paragraph one interceptor salvo is lost. On the other hand if the first interceptor had made both of its attacks before the second interceptor had made one, no salvos would be lost. Thus at least in this case the above assumption concerning interceptor operation is not desirable from the defense's standpoint. However present operational procedures seem to be more in agreement with the initial assumption. Present interceptor speeds are so great that it is very time consuming for an interceptor to turn and reacquire the bomber for the second attack. It is
highly likely that the other interceptors in the group will make their attacks before the first can return. Thus a model based on the assumption that all interceptors make an initial attack before any make the second, etc., seems reasonable.

**Probability of the Interceptors Reaching Bombers**

The model developed in this chapter is based on the assumption that the probability of interceptors reaching bombers (a first bomber, a second bomber, etc.) is known. These probabilities could be specified in any manner desired. Some form of analytical or empirical model might be used. Rather than using a model the analyst might use expert judgement about the capability of interceptors to find bombers.

One possible analytical approach to specifying the probability of interceptors reaching bombers will be outlined below (see Reference 2 for a similar development). Assume that there are \( N_D \) bombers presently in the battle \( N_{bl} \) of which have been selected as initial targets (and will always be reached) by the groups of interceptors (thus the defense has committed \( N_{bl} \) groups of interceptors). Therefore \( N_D - N_{bl} \) bombers are potential candidates as secondary targets for the interceptors if no bomber can be attacked by more than
one group of interceptors. Assume that the defense has located these bombers within an area of size \( A \), (\( A \) may represent the whole area being defended) and that any interceptors who are able to make additional attacks after the primary target (bomber) is destroyed will initiate a random search within this area. Note, however, that no two searches will be allowed to overlap. Assume further that the interceptors are predicted to have the capability of flying \( r \) miles while in random search with the ability of detecting any bomber that comes within \( w \) miles of the flight path. Thus the interceptor group can investigate an area of \( 2rw \) square miles within the area \( A \) as shown in Figure 2.

If \( w \) is much smaller than \( r \) (normally true), then it is approximately true that any bombers seen by the interceptors can be reached without exceeding the range limitations of the interceptors. Notice that in reality if a bomber were observed at the corner of the region searched as shown in Figure 3, the interceptors cannot reach this point without exceeding their range \( r \). However since it is assumed that \( w \) is much smaller than \( r \), little error is introduced by assuming that any target found in the area searched can be attacked.

The probability that a given bomber in the group of possible candidates for a second attack is found is
Initial Attack

FIGURE 2  MAXIMUM AREA SEARCHED

FIGURE 3  AN UNREACHABLE BOMBER
simply $2rw/A$ (note that it is assumed that bombers might have moved into the region swept out by the interceptors in reaching the first bomber and thus bombers may be anywhere in A). Thus the probability of finding exactly $n$ additional bombers is the same as the probability associated with finding $n$ successes from a binomial sample with $N_b - N_{bl}$ trials with parameter $2rw/A$. The probability of finding $n$ additional bombers would thus be:

$$\text{Probability of Finding n Additional Bombers}$$

$$= \binom{N_b - N_{bl}}{n} (2rw/A)^n (1-2rw/A)^{N_b - N_{bl} - n} \tag{95}$$

Example Problem (Continued)

The model developed in this chapter was used in analyzing the example problem introduced in Chapter III. Only those weapon loadings that were still considered candidates for an optimum solution after examining the upper and lower bound solutions were considered.

Until this point in the analysis of the sample problem no attempt had been made to predict the defense's capability to reach bombers after making the first intercept. It had, however, been assumed that the
defense could always direct interceptors to the first bomber to be attacked as long as no cross-targeting (assignment of one bomber to two or more groups of interceptors) was allowed. The example problem was continued assuming that each interceptor group could reach 2.5 per cent of the remaining bomber force (surviving bombers at start of cycle) as long as cross-targeting did not result. This rule might be typical of the form taken by expert judgement type rules. For the first cycle \((B(0) = 100)\) each group of interceptors could reach 2.5 bombers. This was modeled as being equivalent to the statement that all interceptor groups could reach two bombers and half of the groups can reach a third. If some \(B(n-1)\) was found to be less than forty (therefore \(0.025B(n-1)\) is less than one), it was assumed that each interceptor group could reach exactly one bomber.

Since the initial bomber force contains 100 bombers, the defense was not allowed to commit more than 100 interceptor groups in cycle one. Thus if \(N_1\) equals one, only 100 of the 200 interceptors got into action. If \(N_1\) equals two, the defense was again assumed to form 100 groups which is obviously better from the defense's standpoint than when \(N_1\) equals one. When \(N_1\) equaled two, each interceptor group was not allowed to make more
than one attack during cycle one (even though \(0.025B(0)\) is greater than one) in order to retain the assumption of no cross-targeting.

The results obtained applying these ground rules to the example problem are found in Table 13 (a more complete description of the results can be found in Appendix C). Since it is assumed for this example problem that the defense has perfect information concerning what the defensive loading carried by the bombers is, and that the defense has to select a group of constant size for the duration of the battle, the optimum solution occurs when the bombers carry eight number six missiles and the interceptors respond by operating in groups of size eleven or twelve. The solution procedure that should be used if the defense did not know what weapons are carried by the bombers is outlined in Appendix B.

It is interesting to note that the optimum group size using the model developed in this section was generally larger but never lower than that selected for the upper bound solution. This result is not surprising since the present model allows interceptor groups to attack more than one bomber per cycle while the upper bound model limited the attacks to one bomber per interceptor group per cycle. Since more targets (bombers)
Table 13

Number of Offensive Weapons Delivered as a Function of
Defensive Weapons Carried and Interceptor Group Size

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<th>Miss Type</th>
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Min. Sol. 125
are available per group, it is not surprising that larger groups become more practical. However it is not clear that this result will always hold in all cases. Since the model developed in this chapter also considers the effect of a gradual depletion of bomber defensive weapons from cycle to cycle, there may be situations in which the optimum group size is less than that found for the upper bound solution.

It should also be noted that if the number of offensive weapons delivered is plotted against the interceptor group size, for some defensive weapon loadings more than one group size yield local maxima. There are apparently two reasons for this result. First there may be a group size that is effective against the first bomber attacked while another grouping is effective against the first and the second or third bomber to be attacked. Thus the situation is too complex to expect unimodal behavior. A second reason for the multiple local maxima occurring concerns the determination of the number of groupings formed. No partial groups are allowed. Thus if a group size did not allow the division of the 200 available interceptors into an integer number of groups, some interceptors were discarded. For example when $N_1$ equaled eleven, it was assumed that eighteen groups were available thus
accounting for but 198 interceptors. If two group sizes yield nearly equal effectiveness values assuming all 200 interceptors are used (maybe allowing formation of a partial group), the number of interceptors discarded in each case might determine which grouping actually results in greater effectiveness. This factor could conceivably result in a great number of local maxima.

Conclusions

At this point the derivation of the basic models is complete. As seen in the example problem solution, we now have the tools needed to analyze the bomber effectiveness problem. Of course this assumes that a complete definition of the problem is available. That is the models require as inputs the kill probabilities, the interceptors' capability to reach bombers, etc. as well as a determination of what assumptions should be made. For example the user must decide whether or not the defense will have an intelligence advantage (see Appendix B) and whether or not the interceptor forces can be regrouped after each cycle of the battle.

When the lower bound model was formulated in Chapter III, several approximations were made that could conceivably invalidate the bound. That is the solution
obtained might not actually be a lower bound. However it was found that for the load mixes evaluated using the expected value model, the lower bound solutions were always legitimate bounds. If the expected value solution, for the loading having the largest lower bound solution, had been less than the lower bound solution, the bound would have to be rejected as invalid. In that case, the expected value solution should be used in place of the lower bound solution and the comparison with the upper bounds for all loadings re-evaluated.

The solution obtained to the example problem is based upon the assumption that both the offense and the defense select the "optimum" tactics and that the problem parameters are known exactly. The next chapter will explore the effect on the solution when these assumptions are not true. First the sensitivity of the effectiveness of the bomber force to the tactics chosen by both sides will be determined. Then the sensitivity of the selection of the optimum defensive weapon to some of the problem parameters will be explored.

The next chapter will also consider the question of how good the bounding analysis is. This will involve a comparison of the lower and upper bound solutions to determine for which defensive weapon loadings the two bounds were reasonably close in their prediction of
bomber effectiveness.
CHAPTER VI

SENSITIVITY ANALYSES

Three types of sensitivity analysis will be explored in this chapter. First consideration will be given to the sensitivity of the bomber's effectiveness (as measured by the number of bombs delivered) to deviations from the interceptor and bomber courses of action specified by the solution given for the example problem in Chapter V. This analysis will assume no variation in the problem formulation (force sizes, weapon effectiveness, etc.) specified in Chapter III. The sensitivity of bomber effectiveness to the class and number of defensive weapons carried by each bomber and the operating group size of the interceptors \((N_i)\) will be investigated. This analysis will point out the value of not deviating from the solution found in Chapter V.

The next section of this chapter will consider the effect of changes in the problem scenario such as
Interceptor force size, defensive weapon effectiveness, etc. Whenever there is some uncertainty as to the correct values that should be used in a particular study, an evaluation of the sensitivity of the results to changes in these values is needed. For example if the analysis is to be used in determining which defensive weapon is to be developed the battle situation to be studied will presumably be defined to represent expected situations that might be encountered in the future. In this case there is bound to be a great amount of uncertainty as to the exact situation that should be studied. It is also likely that the defensive weapon characteristics will be estimated and so are also uncertain. This naturally results in the need for some consideration of the effect of changes in the problem definition. This consideration should point out which parameters are most likely to change the solution. The estimates used for these critical parameters might then be reconsidered and hopefully improved upon. This procedure should decrease the possibility of a wrong answer resulting from the analysis. In any event, by comparing the defensive weapon selected as best in the solution with the other candidates, it is possible to determine what changes in the problem formulation will result in a change in the rankings of the candidates.
This information might then be used to predict the likelihood of the designated system (in the original analysis) not actually being 'best'. This would normally require a prediction of the probability of the parameters of interest taking on various values.

The sensitivity analysis of the bomber effectiveness to the problem inputs might also point out a defensive weapon that although possibly not best for the most likely situation is relatively insensitive to parameter variation. Thus one of the candidate weapons may be satisfactory over a wider range of possible situations than the weapon identified as best for the most likely situation and thus could be more acceptable to management.

The third type of sensitivity analysis will consider the amount of uncertainty remaining in the expected number of bombs delivered after consideration of the bounding models developed in Chapters III and IV. The amount of uncertainty remaining is one measure of worth of the bounding solutions. The value of the bounding solutions obviously increases as the lower bound solution becomes closer to the upper bound solution. Thus an indication of those situations in which the lower bound solution is close to the upper bound solution should tend to indicate what type of problems will
benefit most from the bounding analysis.

**Deviations From the Solution Found in Chapter V**

The solution given in Chapter V for the example problem calls for each bomber to carry eight 90 per cent effective anti-missile missiles and the interceptors to counter by operating in groups of size eleven or twelve. Consider the other defensive weapons that the bombers could carry. The results presented in Chapter V show that the 75 per cent effective anti-missile missiles or decoys offer the next best choice for a defensive weapon for the bombers. Figure 4 shows the number of bombs delivered by the bombers when various numbers of decoys, 75 per cent effective anti-missile missiles, or 90 per cent effective anti-missile missiles are carried by the bombers. In each loading only one type of defensive weapon was carried. However the number and type of defensive weapons carried was varied. The values shown in this figure are based on the assumption that the defense knows what weapons the bombers are carrying and groups so as to minimize bomber effectiveness. As in the previous analysis of the example problem, the number of bombs delivered is considered to be the same as the number of bombs surviving the air battle. It is obvious that selecting the more effective anti-missile missile
FIGURE 4 THE EFFECT OF THE NUMBER OF DEFENSIVE WEAPONS CARRIED ON THE NUMBER OF BOMBS DELIVERED
greatly improves bomber effectiveness if the best loading is also selected. Carrying any other defensive weapon will result in a loss of at least fifty bombs on target when compared with the loading of eight 90 per cent missiles. However bomber effectiveness is not as sensitive to the number of these weapons carried. Carrying six or ten of these missiles will not greatly reduce effectiveness. Since adding more defensive missiles will obviously improve bomber survivability (but not necessarily bomber effectiveness) it might be decided to sacrifice effectiveness slightly in order to improve survivability and thus select the loading of ten missiles.

Next consider the effect of variation in the interceptor group size \( (N_i) \). Assuming that the bombers choose to carry eight anti-missile missiles (90 per cent effective), Figure 5 shows the effectiveness of the bombers for various interceptor groupings. Note that this curve has several local maxima and minima. As mentioned in Chapter V the defense was assumed to form an integer number of groupings and thus might not use all of the available 200 aircraft for all grouping arrangements. This could account for some minor fluctuations in interceptor effectiveness. However the main reason for the multiple peaks in Figure 5 is the
Note: Each Bomber is Assumed to Carry Eight Number Six Missiles.

FIGURE 5  THE EFFECT OF INTERCEPTOR GROUP SIZE ($N_i$) ON THE NUMBER OF BOMBS DELIVERED
various ways in which the defense could deplete the bombers' weapon supply and as mentioned previously improve their effectiveness. Since the bombers carry eight weapons and the battle is to be fought in two cycles, the defense could attack in groups of five or more and hopefully deplete the defensive weapon supply during the second cycle. On the other hand, groupings of nine or more would cause depletion from the first cycle. However larger groupings generally will result in fewer bombers being attacked and thus the best policy for the defense to follow is not obvious. Thus in Figure 5 we see that bomber effectiveness yields local minima for defensive groupings of six, and eleven (really the big changes in effectiveness occurs when $N_i$ changes from four to five and from eight to nine).

Figure 5 indicates that the solution is not too sensitive to the interceptor group size as long as it is close to the optimum of eleven. However it is evident that too small a grouping ($N_i$ equal to two, three, or four) will not deplete the bombers' weapon supply and significantly reduces the defense's effectiveness.

Assuming that the example problem formulation is correct (that is, force sizes, weapon effectiveness, etc., are considered to be accurate representations of the situations in which these weapons might be used) then
it is apparent that the offense would be very inefficient if the optimum defensive weapon is not used. The bombers should also carry approximately eight of these weapons if they are to operate near their peak effectiveness. The defense would greatly decrease their effectiveness if groupings in rather large numbers (near eleven) are not used. The defense definitely benefits from large groupings that exhaust the bombers' defensive weapon supply.

Variation of the Problem Formulation

Assume that the analysis of the example problem found in Chapters III, IV, and V is aimed at determining which of the several candidate defensive weapons should be developed and eventually produced. The problem scenario is intended to represent the type of situation in which these weapons might be used at some point in the future. As such the problem parameters are necessarily estimated and thus potentially highly uncertain. Since the answer obtained in Chapter V may change if the problem parameters are altered, the analysis should contain a determination of which parameters are most likely to change the results. These parameters might then be re-evaluated and hopefully improved.

Assume that the type of bomber which is to carry these weapons is an existing operational vehicle. It is
reasonable to assume that in this case the weapon carrying capability of these bombers is essentially fixed. It also is likely that the number of such aircraft that will be available in the time period of interest is reasonably well known. However, it is generally true that the available forces and capability of the opposing defensive aircraft is highly uncertain. Of course the characteristics of the proposed defensive weapons might also be highly uncertain. Thus the sensitivity of the solution should be found for changes in the following parameters:

a) The bomber's defensive weapon space trade-off with offensive weapons.

b) The bomber's defensive weapon effectiveness.

c) Number of opposing interceptors.

d) The interceptor's probability of killing a bomber given an encounter.

e) The interceptor's ability to reach secondary targets (bombers).

In all cases the effectiveness of the bombers armed with 90 per cent effective anti-missile missiles was
compared with the effectiveness of the bombers armed with decoys since the decoy was indicated in the analysis of the example problem as the next best choice as a defense for bombers. Of course it would be desirable to consider the other possible defensive weapons also, since it is possible that a third or fourth choice for the original problem would be best if the problem were sufficiently altered. However, to simply illustrate the type of sensitivity analysis that should be made, consideration is limited to decoys as an alternative defensive weapon.

The following figures show the sensitivity of bomber effectiveness to variations in each of the parameters listed previously. In all cases the interceptor force grouping \((N_i)\) was selected to minimize bomber effectiveness. Thus these solutions assume full knowledge by the defense of the bomber force size, weapon load mix, and objectives. The number of defensive weapons carried was selected to maximize bomber effectiveness assuming the interceptors grouped to minimize effectiveness.

The relative rankings of the anti-missile missile and the decoy are very sensitive to the missile effectiveness (Figure 8). Reducing the estimated effectiveness by more than about six per cent changes the
rankings of effectiveness of these two defensive weapons. The rankings are also quite sensitive to the space trade-off of either defensive weapon (Figures 6 and 7).

The effectiveness rankings of the decoy and the missile are not very sensitive to the interceptor force size or the interceptor kill probability \((P_{kb/w})\). Figures 9 and 10 show a preference for the anti-missile missile over a large variation in either of these parameters.

The rankings of the offensive weapons did not change appreciably as the ability of the interceptors to reach bombers changed (see Figure 11). Notice that the decoy effectiveness is not at all sensitive to this parameter. This follows from the fact that regardless of the interceptor's ability to reach bombers the optimum tactic for the defense is to form as small groups as possible, usually of size one, in a heavily decoyed environment. Since it is assumed that an interceptor group will always reach a first bomber or decoy no matter what value is chosen for the per cent of bombers reached (see Example Problem Formulation), a single interceptor is never limited by its ability to reach bombers. Thus in this case the interceptors cannot take advantage of increased capability to reach bombers.
Expected Number of Bombs Delivered

142

500

400

300

200

100

0

Predicted Decoy Effectiveness

Space Trade-Off Ratio
Anti-Missile Missile (90 Per Cent Effective)
with Offensive Weapon

FIGURE 6 THE EFFECT OF MISSILE SIZE ON BOMBER EFFECTIVENESS
FIGURE 7  THE EFFECT OF DECOY SIZE ON BOMBER EFFECTIVENESS
FIGURE 8 THE EFFECT OF MISSILE EFFECTIVENESS
ON THE NUMBER OF BOMBS DELIVERED
FIGURE 9  THE EFFECT OF INTERCEPTOR FORCE SIZE ON THE NUMBER OF BOMBS DELIVERED
Probability of an Interceptor Killing an Unarmed Bomber Given an Attack \( (P_{kb/w}) \)

**FIGURE 10** THE EFFECT OF INTERCEPTOR KILL PROBABILITY \( (P_{kb/w}) \) ON THE NUMBER OF BOMBS DELIVERED
FIGURE 11 THE EFFECT OF THE INTERCEPTORS' ABILITY TO REACH BOMBERS ON THE NUMBER OF BOMBS DELIVERED
If each interceptor group can reach 2.5 per cent of the bomber fleet and anti-missile missiles are used, we found that there is an expectation of 300 bombs surviving. As the per cent of bombers that can be reached by each group of interceptors goes up, the number of bombs surviving will decrease. However, even if an interceptor group could reach 100 per cent of the bombers, the number of surviving bombs cannot be lower than the lower bound found in Chapter III. Thus no matter what per cent of bombers can be reached by a group of interceptors, an anti-missile missile defense will be more effective than a decoy defense.

The sensitivity analysis considered in this section will enable the analyst to determine which factors are most likely to affect the results. These factors can then be reconsidered if the estimated values are not considered accurate enough. In the example problem it was found that the solution is very sensitive to the predicted defensive weapon characteristics but not to the enemy capabilities. It should be noted however that the effect of more than one parameter changing simultaneously was not considered. Although it would be desirable to explore the effect of possible variation interactions, this is beyond the scope of this volume.
Uncertainty Remaining After the Bounding Analysis

The upper and lower bound analyses developed in Chapters III and IV proved to be quite effective in eliminating defensive weapon options. However, it is desirable to have a method of evaluating the goodness of the bounds themselves beyond their effectiveness in solving a particular problem. The difference in the bomber effectiveness as predicted by these two models would be one measure of their goodness. Since the actual difference in the number of bombs delivered is a function of the size of the bomber force and the number of bombs carried by each bomber, this difference is sensitive to the magnitude of the problem. Therefore, it seems reasonable to normalize the difference by dividing by the upper bound value. The resulting measure of effectiveness indicates the maximum percentage that the upper bound might overestimate bomber effectiveness (which will be referred to as the per cent of uncertainty remaining in the upper bound).

Figures 12, 13, and 14 indicate the per cent of uncertainty remaining in the upper bound solution for the defensive weapon loads considered in the example problem analyses. Notice that although the bounding analysis was effective in the example problem, the amount of uncertainty was very large for all of the loadings using
FIGURE 12 UNCERTAINTY IN THE BOUNDING ANALYSIS VS THE NUMBER OF DECOYS/BOMBER
FIGURE 13 UNCERTAINTY IN THE BOUNDING ANALYSIS VS THE NUMBER OF ANTI-INTERCEPTOR MISSILES/BOMBER
FIGURE 14  UNCERTAINTY IN THE BOUNDING ANALYSIS VS
THE NUMBER OF ANTI-MISSILE MISSILES/BOMBER
missiles for a defense. However, the amount of uncertainty is seen to decrease as the missile effectiveness increases for both types of defensive missiles considered. The amount of uncertainty also generally decreased as the number of missiles increased. Since modern weapons are becoming increasingly more effective (and more compact when effectiveness is unchanged) it is likely that this analysis is appropriate for future as well as present uses.

Conclusions

An analysis of the sensitivity of the effectiveness of the bombers to changes in the courses of action taken by both the offense and the defense was explored in this chapter. It was found that for the example problem bomber effectiveness was very sensitive to some decisions (such as the defensive weapon chosen) but was not too sensitive to other decisions (such as the number of defensive weapons chosen).

The effect of uncertainty about the problem parameter values was also considered in this chapter. The example problem analysis indicated that the best defensive weapon to be chosen is highly dependent on some parameters but not on others. Thus it is possible to point out the critical problem parameters that should
be more carefully evaluated.

The final type of sensitivity analysis concerned the amount of uncertainty remaining after evaluating the bounding solutions developed in Chapters III and IV. It was found that the example problem analysis based on the bounding solutions contained a high degree of uncertainty although many weapon mixes were logically eliminated using these results. It was also found that the amount of uncertainty decreased as the defensive missile effectiveness and quantity increased. Since it is very probable that future weapon systems will become increasingly more effective and compact (due to advances in technology), it is reasonable to assume that the bounding model approach will become more effective in future applications.
CHAPTER VII

MIXED DEFENSIVE WEAPON LOADS

The previous analysis was based upon the restriction that the bombers would carry at most one type of defensive weapon. However, it may be more desirable to in some situations carry mixed defensive weapon loads. For example it may be more effective for a bomber to use both anti-interceptor and anti-missile missiles in its defensive arsenal rather than relying on either weapon alone. The question of mixed defensive weapon loads will be considered in this chapter. However, determining typical results (for instance for the example problem introduced in Chapter III) using the models suggested in this chapter is beyond the scope of this paper. The purpose of this chapter is merely to show how the previous analysis could be extended to include nonhomogeneous defensive weapon loads.

One problem that must be faced when analyzing potential mixed defensive weapon loads is the great number of combinations that are possible. Even if one
limits the options to but one candidate weapon in each weapon class (the classes of weapons being anti-interceptor missiles, anti-missile missiles, and decoys) there still may be as many as three classes of weapons carried. If the bombers can carry $N_{ow}$ offensive weapons then any combination of the above three classes of weapons using $N_{ow} - 1$ or fewer offensive weapon spaces would be a possible candidate (although at least two spaces must be used for defensive weapons if no one space can be used by more than one class of defensive weapon and the defensive loading is in fact mixed). If in addition each class of defensive weapon contained several candidates (such as in the example problem) the number of combinations to be considered increases still further. Of course if the non-mixed (homogeneous) defensive weapon loadings are considered first the number of combinations still to be considered will hopefully be reduced. If a homogeneous (or for that matter a heterogeneous) defensive weapon loading would result in an expectation of more offensive weapons being delivered than the bombers would start with when each bomber carries $n$ bombs, then any loading to be better (result in more bombs being delivered) must initially contain more than $n$ bombs per bomber, thus potentially...
reducting the number of defensive weapon combinations that must be considered. For example, in the sample problem 300 offensive weapons were expected to be delivered by the postulated bomber force when the bombers use only missile #6 as a defense. Thus any defensive load resulting in less than four offensive weapons per bomber need not be considered.

Another complication connected with mixed defensive weapon loadings concerns their utilization. If a bomber carries only one type of missile it is obvious that at least one should be used against each threat until the supply is exhausted. If it is impossible to use more than one weapon of a given type in each encounter (as was assumed in the example problem) then deployment policy need not be investigated. However, if a bomber has more than one type of defensive weapon, it may be possible and desirable to use one weapon of each type in each encounter. On the other hand it may be more effective for the bomber to use only one weapon per encounter and save the other weapon for future needs. Thus there is more than one deployment possibility that should be considered.
Determination of the Number of Possible Weapon Loadings

Assume that \( N_{dw} \) offensive weapon spaces are to be used for defensive weapons. Also assume that each defensive weapon is no larger than an offensive weapon and that an offensive weapon space is not to be used to house more than one defensive weapon class. If \( i \) weapon spaces are assigned to one class of defensive weapon \((i=1, \ldots, N_{dw}-1)\), then there are \( N_{dw} - i + 1 \) combinations of the other two classes of defensive weapons that could be used to fill the defensive weapon loading. If no weapons of the first class are used then there are \( N_{dw} - 1 \) combinations of the other two classes of defensive weapons that would result in a mixed defensive loading that uses up the available space. Thus the number of combinations of mixed defensive weapon loadings that can be used to fill the available space is:

\[
\text{Number of Combinations} = \sum_{i=1}^{N_{dw}-1} (N_{dw} - i + 1) + N_{dw} - 1
\]

(96)

Since \( N_{dw} \) can vary from two to some maximum value \( N_{d_{\text{max}}} \) and still result in mixed defensive weapon loadings, the
The total number of possible combinations is:

$$\text{Total Number of Combinations} = \sum_{N_{dw} = 2}^{N_{d \text{max}}} \left[ \sum_{i = 1}^{N_{dw} - 1} (N_{dw} - i + 1) + N_{dw} - 1 \right]$$

$$\sum_{N_{dw} = 2}^{N_{d \text{max}}} \left[ (N_{dw} + 2)(N_{dw} - 1) - \sum_{i = 1}^{N_{dw} - 1} i \right]$$ (97)

If $N_{d \text{max}}$ were ten there would be 255 mixed defensive weapon loadings that would have to be evaluated. However, if as in the example problem $N_{d \text{max}}$ could be reduced to six (by establishing an upper bound on the defensive weapon load size to be considered using the results from the homogeneous loadings) then the number of combinations to be evaluated is reduced to sixty-five.

Of course if one of the weapon class candidates could not fit into one offensive weapon spacing then the number of combinations is reduced. For example assume that one of the candidates trades one for two in space with an offensive weapon (the other two weapon
classes still fit into one offensive weapon space).
Then the total number of combinations that must be considered is:

\[
\text{Total Number of Combinations} = \sum_{N_{dw} = 2, 4, 6}^{N_{dw} - 1} \left[ \sum_{i = 1, 3, 5}^{N_{dw} - i + 1} / 2 + \sum_{i = 2, 4, 6}^{N_{dw} - 2} \right] + \sum_{N_{dw} = 3, 5, 7}^{N_{dw} - 1} \left[ \sum_{i = 1, 3, 5}^{N_{dw} - 2} \right]
\]

\[
\left( N_{dw} - i + 2 \right) / 2 + \left( N_{dw} - 2 \right) / 2
\]

\[
\left( N_{dw} - i + 2 \right) / 2 + \sum_{i = 2, 4, 6}^{N_{dw} - 1} \left( N_{dw} - i + 1 \right) / 2 + \left( N_{dw} - 1 \right) / 2
\]

(98)

In this case if \( N_{d_{\text{max}}} \) is ten there are 135 combinations to be considered while reducing \( N_{d_{\text{max}}} \) to six reduces the number of combinations to thirty-four. Thus, again a reduction in \( N_{d_{\text{max}}} \) can greatly reduce the required analysis.
Given the number of loadings to be considered, the next step is to analyze the possible candidates using the bounding models and the expected value model for bomber effectiveness. The revisions required to incorporate mixed defensive weapon loadings in these models are developed in the remainder of this chapter.

A Lower Bound on Bomber Survivability for Mixed Defensive Weapon Loadings

Assume that each bomber carries $i$ anti-interceptor missiles, $j$ anti-missile missiles, and deploys $N_d$ decoys. Also assume that the interceptors will attack a decoy with the same probability as a bomber. Thus if a target is attacked by an interceptor, the probability that it is a bomber is $1/(N_d+1)$ and the probability that it is a decoy is $N_d/(N_d+1)$. If the probability of killing a decoy given an attack is $P_{kd/w}$ then the upper bound on the expected number of attacks per decoy killed (note that the upper bound on attrition yields a lower bound on bomber effectiveness) follows from equation (69) directly and is:

Upper Bound Expected (# attacks/decoys killed)

$$= \frac{L}{P_{kd/w}}$$
Next consider the upper bound on the expected number of attacks per bomber killed \( (ENA_D) \) given the assumptions of Chapter III but allowing the bombers to have both classes of defensive missiles. The bombers may choose to use both available defensive missile classes in each encounter until one or both classes are exhausted or they may decide to use one class until the supply runs out and then switch to the other.\(^1\) For either of the deployment possibilities outlined above \( P_{kb/w} \) will take on one value for the first \( n \) encounters \( (n \text{ i or j}) \) and then another value for the next \( m \) encounters \( (m+n \text{ i,j}, \text{ or i+j}) \). Note that if only one class of defensive weapon is used at a time it is possible for \( P_{kb/w} \) to be the same for the total \( n \) plus \( m \) encounters. However, if anti-interceptor missiles are used first, interceptors might be killed during the

\(^1\) In the case of a bomber using one type of missile first and then switching to the other it would seem more reasonable to use the available anti-interceptor missiles first and then use the anti-missile missiles. If it is more effective to carry anti-interceptor missiles and thus reduce the threatening force of interceptors, then it seems reasonable to use these missiles at the earliest possible moment.
first n encounters but during the last m encounters only the interceptor missiles will be attacked.

Let:

\[ P_{kb/w_1} \] - the probability a bomber is killed during each of the first n encounters

\[ P_{kb/w_2} \] - the probability a bomber is killed during each of the next m encounters

Therefore:

Upper Bound Expected (# attacks/bombers killed) =

\[ ENA_b = 1 + \frac{1}{n} (1 - P_{kb/w_1}) + 1 \left(1 - P_{kb/w_1}\right)^2 \]

\[ + \frac{1}{m} (1 - P_{kb/w_2}) + 1 \left(1 - P_{kb/w_1}\right)^n \left(1 - P_{kb/w_2}\right) \]

\[ + 1 \left(1 - P_{kb/w_1}\right)^n \left(1 - P_{kb/w_2}\right)^m \]

\[ + 1 \left(1 - P_{kb/w_1}\right)^n \left(1 - P_{kb/w_2}\right)^m \left(1 - P_{kb/w_2}\right)^m \]
Bomber attrition can now be found by substituting expressions (99) and (100) into Equation (80).

Next we need to determine the attrition of the interceptor force. There are three situations that could be of interest when an interceptor attacks a
bomber (assuming one never uses anti-missile missiles first and then anti-interceptor missiles):

a) Only anti-interceptor missiles are used in the first \( n \) encounters and then only anti-missile missiles in the next \( m \) encounters.

b) Both anti-interceptor and anti-missile missiles are used in the first \( n \) encounters and then only anti-missile missiles in the next \( m \) encounters.

c) Both anti-interceptor and anti-missile missiles are used in the first \( n \) encounters and then only anti-interceptor missiles in the next \( m \) encounters.

For situations a and b, Equation (71) applies with \( P_{kb/w} \) equal to \( P_{kb/w_1} \) (however the value used for \( P_{kb/w_1} \) will depend upon which situation is being represented). For situation c the expected number of
interceptors killed per bomber killed is:

Expected Number of Interceptors Killed/Bomber Killed =

\[ P_{ki} + (1-P_{kb/w_1}) P_{ki} \quad \text{---} \quad + (1-P_{kb/w_1})^n P_{ki} \]

\[ + (1-P_{kb/w_1})^n (1-P_{kb/w_2}) P_{ki} \quad \text{---} \quad + (1-P_{kb/w_1})^n (1-P_{kb/w_2})^{m-1} P_{ki} \]

\[ = P_{ki} \sum_{i=0}^{n-1} (1-P_{kb/w_1})^i + P_{ki} (1-P_{kb/w_1})^n \sum_{i=0}^{m-1} (1-P_{kb/w_2})^i \]

\[ = P_{ki} \frac{1-(1-P_{kb/w_1})^n}{P_{kb/w_1}} \]

\[ + P_{ki} \frac{(1-P_{kb/w_1})^n(1-(1-P_{kb/w_2})^m)}{P_{kb/w_2}} \]

Either Equation (71) or the above equation (whichever applies for the situation represented) can be used in Equation (72) to determine the number of interceptors killed.
An Upper Bound For Mixed Defensive Weapon Loadings

The upper bound model introduced in Chapter IV was defined by the following two factors (see Equation 81):

a) The number of bombers attacked.
b) The probability an attacked bomber does not survive.

For a given interceptor force size starting the cycle and a given interceptor group size, the first of the above factors is unaffected by the number of defensive missiles carried by the bombers but is greatly affected by the number of decoys deployed (see Equation (83)). On the other hand the second factor is a function of the number of defensive missiles available but is not a function of the number of decoys deployed.

As in the previous section of this chapter, assume that one defensive missile deployment is used in the first n encounters and another is used in the next m encounters. After n plus m encounters a bomber is assumed to have run out of defensive missiles. Also let \( N_d \) be the number of decoys deployed by each bomber. Then the upper bound on expected bomber attrition as
described in Chapter IV becomes:

\[ \Delta B(n) = (\text{Min}(I(n-1)/(N_i(N_d+1)), B(n-1))) \]

\[ (1-(1-P_{kb/w_1}^{N_i})) \quad N_i \leq n \quad (103) \]

\[ = (\text{Min}(I(n-1)/(N_i(N_d+1)), B(n-1))) \]

\[ (1-(1-P_{kb/w_1}^n(1-P_{kb/w_2}^{N_i-n})) \quad n < N_i \leq n+m \quad (104) \]

\[ = (\text{Min}(I(n-1)/(N_i(N_d+1)), B(n-1))) \]

\[ (1-(1-P_{kb/w_1}^n(1-P_{kb/w_2}^m(1-P_{kb/w}^{N_i-n-m})) \quad (105) \]

\[ N_i > n+m \]

The next step will be to determine the interceptor attrition. The number of interceptors killed by each bomber attacked will depend upon the weapon deployment doctrine used as well as the number of available weapons. The possible deployments of defensive weapons by the bombers and the resulting interceptor attrition follows:

a) If anti-interceptor missiles are used during the first n encounters but not
thereafter the interceptor attrition per bomber attacked is:

\[ \Delta I = P_{ki} (1 - (1 - P_{kb/w_1})^{N_i}) / P_{kb/w_1} \quad \text{if } N_i \leq n \]  
\[ (106) \]

\[ \Delta I = P_{ki} (1 - (1 - P_{kb/w_1})^{n}) / P_{kb/w_1} \quad \text{if } N_i > n \]  
\[ (107) \]

b) If anti-interceptor missiles are used during the first \( n + m \) encounters the interceptor attrition per bomber attacked is:

\[ \Delta I = P_{ki} (1 - (1 - P_{kb/w_1})^{N_i}) / P_{kb/w_1} \quad \text{if } N_i \leq n \]  
\[ (108) \]

\[ \Delta I = P_{ki} (1 - (1 - P_{kb/w_1})^{n}) / P_{kb/w_1} \]

\[ + (1 - P_{kb/w_1})^{n} (1 - (1 - P_{kb/w_2})^{N_i - n} / P_{kb/w_2}) \]

\[ n < N_i < n + m \]

\[ \Delta I = P_{ki} (1 - (1 - P_{kb/w_1})^{n}) / P_{kb/w_1} \]

\[ + (1 - P_{kb/w_1})^{n} (1 - (1 - P_{kb/w_2})^{m} / P_{kb/w_2}) \]

\[ N_i > n + m \]
Since there is an expectation of $1/(N_d + 1)$ of the interceptor groups selecting bombers and not decoys to attack, the interceptor attrition is:

$$\Delta I(n) = \text{Min}(I(n-1)/(N_i(N_d + 1)), B(n-1))$$

An Expected-Value Solution for Mixed Defensive Weapon Loadings

The final task in incorporating mixed defensive weapon loads in the bomber attrition models developed previously will require expanding the expected value model developed in Chapter V to include the effects of heterogeneous defensive weapon loadings. If the bombers use both types of defensive missiles but no decoys, the change is relatively minor. Consider Table 6. Column (1) will have to be adjusted to account for the nonhomogeneous deployment of defensive weapons by the bombers, which is a straightforward correction.

The state of a bomber will now be defined by the number of defensive responses that can still be made (originally the bombers can make $n + m$ such responses). The state of all bombers other than the first one attacked by each interceptor group will be found in the column headed $PW(n-i)$ as before which will now have $n$
plus m entries. The bomber attrition analysis can thus proceed as before. The interceptor attrition can also be found in the same manner as shown in Chapter V (note however a correction in equations (92) and (93) will be needed if anti-interceptor missiles are used after the first n encounters).

The inclusion of decoys in the defensive weapon loading will result in more extensive changes to the expected value model. This is true even if no defensive missiles are carried. One possible way of including decoys in the model might be to think of them as unarmed bombers. Then the bomber force might be increased in size by adding \( N_d B(n-1) \) units each with no defensive missiles. The model could then be used as originally formulated. However the results would have to be analyzed and the decoy attrition separated from the bomber attrition. It appears very difficult to separate these attritions and no ready procedure to accomplish this is apparent. Thus we would seem to have to increase the scope of the model so that it can determine both attrition rates separately.

Every time the interceptors select an object to attack, there is some possibility that it is a bomber and some possibility that it is a decoy (with a known probability for each possibility). The proposed model
will therefore *tabulate* each possibility separately and then combine the results to define the probability distribution describing the number of interceptors available to attack the next target (bomber or decoy). The resulting tabulated data similar to that shown in Table 6 for the homogeneous case is shown in Table 14. In constructing Table 14, it was assumed that $N_d = 1$ and the probability of killing a decoy given an attack by an interceptor is 0.5.

The interceptor attrition might best be found by combining the results for bombers and decoys and treating the decoys as additional unarmed bombers. The procedure as outlined in Chapter V with the possible correction of Equations (92) and (93) as mentioned earlier could then be used.

**Conclusions**

In order to investigate heterogeneous defensive weapon loadings one will have to explore many possible load mixes as well as several possible weapon deployment doctrines. However, if the homogeneous loadings are considered first it is possible that a bound on the ultimate best loading can be found that greatly reduces the amount of investigation required. In fact if heterogeneous mixes allowing large numbers of offensive
Table 14

Analysis of One Battle Cycle

<table>
<thead>
<tr>
<th></th>
<th>1st Target</th>
<th></th>
<th>2nd Target</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Bomber</td>
<td></td>
<td>Bomber</td>
<td></td>
</tr>
<tr>
<td></td>
<td>i TK(i)</td>
<td>PS(i)</td>
<td>P(i)*</td>
<td>PS</td>
</tr>
<tr>
<td>0</td>
<td>1.000</td>
<td>1.000</td>
<td>0.005</td>
<td>1.000</td>
</tr>
<tr>
<td>1</td>
<td>0.050</td>
<td>0.950</td>
<td>0.004</td>
<td>0.903</td>
</tr>
<tr>
<td>2</td>
<td>0.048</td>
<td>0.903</td>
<td>0.009</td>
<td>0.903</td>
</tr>
<tr>
<td>3</td>
<td>0.025</td>
<td>0.250</td>
<td>0.016</td>
<td>0.250</td>
</tr>
<tr>
<td>4</td>
<td>0.125</td>
<td>0.125</td>
<td>0.016</td>
<td>0.125</td>
</tr>
<tr>
<td>5</td>
<td>0.500</td>
<td>0.500</td>
<td>0.016</td>
<td>0.500</td>
</tr>
<tr>
<td>6</td>
<td>0.016</td>
<td>0.016</td>
<td>0.016</td>
<td>0.016</td>
</tr>
<tr>
<td>7</td>
<td>0.008</td>
<td>0.008</td>
<td>0.008</td>
<td>0.008</td>
</tr>
<tr>
<td>8</td>
<td>0.004</td>
<td>0.004</td>
<td>0.004</td>
<td>0.004</td>
</tr>
<tr>
<td>9</td>
<td>0.002</td>
<td>0.002</td>
<td>0.002</td>
<td>0.002</td>
</tr>
<tr>
<td>10</td>
<td>0.001</td>
<td>0.001</td>
<td>0.001</td>
<td>0.001</td>
</tr>
</tbody>
</table>

*P(i)* is determined by the preceding PK(i) for both bomber and decoy.
weapons per bomber are considered first, it may be possible to establish a still better bound on the problem and further reduce the number of loadings considered. When a defensive loading (either homogeneous or heterogeneous) is found that has an expected return (in bombs delivered) that exceeds the maximum possible return one could obtain from other candidate loadings, then these other candidates do not have to be evaluated.

Corrections to both the bounding models and the expected value model to allow incorporation of mixed defensive weapon loadings were outlined in this chapter. The resulting bounding models although more complicated are still basically quite simple. The changes to the expected value model were quite trivial unless the loadings included decoys. If mixed defensive weapon loadings that contain decoys are to be investigated the model will have to determine the attrition of both bombers and decoys separately and thus will become more complicated.
The analysis of bomber effectiveness in the previous chapters has centered on determining bomber survival. The manner in which offensive weapons (bombs) are assigned to targets (ground targets) however, has not been considered. It has been assumed that either there exists a given allocation rule, or that the targets are reached by the bombers after the air battle (this was the assumption used in the example problem) and, therefore, the optimum load of offensive and defensive weapons is determined by maximizing the number of offensive weapons surviving the air battle. In the latter case there is no need to consider how the bombs might be used if one merely wants to select a defensive weapon that maximizes bomber effectiveness. However in many situations the best allocation policy is not obvious and there exists a need to determine the manner in which the bombs should be allocated to ground targets. If targets are reached during the course of the air battle, the allocation problem should be
considered in conjunction with the selection of the weapon load mix that is to be carried by the bombers. An arbitrary allocation rule (for example a uniform division of bombs among targets) may not allow a fair comparison. The rule may be reasonable for some weapon loadings but not for others. In order to properly select a weapon mix we should evaluate effectiveness based on a reasonable (and, hopefully, near optimal) policy for the use of the bombs carried in each candidate loading. Thus a different bomb deployment policy may be required for each load mix considered.

The purpose of this chapter is to establish a method of defining a reasonable (and, hopefully, near optimal) policy for the use of offensive weapons. The resulting policy can then be used to establish the effectiveness of a particular bomber weapon loading. In the general case a sensitivity analysis such as that contained in Chapter VI would include a consideration of the offensive weapon allocation problem.

**Problem Definition**

The procedures developed in Chapters III through V give the probability of survival of the bombers after each cycle of the battle. These cycles are equivalent to periods of time during which some ground targets may be
reached by the bombers (see Reference 5). The method to be developed to allocate bombs to targets will require an estimate of the probability of reaching each target. It will be assumed in this analysis that the same probability of survival can be used to evaluate each target encountered during a given cycle.¹ That is, the bombs will be allocated assuming the same probability of survival to each target in the cycle. Using the probability of survival at the start and the end of each cycle, upper and lower bounds can be placed on the expected amount of target damage for a given bomb deployment policy. In the final analysis, however, the best value to use between these two extremes will probably have to be selected arbitrarily. One likely choice might be the average of the initial and final probabilities of survival for a given cycle.

¹ The method to be developed will still apply if the targets are broken down into smaller groups than those encountered during each cycle. The only requirement is that the probability of surviving to a target is treated as the same for each member of each group. In fact the groups could be reduced to single targets. However, this would require estimating or predicting the probability of survival to each target. Due to the likely lack of knowledge about the change of bomber survivability within a cycle, it seems that in most cases it would be more reasonable to divide the targets into large groups and determine a reasonable way to allot bombs among the groups.
The amount of damage done to the jth target in cycle i (target i,j) will be a function of the number of bombs assigned to the target and the probability of a given bomber reaching the target. The problem is thus to maximize the following:

\[ Z = \sum_{i} \sum_{j} K_{ij} f(n_{ij}, p_i) \]  

(112)

where: \( K_{ij} \) is a weighting constant signifying the value of target i,j

\( n_{ij} \) is the number of bombs assigned to target i,j

\( p_i \) is the probability of each bomber reaching a target in cycle i

\( f(n_{ij}, p_i) \) is some function of \( n_{ij} \) and \( p_i \) that gives the expected damage done to target i,j

Any allocation of bombs to targets must also satisfy the following constraints:

\[ \sum_{i} \sum_{j} n_{ij} \leq N_{\text{ow}} \]  

(113)
\[ n_{ij} \] is a non-negative integer for all \( i, j \)

where: \( N_{ow} \) is the number of bombs carried by
the total bomber force.

Since in any actual problem it would never be
disadvantageous to assign an available bomb, the first
constraint can really be considered an equality constraint.

It will be assumed in the analysis that follows
that the bombers choose to cross-target the available
bombs. That is, if more than one weapon is assigned to
a target, only one bomb will be delivered from a given
bomber, rather than all of the bombs being assigned
from one bomber. If all the bombs assigned to a target
are carried by one bomber, either all are delivered or
none are delivered. When cross-targeting is used, the
same expected number of bombs would be delivered, but
the probability of at least one being delivered on a
given target will rise. Thus cross-targeting will tend
to distribute the surviving bombs among the targets.
Since there is usually a diminishing return from
additional bombs arriving at a given target, cross-
targeting will tend to increase total bomber
effectiveness.
The following function is commonly used to describe the amount of damage to target \( i, j \) in cross-targeting situations and will be used in this analysis:

\[
f(n_{ij}, p_i) = 1 - (1 - p_i k)^{n_{ij}} \quad 0 < k \leq 1
\]

(114)

\( k \) can either be interpreted as the fraction of the remaining target value destroyed by each successive delivered bomb or as the probability that the target is totally destroyed by a given bomb. If it is further assumed that each target to be attacked is of equal value, the allocation problem becomes:

\[
P_1 \quad \text{Maximize } Z' = \sum_{i} \sum_{j} (1 - (1 - p_i k)^{n_{ij}}) \quad (115)
\]

Subject to:

\[
\sum_{i} \sum_{j} n_{ij} = N_{ow}
\]

\( n_{ij} \) a non-negative integer for all \( i, j \)

This is the problem that will be solved in this chapter.
The Bomb Allocation Model

Equation (114) gives the total expected damage done to target \( i,j \) by all assigned bombs. This can be rewritten as the sum of the incremental damages resulting from the assignment of each bomb. Therefore:

\[
1 - (l-p_i k)^n_{ij} = \sum_{m=1}^{\infty} (1-p_i k)^{m-1} p_i k \ q_{ij}(m) \quad (116)
\]

where \( q_{ij}(m) \) indicates whether or not target \( i,j \) is assigned \( m \) or more weapons. That is \( q_{ij}(m) \) equals one if target \( i,j \) is assigned \( m \) or more weapons and is zero otherwise.

If this form is used the following constraints must be added to the problem:

\[
q_{ij}(m) \quad 0 \text{ or } 1 \quad \text{for all } i,j,m
\]

\[
q_{ij}(m) \geq q_{ij}(n) \quad \text{if } m < n
\]

(117)
Thus $P_1$ can be rewritten as:

$$P_1' \text{ Maximize } Z' = \sum_{i} \sum_{j} \left( \sum_{m=1}^{\infty} (1-p_{ik})^{m-1} p_{ik} q_{ij}(m) \right)$$

Subject to:

$$\sum_{i} \sum_{j} \sum_{m} q_{ij}(m) = N_{ow}$$

$$q_{ij}(m) = 0 \text{ or } 1 \text{ all } i, j, m$$

$$q_{ij}(m) \geq q_{ij}(n) \text{ if } m < n$$

Letting $\sum_{j} q_{ij}(m) = r_{im}$, $P_1'$ becomes:

$$P_1'' \text{ Maximize } Z'' = \sum_{i} \sum_{m=1}^{\infty} (1-p_{ik})^{m-1} p_{ik} r_{im}$$

$$\sum_{i} \sum_{m} a_{im} r_{im}$$

(where: $a_{im} = (1-p_{ik})^{m-1} p_{ik}$)

Subject to:

$$\sum_{i} \sum_{m} r_{im} = N_{ow}$$

$$0 \leq r_{im} \leq N(i)$$

$r_{im}$ an integer
\[ r_{im} \geq r_{in} \quad \text{if } m < n \]

where \( N(i) \) is the number of targets in cycle \( i \). Notice that the last constraint on \( P1'' \) (119) will allow the last constraint on \( P1' \) (118) to hold if \( r_{im} \) is properly interpreted. That is when \( r_{im} \) is reconverted to \[ \sum_j q_{ij}(m) \]
care must be taken to ensure that the last constraint on \( P1' \) is not violated. However, as will be shown, the solution to be obtained will automatically yield a feasible solution to \( P1' \).

The problem is thus to select the \( r_{im} \)'s to maximize \( Z'' \) subject to the constraints given in \( P1'' \). A procedure to identify the optimum feasible solution results directly from the following theorems:

**Theorem:** In an optimum feasible solution \( r_{im} = N(i) \) if \( r_{st} \) is greater than zero and \( a_{im} \) is greater than \( a_{st} \).

**Proof:** Assume that in an optimum feasible solution \( r_{st} \) is greater than zero and \( r_{im} \) is less than \( N(i) \). Assume also that \( a_{im} \) is greater than \( a_{st} \). Consider a new assignment:

\[ r_{im}^* = r_{im} - d \leq N(i) \quad d > 0 \] (120)
Obviously this is also a feasible solution. The change in $Z''$ is:

$$d a_{im} - d a_{st} = d (a_{im} - a_{st}) > 0 \quad (121)$$

Therefore the original solution was not optimal and the proof is completed.

Theorem: If $a_{im}$ equals $a_{st}$ and an optimum feasible solution is obtained, then the assignments to $r_{im}$ and $r_{st}$ can be changed as long as $r_{im}$ plus $r_{st}$ is unchanged and the constraints of the problem are not violated, and a new optimum feasible solution is formed.

The proof of this theorem is similar to the proof of the preceding theorem and is not given here.

Thus an optimum feasible solution is found by simply selecting the largest $a_{im}$'s in order and assigning as much as possible to the corresponding $r_{im}$'s subject to the constraints of $P1''$. This process is continued until all available bombs are assigned. If there is a tie for next largest $a_{im}$, the second theorem guarantees that the tie can be broken arbitrarily. Since $N_{cw}$ and all $N(i)$'s
are integers, this procedure will insure that the \( r_{im} \)'s
are in fact integers.

A solution to Pl'' is given by the following
procedure which is computationally easy to implement.
This will yield an optimum feasible solution since:

\[
\begin{align*}
\text{Step 1} & \quad \text{Assign one offensive weapon to each} \\
& \quad \text{target in cycle 1.} \\
\text{Step 2} & \quad \text{Compare the expected damage resulting} \\
& \quad \text{from an additional weapon on a target} \\
& \quad \text{in cycle 1 with the expected damage} \\
& \quad \text{resulting from the first weapon on} \\
& \quad \text{a target in cycle 2. Assign one}
\end{align*}
\]
offensive weapon to each target in the cycle that yields the highest expected damage per weapon.

Step i Compare the expected damage resulting from an additional weapon on a target in each cycle to which weapons have been previously assigned and from an initial assignment in the next cycle not previously assigned weapons (if such exists). Assign one weapon to each target in that cycle that gives the highest expected damage per weapon.

Repeat Step i until all weapons are assigned.

An Example Problem

Suppose it was decided to develop the number six missile defined in the example problem introduced in Chapter III. After adding this missile to the arsenal a situation is encountered in which targets will be reached during the air battle. It must now be decided how many of these defensive missiles should be carried in the new mission and how the bombers should assign
the available bombs. Assume that the threat is the same as that described in the example problem in Chapter III. Further assume that the battle is to last for three cycles with seventy-five targets of equal value being reached during each cycle, and the bombs are predicted to be capable of destroying 50 per cent of the remaining target value (thus \( k = 0.5 \)).

Tables 23, 24, and 25 in Appendix C show the expected number of bombers surviving each of the three cycles when the bombers carry various loadings of the number six missiles and the interceptors work in various sized groups. Assume it has been decided to use the average of the probabilities of survival before and after the cycle in allocating bombs to targets in a given cycle. Thus if four defensive missiles are carried by each bomber and the interceptors choose to form groups of size six then the probabilities associated with reaching the targets are:

\[
P_1 = \left( \frac{B(0)}{B(0)} + \frac{B(1)}{B(0)} \right)/2 = 0.815
\]

\[
P_2 = 0.383
\]

\[
P_3 = 0.070
\]
Since $p_i \ (i = 1, 2, 3)$ is minimized for the case of bombers carrying four defensive missiles when the interceptors form groups of size six, this is the defense's optimum tactic. Thus if the defense is assumed to know the bomber loadings, the offensive weapons allocation should be based on an interceptor grouping of size six.

Table 15 gives the weapon allocation and the resulting expected damage for various loadings of defensive missiles. These results are based on the procedure given in the previous section of this chapter. In all cases the interceptor groupings are chosen to minimize expected target damage assuming that the defense knows what weapons are carried by the bombers. Notice that the bombs are not always equally distributed among cycles. The allocation policy varies radically with the weapon loading.
Table 15

Optimum Allocation of Bombs for Various Weapon Loadings

<table>
<thead>
<tr>
<th>Number of Defensive Missiles per Bomber</th>
<th>Allocation of Bombs to Cycle i</th>
<th>Expected Damage $Z'$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$i = 1$</td>
<td>$i = 2$</td>
</tr>
<tr>
<td>4</td>
<td>300</td>
<td>500</td>
</tr>
<tr>
<td>6</td>
<td>225</td>
<td>250</td>
</tr>
<tr>
<td>8</td>
<td>225</td>
<td>225</td>
</tr>
<tr>
<td>10</td>
<td>150</td>
<td>200</td>
</tr>
<tr>
<td>12</td>
<td>150</td>
<td>150</td>
</tr>
</tbody>
</table>

For this mission a loading of either six or eight defensive missiles will give the same expected return. However, since the loading of eight missiles yields a higher probability of bomber survival it would probably be chosen. Note it may be decided to sacrifice some effectiveness to improve survivability and carry even more defensive missiles. This would of course be strictly a management decision. All that this procedure can do is expose the consequences of the alternative courses of action.
Of course there may be some question as to the validity of using the average probability of survival for each cycle in allotting bombs. Targets may tend to occur either early or late in a given cycle. Attrition may not be uniform during the battle but may tend to occur either at the start or the end of cycles. In either case other estimates for the probability of reaching targets may be more acceptable. By considering the actual situation it may be possible to arrive at better values to use. However, this will require a rather precise definition of the problem which is often impossible. What can be done is to place upper and lower bounds on bomber effectiveness which will help in establishing whether or not the selected loading is really adequate for the given task. This can be done by considering the bounds on the probability of reaching targets within each cycle.

When the bombers carry eight defensive missiles and the interceptors work in groups of eleven, the values for $p_i$ given in Table 16 will yield the desired bounds. The solutions obtained using these bounding probabilities and the average probabilities are given in Table 17.
Table 16

Upper and Lower Bounds on Reaching Targets

<table>
<thead>
<tr>
<th>Survival Probabilities</th>
<th>Upper Bound</th>
<th>Lower Bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1$</td>
<td>1.000</td>
<td>.734</td>
</tr>
<tr>
<td>$P_2$</td>
<td>.734</td>
<td>.500</td>
</tr>
<tr>
<td>$P_3$</td>
<td>.500</td>
<td>.304</td>
</tr>
</tbody>
</table>

*a* Using the probability of survival at the start of each cycle.

*b* Using the probability of survival at the end of each cycle.

Table 17

Optimum Allocation of Bombs

for Bounding and Average Solutions

<table>
<thead>
<tr>
<th>Allocation of Bombs to Cycle $i$</th>
<th>Expected Damage $Z'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i=1$ $i=2$ $i=3$</td>
<td></td>
</tr>
<tr>
<td>Upper Bound</td>
<td>150 225 225</td>
</tr>
<tr>
<td>Average Solution</td>
<td>225 225 150</td>
</tr>
<tr>
<td>Lower Bound</td>
<td>225 225 150</td>
</tr>
</tbody>
</table>
Notice that the allocation of bombs depends on the values used for $p_i$ ($i = 1, 2, 3$). However the allocations $(225, 225, 150)$ which are optimal for the average solution and the lower bound solution will result in an expectation of killing 155 targets when the upper bound probabilities are used. Thus, this allocation is optimal or close to optimal for the three cases considered above and would probably be chosen for actual use. If the bounding allocations had been radically different, a more detailed analysis should be considered.

**Conclusions**

A procedure for allocating bombs to targets was developed in this chapter. In the development it was assumed that each target in a given zone (corresponding to the area being penetrated during one cycle of the battle) had the same probability of being reached. Since the best value to use for the probability of reaching a target in a given zone is generally unknown, bounding solutions were suggested to indicate the amount of uncertainty that might be associated with a given solution. Even if the assumption of a single probability for reaching targets within a zone is not acceptable, the bounding solutions would still indicate the possible
variation in effectiveness.

If the rate of attrition is low, the bounding solutions will not be very different and the solution found by using the average probabilities of reaching targets should be quite acceptable. However in situations in which the rate of attrition is high, the possible inaccuracy that can result from using the average probabilities of reaching targets can become significant. High rates of attrition will be encountered if the kill probability of an interceptor against a bomber is high and/or the number of available interceptors is large. Whether or not the average solution may be very inaccurate can best be determined by evaluating the attrition rates. The bounding solutions obtained for the example problem considered in this chapter were not too different (see Table 17), and so, the average solution should be acceptable.

The procedure for allocating bombs shows how many bombs should be assigned to a given target. These results can then be used to determine the expected target damage resulting from the bomber force attacking with a given weapon loading. Since the optimum allocation policy can change radically from loading to loading, it is necessary to determine the best policy for each loading if
effectiveness of the loadings is to be compared. Offensive weapon allocation is a necessary consideration in evaluating the relative merits of weapon mixes in all situations where targets are reached during the course of the air battle.
CHAPTER IX

CONCLUDING REMARKS

Airpower has become essential in modern military concepts. Present day aircraft have proven to be capable of having a great effect on the course of wars. Thus both the offense and the defense generally will devote a large portion of their resources to attempt to win the air battle. Present trends seem to indicate that the importance of airpower in military planning will continue to increase in the future. As has been shown in this volume, the actual effectiveness of airpower may hinge on the tactics used by both sides (operating group size chosen by the defense and weapon loading chosen by the offense). Thus it is extremely important that military planners attempt to identify the best tactics to use in the air battle. Only by optimizing the tactics for the use of the equipment carried as well as the selection of the equipment itself, can the best strategy be found.
The problem of determining the best tactics for both the offense and the defense is very complicated. Modern technology can offer a great variety of weapons of varying effectiveness to use in fighting the battle. Thus the number of choices for weapon loadings can be quite large. Since the optimum tactics will possibly change when the weapon loading changes, the choice of tactics must be re-evaluated for each weapon loading. In addition there is a tendency for modern weapons to become very large, which leads to the possibility that an aircraft cannot carry enough defensive weapons to ensure against exhaustion. It is important that this basic weakness (the inability to carry large number of weapons) associated with some advanced weapons be recognized and included in any analysis of the problem of determining weapon (and thus aircraft) effectiveness. A more effective weapon may not always be best if the supply is too limited. This is especially true if the defense recognizes this shortcoming and uses tactics that take advantage of the supply limitation. In solving the example problem introduced in Chapter III, it became evident that the defense could greatly improve their effectiveness by selecting the best groupings to use in deploying interceptor aircraft when the supply of defensive
weapons was limited. Of course, if the defense did not know how many defensive weapons were carried, the interceptors might not be capable of selecting the best groupings.

The existing literature on force effectiveness modeling does not offer analytical models that do justice to the problem of analyzing aircraft with a limited supply of weapons. Therefore military planners have been forced to resort to Monte-Carlo battle simulations to determine the effectiveness of modern aircraft. These simulations have unfortunately tended to become quite time consuming computationally. If the number of situations to be considered is very large due to multiple choices of tactics by either or both combatants or due to the need to consider possible variations in the basic problem definition resulting from uncertainty, the use of Monte-Carlo simulations may not be practical.

Another basic problem with Monte-Carlo simulations is that they tend to obscure cause and effect relationships. It may be easy to obtain a result from such a simulation but hard to explain what factors were primarily responsible for the particular answer obtained. Thus Monte-Carlo simulations may fail to suggest to the planner the type of changes to the problem that could radically affect the results, and so may not provide much guidance in selecting
possible tactics that should be considered in the future. Without such guidance the task of searching for better solutions to the problem (such as finding new weapon types) may be very difficult. It should be pointed out however that a properly constructed Monte-Carlo simulation (if it is not too complex) might be programmed to provide some guidance for the task of searching for better solutions.

The above discussion shows that there is a definite need for alternative approaches to Monte-Carlo simulation of the aircraft effectiveness problem such as were developed in this volume. The models that were developed are basically quite simple. The computational effort required to study this problem using these models is minimal. In fact any of the solutions can be obtained without the use of a computer (however if many options are to be considered, a computer will probably be necessary).

The simplicity of the models developed in this volume should also give the planner some insight into the cause and effect relationships yielding the answers. Since the bounding-model solutions are in the form of a pair of attrition equations, the effect of changes in the problem parameters is especially easy to observe. Since improving the bounding solutions should tend to improve the expected
solution, some insight into the problem should be obtained by just considering the effect of changes on the bounds.

Although the expected-value model introduced in Chapter V is not as simple as the bounding models, it too will tend to shed some light on cause and effect relationships. As discussed in Appendix C, the expected value model traces the way in which the defensive weapon supply is exhausted during the battle. This should indicate the importance of the weapon supply limitation on the problem and thus may suggest a better course of action. For example, if it is found that attrition of the bombers is occurring because of a defensive weapon shortage, this suggests a need to decrease the size of the bomber's defensive weapons and thus increase the supply without reducing the number of available offensive weapons. On the other hand if the weapon supply is not being exhausted prior to kills by the defense, this suggests that a more effective defensive weapon is needed by the bomber. Increasing the number of defensive weapons without improving their effectiveness will not appreciably improve the second situation.

The bounding solutions found in Chapters III and IV were effective in eliminating from further consideration possible weapon load candidates proposed in the example
problem. These candidates were proven to be inferior to other candidates by comparing the bounding solutions. However, it should be pointed out that the particular bounding models used in this volume that were so effective in analyzing the example problem may not be well suited to some other problems. Although the solutions developed seem to be well suited to the air battle problem, better bounds may be found for particular battle scenarios. The bounds suggested in this volume are not claimed to be absolute bounds. Larger lower bounds or smaller upper bounds for a particular problem might be obtained from other formulations. It is important, however, to recognize that in a particular problem, a bounding model approach can greatly reduce the amount of required detailed analysis. If the bounds suggested in this volume are not well suited to a given problem, a search for other bounding solutions should be considered.

The solutions arrived at by using the models described in this volume define courses of action by both the defense and the offense that reflect the characteristics of the weapons used in fighting the battle. The bombers' weapon loadings and the defense's deployment policy are determined from the size and effectiveness of the bombers' defensive weapons rather than from some arbitrary rule that may not suit the
actual situation. Thus both combatants are allowed to react in an intelligent manner rather than as defined by some fixed policy. The resulting analysis should be more representative of how wars should be fought. Since modern military tactics are highly adaptive and well thought out, it seems likely that battles will in fact be fought in a near optimum manner. Thus the analysis suggested here may also be representative of how wars will be actually fought.

The models developed in Chapters III, IV, and V were extended in Chapter VII to account for the use of nonhomogeneous defensive weapon loads. The required changes to the models were relatively straightforward. The resulting models present no real computational problems. However, although the models are still quite simple, the consideration of mixed defensive loadings can greatly increase the computational burden since the number of candidate loadings to be evaluated tends to increase. Also more possible deployment policies per candidate loading will normally have to be considered. Therefore, although there is no real problem in evaluating a given loading and deployment policy for nonhomogeneous defensive weapon loads, the large number of such candidates that may have to be included in the analysis can create computational problems. However the use of a
digital computer should relieve such problems.

One possibility that was not explored in this volume is a changing policy for the deployment of missiles. It certainly would be possible using basically the models as developed to explore the possibility that the number of defensive missiles used per encounter by the bombers should change with time. It would also be possible to consider a changing policy for the use of missiles by the interceptors. As in the case of nonhomogeneous defensive weapon loadings, a study of changing deployment policies may require an evaluation of a large number of possible situations. However, a study of variable weapon deployment policies would be a logical extension of this volume. This is certainly a little understood area that deserves consideration.

Limitations of the Solution Procedure

The solutions presented in this volume are deterministic in nature and thus may not really yield an unbiased answer for the expected solution. Just as Lanchester's solutions have been shown to yield a bias in some cases, so might these solutions. The attrition rate predicted from the models developed in this volume
will probably be biased from the point in time when either one of the forces could be annihilated. The greater the probability of such annihilation, the larger the bias. A discussion of the source of this bias in deterministic battle models is contained in Chapter II. As is the case with Lanchester's formulations, the percentage of bias should tend to decrease as the force sizes increase (Reference 13). However it should be realized that the solutions developed in this volume include factors that are ignored in Lanchester's solutions such as weapon supply limitations, defensive deployments and the time at which kills occur. The additional complexity that would probably result from a stochastic solution (if one can be obtained) to the more complicated problem considered in this volume may make its worth questionable. However, the development of a stochastic solution for the problem considered in this volume would be a logical extension of this effort. In addition to resolving the question of bias, a stochastic solution would yield insight into the effect of the random nature of the situation. A solution that optimizes the average outcome but has a large variance may not actually be desirable. An acceptable solution
should result in a high probability of winning the battle, rather than simply requiring that the most likely battle is won.

The expected value model developed in Chapter V lacks the simplicity of the bounding models and thus is not as easy to use. The expected value solution would be more convenient if it could be reduced to a set of attrition equations like those in the bounding solutions. It would then be easier to visualize the problem, and thus, gain insight from the model. Another goal of future effort might be to obtain such a simpler solution for the expected value model. If this could be done, the bounding analysis might then be eliminated since the expected value solutions might be as easy to work with. Of course the bounding solutions might still offer further insight into the problem and so might be worth obtaining even when the expected solution is easily calculated.

The expected value model requires as input the expected number of bombers that a group of interceptors can reach. However to predict the expected number of bombers that can be reached by an interceptor group, it is probably necessary to assume that no more than one interceptor group might reach a given bomber. In this case it is assumed that each group of interceptors
limits its actions to a different area of the battle scenario. Then the number of bombers found is independent of bomber attrition during the battle cycle being considered, and it should be possible to predict the interceptors' capability to find bombers. If it is not assumed that a given bomber is never a target for more than one group of interceptors, it should be much more difficult to predict the expected number of bombers found. Although the expected value model could handle situations in which more than one group of interceptors might attack a given bomber if the expected number of bombers that can be reached by a group of interceptors could be estimated, it would be difficult to obtain such an estimate.

The models developed in this volume are based on the assumption that the battle is made up of a series of engagements between single units from both sides. Thus the battle is pictured as a large number of one-on-one engagements rather than many-on-one or many-on-many engagements. Although present battles do appear to be fought on a one-on-one basis, it might be desirable to include many-on-many capability. This could result in computational advantages in determining battle effectiveness since a campaign would involve fewer
engagements. However, the present single-engagement air battle models are generally limited to one-on-one situations as is much of presently available combat data, and so, it would probably be difficult to obtain predictions of the results of many-on-many engagements either analytically or from experience. Therefore, the analysis in this volume has assumed one-on-one engagements (which is also true of other force attrition models such as Lanchester's (Reference 1), Thomas' (Reference 15) and Fawcett and Jones' (Reference 5)).
APPENDIX A

It was claimed in Chapter I that simultaneous attacking by interceptors is not a good tactic when the bomber can simultaneously deploy all of its defensive missiles. A proof of this claim follows. Let:

\[ P_{sb/w} \] = probability of survival of a bomber when attacked by a single interceptor assuming the bomber uses a defensive weapon.

\[ P_{sb/w^-} \] = probability of survival of a bomber when attacked by a single interceptor assuming the bomber uses no defensive weapon.

\[ k \] = number of defensive weapons carried by a bomber.

\[ n_i \] = number of interceptors within a subgroup.
$P_{ki}$ = probability of kill of single interceptor attacking an armed bomber.

Whether the interceptors attack in a group or in series, the probability of survival of the attacked bomber (assuming all defensive weapons can be deployed simultaneously) is:

\[
\text{Probability (Survival of the Bomber)} = P_{sb/w}^{n_i} \quad n_i \leq k
\]

\[
= P_{sb/w}^{k} P_{sb/w}^{n_i-k} \quad n_i > k
\]

(124)

Now if the interceptors attack in a group, the expected number of interceptors lost is:

\[
\text{Expected (Number of Interceptors Lost)} = n_i P_{ki} \quad n_i \leq k
\]

\[
= k P_{ki} \quad n_i > k
\]

(125)
If the interceptors attack in series:

**Expected (Number of Interceptors Lost)**

$$P_{ki} = (P_{sb/w})^{P_{ki}}$$

$$= (P_{sb/w})^{n_{i}-1}P_{ki}$$

$$= P_{ki}\sum_{i=0}^{n_{i}-1}P_{sb/w}$$

$$= P_{ki}(1-P_{sb/w})^{n_{i}}/(1-P_{sb/w})$$ \( n_{i} \geq k \)

(126)

**Expected (Number of Interceptors Lost)**

$$P_{ki}(1-P_{sb/w})^{k}/(1-P_{sb/w})$$ \( n_{i} > k \)

Consider the term \((1-P_{sb/w})^{j}/(1-P_{sb/w})\) where \(j \geq 0\) (since \(k\) and \(n_{i}\) would both be nonnegative). If \(P_{sb/w} = 1\), the above ratio takes on a value of \(j\). If \(P_{sb/w} = 0\), the above ratio becomes unity. The number of interceptors lost will obviously be the same for both policies (single or simultaneous attacks) unless \(n_{i}\) and \(k\) are both greater than one. Therefore the only cases of interest occur when \(j \geq 2\) or more. In these cases the
above ratio will increase monotonically as $P_{sb/w}$ increases over its possible range of values, zero to one. Since the ratio is monotonically increasing for $P_{sb/w}$ in the range zero to one and we know it equals $j$ at $P_{sb/w} = 1$, it is concluded that:

$$\frac{(1-P_{sb/w})^j}{(1-P_{sb/w})} \leq j \quad 0 \leq P_{sb/w} \leq 1$$

(127)

Thus if the bomber can simultaneously respond to all members of the group up to the limit of the number of defensive weapons carried, it is clear that simultaneous attacks will be at least as costly in interceptors lost as independent attacks would be. Since these two possible modes of operation of the interceptors result in the same survival probability for the bomber and since simultaneous attacks maximize the number of interceptor weapons used, simultaneous attacking would not be desirable from the interceptor's standpoint when the bombers can simultaneously launch counter weapons.
APPENDIX B

In the analysis of the example problem in Chapters III through V it was assumed that the defense knew what weapon loading the bombers selected. Thus the interceptors were assumed to group so as to minimize bomber effectiveness for every load considered. However, it may be that the defense will not know what weapon load is selected by the bombers, and so, must act in the face of uncertainty. The analysis of the problem of selecting tactics when the defense is uncertain of the offensive response will be considered in this section.

The basic scenario outlined in the example problem introduced in Chapter III will be used in this analysis. However, it will be assumed that only the number six defensive missiles will be available for defensive purposes to the bombers. It will further be assumed that the defense knows that the bombers will carry at least four defensive missiles but no more than twelve. These assumptions are made to keep the problem
from becoming unnecessarily complicated.

The effect of uncertainty on the upper bound and expected value models will be considered. Since these two solutions are dependent upon the assumed interceptor group size \( N_i \), they will depend upon the tactics selected by the defense which are in turn dependent upon the defense's state of uncertainty. Defensive uncertainty about bomber weapon loadings will not affect the lower bound solutions since interceptor group size is not reflected in the lower bound analysis.

**Perfect Information Available to the Offense**

Assume first that the defense does not know what weapons the bombers will carry but that the offense will know the group size selected by the defense. The offense may be able to obtain this information from intelligence sources.

**Upper Bound Solution**

Consider first the upper bound model. Table 18 gives the upper bound on the number of bombs surviving for various loadings of number six defensive missiles and various interceptor groupings. This data was generated by using Equation (81).
Table 18

Upper Bound on the Number of Offensive Weapons Delivered as a Function of the Number of Number Six Defensive Missiles Carried and Interceptor Group Size

<table>
<thead>
<tr>
<th>Interceptor Group Size</th>
<th>Number of Defensive Missiles/Bomber</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>683</td>
</tr>
<tr>
<td>2</td>
<td>584</td>
</tr>
<tr>
<td>3</td>
<td>575</td>
</tr>
<tr>
<td>4</td>
<td>585</td>
</tr>
<tr>
<td>5</td>
<td>278</td>
</tr>
<tr>
<td>6</td>
<td>291</td>
</tr>
<tr>
<td>7</td>
<td>348</td>
</tr>
<tr>
<td>8</td>
<td>400</td>
</tr>
<tr>
<td>9</td>
<td>445</td>
</tr>
<tr>
<td>10</td>
<td>480</td>
</tr>
<tr>
<td>11</td>
<td>509</td>
</tr>
<tr>
<td>12</td>
<td>533</td>
</tr>
<tr>
<td>13</td>
<td>554</td>
</tr>
<tr>
<td>14</td>
<td>570</td>
</tr>
<tr>
<td>15</td>
<td>586</td>
</tr>
<tr>
<td>16</td>
<td>600</td>
</tr>
</tbody>
</table>

Upper Bounds 445 445 445 427 342
Since the offense will know the grouping selected by the defense and will presumably arm the bombers accordingly, the defense's optimum tactic is to select that grouping that minimizes the number of bombs surviving when the offense maximizes its effectiveness for the selected grouping. However, since Table 18 contains only the upper bounds on the possible solutions, it is not possible to use Table 18 to determine what decision the defense will make when the expected value solutions are available. What is needed is a bound on the solution that will be obtained when the defense chooses the group size.

Consider the values in Table 18. If the defense selects groups of size nine, the bombers will be able to deliver no more than 445 bombs. If any other grouping is selected, an upper bound for some loading can be found that is greater than 445. Thus 445 is the desired solution bound. There is no assurance that the defense will ultimately select groupings of size nine. However, if the grouping that minimizes maximum bomber effectiveness is selected, the bombers will not be able to deliver more than 445 bombs.

What then are the upper bounds on weapons delivered for all of the possible loadings? If six defensive missiles are carried and the defense forms
interceptor groups of size nine then we see from Table 18 that no more than 392 bombs will be delivered. But the final solution using the expected value results might indicate another grouping of interceptors such as of size six. However this will not happen if the expected solution for groups of size six is greater than 445, since in that case groups of size nine would have to be a better choice for the defense. Thus the upper bound on the loading with six defensive missiles is 445.

Consider next the case where each bomber has twelve defensive missiles. From Table 18 it is seen that the number of bombs delivered cannot exceed 342 for any of the defensive groupings considered. Thus 342 is the desired bound. No matter what grouping is selected by the defense the offense could not deliver more than 342 bombs if twelve defensive missiles are carried by each bomber.

The upper bound for each loading is:

Upper Bound = Min (445, largest bound for the loading)

The upper bounds for all of the loadings can be found in the last row of Table 18. For this problem all of the upper bounds exceed the maximum lower bound (see Chapter III) and so none of the loadings can be rejected
without finding the expected value solutions.

Expected Value Solution

Table 19 contains the expected value solutions for various loadings of number six defensive missiles and various interceptor groupings. These values were obtained from the bomber survivability data found in Appendix C. Since perfect information is assumed to be available to the offense, the defense should select its groupings to minimize the maximum bomber effectiveness. If the defense organizes in groups of size eleven, the bombers could expect to have no more than 321 bombs surviving. Any other group size would result in more than 321 bombs surviving if the bombers selected the most effective weapon loading (assuming knowledge by the offense of the defensive group size) and so the grouping of size eleven should be chosen by the defense. The bombers should then decide to carry six defensive missiles resulting in an expectation of 321 bombs surviving. When the defense was assumed to have an intelligence advantage, there was an expectation of 300 bombs surviving (see Chapter V). Thus giving the offense the intelligence advantage increased the number of bombs surviving by 21.
### Table 19

**Expected Number of Offensive Weapons Delivered as a Function of the Number of Number Six Defensive Missiles Carried and Interceptor Group Size**

<table>
<thead>
<tr>
<th>Interceptor Group Size</th>
<th>Number of Defensive Missiles/Bomber</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>578</td>
</tr>
<tr>
<td>3</td>
<td>288</td>
</tr>
<tr>
<td>4</td>
<td>305</td>
</tr>
<tr>
<td>5</td>
<td>206</td>
</tr>
<tr>
<td>6</td>
<td>109</td>
</tr>
<tr>
<td>7</td>
<td>238</td>
</tr>
<tr>
<td>8</td>
<td>244</td>
</tr>
<tr>
<td>9</td>
<td>282</td>
</tr>
<tr>
<td>10</td>
<td>254</td>
</tr>
<tr>
<td>11</td>
<td>287</td>
</tr>
<tr>
<td>12</td>
<td>314</td>
</tr>
<tr>
<td>13</td>
<td>334</td>
</tr>
<tr>
<td>14</td>
<td>354</td>
</tr>
</tbody>
</table>
Lack of Intelligence to Both Sides

Assume next that not only will the defense not know what weapons the bombers will carry, but that the offense will not know the group size selected by the defense. Thus both sides are faced with uncertainty and a two-player, zero-sum game theory problem results (Reference 4). Each side will have to select its tactics without knowing what choice will be made by the opponent.

Since this problem has a two-player, zero-sum game structure, the only way in which a strategy can be eliminated from further consideration is by showing that it would be dominated when the expected value solutions are available. This could be done if the lower bound for one strategy exceeded the upper bounds of another strategy for all possible strategies of the opponent. This would seem to be very unlikely, and so, it might be best to go directly to the expected value solutions when both sides must act without knowledge of the other side's tactics. The bounding solutions may not offer much advantage in this case.

The game theory solution can be determined using Table 19 as the payoff matrix. When dominated strategies are eliminated, the payoff matrix shown in Table 20 remains. This particular game has no saddle
Table 20

Reduced Payoff Matrix for the Two-Player Zero-Sum Game Problem

<table>
<thead>
<tr>
<th>Interceptor Group Size</th>
<th>Number of Defensive Missiles/Bomber</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>6</td>
</tr>
<tr>
<td>8</td>
<td>286</td>
</tr>
<tr>
<td>9</td>
<td>294</td>
</tr>
<tr>
<td>10</td>
<td>311</td>
</tr>
<tr>
<td>12</td>
<td>321</td>
</tr>
</tbody>
</table>
point and so mixed strategies are required for a solution. The solution to this game is obtained by solving the following linear programming problem:

Minimize \(-v\)

Subject to:

\[\begin{align*}
286x_6 + 362x_8 + 309x_{10} - v & \geq 0 \\
294x_6 + 313x_8 + 325x_{10} - v & \geq 0 \\
311x_6 + 304x_8 + 323x_{10} - v & \geq 0 \\
321x_6 + 300x_8 + 290x_{10} - v & \geq 0 \\
x_6 + x_8 + x_{10} & = 1 \\
x_6, x_8, x_{10}, v & \geq 0
\end{align*}\]

Where: \(v\) is the value of the game

\(x_i\) is the probability of the bombers carrying \(i\) defensive missiles

The solution to the above defines the optimum tactics for the offense - the values for the \(x_i\)’s. The optimum tactics for the defense can be found by solving the dual of the above linear programming problem. A solution
to this game (as well as the solutions to the other two situations considered) is given in Table 21. In this case having an intelligence advantage is not too important for either opponent. The expected number of bombs surviving did not vary greatly in the three situations considered. However the optimum tactics did change considerably from situation to situation.
Table 21

Summary of Solutions for the Three Situations Considered

<table>
<thead>
<tr>
<th>Situation</th>
<th>Offensive Tactics(^a)</th>
<th>Defensive Tactics(^b)</th>
<th>Expected Number of Bombs Surviving</th>
</tr>
</thead>
<tbody>
<tr>
<td>Defense has intelligence advantage</td>
<td>(x_6 = 0.000)</td>
<td>(y_8 = 0.000)</td>
<td>300.</td>
</tr>
<tr>
<td></td>
<td>(x_8 = 1.000)</td>
<td>(y_9 = 0.000)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(x_{10} = 0.000)</td>
<td>(y_{10} = 0.000)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(y_{11} = 1.000)</td>
<td></td>
</tr>
<tr>
<td>Intelligence not available to either opponent.</td>
<td>(x_6 = 0.499)</td>
<td>(y_8 = 0.016)</td>
<td>307.</td>
</tr>
<tr>
<td></td>
<td>(x_8 = 0.185)</td>
<td>(y_9 = 0.486)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(x_{10} = 0.316)</td>
<td>(y_{10} = 0.000)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(y_{11} = 0.498)</td>
<td></td>
</tr>
<tr>
<td>Offense has intelligence advantage.</td>
<td>(x_6 = 1.000)</td>
<td>(y_8 = 0.000)</td>
<td>321.</td>
</tr>
<tr>
<td></td>
<td>(x_8 = 0.000)</td>
<td>(y_9 = 0.000)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(x_{10} = 0.000)</td>
<td>(y_{10} = 0.000)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(y_{11} = 0.000)</td>
<td></td>
</tr>
</tbody>
</table>

\(^a\) \(x_i\) = the probability the bombers will carry \(i\) bombs per bomber

\(^b\) \(y_j\) = the probability the interceptors will form groups of size \(j\)
APPENDIX C

The model used to determine an expected value solution for the survival of a force of penetrating bombers was introduced in Chapter V. A more detailed description of the logic used in the computer program based on this model is given in this appendix. This model requires the following inputs for each battle cycle simulated:

**Model Inputs**

- Probability a bomber is killed given an attack is made and a defensive weapon is used ($P_{kb/w}$)

- Probability a bomber is killed given an attack is made and no defensive weapon is used ($P_{kb/w}$)

- Initial number of defensive weapons available to the bomber ($k_{max}$)

- Probability a given bomber has $k$ remaining defensive weapons at the start of the cycle ($FW(k)$, $k = 0, --, k_{max}$)
Number of interceptors in each group ($N_i$) at the start of a cycle

Probability that the interceptors can reach at least $j$ bombers ($PR(j)$ $j = 1, \ldots, N_i$)

A description of the logic found in the computer program used to study this problem follows. This does not include the procedure for generating the inputs from cycle to cycle which might be unique for each particular problem being a function of such factors as the surviving interceptor force and its ability to regroup.

**General Procedure**

**Step 1** Calculate the probability a bomber will survive $i$ attacks and have $k$ weapons left ($PSW(i,k)$ $i = 0, \ldots, N_i$ $k = 0, \ldots, \text{k}_{\text{max}}$).

Calculate the probability a bomber is killed after exactly $i$ attacks ($PK(i)$ $i = 0, \ldots, N_i$).

**Step 2** Calculate the probability that the $j$th bomber attacked is killed on the $q$th attack by the interceptor group ($PKT(j,q)$ $j = 1, \ldots, N_i$ $q = 1, \ldots, N_i$).
Step 3 Calculate the probability that the jth bomber to be attacked is reached by m interceptors (interceptors that have not already used up their weapons) \( PA(j,m) \), \( j = 1, \ldots, N_i \) \( m = 0, \ldots, N_i \).

Calculate the probability that the jth bomber to be attacked survives with k weapons remaining \( PSWT(j,k) \), \( j = 1, \ldots, N_i \) \( k = 0, \ldots, k_{\text{max}} \).
FIGURE 15  STEP ONE OF THE MODEL

Notes:  
k signifies the number of defensive weapons available to the bomber.

i signifies the number of attacks made upon a bomber.

PK(i) = probability a bomber is killed by the ith attacker.

PSW(i,k) = probability a bomber is attacked i times and survives with k weapons.
ENTER

\[ k = 0 \]

Set \( PK(i) = 0 \)
\( PSW(i,0) = 0 \)
for \( i = 0, \ldots, N_i \)

\[ PSW(0,k) = PW(k) \]

\[ i = i + 1 \]

if \( i > k \)

yes

\[ PK(i) = (1 - P_{kb/w})^k (1 - P_{kb/w})^{i-k-1} P_{kb/w} \cdot PW(k) \]

\[ PSW(i,0) = (1 - P_{kb/w})^k (1 - P_{kb/w})^{i-k-1} PW(k) \]

no

\[ PK(i) = (1 - P_{kb/w})^{i-1} P_{kb/w} PW(k) \]

\[ PSW(i,k-i) = (1 - P_{kb/w})^i PW(k) \]

\[ i = N_i \]

if \( i = N_i \)

yes

\[ k = k_{\text{max}} \]

no

EXIT
FIGURE 16  STEP TWO OF THE MODEL

Notes:

j signifies that this is the jth bomber to be attacked.

q signifies the total number of attacks that have been made against all bombers.

PKT(j,q) = probability the jth bomber is killed on the qth attack of the battle.

ii signifies attack of the battle that results in kill of the (j-1)th bomber.
Set PKT(1,q) = PK(q) for q = 0, ..., N_i

j = 2

q = 0

PKT(j,0) = 0

q = q + 1

ii = 1

PKT(j,q) = 0

PKT(j,q) = PKT(j,q) + PKT(j-1,ii) * PK(q-ii)

ii = ii + 1

if ii < q - 1

if q = N_i

j = j + 1

if j = N_i

EXIT
FIGURE 17  STEP THREE OF THE MODEL

Notes:

j signifies that this is the jth bomber to be attacked.

m signifies that m attackers reach the jth bomber.

$PA(j,m) = \text{probability that } m \text{ attackers reach the } j\text{th bomber.}$

k signifies the number of defensive weapons carried by a bomber.

$PSWT(j,k) = \text{probability jth bomber survives with } k \text{ defensive weapons left.}$
ENTER

PA(1,0) = 1 - PR(1)

Set PA(1,m) = 0 for m = 1, -N_i - 1

PA(1,N_i) = PR(1)

j = 2

PA(j,0) = 1

m = 1

PA(j,m) = PR(j-1,N_i-m) PR(j)

PA(j,0) = PA(j,0) - PA(j,m)

if m = N_i

j = j + 1

if j = N_i

m = m + 1

if m = N_i

k = k + 1

PSWT(j,k) = PSWT(j,k) + PA(j,m) PSW(m,k)

if m = N_i

k = k_max

if j = N_i

Print PSWT(j,k) for j = 1, -N_i, k = 0, -k_max

EXIT
Discussion

A computer program based on the preceding logic was written in Fortran and exercised on the IBM 7094 machine. This program is capable of analyzing several defensive weapon load configurations (each for several interceptor group sizes) using but a few seconds of computer time. In addition to determining the expected number of surviving bombers after each battle cycle (see Tables 23, 24 and 25 for the example problem results), this model determines the expected probability distribution describing the number of defensive weapons available to each bomber after each cycle (for a particular initial defensive weapon loading and a particular interceptor group size). The way in which defensive weapons are depleted from cycle to cycle is thus easily observed. This information may yield insight into the mechanism behind the final results. For example consider the case where the bomber carries five number five missiles and the interceptors choose to operate in groups of size four. Table 22 describes the number of defensive missiles available to surviving bombers at the start of each cycle.
<table>
<thead>
<tr>
<th>Weapon Count</th>
<th>Cycle 1</th>
<th>Cycle 2</th>
<th>Cycle 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>PW (5)</td>
<td>1.00</td>
<td>0.42</td>
<td>0.10</td>
</tr>
<tr>
<td>PW (4)</td>
<td>0.00</td>
<td>0.08</td>
<td>0.05</td>
</tr>
<tr>
<td>PW (3)</td>
<td>0.00</td>
<td>0.08</td>
<td>0.08</td>
</tr>
<tr>
<td>PW (2)</td>
<td>0.00</td>
<td>0.08</td>
<td>0.07</td>
</tr>
<tr>
<td>PW (1)</td>
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<td>0.34</td>
<td>0.57</td>
</tr>
<tr>
<td>PW (0)</td>
<td>0.00</td>
<td>0.00</td>
<td>0.13</td>
</tr>
</tbody>
</table>
Table 23

Expected Number of Bombers Surviving Cycle One as a Function of Defensive Weapons Carried and Interceptor Group Size

<table>
<thead>
<tr>
<th>Miss Type</th>
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<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
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</thead>
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<td>63.0</td>
<td>63.0</td>
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<td>63.4</td>
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<td>63.0</td>
<td>65.0</td>
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<td>63.0</td>
<td>65.0</td>
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<td>67.9</td>
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<td>84.0</td>
<td>84.0</td>
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<td>84.0</td>
<td>84.6</td>
<td>84.0</td>
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<td>85.1</td>
<td>84.0</td>
<td>84.6</td>
<td>84.0</td>
<td>84.8</td>
<td>82.5</td>
<td>84.3</td>
<td>84.8</td>
<td>83.9</td>
<td>85.1</td>
<td>81.8</td>
<td>81.0</td>
</tr>
</tbody>
</table>
Table 24

Expected Number of Bombers Surviving Cycle Two as a Function of Defensive Weapons Carried and Interceptor Group Size

<table>
<thead>
<tr>
<th>Miss Type</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
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</thead>
<tbody>
<tr>
<td>5</td>
<td>43.5</td>
<td>16.5</td>
<td>19.1</td>
<td>20.0</td>
<td>19.0</td>
<td>23.2</td>
<td>25.8</td>
<td>31.4</td>
<td>31.6</td>
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<td>39.7</td>
<td>40.8</td>
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<tr>
<td>8</td>
<td>43.5</td>
<td>34.9</td>
<td>31.5</td>
<td>26.7</td>
<td>30.7</td>
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<td>35.1</td>
<td>36.2</td>
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<td>72.3</td>
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<td>41.7</td>
<td>44.3</td>
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<td>72.3</td>
<td>69.6</td>
<td>50.2</td>
<td>51.1</td>
<td>55.4</td>
<td>44.7</td>
<td>40.9</td>
<td>42.0</td>
<td>44.4</td>
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<td>51.7</td>
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<td>68.5</td>
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<td>65.0</td>
<td>64.6</td>
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<tr>
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<td>69.6</td>
<td>68.5</td>
<td>68.7</td>
<td>68.0</td>
<td>63.0</td>
<td>63.7</td>
<td>66.1</td>
<td>66.4</td>
<td>65.9</td>
<td>66.0</td>
<td>62.0</td>
<td>61.7</td>
</tr>
</tbody>
</table>
Table 25

Expected Number of Bombers Surviving Cycle Three as a Function of Defensive Weapons Carried and Interceptor Group Size

<table>
<thead>
<tr>
<th>Miss Type</th>
<th>Interceptor Group Size</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>10</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>12</td>
</tr>
</tbody>
</table>
Since the bombers initially have five defensive weapons and the interceptors are assumed to attack in groups of size four there is no possibility of the bombers defensive weapons being depleted during cycle one. Most surviving bombers after cycle one have either one defensive weapon remaining (signifying that the bomber was the first bomber attacked by the interceptors and that the bomber survived) or five defensive weapons remaining (signifying that the bomber was not attacked). As the battle progresses, surviving bombers are seen to have fewer and fewer defensive weapons available.
One departure from the ground rules used in the analysis of the example problem that might be of interest concerns the regrouping of the interceptor forces. In the analysis in Chapters IV and V it was assumed that the defenses could be regrouped to form units of the same size as those used in the first cycle. However it may happen that the defense cannot regroup and that the survivors of an original group will constitute a group for the next cycle. In this case the number of groups will not change from cycle to cycle but the size of the groups will if the bombers use weapons that are lethal to interceptors. If anti-interceptor weapons are not used then this departure will have no effect on the results.

Table 26 gives the upper bound solutions for the example problem introduced in Chapter III that are obtained when the interceptors are not assumed to be able to regroup. Loadings not containing missiles #1, #2, or #3 will be unaffected by this change in assumption.
### Table 26

Upper Bounds on Bomber Effectiveness When Defensive Missiles are Used

<table>
<thead>
<tr>
<th>Defensive Missile</th>
<th>Number/ Bomber</th>
<th>Bombs/ Bomber</th>
<th>( N_i )</th>
<th>Upper Bound on Surv. B(1)</th>
<th>Upper Bound on Bombs Surv.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Missile #1</td>
<td>1</td>
<td>9</td>
<td>4</td>
<td>50.6</td>
<td>17.1</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>8</td>
<td>4</td>
<td>51.5</td>
<td>64.0</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>7</td>
<td>4</td>
<td>54.3</td>
<td>166.6</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>6</td>
<td>2</td>
<td>49.0</td>
<td>205.8</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>5</td>
<td>2</td>
<td>49.0</td>
<td>166.5</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>4</td>
<td>2</td>
<td>49.0</td>
<td>137.2</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>3</td>
<td>2</td>
<td>49.0</td>
<td>102.9</td>
</tr>
<tr>
<td>Missile #2</td>
<td>2</td>
<td>7</td>
<td>4</td>
<td>52.0</td>
<td>139.3</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>5</td>
<td>4</td>
<td>56.6</td>
<td>191.5</td>
</tr>
<tr>
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<td>4</td>
<td>4</td>
<td>5</td>
<td>64.4</td>
<td>187.2</td>
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<tr>
<td>Missile #3</td>
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<td>50.2</td>
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<tr>
<td></td>
<td>2</td>
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<tr>
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<td>3</td>
<td>4</td>
<td>6</td>
<td>67.4</td>
<td>190.8</td>
</tr>
</tbody>
</table>
concerning regrouping and so are not included in Table 26. Comparing these solutions with those given in Table 4 (where regrouping is allowed), it is seen that bomber effectiveness, although somewhat different, is not greatly changed.

Notice that the optimum group size \( N_i \) is often larger but never smaller than that given in Table 4. This is not surprising since when regrouping is not allowed the groups will be smaller in the second cycle than in the first, and so, a larger initial grouping is needed to insure the grouping never fails below a critical level such as that required to guarantee defensive weapon depletion of the bombers.

In calculating these results it was assumed that each group was decreased by the number of interceptors killed divided by the total number of groups. Thus the groups were still equal in size although smaller. If the new group size was not an integer, its effectiveness was modeled as a weighted summation of the effectiveness of the nearest two integer groupings.

When these upper bound solutions are compared with the lower bound found for the loading of eight number six missiles, it is still possible to reject all of the anti-interceptor missile loadings. Thus this departure will have no effect on the solution for the example problem found in Chapter V.
LIST OF REFERENCES


