STOCK, Suzanne Jane Foster, 1943-
A COMPARISON OF AN ABSTRACT DEDUCTIVE AND A
CONCRETE INDUCTIVE APPROACH TO TEACHING THE
CONCEPTS OF LIMITS, DERIVATIVES, AND
CONTINUITY IN A FRESHMAN CALCULUS COURSE.

The Ohio State University, Ph.D., 1971
Education, teacher training

University Microfilms, A XEROX Company, Ann Arbor, Michigan

© 1971
SUZANNE JANE FOSTER STOCK
ALL RIGHTS RESERVED

THIS DISSERTATION HAS BEEN MICROFILMED EXACTLY AS RECEIVED
A COMPARISON OF AN ABSTRACT DEDUCTIVE AND A CONCRETE
INDUCTIVE APPROACH TO TEACHING THE CONCEPTS OF
LIMITS, DERIVATIVES, AND CONTINUITY IN A
FRESHMAN CALCULUS COURSE

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

Suzanne Foster Stock, B.Ed., M.A.

The Ohio State University
1971

Approved by

[Signature]
Adviser
College of Education
ACKNOWLEDGMENTS

Although departing from the normal format for acknowledgments, I wish to begin by expressing my deepest thanks and gratitude to my husband, Carl. His understanding and encouragement were indispensable throughout the study.

I wish to express my sincere appreciation to my adviser, Dr. F. Joe Crosswhite, for so generously offering of his time, guidance and inspiration during the entire course of study. Dr. Harold Trimble and Dr. Leslie Miller, members of my committee, are to be thanked for their helpful suggestions and encouragement.

I would like to thank Dr. Arthur White for his assistance in the statistical analysis utilized in the study.

I wish to thank Dr. Ronald Shelton and Dr. Lois Lackner for allowing me the use of their programmed materials.

I wish to thank Mr. James Schultz for his invaluable assistance in arranging the classes and teaching assignments involved in the study. I am also indebted to the willingness and cooperation of the participating instructors.

Finally, I wish to thank my parents. Their confidence in me has offered much encouragement. Also to be included are my two daughters, Lisa and Amy, for their cooperation, love and understanding.
August 18, 1943  Born - Platteville, Wisconsin

1964 ....... B.Ed. Degree, Wisconsin State University, Whitewater, Wisconsin

1964-65 .... Graduate Fellow, Western Michigan University, Kalamazoo, Michigan

1965-66 .... Graduate Assistant, Western Michigan University, Kalamazoo, Michigan

1966 ....... M.A. Degree, Western Michigan University, Kalamazoo, Michigan

1966 ....... Teaching Associate, Western Michigan University, Kalamazoo, Michigan

1966 ....... Mathematics Teacher, Portage High School, Portage, Michigan

1967 ....... Computer Programmer Trainee, The Ohio State University, Columbus, Ohio

1967-1970 ... Teaching Associate, Department of Mathematics, The Ohio State University, Columbus, Ohio

FIELDS OF STUDY

Major Field: Mathematics Education

Dr. F. Joe Crosswhite, Adviser
Dr. Harold C. Trimble

Mathematics

Dr. Leslie H. Miller
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACKNOWLEDGMENTS</td>
<td>11</td>
</tr>
<tr>
<td>VITA</td>
<td>iii</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>vii</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>x</td>
</tr>
<tr>
<td>Chapter</td>
<td></td>
</tr>
<tr>
<td><strong>I. NATURE OF THE STUDY</strong></td>
<td>1</td>
</tr>
<tr>
<td>Introduction</td>
<td></td>
</tr>
<tr>
<td>Problem Statement</td>
<td></td>
</tr>
<tr>
<td>Significance of the Study</td>
<td></td>
</tr>
<tr>
<td>Definition of Terms</td>
<td></td>
</tr>
<tr>
<td>Statement of Hypotheses</td>
<td></td>
</tr>
<tr>
<td>Pilot Study</td>
<td></td>
</tr>
<tr>
<td>Research Design</td>
<td></td>
</tr>
<tr>
<td>Overview of Remaining Chapters</td>
<td></td>
</tr>
<tr>
<td><strong>II. RELATED LITERATURE</strong></td>
<td>12</td>
</tr>
<tr>
<td>Orientation to the Problem</td>
<td></td>
</tr>
<tr>
<td>Elementary Level Literature in Related Fields</td>
<td></td>
</tr>
<tr>
<td>Junior High Level Literature in Related Fields</td>
<td></td>
</tr>
<tr>
<td>High School Level Literature in Related Fields</td>
<td></td>
</tr>
<tr>
<td>College Level Literature in Related Fields</td>
<td></td>
</tr>
<tr>
<td>Elementary Level Literature in Mathematics</td>
<td></td>
</tr>
<tr>
<td>Junior High Level Literature in Mathematics</td>
<td></td>
</tr>
<tr>
<td>High School Level Literature in Mathematics</td>
<td></td>
</tr>
<tr>
<td>College Level Literature in Mathematics</td>
<td></td>
</tr>
<tr>
<td>Directly Related Studies</td>
<td></td>
</tr>
</tbody>
</table>

iv
### TABLE OF CONTENTS (Continued)

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>III. DEVELOPMENT OF TREATMENT AND EVALUATION INSTRUMENTS</td>
<td>42</td>
</tr>
<tr>
<td></td>
<td>Selection of Material to be Learned</td>
</tr>
<tr>
<td></td>
<td>Development of the Concrete Inductive Treatment</td>
</tr>
<tr>
<td></td>
<td>Development of the Abstract Deductive Treatment</td>
</tr>
<tr>
<td></td>
<td>Preliminary Study</td>
</tr>
<tr>
<td></td>
<td>Development of the Evaluation Instruments</td>
</tr>
<tr>
<td>IV. THE EXPERIMENTAL DESIGN</td>
<td>65</td>
</tr>
<tr>
<td></td>
<td>Population and Sampling</td>
</tr>
<tr>
<td></td>
<td>The Design</td>
</tr>
<tr>
<td></td>
<td>Procedures</td>
</tr>
<tr>
<td></td>
<td>The Pretest</td>
</tr>
<tr>
<td></td>
<td>The Achievement Test</td>
</tr>
<tr>
<td></td>
<td>The Retention Test</td>
</tr>
<tr>
<td>V. EXPERIMENTAL RESULTS</td>
<td>82</td>
</tr>
<tr>
<td></td>
<td>Summary of Procedure</td>
</tr>
<tr>
<td></td>
<td>The Covariates</td>
</tr>
<tr>
<td></td>
<td>The Analysis of Covariance Model</td>
</tr>
<tr>
<td></td>
<td>Test Results</td>
</tr>
<tr>
<td></td>
<td>Additional Statistical Analyses</td>
</tr>
<tr>
<td>VI. SUMMARY, RESULTS AND INTERPRETATIONS</td>
<td>96</td>
</tr>
<tr>
<td></td>
<td>Summary</td>
</tr>
<tr>
<td></td>
<td>Results</td>
</tr>
<tr>
<td></td>
<td>Limitations and Delimitations</td>
</tr>
<tr>
<td></td>
<td>Interpretations and Implications for Further Research</td>
</tr>
</tbody>
</table>

**Appendix**

<table>
<thead>
<tr>
<th></th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. QUESTIONNAIRES</td>
<td>105</td>
</tr>
<tr>
<td>B. PRETEST, ACHIEVEMENT TEST, AND RETENTION TEST</td>
<td>111</td>
</tr>
<tr>
<td>Appendix</td>
<td>Page</td>
</tr>
<tr>
<td>----------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>C. SAMPLE COMPOSITION TABLES</td>
<td>124</td>
</tr>
<tr>
<td>D. PRETEST, ACHIEVEMENT TEST, AND RETENTION</td>
<td>130</td>
</tr>
<tr>
<td>TEST DATA</td>
<td></td>
</tr>
<tr>
<td>E. ABSTRACT DEDUCTIVE LESSON PLANS</td>
<td>141</td>
</tr>
<tr>
<td>F. CONCRETE INDUCTIVE LESSON PLANS</td>
<td>165</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>194</td>
</tr>
<tr>
<td>Table</td>
<td>Title</td>
</tr>
<tr>
<td>-------</td>
<td>-------------------------------------------</td>
</tr>
<tr>
<td>1</td>
<td>The Experimental Design</td>
</tr>
<tr>
<td>2</td>
<td>Pilot Study Analysis of Covariance</td>
</tr>
<tr>
<td>3</td>
<td>Pretest Item Classification</td>
</tr>
<tr>
<td>4</td>
<td>Achievement Test Item Classification</td>
</tr>
<tr>
<td>5</td>
<td>Content Analysis of the Achievement Test</td>
</tr>
<tr>
<td>6</td>
<td>Retention Test Item Classification</td>
</tr>
<tr>
<td>7</td>
<td>Content Analysis of the Retention Test</td>
</tr>
<tr>
<td>8</td>
<td>Sample Composition: Males and Females</td>
</tr>
<tr>
<td>9</td>
<td>Progressive Sample Size</td>
</tr>
<tr>
<td>10</td>
<td>The Experimental Design</td>
</tr>
<tr>
<td>11</td>
<td>Background of Instructors</td>
</tr>
<tr>
<td>12</td>
<td>Pretest Item Difficulty Distribution</td>
</tr>
<tr>
<td>13</td>
<td>Pretest Item Discrimination Distribution</td>
</tr>
<tr>
<td>14</td>
<td>Achievement Test Summary Statistics</td>
</tr>
<tr>
<td>15</td>
<td>Achievement Test Item Difficulty Distribution</td>
</tr>
<tr>
<td>16</td>
<td>Achievement Test Item Discrimination</td>
</tr>
<tr>
<td>17</td>
<td>Retention Test Summary Statistics</td>
</tr>
<tr>
<td>18</td>
<td>Retention Test Item Difficulty Distribution</td>
</tr>
<tr>
<td>19</td>
<td>Retention Test Item Discrimination</td>
</tr>
<tr>
<td>20</td>
<td>Analysis of Variance on Pretest Scores</td>
</tr>
<tr>
<td>21</td>
<td>Analysis of Variance on Quality Points</td>
</tr>
<tr>
<td>Table</td>
<td>Page</td>
</tr>
<tr>
<td>-------</td>
<td>------</td>
</tr>
<tr>
<td>22. Correlation Matrix</td>
<td>84</td>
</tr>
<tr>
<td>23. Two-Way Analysis of Covariance on Achievement Test</td>
<td>86</td>
</tr>
<tr>
<td>24. Analysis of Covariance on Achievement Test</td>
<td>88</td>
</tr>
<tr>
<td>25. Two-Way Analysis of Covariance on Retention Test</td>
<td>88</td>
</tr>
<tr>
<td>26. Analysis of Covariance on Retention Test</td>
<td>89</td>
</tr>
<tr>
<td>27. Analysis of Covariance (Achievement Test—Upper 25 Percent)</td>
<td>90</td>
</tr>
<tr>
<td>28. Analysis of Covariance (Retention Test—Upper 25 Percent)</td>
<td>91</td>
</tr>
<tr>
<td>29. Adjusted Means (Upper 25 Percent on Retention)</td>
<td>92</td>
</tr>
<tr>
<td>30. Analysis of Covariance (Achievement Test—Lower 25 Percent)</td>
<td>94</td>
</tr>
<tr>
<td>31. Analysis of Covariance (Retention Test—Lower 25 Percent)</td>
<td>95</td>
</tr>
<tr>
<td>32. Sample Composition: Age</td>
<td>125</td>
</tr>
<tr>
<td>33. Sample Composition: Rank</td>
<td>126</td>
</tr>
<tr>
<td>34. High School Mathematics</td>
<td>127</td>
</tr>
<tr>
<td>35. Average Grade in High School Mathematics</td>
<td>128</td>
</tr>
<tr>
<td>36. Grade in Mathematics</td>
<td>129</td>
</tr>
<tr>
<td>37. Pretest Item Analysis</td>
<td>131</td>
</tr>
<tr>
<td>38. Pretest Score Distribution</td>
<td>132</td>
</tr>
<tr>
<td>39. Achievement Test Statistics for Individual Classes</td>
<td>133</td>
</tr>
<tr>
<td>Table</td>
<td>Page</td>
</tr>
<tr>
<td>----------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>40. Achievement Test Score Distribution</td>
<td>134</td>
</tr>
<tr>
<td>41. Achievement Test Item Analysis</td>
<td>135</td>
</tr>
<tr>
<td>42. Achievement Test Group Statistics</td>
<td>136</td>
</tr>
<tr>
<td>43. Retention Test Statistics for Individual Classes</td>
<td>137</td>
</tr>
<tr>
<td>44. Retention Test Score Distribution</td>
<td>138</td>
</tr>
<tr>
<td>45. Retention Test Item Analysis</td>
<td>139</td>
</tr>
<tr>
<td>46. Retention Test Group Statistics</td>
<td>140</td>
</tr>
</tbody>
</table>
LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Flow Chart for the Development of the Concrete Inductive Unit</td>
<td>48</td>
</tr>
<tr>
<td>2.</td>
<td>Flow Chart for the Development of the Abstract Deductive Unit</td>
<td>50</td>
</tr>
<tr>
<td>3.</td>
<td>Mean Achievement Test Scores</td>
<td>77</td>
</tr>
<tr>
<td>4.</td>
<td>Mean Retention Test Scores</td>
<td>81</td>
</tr>
</tbody>
</table>
CHAPTER I

NATURE OF THE STUDY

Introduction

In recent years, much educational research has been conducted to investigate the various means by which mathematical concepts are learned. Some of these studies have attempted to determine which teaching methods are most effective at several levels of mathematical and academic ability. Concern also grows for what can and what should be taught at these levels. However, research on various approaches to the teaching of advanced topics of a discipline is limited. Sharp, as cited by Lackner, points out several reasons why undergraduate students are often neglected by their college instructors:

... Despite pious declarations in faculty handbooks and administrative speeches good teaching is not rewarded. If one wants to get ahead, one publishes; dull or disorganized teaching will not hurt one's chances much, and successful teaching will be noticed only by the students (26, p. 5).

This investigator believes that, even at the college level, one teaching method may be more effective than another in the discipline of mathematics.
Problem Statement

The purpose of this study was to investigate the effectiveness of teaching limit, continuity, and derivative concepts by two different approaches. One treatment group was taught by a concrete inductive method while the other was taught by an abstract deductive method.

The study is an outgrowth of related studies by Shelton (39), Lackner (26), and Caruso (11). Shelton compared two methods of teaching the limit concept in beginning calculus. Lackner extended Shelton's study to include the derivative concept. She compared results of teaching these two concepts by four combinations of inductive and deductive methods. Both Shelton and Lackner used programed materials as their means of instruction. Caruso, who worked in the area of abstract algebra, devised detailed lesson plans for teaching by two different methods, one inductive, the other deductive.

As in the experiment by Caruso, detailed lesson plans were used in this study. Shelton's programed materials on limits and Lackner's programed materials on derivatives were modified and adapted into these lesson plans. The concept of continuity was added. Both the inductive and the deductive approaches included the same mathematical content. In addition to comparing the treatment groups in mathematical achievement, this study made comparative
measures for retentive achievement. This was not done in two of the three studies mentioned above.

Significance of the Study

Allendoerfer expresses the importance of the limit concept in calculus:

The essential idea in calculus is that of limit, and without a clear exposition of limits, any calculus course is a failure. . . . There are those, however, who begin the course with a brief, but full dress, treatment of limits, using the epsilon-delta technique. This almost universally is wasted on the class, for they are confronted with a difficult new idea without an intuitive preparation (1, p. 484).

Expanding on this position, Taylor indicates that a beginning calculus student rarely grasps and understands the notion of a limit merely by having the basic theorems about limits of sums, products, and quotients presented to him in a precise logical manner. Nevertheless, states Taylor, "there is always a revolutionary avantgarde among calculus teachers with the doctrine that calculus should begin with a strict and complete revelation of the arithmetic theory of limits, via set theory, functions, and inequalities, with no clutter of intuitive language, no appeal to physical ideas about velocity, or the like" (41, p. 3).

Although this deductive approach to teaching finds support among a certain group of instructors, it does not stand unchallenged in the field of teaching methods. Even as far back as the early 1900's, such people as Young felt
that a learning sequence should consist of a series of specific problems which lead to a generalization (46, p. 69). Thus, Young favored an inductive approach over a deductive approach. The relative effectiveness of deductive and inductive teaching methods is the area of investigation of this study.

Several factors warrant further study in the teaching of the concept of limit and the related concepts of continuity and derivatives in the calculus. Most studies comparing inductive and deductive teaching methods have been conducted on the elementary or high school level. Results may differ considerably with college level students. Results could also be affected by using "programed instructors" rather than programed materials as was done in directly related studies at the college level. Finally, a study of retention pertaining to teaching methods could reveal significant implications. If one method of teaching revealed itself as more effective than the other, and the technique of programing instructors proved effective, this could have impact at a large university such as The Ohio State University. Such a procedure would provide the same lecture material to many classes without the loss of personal contact between teacher and student.
Definition of Terms

The following definitions indicate the interpretation intended for terms used in this study.

Mathematical achievement—achievement pertaining to calculus concepts of limits, continuity, and derivatives as measured by scores on a midterm examination.

Retentive achievement—achievement pertaining to the basic calculus concepts of limits, continuity, and derivatives as measured by scores on a final examination.

Achievement test—a midterm examination designed to measure a student's grasp of the basic calculus concepts of limits, continuity, and derivatives.

Retention test—a final examination designed to measure a student's grasp of the basic calculus concepts of limits, continuity, and derivatives.

Quality points—numbers representing a student's average grade (A-4, B-3, C-2, D-1, F-0) in high school mathematics (beginning with Algebra I) multiplied by the number of years of high school mathematics since Algebra I.

Concrete inductive approach—a presentation of a sequence of items leading from specific numerical examples to the general case (39, p. 2613).

Abstract deductive approach—a presentation of a sequence of items leading from the abstract to the particular (39, p. 2613).

The instructional pattern for the abstract deductive method consisted of formulating a generalization and following it with various illustrative examples. The concrete inductive method first considered examples which led to a generalization. A statement of the generalization then preceded more examples. For example, in the concrete inductive approach, the theorem on the limit of the sum of
two functions is taught by first discussing problems such as: \( \lim_{x \to 4} x = 4 \) and \( \lim_{x \to 4} 3 = 3 \). Then a "guess" is made as to \( \lim_{x \to 4} (x + 3) \). This is followed by a consideration of \( \lim_{x \to 4} x + \lim_{x \to 4} 3 = 7 \). After it is noted that \( \lim_{x \to 4} (x + 3) = \lim_{x \to 4} x + \lim_{x \to 4} 3 \), the generalization is formulated and the theorem on the limit of the sum of two functions is stated. This is followed by a proof of the theorem and more examples.

In the abstract deductive approach, this theorem is taught by first making a formal statement of the theorem with no sequence of problems leading into it. This is followed by a proof of the theorem and numerous examples such as that above:

\[
\lim_{x \to 4} (x + 3) = \lim_{x \to 4} x + \lim_{x \to 4} 3 = 4 + 3 = 7
\]

For further clarification of the two teaching methods employed, see the complete sets of lesson plans in Appendixes E and F.

Statement of Hypotheses

The hypotheses tested in this study were:

1. There is no significant difference in the mathematical achievement of the Concrete Inductive Group and the Abstract Deductive Group.

2. There is no significant difference in the mathematical achievement of the Concrete Inductive Group and the Control Group.
3. There is no significant difference in the mathematical achievement of the Abstract Deductive Group and the Control Group.

4. There is no significant difference in the retentive achievement of the Concrete Inductive Group and the Abstract Deductive Group.

5. There is no significant difference in the retentive achievement of the Concrete Inductive Group and the Control Group.

6. There is no significant difference in the retentive achievement of the Abstract Deductive Group and the Control Group.

The Pilot Study

During the Spring Quarter, 1970, a Pilot Study was conducted. Twelve classes of Mathematics 117 at The Ohio State University participated. Six instructors, each teaching two classes, taught one class by the concrete inductive approach and the other class by the abstract deductive approach. At the completion of the units on limits, continuity, and derivatives, all classes were given an achievement test. Due to campus disorders during the quarter, it was not feasible to administer a retention test. For analysis of covariance on the achievement test, total numerical scores from Mathematics 116 were used as the covariate. No differences in achievement were found.

The Pilot Study served as a testing ground for various materials used in the Main Study. The feasibility of using lesson plans to program instructors was determined. Feedback from the instructors and achievement test
results indicated that these modes of instruction are acceptable. The lesson plans themselves were evaluated by the instructors and corrections were made accordingly. The achievement test used in the Pilot Study was revised for the Main Study. Certain problems that were statistically poor were replaced or altered. The remaining items were either used in the final version of the achievement test or replaced by equivalent items. A more complete discussion of the development of the lesson plans and the achievement test appears in the ensuing chapters.

Research Design

The subjects in the study consisted of students enrolled in the course, Mathematics 117 (Mathematics for the Behavioral, Economic, and Social Sciences II), at The Ohio State University, Columbus, Ohio during the Fall Quarter, 1970. Two hundred and fifty-six students were involved in the study. Each student reported to a dispatcher on the first day of the quarter at either 12:00 noon or 4:00 P.M. (depending on his scheduled time). The dispatcher handed each student a card with the room number (randomly assigned) to which he was to report. Four classes comprised each of the two treatment groups and two additional classes made up the Control Group. All classes met five days each week for a 48-minute period.
Topics normally included in Mathematics 117 are analytic geometry, calculus, linear algebra, linear programming and graph theory. The calculus topics comprised the content of this study.

Four instructors taught two sections each; one class of each pair received the concrete inductive treatment and the other member of each pair received the abstract deductive treatment. All four instructors were provided with a notebook containing detailed lesson plans for each approach. Two separate classes, each taught by an additional instructor, served as a control group. The experimental treatment lasted approximately one-third of the ten week quarter.

The experimental design utilized in the study is indicated in Table 1. $X_1$ represents the deductive treatment, $X_2$ represents the inductive treatment, and $O_1$, $O_2$, $O_3$ and $O_4$ indicate quality points, the pretest, the achievement test, and the retention test scores, respectively.

<table>
<thead>
<tr>
<th>Group 1</th>
<th>$O_1$</th>
<th>$O_2$</th>
<th>$X_1$</th>
<th>$O_3$</th>
<th>$O_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Group 2</td>
<td>$O_1$</td>
<td>$O_2$</td>
<td>$X_2$</td>
<td>$O_3$</td>
<td>$O_4$</td>
</tr>
<tr>
<td>Group 3</td>
<td>$O_1$</td>
<td>$O_2$</td>
<td></td>
<td>$O_3$</td>
<td>$O_4$</td>
</tr>
</tbody>
</table>
Prior to beginning the experimental instructional units, a pretest (Appendix B) was given. This test covered material studied in Mathematics 116, the course prerequisite to Mathematics 117.

Upon completion of the unit on limits, continuity, and derivatives, a second test (Appendix B) was administered to determine achievement on these topics. Approximately seven weeks after completion of the achievement test, a third test was administered to determine retentive achievement (Appendix B). Analysis of covariance, using the pretest and quality points as covariates, was used to compare the achievement of each treatment group. Analysis of covariance was used again to compare the retentive achievement of each treatment group; the pretest and quality points were used as covariates.

Overview of Remaining Chapters

Chapter II of this study presents a review of the literature related to the experiment. Studies in other fields as well as studies in mathematics are included; experiments range from the elementary school through the college level.

Chapter III provides a description of the development of both the deductive and inductive lesson plans. A detailed discussion of the Pilot Study also is included. The latter part of the chapter discusses the development
of the achievement test and the retention test.

The experimental design employed in the study is detailed in Chapter IV. Included in this chapter is a description of the sample. This is followed by a thorough discussion of specific procedures. Statistical descriptions of the pretest, achievement test, and retention test also appear.

In Chapter V are the statistical results of both the achievement test and the retention test. Results of an analysis of variance on the pretest and quality points are presented. In addition, results of analysis of covariance on both the achievement test and the retention test are given.

Chapter VI provides a summary of the study. Limitations of the study are discussed. Implications for further research and possibilities for additional studies are outlined.
CHAPTER II

RELATED LITERATURE

Orientation to the Problem

One of the main areas of investigation in educational research during the past decade has been the teaching method frequently referred to as the "discovery method." The increase in research in this area has been accompanied by an increasing number of attempts to teach by this technique (45, p. 223).

As any new concept coming into being, "discovery" has inspired numerous interpretations. Lowry classifies a discovery as "any activity through which the learner creates, invents, finds, or gains understanding of some mathematical principle, concept or procedure through his own efforts—efforts which may, however, be directed or aided by the teacher and/or the text materials. The essence is that the thing learned is not received solely by communication from teacher or text to student" (27, p. 255).

In a similar vein, Bittinger describes learning by discovery as "any learning situation in which the learner completes a learning task without extensive help from the teacher. In discovery learning the teacher's role may vary
from careful guidance to no guidance at all" (5, p. 140).

In a less restrictive manner, Bruner describes discovery as "a matter of rearranging or transforming evidence in such a way that one is enabled to go beyond the evidence so reassembled to new insights" (7, pp. 21-32).

To Ausubel, the distinguishing feature of discovery learning is that "the principal content of what is to be learned is not given but must be discovered by the learner before he can internalize it" (2, p. 126).

We find some of these views prevalent in the early 1900’s. At that time, Young described the heuristic method as one in which, in a sense, the pupil must rediscover the subject "though not without profit from the fact that the race had already discovered it. . . ." The student is led to formulate his own definitions while the teacher sees that they agree with current definitions; when necessary, the teacher also gives the pupil the conventional terms and symbols (46, p. 69).

Advocates of discovery believe it is a superior method. They feel that retention and transfer of concepts, pupil motivation, and the heuristics of discovery are greatly enhanced by discovery. Continued use of this method, by the majority of teachers who have tried the technique, adds to its support (45, p. 223).

Although the discovery method finds a great deal of
support, it is not without critics. Many of them feel that
good expository teaching offers as much to the student as
discovery does. They seem to feel discovery teaching is
pedagogically impractical.

To further enhance this diversity in support of
teaching methods, consider the following remarks by Bruner
and Ausubel, two seemingly opposed forces in this area.
Bruner feels that "if man's intellectual excellence is the
most his own among his perfections, it is also the case
that the most uniquely personal of all that he knows is
that which he has discovered himself" (4, p. 145). He
feels that discovery contributes to an increase in intel-
lectual potency, intrinsic reward, conservation of memory,
and learning the heuristics of discovery (4, p. 145).

In contrast, Ausubel relates that "most of what
anyone really knows consists of insights discovered by
others which have been communicated to him in meaningful
fashion" (4, p. 145). He contends there is much time saved
in communicating a meaningful explanation of an idea rather
than having each rediscover the idea himself.

The crucial points at issue, however, are not
whether learning by discovery enhances learning,
retention, and transferability, but (a) whether it
does so sufficiently, for learners who are capable
of learning principles meaningfully without it, to
warrant the vastly increased expenditure of time
it requires; and (b) whether, in view of this time
consideration, the discovery method is a feasible
technique for transmitting the substantive content
of an intellectual or scientific discipline to cognitively mature students who have already mastered its rudiments and basic vocabulary (3, p. 25).

Even Bruner bends in this direction when he states: "One cannot wait forever for discovery. One cannot leave the curriculum and let discovery flourish willy-nilly wherever it may occur" (6, p. 142).

The literature reveals research studies with results supporting both advocates and critics of the discovery method. As noted by Worthen, Henderson states:

Before turning to other kinds of research on teaching secondary school mathematics, let us consider what can be concluded about the consequences of tell-and-do methods versus heuristic methods. One conclusion is that the evidence is not conclusive. . . (45, pp. 223-24).

For example, studies by Craig (12, pp. 223-34), Kittell (25, p. 402) and Sassenrath (37, pp. 207-08) found that principles directly told to the students led to more transfer and were better retained than those principles which were individually derived (i.e., discovered). On the other hand, findings of Hendrix (22, p. 198) and Meyers (20, p. 296) indicate superior results when the students derived the principles individually.

Some of these discrepancies can be, in part, accounted for when one considers the various types of discovery employed. Bittinger contends that the discovery method entails many methods: (1) the inductive method, (2) the nonverbal awareness method, (3) the incidental
learning method, (4) the deductive method, and (5) the variation method (5, p. 140). Through the inductive method "a student is led into the knowledge of a generalization by various examples." Success is indicated when the student is able to verbalize the generalization. In the nonverbal awareness method, the inductive method is reduced to not requiring a verbalization by the student. What the student should learn through the incidental method is a by-product of other learning. Hendrix describes deduction as "manipulating sentences by logical rules." For example, proving something by one's self is discovery (22, p. 205). Heinke defines variation as "a process of changing elements of the data, or conclusion, or both, of a geometrical (in our case, mathematical) statement, which has been proven to be true or is accepted as true, with a view to obtaining a new set of data, or a new conclusion, or both, resulting in a new statement" (21, p. 148).

In addition to the various methods of discovery, there are several modes of instruction frequently employed. For example, a specially prepared syllabus or textbook may be used along with the recitation mode. Subject matter may be developed genetically with a class while an ordinary text is used. Here we might find the class as a whole discovering generalizations. However, if each pupil is to discover independently, then the class might discuss work
already completed and outline upcoming work (46, p. 72).

In contrast to discovery learning, we find reception or expository learning: "The principal content of what is to be learned is presented to the learner in more or less final form" (2, p. 126). Discovery is not involved; the student must only internalize the material or incorporate it into his cognitive structure.

As a border-line subject with respect to rigor, calculus is preceded by courses concerned with developing concepts, acquiring rules for operations, and perfecting skills. However, courses following calculus tend to treat subject matter more rigorously and formally. Thus, whether or not calculus is a terminal mathematics course for a student, it should provide some opportunity for a rigorous treatment of certain topics (9, pp. 536-37).

With regard to rigor, Dubisch feels "there is general agreement that the principle rather than the manipulational aspects of calculus are the important features not only for the prospective mathematician but also for the prospective engineer, physicist, etc" (19, p. 81).

As indicated by Taylor, it is Bell's conception "that the first course in calculus should be entirely devoted to formal manipulative procedures. Only after that should theory be examined with rigor" (41, p. 6).

Commenting on this controversy at the turn of the century, Young related that in the classroom, inductive work
should assume a more prominent role. He felt that it is not the best approach to state a theorem, prove it in a strict deductive fashion, and finally, apply it to solve problems. Instead, he proposed giving the pupil some specific problems which foreshadow the upcoming theorem. One should continue this until the pupil can relate the theorem and feels a need for a rigorous proof. More application should follow the proof. Thus, we see that the controversy over rigor has stood the test of time (46, pp. 57-58).

With his views coinciding with Young's, Keedy outlines the steps in learning as: "(1) Induction; (2) Deduction; and (3) Possible applications." He points out that deduction should enter into the learning situation as soon as the pupil is intellectually mature (23, p. 454).

Kemeny, also holding this view, relates, "I feel very strongly that, although a degree of rigor is important in teaching because a student should be able to understand what a proof is, it is vastly more important to emphasize basic ideas and to build up the intuition possessed by the student" (24, p. 70).

Research in the teaching of advanced topics of a discipline by discovery as opposed to exposition learning is limited. This study was concerned with deductive receptive or expository methods versus the inductive discovery method. Specifically, it dealt with the elementary calculus topics of limit, continuity, and differentiation
as taught on the freshman level at The Ohio State University in Columbus.

Elementary Level Literature in Related Fields

The literature reveals a limited amount of research in teaching methods at advanced levels in mathematics. However, several related studies in other areas stand out. In economics, although not at the college level, we find a relevant study. Dooley (1968) compared inductive and deductive materials for teaching economics concepts to culturally disadvantaged fourth grade students. Subjects were randomly assigned to classes in which economics was taught either inductively or deductively. Statistical analyses revealed consistently more effective results from the inductive method than the deductive method (17, p. 123).

In 1962 Scandura conducted several studies comparing the expository and the discovery methods of instruction. His subjects, in each case, were taught to solve certain routine abstract problems for which they had no preparational background. The Pilot Study utilized two sixth grade classes. Procedures were given directly in the expository class while indirect problem-solving guidance was applied in the discovery class. On the post-tests, which consisted of both routine and novel problems, the discovery class performed better than the expository class (38, p. 2798).
For the Main Study, a group of fourth and fifth graders were randomly assigned to the expository and discovery classes. On a test comprised of routine problems, the two mean scores showed no significant difference. However, on a test of novel problems, results favored the expository group although not significantly. On a retention test, again the expository class did better to a greater extent, but again not significantly (38, p. 2798).

In the Post Experiment, a shortened version of the material was taught to two classes of gifted fourth and fifth graders. They were taught and tested in one meeting. In general, the discovery students performed better on the novel problems while the expository students were favored on the routine problems (38, p. 2798).

Kittell investigated "the relative effects of three amounts of direction to learners in their discovery of established principles on transfer to differing situations and on retention of learned principles" (25, p. 403). In discovering the principles to determine the solutions of multiple-choice verbal items, each of three groups of sixth graders were directed to a different degree. It was found that the group receiving an intermediate amount of direction learned and transferred an equal or greater number of principles to several different situations than the groups receiving more or less direction. In testing for retention
both two and four weeks later, the intermediate group also retained a larger proportion of the learned principles than either of the other two groups (25, pp. 403-04).

Junior High Level Literature in Related Fields

Theofanis compared the effectiveness of two methods of programed instruction as part of his doctoral dissertation in 1964. His subjects consisted of sixty eighth grade pupils who were randomly assigned to one of the two treatments. The subject matter studied was magnetism and electro-magnetism while the programs were Skinnerian response type. The following conclusions resulted:

1. Students of low and superior mental ability learn the principles of magnetism and electro-magnetism more effectively when programs are sequenced inductively.

2. Students of average and above average ability learn the principles of magnetism and electro-magnetism irrespective of the order in which examples or principles are presented to them.

3. Intelligence is significantly related to the amount of material students learn from programed instruction.

4. Pretest scores are better predictors than IQ scores of amount learned when programs are inductively sequenced.

5. Intelligence does not affect the time students require to complete programs organized into frames that contain examples and principles in consecutive order.

6. The content of a unit of magnetism and electro-magnetism can be logically analyzed into examples and principles, constructed into programed format and taught effectively to eighth grade students.
7. The findings support B. F. Skinner's hypothesis as to the nature of concept formation proceeding from concrete to abstract (42, p. 7092).

Ray hypothesized that students who become "active" in learning might possibly increase their retention. Further, he felt that students who are allowed to discover methods and relationships for themselves, formulate their own generalizations, and draw their own conclusions would perhaps be better equipped to apply the material learned (36, p. 271).

To test these possibilities, Ray conducted a study in which he compared the "directed discovery" method with results of the "tell and do" method of instruction. In this study, 117 ninth grade boys were taught skills and principles of micrometer measurement. Using a treatments by levels design, Ray concluded the following. He found no significant difference between the direct and detailed and the directed discovery methods of teaching with regard to initial learning of micrometer principles and skills. He found no significant difference between the two methods with reference to retention as measured one week after instruction. However, after six weeks, he found the directed discovery method to be superior to the direct and detailed instruction with respect to retention of material (36, p. 280).

As indicated by Ray, this third conclusion finds further support from studies by McConnell; Swenson,
Anderson, and Stacey; and Thiele. They found that for actively involved learners, retention is enhanced (36, p. 271).

High School level Literature in Related Fields

As an extension of Ray's study, Moss compared two methods of verbal instruction. The "direct-detailed, incorporating many implications from Thorndike's Connectionism, consisted of positive, continuous, and detailed presentation of all the material the students were to learn and understand. Sample problems were posed to illustrate the application of the information provided, but these were immediately solved by the instructor in a step-by-step fashion" (30, p. 51). The directed discovery method "was derived from Gestalt learning theory. A minimum of basic information was presented in a direct manner. With this essential material as a foundation, many carefully structured and ordered questions and hints were utilized to assist pupil discovery of the remaining information and functional relationships to be acquired and understood" (30, p. 51).

Moss's sample was made up of 106 male high school juniors and seniors. Using a treatments by levels analysis of variance design, Moss found, for the low ability groups, that the directed discovery method was at least as effective as the direct-detailed method. A third (control) group received no instruction. Also, between the immediate initial
test and a retention test, results indicated an increase in test scores for the control group (30, p. 51).

In a study conducted at the secondary level, Driscoll compared the relative effectiveness of a traditional deductive-descriptive teaching method and a problem-solving method. The subject of basic electrical circuit principles was taught. Under the first method, the teacher selected the learning materials; principles and generalizations were presented early in the learning sequence. The second teaching method was characterized by the cooperation of students and teachers in choosing problems and problem-solving activities. The students, through study and experimentation, derived the principles and generalizations. Driscoll concluded with no significant difference in achievement between the two methods. However, due to the time-cost factor, the deductive-descriptive method was found more efficient since, after an average of ten weeks of treatment, it produced a significant difference in achievement while average duration of treatment was thirteen weeks for the problem-solving method (18, p. 4573).

Also as a dissertation, Twelker "investigated the interaction of two types of prompts (i.e., rules and answers) and feedback which specified both rules and answers upon subjects' performance on tests of learning, retention, and transfer" (43, p. 5131). His sample was comprised of two hundred forty-five secondary school
students who were randomly assigned to each of eight experimental groups and one control group.

The subjects were taught to decipher one type of transpositional cryptogram by one of eight experimental treatments. "Four kinds of treatments were formed which represented the manipulation of the two variables. These four were: answer given, rule given; answer given, rule not given; answer not given, rule given; and answer not given, rule not given. In addition, each of these four groups was split into two subgroups on the basis of the feedback variable: feedback given and feedback not given. A control group was used to assess chance performance" (43, p. 5131).

In a study of 1958, O'Connell investigated the achievement of high school students who had done inductive laboratory work and deductive descriptive work in chemistry. The study utilized three schools involving twelve classes, 420 students, and three teachers. Six of these classes were taught inductively while the remaining six were taught deductively, both for a complete school year. Results of a t-test indicated a significant superiority of the inductive method (33, p. 1679).

A second phase of O'Connell's experiment compared inductive and deductive methods of teaching a unit on chemical equation-balancing. Her sample consisted of
thirty-two schools involving fifty-six classes and forty teachers. Half the classes were taught inductively and the remainder deductively. Again an application of a t-test indicated superiority of the inductive method (33, p. 1679).

In 1965, Wallace compared two self-instructional methods for improving spelling. Six hundred and six high school and college students in twenty-six paired experimental classrooms participated. One class of each pair was given a traditional-deductive presentation of spelling rules and lists of words exemplifying each rule. The remaining class in each pair received a programed-inductive presentation leading the learner to observe the spelling behavior of words and to make generalizations. Also involved were an additional unpaired nine classrooms. Results yielded no significant difference for method alone. Boys using programed materials made higher scores. Girls were better spellers before, during, and after instruction than boys. Finally, general improvement in spelling occurred without regard to method (44, p. 5801).

**College Level Literature in Related Fields**

In an attempt to determine the effect of directed discovery on retention and the ability to discover new relations, Craig designed a study. Different amounts of direction during discovery of means to determine solutions of multiple-choice verbal items were provided to each of
two groups of college students. In each of three trials, the group receiving the most direction learned more relations. The two groups did not differ in the proportion of learned relations retained both three days and seventeen days after learning. However, after thirty-one days, the directed group again retained a larger proportion of principles than the discovery group (12, pp. 23-34).

In a related college level study of 1957, Della-Piana studied the learning effects of two methods of feedback. Of the two treatments, the "searching treatment" involved only telling the student to continue trying until he discovered the correct answer; the "dependency treatment" involved simply telling the correct answer when an incorrect one was given. In an inductive concept identification task, the "subjects were to learn the names (nonsense syllables) of concept drawings presented singly but successively in series after series of drawings." Results indicated that a significantly greater number of meanings were recalled and recognized by students in the searching treatment than in the dependency treatment. The learning of meanings seemed to be enhanced through the searching treatment (15, p. 2.13).
Elementary Level Literature in Mathematics

The area of mathematics reveals the following related research. To partially fulfill the requirements for the Master of Science degree at the University of Utah, Worthen completed a study of the relative effectiveness of two methods of teaching certain mathematical concepts to fifth and sixth grade pupils (1965). In one treatment, the expository, the initial step in the instructional sequence consisted of verbalizing the mathematical concept or generalization. After this verbalization and a written presentation of the principle were given, the subject was to solve problems which exemplified the principle. As the second treatment, discovery, verbalization of the concept or principle followed the instructional sequence used. To facilitate discovering the generalization, problems enhancing the principle were studied before the verbalization occurred (15, p. 5.01).

To evaluate outcomes, tests of initial learning, retention, and transfer of the mathematical concepts studied; tests for transfer of heuristics; and measures of attitude toward subject content were administered. The sample consisted of 538 fifth and sixth grade pupils in Salt Lake City. Four hundred and thirty-two of these students were divided equally among eight elementary schools. Two classes in each of these schools were taught by the same
teacher, one class by the discovery method and the other by the expository (45, pp. 223-42).

On the initial criterion test, the expository treatment yielded significantly better results \( (p < .01) \) than the discovery treatment. However, for the concept retention test given five weeks and eleven weeks after instruction, the discovery method produced superior results to the expository method \( (p < .05 \text{ for the first administration and } p < .025 \text{ for the second}) \). Results also suggested that pupils in the discovery treatment transferred the mathematical concepts learned during the instructional sequence more readily than the subjects in the expository treatment \( (p < .08) \). For negative transfer between the treatments, there were no differences \( (45, \text{ pp. 223-42}) \).

Junior High Level Literature in Mathematics

In a study by Neuhausser in 1964, a comparison of three methods of teaching a unit on exponents to eighth grade students was made. Basically, the first method stated a rule and a rationale for the rule. This was followed by illustrations and examples to be worked by the students. Both the second and third methods were discovery. In the second, rules were not verbalized; however, in the third, rules were stated by the pupil after discovery. Students were randomly assigned to one of the
three treatments resulting with 39 in the first, 38 in the second, and 40 in the third. The over-riding conclusion of this experiment is that "a nonverbal directed discovery method is better, at this level and for material of this type, than a nondiscovery method. That is, students taught by a nonverbal directed discovery method probably take no longer to learn, have at least as much manipulative ability, more understanding, more ability to transfer, and much more retention than students taught by a non-discovery method" (31, p. 5027).

A study conducted by Michaels compared a discovery group and a drill group both of which were concerned with various manipulations of signed numbers. In the drill group, a formal statement of each rule was first made and followed by drill. There was no explanation of why the rules worked. No verbalized statement of rules was made in the discovery group. Instead, the group was expected to discover and understand the basic principles and relationships to be learned by working through various exercises. In general, the deductive method was favored in the results. However, the inductive method produced significantly greater gains in the area of multiplication (29, p. 83).

On the ninth grade algebra level, Denmark compared certain variations of the deductive and inductive approaches
to problem solving with respect to their effectiveness in teaching the use of algebraic technique to solve verbal problems. "The algebraic technique involves the use of the solution of an equation, derived from the conditions of the problem, in determining the solution of the problem" (16, p. 5295). In the deductive variation, tables were constructed to identify and organize the parts of the problems. From the table an equation was derived. The inductive approach employed development of a pattern through trial and error attempts to solve a problem. Equations were then derived from the pattern.

Denmark's sample consisted of seven pairs of elementary algebra classes. One instructor taught both members of each pair except for one pair. One pair of classes was comprised of eighth graders. The remaining classes were classified as "average" ninth grade algebra classes. The instructional materials used were programmed textbooks. In teaching students to use algebraic techniques to solve verbal problems, the deductive method was more effective. However, when one considered only the number of correct problem solutions, the inductive method was more effective. In comparing achievement means, there was no significant difference for the two eighth grade classes. It was concluded that for the development of a variety of problem solving techniques, the inductive method is more
appropriate. For teaching algebraic problem solving techniques, the deductive method proved more suitable (16, p. 5296).

The effect of presenting programed conceptual learning materials by three related methods was investigated in 1961 by Gagne and Brown. The rule and example treatment "received a verbal statement of the principle and formal practice in the application of each rule through a series of examples." The guided discovery treatment "consisted of giving the subjects examples and guiding them through a series of directed questions to a final formulation of the rule" (15, 2.18). Independent discovery of the rules from several different series of examples comprised the discovery treatment. The sample consisted of boys studying ninth grade algebra.

A transfer test was administered almost immediately after the learning period. Although the guided discovery method revealed itself as most effective in all cases, it was only slightly above the discovery group. Time required for the discovery group was less than that for either of the other groups. It was concluded that both discovery treatments were superior to the rule and example method. Gagne and Brown pointed out (as indicated by Della-Piana):

However, our results emphasized the importance of "what is learned" rather than "how it is learned" as the crucial factor in learning effectiveness. Discovery as a method appears
To gain its effectiveness from the fact that it requires the individual learner to reinstate (and in this sense to practice) the concepts he will later use in solving new problems (15, p. 2.18).

To continue Gagne and Brown's investigations into how children learn mathematics and how this knowledge is used in problem-solving situations, Meconi (1967) used programmed materials to study results of teaching by the guided discovery, pure discovery, and rule and examples methods. The hypotheses tested by Meconi were:

1. With relative high-ability subjects there are no differences in immediate transfer value of "pure discovery" versus "guided discovery" versus "rule and example" methods of presenting number-sequence generalizations or concepts.

2. There are no significant differences between a discovery method, either "pure discovery" or "guided discovery," and a "rule and example" method when investigating retention after a period of approximately four weeks (28, p. 51).

Forty-five relatively high-ability eighth and ninth grade boys and girls comprised the sample for the two experiments. Intelligence quotient tests and mathematical and achievement tests were used to determine general academic and mathematical ability. Meconi worked with three subjects at a time in both experiments. The subjects were randomly placed in one of the three experimental groups before entering the experiment.

With some revision and modification, the programmed materials designed by Gagne and Brown were used in both experiments. The subject content consisted of principles
for simple number sequences. To measure performance, all three groups took the same test. The students were to find a rule or formula for finding the sum of any number of terms for each of four number sequences which comprised the test. Four hint cards followed each problem. To test retention approximately four weeks after the initial test, a second test was developed. It consisted of two new sequences and two sequences previously used in the problem-solving test.

All three treatment groups showed significant learning gains (p < .02). Using an analysis of variance, there were no significant differences at the .05 level between the three treatments on either the initial transfer test or the retention test. However, the rule and example method seemed to be least effective with respect to time and the pure discovery method most effective (28, p. 57).

High School Level Literature in Mathematics

Nichols undertook a study "to assess the effectiveness of learning of certain geometric topics as related to the method by which they were taught" (32, p. 2107). In the first method, the deductive, the teacher stated the assumptions, theorems, definitions and verbalized principles. Through mensuration and through sequences of concrete exercises and drawings of geometric figures, students
discovered their own relationships under the inductive method. The subjects involved were matched on criterion test scores, IQ, sex and age. They comprised two groups of twenty-one students each.

Results from Nichol's study showed that both approaches were equally effective in terms of the criterion test. With regard to both superior IQ and average IQ, both methods were equally effective in terms of the criterion test (32, p. 2107).

To determine whether or not high school pupils can profit from studying the limit concept in mathematics and to determine the relative importance of experience, as compared to maturity, in acquiring the limit concept, Smith investigated 578 pupils in grades seven through twelve. He found that these students can acquire the limit concept. He also concluded, "at this age level, experience is the important factor in determining the extent to which an individual has conceptualized the limit in mathematics" (40, p. 344). Smith also indicated that, more than any other single factor, the nature of a pupil's background of experience is more significant in determining his success with problems concerned with the limit (40, p. 344).

College Level Literature in Mathematics

The area of differential calculus provides a study which investigates "the preciseness of the concepts formed as a
result of a concept-emphasized approach" (8, p. 2102). In this experiment, Burkhart utilized 235 students from nine regular differential calculus classes. The conventional teaching method emphasizing skills and problem solving was used in two classes. The remaining classes received experimental treatment with emphasis on understanding the calculus concepts.

Results of Burkhart's study indicated that "comparison between the preciseness of the concepts acquired by students in the concept approach to the calculus and the preciseness of the concepts acquired incidentally by students in the skill approach to the calculus shows that the differences observed are primarily those of quality of concept. While the differences in distribution of excellent and adequate responses are significant for the concept of variable and for the applications of the calculus in solving problems, other differences point toward the effectiveness of the concept approach as a method of teaching the calculus." These differences imply "that increased emphasis on understanding the concepts of the calculus contributes toward improvement in learning, toward greater retention and toward increased efficiency in the use of the calculus in related subject-matter areas" (8, p. 2102).

In a study using the discovery approach to teach calculus, Cummins tested the hypothesis, "a student
experience-discovery approach to calculus in the university yields results every bit as promising as those under similar conditions on the secondary level" (14, p. 163). The experimental groups were taught by the discovery method while the control groups were taught in a traditional manner. Two tests, one for the discovery groups and one for the traditional groups, were designed. Scores from the first test were significantly higher for the discovery groups than the traditional. There was no significant difference in results from the second test. Overall results imply that discovery trained students will perform better on a discovery test.

Directly Related Studies

Another study involving calculus, conducted by Shelton in 1965, compared achievement as a result of teaching the limit concept by two different methods. The two methods were classified as a concrete inductive approach and an abstract deductive approach. Shelton defined the concrete inductive approach as "a presentation of a sequence of items leading from specific numerical examples to the general case"; he defined the abstract deductive approach as "a presentation of a sequence of items leading from the abstract to the particular" (39, p. 2613). Mathematically, the content was the same for both approaches.

Shelton's sample consisted of two different college level mathematics classes, a beginning calculus class and an
analytic geometry class. Using scores on a pretest, the subjects were divided into high and low levels. Six 50-minute class periods of programed instructional material comprised the treatments. Both treatments physically took the form of a programed textbook. The conclusions of Shelton's experiment were:

1. There were no statistically significant differences in achievement between the two treatment groups shown by the criterion test.

2. There were no statistically significant differences in achievement shown by the criterion test between the two levels used in the study.

3. There was no statistically significant interaction between the treatments and levels as measured by the criterion test (39, p. 2614).

In an experiment at New York University in 1966, Caruso studied the effectiveness of teaching the mathematical theory of groups, rings, and fields to college freshmen by two methods, an abstract and a concrete approach. The abstract approach consisted of teaching deductively by first giving rigorous definitions of the mathematical systems studied followed by illustrations of their properties and finally by giving examples of these systems. The concrete approach consisted of teaching inductively by first presenting specific concrete examples of the systems studied, enhancing familiarity with them through problem solving and then by studying their properties and introducing the definitions and properties of groups, rings and fields to
the students. The hypothesis tested by Caruso were:

1. The experimental group, taught using the abstract approach, will show a higher achievement in the learning of the theory of groups, rings and fields than the control group taught by the concrete approach.

2. The experimental group will show a higher achievement in delayed recall of the theory of groups, rings, and fields than the control group (11, p. 3769).

Caruso developed two sequences of lesson plans: each consisted of eleven fifty minute lessons and extended over four weeks. In one, the abstract approach was stressed while the concrete approach was stressed in the other. The sample consisted of five control (concrete) and five experimental (abstract) classes comprised of community college freshmen. Five instructors, each with one experimental and one control class, taught the materials.

To test the hypotheses, two test forms were constructed—one to be given at the conclusion of the eleven lessons and the other to be administered nine weeks later as a measure of retention. Using a one-tailed t-test, no significant difference was found between the two groups on the first test. The experimental group revealed a significantly higher achievement on retention than did the control group.

In a replication of Shelton's experiment, Lackner (1968) undertook an investigation consisting of three studies. In the first study she compared an inductive
(discovery) and deductive (expository) teaching of the limit concept in beginning calculus. In the second she did the same using the derivative concept. In the third study she compared the following four paired, ordered combinations: inductive limit-inductive derivative, deductive limit-deductive derivative, inductive limit-deductive derivative, and deductive limit-inductive derivative (26, p. 2150).

Lackner utilized Shelton's two programed units on the limit; one inductive, the other deductive. She constructed two derivative units, also inductive and deductive. In the abstract deductive unit, Lackner used a "rule-to-example" programing technique. After stating a definition or stating and proving a theorem or corollary, examples were used to illustrate. For the concrete inductive units she used an "example-to-rule" programing technique in which the student, through sequences of examples, was led to formulate a statement of a theorem, a definition, or a corollary.

Lackner's sample consisted of 400 junior and senior Chicago suburban high school students. On the basis of a pretest, the students were assigned to high and low achievement levels. For the statistical analysis, a two-by-two, treatments by levels, analysis of covariance design was used on the limit and derivative studies. For the total treatment study, a four-by-four, treatments by levels analysis of covariance design was used.
Results of this study revealed no difference in teaching methods for the limit study. A difference in achievement did occur in both the derivative and the total treatment studies. The deductive approach was favored for the derivative study and the deductive limit-deductive derivative in the total treatment study.

In the previous studies, it is to be noted that both Lackner and Shelton used programed materials to compare an abstract deductive approach and a concrete inductive approach in teaching limits and derivatives. Although this helps to control the teacher variable, it also contributes to a rather sterile classroom situation in which the students could become bored. To avoid such a situation and create a more realistic classroom situation, one might use "programed instructors" rather than programed materials. Also, Lackner used high school students as her sample; Shelton and Caruso used college freshmen and community college students, respectively. Thus, one might investigate teaching of the limit and derivative on the college level, exclusively. Finally, neither Lackner nor Shelton tested for retention. Because of the importance of delayed recall, such an investigation appears justified.

In summary, the following study is a replication of Caruso's experiment insofar as experimental design is concerned. The topics involved were the limit and derivative concepts, as in the cases of Shelton and Lackner.
CHAPTER III

DEVELOPMENT OF TREATMENT AND EVALUATION INSTRUMENTS

Selection of Material to be Learned

Lackner and Shelton both utilized programed materials as their primary means of instruction. Shelton worked in the area of limits, while Lackner extended Shelton's study to include derivatives. Since it was desired to conduct a related study employing a different method of instruction, this investigator used detailed lesson plans covering the same mathematical content. Two instructional approaches, a concrete inductive and an abstract deductive, were employed.

In developing the concrete inductive and the abstract deductive units, both Shelton's programed materials on limits and Lackner's program materials on derivatives were utilized. Continuity and applications of the derivative to graphing were added while several topics such as velocity applications and the limits of sequences of both inscribed and circumscribed regular polygons were deleted.

Content of the Units

In both treatments, the concrete inductive and the abstract deductive, the limit concept is taught first, then
continuity, and finally derivatives. Both limits of sequences and functions are considered. However, limits of sequences precede limits of functions in the inductive approach but the order is reversed in the deductive. In the latter approach, sequences are presented as a special case of functions. To enhance understanding of the definition of a limit, absolute values and inequalities, along with various graphs, are studied first. The definition of limit is utilized in the traditional epsilon-delta form as well as with neighborhoods. The basic limit theorems for sum, difference, product, and quotient are considered. Functions for which limits do not exist are also studied.

Continuity follows the presentation on limits. Both the definition for continuity of a function at a point and the definition for continuity of a function over a set are included. Numerous graphs of continuous and discontinuous functions are studied.

The third section, on derivatives, includes the following: the definition of derivative; the application of the derivative as the slope of a tangent line to a curve; and the basic derivative theorems for sums, differences, products, and quotients of functions, constant functions, the independent variable, the independent variable to a real power, and composite functions. Examples and exercises
in both treatments are, for the most part, identical, and otherwise, comparable.

Lesson Plans

All lesson plans for each treatment were organized in the same manner (Appendixes E and F). Each begins with a statement of the objectives for that particular lesson. For example, the objectives of Lesson VII in the concrete inductive approach are:

1. Evaluate the slope of a tangent line to a curve by considering a sequence of slopes of secant lines to the curve through the point of tangency and a nearby point.

2. Evaluate the slope of a tangent line to a curve at any point on the curve.

3. Formally define the slope of a tangent line to a curve.

It is to be noted that the content of each concrete inductive lesson plan and its corresponding abstract deductive lesson plan is not necessarily the same and, as a result, the objectives cannot coincide for each day. This is a result of the variation in sequencing the materials for the two approaches. This difference is clarified by considering the objectives of Lesson VII of the abstract deductive approach and contrasting them with those listed above:

1. Define the derivative of a function.

2. Evaluate the derivative of various functions using the definition.

3. Consider four basic derivative theorems.
Following the objectives is a detailed section entitled "Procedures." Here is found an ordered sequence of concepts and examples.

For each treatment, the content, discussed in detail below, is covered in twelve detailed lesson plans. Each lesson is designed to be covered in one 48-minute class period. To insure completion of all lesson plans by a common date, several free days were allowed for catch-up. As a result, the entire sequence required approximately fourteen to sixteen days.

Development of the Concrete Inductive Treatment

The concrete inductive unit begins with a review of the definition of a sequence and various examples of sequences. Limits of simple sequences such as $1/n$ and $(n + 1)/n$ are "guessed" by considering terms of the sequence for such values of $n$ as 1, 10, 100, 1000, etc. Absolute values are reviewed and utilized in defining a neighborhood. Graphs of basic inequalities are also reviewed.

Next, an intuitive idea of limit is sought by considering linear functions. Limits are discussed in terms of "closeness," inequalities, and graphs. Discussion of the limit of a linear function, for which a single point of its domain has been deleted, leads to the definition of a deleted neighborhood. Again, absolute values and
inequalities are applied. Limits of step functions are discussed as well as limits of functions that increase or decrease without bound.

Finally, the intuitive idea of a limit is generalized and the formal definition of a limit of a function is stated and discussed. The formal definition of the limit of a sequence along with the definitions of convergence and divergence follow. Through consideration of various examples, the four basic limit theorems (sum, difference, product, and quotient of two functions) are derived.

To develop an intuitive idea of continuity, various examples of discontinuous functions are discussed in terms of "jumps," "holes," or the like. Various functions for which \( \lim_{x \to a} f(x) \) does not exist, \( f(a) \) does not exist, and/or \( \lim_{x \to a} f(x) \neq f(a) \) are studied. This results in the definitions of a function continuous at a point and the definition of a function continuous over a set.

The section on derivatives begins with a detailed evaluation of the slope of a tangent line to a curve at a specific point. This is accomplished by evaluating the slopes of a sequence of secant lines to the curve through the point of tangency and a nearby point. Several tables and graphs are utilized to emphasize this. The difference quotient is brought into use and, after careful study of examples, the slope of the tangent line and the limit of
the difference quotient are associated. The formal definition of the slope of a tangent line to a curve follows. The derivative is applied in finding the equation of a tangent line to a curve at a point.

Examples of functions for which the slope of the tangent line does not exist at a point because a) the limit does not exist and b) the tangent line is vertical are considered. The formal definition of the derivative of a function at a point is stated. The basic derivative theorems involving the derivative of a constant, the independent variable, the independent variable to a real power, and the sum and difference of two functions, and the product and quotient of two functions are developed.

Finally, numerous examples of composite functions are studied and their derivatives evaluated. The Chain Rule is discussed and applied to several examples.

The last topic covered includes applications of the derivative to graphing. Through looking at numerous examples, both the First and the Second Derivative tests are derived.

The development of the entire concrete inductive unit is illustrated in Figure 1, an adaptation of Shelton's flow chart for limits and Lackner's flow chart for derivatives. Generally, all topics in the concrete inductive treatment are developed through a definite sequence of steps. After
Limits of sequences.  **Analogy**  Concept of limit used in several types of problems.

Slope of tangent line.  **Analogy**

*Simple enumeration*  Abstract  **Difference and agreement**  properties.

Limits may or may not exist at a point.  **Simple enumeration**  Theorems about limits of sums, products, differences, and quotients (39, p. 28).

**Difference and agreement**  Continuous and non-continuous functions.  **Simple enumeration**  Definition of continuity.

\[ \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]  **Simple enumeration**  Definition of slope of a tangent line to a graph at \( x \).

\[ \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]  **Simple enumeration**  may or may not exist.  Definition of derivative.

Theorems and corollaries for derivatives of sums, differences, products, quotients of functions and other selected functions (26, p. 30).

**Fig. 1.--Flow Chart for the Development of the Concrete Inductive Unit.**
a review of prerequisite knowledge, examples illustrating properties of the new topic are considered. The examples are ordered so the student is led to a generalization, i.e., the definition or theorem desired from the lesson. However, review sections and proofs of theorems are presented through straightforward exposition. This is done to economize on time and to insure completion of the two treatments in approximately the same time span.

Development of the Abstract Deductive Treatment

In contrast to the concrete inductive unit, all topics in the abstract unit are developed deductively. For each new concept, a generalization is stated and is followed by various appropriate examples. Figure 2, compiled from similar flow charts originated by Shelton and Lackner, illustrates the logical development of the deductive unit.

The development of the abstract deductive treatment begins by preparing the students for the definition of a limit. First, consideration is given to the definition of a neighborhood and a deleted neighborhood. These two concepts are compared and differences are pointed out. They are discussed in terms of absolute values and inequalities and in the context of graphs. The formal epsilon-delta definition of a limit is stated and the notation is discussed. Limits of linear functions are considered; proofs
Function \textbf{Exposition} \textgreater \textbf{Limit of Logical Theorems} a function \textbf{deduction} \textgreater \textbf{of limits}

\textbf{Exposition} \textgreater \textbf{One-sided Exposition} \textgreater \textbf{Sequences Logical} \textbf{deduction}

Theorems on \textbf{Exposition} \textgreater \textbf{Uses of limit. Exposition} \textgreater \textbf{Slope of tangent line (39, p. 28)}.

\textbf{Continuity of Exposition} \textgreater \lim_{x \to x_0} f(x) = \lim_{h \to 0} f(x_0 + h)\text{ a function.}

\textbf{Exposition} \textgreater \textbf{Definition Logical} \textbf{deduction} \textbf{of the derivative}

Theorems and corollaries for derivatives \textbf{Exposition} \textgreater \textbf{of sums, differences, products, quotients of functions and other selected functions.}

The derivative may \textbf{Exposition} \textgreater Application of the derivative as slope of a tangent line to a curve (26, p. 30).

\textbf{Exposition} \textgreater Application of derivative to graphing.

\textbf{Fig. 2.--Flow Chart for the Development of the Abstract Deductive Unit.}
of limits of a linear function are also discussed. The inequalities involved with the definition of the limit of a function are graphed. The theorem for the limit of a constant is stated and proved.

In turn, the basic theorems involving the limit of the independent variable, the sum of two functions, the product of two functions, the difference of two functions, and the quotient of two functions are stated, proved, and illustrated with examples. Next, the limit of a function as the independent variable becomes infinite is defined and examples are considered.

Functions for which the limit at a point does not exist because the two one-sided limits either increase or decrease without bound are discussed. This is followed by a study of functions for which the limit at a point does not exist because the two one-sided limits are unequal at that point.

After studying the limit of a function, the limit of a sequence is considered. Prior to this, the definition of a sequence is reviewed. Various examples of sequences and their limits are studied. This is followed by definitions of convergent and divergent sequences. The theorems for the limit of the sum, difference, and product of two sequences are then stated and proved.

The topic of continuity begins with the formal
definition of a function continuous at a point and is followed with examples. Then the definition of a function continuous over a set is stated and examples considered.

As in the case of continuity, the section on derivatives begins with the definition of a derivative of a function. The notation involved is carefully discussed and interpreted. Theorems for the derivative of the independent variable, a constant, the independent variable to a real power, the sum, difference, product, and quotient of two functions are all stated and proved. Each is followed by examples.

Composite functions are reviewed and the Chain Rule is formalized. Next, the derivative of a function at a point is related to the slope of the tangent line at that point. This is applied to finding the equation of the tangent line to various types of curves. Finally, the first and second derivatives are applied to graphing.

Preliminary Study

During the Spring Quarter of 1970 a Pilot Study was conducted at The Ohio State University. The study involved 219 students enrolled in Mathematics 117. Twelve classes comprised the sample for the two experimental treatments. Each treatment involved six of these classes. Students were randomly assigned to sections within a given time period.
Of the two experimental groups, one was taught the limit, continuity, and derivative concepts by the concrete inductive method. The second group was taught these same concepts by the abstract deductive method. All classes met five days a week for a 48-minute class period. Homework and exams in both groups were comparable and pre-determined; both groups took the same achievement test concurrently upon completion of the instructional unit.

The course used in this study, Mathematics for the Behavioral, Economic, and Social Sciences II, is described in The Ohio State University Bulletin, 1970-71 (34, p. 275), as:

The sequence 116, 117 treats topics in mathematics with applications to the nonphysical sciences. Topics will include analytic geometry, calculus, linear algebra, linear programming and graph theory; applications.

The textbook normally used for the course is An Introduction to Calculus with Economic Applications by John Riner and Bert Waites. Topics covered in the units on limit and derivative include:

1. Sequences.
3. Divergence. Special conventions.
   Special sequences.
4. Limits of functions.
5. Properties of limits of functions.
6. Continuity.
7. Definition of the derivative of a function.
8. Graphing.
These topics comprised the content covered by the two experimental groups. Mathematics 117 is usually considered to be a terminal course in mathematics. It is normally taken, along with the preceding course, Mathematics 116, to fulfill a ten credit hour mathematics requirement.

The instructors for the two treatment groups were selected from those who could be scheduled to teach two sections of Mathematics 117 and who were willing to participate. Each instructor taught one section by the abstract deductive approach and one section by the concrete inductive approach. Detailed lesson plans for each approach were distributed to the instructors.

Data was gathered at two points in the Pilot Study. To determine the equivalence of the two treatment groups and the Control Group, total point scores of the prerequisite course, Mathematics 116, were used. Because these Mathematics 116 scores were used as a covariate, only students who had completed Mathematics 116 the previous quarter (Winter 1970) were used in the sample. To determine achievement, a test was administered at the completion of the limit, continuity, and derivative unit. This occurred during the fourth week of the quarter. This test, comprising the first midterm for the course, was given in the evening from 6:00 P.M. to 7:00 P.M. It was given to all Mathematics 117 students concurrently and was constructed from
an item bank supplied by previous examinations given at The Ohio State University in Mathematics 117.

The purpose of the Pilot Study was to refine the teaching materials. Various suggestions were offered by the participating instructors and these were incorporated into the revised version of the lesson plans. It was suggested that answers to assignments be distributed to the students. In the main study this was done and served to release the instructors from spending a great deal of time giving answers to problems.

In the concrete inductive unit in the Pilot Study, the definition of the limit of a function as the independent variable becomes infinite was unintentionally omitted. In the main study, this definition was incorporated into the lesson plans.

A few assignments in the Pilot Study involved some problems which could not logically be solved at that point in the instructional sequence. Such misplaced problems were relocated in the main study. The remaining corrections in the lesson plans consisted primarily of removing typographical errors.

The statistics on the Pilot Study, an analysis of covariance using total numerical scores from Mathematics 116 as the covariate, resulted in no significant differences. Statistics revealed, however, that students were
learning by both treatments. Table 2 summarizes these results.

**TABLE 2**

PILOT STUDY ANALYSIS OF COVARIANCE

<table>
<thead>
<tr>
<th>Source</th>
<th>Sums of Squares (SS)</th>
<th>Degrees of Freedom (df)</th>
<th>Mean Squares (MS)</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Groups</td>
<td>752.887</td>
<td>2</td>
<td>376.444</td>
<td>.139</td>
</tr>
<tr>
<td>Subgroups</td>
<td>66,578.668</td>
<td>21</td>
<td>3,170.413</td>
<td>1.171</td>
</tr>
<tr>
<td>Within Groups</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Error</td>
<td>974,780.895</td>
<td>360</td>
<td>2,707.725</td>
<td></td>
</tr>
<tr>
<td>TOTAL</td>
<td>1,042,112.450</td>
<td>383</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For the Pilot Study achievement test, the maximum score possible was twenty-five. While the range of scores was twenty-two, the highest score attained was twenty-five and the lowest was three. With 536 students taking the test, the resulting mean was 16.94, the median was 17, and the mode was 22.

Due to campus riots, The Ohio State University was temporarily closed during Spring Quarter, 1970. Although the achievement test was administered prior to the onset of disruption, a retention test was not given as originally intended. Even though classes were resumed, the general atmosphere on campus might have invalidated the study.
Development of the Evaluation Instruments

Two questionnaires (Appendix A) were utilized the first day of classes. Their purpose was to provide background information on each student. Quality points were found by Crosswhite (13, p. 165) and again by Paul (35, p. 124) to have high predictive value for achievement in calculus at The Ohio State University. Due to this finding, quality points were evaluated from information indicated on the questionnaires and used as covariates in analyses of covariance.

A pretest (Appendix B) was developed and administered the first full day of classes (during the second class meeting). Students were allowed one hour for completing the test. The test was developed from a bank of items used previously on Mathematics 116 examinations at The Ohio State University and for which item analyses were available. Shelton's pretest was used as a guideline. Two experts in the field of mathematics reviewed the test and made various pertinent suggestions. These recommendations were incorporated into the final version of the pretest. The revisions resulted in a 20-item test in which each item had five choices. Table 3 summarizes the nature of items on the pretest.

The achievement test (Appendix B), which comprised the first midterm for Mathematics 117, was a test
equivalent to the achievement test administered during the Pilot Study. As indicated earlier, the original test was constructed from items used previously on Mathematics 117 examinations and for which item analyses were available. Each item used on the achievement test had previously displayed a discrimination index greater than .40. The poorer items on the Pilot Study achievement test were either eliminated or revised depending on the nature of their discrepancies. The resulting test, the achievement test for the Main Study, contained 25 multiple choice items each of which had five distractors. Table 4 presents a summary of the types of items involved in the achievement test. In a more detailed description, Table 5 displays the content
analysis of the criterion test. This test was administered during the fourth full week of classes from 5:00 P.M. to 6:00 P.M.

**TABLE 4**

**ACHIEVEMENT TEST ITEM CLASSIFICATION**

<table>
<thead>
<tr>
<th>Nature of Problem</th>
<th>Item Number</th>
<th>Percentage of Total Test Items</th>
</tr>
</thead>
<tbody>
<tr>
<td>Limits</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11</td>
<td>44</td>
</tr>
<tr>
<td>Continuity</td>
<td>12</td>
<td>4</td>
</tr>
<tr>
<td>Differentiation</td>
<td>13, 14, 15, 16, 17, 18, 19, 22</td>
<td>32</td>
</tr>
<tr>
<td>Tangent lines</td>
<td>20, 21</td>
<td>8</td>
</tr>
<tr>
<td>Maximums, minimums,</td>
<td>23, 24, 25</td>
<td>12</td>
</tr>
<tr>
<td>points of inflection</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
TABLE 5

CONTENT ANALYSIS OF THE ACHIEVEMENT TEST

<table>
<thead>
<tr>
<th>Item of knowledge needed to complete each question.</th>
<th>Question Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Definition of the limit of a function</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11</td>
</tr>
<tr>
<td>2. Definition of one-sided limits</td>
<td>2, 3</td>
</tr>
<tr>
<td>3. ( \lim_{x \to a} f(x) ) may not be ( f(a) )</td>
<td>1, 3</td>
</tr>
<tr>
<td>4. ( \lim_{x \to a} f(x) ) may exist when ( f(a) ) is not defined</td>
<td>6, 7, 8</td>
</tr>
<tr>
<td>5. ( \lim_{x \to a} f(x) ) may not exist</td>
<td>2, 3, 4, 18</td>
</tr>
<tr>
<td>6. Limit of a sequence</td>
<td>9, 10, 11</td>
</tr>
<tr>
<td>7. Relation between ( \delta ) and ( \varepsilon )</td>
<td>11</td>
</tr>
<tr>
<td>8. Continuity of a function</td>
<td>12</td>
</tr>
<tr>
<td>9. Definition of the derivative</td>
<td>13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24</td>
</tr>
<tr>
<td>10. Definition of the slope of a tangent line</td>
<td>20, 21</td>
</tr>
<tr>
<td>11. The derivative of the independent variable</td>
<td>14, 15, 16, 17, 19, 20</td>
</tr>
<tr>
<td>12. The derivative of a constant function</td>
<td>13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 24</td>
</tr>
<tr>
<td>13. The derivative of a real power of the indepen-</td>
<td>13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 24</td>
</tr>
<tr>
<td>dendent variable</td>
<td></td>
</tr>
<tr>
<td>14. Derivative of a sum</td>
<td>13, 16, 18, 19, 23, 24</td>
</tr>
<tr>
<td>15. Derivative of a difference</td>
<td>13, 14, 15, 16, 17, 19</td>
</tr>
<tr>
<td>16. Derivative of a product</td>
<td>13, 14, 15, 16, 17, 19, 20, 21, 23, 24</td>
</tr>
</tbody>
</table>
The retentive achievement test (Appendix B) was constructed of items equivalent to those used on the achievement test. Several items were identical to those on the achievement test for the Pilot Study. Due to a time limitation factor for taking this examination, the test was limited to 20 multiple choice items each with five possible choices. Tables 6 and 7 give descriptive analyses of the retentive achievement test. The retention test comprised a portion of the final examination and was administered during final examination week. This part of the examination was allotted a total of 50 minutes and was given from 8:00 P.M. to 8:50 P.M. on the pre-scheduled examination day.
<table>
<thead>
<tr>
<th>Nature of Problem</th>
<th>Item Number</th>
<th>Percentage of Total Test Items</th>
</tr>
</thead>
<tbody>
<tr>
<td>Limits</td>
<td>1, 2, 3, 4, 5, 6</td>
<td>30</td>
</tr>
<tr>
<td>Continuity</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>Differentiation</td>
<td>8, 9, 10, 11, 12, 13</td>
<td>30</td>
</tr>
<tr>
<td>Tangent lines</td>
<td>14, 15</td>
<td>10</td>
</tr>
<tr>
<td>Maximums, minimums, points of inflection</td>
<td>16, 17, 18, 19, 20</td>
<td>25</td>
</tr>
<tr>
<td>Item of knowledge needed to complete each question.</td>
<td>Question Number</td>
<td></td>
</tr>
<tr>
<td>------------------------------------------------</td>
<td>----------------</td>
<td></td>
</tr>
<tr>
<td>1. Definition of the limit of a function</td>
<td>1, 2, 3, 4, 5, 6</td>
<td></td>
</tr>
<tr>
<td>2. Definition of one-sided limits</td>
<td>1, 2</td>
<td></td>
</tr>
<tr>
<td>3. ( \lim_{x \to a} f(x) ) may not be ( f(a) )</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>4. ( \lim_{x \to a} f(x) ) may exist when ( f(a) ) is not defined</td>
<td>3, 4</td>
<td></td>
</tr>
<tr>
<td>5. ( \lim_{x \to a} f(x) ) may not exist ( x \to a )</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>6. Limit of a sequence</td>
<td>6, 7</td>
<td></td>
</tr>
<tr>
<td>7. Continuity of a function</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>8. Definition of the derivative</td>
<td>8, 9, 10, 11, 12, 13, 14, 15, 18, 19</td>
<td></td>
</tr>
<tr>
<td>9. Definition of the slope of a tangent line</td>
<td>14, 15</td>
<td></td>
</tr>
<tr>
<td>10. The derivative of the independent variable</td>
<td>8, 9, 10, 11, 13, 14</td>
<td></td>
</tr>
<tr>
<td>11. The derivative of a constant function</td>
<td>8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19</td>
<td></td>
</tr>
<tr>
<td>12. The derivative of a real power of the indepen­dent variable</td>
<td>8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19</td>
<td></td>
</tr>
<tr>
<td>13. Derivative of a sum</td>
<td>8, 10, 11, 12, 13, 14, 15, 18, 19</td>
<td></td>
</tr>
<tr>
<td>14. Derivative of a difference</td>
<td>9, 10, 14, 15, 17</td>
<td></td>
</tr>
<tr>
<td>15. Derivative of a product</td>
<td>8, 9, 10, 11, 13, 14, 15, 18, 19</td>
<td></td>
</tr>
<tr>
<td>Item of knowledge needed to complete each question.</td>
<td>Question Number</td>
<td></td>
</tr>
<tr>
<td>---------------------------------------------------</td>
<td>-----------------</td>
<td></td>
</tr>
<tr>
<td>16. Derivative of a quotient</td>
<td>10, 11</td>
<td></td>
</tr>
<tr>
<td>17. Derivative of a constant multiplied by a function of x</td>
<td>8, 9</td>
<td></td>
</tr>
<tr>
<td>18. Derivative of a composite function</td>
<td>8, 9, 11, 12, 13</td>
<td></td>
</tr>
<tr>
<td>19. Applications of the derivative</td>
<td>14, 15, 16, 17, 18, 19, 20</td>
<td></td>
</tr>
</tbody>
</table>
CHAPTER IV

THE EXPERIMENTAL DESIGN

Population and Sampling

The subjects participating in this study were students at The Ohio State University, Columbus, Ohio during the Fall Quarter, 1970. All subjects were enrolled in Mathematics 117 (An Introduction to Calculus with Economic Applications). On the first day of classes, all subjects enrolled in Mathematics 117 at a specific time reported to a dispatcher who handed each student a card indicating the room to which he was to report. Since teaching approaches for each instructor were predetermined for each class and since students were randomly distributed to classes within each time period, treatments were assumed to be randomly assigned.

Initially, the number of students involved was 256; these students comprised ten individual classes. The Abstract Deductive Treatment Group consisted of 107 students (four classes), the Concrete Inductive Treatment Group consisted of 106 students (four classes), and the Control Group contained 43 students (two classes). Four of these classes were offered at 12:00 noon and the
remaining six were scheduled at 4:00 P.M. Two instructors taught inductively at noon and deductively at 4:00 P.M.; the remaining two instructors taught in the reverse order.

The number and percent of males and females involved in each group are indicated in Table 8 below. It is to be noted that, for the complete sample, the ratio of the number of males to the number of females is almost 2:1. Most students in the sample were 19 years of age and Sophomores constituted the most populous level. For a further breakdown of the sample in terms of age and class level, see Appendix C.

<table>
<thead>
<tr>
<th>Treatment Group</th>
<th>Number of Males</th>
<th>Percent of Males</th>
<th>Number of Females</th>
<th>Percent of Females</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abstract Deductive</td>
<td>78</td>
<td>73</td>
<td>29</td>
<td>27</td>
</tr>
<tr>
<td>Concrete Inductive</td>
<td>65</td>
<td>61</td>
<td>41</td>
<td>39</td>
</tr>
<tr>
<td>Control</td>
<td>27</td>
<td>63</td>
<td>16</td>
<td>37</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>170</td>
<td>66</td>
<td>86</td>
<td>34</td>
</tr>
</tbody>
</table>

While 47% of the sample had completed three years of high school mathematics, less than 1% (only one in 249) had completed five or more years. As would be expected, the majority of the subjects (58%) in the total sample had earned a "C" average in high school mathematics courses.
See Appendix C for more detailed information concerning the mathematical background of the subjects.

Mathematics 116 is a course prerequisite to Mathematics 117. For Mathematics 116, most students in both treatment groups, the control group, and the total sample had earned the grade of "C." During the Spring Quarter, 1970, all students were given the option of receiving a letter grade for a quarter's work in each course or being evaluated on a "Pass" or "Non-Pass" basis. Twenty-three percent of the total sample received a "Pass" for Mathematics 116. Appendix C provides further information on grades received in Mathematics 116. Of the original 256 students in the experimental sample, 87% indicated they were taking Mathematics 117 for the first time. Eleven percent indicated they were repeating Mathematics 117.

Table 9 indicates the loss of subjects in the treatment groups and the Control Group as the experiment.

<table>
<thead>
<tr>
<th>Test Completed</th>
<th>Inductive Treatment</th>
<th>Deductive Treatment</th>
<th>Control Group</th>
<th>Total Sample</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pretest</td>
<td>99</td>
<td>101</td>
<td>46</td>
<td>246</td>
</tr>
<tr>
<td>Achievement Test</td>
<td>83</td>
<td>80</td>
<td>42</td>
<td>205</td>
</tr>
<tr>
<td>Retention Test</td>
<td>54</td>
<td>61</td>
<td>31</td>
<td>146</td>
</tr>
</tbody>
</table>
progressed. For the most part, loss was due to students dropping Mathematics 117. However, some loss was due to students who took one or more of the tests at a time other than when scheduled; these students were eliminated from the study. Because graduating seniors were not required to take final examinations, a nominal number of students were omitted for this reason.

The Design

The reader will recall that the subjects of this study were divided into three groups: two treatment groups and a control group. One treatment group was taught by an abstract deductive approach while the other treatment group was taught by a concrete inductive approach. Basically, the abstract deductive approach is an instructional approach which proceeds from the general to the specific; a generalization (such as a theorem or a definition) is stated and followed by numerous illustrative examples. The concrete inductive instructional approach proceeds from the specific to the general; various ordered sequences of examples are studied which lead to a generalization. The control group received no special treatment.

Table 10 exhibits the experimental design employed in this experiment: $O_1, O_2, O_3$ and $O_4$ represent quality points, the pretest, the achievement test, and the retention test scores, respectively; $X_1$ indicates Treatment 1,
or the concrete inductive treatment, while $X_2$ indicates Treatment 2, the abstract deductive approach.

### TABLE 10

**THE EXPERIMENTAL DESIGN**

<table>
<thead>
<tr>
<th>Inductive Treatment</th>
<th>$O_1$</th>
<th>$O_2$</th>
<th>$X_1$</th>
<th>$O_3$</th>
<th>$O_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deductive Treatment</td>
<td>$O_1$</td>
<td>$O_2$</td>
<td>$X_2$</td>
<td>$O_3$</td>
<td>$O_4$</td>
</tr>
<tr>
<td>Control</td>
<td>$O_1$</td>
<td>$O_2$</td>
<td></td>
<td>$O_3$</td>
<td>$O_4$</td>
</tr>
</tbody>
</table>

The controlled variables for the analyses were:

1. administration of the limit and derivative treatments, the pretest, the achievement test, and the retention test;
2. length of the limit and derivative treatments, the pretest, the achievement test, and the retention test; and
3. content of the limit and derivative treatments, the pretest, the achievement test, and the retention test.

The hypotheses tested in the experiment are:

1. There is no significant difference in the mathematical achievement of the Concrete Inductive Group and the Abstract Deductive Group.
2. There is no significant difference in the mathematical achievement of The Concrete Inductive Group and the Control Group.
3. There is no significant difference in the mathematical achievement of the Abstract Deductive Group and the Control Group.
4. There is no significant difference in the retentive achievement of the Concrete Inductive Group and the Abstract Deductive Group.
5. There is no significant difference in the retentive achievement of the Concrete Inductive Group and the Control Group.

6. There is no significant difference in the retentive achievement of the Abstract Deductive Group and the Control Group.

Procedures

Lesson plans for both the concrete inductive and the abstract deductive approaches were furnished for each instructor. Both sequences were bound in notebook form and the teachers received them prior to the beginning of classes. A schedule was also given to each instructor. Since instructors were not expected to complete each lesson at exactly the same time, several days were allowed for catch-up purposes. All twelve lesson plans were to be completed within 22 class periods. This was to be followed by two class periods of reviewing for the criterion test.

Each of the classes comprising the treatment groups, with the exception of the two taught by the investigator, were visited by an outside observer. This was done to assure the investigator that each instructor was effectively following the designated lesson plans. Frequent personal consultations took place between the investigator and the participating instructors; questions or problems were discussed at these meetings. Table 11 briefly outlines the educational background of the six instructors involved in the study.
TABLE 11
BACKGROUND OF INSTRUCTORS

<table>
<thead>
<tr>
<th>Instructor</th>
<th>Major Field</th>
<th>Education Credits*</th>
<th>Mathematics Credits</th>
<th>Teaching Experience</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Undergrad Grad</td>
<td>Undergrad Grad</td>
<td>High School College</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Experimental</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Mathematics Education</td>
<td>40  42  50  130</td>
<td>17  3</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Mathematics Education</td>
<td>30  60  38  70</td>
<td>1  3</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Mathematics Education</td>
<td>36  100  51  105</td>
<td>18  12</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Mathematics Education</td>
<td>0  33  40  65</td>
<td>0  5</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>Mathematics Education</td>
<td>0  11  60  30</td>
<td>0  4</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>Mathematics</td>
<td>3  0  50  24</td>
<td>0  1</td>
<td></td>
</tr>
</tbody>
</table>

*Credits are indicated in terms of semester values.

For a period of one week during the treatment, it was not possible for the investigator to teach her own two sections. Another instructor took over both classes and followed the appropriate lesson plans. Because the same instructor taught both sections, it was assumed that results of the experiment were not significantly affected.
Questionnaires (Appendix A) were completed by each student during the first class meeting of the Fall Quarter, 1970 (September 30). The items on the questionnaire were designed to give background information and general characteristics of the subjects participating in the study.

The Pretest

A pretest (Appendix B) was administered by the instructors during the second class period (October 1, 1970). This test consisted of twenty multiple-choice items each with five alternatives. Each student was allowed a maximum of 48 minutes to complete the test. These tests were machine scored. The mean for the 249 students completing the test was 5.84; the 56 students comprising the upper 27.5 percent obtained a mean score of 9.964 and the lower 27.5 percent scored a mean of 2.927. The median was 6 while the mode was five. The scores ranged from zero to sixteen. The standard deviation for the pretest was 2.88.

Table 12 relates the item difficulty distribution. The pretest mean item difficulty index was .708.

Table 13 gives the item discrimination distribution for the pretest. The pretest produced a mean discrimination index of .352. For a detailed listing of the difficulty indices and discrimination indices for each item of the pretest and for the distribution of test scores from the pretest, see Appendix D. The Kuder-Richardson Formula
20 yielded a reliability coefficient of .552 while the Kuder-Richardson Formula 21 indicated a reliability coefficient of .528.

**TABLE 12**

<table>
<thead>
<tr>
<th>Range</th>
<th>Number of Items</th>
<th>Percentage of Items</th>
</tr>
</thead>
<tbody>
<tr>
<td>.81-.100</td>
<td>3</td>
<td>15</td>
</tr>
<tr>
<td>.61-.80</td>
<td>15</td>
<td>75</td>
</tr>
<tr>
<td>.41-.60</td>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>.21-.40</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>.00-.20</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**TABLE 13**

<table>
<thead>
<tr>
<th>Range</th>
<th>Number of Items</th>
<th>Percentage of Items</th>
</tr>
</thead>
<tbody>
<tr>
<td>.81-.100</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>.61-.80</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>.41-.60</td>
<td>6</td>
<td>30</td>
</tr>
<tr>
<td>.21-.40</td>
<td>13</td>
<td>65</td>
</tr>
<tr>
<td>.00-.20</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>Below .00</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Since only three subjects in 249 (approximately 1%) scored 75% or better on the pretest, it was presumed that very few students displayed an in-depth knowledge of the mathematical content tested. In fact, 92% of the subjects scored 10 or less (50%) and approximately 50% of the students scored 5 or below (25%).

The Achievement Test

The achievement test (Appendix B) on the limit and derivative concepts was administered on October 29, 1970 after completion of the instructional treatment units. The test was given at 5:00 P.M. and one hour was allowed for its completion. Each item of this 25-item test had five alternatives and was worth one point in grading. Thus, the maximum possible score was 25. With the exception of one instructor, each teacher administered the criterion test to his own students. In this case, the instructor for one of the control groups administered the test to both of this instructor's sections as well as to his own.

Table 14 summarizes the statistics for the achievement test. The highest attained score was 24 and the lowest was 2. The mean for the total sample was 14.19 and the standard deviation was 4.91. Both the Kuder-Richardson 20 and the Kuder-Richardson 21 yielded a reliability coefficient of (approximately) 0.8. Appendix D provides more complete statistics for each class within the treatments.
### TABLE 14

**ACHIEVEMENT TEST SUMMARY STATISTICS**

<table>
<thead>
<tr>
<th></th>
<th>Inductive</th>
<th>Deductive</th>
<th>Control</th>
<th>Total Sample</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Students</td>
<td>88</td>
<td>91</td>
<td>46</td>
<td>225</td>
</tr>
<tr>
<td>Mean</td>
<td>13.69</td>
<td>14.22</td>
<td>15.09</td>
<td>14.19</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>4.70</td>
<td>4.98</td>
<td>5.06</td>
<td>4.91</td>
</tr>
<tr>
<td>Median</td>
<td>14</td>
<td>14</td>
<td>15</td>
<td>14</td>
</tr>
<tr>
<td>Mode</td>
<td>12</td>
<td>18</td>
<td>9</td>
<td>18</td>
</tr>
<tr>
<td>Lowest Score</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>Highest Score</td>
<td>24</td>
<td>24</td>
<td>24</td>
<td>24</td>
</tr>
<tr>
<td>Range of Scores</td>
<td>22</td>
<td>21</td>
<td>19</td>
<td>22</td>
</tr>
<tr>
<td>Mean Item Difficulty</td>
<td>.452</td>
<td>.431</td>
<td>.397</td>
<td>.432</td>
</tr>
<tr>
<td>Mean Item Discrimination</td>
<td>.489</td>
<td>.490</td>
<td>.506</td>
<td>.491</td>
</tr>
<tr>
<td>Kuder-Richardson 20</td>
<td>.778</td>
<td>.809</td>
<td>.825</td>
<td>.801</td>
</tr>
<tr>
<td>Kuder-Richardson 21</td>
<td>.749</td>
<td>.784</td>
<td>.798</td>
<td>.777</td>
</tr>
</tbody>
</table>

For a further breakdown in the item analysis, Table 15 provides the item difficulty distribution and Table 16 gives the item discrimination distribution. Figure 3 illustrates the mean achievement test scores for the three groups.
TABLE 15

ACHIEVEMENT TEST ITEM DIFFICULTY DISTRIBUTION

<table>
<thead>
<tr>
<th>Range</th>
<th>Number of Items</th>
<th>Percentage of Items</th>
</tr>
</thead>
<tbody>
<tr>
<td>.81-1.00</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>.61-.80</td>
<td>4</td>
<td>16</td>
</tr>
<tr>
<td>.41-.60</td>
<td>10</td>
<td>40</td>
</tr>
<tr>
<td>.21-.40</td>
<td>9</td>
<td>36</td>
</tr>
<tr>
<td>.00-.20</td>
<td>2</td>
<td>8</td>
</tr>
</tbody>
</table>

Mean Item Difficulty: .432.

TABLE 16

ACHIEVEMENT TEST ITEM DISCRIMINATION DISTRIBUTION

<table>
<thead>
<tr>
<th>Range</th>
<th>Number of Items</th>
<th>Percentage of Items</th>
</tr>
</thead>
<tbody>
<tr>
<td>.81-1.00</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>.61-.80</td>
<td>5</td>
<td>20</td>
</tr>
<tr>
<td>.41-.60</td>
<td>13</td>
<td>52</td>
</tr>
<tr>
<td>.21-.40</td>
<td>7</td>
<td>28</td>
</tr>
<tr>
<td>.00-.20</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Below .00</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Mean Item Discrimination: .491.
Fig. 3.—Mean Achievement Test Scores.
Fig. 3.—Mean Achievement Test Scores.
The Retention Test

On December 15, 1970 the retentive achievement test (Appendix B) was administered. This test comprised one-half of the final examination for the Mathematics 117 course. It consisted of 20 multiple choice questions, each with five alternatives. This test was given at 8:00 P.M. and time was limited to 50 minutes duration. Each item was scored on the basis of one point; maximum total score possible was 20. Each instructor administered the test to his own students. All tests in this experiment were machine scored to assure uniformity in grading.

Statistics for the retentive achievement test are summarized in Table 17. The highest attained score was 20 and the lowest was 2. The retention test had a mean of 13.45 and a standard deviation of 4.03 for the entire sample. The reliability coefficient for both the Kuder Richardson 20 and the Kuder Richardson 21 was (approximately) .8. For a more detailed presentation of summary statistics on the retention test, see Appendix D.

Table 18 provides the retention test item difficulty distribution; Table 19 indicates the retention test item discrimination distribution. Figure 4 graphically illustrates the means of the treatment groups, the control group, and the total sample.
<table>
<thead>
<tr>
<th></th>
<th>Inductive</th>
<th>Deductive</th>
<th>Control</th>
<th>Total Sample</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Students</td>
<td>57</td>
<td>74</td>
<td>32</td>
<td>163</td>
</tr>
<tr>
<td>Mean</td>
<td>13.26</td>
<td>13.30</td>
<td>14.16</td>
<td>13.45</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>3.98</td>
<td>4.04</td>
<td>4.02</td>
<td>4.03</td>
</tr>
<tr>
<td>Median</td>
<td>13</td>
<td>14</td>
<td>16</td>
<td>14</td>
</tr>
<tr>
<td>Mode</td>
<td>10</td>
<td>13</td>
<td>16</td>
<td>13</td>
</tr>
<tr>
<td>Lowest Score</td>
<td>4</td>
<td>2</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>Highest Score</td>
<td>20</td>
<td>19</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>Range of Scores</td>
<td>16</td>
<td>17</td>
<td>16</td>
<td>18</td>
</tr>
<tr>
<td>Mean Item Difficulty</td>
<td>.337</td>
<td>.335</td>
<td>.292</td>
<td>.327</td>
</tr>
<tr>
<td>Mean Item Discrimination</td>
<td>.483</td>
<td>.490</td>
<td>.483</td>
<td>.509</td>
</tr>
<tr>
<td>Kuder-Richardson 20</td>
<td>.781</td>
<td>.791</td>
<td>.824</td>
<td>.792</td>
</tr>
<tr>
<td>Kuder-Richardson 21</td>
<td>.756</td>
<td>.765</td>
<td>.784</td>
<td>.767</td>
</tr>
</tbody>
</table>
### TABLE 18
**RETENTION TEST ITEM DIFFICULTY DISTRIBUTION**

<table>
<thead>
<tr>
<th>Range</th>
<th>Number of Items</th>
<th>Percentage of Items</th>
</tr>
</thead>
<tbody>
<tr>
<td>.81-1.00</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>.61-.80</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>.41-.60</td>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>.21-.40</td>
<td>14</td>
<td>70</td>
</tr>
<tr>
<td>.00-.20</td>
<td>3</td>
<td>15</td>
</tr>
</tbody>
</table>

Mean Item Difficulty: .327.

### TABLE 19
**RETENTION TEST ITEM DISCRIMINATION DISTRIBUTION**

<table>
<thead>
<tr>
<th>Range</th>
<th>Number of Items</th>
<th>Percentage of Items</th>
</tr>
</thead>
<tbody>
<tr>
<td>.81-1.00</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>.61-.80</td>
<td>5</td>
<td>25</td>
</tr>
<tr>
<td>.41-.60</td>
<td>11</td>
<td>55</td>
</tr>
<tr>
<td>.21-.40</td>
<td>4</td>
<td>20</td>
</tr>
<tr>
<td>.00-.20</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Below .00</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Mean Item Discrimination: .509.
Fig. 4.--Mean Retention Test Scores.
CHAPTER V

EXPERIMENTAL RESULTS

Summary of Procedure

Analysis of covariance was the statistical tool utilized in this study. A comparison between inductive and deductive teaching of limits and derivatives was made by considering the achievement of students in these two treatments. Differences in the retentive achievement of the two treatment groups were also studied. As secondary tests, achievement and retentive achievement of the inductive and deductive treatments were also compared with the achievement and retention of a control group.

A pretest, achievement test, and a retention test were administered to all three groups. The pretest scores and quality points served as covariates for the comparison of achievement of the groups. The pretest and quality points were again utilized as covariates in the comparison of groups with regard to retention.

The Covariates

Analysis of Variance

Analysis of variance was used to determine if the two treatment groups and the control group were equivalent
prior to treatment. Both the pretest scores and the quality points were analyzed. The analysis of variance on the pretest yielded an $F$ of 0.6487 (Table 20). Since this is not significant at the 0.05 level, the three groups were regarded as equivalent with respect to the pretest scores. An $F$ of 1.0935 (Table 21), from the analysis of variance on the quality points, was not significant at the 0.05 level. Again the two treatment groups and the control group were regarded as equivalent with respect to quality points.

**TABLE 20**

**ANALYSIS OF VARIANCE ON PRETEST SCORES**

<table>
<thead>
<tr>
<th>Source</th>
<th>Sum of Squares</th>
<th>df</th>
<th>Mean Squares</th>
<th>$F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Between groups</td>
<td>11.5714</td>
<td>2</td>
<td>5.7857</td>
<td>0.6487</td>
</tr>
<tr>
<td>Within groups</td>
<td>1721.4109</td>
<td>193</td>
<td>8.9192</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>1732.9822</td>
<td>195</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**TABLE 21**

**ANALYSIS OF VARIANCE ON QUALITY POINTS**

<table>
<thead>
<tr>
<th>Source</th>
<th>Sum of Squares</th>
<th>df</th>
<th>Mean Squares</th>
<th>$F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Between groups</td>
<td>24.2097</td>
<td>2</td>
<td>12.1049</td>
<td>1.0935</td>
</tr>
<tr>
<td>Within groups</td>
<td>2136.4832</td>
<td>193</td>
<td>11.0699</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>2160.6929</td>
<td>195</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Correlation Matrix

For further consideration of the two covariates, a correlation matrix was evaluated among the pretest, quality points, the achievement test, and the retention achievement test. Results are found in Table 22. Although there was not a high correlation between the covariates (pretest and quality points) and either the achievement test (.32 and .38, respectively) or the retentive achievement test (.35 and .40, respectively), these correlations were deemed high enough to warrant use of both the pretest and the quality points as covariates in analyses of covariance.

**TABLE 22**

**CORRELATION MATRIX**

(Sample Sizes in Parentheses)

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pretest (1)</td>
<td>1.00000</td>
<td>0.37032</td>
<td>0.32055</td>
<td>0.34818</td>
</tr>
<tr>
<td></td>
<td>(207)</td>
<td>(196)</td>
<td>(207)</td>
<td>(149)</td>
</tr>
<tr>
<td>Quality Points (2)</td>
<td>1.00000</td>
<td>0.38421</td>
<td>0.40837</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(196)</td>
<td>(196)</td>
<td>(142)</td>
<td></td>
</tr>
<tr>
<td>Achievement Test (3)</td>
<td>1.00000</td>
<td></td>
<td>0.56898</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(207)</td>
<td></td>
<td>(149)</td>
<td></td>
</tr>
<tr>
<td>Retentive Achievement Test (4)</td>
<td></td>
<td></td>
<td>1.00000</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(149)</td>
<td></td>
</tr>
</tbody>
</table>
The Analysis of Covariance Model

Analysis of covariance was used on the following data collected from students completing Mathematics 117 during the Fall Quarter, 1971.

1. Subject's quality points (independent variable).
2. Subject's pretest score (independent variable).
3. Subject's achievement test score (dependent variable).
4. Subject's final examination score (dependent variable).

The model for the analysis of covariance is given by:

\[ Y_{i,j,k} = \mu + \alpha_1 + \beta_j + \alpha_1\beta_{1,j} + \gamma_1x_{i,j,k} + \gamma_2y_{i,j,k} + \epsilon_{i,j,k} \]

where \( Y_{i,j,k} \) is the response of the \( k^{th} \) student, with instructor \( i \) and method \( j \), and where

1. \( \mu \) is the overall mean (achievement or retentive achievement).
2. \( \alpha_1 \) is the effect for the \( i^{th} \) method (\( i = 1, 2 \)).
3. \( \beta_j \) is the effect for the \( j^{th} \) instructor (\( j = 1,2,3,4 \)).
4. \( \alpha_1\beta_{1,j} \) is the interaction effect (the effect of method \( i \) with instructor \( j \)).
5. \( \gamma_1 \) is the regression coefficient of \( x \).
6. \( x_{i,j,k} \) is the quality points for the \( k^{th} \) student in method \( j \) with instructor \( i \).
7. \( \gamma_2 \) is the regression coefficient of \( y \).
8. \( y_{i,j,k} \) is the pretest score for the \( k^{th} \) student in method \( j \) with instructor \( i \).
Test Results

Four teachers were assigned classes across the two treatment groups while two additional teachers taught the control groups. Because of this arrangement, it was necessary to run a two-way analysis of covariance comparing just the two treatment groups as well as a one-way analysis of covariance comparing the two treatment groups and the control group. Results of these tests follow.

Hypothesis 1. There is no significant difference in the mathematical achievement of the Concrete Inductive Group and the Abstract Deductive Group.

Results of the two-way analysis of covariance on the achievement test, using the pretest and quality points as covariates, are summarized in Table 23. In checking for

<table>
<thead>
<tr>
<th>Source</th>
<th>SS</th>
<th>DF</th>
<th>MS</th>
<th>F</th>
<th>P less than</th>
</tr>
</thead>
<tbody>
<tr>
<td>Within Cells</td>
<td>2527.802</td>
<td>147</td>
<td>17.196</td>
<td>0.001</td>
<td>0.001</td>
</tr>
<tr>
<td>Regression</td>
<td>904.288</td>
<td>2</td>
<td>452.144</td>
<td>26.294</td>
<td>0.001</td>
</tr>
<tr>
<td>Methods</td>
<td>1.278</td>
<td>1</td>
<td>1.278</td>
<td>0.074</td>
<td>0.786</td>
</tr>
<tr>
<td>Teachers</td>
<td>121.478</td>
<td>3</td>
<td>40.493</td>
<td>2.355</td>
<td>0.074</td>
</tr>
<tr>
<td>Interactions</td>
<td>98.192</td>
<td>3</td>
<td>32.731</td>
<td>1.903</td>
<td>0.132</td>
</tr>
</tbody>
</table>
interactions between teachers and methods, an F ratio of 1.903 was derived. This value is not significant at the 0.05 level. Consequently, interactions between teachers and methods were not shown to be appreciable. Although Hypothesis 1 did not pertain to teacher differences, it is interesting to note that the F ratio of 2.355 is not significant at the 0.05 level; thus, no significant teacher differences were found.

The test comparing the achievement of the concrete inductive and the abstract deductive groups yielded an F ratio of 0.074; this number is not significant at the 0.05 level. Hence, at the 5% level we cannot reject Hypothesis 1.

Hypothesis 2. There is no significant difference in the mathematical achievement of the Concrete Inductive Group and the Control Group.

Hypothesis 3. There is no significant difference in the mathematical achievement of the Abstract Deductive Group and the Control Group.

To compare achievement of the two treatment groups and the Control Group, a one-way analysis of covariance was used on the achievement test using the pretest and quality points as covariates. This analysis produced an F of 0.6470 (Table 24). Since this F is not significant at the 0.05 level, neither Hypotheses 2 nor 3 can be rejected.
TABLE 24
ANALYSIS OF COVARIANCE ON ACHIEVEMENT TEST

<table>
<thead>
<tr>
<th>Sum of Squares</th>
<th>df</th>
<th>Mean Squares</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equality of adjusted cell means</td>
<td>25.1716</td>
<td>2</td>
<td>12.5858</td>
</tr>
<tr>
<td>Zero slope</td>
<td>870.8250</td>
<td>2</td>
<td>435.4124</td>
</tr>
<tr>
<td>Error</td>
<td>3715.7258</td>
<td>191</td>
<td>19.4541</td>
</tr>
<tr>
<td>Equality of slopes</td>
<td>75.8325</td>
<td>4</td>
<td>18.9581</td>
</tr>
<tr>
<td>Error</td>
<td>3639.8933</td>
<td>187</td>
<td>19.4647</td>
</tr>
</tbody>
</table>

Hypothesis 4. There is no significant difference in the retentive achievement of the Concrete Inductive Group and the Abstract Deductive Group.

Results of a two-way analysis of covariance on the final examination are summarized in Table 25. It is

TABLE 25
TWO-WAY ANALYSIS OF COVARIANCE ON RETENTION TEST

<table>
<thead>
<tr>
<th>Source</th>
<th>SS</th>
<th>DF</th>
<th>MS</th>
<th>F</th>
<th>P less than</th>
</tr>
</thead>
<tbody>
<tr>
<td>Within Cells</td>
<td>1309.902</td>
<td>102</td>
<td>12.842</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Regression</td>
<td>375.427</td>
<td>2</td>
<td>187.713</td>
<td>14.617</td>
<td>0.001</td>
</tr>
<tr>
<td>Methods</td>
<td>2.366</td>
<td>1</td>
<td>2.366</td>
<td>0.184</td>
<td>0.669</td>
</tr>
<tr>
<td>Teachers</td>
<td>14.660</td>
<td>3</td>
<td>4.887</td>
<td>0.381</td>
<td>0.767</td>
</tr>
<tr>
<td>Interactions</td>
<td>5.717</td>
<td>3</td>
<td>1.906</td>
<td>0.148</td>
<td>0.931</td>
</tr>
</tbody>
</table>
obvious from this table that neither methods ($F = 0.184$),
teachers ($F = 0.381$), nor interactions ($F = 0.148$) dis-
played significant differences at the 0.05 level. Hence,
Hypothesis 4 cannot be rejected.

Hypothesis 5. There is no significant difference in the
retentive achievement of the Concrete Inductive Group and
the Control Group.

Hypothesis 6. There is no significant difference in the
retentive achievement of the Abstract Deductive Group and
the Control Group.

In comparing the retentive achievement of the two
treatment groups and the Control Group, a one-way analysis
of covariance was utilized using the pretest scores and
quality points as covariates. Results of this test yielded
an $F$ value of 0.7404 (Table 25). Since this value is not

<table>
<thead>
<tr>
<th>Source</th>
<th>Sum of Squares</th>
<th>df</th>
<th>Mean Squares</th>
<th>$F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equality of adjusted cell means</td>
<td>19.2429</td>
<td>2</td>
<td>9.6215</td>
<td>0.7404</td>
</tr>
<tr>
<td>Zero slope</td>
<td>466.8853</td>
<td>2</td>
<td>233.4426</td>
<td>17.9632</td>
</tr>
<tr>
<td>Error</td>
<td>1767.4063</td>
<td>136</td>
<td>12.9956</td>
<td></td>
</tr>
<tr>
<td>Equality of slopes</td>
<td>25.6118</td>
<td>4</td>
<td>6.4030</td>
<td>0.4852</td>
</tr>
<tr>
<td>Error</td>
<td>1741.7944</td>
<td>132</td>
<td>13.1954</td>
<td></td>
</tr>
</tbody>
</table>

significant at the 0.05 level, we cannot reject Hypotheses
5 or 6.
Additional Statistical Analyses

Because most of the previous statistical results indicated no significant differences in achievement and retentive achievement of the three groups involved, it was of interest to investigate differences in achievement and retentive achievement among subsets of the original sample. A one-way analysis of covariance was used to compare the achievement and the retentive achievement of all students scoring among the upper 25 percent on the achievement test and again on the retention test; both the pretest and quality points were used as covariates.

Table 27 shows the results of the analysis of covariance for the achievement test. Since an $F$ of 1.4942 is not significant at the 0.05 level, there were no significant differences in achievement for the three groups.

| TABLE 27 |
| ANALYSIS OF COVARIANCE |
| (ACHIEVEMENT TEST—UPPER 25 PERCENT) |

<table>
<thead>
<tr>
<th>Source</th>
<th>Sum of Squares</th>
<th>df</th>
<th>Mean Squares</th>
<th>$F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equality of adjusted cell means</td>
<td>16.5144</td>
<td>2</td>
<td>8.2572</td>
<td>1.4942</td>
</tr>
<tr>
<td>Zero slope</td>
<td>41.4976</td>
<td>2</td>
<td>20.7488</td>
<td>3.7547</td>
</tr>
<tr>
<td>Error</td>
<td>303.9341</td>
<td>55</td>
<td>5.5261</td>
<td></td>
</tr>
<tr>
<td>Equality of slopes</td>
<td>14.1870</td>
<td>4</td>
<td>3.5468</td>
<td>.6243</td>
</tr>
<tr>
<td>Error</td>
<td>289.7471</td>
<td>51</td>
<td>5.6813</td>
<td></td>
</tr>
</tbody>
</table>
In comparing the retentive achievement of those students scoring among the upper 25 percent on the final examination, an F of 3.7253 was found (Table 28). This value was significant at the 0.05 level. Because this indicated a significant difference in achievement among the three groups, Tukey's method of comparing the adjusted means of any two groups was applied.

### Table 28

**Analysis of Covariance (Retention Test—Upper 25 Percent)**

<table>
<thead>
<tr>
<th>Source</th>
<th>Sum of Squares</th>
<th>df</th>
<th>Mean Squares</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equality of adjusted cell means</td>
<td>4.8777</td>
<td>2</td>
<td>2.4389</td>
<td>3.7253</td>
</tr>
<tr>
<td>Zero slope</td>
<td>7.6759</td>
<td>2</td>
<td>3.8379</td>
<td>5.8623</td>
</tr>
<tr>
<td>Error</td>
<td>20.9497</td>
<td>32</td>
<td>0.6547</td>
<td></td>
</tr>
<tr>
<td>Equality of slopes</td>
<td>3.3977</td>
<td>4</td>
<td>0.8494</td>
<td>1.3550</td>
</tr>
<tr>
<td>Error</td>
<td>17.5521</td>
<td>28</td>
<td>0.6269</td>
<td></td>
</tr>
</tbody>
</table>

To determine if implicated method differences were significant, Tukey's test of multiple pairwise comparisons was employed. This test is applicable only after a significant F value results from a main effect (in this case, methods) from an analysis of variance. This method engages the studentized range statistics (q) which is defined by:
\[ q = \frac{|X_1 - X_2|}{\sqrt{\frac{MS_{error}}{n}}} \]

where \( X_1 \) is the adjusted mean of the first group and \( X_2 \) is the adjusted mean of the remaining group. \( MS_{error} \) is the mean square error appearing on the analysis of covariance table and \( n = (n_1 + n_2)/2 \) while \( n_1 \) is the number of observations in group one and \( n_2 \) is the number of observations in the second group. The number of groups is the degrees of freedom for the numerator; the degrees of freedom of the denominator are the same as for \( MS_{error} \).

Table 29 indicates the means and the adjusted means for the three groups under consideration. Using these

| TABLE 29 |
| ADJUSTED MEANS |
| (UPPER 25 PERCENT ON RETENTION) |

<table>
<thead>
<tr>
<th>Group</th>
<th>N</th>
<th>Group Mean</th>
<th>Adjusted Group Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deductive</td>
<td>15</td>
<td>17.73328</td>
<td>17.58560</td>
</tr>
<tr>
<td>Inductive</td>
<td>13</td>
<td>18.15381</td>
<td>18.22383</td>
</tr>
<tr>
<td>Control</td>
<td>9</td>
<td>18.33328</td>
<td>18.47818</td>
</tr>
</tbody>
</table>

adjusted mean values and the mean square for error as indicated in the analysis of variance table, a \( q \) for Tukey's test was evaluated for each pair of groups. In comparing the inductive treatment with the deductive treatment, a
q of 2.90 was arrived at. This value is significant at the 0.05 level indicating a significant difference in the two measures as determined by adjusted retentive achievement test means. Since the inductive group produced the higher adjusted mean, it is concluded that, for students scoring in the upper 25 percent on the retentive achievement test, students taught by an inductive approach achieve higher than students taught by a deductive approach.

A similar comparison between the inductively taught group and the Control Group yielded a q of 1.0 which is not significant (p<0.05). Hence, there was no significant difference in retentive achievement between the Concrete Inductive Group and the Control Group.

A q of 3.82 resulted when the deductive group and the Control Group were compared. Since this value is significant (p<0.05), the adjusted means for these two approaches were again considered. Because the adjusted mean of the Control Group was greater than that of the Deductive Treatment Group, it was concluded that, for students scoring among the upper 25 percent on the final examination, the Control Group achieved significantly higher than the Deductive Treatment Group.

Thus, for all students scoring in the upper 25 percent on the retentive achievement test, students in both the Concrete Inductive Group and the Control Group achieved significantly higher than students in the Abstract Deductive
There was no significant difference in the retentive achievement of the inductive treatment group and the Control Group.

In addition to investigating the achievement of students scoring among the upper 25 percent on the achievement and retentive achievement tests, a study was made of those students scoring in the lower 25 percent.

For those students scoring among the lower 25 percent on the achievement test, a one-way analysis of covariance was used to test results. Again the pretest scores and quality points were used as covariates. Table 30 illustrates the results of this test. An $F$ of 0.9673, not significant at the 0.05 level, indicates no significant difference in achievement among the three groups.

<table>
<thead>
<tr>
<th>Source</th>
<th>Sum of Squares</th>
<th>df</th>
<th>Mean Squares</th>
<th>$F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equality of adjusted cell means</td>
<td>6.7914</td>
<td>2</td>
<td>3.3957</td>
<td>0.9673</td>
</tr>
<tr>
<td>Zero slope</td>
<td>8.9796</td>
<td>2</td>
<td>4.4808</td>
<td>1.2790</td>
</tr>
<tr>
<td>Error</td>
<td>133.3985</td>
<td>38</td>
<td>3.5105</td>
<td></td>
</tr>
<tr>
<td>Equality of slopes</td>
<td>38.8964</td>
<td>4</td>
<td>9.7241</td>
<td>3.4985</td>
</tr>
<tr>
<td>Error</td>
<td>94.5022</td>
<td>34</td>
<td>2.7795</td>
<td></td>
</tr>
</tbody>
</table>
Finally, a one-way analysis of covariance was again used to determine differences in retentive achievement for the two treatment groups and the Control Group for those scoring among the lower 25 percent on the final examination. The analysis yielded an $F$ of 2.0299 (Table 31). This was not significant at the 0.05 level; there were no significant differences in retentive achievement among the three groups.

**TABLE 31**

ANALYSIS OF COVARIANCE
(RETENTION TEST—LOWER 25 PERCENT)

<table>
<thead>
<tr>
<th>Source</th>
<th>Sum of Squares</th>
<th>df</th>
<th>Mean Squares</th>
<th>$F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equality of adjusted cell means</td>
<td>16.8520</td>
<td>2</td>
<td>8.4260</td>
<td>2.0299</td>
</tr>
<tr>
<td>Zero slope</td>
<td>0.3951</td>
<td>2</td>
<td>0.1975</td>
<td>0.0476</td>
</tr>
<tr>
<td>Error</td>
<td>132.8281</td>
<td>32</td>
<td>4.1509</td>
<td></td>
</tr>
<tr>
<td>Equality of slopes</td>
<td>4.1168</td>
<td>4</td>
<td>1.0292</td>
<td>0.2239</td>
</tr>
<tr>
<td>Error</td>
<td>128.7112</td>
<td>28</td>
<td>4.5968</td>
<td></td>
</tr>
</tbody>
</table>
CHAPTER VI

SUMMARY, RESULTS AND INTERPRETATIONS

Summary

The purpose of this study was to evaluate achievement and retentive achievement in learning limit, derivative, and continuity concepts as taught by an inductive approach compared to a deductive approach. The experiment was an extension of the studies conducted by Lackner (26) and Shelton (39).

Two treatments were utilized. In the first, instructors taught the concepts, limits, continuity, and derivatives, by a deductive approach; in the second, an inductive instructional approach was used to teach these same concepts. To control the teacher variable, each teacher taught two sections, one using each approach. To insure uniformity within a treatment, instructors followed lesson plans provided by the experimenter. A Control Group employing no special treatment was also included in the study.

This study took place at The Ohio State University, Columbus, Ohio during the Fall Quarter, 1970. Students enrolled in the freshman level course, Mathematics 117,
Introduction to Calculus with Economic Applications, participated in the experiment. The students had previously completed the same one-quarter prerequisite mathematics course; thus, each student was assumed to have had the necessary mathematical background for learning the limit, continuity, and derivative concepts. Students were randomly assigned to treatments.

To better understand the pertinent characteristics of the sample, questionnaires providing background information were completed by each student involved. These were filled out on the first day of class. A pretest covering prerequisite material was administered during the regular class period (either 12:00 noon or 4:00 P.M.) on the second day of the class. In addition to the pretest scores, quality points were collected for each student.

An achievement test on the limit, continuity, and derivative concepts was used to test for achievement. Analysis of covariance was applied with the pretest and quality points serving as covariates. This achievement test was developed from the achievement test used in the Pilot Study. The achievement test for the Pilot Study consisted of 25 multiple choice items. Each item had five possible choices, one of which was correct. These 25 items were selected from equivalent items used on previous midterms and examinations for Mathematics 117. Items were selected on
the bases of their discrimination index and difficulty level.

The achievement test was administered to all students from 5-6:00 P.M. on the twenty-second class day of the quarter. It consisted of 25 multiple choice items each of which was equivalent to an item used on the achievement test in the Pilot Study. However, five items were replaced or revised because their discrimination indices were poor.

To compare retentive achievement of the three groups, a retention test was administered. Analysis of covariance was utilized with the pretest and quality points as covariates. The retention test, which comprised part of the final examination for the Mathematics 117 course, was given to all students from 8-10:00 P.M. forty-five days after the criterion test. The retention test consisted of 20 multiple choice questions each with five alternates. One-half the items were taken directly from the achievement test on limits, continuity, and derivatives; the second half consisted of items equivalent to 10 different items on the achievement test.

Results

Main Study

Analysis of covariance was used to test six hypotheses in this experiment. In the test for achievement, results indicated the following:
1. There was no significant difference in the mathematical achievement of the Concrete Inductive Group and the Abstract Deductive Group.

2. There was no significant difference in the mathematical achievement of the Concrete Inductive Group and the Control Group.

3. There was no significant difference in the mathematical achievement of the Abstract Deductive Group and the Control Group.

In the test for retentive achievement, the researcher found:

1. There was no significant difference in the mathematical retentive achievement of the Concrete Inductive Group and the Abstract Deductive Group.

2. There was no significant difference in the mathematical retentive achievement of the Concrete Inductive Group and the Control Group.

3. There was no significant difference in the mathematical retentive achievement of the Abstract Deductive Group and the Control Group.

Additional Analyses

In additional analyses, analysis of covariance was used to compare the achievement and the retentive achievement of those students scoring among the upper 25 percent on the achievement test and on the final examination.
Results for this subgroup of the original sample were:

1. There was no significant difference in the mathematical achievement of the Concrete Inductive Group and the Abstract Deductive Group.

2. There was no significant difference in the mathematical achievement of the Concrete Inductive Group and the Control Group.

3. There was no significant difference in the mathematical achievement of the Abstract Deductive Group and the Control Group.

4. The Concrete Inductive Group scored significantly higher in mathematical retentive achievement than the Abstract Deductive Group.

5. There was no significant difference in the mathematical retentive achievement of the Concrete Inductive Group and the Control Group.

6. The Control Group scored significantly higher in mathematical retentive achievement than the Abstract Deductive Group.

Comparisons of all students in the three groups scoring among the lower 25 percent were:

1. There was no significant difference in the mathematical achievement of the Concrete Inductive Group and the Abstract Deductive Group.
2. There was no significant difference in the mathematical achievement of the Concrete Inductive Group and the Control Group.

3. There was no significant difference in the mathematical achievement of the Abstract Deductive Group and the Control Group.

4. There was no significant difference in the mathematical retentive achievement of the Concrete Inductive Group and the Abstract Deductive Group.

5. There was no significant difference in the mathematical retentive achievement of the Concrete Inductive Group and the Control Group.

6. There was no significant difference in the mathematical retentive achievement of the Abstract Deductive Group and the Control Group.

Limitations and Delimitations

The limitations of this study involved certain aspects of the sample, the instructors, and the instruments. The two instructional approaches, inductive and deductive, might have been totally new to the students. Results might differ if the students had had previous experience with these methods. Results might also be affected if these same methods were applied to students with no previous college mathematics courses; the students in this study had previously taken one or two college mathematics courses.
Students could have discussed the material among themselves or with students in the other treatment groups or the Control Group. They might have consulted other instructors, tutors, or numerous textbooks. The mathematical maturity of the subjects might also have affected results. Perhaps if students enrolled in more sophisticated mathematics courses were instructed by the abstract deductive and concrete inductive approaches over the same or different concepts as in this study, results would more clearly favor one approach to the other.

An attempt was made to control the teacher variable by having instructors teach one class in each treatment group. Because instructors for the experimental groups were involved in teaching both approaches, daily teacher preparation was greatly increased. Instructors indicated that, at times, it was difficult to recall how certain topics were approached for each class. This confusion might have affected the experimental results.

The achievement test and the retentive achievement test might not have measured changes in achievement resulting from instructional methods. In addition, an extension of the treatments over a longer period of time or over more concepts might find one teaching approach more effective than the other.
Interpretations and Implications for Further Research

In this experiment, instructors followed lesson plans written from both a concrete inductive approach and an abstract deductive approach. Results of the Main Study clearly indicate that students taught by either of these methods did achieve as well as students learning the same mathematical concepts under traditional instruction. Results for achievement showed no significance in favor of one method over another. Students taught under the inductive method or under the deductive method performed as well on a retentive achievement test as did students in the Control Group. Additional related studies in mathematics on the college level should be performed.

This study, covering only limits, continuity, and derivatives, was rather limited in the scope of its mathematical content. A study could be conducted with more or different mathematical topics such as matrices, number theory, analysis, etc. Or, rather than having one treatment exclusively inductive and the other exclusively deductive, one could develop treatments using various combinations of methods.

This study could be replicated with all high or low ability students. Consideration of such things as sex differences, grade level differences and attitudes, or subjects of different ability levels could be incorporated
into the study.

One might also replicate this study by assigning instructors to teach one treatment group, rather than two. This would alleviate problems of double preparation for the same content. An extension of the treatment period could be implemented. Rather than college freshmen, mathematics majors or graduate students might comprise the sample. Or one could conduct an experiment using only instructors who obviously favor one method over another.
APPENDIX A

QUESTIONNAIRES
Questionnaire 1

Name ____________________ SSN__________ __ Male __ Female

OSU Address _________________ Phone ________________ Age________

Entered OSU ______ (Quarter), ________ (Year).

Rank _Fr _ S _ Jr _Sr

Transferred from another university or college to OSU:

__Yes __ NO

H.S. Background: __________ (HS), __________ (City), __________ (St)

<table>
<thead>
<tr>
<th>Course</th>
<th># of Sem</th>
<th>Grade</th>
<th>Other Math Courses: (Describe Content)</th>
<th># of Sem.</th>
<th>Grade</th>
</tr>
</thead>
<tbody>
<tr>
<td>Algebra I</td>
<td>________</td>
<td>______</td>
<td>________________________________________</td>
<td>________</td>
<td>______</td>
</tr>
<tr>
<td>Algebra II</td>
<td>________</td>
<td>______</td>
<td>________________________________________</td>
<td>________</td>
<td>______</td>
</tr>
<tr>
<td>Geometry</td>
<td>________</td>
<td>______</td>
<td>________________________________________</td>
<td>________</td>
<td>______</td>
</tr>
<tr>
<td>Trigonometry</td>
<td>________</td>
<td>______</td>
<td>________________________________________</td>
<td>________</td>
<td>______</td>
</tr>
<tr>
<td>Prob. &amp; Stat.</td>
<td>________</td>
<td>______</td>
<td>________________________________________</td>
<td>________</td>
<td>______</td>
</tr>
</tbody>
</table>

List any previous college math courses, grades and, if not OSU, school.

__________________________________________________________

When did you complete your last math course (H.S. or college)?

__________ (Mo.) __________ (Yr.)

List academic area(s) in which you plan to major ____________________________

are interested ____________________________

have most difficulty ____________________________
Indicate when you took the mathematics placement exam. 19_____
___ Autumn ___ Winter ___ Spring ___ Summer
Indicate when you (last) took Mathematics 116. 19_____
___ Autumn ___ Winter ___ Spring ___ Summer
Give your instructor's name for Mathematics 116. _______
Indicate the grade you received in Mathematics 116.
___ A ___ B ___ C ___ D ___ E ___ PA ___ NP
Are you repeating Mathematics 117? ___ Yes ___ No
I. Identification Data

A. Print your name in the space provided on the response sheet and darken the matching grid below your name.

B. Write the numeric portion (five numbers) of your student number (found on your schedule or fee cards) in the first five columns of the Student Number section. Enter a zero in column six. Leave columns seven and eight blank. Fill in the grid.

C. Your classroom teacher will provide you with the section number. Enter this number in the space provided and fill in the grid.

D. Identify your sex. Omit the section on "Test Form."

E. Use the section as shown to identify your first quarter of enrollment at OSU. In filling out the grid use the following key for the Quarter you entered: 1) Summer, 2) Autumn, 3) Winter, 4) Spring

F. Indicate your course number and instructor's name.

G. In the blank marked "Campus" identify the college you are now enrolled in.

II. Background Information

In responding to questions in this section, use items 1-25 on the response sheet.

1. Did you transfer to Ohio State from some other college or university? a) yes, b) no

2. How many years of high school mathematics (starting with Algebra I) have you taken? a) 1, b) 2, c) 3, d) 4, e) 5 or more
3. What was your average grade in high school mathematics courses: a) A, b) B, c) C, d) D, e) F

4-5. When was the last time you took a math. course (high school or college)?
   4. a) 1970, b) 1969, c) 1968, d) 1967, e) 1966 or before
   5. a) Autumn, b) Summer, c) Spring, d) Winter

6. Have you taken any previous math courses in college?
   a) yes, b) no If your answer is b), go to Section III

7-26. In the items which follow you are asked to indicate math courses (identified by OSU course number) which you have taken at the college level. Only answer items for courses you have taken. Leave other blank. Each item consists of two sections with format as follows:
   (Course) a) took at OSU for grade, b) took at another school for grade, c) audited or enrolled as PA/NP, d) audited at another school, e) received proficiency credit.

   (Grade) a) A, b) B, c) C, d) D, e) E Answer this part only if answer to first part is a) or b).

   If you took a course more than once, provide information for the last time you took a given course.

   7-8  101    17-18  122
   9-10 105    19-20  123
   11-12 116    21-22  150
   13-14 117    23-24  151
   15-16 121    25-26  152

III. The Department of Mathematics is interested in providing experiences and opportunities which are responsive to individual student's interests, strengths, and needs. Listed below are academic areas of study available to students at OSU. You are asked to respond to individual items according to whether you
   a) are interested as a possible major area of study
   b) do not plan to major, but have a special interest or liking
   c) have no special interest or difficulty
   d) have some difficulty or dislike
   e) have considerable difficulty or dislike
<table>
<thead>
<tr>
<th>Number</th>
<th>Subject</th>
</tr>
</thead>
<tbody>
<tr>
<td>41</td>
<td>Mathematics, stat or CIS</td>
</tr>
<tr>
<td>42</td>
<td>Physical Sciences</td>
</tr>
<tr>
<td>43</td>
<td>Engineering</td>
</tr>
<tr>
<td>44</td>
<td>Biological Sciences</td>
</tr>
<tr>
<td>45</td>
<td>Adm. Sciences</td>
</tr>
<tr>
<td>46</td>
<td>Social Sciences</td>
</tr>
<tr>
<td>47</td>
<td>Behavioral Sciences</td>
</tr>
<tr>
<td>48</td>
<td>Fine Arts</td>
</tr>
<tr>
<td>49</td>
<td>English, Speech, Journalism</td>
</tr>
<tr>
<td>50</td>
<td>Foreign Languages</td>
</tr>
<tr>
<td>51</td>
<td>Humanities</td>
</tr>
<tr>
<td>52</td>
<td>Elementary Education</td>
</tr>
<tr>
<td>53</td>
<td>Secondary Education</td>
</tr>
<tr>
<td>54</td>
<td>Agriculture</td>
</tr>
<tr>
<td>55</td>
<td>Veterinary Medicine</td>
</tr>
<tr>
<td>56</td>
<td>Dentistry</td>
</tr>
<tr>
<td>57</td>
<td>Medicine</td>
</tr>
<tr>
<td>58</td>
<td>Optometry</td>
</tr>
<tr>
<td>59</td>
<td>Pharmacy</td>
</tr>
<tr>
<td>60</td>
<td>Law</td>
</tr>
<tr>
<td>61</td>
<td>Physical Education</td>
</tr>
<tr>
<td>62</td>
<td>Military, Air, or Naval Sciences</td>
</tr>
</tbody>
</table>
APPENDIX B

PRETEST, ACHIEVEMENT TEST, AND RETENTION TEST
1. Let \( e = 0.056 \). Then a positive number less than \( e \) is
   (a) \( 2e \) (b) \( e/0.1 \) (c) \( e - 0.1 \) (d) \( e^2 \) (e) None of these

2. A rational number between \( 1/7 \) and \( 1/8 \) is
   (a) \( 15/112 \) (b) \( 11/96 \) (c) \( 16/105 \) (d) \( 16/129 \) (e) None of these

3. The solution set for \( (x + 1)/x \leq 0 \) is:
   (a) \( \{x \leq -1\} \) (b) \( \{x > -1\} \) (c) \( \{x < 0\} \)
   (d) \( \{x \leq -1 \text{ or } x > 0\} \) (e) \( \{1 \leq x \leq 0\} \)

4. The solution set for \( x^2 \geq 25 \) is:
   (a) \( \{x > 5\} \) (b) \( \{x > -5\} \) (c) \( \{-5 \leq x \leq 5\} \)
   (d) \( \{x > 5 \text{ or } x < -5\} \) (e) None of these

5. The solution set for \( \frac{2x + 1}{x - 1} \leq 1 \) is
   (a) \( \{x \leq 1\} \) (b) \( \{x \leq -2\} \) (c) \( \{x \geq -2\} \)
   (d) \( \{-2 \leq x \leq 1\} \) (e) None of these

6. The solution for \( |x^2 - x - 2| = x^2 - x - 2 \) is
   (a) \( \{x \leq -1\} \) (b) \( \{-1 \leq x \leq 2\} \) (c) \( \{x \leq -1 \text{ or } x \geq 2\} \)
   (d) \( R \) (e) \( \emptyset \)

7. If \( y = 1/x \), then for all \( x \) such that \( 1.9 < x < 2.1 \)
   (a) \( 10/21 < y < 10/19 \) (b) \( y > x \) (c) \( y < 1/2 \) (d) \( y = 2 \)
   (e) None of these

8. If \( f(x) = 3x + 7 \) and \( g(x) = x^2 \), then \( f[g(x)] \) equals
   (a) \( 3x^2 + 7 \) (b) \( 9x^2 + 42x + 49 \) (c) \( 3x^3 + 7x^2 \)
   (d) \( x^2 + 3x + 7 \) (e) None of these

9. If \( f(x) = x^2 \), the \( \frac{f(x + h) - f(x)}{h} \) equals (if \( h \neq 0 \))
   (a) \( 1 \) (b) \( h(2x + 1) \) (c) \( h \) (d) \( 2x + h \) (e) None of these

10. The domain of \( f(x) = \sqrt{x^2 - 4} \) is
    (a) \( \{x \geq 2\} \) (b) \( R \) (c) \( \{x \geq 2\} \cup \{x \geq -2\} \)
    (d) \( \{-2 \leq x \leq 2\} \) (e) None of these

11. The domain of \( H(x) = \frac{x^2 + 3x - 1}{x - 2} \) is
    (a) \( \{x < 2\} \) (b) \( \{x > 2\} \) (c) \( R \) (d) \( \emptyset \) (e) None of these
12. The shaded region is represented by
(a) \(|x - 1| < 0.3\)
(b) \(x > 0.7\)
(c) \(x < 1.3\)
(d) \(|x| < 0.3\)
(e) None of these

13. The graph of \(y = |x + 3|\) is best represented by
(a) 
(b) 
(c) 
(d) 
(e) 

14. The graph of \(y + |x| > 3\) is best represented by
(a) 
(b) 
(c) 
(d) 
(e)
15. The slope of the line perpendicular to \(2y + x = 5\) is
   (a) \(-\frac{1}{2}\) (b) \(\frac{1}{2}\) (c) \(2\) (d) \(-2\) (e) None of these

16. The equation of the line through \((3, -1)\) which is perpendicular to \(y + 3x = 5\) is
   (a) \(x + 3y = 0\) (b) \(3y - x + 6 = 0\) (c) \(3y - x = 0\)
   (d) \(3y - x + 4 = 0\) (e) None of these

17. The equation of the line through \((2, 5)\) which is parallel to the line \(y + 2x = 3\) is
   (a) \(y - 2x + 5 = 0\) (b) \(y + 2x = 5\) (c) \(y + 2x = 15\)
   (d) \(y - 2x + 15 = 0\) (e) None of these

18. Consider the sequence \(\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}\) where the \(n^{th}\) number, \(a_n = \frac{1}{n}\) for all positive integers \(n\).
   Then for \(n > 100\),
   (a) \(a_n > \frac{1}{10}\) (b) \(|a_n + 1 - a_n| > 0.1\) (c) \(a_n < \frac{1}{100}\)
   (d) \(a_n + \frac{1}{a_n} > 1\) (e) None of these

19. The solution set for \(|x - 3| \geq -2\) is
   (a) \(x \geq 1\) (b) \(x \leq -5\) (c) \(R\) (d) \(\emptyset\)
   (e) None of these

20. The region(s) where both inequalities \(y > x^2 - 4\) and \(x < y - 1\) hold is (are)
   (a) A
   (b) B
   (c) C
   (d) D and E
   (e) None of these
ACHIEVEMENT TEST

Directions: Select the one best response to each of the following questions. In all questions involving limits, if the limit does not exist (D.N.E.), select $\infty$ or $-\infty$ if applicable. Be sure your name, section number, and student number are coded in the proper area on the answer sheet.

In problems 1, 2 and 3 $f: [1, 5] \to \mathbb{R}$ as given in the figure.

1. $\lim_{x \to 2} f(x) =$ a) 1 b) $f(2)$ c) 2 d) D.N.E. e) None of these

2. $\lim_{x \to 3} f(x) =$ a) 1 b) $f(3)$ c) 2 d) $-\infty$ e) D.N.E.

3. $\lim_{x \to 4} f(x) =$ a) 2 b) 3 c) 1 d) $\infty$ e) D.N.E.

In problems 4 and 5 $h(x)$ is defined on $[1, 5]$ by

$$h(x) = \begin{cases} \lfloor x \rfloor & \text{if } 1 \leq x \leq 3 \\ x + 1 & \text{if } 3 < x < 5 \\ 4 & \text{if } x = 5 \end{cases}$$

4. $\lim_{x \to 2} h(x) =$ a) 1 b) 2 c) 3 d) 4 e) D.N.E.

5. $\lim_{x \to 5} h(x) =$ a) 6 b) 5 c) 4 d) 3 e) D.N.E.
In problems 6-10, find the indicated limits

6. \( \lim_{x \to 6} \frac{x - 6}{x^2 - x - 30} \)  
   a) 1/11  b) 1  c) \( \infty \)  d) D.N.E.  
   e) None of these

7. \( \lim_{h \to 0} \frac{(x + h)^2 - x^2}{h} \)  
   a) 0  b) 2x  c) \( \infty \)  d) D.N.E.  
   e) None of these

8. \( \lim_{x \to 4} \frac{x^2 - x - 12}{(x - 4)^2} \)  
   a) \( \infty \)  b) \(-\infty\)  c) 0  d) 1  e) D.N.E.

9. \( \lim_{n \to \infty} \frac{4 - 3n + 2n^2}{n^3 + n - 3} \)  
   a) 2  b) -4/3  c) 0  d) \( \infty \)  e) D.N.E.

10. \( \lim_{n \to \infty} \frac{4n^3 + 2n - 3}{3 - n^3} \)  
    a) 4  b) -4  c) \( \infty \)  d) 0  e) -1

11. Recall the definition:

A sequence \( \{a_n\} \) approaches a number \( A \) as a limit if for every \( \varepsilon > 0 \) there is a positive integer \( N \) such that for all \( n > N \), \( |a_n - A| < \varepsilon \).

Consider \( \lim_{n \to \infty} \frac{4}{n} = 0 \). If \( \varepsilon = 1/5 \), which of the following is the first natural number that could be used as \( N \)?
   a) 1  b) 5  c) 6  d) 8  e) 21

12. The graph of \( y = \frac{x^2 - 9}{x^2 - 36} \) is discontinuous at the points with x-coordinates of:
   a) 3 and -6  b) -3 and 6  c) 3 and -3  d) 6 and -6  
   e) 3, -3, 6, and -6
13. If \( y = -3x^{-2/3} + x^{1/3} + 3x^2 \), then \( \frac{\partial^2 y}{\partial x^2} = \)

a) \[ 2x^{1/3} + \frac{1x^{4/3}}{3} + 6x^3 \]

b) \[ 2x^{-5/3} + \frac{1x^{-1/3}}{3} + 6x \]

c) \[ 2x^{1/3} + \frac{1x^{-2/3}}{3} + 6x \]

d) \[ 2x^{-5/3} + \frac{1x^{-2/3}}{3} + 6x \]

e) \[ -3x^{-5/3} + x^{-2/3} + 3x \]

14. If \( y = 2x(3 - x^2)^8 \), then \( y' = \)

a) \[ 16x(3 - x^2)^7 \]

b) \[ -32x^2(3 - x^2)^7 \]

c) \[ 2x(3 - x^2)^7 + (3 - x^2)^8(2x) \]

d) \[ -4x^2(3 - x^2)^7 + (3 - x^2)^8(2) \]

e) \[ -32x^2(3 - x^2)^7 + 2(3 - x^2)^8 \]

15. If \( y = \frac{1}{4}(3t - 1)^8 \), then \( y' = \)

a) \[ 2(3t - 1)^7 \]

b) \[ 6(3t - 1)^7 \]

c) \[ 6(3t - 1) \]

d) \[ 2(3t - 1)^9 \]

e) None of these

16. If \( y = \frac{2x - 5}{3x + 7} \), then \( \frac{\partial^2 y}{\partial x^2} = \)

a) \[ \frac{29}{(3x + 7)^2} \]

b) \[ \frac{(2x - 5) \cdot 3 - (3x + 7) \cdot 2}{(2x - 5)^2} \]

c) \[ \frac{(3x + 7) \cdot 2 + (2x - 5) \cdot 3}{(3x + 7)^2} \]

d) \[ \frac{(3x + 7) \cdot 2 - (2x - 5) \cdot 3}{9x^2 + 49} \]

e) \[ 2/3 \]

17. If \( y = \frac{(5x - 2)^3}{x^2} \), then \( y' = \)

a) \[ \frac{5(5x - 2)^2}{x} \]

b) \[ \frac{(5x - 2)(2x) - x^2(3)(5x - 2)^2}{x^4} \]

c) \[ \frac{x^2(5x - 2)^2 - (5x - 2)^3}{x^4} \]

d) \[ \frac{15x(5x - 2)^2 - 2(5x - 2)^3}{x^3} \]

e) \[ \frac{2(5x - 2)^3 - 15x(5x - 2)^2}{x^3} \]
18. If \( f(x) = \frac{4}{\sqrt{(x^2 + 5)^3}} \), then \( f'(x) = \)

a) \( \frac{4}{3}(x^2 + 5)^{1/3}(2x) \)

b) \( \frac{3}{4}(x^2 + 5)^{2}(2x)^2(2) \)

c) \( \frac{3}{4}(x^2 + 5)^{-1/4}(2x) \)

d) \( \frac{3}{4}(x^2 + 5)^{-1/4} \)

e) None of these

19. If \( y = x^2 + (3x - 1)^2 \), \( y'' = \)

a) 20  
b) 4  
c) 20x - 6  
d) 8x - 2  
e) 8

20. The slope of the line tangent to \( y = 3 + 2x - 5x^3 \) at the point \((-1, 6)\) is:

a) 6  
b) -13  
c) 13  
d) -6  
e) Not defined.

21. The equation of the line tangent to \( y = x^3 - 2x^2 + 3 \) at \((2, f(2))\) is:

a) \( y - 2x + 3 = 0 \)  
b) \( y - 4x + 5 = 0 \)  
c) \( y - 2x + 11 = 0 \)  
d) \( y - 4x - 11 = 0 \)  
e) None of these.

22. The derivative of \( y = f(x) \) can be defined by:

a) \( \lim_{x \to 0} \frac{f(x + h) - f(x)}{h} \)  
b) \( \lim_{h \to 0} \frac{f(x + h) - f(h)}{x} \)  
c) \( \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \)  
d) \( \lim_{h \to \infty} \frac{f(x + h) - f(h)}{h} \)  
e) \( \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \)

For problems 23 - 25, let \( y = f(x) = x^3 + 6x^2 + 2 \).
23. The x-coordinate(s) of the relative maximum point(s) for \( f(x) \) is (are):
   a) 0  b) -4  c) -2  d) 0, -2  e) -2, -4

24. The x-coordinate(s) of the point(s) of inflection is (are):
   a) 0  b) -4  c) -2  d) 0, -2  e) -2, -4

25. Disregarding any scale, the shape which best describes the graph of \( y = f(x) \) is:
   a) 
   b) 
   c) 
   d) 
   e)
Directions: Select the one best response to each of the following questions. In all questions involving limits, if the limit does not exist (D.N.E.), select $\infty$ or $-\infty$ if applicable. Be sure your name, section number, and student number are coded in the proper area on the answer sheet.

In problems 1 and 2, $f(x)$ is defined on $[0, 4]$ by

$$f(x) = \begin{cases} 
0 & \text{if } x = 0 \\
\frac{1}{x} & \text{if } 0 < x \leq 1 \\
1 & \text{if } 1 \leq x \leq 4 
\end{cases}$$

1. $\lim_{x \to 0} f(x) = \begin{cases} 
a) 0 & b) +\infty & c) 1 & d) \text{D.N.E.} \\
e) \text{None of these.}
\end{cases}$

2. $\lim_{x \to 1} f(x) = \begin{cases} 
a) 0 & b) +\infty & c) 1 & d) \text{D.N.E.} \\
e) \text{None of these.}
\end{cases}$

In problems 3-6, find the indicated limits.

3. $\lim_{x \to 5} \frac{x - 5}{x^2 - x - 20} = \begin{cases} 
a) 0 & b) 1 & c) \frac{1}{9} & d) \infty & e) \text{D.N.E.} 
\end{cases}$

4. $\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \begin{cases} 
a) 0 & b) \infty & c) 2x & d) \text{D.N.E.} \\
e) \text{None of these.}
\end{cases}$

5. $\lim_{n \to \infty} \frac{2n^2 - 3n - 4}{7 - n + n^3} = \begin{cases} 
a) \frac{-4}{7} & b) 0 & c) \infty & d) \text{D.N.E.} \\
e) \text{None of these.}
\end{cases}$

6. $\lim_{n \to \infty} \frac{5n^3 + 3n - 4}{4 - n^3} = \begin{cases} 
a) -5 & b) -1 & c) 5 & d) \infty \\
e) \text{None of these.}
\end{cases}$
7. The graph of \( \frac{x^2 - 25}{x^2 - 9} \) is discontinuous at the points with x-coordinates of:

a) 5 and -5  b) 3 and -3  c) -5 and 3  d) 5, -5, 3 and -3  
e) None of these.

8. If \( y = 3x(x^3 + 5)^4 \), then \( D_x y = \)

a) \( 36x^3(x^3 + 5)^3 \)  
b) \( 12x(x^3 + 5)^3 \)  
c) \( 12x^2(x^3 + 5)^3 \)  
d) \( 36x^3(x^3 + 5)^3 + 3(x^3 + 5)^4 \)  
e) None of these.

9. If \( y = \frac{1}{3}(5t - 1)^6 \), then \( y' = \)

a) \( 2(5t - 1)^7(5) \)  
b) \( 10(5t - 1) \)  
c) \( 10(5t - 1)^5 \)  
d) \( 2(5t - 1)^5 \)  
e) None of these.

10. If \( y = \frac{3x + 5}{2x - 7} \), then \( D_x y = \)

a) \( \frac{3}{2} \)  
b) \( \frac{(2x - 7)(3) - (3x + 5)(2)}{(3x + 5)^2} \)  
c) \( \frac{(2x - 7)(3) + (3x + 5)(2)}{(2x - 7)^2} \)  
d) \( \frac{(2x - 7)(3) - (3x + 5)(2)}{(2x - 7)^2} \)  
e) None of these.

11. If \( y = \frac{(4x + 1)^3}{x^2} \), \( y' = \)

a) \( 3(4x + 1)^2 \)  
b) \( \frac{2(4x + 1)^3 - 12x(4x + 1)^2}{x^3} \)  
c) \( \frac{x^2(4x + 1)^2 - (4x + 1)^3}{x^4} \)  
d) \( \frac{12x(4x + 1)^2 - 2(4x + 1)^3}{x^3} \)  
e) \( \frac{(4x + 1)2x - x^2(3)(4x + 1)^2}{x^4} \)
12. If \( f(x) = \sqrt[5]{(x^2 + 4)^3} \), then \( f'(x) = \)

a) \( \frac{5}{3}(x^2 + 4)^{2/3} \)  
b) \( \frac{3}{5}(x^2 + 4)^{-2/3} \)

c) \( \frac{3}{5}(x^2 + 4)^{-2/5}(2x) \)  
d) \( \frac{5}{3}(x^2 + 4)^{2/3}(2x) \)

e) None of these.

13. If \( y = (3x + 2)^2 + x^2 \), \( y'' = \)

a) 20  
b) 20x + 12

c) 8  
d) 8x + 4  
e) None of these.

14. The slope of the line tangent to \( y = 2 - 6x + 3x^3 \) at the point \((-1, 5)\) is:

a) -5  
b) -3  
c) 3  
d) 5  
e) Not defined.

15. The equation of the line tangent to \( y = x^3 - 2x^2 + 3 \) at \((2, f(2))\) is:

a) \( y - 2x + 3 = 0 \)  
b) \( y - 2x + 11 = 0 \)

c) \( y - 4x - 11 = 0 \)  
d) \( y - 4x + 5 = 0 \)

e) None of these.

16. A function \( f(x) \) has a local maximum when:

a) \( f'(x) = 0 \) and \( f''(x) > 0 \)  
b) \( f'(x) = 0 \) and \( f''(x) = 0 \)

c) \( f'(x) = 0 \) and \( f''(x) = 0 \)

d) \( f'(x) > 0 \) and \( f''(x) = 0 \)  
d) \( f'(x) < 0 \) and \( f''(x) = 0 \)

17. The maximum value of \( f(x) = 2 - x^2 \) in \([1, 3]\) is:

a) 0  
b) 1  
c) 2  
d) There is no maximum value  
e) None of these.
For problems 18 - 20, let \( y = f(x) = x^3 + 3x^2 + 2 \).

18. The x-coordinate(s) of the relative minimum point(s) for \( f(x) \) is (are):
   a) -2  b) -1  c) 0  d) -1 and 0  e) -2 and -1

19. The x-coordinate(s) of the point(s) of inflection is (are):
   a) -2  b) -1  c) 0  d) -1 and 0  e) -2 and -1.

20. Disregarding any scale, the shape which best describes the graph of \( y = f(x) \) is:
   a) ![Graph A]
   b) ![Graph B]
   c) ![Graph C]
   d) ![Graph D]
   e) ![Graph E]
APPENDIX C

SAMPLE COMPOSITION TABLES
### TABLE 32

**SAMPLE COMPOSITION: AGE**

<table>
<thead>
<tr>
<th>Treatment Group</th>
<th>Age in Years</th>
<th>17-18</th>
<th>19</th>
<th>20</th>
<th>21</th>
<th>22-29</th>
<th>Over 30</th>
<th>No Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abstract</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Deductive</td>
<td>11</td>
<td>10</td>
<td>46</td>
<td>43</td>
<td>14</td>
<td>13</td>
<td>10</td>
<td>9</td>
</tr>
<tr>
<td>Concrete</td>
<td>14</td>
<td>13</td>
<td>37</td>
<td>35</td>
<td>19</td>
<td>18</td>
<td>9</td>
<td>8</td>
</tr>
<tr>
<td>Inductive</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Control</td>
<td>3</td>
<td>7</td>
<td>12</td>
<td>28</td>
<td>9</td>
<td>21</td>
<td>8</td>
<td>19</td>
</tr>
<tr>
<td>Total Sample</td>
<td>28</td>
<td>11</td>
<td>95</td>
<td>37</td>
<td>42</td>
<td>15</td>
<td>27</td>
<td>11</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>#</th>
<th>%</th>
<th>#</th>
<th>%</th>
<th>#</th>
<th>%</th>
<th>#</th>
<th>%</th>
<th>#</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abstract Deductive</td>
<td>11</td>
<td>10</td>
<td>46</td>
<td>43</td>
<td>14</td>
<td>13</td>
<td>10</td>
<td>9</td>
<td>24</td>
<td>22</td>
</tr>
<tr>
<td>Concrete Inductive</td>
<td>14</td>
<td>13</td>
<td>37</td>
<td>35</td>
<td>19</td>
<td>18</td>
<td>9</td>
<td>8</td>
<td>25</td>
<td>24</td>
</tr>
<tr>
<td>Control</td>
<td>3</td>
<td>7</td>
<td>12</td>
<td>28</td>
<td>9</td>
<td>21</td>
<td>8</td>
<td>19</td>
<td>10</td>
<td>23</td>
</tr>
</tbody>
</table>
TABLE 33

SAMPLE COMPOSITION: RANK

<table>
<thead>
<tr>
<th></th>
<th>Freshmen</th>
<th>Sophomores</th>
<th>Juniors</th>
<th>Seniors</th>
<th>Others</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>#</td>
<td>%</td>
<td>#</td>
<td>%</td>
<td>#</td>
</tr>
<tr>
<td>Abstract Deductive</td>
<td>25</td>
<td>23</td>
<td>46</td>
<td>43</td>
<td>19</td>
</tr>
<tr>
<td>Concrete Inductive</td>
<td>28</td>
<td>26</td>
<td>41</td>
<td>39</td>
<td>12</td>
</tr>
<tr>
<td>Control</td>
<td>10</td>
<td>23</td>
<td>16</td>
<td>37</td>
<td>8</td>
</tr>
<tr>
<td>Total Sample</td>
<td>63</td>
<td>25</td>
<td>103</td>
<td>40</td>
<td>39</td>
</tr>
<tr>
<td>Treatment Group</td>
<td>One Year</td>
<td>Two Years</td>
<td>Three Years</td>
<td>Four Years</td>
<td>Five or More Years</td>
</tr>
<tr>
<td>-----------------------</td>
<td>----------</td>
<td>-----------</td>
<td>-------------</td>
<td>------------</td>
<td>-------------------</td>
</tr>
<tr>
<td>Abstract Deductive</td>
<td>9</td>
<td>9</td>
<td>17</td>
<td>16</td>
<td>47</td>
</tr>
<tr>
<td>Concrete Inductive</td>
<td>2</td>
<td>2</td>
<td>34</td>
<td>33</td>
<td>50</td>
</tr>
<tr>
<td>Control</td>
<td>0</td>
<td>0</td>
<td>5</td>
<td>12</td>
<td>21</td>
</tr>
<tr>
<td>Total Sample</td>
<td>11</td>
<td>4</td>
<td>56</td>
<td>22</td>
<td>118</td>
</tr>
<tr>
<td>Grade</td>
<td>A</td>
<td>B</td>
<td>C</td>
<td>D</td>
<td>No Response</td>
</tr>
<tr>
<td>---------------</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>-------------</td>
</tr>
<tr>
<td>Treatment Group</td>
<td># %</td>
<td># %</td>
<td># %</td>
<td># %</td>
<td># %</td>
</tr>
<tr>
<td>Abstract Deductive</td>
<td>7 7</td>
<td>31 30</td>
<td>63 60</td>
<td>4 4</td>
<td>0 0</td>
</tr>
<tr>
<td>Concrete Inductive</td>
<td>5 5</td>
<td>28 28</td>
<td>60 58</td>
<td>9 9</td>
<td>1 1</td>
</tr>
<tr>
<td>Control</td>
<td>0 0</td>
<td>13 32</td>
<td>22 54</td>
<td>6 15</td>
<td>0 0</td>
</tr>
<tr>
<td>Total Sample</td>
<td>12 5</td>
<td>72 29</td>
<td>145 58</td>
<td>19 8</td>
<td>1 -</td>
</tr>
</tbody>
</table>
## TABLE 36
GRADE IN MATHEMATICS 116

<table>
<thead>
<tr>
<th>Grade</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>Pass</th>
<th>Other</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatment Group</td>
<td>#</td>
<td>%</td>
<td>#</td>
<td>%</td>
<td>#</td>
<td>%</td>
</tr>
<tr>
<td>Abstract Deductive</td>
<td>13</td>
<td>12</td>
<td>22</td>
<td>21</td>
<td>28</td>
<td>26</td>
</tr>
<tr>
<td>Concrete Inductive</td>
<td>11</td>
<td>10</td>
<td>16</td>
<td>15</td>
<td>34</td>
<td>32</td>
</tr>
<tr>
<td>Control</td>
<td>1</td>
<td>2</td>
<td>15</td>
<td>35</td>
<td>8</td>
<td>19</td>
</tr>
<tr>
<td>Total Sample</td>
<td>25</td>
<td>10</td>
<td>53</td>
<td>21</td>
<td>70</td>
<td>27</td>
</tr>
</tbody>
</table>
APPENDIX D

PRETEST, ACHIEVEMENT TEST, AND RETENTION TEST DATA
<table>
<thead>
<tr>
<th>Item Number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Discrimination Index</td>
<td>47</td>
<td>36</td>
<td>37</td>
<td>38</td>
<td>42</td>
<td>38</td>
<td>37</td>
<td>43</td>
<td>25</td>
<td>26</td>
<td>47</td>
<td>28</td>
<td>52</td>
<td>12</td>
<td>29</td>
<td>25</td>
<td>36</td>
<td>35</td>
<td>43</td>
<td>29</td>
</tr>
<tr>
<td>Item Relative Difficulty</td>
<td>.64</td>
<td>.57</td>
<td>.78</td>
<td>.41</td>
<td>.62</td>
<td>.81</td>
<td>.74</td>
<td>.72</td>
<td>.84</td>
<td>.80</td>
<td>.70</td>
<td>.81</td>
<td>.64</td>
<td>.75</td>
<td>.72</td>
<td>.80</td>
<td>.72</td>
<td>.72</td>
<td>.71</td>
<td>.68</td>
</tr>
<tr>
<td>Percent correct in top 27.5 percentile</td>
<td>64</td>
<td>61</td>
<td>46</td>
<td>80</td>
<td>63</td>
<td>46</td>
<td>52</td>
<td>55</td>
<td>34</td>
<td>38</td>
<td>57</td>
<td>38</td>
<td>59</td>
<td>29</td>
<td>45</td>
<td>36</td>
<td>46</td>
<td>52</td>
<td>54</td>
<td>43</td>
</tr>
<tr>
<td>Percent correct in lower 27.4 percentile</td>
<td>17</td>
<td>24</td>
<td>10</td>
<td>43</td>
<td>21</td>
<td>9</td>
<td>15</td>
<td>12</td>
<td>9</td>
<td>11</td>
<td>10</td>
<td>10</td>
<td>7</td>
<td>17</td>
<td>16</td>
<td>11</td>
<td>11</td>
<td>17</td>
<td>11</td>
<td>13</td>
</tr>
<tr>
<td>Percent correct of total</td>
<td>36</td>
<td>43</td>
<td>22</td>
<td>59</td>
<td>38</td>
<td>19</td>
<td>27</td>
<td>29</td>
<td>16</td>
<td>20</td>
<td>30</td>
<td>19</td>
<td>37</td>
<td>25</td>
<td>28</td>
<td>20</td>
<td>28</td>
<td>28</td>
<td>29</td>
<td>33</td>
</tr>
<tr>
<td>Raw Score</td>
<td>Frequency</td>
<td>Cumulative Frequency</td>
<td>Percentile Rank</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-----------</td>
<td>-----------</td>
<td>---------------------</td>
<td>----------------</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>1</td>
<td>249</td>
<td>99.8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>2</td>
<td>248</td>
<td>99.2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>1</td>
<td>246</td>
<td>98.6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>6</td>
<td>245</td>
<td>97.2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>1</td>
<td>239</td>
<td>95.8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>8</td>
<td>238</td>
<td>94.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>6</td>
<td>230</td>
<td>91.2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>12</td>
<td>224</td>
<td>87.6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>19</td>
<td>212</td>
<td>81.3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>30</td>
<td>193</td>
<td>71.5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>40</td>
<td>163</td>
<td>57.4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>41</td>
<td>123</td>
<td>41.2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>30</td>
<td>82</td>
<td>26.9</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>26</td>
<td>52</td>
<td>15.7</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>18</td>
<td>26</td>
<td>6.8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>8</td>
<td>2.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>2</td>
<td>0.4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Class</td>
<td>Number of Subjects</td>
<td>Mean</td>
<td>Standard Deviation</td>
<td>Treatment</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-------</td>
<td>--------------------</td>
<td>------</td>
<td>--------------------</td>
<td>-----------</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>22</td>
<td>12.64</td>
<td>4.62</td>
<td>Deductive</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>24</td>
<td>14.00</td>
<td>5.27</td>
<td>Deductive</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>26</td>
<td>14.54</td>
<td>5.46</td>
<td>Deductive</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>19</td>
<td>15.89</td>
<td>3.48</td>
<td>Deductive</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>21</td>
<td>13.95</td>
<td>4.86</td>
<td>Inductive</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>15</td>
<td>12.20</td>
<td>4.59</td>
<td>Inductive</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>27</td>
<td>13.07</td>
<td>4.21</td>
<td>Inductive</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>25</td>
<td>15.04</td>
<td>4.73</td>
<td>Inductive</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>26</td>
<td>15.96</td>
<td>4.71</td>
<td>Control</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>20</td>
<td>12.65</td>
<td>4.42</td>
<td>Control</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Raw Score</td>
<td>Frequency</td>
<td>Cumulative Frequency</td>
<td>Percentile Rank</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-----------</td>
<td>-----------</td>
<td>----------------------</td>
<td>-----------------</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>4</td>
<td>225</td>
<td>99.1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>23</td>
<td>8</td>
<td>221</td>
<td>96.4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>22</td>
<td>5</td>
<td>213</td>
<td>93.6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>5</td>
<td>208</td>
<td>91.3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>14</td>
<td>203</td>
<td>87.1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>9</td>
<td>189</td>
<td>82.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>20</td>
<td>180</td>
<td>75.6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>14</td>
<td>160</td>
<td>68.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>16</td>
<td>146</td>
<td>61.3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>13</td>
<td>130</td>
<td>54.9</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>14</td>
<td>117</td>
<td>48.9</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>14</td>
<td>103</td>
<td>42.7</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>19</td>
<td>89</td>
<td>35.3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>14</td>
<td>70</td>
<td>28.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>13</td>
<td>56</td>
<td>22.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>13</td>
<td>43</td>
<td>16.2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>7</td>
<td>30</td>
<td>11.8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>12</td>
<td>23</td>
<td>7.6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>11</td>
<td>4.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>7</td>
<td>2.4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>4</td>
<td>1.6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>3</td>
<td>0.9</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0.2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Item Number</td>
<td>Discrimination Index</td>
<td>Relative Difficulty</td>
<td>Percent Correct in Top 27.5 Percentile</td>
<td>Percent Correct in Lower 27.5 Percentile</td>
<td>Percent Correct of Total</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-------------</td>
<td>----------------------</td>
<td>---------------------</td>
<td>----------------------------------------</td>
<td>----------------------------------------</td>
<td>--------------------------</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>32.8</td>
<td>.440</td>
<td>74</td>
<td>41</td>
<td>56</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>45.6</td>
<td>.244</td>
<td>94</td>
<td>48</td>
<td>76</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>53.8</td>
<td>.373</td>
<td>88</td>
<td>34</td>
<td>63</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>50.1</td>
<td>.560</td>
<td>66</td>
<td>16</td>
<td>44</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>30.3</td>
<td>.738</td>
<td>45</td>
<td>14</td>
<td>26</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>34.9</td>
<td>.520</td>
<td>62</td>
<td>27</td>
<td>48</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>69.6</td>
<td>.400</td>
<td>89</td>
<td>20</td>
<td>60</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>45.0</td>
<td>.436</td>
<td>75</td>
<td>30</td>
<td>56</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>29.8</td>
<td>.218</td>
<td>92</td>
<td>63</td>
<td>78</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>44.8</td>
<td>.382</td>
<td>88</td>
<td>43</td>
<td>62</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>39.5</td>
<td>.262</td>
<td>88</td>
<td>48</td>
<td>74</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>52.9</td>
<td>.622</td>
<td>71</td>
<td>18</td>
<td>38</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>25.5</td>
<td>.133</td>
<td>97</td>
<td>71</td>
<td>87</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>50.3</td>
<td>.609</td>
<td>65</td>
<td>14</td>
<td>39</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>62.1</td>
<td>.502</td>
<td>80</td>
<td>18</td>
<td>50</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>46.6</td>
<td>.338</td>
<td>88</td>
<td>41</td>
<td>66</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>57.8</td>
<td>.569</td>
<td>74</td>
<td>16</td>
<td>43</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>54.3</td>
<td>.342</td>
<td>85</td>
<td>30</td>
<td>66</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>70.6</td>
<td>.507</td>
<td>83</td>
<td>13</td>
<td>49</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>59.4</td>
<td>.467</td>
<td>96</td>
<td>27</td>
<td>53</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>47.3</td>
<td>.653</td>
<td>62</td>
<td>14</td>
<td>35</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>22</td>
<td>38.2</td>
<td>.182</td>
<td>95</td>
<td>57</td>
<td>82</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>23</td>
<td>65.5</td>
<td>.427</td>
<td>92</td>
<td>27</td>
<td>57</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>72.2</td>
<td>.396</td>
<td>95</td>
<td>23</td>
<td>60</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>48.4</td>
<td>.489</td>
<td>77</td>
<td>29</td>
<td>51</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Percent Students</td>
<td>Number Students</td>
<td>Mean Score</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>----------------</td>
<td>------------------</td>
<td>-----------------</td>
<td>------------</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Deductive</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>100.00</td>
<td>91</td>
<td>14.220</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Upper</td>
<td>29.67</td>
<td>27</td>
<td>20.111</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lower</td>
<td>25.27</td>
<td>23</td>
<td>7.870</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Inductive</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>100.00</td>
<td>88</td>
<td>13.693</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Upper</td>
<td>23.86</td>
<td>21</td>
<td>19.762</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lower</td>
<td>25.00</td>
<td>22</td>
<td>7.545</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Control</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>100.00</td>
<td>46</td>
<td>15.087</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Upper</td>
<td>26.09</td>
<td>12</td>
<td>21.333</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lower</td>
<td>28.26</td>
<td>13</td>
<td>8.692</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total Sample</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>100.00</td>
<td>225</td>
<td>14.191</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Upper</td>
<td>28.89</td>
<td>65</td>
<td>20.092</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lower</td>
<td>24.89</td>
<td>56</td>
<td>7.821</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
TABLE 43
RETENTION TEST STATISTICS FOR INDIVIDUAL CLASSES

<table>
<thead>
<tr>
<th>Class</th>
<th>Number of Subjects</th>
<th>Mean</th>
<th>Standard Deviation</th>
<th>Treatment</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>15.30</td>
<td>4.10</td>
<td>Deductive</td>
</tr>
<tr>
<td>2</td>
<td>18</td>
<td>12.33</td>
<td>4.51</td>
<td>Deductive</td>
</tr>
<tr>
<td>3</td>
<td>21</td>
<td>13.38</td>
<td>3.47</td>
<td>Deductive</td>
</tr>
<tr>
<td>4</td>
<td>19</td>
<td>12.95</td>
<td>3.93</td>
<td>Deductive</td>
</tr>
<tr>
<td>5</td>
<td>15</td>
<td>14.13</td>
<td>4.51</td>
<td>Inductive</td>
</tr>
<tr>
<td>6</td>
<td>11</td>
<td>11.82</td>
<td>3.83</td>
<td>Inductive</td>
</tr>
<tr>
<td>7</td>
<td>15</td>
<td>13.40</td>
<td>3.86</td>
<td>Inductive</td>
</tr>
<tr>
<td>8</td>
<td>16</td>
<td>13.31</td>
<td>3.35</td>
<td>Inductive</td>
</tr>
<tr>
<td>9</td>
<td>18</td>
<td>15.56</td>
<td>3.35</td>
<td>Control</td>
</tr>
<tr>
<td>10</td>
<td>14</td>
<td>12.36</td>
<td>4.10</td>
<td>Control</td>
</tr>
</tbody>
</table>
### TABLE 44

**RETENTION TEST SCORE DISTRIBUTION**

<table>
<thead>
<tr>
<th>Raw Score</th>
<th>Frequency</th>
<th>Cumulative Frequency</th>
<th>Percentile Rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>3</td>
<td>163</td>
<td>99.1</td>
</tr>
<tr>
<td>19</td>
<td>8</td>
<td>160</td>
<td>95.7</td>
</tr>
<tr>
<td>18</td>
<td>19</td>
<td>152</td>
<td>87.4</td>
</tr>
<tr>
<td>17</td>
<td>15</td>
<td>133</td>
<td>77.0</td>
</tr>
<tr>
<td>16</td>
<td>16</td>
<td>118</td>
<td>67.5</td>
</tr>
<tr>
<td>15</td>
<td>13</td>
<td>102</td>
<td>58.6</td>
</tr>
<tr>
<td>14</td>
<td>13</td>
<td>89</td>
<td>50.6</td>
</tr>
<tr>
<td>13</td>
<td>19</td>
<td>76</td>
<td>40.8</td>
</tr>
<tr>
<td>12</td>
<td>8</td>
<td>57</td>
<td>32.5</td>
</tr>
<tr>
<td>11</td>
<td>8</td>
<td>49</td>
<td>27.6</td>
</tr>
<tr>
<td>10</td>
<td>11</td>
<td>41</td>
<td>21.8</td>
</tr>
<tr>
<td>9</td>
<td>9</td>
<td>30</td>
<td>15.6</td>
</tr>
<tr>
<td>8</td>
<td>4</td>
<td>21</td>
<td>11.7</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>17</td>
<td>8.3</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>10</td>
<td>4.9</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>6</td>
<td>3.1</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>4</td>
<td>1.5</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0.3</td>
</tr>
</tbody>
</table>


<table>
<thead>
<tr>
<th>Item Number</th>
<th>Discrimination Index</th>
<th>Item Relative Difficulty</th>
<th>Percent Correct in Top 27.5 Percentile</th>
<th>Percent Correct in Lower 27.5 Percentile</th>
<th>Percent Correct of Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>44.3</td>
<td>.540</td>
<td>71</td>
<td>27</td>
<td>46</td>
</tr>
<tr>
<td>2</td>
<td>28.8</td>
<td>.331</td>
<td>80</td>
<td>51</td>
<td>67</td>
</tr>
<tr>
<td>3</td>
<td>66.3</td>
<td>.356</td>
<td>96</td>
<td>29</td>
<td>64</td>
</tr>
<tr>
<td>4</td>
<td>41.7</td>
<td>.215</td>
<td>98</td>
<td>56</td>
<td>79</td>
</tr>
<tr>
<td>5</td>
<td>34.4</td>
<td>.141</td>
<td>98</td>
<td>63</td>
<td>86</td>
</tr>
<tr>
<td>6</td>
<td>34.6</td>
<td>.221</td>
<td>96</td>
<td>61</td>
<td>78</td>
</tr>
<tr>
<td>7</td>
<td>53.2</td>
<td>.405</td>
<td>80</td>
<td>27</td>
<td>60</td>
</tr>
<tr>
<td>8</td>
<td>54.7</td>
<td>.350</td>
<td>89</td>
<td>34</td>
<td>65</td>
</tr>
<tr>
<td>9</td>
<td>54.3</td>
<td>.239</td>
<td>93</td>
<td>39</td>
<td>76</td>
</tr>
<tr>
<td>10</td>
<td>41.7</td>
<td>.153</td>
<td>98</td>
<td>56</td>
<td>85</td>
</tr>
<tr>
<td>11</td>
<td>48.8</td>
<td>.209</td>
<td>99</td>
<td>51</td>
<td>79</td>
</tr>
<tr>
<td>12</td>
<td>36.6</td>
<td>.147</td>
<td>99</td>
<td>63</td>
<td>85</td>
</tr>
<tr>
<td>13</td>
<td>59.6</td>
<td>.344</td>
<td>89</td>
<td>29</td>
<td>66</td>
</tr>
<tr>
<td>14</td>
<td>66.3</td>
<td>.319</td>
<td>96</td>
<td>29</td>
<td>68</td>
</tr>
<tr>
<td>15</td>
<td>71.2</td>
<td>.393</td>
<td>96</td>
<td>24</td>
<td>61</td>
</tr>
<tr>
<td>16</td>
<td>61.6</td>
<td>.337</td>
<td>93</td>
<td>32</td>
<td>66</td>
</tr>
<tr>
<td>17</td>
<td>43.1</td>
<td>.712</td>
<td>58</td>
<td>15</td>
<td>29</td>
</tr>
<tr>
<td>18</td>
<td>67.4</td>
<td>.509</td>
<td>84</td>
<td>17</td>
<td>49</td>
</tr>
<tr>
<td>19</td>
<td>59.2</td>
<td>.288</td>
<td>93</td>
<td>34</td>
<td>71</td>
</tr>
<tr>
<td>20</td>
<td>49.6</td>
<td>.337</td>
<td>91</td>
<td>41</td>
<td>66</td>
</tr>
<tr>
<td></td>
<td>Percent Students</td>
<td>Number Students</td>
<td>Mean Score</td>
<td></td>
<td></td>
</tr>
<tr>
<td>----------------</td>
<td>------------------</td>
<td>----------------</td>
<td>------------</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Deductive</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>100.00</td>
<td>74</td>
<td>13.297</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Upper</td>
<td>27.03</td>
<td>20</td>
<td>17.700</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lower</td>
<td>28.38</td>
<td>21</td>
<td>7.905</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Inductive</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>100.00</td>
<td>57</td>
<td>13.263</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Upper</td>
<td>28.07</td>
<td>16</td>
<td>18.125</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lower</td>
<td>29.82</td>
<td>17</td>
<td>8.471</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Control</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>100.00</td>
<td>32</td>
<td>14.156</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Upper</td>
<td>28.13</td>
<td>9</td>
<td>18.333</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lower</td>
<td>28.13</td>
<td>9</td>
<td>8.667</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>100.00</td>
<td>163</td>
<td>13.454</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sample</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Upper</td>
<td>27.61</td>
<td>15</td>
<td>17.978</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lower</td>
<td>25.15</td>
<td>11</td>
<td>7.805</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
LESSON I

OBJECTIVES

1. Relate the concepts of neighborhood and deleted neighborhood.

2. Define and discuss the limit of a function.

3. Derive a basic limit theorem (limit of a constant).

PROCEDURES

1. Define neighborhood:
   Definition. The set of numbers $x$ satisfying an expression of the form $|x - a| < d$ where $d$ is any positive number is called a neighborhood of the number $a$.
   a) Examples: $|x - 2| < 0.1$ is a neighborhood of 2.

   $|x + 3| < 0.5$ is a neighborhood of -3.

   Give examples of numbers that belong to each of these neighborhoods.

   Describe these neighborhoods without absolute value, as: $1.9 < x < 2.1$ and $-3.5 < x < -2.5$.

   b) Graph $|x - 2| < 0.1$ iff $1.9 < x < 2.1$.

   c) Graph $|x - 2| < 0.1$ on the $xy$-plane:

   Give examples of ordered pairs in this strip and not in this strip.

   d) Repeat the above with $|y - 1| < 0.2$.

   f) Graph $|x - 2| < 0.07$ and $|y + 1| < 0.1$ together.

   Recall that this is all points less than 0.07 from 2, etc.

   Point out that the rectangle represents all points satisfying both inequalities.

2. a) Point out the difference between $|x - a| < d$ and $0 < |x - a| < d$.

   b) Make note that the expression $0 < |x - a| < d$ is called a "deleted neighborhood" of the number $a$ since the number $a$, which is an element of the neighborhood $|x - a| < d$, is not included in the set $0 < |x - a| < d$.

   c) Example: $0 < |x - 2| < 0.3$ is a deleted neighborhood of 2.
3. Define the limit of a function:

**Definition.** The limit of \( f(x) \) as \( x \) approaches the number \( a \) is a number \( L \), if for each positive number \( \varepsilon \) there is a positive number \( \delta \) such that \( |f(x) - L| < \varepsilon \) for all \( x \) such that \( 0 < |x - a| < \delta \).

a) Notation: "\( x \to a \)" means \( x \) approaches \( a \).

b) Restate the definition in terms of neighborhoods:

The limit of \( f(x) \) as \( x \to a \) is a number \( L \), if for each neighborhood, \( N \), of \( L \), there is a deleted neighborhood, \( D \), of \( a \) such that \( f(x) \) is in \( N \) for every \( x \) in \( D \).

c) Example: \( \lim_{x \to 4} (3x - 1) = 11 \). State this verbally.

d) Call attention to the fact that the definition does not tell how to find \( L \).

- \( \varepsilon \) is regarded as given and we must find a such that the required inequalities hold.

- \( \delta \) does not have to be the largest number possible, but a number \( \delta \) must be found.

- we must find a \( \delta \) for any positive \( \varepsilon \).

4. On the same graph, graph:

- \( |f(x) - L| < \varepsilon \) and notice the width of the horizontal strip is \( 2\varepsilon \).

- \( 0 < |x - a| < \delta \) and notice the width of the vertical strip is \( 2\delta \).

a) If \( \lim_{x \to a} f(x) = L \), then for all \( x \) in the vertical strip, except possibly at \( x = a \), the graph of the function must lie within the horizontal strip.

b) Emphasize that, for this to be true, we must find a \( \delta \) for \( f \) at \( x = a \) and any number \( \varepsilon \).

5. Consider \( f(x) = k \), \( k \) a constant.

a) Graph \( f(x) \).

b) Guess the limit as \( k \) for \( x \to a \).

c) Discuss: If \( \varepsilon > 0 \) is given, we then have \( |f(x) - k| < \varepsilon \) and we must find a deleted \( \delta \)-neighborhood of \( x = a \) such that for all \( x \) in this \( \delta \)-neighborhood, \( f(x) \) is in the neighborhood \( |f(x) - k| < \varepsilon \).

But \( |f(x) - k| = 0 \) for all \( x \) so any real number will work for \( \delta \).

6. Generalize with:

**Theorem.** If \( f(x) = k \), then \( \lim_{x \to a} f(x) = k \).

**Example:** \( g(x) = 3 \) so \( \lim_{x \to 7} g(x) = 3 \).
LESSON II

OBJECTIVES

Develop several basic limit theorems and illustrate their use by considering several examples.

PROCEDURES

1. Consider \( f(x) = x \).
   a) Graph \( f(x) \).
   b) Guess that as \( x \to a \), the limit would be \( a \).
   c) Prove this conjecture as follows:
      The definition says that for any given \( \varepsilon > 0 \)
      we must find a \( \delta > 0 \) such that \( |f(x) - a| < \varepsilon \) for all
      \( x \) such that \( 0 < |x - a| < \delta \).
      Since \( f(x) = x \), we have: \( |f(x) - a| = |x - a| < \varepsilon \).
      Thus we should let \( \delta = \varepsilon \).
      d) Interpret this graphically:
         The given \( \varepsilon \)-neighborhood is a horizontal strip centered at \( h(x) = a \).
         The deleted \( \delta \)-neighborhood is a vertical strip centered at \( x = a \) with the line \( x = a \) deleted.
      e) Point out that we have chosen \( \delta \) so that for all \( x \)
         in the deleted \( \delta \)-neighborhood, the graph of \( f \) will
         lie inside the rectangle with vertices \( (a - \delta, a - \varepsilon) \),
         \( (a - \delta, a + \varepsilon) \), \( (a + \delta, a - \varepsilon) \), and \( (a + \delta, a + \varepsilon) \).
      f) Also point out that once we have found a \( \delta' \) for \( \delta \) so
         that \( |f(x) - a| < \varepsilon \) is satisfied for all such \( x \), we
         may choose any positive number smaller than \( \delta' \) and the inequality \( |f(x) - a| < \varepsilon \) will still hold.
      g) We have proved:
         Theorem. For \( f(x) = x \), \( \lim_{x \to a} x = a \).

2. Consider the sum of two functions as follows:
   a) Suppose \( \lim_{x \to a} f(x) = F \) and \( \lim_{x \to a} g(x) = G \). Let \( h(x) = f(x) + g(x) \).
   b) Graph \( f(x) \), \( g(x) \) and \( h(x) \):
   c) Guess \( \lim_{x \to a} h(x) \): As \( x \to a \), \( h(x) \to F + G \).
      If so, the limit of the sum of two functions is the sum of the limits.
3. Generalize with:

**Theorem.** If \( \lim_{x \to a} f(x) = F \) and \( \lim_{x \to a} g(x) = G \), then

\[
\lim_{x \to a} [f(x) + g(x)] = F + G.
\]

Prove as follows:

For any \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that

|\( f(x) - F \)\| < \( \varepsilon \), for all \( x \) such that \( 0 < |x - a| < \delta \).

For all \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that

|\( g(x) - G \)\| < \( \varepsilon \), for all \( x \) such that \( 0 < |x - a| < \delta \).

|\( f(x) + g(x) - (F + G) \)\| < \( \varepsilon \), for all \( x \) such that \( 0 < |x - a| < \delta \).

|\( f(x) + g(x) - (F + G) \)\| < \( \varepsilon \).

Be sure to discuss the choice of \( \delta \) as follows:

Recall that: if \( x \to a \) and \( x \to a \), then \( x \to a \), and

|\( f(x) - F \)\| < \( \varepsilon \) if \( 0 < |x - a| < \delta \).

|\( g(x) - G \)\| < \( \varepsilon \) if \( 0 < |x - a| < \delta \).

Consider the diagram:

\[\begin{array}{c}
| & \delta_2 & | \\
| & \delta & | \\
| \delta_1 & a & | \\
\end{array}\]

Determine where the two deleted neighborhoods of \( x = a \) have a common deleted neighborhood:

This will occur if we choose \( \delta \) equal to the smaller and will have a common neighborhood.

Hence, for all \( x \) such that \( 0 < |x - a| < \delta \), we have \( |f(x) + g(x) - (F + G)| < \varepsilon \).

4. Work examples as follows:
   a) \( \lim_{x \to 4} x = 4 \) and \( \lim_{x \to 4} 3 = 3 \), so \( \lim_{x \to 4} (x + 3) = 7 \).
   b) Since \( \lim_{x \to -3} x = -3 \), we have \( \lim_{x \to -3} 2x = \lim_{x \to -3} x + \lim_{x \to -3} x = -6 \).

5. State:

**Theorem.** If \( \lim_{x \to a} f(x) = F \) and \( \lim_{x \to a} g(x) = G \), then

\[
\lim_{x \to a} [f(x) \cdot g(x)] = F \cdot G.
\]

Do not prove this theorem, but relate that we must show that: For all \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that

|\( f(x) \cdot g(x) - F \cdot G \)\| < \( \varepsilon \), whenever \( 0 < |x - a| < \delta \).

6. Work examples as follows:
   a) \( \lim_{x \to 4} 3x = (\lim_{x \to 4} 3)(\lim_{x \to 4} x) = 3 \cdot 4 = 12 \)
b) \[ \lim_{x \to 4} x^2 = (\lim_{x \to 4} x)(\lim_{x \to 4} x) = 4 \cdot 4 = 16. \]

c) \[ \lim_{x \to 4} x(x + 3) = \lim_{x \to 4} x \cdot \lim_{x \to 4} (x + 3) = 4.7 = 28 \]

7. Consider the difference of two functions as follows:
   a) Let \( h(x) = f(x) - g(x) = f(x) + (-1)g(x) \).
      Since \( \lim_{x \to a} (-1)g(x) = \lim_{x \to a} (-1) \cdot \lim_{x \to a} g(x) = -G \) if
      \[ \lim_{x \to a} g(x) = G, \]
      we have: \( \lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) \).

   b) We have proved:
      Theorem. If \( \lim_{x \to a} f(x) = F \) and \( \lim_{x \to a} g(x) = G \), then
      \[ \lim_{x \to a} [f(x) - g(x)] = F - G. \]

   c) Example: \( \lim_{x \to 2} (x^2 - 3x) = _____. \)

8. Consider the limit of a function as the independent variable gets large.
   a) Consider \( f(x) = \frac{1}{x} \).
   b) Graph \( f(x) \).
   c) Choose various epsilons such as \( \varepsilon = \frac{1}{2}, \frac{1}{4}, ... \)
      Show that for \( \varepsilon = \frac{1}{2} \), that \( |1/x - 0| < \frac{1}{2} \) if \( x > 2 \), etc.
LESSON III

OBJECTIVES

1. Present the basic theorem for the limit of a quotient of two functions.

2. Evaluate the limit of numerous types of quotients.

PROCEDURES

1. Consider the limit of a quotient of two functions as follows:

   **Theorem.** If \( \lim f(x) = F \) and \( \lim g(x) = G \) and \( G \neq 0 \), then

   \[
   \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{F}{G}
   \]

   Do not prove this theorem but note that we must show:

   For all \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that

   \[
   \left| \frac{f(x)}{g(x)} - \frac{F}{G} \right| < \varepsilon
   \]
   whenever \( 0 < |x - a| < \delta \).

   Work examples as follows:

   a) \( \lim_{x \to 2} \frac{x - 1}{x^2 - 1} \):

      Since \( \lim_{x \to 2} (x - 1) = \ldots \) and \( \lim_{x \to 2} (x^2 - 1) = \ldots \).

      We have \( \lim_{x \to 2} \frac{x - 1}{x^2 - 1} = \ldots \)

      OR

      \( \lim_{x \to 2} \frac{x - 1}{x^2 - 1} = \lim_{x \to 2} \frac{1}{x + 1} = \frac{1}{3}, \) if \( x \neq 1 \).

   b) \( \lim_{x \to -2} \frac{3x}{x + 4} = \ldots \).

   c) \( \lim_{x \to 3} \frac{x^2 - 4}{x + 2} = \ldots \).

2. Consider \( \lim_{x \to a} f(x) = f(a) \) for \( f(x) = x^2 - 3x + 4 \) by first noting that \( \lim_{x \to 1} f(x) = 2 \) and \( f(1) = 2 \).

   a) Point out that this is not always true by considering:

      \( f(x) = \frac{x^2 - 4}{x + 2} \).

      \( f(-2) \) is undefined.

      However, \( \lim_{x \to -2} f(x) = \lim_{x \to -2} (x - 2) = -4 \), if \( x \neq 2 \).

      Explain why this is so by recalling the definition of the limit of a function, i.e., we consider values of \( f(x) \) when \( x \) satisfies \( 0 < |x + 2| < \delta \).

      This excludes the possibility that \( x = -2 \). Thus it is legitimate to simplify first.
b) Re-emphasize that in this example we needed to use
the deleted neighborhood of \( x = a \) to find the limit.
Thus, although \( f(-2) \) does not exist, the
\( \lim_{x \to -2} f(x) \) exists.

c) Work the following examples and, in each case, note
that \( f(a) \) is undefined, but \( \lim_{x \to a} f(x) \) exists.

1) \( \lim_{x \to 1} \frac{x - 1}{x^2 - 1} \)
2) \( \lim_{x \to 3} \frac{x^2 - 2x - 3}{x - 3} \)
3) \( \lim_{x \to 2} \frac{1/x - 1/2}{x - 2} \)
4) \( \lim_{x \to 0} \frac{(b + x)^3 - b^3}{x} \)
5) \( \lim_{x \to 0} \frac{\sqrt{7} + x - \sqrt{7}}{2x} \)

3. Point out that in the above examples both \( F \) and \( G \) equal 0.
Suppose \( G = 0 \) but \( F \neq 0 \). Analyze \( f(x)/g(x) \) as follows:
a) As \( x \to a \), \( g(x) \) gets close to 0, and \( f(x) \) takes on
values close to \( F \).
b) \( f(x)/g(x) \), as a result, will get larger in absolute
value.
c) To indicate this, we write \( \lim_{x \to a} \frac{f(x)}{g(x)} = +\infty \), i.e., the
\( \lim_{x \to a} g(x) \)
limit does not exist.
d) Consider \( \lim_{x \to -1} \frac{x^2 - x - 6}{x + 1} \): \( \lim_{x \to -1} (x^2 - x - 6) = -4 \)
\( \lim_{x \to -1} (x + 1) = 0 \).
Thus, the quotient of the limits is undefined.
We write: \( \lim_{x \to -1} \frac{|x^2 - x - 6|}{|x + 1|} \to \infty \) to mean the limit
as \( x \) approaches -1 does not exist but increases
without bound.
\( \lim_{x \to 0} 1 \)
4. Consider \( \lim_{x \to 0} \frac{x}{\sqrt{x}} = \infty \)
Again, as above, the limit does not exist.

5. Point out that the previous functions are examples of
functions where the limit fails to exist at a point be­
cause the value of the function increases without bound.
Also, the limit of a function may fail to exist at
a point even though the function is defined in a neigh­
borhood of the point and the value of the function is
always a real number.
LESSON IV

OBJECTIVES

1. Consider, in general, the limit of a function at a point which does not exist because the two one-sided limits are not equal at that point.

2. Consider, specifically and with various examples, functions for which the limit at a point does not exist because the two one-sided limits are unequal at that point.

PROCEDURES

1. Consider a function with the graph: 

Discuss its limit at a as follows:

a) Suppose the limit $L = C$. If

is small enough then the neighborhood $|f(x) - C|$ fails to contain any numbers $f(x)$ when $x$ is greater than $a$. Thus, no matter how small we choose $\delta$, we fail because when $x > a$ there is no number $f(x)$ such that $|f(x) - C| < \varepsilon$ for every $\varepsilon > 0$.

b) Repeat (a) for $L = D$.

c) Suppose we choose $L$ between $C$ and $D$. Here, if $\varepsilon$ is less than the distance from $L$ to either $C$ or $D$, then $|f(x) - L|$ is greater than $\varepsilon$ for all $x$.

Recall that the definition of limit requires that we find a $\delta$ for any $\varepsilon > 0$ so we could be given such an $\varepsilon$ as mentioned.

d) Thus, the limit of $f$ as $x$ approaches $a$ does not exist.

e) Allow $x$ to approach $a$ from the left only, i.e., consider values of $x$ less than $a$, then, $f(x)$ will approach $C$ for such values of $x$.

Note that this is sometimes called a one-sided limit; to indicate that $x$ approaches $a$ from the negative direction (or left) only, write $x \to a^-$. 

Notation: \( \lim_{x \to a^-} f(x) = C \).

f) Repeat (e) for $x$ approaching $a$ from the right.

g) Be sure to point out that if the two one-sided limits at a point are not equal, we say that the limit of the function at that point does not exist.

h) Also point out that if the limit of a function $g$ does exist at some point $x = b$, then the one-sided limits of $g$ at $x = b$ will be equal.
2. To illustrate the above, consider \( f(x) = \frac{2(x+1)}{|x+1|} \).

a) Note that \( f(x) \) is defined except at \( x = -1 \).

b) Rewrite \( f(x) \) as follows:
\[
f(x) = \begin{cases} 
2 & \text{if } x > -1 \\
-2 & \text{if } x < -1.
\end{cases}
\]

c) Graph \( f(x) \).

d) Indicate the following: \( \lim_{x\to -1^+} f(x) = 2 \) and \( \lim_{x\to -1^-} f(x) = -2 \).

e) Conclude that \( \lim_{x\to -1} f(x) \) is not defined since the two one-sided limits at \( x = -1 \) are not equal.

f) Point out that the limit of the function does exist at other points as follows:
\[
\lim_{x\to 3} f(x) = 2 \quad \text{and} \quad \lim_{x\to -5} f(x) = -2.
\]

3. Consider \( y = \lceil x \rceil \), the greatest integer function.

a) Graph the function.

b) Discuss \( \lim_{x\to 2} f(x) \) by considering:
\[
\lim_{x\to 2^+} f(x) = 2 \quad \text{and} \quad \lim_{x\to 2^-} f(x) = 1.
\]

Conclude that \( \lim_{x\to 2} f(x) \) does not exist.

c) Discuss \( \lim_{x\to 3/2} f(x) \) by considering:
\[
\lim_{x\to 3/2^+} f(x) = 1 \quad \text{and} \quad \lim_{x\to 3/2^-} f(x) = 1.
\]

To verify that \( \lim_{x\to 3/2} f(x) = 1 \), recall that for every \( \epsilon > 0 \) we must find a \( \delta > 0 \) such that \( |f(x) - 1| < \epsilon \) for all \( x \) such that \( 0 < |x - 3/2| < \delta \).

Suppose that \( \epsilon = 1 \). Then proceed to find a corresponding \( \delta \) such that \( |f(x) - 1| < 1 \) whenever \( 0 < |x - 3/2| < \delta \).

Recall that \( f(x) = 1 \) for all \( x \) such that \( 1 \leq x < 2 \). So, if \( 1 \leq x < 2 \), it follows that \( |x - 3/2| < \frac{1}{2} \) so we may choose for \( \delta \) any positive real number less than or equal to \( \frac{1}{2} \).

4. Summarize \( f(x) = \lceil x \rceil \) by pointing out that \( f(x) \) does have a limit when \( x \) is any non-integer for which \( f(x) \) is defined. But, \( f(x) \) does not have a limit when \( x \) is any integer for which \( f \) is defined.
5. Recall the definition of sequence:
   a) **Definition.** An infinite sequence \( \{a_1, a_2, \ldots, a_n, \ldots\} \)
   is a function of \( n \) whose domain is the set of positive integers.
   Thus, a sequence associates a number \( a_n \) with each positive integer \( n \).
   b) **Example:** \( \{1, 1/2, 1/3, \ldots, 1/n, \ldots\} \) where the tenth term would be \( 1/10 \).
   c) **Notation:** \( \{a_n\} \) represents the entire sequence while \( a_n \) represents the \( n \)-th term.
   d) Write out various terms for the sequence \( 1/2^n \).

6. Define the limit of a sequence:
   a) **Definition:** A sequence \( \{a_n\} \) approaches a number \( A \) as a limit if for every \( \varepsilon > 0 \) there is a positive integer \( N \) such that for all \( n > N, \ |a_n - A| < \varepsilon \).
   b) Explain that this means that \( a_n \) gets closer to \( A \) as \( n \) increases.
   c) **Notation:** \( \lim_{n \to \infty} a_n = A \).
   d) **Compare** the definition of limit for a sequence with that of limit for a function:
      \( \varepsilon \) plays the same role in both definitions and \( A \) is similar to \( L \).
      Rather than find a deleted \( \delta \)-neighborhood of \( x = a \) as before, we must find an integer \( N \) such that for all \( n > N \) the inequality, \( |a_n - A| < \varepsilon \) holds.
OBJECTIVES

1. Consider various examples of sequences and evaluate their limits as n gets large.

2. State and illustrate two basic theorems for the limits of sequences.

PROCEDURES

1. Consider the sequence \{1/n\}.
   a) Write out the first few terms of the sequence:
   b) Guess that as n increases, 1/n decreases and approaches 0.
      To show this, one must show that for any \( \epsilon > 0 \) there is an integer N such that \(|1/n - 0| < \epsilon\) for all n > N.
      Note that \(|1/n - 0| = 1/n\) since n > 1. Then, 1/n < \( \epsilon \) if \( 1 < n\epsilon \Rightarrow n > 1/\epsilon \).
      So, retracing our steps, if n > 1/\( \epsilon \), then |1/n - 0| < \( \epsilon \) so we may choose N as any integer larger than or equal to 1/\( \epsilon \).
   c) For example:
      If \( \epsilon = 0.1 \), we must choose N > 10.
      If \( \epsilon = 0.01 \), N > 100.
      If \( \epsilon = 0.03 \), give examples of integers that will/will not work for N.

2. Repeat #1 with the sequence \{(n + 1)/n\}.

3. Repeat #1 with the sequence \{((-1)^n)/n\}.

4. Define a convergent (divergent) sequence:
   a) Definition. A sequence \( a_n \) that has a limit A is said to be convergent.
      We write: \( \{a_n\} \) converges to A.
   b) Definition. If a sequence does not have a limit, we say it is divergent.
   c) Consider the examples in numbers 1, 2, and 3 and determine if they converge or diverge.

5. Consider \{2n\}.
   a) Write out a few terms and note that as n gets larger, 2n gets larger.
   b) Guess 1,000,000 as a possible limit.
      We must show that for any given \( \epsilon > 0 \) we can find an integer N such that \(|2n - 1,000,000| < \epsilon\) for all n > N.
Since \(2n = 1,000,000\) for 500,000, we might choose \(N = 500,000\). But if we choose \(n = 700,000\), our "guess" fails since \(|2n - 1,000,000| = 400,000\) which is hardly small.

Point out that the larger we choose \(n\), the greater the difference between \(2n\) and 1,000,000. Note that any choices we make for \(A\) and \(N\) will produce the same results so we conclude that the sequence \(2n\) has no limit and is divergent.

6. Consider \((-1)^n\).
   a) Write out several terms.
   b) Discuss the limit of the function as follows:
      If we choose \(A = 1\), we must find an integer \(n\) such that \(|(-1)^n - 1| < \varepsilon\) for all \(n > N\).
      But \(|(-1)^n - 1| = 0\) if \(n\) is even and \(|(-1)^n - 1| = 2\) if \(n\) is odd.
      Clarify that no matter what integer we choose for \(N\), we can find an odd integer \(n > N\) so that \(|(-1)^n - 1|\) will be larger than \(\varepsilon\) (if \(\varepsilon < 2\)) for such an integer.
      Similarly, for any choice of \(A\) so \((-1)^n\) is divergent since it has no limit.

7. Consider \(a_n = b_n + c_n\) for all positive integers \(n\).
   a) Guess \(\lim a_n = \lim (b_n + c_n) = \lim b_n + \lim c_n\)
      provided these limits exist.
      To show \(\lim (b_n + c_n) = B + C\) if \(\lim b_n = B\) and \(\lim c_n = C\), we must show that for every \(\varepsilon > 0\) there is an integer \(N\) such that \(|(b_n + c_n) - (B + C)| < \varepsilon\) for all \(n > N\).
   b) State and prove:
      Theorem. If \(\lim b_n = B\) and \(\lim c_n = C\), then for \(a_n = b_n + c_n\), \(\lim a_n = B + C\).
   c) State:
      Theorem. If \(\lim b_n = B\) and \(\lim c_n = C\), then \(\lim (b_n \cdot c_n) = (\lim b_n)(\lim c_n) = B \cdot C\).
      Verbally, this means that the limit of a product of two convergent sequences is the product of the limits.
LESSON VI

OBJECTIVES

1. Formally define continuity of a function at a point.
2. Formally define continuity of a function on a set.
3. Consider various examples of continuous and discontinuous functions.

PROCEDURES

1. Define continuity at a point.
   a) Definition. A function $f$ is continuous at a point $a$ if and only if $\lim_{x \to a} f(x) = f(a)$.
   b) Give various examples of continuous functions and point out that in each case, $\lim_{x \to a} f(x) = f(a)$.
   c) Discuss the meaning "$\lim_{x \to a} f(x) = f(a)$," i.e., the function must be defined at $a$, the limit must exist and must equal $f(a)$.
   d) Graph and discuss various kinds of discontinuous functions such as those listed below.
      
      $f(x) = [x]$  
      $f(x) = \begin{cases} 2x + 1 & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$  
      $f(x) = \frac{1}{x}$  
      
      For each of these functions, point out the ones for which (1) $\lim_{x \to a} f(x)$ does not exist, (2) $f(a)$ does not exist, or (3) $\lim_{x \to a} f(x) \neq f(a)$.

2. Define continuity over a set.
   a) Definition. A function is continuous on a set $X$ if and only if it is continuous at each point of $X$.
   b) Give several examples to illustrate this definition.
   c) Consider a function such as $f(x) = x^2$ over $[0, 1]$. discuss continuity at the endpoints.
LESSON VII

OBJECTIVES

1. Define the derivative of a function.

2. Evaluate the derivative of various functions using the definition.

3. Consider four basic derivative theorems.

PROCEDURES

1. Define the derivative of a function $f$:

   Definition. The derivative of the function $f$, evaluated at $x$, is \( \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \), provided this limit exists.

   a) Note: the derivative is denoted by $f'(x)$ and is a function itself.

   b) Note: other notations for derivative are $y'$, $D_x y$, $\frac{dy}{dx}$.

   c) Note: $\frac{f(x+h) - f(x)}{h}$ is called a difference quotient. Explain why.

   d) Point out that, in the definition, the limit of the difference quotient must exist. Recall that this means that the two one-sided limits must exist and be equal.

2. Outline the four steps to computing the derivative of a function $f$ at $x$ as follows:

   a) (1) Express $f(x + h)$ and $f(x)$
      (2) Evaluate the difference $f(x + h) - f(x)$
      (3) Compute the difference quotient $\frac{f(x + h) - f(x)}{h}$
      (4) Determine $\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$

   b) Using these four steps, find the derivative for:
      (1) $f(x) = x$
      (2) $f(x) = k$, $k$ a constant
      (3) $f(x) = x^2$
      (4) $f(x) = x - x^2$

3. In example #1 of 2b, we have proved:
   Theorem. The derivative of the independent variable is 1, i.e., if $f(x) = x$, then $f'(x) = 1$.

4. In 2b, #2, we have also proved:
   Theorem. The derivative of the constant function is zero.
Thus, if \( f(x) = k \), \( k \) a constant, then \( f'(x) = 0 \).
Examples: \( f(x) = 7 \), \( f'(x) = 0 \).
\( f(x) = -1/5 \), \( f'(x) = 0 \).

5. Consider

Theorem. If \( f(x) = x^n \), \( n \) a non-negative integer, then \( f'(x) = n \cdot x^{n-1} \).

Do not prove this theorem, but point out that the proof is by induction.
Examples: \( f(x) = x^3 \), \( f'(x) = 3x^2 \).
\( f(x) = x^9 \), \( f'(x) = 9x^8 \).

a) Apply theorem to \( f(x) = x \) and \( f(x) = 1 \).
b) Note that this theorem can be proved for any real number but the proof is beyond the scope of this course. However, we will use the theorem as it applies to real numbers.
Example:
\( f(x) = x^{5/4} \).

6. Since a derivative is a limit, point out that we would expect limit theorems to apply.

Theorem. If \( f(x) = g(x) + h(x) \), where \( g \) and \( h \) are functions of \( x \) as indicated, and \( g'(x) \) and \( h'(x) \) exist, then \( f'(x) = g'(x) + h'(x) \).

a) Point out that both \( g' \) and \( h' \) must exist.
b) Proof: Using the four steps for evaluating a derivative, prove this if the time permits.
c) Work examples as follows:
\( f(x) = x + 7 \) \( f(x) = x - x^2 \)
\( f(x) = x^3 - 6 \) \( f(x) = x^9/7 - (\sqrt{2}) 7 \)
\( f(x) = x^4 + x \) \( f(x) = x^9/7 - x^2 \)
\( f(x) = x^8/7 - x^2 \) \( f(x) = \pi - x \)
Be sure to point out \( g \) and \( h \) for each of these functions.

7. Point out that if \( f = g + h \), that \( f' \) can exist even though \( g' \) and \( h' \) do not. Consider the example where \( g(x) = |x| \) and \( h(x) = -|x| \) so that \( f(x) = 0 \).
Thus \( f'(x) = 0 \).
LESSON VIII

OBJECTIVES

1. Consider the derivative of the product of two functions.
2. Consider the derivative of the quotient of two functions.
3. Evaluate the derivatives for various products and quotients.

PROCEDURES

1. Consider the derivative of a product as follows:
Theorem. If \( f(x) = g(x) \cdot h(x) \), then \( f'(x) = g(x) \cdot h'(x) + h(x) \cdot g'(x) \) provided \( g'(x) \) and \( h'(x) \) exist.
Prove this in class.

2. Illustrate with examples as follows:
a) \( f(x) = g(x) \cdot h(x) = (x^2) \cdot (x^3) \), where \( g(x) = x^2 \) and \( h(x) = x^3 \).
   \( g'(x) = 2x \) and \( h'(x) = 3x^2 \)
   Thus, \( f'(x) = g'(x) \cdot h(x) + h'(x) \cdot g(x) = 2x \cdot x^3 + 3x^2 \cdot x^2 = 5x^4 \).
   Point out also that \( f(x) = x^2 \cdot x^3 = x^5 \) so \( f'(x) = 5x^4 \).
b) \( f(x) = g(x) \cdot h(x) = x(1 - x) \) where \( g(x) = x \) and \( h(x) = 1 - x \).
   \( g'(x) = 1 \) and \( h'(x) = -1 \)
   Thus, as above, \( f'(x) = g(x) \cdot h'(x) + h(x) \cdot g'(x) \).
   Again consider \( f(x) = x - x^2 \) so \( f'(x) = 1 - 2x \).
c) Repeat this procedure with
   \( f(x) = g(x) \cdot h(x) = (x^2 + 1)(x^3 + 1) \) and
   \( f(x) = x^5 + x^3 + x^2 + 1 \).

3. Discuss the possibility of having two functions that do not have a derivative at a point but their product has a derivative at that point.
   Consider \( f(x) = g(x) \cdot h(x) = |x| [- |x|] \).

4. Consider
   Corollary. If \( f(x) = k \cdot g(x) \), where \( k \) is a constant, and \( g'(x) \) exists, then \( f'(x) = k \cdot g'(x) \)
   Prove this in class using the previous theorem.
   Verbalize this as the derivative of a constant and a function is the constant times the derivative of the function, provided the derivative exists.
5. Work examples as follows:
   \[ f(x) = 7x^5 \]
   \[ f(x) = 3(x^3 + x^4) \]
   \[ f(x) = -6x(x + 1) = -6[x(x + 1)] \]

6. Discuss
   Theorem. If \( f(x) = g(x)/h(x) \) and \( g'(x) \) and \( h'(x) \) exist, then
   \[ f'(x) = \frac{h(x) \cdot g'(x) - g(x) \cdot h'(x)}{h^2(x)} \]
   provided \( h(x) \neq 0 \).

   Verbalize this and prove it using the four steps for evaluating a derivative by the definition if time permits.

7. Work examples:
   \[ f(x) = g(x)/h(x) = x^7/x^2, \ x \neq 0 \]—compare with the derivative of \( f(x) = x^5 \).
   \[ f(x) = g(x)/h(x) = (x + 1)/(x - 3), \ x \neq 3 \]
   \[ f(x) = (x^2 + 3x)/(x^3 + 1), \ x^3 + 1 \neq 0 \]
   \[ f(x) = (x^3 + 2x^2 + x - 1)/(x^4 + 1), \ x^4 + 1 \neq 0. \]
LESSON IX

OBJECTIVES

1. Consider the Chain Rule and illustrate its use by applying it to several examples.

2. Consider the derivative of a curve at a point as the slope of the tangent line at that point.

PROCEDURES

1. Review the definition of a function—a set of ordered pairs such that no two distinct ordered pairs have the same first element.

2. A composite function is given by \( f \circ g(x) = (f \cdot g)(x) \).
   For example: \( f \circ g(x) = (f \cdot g)(x) = (x^2 + 1)^2 \)
   \[ z = g(x) = x^2 + 1 \text{ and } f(z) = f[g(x)] = z^2. \]

3. Consider
   Theorem: If \( F(x) = (f \circ g)(x) = f]\circ g(x)\], and \( f'(g(x)) \) and \( g'(x) \) exist, then \( F'(x) = f]\circ g(x)\].\( g'(x) \).
   a) Note—this is the Chain Rule.
   b) Verbalize this statement.
   c) Prove this theorem following the four steps for finding the derivative of a function.

4. Corollary: If \( F(x) = [g(x)]^n \), and \( g'(x) \) exists, \( F'(x) = n[g(x)]^{n-1} \cdot g'(x) \).
   a) Do not prove in class.
   b) Being careful to point out \( g(x) \), \( f]\circ g(x)\], \( g'(x) \), and \( f]\circ g(x)\], carefully evaluate the derivatives for each of the following:
   \[ F(x) = \sqrt{25 - x^2}, -5 \leq x \leq 5. \]
   \[ F(x) = (x^2 - 2x - 3)^{7/2} \]
   \[ F(x) = 1/(x^4 - 1)^2, x^4 - 1 \neq 0 \]
   \[ F(x) = (x - 2x^2)^{1/2} \]

5. Consider \( f(x) = |x - 1| \).
   a) Graph \( f(x) \).
   b) Using the four steps for finding a derivative, determine \( f'(x) \).
   c) Find \( f'(1) \) and simplify until you have \( \lim_{h \to 0} \frac{|h|}{h} \).
   d) Evaluate the two one sided limits and note they are unequal.
6. Consider \( f(x) \) as follows:
   a) Graph a curve, mark two points A and B, and draw the secant AB.
   b) Write the slope of AB as:
      \[ m = \frac{f(x + h) - f(x)}{(x + h) - x} \]
   c) Note that as B moves close to A that \( h \to 0 \). Thus the slope of AB approaches the slope of the tangent line at A. Thus,
      \[ m(\text{tangent}) = \lim_{h \to 0} m(\text{secant}) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]
   d) Since this is the same as the derivative, the slope of the tangent line to the graph of \( f \) at a point \( x \) is \( f'(x) \).

7. Consider \( f(x) = x^2 \).
   a) \( f'(x) = 2x \)
   b) The slope at \((1, 1)\) is 2.1 or 2.
   c) Thus, evaluate the slope for a few other points as \((2, 4), (0, 0), \left(\frac{7}{2}, \frac{49}{4}\right)\).

8. Briefly review the equation of a line and recall the two-point formula. Point out what is needed to use this formula.
   Apply this to \( f(x) = x^2 \) by finding the equation of the tangent line at \((-5, 25)\) for example.
LESSON X

OBJECTIVES

Determine the equation of the tangent line to various types of curves.

PROCEDURE

1. Determine the equation of the tangent line at the indicated point for each of the following:
   a) \( F(x) = \frac{\sqrt{25 - x^2}}{x} \) at (4, 3)—graph this function.
   b) \( f(x) = x^2 + x - 6 \) at (2, 0).
   c) \( f(x) = x^3 - 2x^2 + 5x - 1 \) at (-1, -3).

2. For each of the following, determine if the derivative exists by considering the definition of the derivative. To do this look at the two one-sided limits in each case.
   a) \( f(x) = |x - 1| \) at \( x = 1 \).
   b) \( f(x) = x \).
   c) \( f(x) = k \).

3. Recall that vertical lines have no slope. Since the slope \( m \) is undefined, we expect \( f' \) to not exist.
   a) Graph \( F(x) = \frac{\sqrt{25 - x^2}}{x} \) and give its domain.
      Point out that we have vertical tangents at (5, 0) and (-5, 0) and the derivative does not exist at these two points.
   b) Evaluate \( F'(x) \) and determine \( F'(5) \) and \( F'(-5) \). Since \( F'(5) \) and \( F'(-5) \) do not exist, the tangent lines are \( x = 5 \) and \( x = -5 \).
   c) Consider \( G(x) = x^{1/4}, x \geq 0 \).
   d) Give the domain and range of \( G(x) \) and graph it.
   e) Note that there is a vertical tangent at (0, 0) and the slope does not exist at (0, 0).
   f) Evaluate \( G'(x) \) and determine \( G'(0) \).
   g) Thus, the equation of the tangent line at (0, 0) is \( x = 0 \).
   h) Consider \( f(x) = \sqrt{x - 2}, x \geq 2 \) and determine the equation of the tangent line at (2, 0) as in the preceding examples.

4. Consider \( f(x) = \frac{1}{x} \).
   a) Graph \( f(x) \) and give its domain and range.
   b) Determine \( f'(x) \) and then \( f'(0) \).
c) Be sure to point out that we cannot write the equation for the tangent line because \( f(0) \) is not defined, i.e., we have no second element in our ordered pair.

5. Summarize this by saying that the derivative does not exist if:

a) \( \lim_{h \to 0^+} \frac{f(x + h) - f(x)}{h} \neq \lim_{h \to 0^-} \frac{f(x + h) - f(x)}{h} \)

b) \( \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \) becomes infinite at \( x \) even though \( f(x) \) exists.

c) If the function and its derivative do not exist at some point \( x \), then the tangent line does not exist.
LESSON XI

OBJECTIVES

Introduce and apply the First Derivative Test for graphing.

PROCEDURES

1. State: For a function $f(x)$,
   - If $f'(x) > 0$, the graph increases from left to right.
   - If $f'(x) < 0$, the graph decreases from left to right.
   - If $f'(x) = 0$, the graph has a horizontal tangent.

2. For familiar functions such as $f(x) = x^2$, $f(x) = 1/x$, $f(x) = x^3$, and $f(x) = -1/x$
   a) Graph $f(x)$.
   b) Determine $f'(x)$.
   c) Determine for which values of $x$ that $f'(x)$ is positive, negative, or zero.
   d) Point out that these values correspond to where the graph increases, decreases, or has a horizontal tangent, respectively.

3. Work an example such as $f(x) = x^2 - 1$.
   a) Find the zeros.
   b) Find the $y$-intercepts.
   c) Determine if the graph has a horizontal tangent.
   d) Determine where the graph increases.
   e) Determine where the graph decreases.
   f) Graph $f(x)$.
   g) To check against this $\square$, plot extra points.

4. Following the steps in #3, graph $f(x) = 2x^3 - 9x^2 + 12x$. 

```
\begin{tikzpicture}
    \draw[->] (-1,0) -- (5,0) node[right] {$x$};
    \draw[->] (0,-1) -- (0,5) node[above] {$y$};
    \draw (1.5,0) .. controls (2,1) and (3,-1) .. (4,0);
    \node at (2.5,1) {$f(x)$};
\end{tikzpicture}
```
LESSON XII

OBJECTIVES

Introduce the Second Derivative Test for graphing.

PROCEDURES

1. Discuss $f''(x)$ as being the derivative of $f'(x)$ by first noting that $f'(x)$ is a function.
   a) Point out the notation: $f''(x)$, $y''$, $D^2_y$.
   b) Find $f''(x)$ for such functions as
      
      \[
      y = f(x) = 3x^3 + 2x^2 - 5x
      \]
      and
      
      \[
      y = (x^2 + 2)^2
      \]

2. Apply the second derivative test to graphing.
   a) Note that the first derivative was interpreted as the slope of the tangent line at a point and we interpret the second derivative as the rate of change of the slopes of the tangent lines.
   b) Point out: If $y'' > 0$, the slopes are increasing.
      If $y'' < 0$, the slopes are decreasing.
      If $y'' = 0$, there is no change in slope.
   c) Discuss this using diagrams as follows.
      If $y'' > 0$, the graph is concave up.
      \[\text{[Diagram of upward concavity]}\]
      If $y'' < 0$, the graph is concave down.
      \[\text{[Diagram of downward concavity]}\]
      If $y'' = 0$, we have a point of inflection if $y''$ changes signs as we pass through this point, i.e., if the concavity changes.
   d) Point out:
      If $y' = 0$ (horizontal tangent) and if $y'' > 0$ (concave up), then we have a local minimum.
      If $y' = 0$ and $y'' < 0$, then we have a local maximum.

3. Graph some examples such as $y = f(x) = 2x^3 + 3x^2 - 12x$.

   Horizontal tangents: $(-2, 20), (1, -7)$
   Local minimum: $(1, -7)$
   Local maximum: $(-2, 20)$
   Point of inflection: $(-1/2, 13/2)$
APPENDIX F

CONCRETE INDUCTIVE LESSON PLANS
LESSON I

OBJECTIVES:

1. Review the definition of sequence, consider examples of sequences, write out terms of sequences, and determine limits of simple sequences.

2. Review graphs of inequalities such as \(|x - a| \leq p\) and \(|y - b| \leq q\) and the graphs of intersections of such graphs.

PROCEDURE:

1. a) Review the definition of a sequence:
   Definition. An infinite sequence is a collection of terms \([a_1, a_2, \ldots, a_n, \ldots]\) where \(a_n\) is a function of the positive integer \(n\). (It might be necessary to briefly review "function" at this point.)
   Examples:
   \[
   \left\{ \frac{1}{2^n} \right\}, \left\{ \frac{1}{n} \right\}
   \]
   Notation: We will use \(\{a_n\}\) to represent a sequence where \(a_n\) is the \(n\)-th term.
   Write out various terms, as tenth, hundredth, thousandth and note what happens as \(n\) gets large.

   b) Point out that we can get \(1/n\) as close to 0 as we like by taking \(n\) large enough.
   Example:
   If we want \(1/n\) smaller than .1, we can accomplish this by choosing \(n\) larger than \(1/10\) or 10.
   Example:
   \(1/n\) less than .05 from 0 implies \(n > 20\).
   \(1/n\) less than .001 from 0 implies \(n > 1000\).
   c) Consider \([2, 3/2, 4/3, \ldots]\) where the \(n\)-th term is \(\frac{n}{n+1}\).
   Write out some terms as: 10-th, 100-th, \(a_{1000}\).
   Plot some points to discover what happens as \(n\) gets large (\(n \to \infty\)).

   \[
   \begin{array}{cccccc}
   1 & \frac{1}{100} & \frac{2}{6} & \frac{3}{5} & \frac{4}{4} & \frac{5}{3} & \frac{6}{2} & 2 \\
   \end{array}
   \]
   So as \(n \to \infty\), \((n + 1)/n \to 1\).

2. a) Review the meaning of \(a_n\) "being close to" the number 1.
   As: \(a_n\) is less than .1 from 1 when either:
   \[
   a_n - 1 < .1 \quad \text{or} \quad 1 - a_n < .1 \Rightarrow |a_n - 1| < .1
   \]
b) Note that any two numbers are close to each other if \( |r - s| \) is very small.

3. Again consider \( a_n = (n + 1)/n \).
   Note that \( |a_n - 1| = \left| \frac{n + 1}{n} - 1 \right| = \frac{1}{n} \).
   Thus, \( |a_n - 1| < .1 \) when \( |1/n| < .1 \) but since \( 1/n > 0 \),
   \( 1/n < .1 \) when \( n > 10 \).
   Thus, \( \left| \frac{n + 1}{n} - 1 \right| < .02 \) when \( n > \) _______.
   If \( n > 250 \), \( \left| \frac{n + 1}{n} - 1 \right| < \) _______.

4. a) Consider \( \{-1, 1/2, -1/3, 1/4, \ldots\} \), where \( a_n = (-1)^n/n \).
   Note that as \( n \to \infty \), \( a_n \to 0 \).
   Again consider \( \left| (-1)^n/n - 0 \right| = \frac{1}{n} \) for all \( n \in \mathbb{I} \).
   Thus, \( \left| (-1)^n/n - 0 \right| < .1 \) if \( n > 10 \).
   b) Make note that the number "0" is called the limit of the sequence.

5. Have students graph on the number line all numbers that
differ from 2 by an amount less than 0.1.
   a) Draw graph on chalkboard.

5. a) Graph:

5. b) Point out various numbers which are included in this
    set such as 1.91, 1.95, 1.99, 2.01, 2.09, etc.

6. Have the students graph in the xy-plane all the points
   that have x-coordinate that differs from 2 by an amount
   less than 0.1.
   a) Draw graph:

6. a) Graph:

6. b) Point out various points satisfying this as:
    \((2.01, 2.01), (1.95, 3)\).

7. Interpret these stipulations both algebraically and in
terms of distance from 2:
   a) \( 1.9 < x < 2.1 \)
   b) \( |x - 2| < 0.1 \)

8. Repeat #5, #6, #7 for "all numbers x less than .05 from
   -3."

9. Point out that a set of numbers x such that x is close to
   a number a is a neighborhood of the number a.
   Example: \( |x - 4| < 0.1 \)
   \( |x + 2| < 0.06 \)
10. Define a neighborhood:
Definition: A neighborhood of a number $a$ is the set of numbers $x$ such that $|x - a|$ is less than some positive number.

11. Apply the above to the $y$-axis.
Example: Shade in the region for which the $y$-coordinate is less than 0.2 from 1.
Again give the algebraic interpretations: $|y - 1| < 0.2$ or $0.8 < y < 1.2$.

12. Combining the above, graph the set of points whose $x$-coordinate is less than 0.07 from 2 and $y$-coordinate is less than 0.1 from -1.

a)

```
    2
   /|
 /  |
|  |
|  |
|  |
  |
  |
-4
```

b) Algebraically, write this as $|x - 2| < 0.07$ and $|y + 1| < 0.1$.

c) Repeat with another example as $|x + 1| < 0.3$ and $|y - 2/3| < 1/6$. 

LESSON II

OBJECTIVES

1. Develop in the students an intuitive idea of limit using a linear example.

2. Evaluate algebraically the limit of several functions: a function defined over the real numbers, a function undefined at a point and with no limit at that point, and a function having a limit at an undefined point.

PROCEDURES

1. Consider \( f(x) = 2x + 1 \).
   a) Graph \( f(x) \).
   b) Point out that \( f(x) \) increases as \( x \) increases.
   c) Explain that if \( 0.9 < x < 1.1 \), then \( f(x) > 2.8 \) and \( f(x) < 3.2 \).
   d) Write this algebraically: For the function \( f(x) = 2x + 1 \), for all \( x \) such that \( |x - 1| < 0.1 \), we have \( |f(x) - 3| < 0.2 \).
   e) Verbalize this as: As \( x \) is close to 1, \( f(x) \) is close to 3.
   f) Have students work an example with you such as:
      If \( |x - 1| < 0.05 \), then \( |f(x) - 3| < 0.1 \).
   g) Generalize this to: If \( |x - 1| < d \), then \( |f(x) - 3| < 2d \) for some distance \( d \).
      Again point out that if \( d \) gets smaller, \( x \) gets closer to 3 since \( 2d \) also gets smaller.

2. Again consider \( f(x) = 2x + 1 \) as \( x \) gets close to 1. Conclude, with the help of the students, that the limit of \( f(x) \) as \( x \) approaches 1 is 3.

3. Demonstrate that to make \( x \) closer to some number \( a \) we decrease the width of the vertical strip and to make \( f(x) \) closer to \( f(a) \), we decrease the width of the horizontal strip.

4. Consider \( f(x) = \frac{2(x + 1)}{|x + 1|} \).
   a) Recall that \( f(x) = \begin{cases} 2 & \text{if } x > -1 \\ -2 & \text{if } x < -1 \end{cases} \)
   b) Graph \( f(x) \).
5. a) Discuss \( f(x) \) as \( x \) approaches 3. Point out that if \( |x + 3| < 0.1 \), then \( |f(x) - 2| \) is less than any positive number since \( f(x) = 2 \) for all \( x > -1 \).

b) Have students give possible values for \( d \) such that if \( |x - 3| < d \), then \( |f(x) - 2| < 0.5 \).

\[
\frac{2(x + 1)}{|x + 1|}
\]

6. Discuss \( f(x) = \frac{2(x + 1)}{|x + 1|} \) for values close to -1.
   a) Ask students to find a number \( L \) which \( f(x) \) gets close to as \( x \) gets close to -1.
   b) Point out that the smallest width of the horizontal strip is 4 since the maximum \( f(x) \) is 2 and the minimum \( f(x) \) is -2. Thus, no matter how narrow we make the vertical strip, the horizontal strip still has a width of 4.
   c) Point out that as \( x \) gets close to -1, we cannot force \( f(x) \) close to any unique number. Conclude that the limit of \( f(x) = \frac{2(x + 1)}{|x + 1|} \) as \( x \) approaches -1 does not exist.

7. Consider \( g(x) = \frac{x^2 - 4}{x - 2} \) which is not defined at \( x = 2 \).
   a) Discuss the limit of \( g(x) \) as \( x \) approaches 1 as follows: \( g(x) = x + 2 \) if \( x \neq 2 \).
      
      \[
      g(0.9) = 2.9 \quad g(0.99) = 2.99 \\
      g(1.1) = 3.1 \quad g(1.01) = 3.01
      \]
      Thus, as \( x \) approaches 1, \( g(x) \) approaches 3.
      Also note that \( g(1) = 3 \).
      Conclude that the limit of \( g(x) \) as \( x \) approaches 1 is 3.
   b) Discuss \( g(x) \) as \( x \to 2 \) as follows:
      
      \[
      g(x) = x + 2 \text{ if } x \neq 2.
      \]
      \[
      g(1.9) = 3.9 \quad g(1.99) = 3.99 \\
      g(2.1) = 4.1 \quad g(2.01) = 4.01
      \]
      Thus, the limit of \( g(x) \) as \( x \) approaches 2 is 4.
   c) Be sure to point out that we do not consider \( g(x) \) at 2 since it is undefined there. Thus, we will delete 2 from the neighborhood \( |x - 2| < d \) by writing \( 0 < |x - 2| < d \). This is a deleted neighborhood of 2.
      Give another example such as: \( 0 < |x - 4| < 0.1 \).
   d) Graph \( g(x) \). Indicate all \( x \) so that \( 0 < |x - 2| < 0.2 \).
e) Discuss the bounds on $g(x)$ for $0 < |x - 2| < 0.2$, i.e.,
$3.8 < g(x) < 4.2$ or $|g(x) - 4| < 0.2$.
f) Conclude that for all $x$ such that $0 < |x - 2| < 0.2$, we have $|g(x) - 4| < 0.2$.
g) Verbalize this as: as $x$ gets close to 2, $g(x)$ gets close to 4.
h) Point out that the limit of $g(x)$ as $x$ approaches 2 is 4 even though $g(2)$ is not defined.

8. If there is time, consider $f(x) = \frac{1}{x}$. Discuss the limit as $x$ approaches some number such as 3 and then as $x$ approaches 0. Discuss in the same manner as the preceding examples.
If there is not time, assign this function as a problem.
LESSON III

OBJECTIVES

Discuss the limit of various functions that do not have a limit at a certain point.

PROCEDURES

1. Consider \( f(x) = \lfloor x \rfloor \) at \( x \) close to 2.
   a) Place the following table on the board:
   
   \[
   \begin{array}{c|c|c|c|c|c}
   x & 1.9 & 1.99 & 2 & 2.01 & 2.1 \\
   f(x) & 1 & 1 & 2 & 2 & 2
   \end{array}
   \]
   b) Point out that for values of \( x \) within 0.1 of 2, i.e., \(|x - 2| < 0.1\), that \( f(x) \) is either 1 or 2.
   c) In terms of a horizontal strip, if we decrease its width, \( f(x) \) still has a value of 1 or 2.
   d) Conclude, with students, that as \( x \) approaches 2, the limit of \( f(x) \) does not exist since the corresponding values of \( f(x) \) do not get as close as we like to any unique number.

2. Consider \( f(x) = \lfloor x \rfloor \) as \( x \) approaches 2 from the right and then left.
   a) Establish the notation: \( x \to 2^+ \) and \( x \to 2^- \)
   b) Note that: \( \{ \text{as } x \to 2^+, f(x) = 2 \text{ and } \}
   \{ \text{as } x \to 2^-, f(x) = 1. \}
   c) Generalize these results as: if the two one-sided limits at a point are not equal, the limit at that point does not exist.
   d) Discuss the converse of (c) and emphasize that if the limit of a function \( f \) does exist at some point \( a \), then the two one-sided limits at \( a \) are equal.

3. Consider \( f(x) = \lfloor x \rfloor \) as \( x \to 1.5 \).
   a) Discuss: \( |x - 3/2| < 0.1 \iff 1.4 < x < 1.6 \) and \( f(x) = 1 \) for all such \( x \)'s.
   b) Conclude that if \( |x - 1.5| < 0.1 \), then \(|f(x) - 1|\) is less than any positive number.
   Verbalize this as: \( x \to 1.5, f(x) \) is close to 1.

4. Work the following as examples:
   a) As \( x \to 2, 3x^2 - 2x + 7 \to \)
   b) As \( x \to 2, 3x/(x + 4) \to \)
   c) As \( x \to 3, (x^2 - 2x - 3)/(x - 3) \to \)
   d) As \( x \to 2, (1/x - 1/2)/(x - 2) \to \)
   e) As \( x \to 0, (b + x)^3 - b^3 \to \)
   f) As \( x \to 0, \sqrt[2]{x} \to \sqrt[2]{x} \to \)
5. Consider \((x^2 - x - 6)/(x + 1)\) as \(x \to -1\) by looking at:
\[
\begin{align*}
\text{As } x \to -1, \quad (x + 1) & \to 0 \\
\hline
x & x^2 - x - 6 & x + 1 & \frac{x^2 - x - 6}{x + 1} \\
-0.9 & -4.29 & -42.9 & -1.1 & -3.69 & -0.1 & 36.9 \\
-0.99 & -4.0299 & -402.99 & -1.01 & -3.9699 & -0.01 & 396.99
\end{align*}
\]

a) Complete and discuss the following:

b) As \(x \to -1^-\), it appears that \((x^2 - x - 6)/(x + 1)\) is negative and larger in absolute value.

b) As \(x \to -1^+\), it appears that \((x^2 - x - 6)/(x + 1)\) is becoming larger.

c) Conclude that as \(x \to -1\), \((x^2 - x - 6)/(x + 1)\) does not have a limit as \(\left|\frac{x^2 - x - 6}{x + 1}\right|\) increases without bound.

6. Consider \(f(x) = \frac{x - 3}{x^2 - 9}\)

a) Note that for \(x > 3\) or \(x < -3\), \(|x^2 - 9| = x^2 - 9\) and for \(-3 < x < 3\), \(|x^2 - 9| = -(x^2 - 9)\).

b) Rewrite \(f(x)\) as:
\[
f(x) = \begin{cases} 
  x + 3 & \text{if } x > 3 \text{ or } x < -3 \\
  -(x + 3) & \text{if } -3 < x < 3
\end{cases}
\]

c) Evaluate: As \(x \to 3^+\), \(f(x) \to 6\)

As \(x \to 3^-\), \(f(x) \to -6\)

d) Recall that the limit of \(f(x)\) as \(x \to 3\) does not exist since the two one-sided limits are not equal.
LESSON IV

OBJECTIVES

1. Generalize the intuitive idea of limit and formally define the limit of a function.
2. Define the limit of a sequence.

PROCEDURES

1. Explain to the students that today we are going to generalize much of what has been said in the past three days.
   a) Point out that we are considering the values of \( f(x) \) in some neighborhood of a point \( x = a \) except possibly at \( x = a \).
   b) Review meaning of \( 0 < |x - a| < \delta \) (some positive number)
   c) Consider:
      \[
      \begin{array}{c}
      \text{Discuss the idea that}
      \end{array}
      \]
      \[
      \text{L appears to be the}
      \]
      \[
      \text{limit of } f(x) \text{ as } x \to a.
      \]
      \[
      \text{In other words, as}
      \]
      \[
      x \to a, f(x) \text{ gets as}
      \]
      \[
      \text{close as we please to}
      \]
      \[
      L \text{ for all } x \text{ sufficiently close to } a.
      \]
      \[
      \text{Restate: all } f(x) \text{ is in a neighborhood of } L \text{ or all } f(x) \text{ such that } |f(x) - L| < \varepsilon \text{ for any } \varepsilon > 0.
      \]
   d) Discuss making \( \varepsilon \) smaller, i.e., we must be closer to \( a \) to guarantee \( |f(x) - L| < \varepsilon \). Point out that we must choose \( \delta \) such that \( 0 < |x - a| < \delta \).

2. Consider \( g(x) \) where \( g(a) \) is not defined.
   a) \[
   \begin{array}{c}
   \text{Consider picking } \delta' \text{ such that } 0 < \delta' < \delta. \text{ Point out that it is still true that } |g(x) - L| < \varepsilon \text{ for all } x \text{ such that } 0 < |x - a| < \delta'.
   \end{array}
   \]
d) Consider picking \( d'' \) such that \( d'' > d' \). Demonstrate that it is not now true that \( |g(x) - L| < \epsilon \) for all \( x \) satisfying \( 0 < |x - a| < d'' \).

e) Point out that many \( d' \)'s work and we need only find one.

3. Define limit of a function:
   a) Definition. The limit of a function \( f(x) \) as \( x \) approaches \( a \) is a number \( L \) if \( |f(x) - L| < \epsilon \) for all \( x \) such that \( 0 < |x - a| < d' \).
   b) The notation used is: \( \lim_{x \to a} f(x) = L \).
   c) Discuss the definition as to what is given and what is not given:
      - \( \epsilon \) is given and may be any positive number.
      - We must find a corresponding \( d' \) which satisfies the required inequalities for any \( \epsilon \) given us. We need not find a best \( d' \) but any one that works.
      - \( f(a) \) may or may not be defined.
      - The definition does not tell us how to find the limit \( L \).

4. Define the limit of a sequence:
   a) Definition. The limit of a sequence \( \{a_n\} \) is the number \( A \) if given any \( \epsilon > 0 \), there is an integer \( N \) such that \( |a_n - A| < \epsilon \) for all \( n > N \).
   b) Definition. If a sequence has a limit, we say that it converges.
   c) Definition. If a sequence does not converge, we say it diverges.

5. Consider \( \{a_n\} = \{(-1)^n\} = \{-1, 1, -1, 1, \ldots\} \) and discuss the \( \lim_{n \to \infty} a_n \).
   a) Since \( |a_n - 1| > \epsilon \) for some \( n \) larger than any positive integer, if \( \epsilon < 2 \), there is no limit.
   b) Note that 0 is not a limit since \( |a_n - 0| > 0.9 \) for any \( n \).
LESSON V

OBJECTIVES

Introduce four basic limit theorems.

PROCEDURES

1. Consider \( f(x) = k \), \( k \) a constant.
   a) Graph \( f(x) \).
   b) Guess that \( \lim_{x \to a} f(x) = k \) since \( f(x) = k \) for all \( x \).
   c) Discuss \( |f(x) - k| < \epsilon \) for all \( x \) satisfying \( 0 < |x - a| < \delta \).
   Suppose \( \epsilon = 0.1 \). Note that, for any choice of \( \delta \), \( |f(x) - k| < 0.1 \) for all \( x \) satisfying \( 0 < |x - a| < \delta \).

2. Consider \( f(x) = x \).
   a) Graph \( f(x) \).
   b) Guess that \( \lim_{x \to a} f(x) = a \).
   c) Again for any \( \epsilon > 0 \), find a \( \delta > 0 \) such that if \( 0 < |x - a| < \delta \), then \( |f(x) - f(a)| < \epsilon \).
   Discuss \( |f(x) - f(a)| = |x - a| \) so we can choose or any positive smaller number.

3. Consider \( f(x) = x+3 \).
   a) Guess that as \( x \to 4 \), \( x+3 \) approaches 7.
   b) Point out that: \( \lim_{x \to 4} x = 4 \) and \( \lim_{x \to 4} 3 = 3 \).
   c) Point out that: \( \lim_{x \to 4} (x+3) = \lim_{x \to 4} x + \lim_{x \to 4} 3 \).

4. Generalize with:
   Theorem. If \( \lim_{x \to a} f(x) = F \) and \( \lim_{x \to a} g(x) = G \), then \( \lim_{x \to a} (f(x) + g(x)) = F + G \).
   a) Proof:
      For any \( \epsilon_1 > 0 \) there exists a \( \delta_1 > 0 \) such that \( |f(x) - F| < \epsilon_1 \) for all \( x \) such that \( 0 < |x - a| < \delta_1 \).
      For all \( \epsilon_2 > 0 \) there exists a \( \delta_2 > 0 \) such that \( |g(x) - G| < \epsilon_2 \) for all \( x \) such that \( 0 < |x - a| < \delta_2 \).
      \[ |f(x) + g(x) - (F + G)| = |f(x) - F + g(x) - G| \leq |f(x) - F| + |g(x) - G| \]
      \[ \leq \epsilon_1 + \epsilon_2 \]

Be sure to discuss the choice of \( \delta \) as follows:
Recall that: \( |f(x) - F| < \epsilon_1 \) if \( 0 < |x - a| < \delta_1 \)
\( |g(x) - G| < \epsilon_2 \) if \( 0 < |x - a| < \delta_2 \).
Consider the diagram:

\[ \begin{align*}
\text{Determine where the two deleted neighborhoods of } x = a \text{ have a common deleted neighborhood.} \\
\text{This will occur if we choose } \delta \text{ equal to the smaller of } \delta_1 \text{ and } \delta_2, \text{ as we will then satisfy both inequalities and will have a common neighborhood.} \\
\text{Hence, for all } x \text{ such that } 0 < |x - a| < \delta', \text{ we have } |f(x) + g(x) - (F + G)| < \varepsilon.
\end{align*} \]

5. Consider \( f(x) = 3x \).
   a) Guess \( \lim_{x \to 4} f(x) = 12 \) after substituting numbers near 4.
   b) Note: \( \lim_{x \to 4} 3 = 3 \) and \( \lim_{x \to 4} x = 4 \).
   c) Note: \( \lim_{x \to 4} (3x) = (\lim_{x \to 4} 3)(\lim_{x \to 4} x) = 12 \).

6. Consider \( g(x) = x^2 \).
   a) The \( \lim_{x \to 4} x^2 \) for values close to 4 is 16.
   b) Again, \( \lim_{x \to 4} x \cdot x = (\lim_{x \to 4} x)(\lim_{x \to 4} x) = 16 \).

7. Consider \( g(x) = x(x + 3) = x^2 + 3x \).
   a) \( \lim_{x \to 4} g(x) = (\lim_{x \to 4} x)(\lim_{x \to 4} (x + 3)), \) etc.
   b) \( \lim_{x \to 4} g(x) = \lim_{x \to 4} x^2 + \lim_{x \to 4} 3x, \) etc.

8. Generalize with:
   Theorem. If \( \lim_{x \to a} f(x) = F \) and \( \lim_{x \to a} g(x) = G \), then
   \[ \lim_{x \to a} [f(x) \cdot g(x)] = F \cdot G. \]
   Do not prove this theorem, but relate that we must show: For all \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that \( |f(x) \cdot g(x) - F \cdot G| < \varepsilon \) whenever \( 0 < |x - a| < \delta \).
9. Consider \( f(x) = x^2 - 3x \).

Discuss: \( \lim_{x \to a} (x^2 - 3x) = \lim_{x \to a} x^2 + \lim_{x \to a} (-1)(3x) \)

\( \lim_{x \to a} x^2 + (-1)\lim_{x \to a} 3x \)

\( \lim_{x \to a} x^2 - \lim_{x \to a} 3x \)

10. Generalize with:

**Theorem.** If \( \lim_{x \to a} f(x) = F \) and \( \lim_{x \to a} g(x) = G \), then

\( \lim_{x \to a} [f(x) - g(x)] = F - G \).

Do not prove here.

11. Consider \( f(x) = \frac{x^2 - 4}{x + 2} \)

a) Discuss \( \lim_{x \to 3} f(x) \) as follows:

\( f(3) = 1, \quad f(3.1) = 1.1, \) and \( f(2.9) = 0.9 \).

It appears that as \( x \to 3 \), \( f(x) \to 1 \).

Note that \( \lim_{x \to 3} (x^2 - 4) = 5 \) and \( \lim_{x \to 3} (x + 2) = 5 \).

Conclude that it appears that the limit is the quotient of the limits.

b) Discuss the \( \lim_{x \to -2} f(x) \):

\( \lim_{x \to -2} f(x) = x + 2 \) if \( x \neq -2 \).

Point out that is a deleted neighborhood of \(-2\),

that \( x - 2 \) is neighborhood of \(-4\).

Thus, \( \lim_{x \to -2} (x^2 - 4)/(x + 2) = -4 \).

Be sure to illustrate: \( \lim_{x \to -2} (x^2 - 4) = 0 \) and

\( \lim_{x \to -2} (x + 2) = 0 \)

so the limit is not the quotient of the limits.

12. Generalize with:

**Theorem.** If \( \lim_{x \to a} f(x) = F \) and \( \lim_{x \to a} g(x) = G \), then

\( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{F}{G} \) unless \( G = 0 \).

Do not prove this theorem but note that we must show:

For all \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that

\[ \left| \frac{f(x)}{g(x)} - \frac{F}{G} \right| < \epsilon \quad \text{whenever} \quad 0 < |x - a| < \delta. \]
Briefly discuss \( \lim_{x \to \infty} f(x) \); point out that they are evaluated in a manner similar to the limit of a sequence.
LESSON VI

OBJECTIVES

1. Develop an intuitive idea of continuity by considering various illustrative examples.

2. Formally define continuity of a function at a point.

3. Formally define continuity of a function on a set.

PROCEDURES

1. Graph and discuss various kinds of discontinuous functions such as those listed below.

   \[
   f(x) = \begin{cases} 
   [x] & x \geq 2 \\
   2x + 1 & x < 2 \\
   1 & x = 2
   \end{cases}
   \]

   f(x) = \begin{cases} 
   1 & x = 2 \\
   \frac{1}{x} & \text{otherwise}
   \end{cases}

   a) With reference to these examples, informally discuss discontinuity in terms of "jumps," "holes," or the like.
   
   b) Then, for each function, point out for which ones

   (1) \( \lim_{x \to a} f(x) \) does not exist, (2) \( f(a) \) does not exist, or (3) \( \lim_{x \to a} f(x) \neq f(a) \).

2. Give examples of continuous functions and point out that in each case, \( \lim_{x \to a} f(x) = f(a) \).

   Discuss the meaning of "\( \lim_{x \to a} f(x) = f(a) \)," i.e., the function must be defined at \( a \), the limit must exist and must equal \( f(a) \).

3. Consider a function such as \( f(x) = x^2 \) over \([0, 1]\).

   Discuss continuity at the endpoints.

4. Define continuity at a point.

   Definition. A function \( f \) is continuous at a point \( a \) if and only if \( \lim_{x \to a} f(x) = f(a) \).

5. Define continuity over a set.

   Definition. A function is continuous on a set \( X \) if and only if it is continuous at each point of \( X \).
LESSON VII

OBJECTIVES

1. Evaluate the slope of a tangent line to a curve by considering a sequence of slopes of secant lines to the curve through the point of tangency and a nearby point.

2. Evaluate the slope of a tangent line to a curve at any point on the curve.

3. Formally define the slope of a tangent line to a curve.

PROCEDURES

1. Consider \( y = x^2 \) and the slope of the tangent line at \( x=1 \).
   a) Graph the function and the tangent line at \((1, 1)\).
   b) Review the formula for the slope of a line:
      \[
      y = \frac{(y_2 - y_1)}{(x_2 - x_1)}.
      \]
      Discuss why this cannot be applied to our tangent line (need two points).
   c) Consider a secant line through point \( P \) at \((1, 1)\) and a close point \( Q \) at \((2, 4)\). Evaluate the slope of this secant line.
   d) Hand out copies of the table below and complete it in class.

<table>
<thead>
<tr>
<th>( x_p )</th>
<th>( y_p = f(x_q) )</th>
<th>( x_q )</th>
<th>( y_q = f(x_q) )</th>
<th>( h = x_q - x_p )</th>
<th>( f(x_q) - y_p )</th>
<th>( m_s = \frac{y_q - y_p}{x_q - x_p} )</th>
<th>( f(x_q) - f(x_p) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(PQ1)</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>(PQ2)</td>
<td>1</td>
<td>1</td>
<td>1.5</td>
<td>2.25</td>
<td>1</td>
<td>3</td>
<td>2.5</td>
</tr>
<tr>
<td>(PQ3)</td>
<td>1</td>
<td>1</td>
<td>1.1</td>
<td>1.21</td>
<td>.1</td>
<td>3</td>
<td>2.5</td>
</tr>
<tr>
<td>(PQ4)</td>
<td>1</td>
<td>1</td>
<td>1.01</td>
<td>1.0201</td>
<td>.01</td>
<td>.0201</td>
<td></td>
</tr>
<tr>
<td>(PQ5)</td>
<td>1</td>
<td>1</td>
<td>1.0001</td>
<td>1.00020001</td>
<td>.001</td>
<td>.00020001</td>
<td></td>
</tr>
</tbody>
</table>
2. Repeat this process in general as follows:
   a) Graph \( y = x^2 \) and mark point A at \((x_1, f(x_1))\).
   b) Choose B to the right of A and with x-coordinate of \(x_1 + h\).
   c) Using \( m = \frac{y_2 - y_1}{x_2 - x_1} \), compute the slope of the line through AB:
      \[ m_s = \frac{(x_1 + h)^2 - x_1}{(x_1 + h) - x_1} = 2x_1 + h. \]
   d) Note that if we choose h very small, \( m_s \) gets very close to \(2x_1\).
      Notation: \( \lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h} \)
   e) Point out that as B gets close to A, the secant line gets close to the tangent line and the slope of the secant line gets close to the slope of the tangent line.
   f) We write: \( \lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h} \) \( \lim_{h \to 0} \frac{2x_1 + h}{h} = 2x_1 \).
      Thus, for the slope \( m_t \) of the tangent line we have \( m_t = 2x_1 \).
   g) Again consider the point \((1, 1)\). \( m_t = 2x_1 \) or \( m_t = 2 \).
   h) Work other examples as \((2, 4)\), \((-5, 25)\), \((0, 0)\).

3. Repeat above with \( y = \sqrt{25 - x^2} \) at \( x_1 = 4 \). However, hand out the table as below:
   a)
b) After finding the slope of the tangent line at (4, 3), find it in general as in #2 above, i.e., consider
\[
\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{-2x + h}{\sqrt{25 - (x+h)^2} + \sqrt{25 - x^2}}
\]
\[
= \frac{-x}{\sqrt{25 - x^2}}.
\]
c) Evaluate this limit for a few specific points as (0, 5), (-4, 3), (1, 24).

4. Repeat the above for a general function f(x) as follows:
   a) Graph f(x).
   b) Set up the difference quotient and explain that \( m_t = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \). (1) Compute f(x + h) and f(x).
   (2) Evaluate the difference f(x + h) - f(x).
   (3) Find the difference quotient \( \frac{f(x+h) - f(x)}{h} \).
   (4) Compute \( \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \).

5. Define the slope of a tangent line.
   a) Definition. The slope of the tangent line to the graph of f, at the point (x,y) is \( \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \), if this limit exists.
   b) Outline the steps for finding \( m_t \).
      (1) Compute f(x + h) and f(x).
      (2) Evaluate the difference f(x + h) - f(x).
      (3) Find the difference quotient \( \frac{f(x+h) - f(x)}{h} \).
      (4) Compute \( \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \).

6. Apply the above to finding the equation of the tangent line by an example such as: \( y = x^2 \) at (1, 1).
   Work other examples as time permits.
LESSON VIII

OBJECTIVES

1. Consider examples of functions for which the slope of the tangent line does not exist at a point because (a) the limit does not exist and (b) the tangent line is vertical.

2. Define the derivative of a function.

3. Develop four basic derivative theorems.

PROCEDURES

1. Consider \( f(x) = |x - 1| \)
   a) Graph \( f(x) \)
   b) Consider point \( P \) at \((1, 0)\) and a point \( B \) to the right of \( P \). Find the slope of the secant line \( PB \) as \( B \) moves closer to \( P \). Note that the secant and tangent lines coincide and the slope is 1.
   c) Do the same on the left and notice that the slope is -1.
   d) Point out that we have:
      \[ \lim_{h \to 0^+} \frac{f(x + h) - f(x)}{h} = 1, \quad \lim_{h \to 0^-} \frac{f(x + h) - f(x)}{h} = -1. \]
      Conclude that \( \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \) does not exist since the two one-sided limits are not equal.

2. Consider \( f(x) = \sqrt{x - 2} \)
   a) Graph \( f(x) \).
   b) Consider the tangent line at \((2, 0)\) by evaluating the two one-sided limits as above.
      Note that the left-hand limit cannot be determined since we are not in the domain.
   c) Find the right hand limit and evaluate it at \( x = 2 \). Note also that \( f(2) \) exists.
   d) Conclude that even though \( f(2) \) is defined, the tangent line has undefined slope and is a vertical line.

3. Consider \( f(x) = \frac{1}{x} \)
   a) Graph \( f(x) \).
   b) Evaluate the one-sided limits at 0 as before and conclude that the limit does not exist.
   c) Point out that \( f(0) \) is undefined.
   d) Since the limit does not exist and \( f(0) \) is undefined, conclude that there is no tangent line at \( x = 0 \).
Define the derivative of a function \( f \) at a point \( x \).

a) Definition. The derivative of \( f \), evaluated at \( x \), is
\[
\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]
if this limit exists.

b) Notation: \( f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \).

Note that \( f'(x), h', D_{xy} \) and \( \frac{dy}{dx} \) are used interchangeably.

c) Point out that this process is called differentiation.

5. Point out that we have already proved the following:

**Theorem.** If \( f(x) = k \), \( f'(x) = 0 \).

In terms of tangent lines this means the slope is 0.

6. We have also proved:

**Theorem.** If \( f(x) = x \), \( f'(x) = 1 \).

Again, the slope of the tangent line is 1.

7. Consider \( f(x) = x^n \).

a) Recall that for \( n=2 \), \( f(x) = x^2 \), and we found
\( f'(x) = 2x \).

b) In an exercise, we found that \( f'(x) = 3x^2 \) for
\( f(x) = x^3 \).

c) Summarizing, we have:
\[
\begin{align*}
\text{f(x)} &= x^1, \quad f'(x) = 1 \cdot x^{1-1} = 1 \cdot x^0 = 1, 1 = 1 \\
\text{f(x)} &= x^2, \quad f'(x) = 2 \cdot x^{2-1} = 2x^1 \\
\text{f(x)} &= x^3, \quad f'(x) = 3 \cdot x^{3-1} = 3x^2 \\
\text{f(x)} &= x^4, \quad f'(x) = 4 \cdot x^{4-1} = 4x^3 \\
& \vdots \\
\text{f(x)} &= x^n, \quad f'(x) = n \cdot x^{n-1}
\end{align*}
\]

d) Summarize with

**Theorem.** If \( f(x) = x^n \), where \( n \) is a positive integer,
\( f'(x) = nx^{n-1} \).

Do not prove this in class but point out that
the proof is by induction.

e) Work examples as \( f(x) = x^5 \), \( f(x) = x^7 \)
f) Point out that this theorem can be proved for real
numbers and not just integers so we will apply it to
real numbers as well.

g) Work examples as \( f(x) = x^{4/5}, \ f(x) = x^{-5}, \ f(x) = x^{-1/7}, \ f(x) = x^{10} \).
8. Consider \( f(x) = x^2 + x - 6 \).
   a) Recall that we found the slope to be \( 2x + 1 = f'(x) \).
   b) Recall the derivative of \( x^2 \) is \( 2x \) and the derivative
      of \( x \) is \( 1 \) and the derivative of \(-6\) is \( 0\).
      Summing, this would give us \( 2x + 1 \).

9. Consider \( f(x) = x + 2 \) and evaluate \( f'(x) \) as in #8.

10. Consider \( f(x) = x^3 + x^2 + x + 11 \) and evaluate \( f'(x) \) as
    in #8.

11. State
    Theorem. If \( f(x) = g(x) + h(x) \), and \( g'(x) \) and \( h'(x) \)
    exist, then \( f'(x) = g'(x) + h'(x) \).
        Prove this theorem if time permits.
        Verbalize this as "the derivative of a sum is the
        sum of the derivatives if they each exist."

12. Work examples such as \( f(x) = x^2 - x + 6 \),
    \( f(x) = x^3 - x^2 - x - 11 \).
LESSON IX

OBJECTIVES

1. Consider the derivative of the product of two functions.
2. Consider the derivative of the quotient of two functions.

PROCEDURES

1. Discuss the derivative of the product of two functions as follows:
   a) Consider \( f(x) = x^6 \).
      Note that \( f'(x) = 6x^5 \).
      Write \( f(x) = x^6 \cdot x^2 \) and note that:
      - If \( g(x) = x^4 \), \( g'(x) = 4x^3 \).
      - If \( h(x) = x^2 \), \( h'(x) = 2x \).
   b) Write this as:
      \[ \begin{array}{c|c|c}
         \text{Function} & \text{Evaluated as } x & \text{Derivative Evaluated at } x \\
         \hline
         g(x) & x^4 & 4x^3 \\
         h(x) & x^2 & 2x
      \end{array} \]
      Point out that we can obtain the derivative of \( f(x) = x^6 \), i.e., \( 6x^5 \), if we take the sum of the products of the terms on the opposite ends of the arrows:
      \[ x^4 \cdot 2x + x^2 \cdot 4x^3 = 6x^5. \]
   c) Repeat this discussion using \( f(x) = x^6 = x \cdot x^5 \) and \( x^3 \cdot x^3 \).

2. Generalize by considering:
   - If \( f(x) = x^6 = x^4 \cdot x^2 \), \( f'(x) = x^4 \cdot 2x + x^2 \cdot 4x^3 = 6x^5 \).
   - If \( f(x) = x^6 = x^5 \cdot x \), \( f'(x) = x^5 \cdot 1 + x \cdot 5x^4 = 6x^5 \).
   - If \( f(x) = x^6 = x^3 \cdot x^3 \), \( f'(x) = x^3 \cdot 3x^2 + x^3 \cdot 2x = 6x^5 \).
   ... 
   - If \( f(x) = g(x) \cdot h(x) \), \( f'(x) = g(x) \cdot h'(x) + h(x) \cdot g'(x) \).
      Verbalize this as the derivative of the product of two functions is the sum of the first function multiplied by the derivative of the second function and the second function multiplied by the derivative of the first function, if the two derivatives exist.
3. State Theorem. If \( f(x) = g(x) \cdot h(x) \) and if \( g'(x) \) and \( h'(x) \) exist, then \( f'(x) = g(x) \cdot h'(x) + h(x) \cdot g'(x) \).

a) Point out that if the derivative of one function does not exist at a point that the derivative of the product will not exist. For example use \( g(x) = x + 2 \) and \( h(x) = 1/(x + 2) \).

b) Using the product rule, differentiate \( f(x) = x^2 \cdot x^3 \) and compare it to the derivative of \( f(x) = x^5 \).

c) Evaluate the derivative of \( f(x) = x(1 - x) = x - x^2 \) by both the product rule and the difference rule.

d) Work additional examples as \( f(x) = x^3 \), \( f(x) = -6(x + 1) \), \( f(x) = 7(x^5 + x^3 + x^2 + 1) \).

e) Rewrite the above as:
\[
\begin{align*}
&\text{If } f(x) = 7x^2, f'(x) = 7 \cdot 5x^4. \\
&\text{If } f(x) = -6(x + 1), f'(x) = -6 \cdot 1. \\
&\text{If } f(x) = 7(x^5 + x^3 + x^2 + 1), f'(x) = 7(5x^4 + 3x^2 + 2x). \\
\end{align*}
\]

f) Generalize these results with:

Theorem. If \( f(x) = k \cdot g(x) \), where \( k \) is a constant, and if \( g'(x) \) exists, then \( f'(x) = k \cdot g'(x) \).

Verbalize this as the derivative of a constant times a function is the constant times the derivative of the function, if this derivative exists.

4. Point out that since the derivative of a product is not the product of the derivatives, we would not expect the derivative of a quotient to be the quotient of the derivatives.

a) Consider

Theorem. If \( f(x) = g(x)/h(x) \), \( h(x) \neq 0 \), and \( g'(x) \) and \( h'(x) \) exist, then \( f'(x) = \frac{h(x) \cdot g'(x) - g(x) \cdot h'(x)}{[h(x)]^2} \)

provided \( h(x) \neq 0 \).

b) Verbalize this theorem. Do not prove it in class.

c) Work examples as follows:

\[
\begin{align*}
f(x) &= (x + 1)/(x - 3) \\
f(x) &= (x^2 + 3x)/(x^3 + 1) \\
f(x) &= (x^2 - 7)/(x + 4), \quad x \neq -4 \\
f(x) &= (3x^3 - 7x^2 + 9x - 4)/(6x^4 - 2x^3 + 7x - 6), \quad \text{provided the denominator is not 0.}
\end{align*}
\]
OBJECTIVES

1. Consider various examples of composite functions and evaluate their derivatives by the definition.

2. Discuss the Chain Rule and apply it to several examples.

PROCEDURES

1. Review the definition of function—a set of ordered pairs such that no two distinct ordered pairs have the same first element.
   a) Consider a composite function defined by \( f[g(x)] \) or \( f \circ g \).
   b) Example: \( f[g(x)] = (x^2 + 1)^2 \)
      
      \[
      z = g(x) = x^2 + 1 \\
      f(z) = f[g(x)] = z^2
      \]
      
      Give examples of ordered pairs belonging to \( f \circ g \) as \((1, 4), (0, 1), (2, 25), (-1, 4), (3, 100)\).
   c) Example: \( f[g(x)] = (f \circ g)(x) = \sqrt{x^3} \)
      
      \[
      z = g(x) = x^3 \\
      f(z) = f[g(x)] = \sqrt{z}
      \]
      
      Give examples of ordered pairs as \((1, 1), (2, \sqrt{8})\).
   d) Point out that a function has an infinite number of representations as a composite function by considering:
      \( (f \circ g)(x) = f[g(x)] = (x^2 + 1)^2 \).
      
      \[
      (p \circ q)(x) = p[q(x)] = (x^2 + 1)^4 \\
      \text{where } z = q(x) = (x^2 + 1)^4 \\
      \text{and } p(z) = p[q(z)] = \sqrt{z}.
      \]
      \[
      (m \circ n)(x) = m[n(x)] = (3\sqrt{x^2 + 1})^6 \\
      \text{where } n(x) = 3\sqrt{x^2 + 1} \\
      \text{and } m(z) = z^6.
      \]

2. Consider \( (f \circ g)(x) = f[g(x)] = \sqrt{x^3}, x \geq 0 \).
   a) Since \( x^3 \sqrt{1/2} = (x^3)^{1/2} = (f \circ g)(x) \) may also be written:
      \( (p \circ q)(x) = p[q(x)] = x^{1/2} \),
      
      \[
      \text{where } z = q(x) = x^{1/2} \\
      \text{and } p(z) = p[q(x)] = z^3.
      \]
   b) \( (f \circ g)(x) \) may also be written:
      \( (m \circ n)(x) = m[n(x)] = 4\sqrt{x^6} \)
      
      \[
      \text{where } n(x) = x^6 \text{ and } m(z) = z^{1/4}.
      \]
3. Discuss examples such as:
   a) \((f \circ g)(x) = \sqrt[3]{x + 1} = (x + 1)^{1/3}, x \neq -1:\)
      \[g(x) = x + 1 \text{ and } f(z) = \sqrt[3]{z}.
   \]
   b) \((p \circ q)(x) = (3 \sqrt[3]{x + 1})^{-1}:
      \[q(x) = \sqrt[3]{x + 1} \text{ and } p(z) = z^{-1}.
   \]
   c) \((m \circ n)(x) = \sqrt[3]{(x + 1)^{-1}}:
      \[n(x) = (x + 1)^{-1} \text{ and } m(z) = z^{1/3}.
   \]

4. Consider \(G(x) = (x - 2)^{1/2}\) where \(G(x) = (f \circ g)(x) = (x - 2)^{1/2}\)
   while \(g(x) = x - 2\) and \(f(z) = z^{1/2}\).
   Recall \(g'(x) = \frac{1}{2\sqrt{x - 2}}\).

5. Consider \(H(x) = (f \circ g)(x) = (x^2 - 2x)^{1/2}\) where \(g(x) = x^2 - 2x\)
   and \(f(z) = z^{1/2}\)
   Find \(H'(x)\) as follows:
   \[
   \lim_{h \to 0} \frac{[(x + h)^2 - 2(x + h)]^{1/2} - (x^2 - 2x)^{1/2}}{h} = \lim_{h \to 0} \frac{2xh + h^2 - 2h}{h \left[(x + h)^2 - 2(x + h)]^{1/2} + (x^2 - 2x)^{1/2}\right]}
   \[
   = \frac{2x - 2}{2\sqrt{x^2 - 2x}} \text{ (if } x^2 - 2x > 0) = \frac{x - 1}{\sqrt{x^2 - 2x}}
   \]

6. Generalize as follows:
   If \(G(x) = (x - 2)^{1/2}, G'(x) = 1/2(x - 2)^{-1/2} \cdot D_x(x - 2).\)
   If \(H(x) = (x^2 - 2x)^{1/2}, H'(x) = 1/2(x^2 - 2x)^{-1/2} \cdot D_x(x^2 - 2x).\)
   
   If \(F(x) = (f \circ g)(x) = [g(x)]^n, F'(x) = n[g(x)]^{n-1} \cdot g'(x).\)
   Point out that this is a special case of:
   Theorem. If \(F(x) = (f \circ g)(x) = f[g(x)], \text{ and } f'(g(x))\) and \(g'(x)\) exist, then \(F'(x) = f'[g(x)] \cdot g(x).\)
   What we have considered is:
Corollary. If \( F(x) = [g(x)]^n \), and \( g'(x) \) exists, \( F'(x) = \frac{n}{n[g(x)]^{n-1}} g'(x) \).

Note that this theorem is called the Chain Rule.

7. Work examples as follows:

\[
F(x) = \sqrt{25 - x^2}
\]

\[
F(x) = (x^2 - 2x - 3)^{7/2}
\]

\[
F(x) = \frac{1}{(x^4 - 1)^2}
\]
LESSON XI

OBJECTIVES

Introduce and apply the First Derivative Test for graphing.

PROCEDURES

1. For the functions \( f(x) = x^2 \), \( f(x) = 1/x \),
\( f(x) = x^3 \), and \( f(x) = -1/x \)
consider each of the following:
a) Graph \( f(x) \)
b) Discuss the slope of the tangent line as to being positive, negative, or zero.
c) Note that wherever the tangent line has a negative slope, the graph decreases from left to right.
d) Note that wherever the tangent line has a positive slope, the graph increases from left to right.
e) Note that wherever the tangent line is horizontal and has a slope of 0, the graph has a low point, a high point, or a "flat" area.

2. After considering these or equivalent examples, generalize to:
   If \( f'(x) > 0 \), the graph increases from left to right.
   If \( f'(x) < 0 \), the graph decreases from left to right.
   If \( f'(x) = 0 \), the graph has a horizontal tangent.

3. Work an example such as \( f(x) = x^2 - 1 \).
a) Find the zeros.
b) Find the y-intercepts.
c) Determine if the graph has a horizontal tangent.
d) Determine where the graph increases.
e) Determine where the graph decreases.
f) Graph \( f(x) \).
g) To check against this, plot extra points.

4. Following the steps in #3, graph \( f(x) = 2x^3 - 9x^2 + 12x \).
LESSON XII

OBJECTIVES

Introduce the Second Derivative Test for graphing.

PROCEDURES

1. Discuss second derivatives of functions as follows.
   a) Consider \( h(x) = 6x + 4 \)
      \( g(x) = 3x^2 + 4x \)
      \( f(x) = x^3 + 2x^2 + 3 \)
   b) Evaluate \( f'(x) \) and note that \( f'(x) = g(x) \).
   c) Evaluate \( g'(x) \) and note that \( g'(x) = h(x) \).
   d) Discuss the relation between \( h(x) \) and \( f(x) \).
      Perhaps note that \( h(x) = g'(x) = Dxf'(x) = f''(x) \).
   e) Make note that this is called the "second derivative" of the function.
   f) Discuss the various notations used: \( f''(x) \), \( y'' \), \( \frac{D^2y}{dx^2} \).

2. Again consider examples such as:
   \( f(x) = x^2 \), \( f(x) = \frac{1}{x} \),
   \( f(x) = x^3 \), and \( f(x) = -\frac{1}{x} \).
   a) Graph the function.
   b) Determine \( f'(x) \).
   c) Determine \( f''(x) \).
   d) Note where the slopes of the tangent lines are increasing (and decreasing).
   e) Determine where the second derivative is positive and note this happens where the slopes of the tangent lines increase.
   f) Determine where the second derivative is negative and note that this occurs where the slopes of the tangent lines decrease.
   g) Generalize with:
      If \( f''(x) > 0 \), the graph of \( f(x) \) is concave up.
      If \( f''(x) < 0 \), the graph of \( f(x) \) is concave down.
      If \( f''(x) = 0 \), the graph of \( f(x) \) has a point of inflection.

3. Consider \( y = -x^2 \).
   a) Graph \( f(x) \).
   b) Note the highest point at \( x = 0 \) and point out that this is a "local maximum."
   c) Note also, that at \( x = 0 \), \( y' = 0 \) and \( y'' > 0 \) (concave down).

4. Completely work other examples such as \( y = f(x) = 2x^3 + 3x^2 - 12x \).


