KARAMANOUKIAN, Zaven Albert, 1941--
ON THE COVERING PROBLEM FOR THE GAUSSIAN
AND EISENSTEIN FIELDS.

The Ohio State University, Ph.D., 1971
Mathematics

University Microfilms, A XEROX Company, Ann Arbor, Michigan
ON THE COVERING PROBLEM FOR THE
GAUSSIAN AND EISENSTEIN FIELDS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree of Doctor of Philosophy in the Graduate
School of The Ohio State University

By

Zaven A. Karamanoukian, B.A., M.A.

The Ohio State University
1971

Approved by

Advisor
Department of Mathematics
ACKNOWLEDGMENTS

I would like to express my gratitude to Professor R. P. Bambah who introduced me to the subject of geometry of numbers through his interesting lectures. I am indebted to Professor T. J. Dickson for several of his suggestions. Special thanks are due to Professor A. C. Woods for suggesting the problem of the dissertation and for his help as advisor during the preparation of the dissertation.
VITA

February 23, 1941 . . . . Born - Aleppo, Syria

1962 . . . . . . . . B.A., Mathematics, American International College, Springfield, Massachusetts


1964-1968 . . . . . . Instructor, Department of Mathematics, Denison University, Granville, Ohio

1968-1971 . . . . . . Assistant Professor, Department of Mathematics, Denison University, Granville, Ohio

FIELDS OF STUDY

Major Field: Mathematics

Studies in Number Theory: Professors R. P. Bambah, K. Mahler

Studies in Geometry of Numbers: Professors R. P. Bambah, A. C. Woods
<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Vertices of ( \Pi (f) ) for ( f ) in ( \Delta' )</td>
<td>16</td>
</tr>
<tr>
<td>2.</td>
<td>Vertices of ( \Pi (f) ) for ( f ) in ( \Delta'' )</td>
<td>18</td>
</tr>
</tbody>
</table>
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACKNOWLEDGMENTS</td>
<td>ii</td>
</tr>
<tr>
<td>VITA</td>
<td>iii</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>iv</td>
</tr>
<tr>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>CHAPTER</td>
<td></td>
</tr>
<tr>
<td>I  PRELIMINARIES</td>
<td>2</td>
</tr>
<tr>
<td>1. Lattice Coverings</td>
<td>2</td>
</tr>
<tr>
<td>2. k-lattices</td>
<td>5</td>
</tr>
<tr>
<td>3. Results of Voronoi</td>
<td>11</td>
</tr>
<tr>
<td>II EXTREME GAUSSIAN LATTICES</td>
<td>20</td>
</tr>
<tr>
<td>III EXTREME EISENSTEIN LATTICES</td>
<td>39</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>62</td>
</tr>
</tbody>
</table>
INTRODUCTION

The two main problems in the study of lattice coverings of n-space by a subset S of $\mathbb{R}_n$ are that of determining $\theta(S)$, the density of the most economical lattice covering of $\mathbb{R}_n$ by S and that of determining the collection of lattices for which the density $\theta(S)$ is attained.

$\theta(S)$ is known only for convex bodies, S when $n=2$ (Fary [8]). When S is a sphere, Bambah [1] and Dickson [6] have obtained $\theta(S)$ for $n=2$ and $n=3$ respectively. For $n \geq 5$ Rogers [9] has estimates for $\theta(S)$ when S is a sphere.

In the following chapters we investigate the problem of lattice coverings by spheres in 2-dimensional complex space. The 2-dimensional complex lattices which we consider are k-lattices (lattices associated with the imaginary quadratic field $\mathbb{Q}(\sqrt{-k})$).

Chapter I is a brief survey of the lattice covering problem together with the necessary definitions and notations needed to formulate the problem. In Chapters II and III we determine $\theta(S)$ where S is a sphere and the lattices under consideration are k-lattices with $k=1$ and 3.
CHAPTER I
PRELIMINARIES

1. LATTICE COVERINGS

Let $L$ be a real $n$-dimensional lattice of determinant $d(L)$ and $S$ a bounded subset of $\mathbb{R}^n$ of volume $V(S)$. $L$ is called a covering lattice for $S$ if $\mathbb{R}^n$ is the union of sets $S+a$ with $a \in L$.

Let $C(r)$ be the cube $|x_i| \leq r$ and $N(r)$ the number of points of $L$ in $C(r)$. If $L$ is a covering lattice for $S$ we define

$$\theta(S, L, r) = \frac{N(r)V(S)}{(2r)^n}$$

and

$$\theta(S, L) = \limsup_{r \to \infty} \theta(S, L, r)$$

since

$$\lim_{r \to \infty} \frac{(2r)^n}{N(r)} = d(L),$$

$$\theta(S, L) = \frac{V(S)}{d(L)}.$$
where the infimum is taken over all covering lattices for $S$.

If $L = \{P_1, \ldots, P_n\}$ and $P$ is the matrix whose columns are
$P_i = (p_{i1}, \ldots, p_{in})$ the positive definite quadratic form associated with $L$ is

$$f(x) = x'P'Px = x'Ax.$$ 

For an arbitrary point $\alpha \in \mathbb{R}^n$ we define

$$m(f, \alpha) = \min_{m \in L_0} f(\alpha + m)$$

where $L_0$ denotes the $n$-dimensional integral lattice,

$$m(f) = \max_{\alpha \in \mathbb{R}^n} m(f, \alpha)$$

and

$$\lambda(f) = \frac{m(f)}{(d(f))^{1/n}}$$

where

$$d(f) = \det (A).$$

If $E$ is the ellipsoid : $f(x) \leq m(f)$ the integral lattice $L_0$ is a covering lattice for $E$. For, if $\alpha \in \mathbb{R}^n$, there exists $m_0 \in L_0$ such that $m(f, \alpha) = f(\alpha + m_0) \leq m(f)$, i.e. $\alpha + m_0 \in E$ hence $\alpha \in E - m_0$. Furthermore $L_0$ is not a covering lattice for $E_\varepsilon : f(x) \leq m(f) - \varepsilon$ for any $\varepsilon > 0$. Because there exists $\alpha \in \mathbb{R}^n$ such that $m(f) = m(f, \alpha) \leq f(\alpha + m)$ for all $m \in L_0$. Hence $\alpha + m \not\in E_\varepsilon$.
for all \( m \in L_0 \).

Let \( T \) be the linear transformation given by \( T(e_i) = P_i \) where \( e_i \) is the point with all coordinates zero except for the \( i \)-th coordinate which is 1. For \( x = (x_1, \ldots, x_n) \)

\[
P_x = \left( \sum_{i=1}^{n} x_i P_{i1}, \ldots, \sum_{i=1}^{n} x_i P_{in} \right) = T(x)
\]

therefore \( f(x) = x'P'Px \leq m(f) \) if and only if \( \|T(x)\| \leq m(f) \).

\( T \) maps \( L_0 \) onto \( L \) and \( E \) onto \( S_n \), the sphere of radius \( \sqrt{m(f)} \) and center 0. Thus spheres of radius \( \sqrt{m(f)} \) and centered at points of \( L \) cover space minimally.

\[
\Theta(S_n, L) = \frac{V(S_n)}{d(L)} = \frac{J_n(m(f))^{n/2}}{d(L)} = \frac{J_n(m(f))^{n/2}}{d(f)^{1/2}} = J_n^\frac{m(f)^{n/2}}{d(f)^{1/2}}
\]

where \( J_n \) is the volume of the unit sphere in \( \mathbb{R}^n \).

It is well known (see for example Rogers [10]) that \( \Theta(S, L) = \Theta(T(S), T(L)) \) for any linear transformation \( T \). Hence for the unit sphere \( S_n \),

\[
\Theta(S_n) = \inf J_n^\frac{m(f)^{n/2}}{d(f)^{1/2}} \text{ where } f \text{ ranges over positive definite quadratic forms in } n \text{ variables.}
\]

Thus the problem of minimizing \( \Theta(S_n, L) \) is equivalent to that of minimizing \( J_n m(f) \).
Since $f^*(f)$ is invariant under unimodular transformations and multiplications of $f$ by positive constants, $f$ and $f'$ (as well as their corresponding lattices) will be considered equivalent if $f'$ is obtained from $f$ by a unimodular transformation or if $f'$ is a positive multiple of $f$.

$f$ is called extreme if $f^*$ has a local minimum at $f$ and absolutely extreme if $\inf f^*(g) = \inf f^*(g)$ where $g$ ranges over the positive definite quadratic forms. A lattice, $L$ is extreme (absolutely extreme) if the form corresponding to it is extreme (absolutely extreme).

Bleicher [5] has shown that $n \sum_{i=1}^{n} x_i^2 - 2 \sum_{i<j} x_i x_j$ is extreme for all $n$. For $n=2$ and $n=3$ Bambah [1] showed that these forms are absolutely extreme and Barnes [3] showed that these are the only classes of extreme forms. Dickson [6] has shown that for $n=4$ there are three classes of extreme forms with $4 \sum_{i=1}^{4} x_i^2 - 2 \sum_{i<j} x_i x_j$ being absolutely extreme.

In the next two chapters we consider the problem of minimizing $\Theta(S_n, L)$ for special classes of 2-dimensional complex lattices.

2. k-LATTICES

Let $k$ be a positive square free integer and $N(\sqrt{-k})$ the ring of algebraic integers in $\mathbb{Q}(\sqrt{-k})$. $L$ is an $n$-dimensional $k$-lattice if

$$L = \left\{ \sum_{j=1}^{n} u_j A_j \mid u_j \in N(\sqrt{-k}) \right\}$$
where \( A_1 = (a_{11}, \ldots, a_{1n}) \) are \( n \) linearly independent points in complex \( n \)-space.

The two cases of \( k \equiv 1 \) or \( 2(\text{mod } 4) \) and \( k \equiv 3(\text{mod } 4) \) are considered separately.

**Case 1: \( k \equiv 1 \) or \( 2(\text{mod } 4) \)**

We can put \( u_j = v_j + i\omega_j \sqrt{k} \). The \( j \)-th coordinate of an arbitrary point \( A = u_1A_1 + \ldots + u_nA_n \) of \( L \) is

\[
\sum_{t=1}^{n} u_t a_{tj} = \sum_{t=1}^{n} (v_t + i\omega_t \sqrt{k}) a_{tj}
\]

\[
= \sum_{t=1}^{n} (v_t \Re(a_{tj}) - \omega_t \sqrt{k} \Im(a_{tj}))
\]

\[
+ \sum_{t=1}^{n} (v_t \Im(a_{tj}) + \omega_t \sqrt{k} \Re(a_{tj}))
\]

The mapping \( T: (a_1, \ldots, a_n) \rightarrow (\Re(a_1), \Im(a_1), \ldots, \Re(a_n), \Im(a_n)) \) from \( \mathbb{C}^n \) into \( \mathbb{R}_{2n} \) transforms \( A \) into

\[
\sum_{t=1}^{n} (\Re(a_{t1}), \Im(a_{t1}), \ldots, \Re(a_{tn}), \Im(a_{tn}))
\]

\[
+ \sqrt{k}w_1 (-\Im(a_{t1}), \Re(a_{t1}), \ldots, -\Im(a_{tn}), \Re(a_{tn}))
\]

\[
+ \ldots
\]

\[
+ v_n (\Re(a_{n1}), \Im(a_{n1}), \ldots, \Re(a_{nn}), \Im(a_{nn}))
\]

\[
+ \sqrt{k}w_n (-\Im(a_{n1}), \Re(a_{n1}), \ldots, -\Im(a_{nn}), \Re(a_{nn})).
\]

Thus \( T(L) \) is a real \( 2n \)-dimensional lattice with basis

\[
\{ x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n \}
\]

where
Suppose now that \( n = 2 \). \( T(L) \) has a basis \( X, \overline{X}, Y, \overline{Y} \) where

\[
X = (x_1, x_2, x_3, x_4), \quad \overline{X} = \sqrt{k} (-x_2, x_1, -x_4, x_3)
\]

\[
Y = (y_1, y_2, y_3, y_4), \quad \overline{Y} = \sqrt{k} (-y_2, y_1, -y_4, y_3).
\]

Let \( P \) be the matrix with columns \( X, \overline{X}, Y, \overline{Y} \) we have:

\[
A = P'P = \\
\begin{pmatrix}
E & 0 & x & \sqrt{k}y \\
0 & kE & -\sqrt{k}y & kx \\
x & -\sqrt{k}y & C & 0 \\
\sqrt{k}y & kx & 0 & kC
\end{pmatrix}
\]

where \( E = X \cdot X, \ x = X \cdot Y, \ y = X \cdot \overline{Y} \) and \( C = Y \cdot Y \).

The positive definite quadratic form \( f \) corresponding to \( T(L) \) is

\[
f(u) = u' Au \text{ or } f(u_1, u_2, u_3, u_4) = E(u_1^2 + ku_2^2) + C(u_3^2 + ku_4^2) + 2x(u_1u_3 + ku_2u_4) + 2y \sqrt{k}(u_1u_4 - u_2u_3).
\]

Putting \( u = u_1 + u_2 \sqrt{k}i \)

\[
v = u_3 + u_4 \sqrt{k}i
\]

\( B = x + iy \)

\[
f(u_1, u_2, u_3, u_4) = f(u,v) = Eu \bar{u} + Bu \bar{v} + \bar{B}uv + Cv \bar{v}.
\]
Case 2: $k \equiv 3 \pmod{4}$

We put $u_j = v_j + w_j \left( \frac{1 + \sqrt{k}i}{2} \right)$ and let $A = u_1 A_1 + \ldots + u_n A_n$ be an arbitrary point of $L$. The $j$-th coordinate of $A$ is

$$
\sum_{t=1}^{n} u_t(a_{tj}) = \sum_{t=1}^{n} \left( v_t + \frac{w_t}{2} + \frac{w_t \sqrt{k}i}{2} \right) a_{tj}
$$

$$
= \sum_{t=1}^{n} \left( (v_t + \frac{w_t}{2}) \text{Re}(a_{tj}) - \frac{w_t \sqrt{k}i}{2} \text{Im}(a_{tj}) \right)
$$

$$
+ i \sum_{t=1}^{n} \left( (v_t + \frac{w_t}{2}) \text{Im}(a_{tj}) + \frac{w_t \sqrt{k}i}{2} \text{Re}(a_{tj}) \right).
$$

The mapping $T: (a_1, \ldots, a_n) \mapsto (\text{Re}(a_1), \text{Im}(a_1), \ldots, + \text{Re}(a_n), \text{Im}(a_n))$ takes $A$ into

$$
v_1(\text{Re}(a_{11}), \text{Im}(a_{11}), \ldots, \text{Re}(a_{1n}), \text{Im}(a_{1n}))
$$

$$
+ \frac{w_1}{2} (\text{Re}(a_{11}) - \sqrt{k} \text{Im}(a_{11}), \sqrt{k} \text{Re}(a_{11}) + \text{Im}(a_{11}), \ldots,
$$

$$
\text{Re}(a_{1n}) - \sqrt{k} \text{Im}(a_{1n}), \sqrt{k} \text{Re}(a_{1n}) + \text{Im}(a_{1n})
$$

$$
+ \ldots \ldots
$$

$$
+ v_n(\text{Re}(a_{n1}), \text{Im}(a_{n1}), \ldots, \text{Re}(a_{nn}), \text{Im}(a_{nn}))
$$

$$
+ \frac{w_n}{2} (\text{Re}(a_{n1}) - \sqrt{k} \text{Im}(a_{n1}), \sqrt{k} \text{Re}(a_{n1}) + \text{Im}(a_{n1}), \ldots,
$$

$$
\text{Re}(a_{nn}) - \sqrt{k} \text{Im}(a_{nn}), \sqrt{k} \text{Re}(a_{nn}) + \text{Im}(a_{nn})).$$
The 2n-dimensional real lattice, T(L), has a basis
\[ \{ x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n \} \]
where
\[ x_j = (\text{Re}(a_{j1}), \text{Im}(a_{j1}), \ldots, \text{Re}(a_{jn}), \text{Im}(a_{jn})) \]
\[ \bar{x}_j = \frac{1}{2} (\text{Re}(a_{j1}) - \sqrt{k} \text{Im}(a_{j1}), \sqrt{k} \text{Re}(a_{j1}) + \text{Im}(a_{j1}), \ldots, \text{Re}(a_{jn}) - \sqrt{k} \text{Im}(a_{jn}), \sqrt{k} \text{Re}(a_{jn}) + \text{Im}(a_{jn})) \]

In particular if n=2, T(L) has a basis
\[ x = (x_1, x_2, x_3, x_4) \]
\[ \bar{x} = \frac{1}{2} (x_1 - \sqrt{k} x_2, \sqrt{k} x_1 + x_2, x_3 - \sqrt{k} x_4, \sqrt{k} x_3 + x_4) \]
\[ y = (y_1, y_2, y_3, y_4) \]
\[ \bar{y} = \frac{1}{2} (y_1 - \sqrt{k} y_2, \sqrt{k} y_1 + y_2, y_3 - \sqrt{k} y_4, \sqrt{k} y_3 + y_4) \]

If P is the matrix with columns X, \( \bar{X} \), Y, \( \bar{Y} \),
\[ A = P'P = \begin{pmatrix} x \cdot x & x \cdot \bar{x} & x \cdot y & x \cdot \bar{y} \\ x \cdot \bar{x} & \bar{x} \cdot \bar{x} & \bar{x} \cdot y & \bar{x} \cdot \bar{y} \\ x \cdot y & \bar{x} \cdot y & y \cdot y & y \cdot \bar{y} \\ x \cdot \bar{y} & \bar{x} \cdot \bar{y} & y \cdot \bar{y} & \bar{y} \cdot \bar{y} \end{pmatrix} \]

Using the simple identities:
\[ x \cdot \bar{x} = \frac{1}{2} x \cdot x \]
\[ \bar{x} \cdot x = (\frac{k+1}{4}) x \cdot x \]
\[ x \cdot \bar{y} = x \cdot y - \bar{x} \cdot y \]
and letting

\[ E = X \cdot X \quad \quad \quad \quad C = Y \cdot Y \]
\[ x = X \cdot Y \quad \quad \quad \quad x \cdot y = \frac{x \cdot y}{2} \]
\[ \bar{X} \cdot Y = \frac{x + y}{2} \]

\[ A = \begin{pmatrix} E & \frac{E}{2} & x & \frac{x \cdot y}{2} \\ \frac{E}{2} & mE & \frac{x + y}{2} & mx \\ x & x + y & C & C \frac{2}{2} \\ \frac{x \cdot y}{2} & mx & C & 2 \frac{mC}{2} \end{pmatrix} \]

The form associated with the lattice \( T(L) \) is

\[ f(u_1, u_2, u_3, u_4) = E u_1^2 + mE u_2^2 + E u_1 u_2 + C u_3^2 + mC u_4^2 + C u_3 u_4 \]

\[ + y(u_2 u_3 - u_1 u_4) + x(2u_1 u_3 + u_1 u_4 + u_2 u_3 + mu_2 u_4) \]

Letting \( u = u_1 + u_2 \left( \frac{1 + \sqrt{k} i}{2} \right) \)

\[ v = u_3 + u_4 \left( \frac{1 + \sqrt{k} i}{2} \right) \]

we get \( f(u_1, u_2, u_3, u_4) = f(u, v) = E u^2 + B u v + B u v + C v^2 \) where

\[ B = x + \frac{y i}{\sqrt{k}} \]

From cases 1 and 2 we see that for \( n=2 \) the problem of minimizing
the density of k-lattice coverings of space by equal spheres is equivalent to that of minimizing $f^{*}(f)$ as $f$ ranges over the positive definite Hermitian forms over the quadratic field $\mathbb{Q}(\sqrt{-k})$. We can now broaden our definition of equivalence of forms to include the transformations of the Picard group.

In the next two chapters we investigate this problem for $k=1$ and 3. 1-lattices are called Gaussian lattices and 3-lattices are called Eisenstein lattices.

3. RESULTS OF VORONOI

In this section we outline some results due to Voronoi ([11] and [12]) concerning the polyhedra associated with positive definite forms and a reduction theorem for positive definite forms.

$\Gamma(f)$, the polyhedron corresponding to a positive definite form $f$ is defined as the set of points $x$ in $\mathbb{R}^n$ which satisfy $f(x) \leq f(x-m)$ for all non-zero integer points $m$. In the definition it suffices to consider only the integer points $m \neq 0$ with the property $f(m) = \min f(x)$ where the minimum, taken over all integral $x \equiv m \pmod{2}$, is attained only at $\pm m$.

At most $2^{n}-1$ pairs of integer points $\pm m$ are needed to define $\Gamma(f)$. In cases where $f$ attains its minimum at more than two points of a congruence class mod 2 of integer points, $\Gamma(f)$ can be defined by less than $2^{n}-1$ pairs of points. Every integer point $m$ needed in the definition of $\Gamma(f)$ describes the $(n-1)$-dimensional face $f(x)=f(x-m)$ of $\Gamma(f)$. $\Gamma(f)$ is a symmetric convex polyhedron bounded by pairs of
parallel faces and \( \overline{\Pi}(f) + m, (m \in L_0) \) fit together to fill all of space.

From previous definitions

\[
m(f, \alpha) = \min_{x \in L_0} f(x + \alpha)
\]

without loss of generality we can assume \( \alpha \in \overline{\Pi}(f) \). Then

\[
m(f, \alpha) = f(\alpha - m_0)
\]

where \( t m_0 \) represent a unique mod 2 minimum of \( f \). Thus

\[
m(f) = \max_{\alpha} m(f, \alpha) = \max_v f(v)
\]

where \( v \) runs over the vertices of \( \overline{\Pi}(f) \). Vertices of \( \overline{\Pi}(f) \) for which \( f(v) \) is a maximum are called maximal vertices.

A vertex \( v \) is on \( n \) or more faces of \( \overline{\Pi}(f) \) hence each vertex is defined by integer points \( m_1, \ldots m_n \) such that

\[
f(v) = f(v - m_i) \quad 1 \leq i \leq n
\]

i.e.

\[
v'Av = (v - m_i)'A(v - m_i) \quad 1 \leq i \leq n
\]

or

\[
2m_i'Av = f(m_i) \quad 1 \leq i \leq n
\]

Two vertices \( v \) and \( v' \) are congruent if \( v - v' \in L_0 \). If \( v \) is determined by the integer points \( m_1, \ldots m_t \) (\( t \geq n \)) then

\[
v_j = v - m_j \quad 1 \leq j \leq t
\]

are vertices of \( \overline{\Pi}(f) \) congruent to \( v \). Since

\[
f(v - m_j) = f(v) - 2m_jAv + f(m_j) = f(v),
\]

we need consider only one vertex from each congruence class of vertices in the determination of \( \max f(v) \).
Voronoi also showed that $\frac{1}{2} n(n+1)$-dimensional coefficient space of positive definite quadratic forms can be partitioned into polyhedral cones ($\Delta$) with the origin as vertex and properties:

i) no two cones have a common interior point.

ii) a unimodular transformation of the variables either leaves a cone invariant or transforms it into another cone of the system.

iii) there are a finite number of cones $\Delta_1, \ldots, \Delta_t$ such that any positive definite form is equivalent to a form in one of $\Delta_1, \ldots, \Delta_t$.

iv) if $f$ and $f'$ are interior to some cone $\Delta$, then $f$ and $f'$ attain their minima at the same integral points $\pm m_1$ in each congruence class mod 2. Furthermore these minima are attained at unique points $\pm m_1$ in each congruence class.

v) if $f$ is in the interior and $f'$ on the boundary of $\Delta$ and if $f$ attains its mod 2 minima at $\pm m_1, \ldots, \pm m_{2^{n-1}}$

then $f'$ also attains mod 2 minima at $\pm m_1', \ldots, \pm m_{2^{n-1}}$.

In the case of $f'$ however $\pm m_1', \ldots, \pm m_{2^{n-1}}$ may not be unique.

For $n=2$ and $n=3$ Voronoi showed that there is only one cone $\Delta$ (i.e. all cones are congruent to $\Delta$). For $n=4$ there are three incongruent cones $\Delta, \Delta'$ and $\Delta''$. For later use we give a more detailed description of $\Delta'$ and $\Delta''$. 
The cone $\Delta'$: $\Delta'$ consists of forms of type

$$f(x) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 + \lambda_4 x_4^2 + \mu_1 w + \mu_2 (x_1-x_3)^2$$

$$+ \mu_3 (x_1-x_4)^2 + \mu_4 (x_2-x_3)^2 + \mu_5 (x_2-x_4)^2 + \mu_6 (x_3-x_4)^2$$

where $\lambda_1 > 0$, $\mu_1 > 0$ and

$$w = 2 \sum_{i=1}^{4} x_i^2 + 2x_1x_2 - 2x_1x_3 - 2x_1x_4 - 2x_2x_3 - 2x_2x_4.$$  

If $f$ is interior to $\Delta'$ i.e. if $\lambda_1 > 0$ and $\mu_1 > 0$, then $f$ attains its mod 2 minima at:

- $m_1 = (1, 0, 0, 0)$
- $m_2 = (0, 1, 0, 0)$
- $m_3 = (0, 0, 0, 1)$
- $m_4 = (0, 0, 0, 1)$
- $m_5 = (1, -1, 0, 0)$
- $m_6 = (1, 0, 1, 0)$
- $m_7 = (1, 0, 0, 1)$
- $m_8 = (0, 1, 1, 0)$
- $m_9 = (0, 1, 0, 1)$
- $m_{10} = (0, 0, 1, 1)$
- $m_{11} = (1, 1, 1, 0)$
- $m_{12} = (1, 1, 0, 1)$
- $m_{13} = (1, 0, 1, 1)$
- $m_{14} = (0, 1, 1, 1)$
- $m_{15} = (1, 1, 1, 1)$

and at $m_{-i} = -m_i$ $(1 \leq i \leq 15)$.

The vertices of an interior form are given in the following table. The symbol $[i_1, i_2, i_3, i_4]$ indicates the vertex on the faces

$$f(x) = f(x - m_{i_j})$$  

$(1 \leq j \leq 4)$
Congruent vertices together with their negatives are grouped together on successive lines. These groupings are indicated in the extreme left hand column by Roman numerals.
<table>
<thead>
<tr>
<th>Vertices of ( \Pi(f) ) for ( f \in \Delta' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
</tr>
<tr>
<td>II</td>
</tr>
<tr>
<td>III</td>
</tr>
<tr>
<td>IV</td>
</tr>
<tr>
<td>V</td>
</tr>
<tr>
<td>VI</td>
</tr>
<tr>
<td>VII</td>
</tr>
<tr>
<td>VIII</td>
</tr>
<tr>
<td>IX</td>
</tr>
<tr>
<td>X</td>
</tr>
<tr>
<td>XI</td>
</tr>
<tr>
<td>XII</td>
</tr>
</tbody>
</table>
The cone $\Delta''$: $\Delta''$ consists of forms of type

$$f(x) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 + \lambda_4 x_4^2 + \mu_1 w + \mu_2 (x_i^2 - x_j)^2$$

$$+ \mu_3 (x_i - x_j)^2 + \mu_4 (x_i - x_j)^2 + \mu_5 (x_i - x_j)^2 + \mu_6 (x_i + x_j - x_k - x_l)^2.$$ 

where $w$ is as before. Interior forms attain their mod 2 minima at

$$m_1 = (1, 0, 0, 0) \quad m_2 = (0, 1, 0, 0)$$
$$m_3 = (0, 0, 1, 0) \quad m_4 = (0, 0, 0, 1)$$
$$m_5 = (1, -1, 0, 0) \quad m_6 = (1, 0, 1, 0)$$
$$m_7 = (1, 0, 0, 1) \quad m_8 = (0, 1, 1, 0)$$
$$m_9 = (0, 1, 0, 1) \quad m_{10} = (0, 0, 1, -1)$$
$$m_{11} = (1, 1, 1, 0) \quad m_{12} = (1, 1, 0, 1)$$
$$m_{13} = (1, 0, 1, 1) \quad m_{14} = (0, 1, 1, 1)$$
$$m_{15} = (1, 1, 1, 1)$$

and at $m_{-i} = -m_i \quad (1 \leq i \leq 15)$

The vertices of interior forms are given in the following table with the same notation as before.
<table>
<thead>
<tr>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
<th>VI</th>
<th>VII</th>
<th>VIII</th>
<th>IX</th>
<th>X</th>
<th>XI</th>
<th>XII</th>
</tr>
</thead>
<tbody>
<tr>
<td>[7, 5, 6, 13]</td>
<td>[-7, -9, 10, 3]</td>
<td>[9, 7, 8, 14]</td>
<td>[-10, -8, -6, 4]</td>
<td>[-3, -14, -3, -13]</td>
<td>[-7, -5, -6, -13]</td>
<td>[7, 9, -10, -3]</td>
<td>[-9, 7, -8, -14]</td>
<td>[10, 8, 6, -4]</td>
<td>[3, 14, 4, 13]</td>
<td>[1, 2, 11, 15]</td>
<td>[-1, -5, 8, 14]</td>
</tr>
</tbody>
</table>
We should note here that if $f$ is interior to some cone $\Delta$, the vertices of $f$ are determined by exactly $n$ integer points. If two sets of integer points $m_1, \ldots, m_n$ and $t_1, \ldots, t_n$ define two distinct vertices of $\Pi(f)$ then $m_1, \ldots, m_n$ and $t_1, \ldots, t_n$ also define vertices of $\Pi(g)$ where $g$ is any boundary form in $\Delta$. In the case of $\Pi(g)$ however these vertices may not be distinct.

As a final remark we mention a result due to Dickson [7]. If $f$ and $f'$ are two forms in a cone $\Delta$, the maximum of $\mu$ on the line segment with endpoints $f$ and $f'$ is either at $f$ or at $f'$. In particular this implies that in a closed convex subset of $\Delta$, $\mu$ has exactly one local minimum.
CHAPTER II
EXTREME GAUSSIAN LATTICES

Lemma 1: Every positive definite binary Hermitian form over \( \mathbb{Q}(\sqrt{-1}) \) is equivalent to a form:

\[ uu + Buv + \bar{B}u\bar{v} + zv\bar{v} \]

where \( z > 1, B = x + iy \) and \(-1/2 \leq x \leq y \leq 0\).

Proof: Let

\[ f' = t'uu + B'u\bar{v} + \bar{B}'u\bar{v} + z'v\bar{v} \]

be a positive definite Hermitian form over \( \mathbb{Q}(\sqrt{-1}) \). There exist \( \alpha, \beta \) such that \( f'(\alpha, \beta) \) is minimal. Since \( (\alpha, \beta) = 1 \), there exist \( \beta_0, \delta_0 \) such that

\[ \alpha \delta_0 - \beta_0 \beta = 1. \]

The solutions of \( \alpha \delta - \beta \beta = 1 \) are:

\[ \beta = \beta_0 + k\alpha, \quad \delta = \delta_0 + k\beta. \]

Choose \( k \) so that \( f'(\beta_0 + k\alpha, \delta_0 + k\beta) \) is minimal and let

\[ f(u, v) = f'(\alpha u + \beta v, \beta u + \delta v) = tuu + Bu\bar{v} + \bar{B}u\bar{v} + zv\bar{v} \]

then

\[ \min f(u, v) = f(1, 0) = t. \]

If \( |v| = 1 \) then

\[ f(u, v) = f'(\alpha u + \beta v, \beta u + \delta v) = f'(\alpha uv^{-1} + \beta, \beta uv^{-1} + \delta) \]

20
\[ f'(uv^{-1} + \beta_0 + k\xi, )'uv^{-1} + s_0 + k' \] 
\[ = f'(uv^{-1} + k) + \beta_0, )'(uv^{-1} + k) + s_0 \]

Thus

\[ \min_{|w| = 1} f(u, v) = f(0, 1) = z. \]

Dividing \( f \) by \( t \) we can suppose that

\[ f = u\bar{u} + B\bar{u}v + Buv + z\bar{v}v \]

where \( z \gg 1 \) and \( B = x + iy \).

\[ z \leq f(1, \pm 1) = 1 \pm 2x + z \]

hence \(|x| \leq 1/2\). Similarly,

\[ z \leq f(1 \pm 1) = 1 \pm 2y + z \]

and \(|y| \leq 1/2\).

Let \( u = u_1 + u_2i \) and \( v = u_3 + u_4i \)

\[ f = u_1^2 + u_2^2 + 2\text{Re}(B\bar{u}v) + z(u_3^2 + u_4^2) \]

\[ = u_1^2 + u_2^2 + z(u_3^2 + u_4^2) + 2x(u_1u_3 + u_2u_4) + 2y(u_1u_4 - u_2u_3). \]

If \( x \) and \( y \) are not of the same sign, we apply the transformation:

\[ u_1 \rightarrow u_2, \quad u_2 \rightarrow u_1, \quad u_3 \rightarrow u_4, \quad u_4 \rightarrow u_3 \]

to get

\[ u_1^2 + u_2^2 + z(u_3^2 + u_4^2) + 2x(u_1u_3 + u_2u_4) - 2y(u_1u_4 - u_2u_3). \]

Thus we can assume \( xy \gg 0 \).

If \( x \) and \( y \) are both positive, we apply the transformation:

\[ u_1 \rightarrow -u_1, \quad u_2 \rightarrow -u_2, \quad u_3 \rightarrow u_3, \quad u_4 \rightarrow u_4 \]

to \( f \) to get
\[ u_1^2 + u_2^2 + z(u_3^2 + u_4^2) - 2x(u_1u_3 + u_2u_4) - 2y(u_1u_4 - u_2u_3). \]

Thus we can assume \( x \leq 0 \) and \( y \leq 0 \).

If \( y < x \), we apply the transformation: \( u_1 \rightarrow u_1, u_2 \rightarrow -u_2, \)
\( u_3 \rightarrow u_4, u_4 \rightarrow u_3 \) to \( f \) to get

\[ u_1^2 + u_2^2 + z(u_3^2 + u_4^2) + 2x(u_1u_3 - u_2u_4) + 2y(u_1u_4 + u_2u_3). \]

**Lemma 2:** All positive definite binary Hermitian forms over \( \mathbb{Q}(\sqrt{-1}) \)
are in a cone equivalent to \( \Delta^* \).

**Proof:** Let \( f \) be a positive definite binary Hermitian form over \( \mathbb{Q}(\sqrt{-1}) \).

By lemma 1, we can write \( f \) as

\[ u_1^2 + u_2^2 + z(u_3^2 + u_4^2) + 2x(u_1u_3 + u_2u_4) + 2y(u_1u_4 - u_2u_3) \]

where \( z > 1 \), and \( -1/2 \leq x \leq y \leq 0 \).

Let \( T \) be the transformation: \( u_1 \rightarrow -u_4, u_2 \rightarrow u_1 - u_3, \)
\( u_3 \rightarrow u_2 - u_4, u_4 \rightarrow -u_3 \)

\[ T(f) = u_4^2 + (u_1 - u_3)^2 + z(u_2 - u_4)^2 + z(u_3^2 + u_4^2) + 2x(u_4^2 - u_{13}^2 + u_{24}^2 - u_{14}^2) \]

\[ + 2y(u_3u_4 - u_1u_2 - u_3u_4 + u_1u_4 + u_2u_3) \]

\[ = (z-1)(u_3^2 + (u_2 - u_4)^2) + (1+2x)(u_3^2 + u_4^2 + (u_1 - u_3)^2 + (u_2 - u_4)^2) \]

\[ + (2y-2x)(u_1 - u_3)^2 + (u_2 - u_4)^2 + u_2u_4 + u_1u_3 \]

\[ - 2y((u_1 - u_3)^2 + (u_2 - u_4)^2 + u_2u_4 + u_1u_3 - u_1u_4 - u_2u_3 + u_2u_4). \]

(1) \( T(f) = (z-1)(u_3^2 + (u_2 - u_4)^2) + (1+2x)(u_3^2 + u_4^2 + (u_1 - u_3)^2 + (u_2 - u_4)^2) \)
\[ + (y-x) \left( \frac{1}{2} \sum_{i=1}^{l} u_i^2 + (u_1-u_3)^2 + (u_2-u_4)^2 \right) \]
\[ - y(2 \sum_{i=1}^{l} u_i^2 + 2u_1u_2-2u_1u_3 - 2u_1u_4 - 2u_2u_3 - 2u_2u_4) \].

Since the coefficients z-1, 1+2x, y-x and -y are non-negative, T(f) is in \( \Delta'' \). More precisely T(f) is on the common boundary of \( \Delta' \) and \( \Delta'' \).

**Lemma 3:** Let \( v = (v_1, \ldots, v_4) \) be a vertex of \( \overline{T}(g) \) defined by the integer points \( m_1, m_2, m_3, m_4 \) where \( g = T(f) \). Then \( g(v) = w'A^{-1}w \) where \( A \) is the matrix of \( f \), \( w = (-t_2, -t_4, t_1, t_2, -t_1, -t_3) \) and \( t \) is defined by \( 2m_1t = g(m_1) \).

**Proof:**

\[ f = u_1^2 + u_2^2 + z(u_3^2 + u_4^2) + 2x(u_1u_3 + u_2u_4) + 2y(u_1u_4 - u_2u_3) \]

has matrix

\[
A = \begin{pmatrix}
1 & 0 & x & y \\
0 & 1 & -y & x \\
x & -y & z & 0 \\
y & x & 0 & z
\end{pmatrix}
\]

and

\[ d(f) = \det A = (z - x^2 - y^2)^2 = D^2 \]

\[
A^{-1} = \frac{1}{D} \begin{pmatrix}
z & 0 & -x & -y \\
0 & z & y & -x \\
x & -y & 1 & 0 \\
y & -x & 0 & 1
\end{pmatrix}
\]

Let \( S \) be the matrix of the transformation \( T \) of lemma 2. The matrix \( B \) of \( g = T(f) \) can be obtained by
Let \( v = (v_1, \ldots, v_4) \) be a vertex of \( \Pi(g) \) defined by the integer points \( m_1, \ldots, m_4 \).

Put

\[
Bv = (Bv)'B^{-1} = t'B^{-1}t
\]

and

\[
w = (S')^{-1}t.
\]

Then

\[
g(v) = (Bv)'B^{-1}(Bv) = t'B^{-1}t = t'B^{-1}t = t'S^{-1}A^{-1}(S')^{-1}t = w'A^{-1}w.
\]

\( m_1, \ldots, m_4 \) define \( v \) implies \( 2m_1'Bv = g(m_1) \) i.e. \( 2m_1't = g(m_1) \)

\[
S = \begin{pmatrix}
0 & 0 & 0 & -1 \\
1 & 0 & -1 & 0 \\
0 & 1 & 0 & -1 \\
0 & 0 & -1 & 0 \\
\end{pmatrix}
\]

\[
(S')^{-1} = \begin{pmatrix}
0 & -1 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & -1 & 0 \\
\end{pmatrix}
\]

\( w = (S')^{-1}t \) implies

\[
w = (-t_2, -t_4, t_1, t_2, -t_1, t_3)
\]
Lemma 4: If \( v \) is of type V then \( 4g(v) = \frac{1}{D} (2z - 4xy)(1+2x+z) \).

Proof: Choose \( v = [1, 2, -3, -4] \) of type V. The integer points defining \( v \) are

\[
m_1 = (1,0,0,0) \quad m_2 = (0,1,0,0) \quad m_3 = (0,0,-1,0)
\]

and \( m_4 = (0,0,0,-1) \). From (1)

\[
g(m_1) = 1 \quad g(m_2) = z \quad g(m_3) = 1+2x+z
\]

\[
g(m_4) = 1+2x+z
\]

Solving \( 2m_1^t = g(m_1) \) for \( t \) we get

\[
2t = (1, z, -1-2x-z, -1-2x-z)
\]

\[
2w = (1+2x, 1, z, z+2x)
\]

\[4 \cdot \text{Dg}(v) = (2w)'(DA^{-1})(2w) = F(w)\]

\[
= z\left(w_1^2 + w_2^2\right) + \left(w_3^2 + w_4^2\right) - 2x(w_1w_3 + w_2w_4) + 2y(w_2w_3 - w_1w_4)
\]

\[
= z(2 + 4x^2 + 4x) + (2z^2 + 4x^2 + 4zx) - 2x(z+2xz+z+2x)
\]

\[
+ 2y(z-z-2x-2xz-4x^2)
\]

\[
= 2z + 4x^2z + 4xz + 2z^2 + 4x^2 + 4xz - 4xz - 4x^2z - 4x^2
\]

\[
- 4xy - 4xyz - 8x^2y
\]

\[
= 2z + 2z^2 + 4xz - 4xy - 4xyz - 8x^2y
\]

\[
= (2z - 4xy)(1+2x+z)
\]
Define \( 4 \overline{\mu}(x, y, z) = \frac{(2z-4xy)(1+2x+z)}{D^{3/2}} \)

where \( D = \sqrt{d(g)} = z - x^2 - y^2 \).

If vertices of type V are maximal then \( \overline{\mu}(x, y, z) = \mu(g) \). In any case \( \overline{\mu}(x, y, z) \leq \mu(g) \) for any \((x, y, z)\) subject to the restriction \( z \geq 1, -\frac{1}{2} \leq x \leq y \leq 0 \).

**Lemma 5:**

i) \( \overline{\mu}_{x} \) and \( \overline{\mu}_{z} \) do not vanish simultaneously unless \( x = -1/2, y = 0 \) and \( z = 1 \)

ii) \( \overline{\mu}(-1/2, y, z) \geq \mu(-1/2, y, 1) \)

iii) \( \overline{\mu}(x, y, 1) \geq \mu(-1/2, y, 1) \)

**Proof:**

i) \( 4 \overline{\mu}_{x} = \frac{1}{D^3} \left\{ D^{3/2} \left(-4y(1+2x+z) + 2(2z-4xy)\right) \right. \\
\left. + 3x D^{1/2} (2z-4xy)(1+2x+z) \right\} \\
\overline{\mu}_{x} = 0 \) implies

(2) \(-4y D (1+2x+z) + 2D (2z-4xy) + 3x(2z-4xy)(1+2x+z) = 0 \).

ii) \( 4 \overline{\mu}_{z} = \frac{1}{D^3} \left\{ D^{3/2} \left(2(1+2x+z) + 2z-4xy\right) \right. \\
\left. - 3/2 D^{1/2} (2z-4xy)(1+2x+z) \right\} \\
\overline{\mu}_{z} = 0 \) implies
(3) \[ 4D(1+2x+z) + 2D(2z-4xy) - 3(2z-4xy)(1+2x+z) = 0. \]

Multiplying (3) by $x$ and adding to (2),

\[ -4yD(1+2x+z) + 2D(2z-4xy) + 4xD(1+2x+z) + 2xD(2z-4xy) = 0. \]

Divide by $D$ and combine terms to get

\[ A(x, y, z) = (1+2x+z)(4x-4y) + (2z-4xy)(2+2x) = 0 \]

\[ \frac{\partial A}{\partial y} = -4(1+2x+z) - 4x(2+2x) \]

\[ = -4 - 16x - 4z - 8x^2 \leq (-8 + 16x) - 8x^2 < 0 \]

Therefore $A(x, y, z) \geq A(x, 0, z)$

\[ = (1+2x+z)(4x) + 2z(2+2x) \]

\[ = (4 + 8x)z + 4x + 8x^2. \]

Since this is an increasing function of $z$,

\[ A(x, y, z) \geq 4 + 12x + 8x^2 = 8(x+\frac{1}{2})(x+1) \geq 0. \]

Thus if both partials vanish then $x = -1/2$, $y = 0$ and $z = 1$.

\[ \ell) \quad \frac{\partial}{\partial z} (z) = \frac{z(2z+2y)}{(z-4y^2)^{3/2}} \]

\[ \frac{\partial A(-1/2, y, z)}{\partial z} \quad \text{has the same sign as} \]

\[ A(y, z) = (z-4y^2)(4z+2y) - 3/2(2z^2 + 2yz) \]

\[ \frac{\partial A(y, z)}{\partial z} = 4z + 2y + 4z - 1 - 4y^2 - 6z - 3y. \]
\[ 2z - 1 - y - 4y^2 \geq 1 - y - 4y^2 \geq 0. \] Hence

\[ A(y, z) \geq A(y, 1) = (3/4 - y^2)(4 + 2y) - 3(1 + y) \]

\[ = -3/2y - 4y^2 - 2y^3 = -y(y+1/2)(2y + 3) \geq 0 \] therefore

\[ \exists \mu \left[ \frac{1}{z} \right. \left. (-1/2, y, z) \right] \geq 0 \] and \[ \exists \mu \left[ (-1/2, y, z) \right] \geq \exists \mu \left[ (-1/2, y, 1) \right]. \]

iii) \[ \exists \mu \left[ (x, y, 1) \right] = \frac{(2-4xy)(2+2x)}{(1-x^2-y^2)^{3/2}} \]

\[ \exists \mu \left[ (x, y, 1) \right] = \frac{1}{(1-x^2-y^2)^{3/2}} \left\{ (1-x^2-y^2)^{3/2} (4-8y-16xy) + 3x(1-x^2-y^2)^{1/2}(2+2x)(2-4xy) \right\}. \]

This has the same sign as

\[ A(x, y) = (1-x^2-y^2)(4-8xy-16xy) + 3x(2+2x)(2-4xy) \]

\[ = (1-x^2-y^2)(4-8xy) - 4y(1-x^2-y^2)(2+2x) + 3x(2+2x)(2-4xy) \]

\[ = (2-4xy)(2-2x^2-2y^2+6x+6x^2) - 4y(1-x^2-y^2)(2+2x) \]

\[ = (2-4xy)(4x^2+6x+2) - 2y^2(2-4xy) + 4y(1-x^2-y^2)(2+2x) \]

the term \( 4x^2+6x+2 = (x+1/2)(4x+4) \geq 0 \) and \( 2 - 4xy > 0 \) therefore \( A(x, y) \geq -4y \left\{ (1-x^2-y^2)(2+2x) + y - 2xy^2 \right\} \)

\[ \geq -4y(1/2 + y - 2xy^2). \]

\[ -4y \geq 0 \] and \( 1/2 + y - 2xy^2 \geq y + 1/2 \geq 0. \)

Hence \( A(x, y) \geq 0. \)
Lemma 6: \( \bar{f}(x, y, z) \) has a minimum of \( \frac{2\sqrt{2}}{9 \sqrt{3}} (1 + \sqrt{3}) \) and this minimum is attained at \((-1/2, -\frac{3 + \sqrt{3}}{4}, 1)\).

**Proof:** By lemma 5 the minimum of \( \bar{f} \) is at a boundary point.

1) The minimum of \( \bar{f} \) on the face \( x=y \) is \( \frac{\sqrt{2}}{2} \).

On the face \( x=y \)

\[
4 \bar{\mu} = \frac{(2z-4x^2)(1+2x+z)}{D^{3/2}}
\]

where \( D^{3/2} = z-2x^2 \).

By lemma 5 the minimum on this face is at a boundary point

a) \( x = 0 \)

\[
4 \bar{\mu} = \frac{2z(1+z)}{z^{3/2}}.
\]

\[
4 \bar{\mu}'(z) = z^{3/2} \frac{(2+4z)-3z}{z^3} \frac{(1+z)}{(1+z)}
\]

\[
= \frac{2+4z-3z}{z^{3/2}} = \frac{z-1}{z^{3/2}} \geq 0.
\]

Hence the minimum is at \( z=1 \) and \( \bar{\mu}(0, 0, 1) = \frac{1}{2} \).

b) \( x = -1/2 \) By ii) of lemma 5,

\[
\bar{\mu}(-1/2, -1/2, z) \geq \bar{\mu}(-1/2, -1/2, 1) = \frac{\sqrt{2}}{2}.
\]

c) \( z = 1 \) By iii) of lemma 5

\[
\bar{\mu}(x, x, 1) \geq \bar{\mu}(-1/2, -1/2, 1) = \frac{\sqrt{2}}{2}.
\]
ii) The minimum of $\tilde{u}$ on the face $y=0$ is $\frac{4\sqrt{3}}{9}$.

$\tilde{u}(x, 0, z)$ must have its minimum at a boundary point by i) of lemma 5

a) $x = 0$. By a) of part i) of this lemma

$\tilde{u}(x, 0, z) \geq \tilde{u}(0, 0, z) = 1$

b) $x = -1/2$

$\tilde{u}(-1/2, 0, z) \geq \tilde{u}(-1/2, 0, 1) = \frac{4\sqrt{3}}{9}$

c) $z = y^*(x, 0, 1) \geq \tilde{u}(-1/2, 0, 1) = \frac{4\sqrt{3}}{9}$.

iii) The minimum of $\tilde{u}$ on the face $x = -1/2$ is $\frac{2\sqrt{2}}{9\sqrt{3}} (1 + \sqrt{3})$

and this minimum is attained at $(-1/2, \frac{-3+\sqrt{3}}{4}, 1)$.

By ii) of lemma 5, $\tilde{u}(-1/2, y, z) \geq$

$\tilde{u}(-1/2, y, 1) = \frac{1 + y}{2(3/4-y^2)^{3/2}}$

$\frac{\partial \tilde{u}(-1/2, y, 1)}{\partial y} = 0$ implies

$2(3/4-y^2)^{3/2} + 6y(1+y)(3/4-y^2)^{1/2} = 0$.

Hence

$0 = 2(3/4-y^2) + 6y + 6y^2 = 3/2 + 4y^2 + 6y$.

This polynomial has roots

$\frac{-6 \pm \sqrt{12}}{8} = \frac{-3 \pm \sqrt{3}}{4}$
\[ y = \frac{-3 - \sqrt{3}}{4} \] lies outside our region therefore

\[ \tilde{\mu}(-1/2, y, 1) \] has a critical point at \( y = \frac{-3 + \sqrt{3}}{4} \)

\[ \tilde{\mu}(-1/2, \frac{-3 + \sqrt{3}}{4}, 1) = \frac{1/4(1 + \sqrt{3})}{2(\frac{3 \sqrt{3}}{8})^{3/2}} \]

\[ = \frac{2^{3/2}(1+\sqrt{3})}{3^{9/4}} = \frac{2 \sqrt{2}}{9^{4} \sqrt{3}} (1 + \sqrt{3}) = 0.6523... \]

\[ \tilde{\mu}(-1/2, 0, 1) = \frac{4 \sqrt{3}}{9} = 0.769... \]

\[ \tilde{\mu}(-1/2, -1/2, 1) = \frac{\sqrt{2}}{2} = 0.70... \]

Therefore there is a minimum at \( y = \frac{-3 + \sqrt{3}}{4} \).

iv) The minimum of \( \tilde{\mu} \) on the face \( z=1 \) is \( \frac{2 \sqrt{2}}{9^{4} \sqrt{3}} (1 + \sqrt{3}) \) attained at \( (-1/2, -3 + \sqrt{3}, 4, 1) \)

By iii) of lemma 5 \( \tilde{\mu}(x, y, 1) \geq \tilde{\mu}(-1/2, y, 1) \) and by the above the minimum is at \( y = \frac{-3 + \sqrt{3}}{4} \).

**Theorem 1:** The extreme positive definite binary Hermitian forms are equivalent to:

\[ g_0 = uu + Bu + Buv + vv \]

where \( B = -1/2 + \frac{(-3 + \sqrt{3})i}{4} \)

and \( \mu(g_0) = \frac{2 \sqrt{2}}{9^{4} \sqrt{3}} (1 + \sqrt{3}). \)
Proof:

In lemma 4 we showed that for vertices of type V \( 4g(v) = \frac{(2z-4xy)(1+2x+z)}{Dv/d(\theta)} \).

\[ 4_{\mu}(g) = \frac{4m(g)}{(d(g))^{1/4}} \geq \frac{4g(v)}{(d(g))^{1/4}} \]

\[ = \frac{(2z-4xy)(1+2x+z)}{(d(g))^{3/4}} \]

\[ = 4_{\mu}(x, y, z). \]

Thus to prove that \( g_0 \) is extreme it suffices to show that vertices of type V are maximal when \((x, y, z) = (-1/2, y, 1)\) and \(-1/2 \leq y \leq 0\).

The uniqueness of \( g_0 \) will follow from Dickson's result that a closed convex subset of a Voronoi cone contains only one extreme form.

Putting \( x = -1/2 \) and \( z = 1 \) in (1) we consider forms:

\[ g = (y+1/2)(\sum_{i=1}^{4} u_i^2 + (u_1-u_3)^2 + (u_2-u_4)^2) \]

\[ -y(2\sum_{i=1}^{4} u_i^2 + 2u_1u_2 - 2u_1u_3 - 2u_1u_4 - 2u_2u_3 - 2u_2u_4) \]

\( m_1 \), the minima modulo 2 and \( g(m_1) \) are summarized in the following table.
\[2t = (1, -1 - 2y, 0, 1 + 2y)\]

From lemma 2, \(Dg(v_1) = w'A^{-1}w = F(w)\)

\[
\begin{align*}
&= \sum_{i=1}^{4} w_i^2 + w_1 w_3 + w_2 w_4 + 2y(w_2 w_3 - w_1 w_4) \\
\end{align*}
\]

and \(w = (-t_2 - t_4, t_1, t_2, -t_1 - t_3)\)

\[2w = (0, 1, -1 - 2y, -1)\]

\[4Dg(v_1) = (2 + 1 + 4y^2 + 4y - 1 + 2y(-1 - 2y))\]

\[= 2 + 2y\]

ii) Choose \(v_2 = [1, 2, 11, 15]\) of type II

\[2t_1 = 2t_2 = 1\]
\[2t_1 + 2t_2 + 2t_3 = 2\]
\[2t_1 + 2t_2 + 2t_3 + 2t_4 = 2 + 2y\]
\[2t = (1, 1, 0, 2y)\]
\[2w = (-1 - 2y, 1, 1, -1)\]

\[4Dg(v_2) = 1 + 4y^2 + 4y + 3 - 1 - 2y - 1 + 2y - 2y - 4y^2\]

\[= 2 + 2y\]

iii) Choose \(v_3 = [1, 6, 11, -4]\) of type III

\[2t_1 = 1 , \quad 2t_1 + 2t_3 = 1 , \quad -2t_4 = 1\]

\[2t_1 + 2t_2 + 2t_3 = 2\]
$$\begin{array}{ccc}
i & m_i & g(m_i) \\
1 & (1, 0, 0, 0) & 1 \\
2 & (0, 1, 0, 0) & 1 \\
3 & (0, 0, 1, 0) & 1 \\
4 & (0, 0, 0, 1) & 1 \\
5 & (1, -1, 0, 0) & 2 + 2y \\
6 & (1, 0, 1, 0) & 1 \\
7 & (1, 0, 0, 1) & 2 + 2y \\
8 & (0, 1, 1, 0) & 2 + 2y \\
9 & (0, 1, 0, 1) & 1 \\
10 & (0, 0, 1, -1) & 2 \\
11 & (1, 1, 1, 0) & 2 \\
12 & (1, 1, 0, 1) & 2 \\
13 & (1, 0, 1, 1) & 2 + 2y \\
14 & (0, 1, 1, 1) & 2 + 2y \\
15 & (1, 1, 1, 1) & 2 + 2y \\
\end{array}$$

i) Choose $v_1 = [7, 5, 6, 13]$ of type I

$2m_1 t = g(m_1)$ implies

$$2t_1 + 2t_4 = 2 + 2y$$
$$2t_1 - 2t_2 = 2 + 2y$$
$$2t_1 + 2t_3 = 1$$
$$2t_1 + 2t_3 + 2t_4 = 2 + 2y$$
\[2t = (1, -1 - 2y, 0, 1 + 2y)\]

From lemma 2, \(Dg(v_1) = w'A^{-1}w = F(w)\)

\[
= \sum_{i=1}^{4} w_i^2 + w_1w_3 + w_2w_4 + 2y(w_2w_3 - w_1w_4)
\]

and

\[w = (-t_2 - t_4, t_1, t_2, -t_1 - t_3)\]

\[2w = (0, 1, -1 - 2y, -1)\]

\[4Dg(v_1) = (2 + 1 + 4y^2 + 4y - 1 + 2y(-1 - 2y))\]

\[= 2 + 2y\]

ii) Choose \(v_2 = [1, 2, 11, 15]\) of type II

\[2t_1 = 2t_2 = 1\]
\[2t_1 + 2t_2 + 2t_3 = 2\]
\[2t_1 + 2t_2 + 2t_3 + 2t_4 = 2 + 2y\]
\[2t = (1, 1, 0, 2y)\]
\[2w = (-1 - 2y, 1, 1, -1)\]

\[4Dg(v_2) = 1 + 4y^2 + 4y + 3 - 1 - 2y - 1 + 2y - 2y - 4y^2\]

\[= 2 + 2y\]

iii) Choose \(v_3 = [1, 6, 11, -4]\) of type III

\[2t_1 = 1, \; 2t_1 + 2t_3 = 1, \; -2t_4 = 1\]

\[2t_1 + 2t_2 + 2t_3 = 2\]
$2t = (1, 1, 0, -1)$
$2w = (0, 1, 1, -1)$
$4Dg(v_3) = 3 - 1 + 2y = 2 + 2y$

iv) Choose $v_4 = [-2, 3, 4, 13]$ of type IV.

$2t_2 = -1, \ 2t_3 = 2t_4 = 1$

$2t_1 + 2t_3 + 2t_4 = 2 + 2y$

$2t = (2y, -1, 1, 1)$
$2w = (0, 2y, -1, -1 - 2y)$
$4Dg(v_4) = 4y^2 + 1 + 1 + 4y^2 + 4y - 2y - 4y^2 - 4y^2$
$= 2 + 2y$

v) Choose $v_5 = [-1, -2, 3, 4]$ of type V.

$4Dg(v_5) = 2 + 2y$

vi) Choose $v_6 = [-2, 5, 7, 13]$

$2t_2 = -1, \ 2t_1 - 2t_2 = 2 + 2y$

$2t_1 + 2t_4 = 2 + 2y$

$2t_1 + 2t_3 + 2t_4 = 2 + 2y$

$2t = (1 + 2y, -1, 0, 1)$
$2w = (0, 1+2y, -1, -1 - 2y)$
$4Dg(v_6) = 1 + 2 + 8y^2 + 8y - 1 - 4y^2 - 4y - 2y - 4y^2$
$= 2 + 2y.$
vii) Choose \( v_7 = [1, -4, -9, 6] \) of type VII

\[
\begin{align*}
2t_1 &= 1, \quad 2t_4 = -1, \quad 2t_3 + 2t_1 = 1 \\
2t_2 + 2t_4 &= -1 \\
2t &= (1, 0, 0, -1) \\
w &= (1, 1, 0, -1) \\
4Dg(v_7) &= 3 - 1 + 2y = 2 + 2y
\end{align*}
\]

viii) Choose \( v_8 = [-1, 3, 4, 14] \) of type VIII

\[
\begin{align*}
2t_1 &= -1, \quad 2t_3 = 2t_4 = 1 \\
2t_2 + 2t_3 + 2t_4 &= 2 + 2y \\
2t &= (-1, 2y, 1, 1) \\
w &= (-1 - 2y, -1, 2y, 0) \\
4Dg(v_8) &= 1 + 4y^2 + 4y + 1 + 4y^2 - 2y - 4y^2 - 4y^2 \\
&= 2 + 2y
\end{align*}
\]

ix) Choose \( v_9 = [1, 2, -3, 12] \) of type IX

\[
\begin{align*}
2t_1 &= 2t_2 = 1, \quad 2t_3 = -1 \\
2t_1 + 2t_2 + 2t_4 &= 2 \\
2t &= (1, 1, -1, 0) \\
w &= (-1, 1, 1, 0) \\
4Dg(v_9) &= 3 - 1 + 2y = 2 + 2y
\end{align*}
\]

x) Choose \( v_{10} = [1, 2, -4, 11] \)

\[
\begin{align*}
2t_1 &= 2t_2 = 1, \quad 2t_4 = -1 \\
2t_1 + 2t_2 + 2t_3 &= 2
\end{align*}
\]
2t = (1, 1, 0, -1)
2w = (0, 1, 1, -1)

$4D_g(v_{10}) = 3 - 1 + 2y = 2 + 2y$

xi) Choose $v_{11} = [-1, 3, 8, 14]$

- $2t_1 = -1, 2t_3 = 1, 2t_2 + 2t_3 = 2 + 2y$
- $2t_2 + 2t_3 + 2t_4 = 2 + 2y$
- $2t = (-1, 1+2y, 1, 0)$
- $2w = (-1 -2y, -1, 1+2y, 0)$

$4D_g(v_{11}) = 2 + 8y^2 + 8y + 1 - 1 - 4y^2 - 4y - 2y - 4y^2$

- $= 2 + 2y$

xii) Choose $v_{12} = [-1, 4, 9, 14]$

- $2t_1 = -1, 2t_4 = 1, 2t_2 + 2t_4 = 1$
- $2t_2 + 2t_3 + 2t_4 = 2 + 2y$
- $2t = (-1, 0, 1+2y, 1)$
- $2w = (-1, -1, 0, -2y)$

$4D_g(v_{12}) = 2 + 4y^2 + 2y - 4y^2 = 2 + 2y$

Formulated in terms of lattices the above theorem becomes:

**Theorem 2**: The extreme 2-dimensional Gaussian lattices are equivalent to

$$L = \left\{(1, 0), (-1/2 - yi, \sqrt{3/4 - y^2})\right\}$$

where

$$y = \frac{-3 + \sqrt{3}}{4}.$$
Furthermore

\[ \varrho(S) = \frac{4(1 + \sqrt{3})^2}{81 \sqrt{3}} \cdot \pi^2 \]

where \( S \) is the unit sphere in complex 2-space.

**Proof:** With \( x = -1/2 \) and \( z = 1 \) the matrix \( A \) of \( f \) can be written as \( P^TP \) with

\[
\begin{pmatrix}
1 & 0 & -1/2 & y \\
0 & 1 & -y & -1/2 \\
0 & 0 & \sqrt{3/h-y^2} & 0 \\
0 & 0 & 0 & \sqrt{3/h-y^2}
\end{pmatrix}
\]

As a real 4-dimensional lattice, the extreme lattice is generated by the columns of \( P \). Hence it has a basis

\[
\{(1, 0), (-1/2 - yi, \sqrt{3/4} - y^2)\}
\]
as a complex 2-dimensional lattice over the Gaussian integers.

It has already been established that \( y = (-3 + \sqrt{3})/4 \) if \( L \) is extreme.

The formula

\[ \varrho(S) = \inf J_n f^2 \]

and

\[ \inf J_n^\mu f = \frac{2 \sqrt{2}}{9 \sqrt{3}} (1 + \sqrt{3}) \]
yield

\[ \varrho(S) = \frac{4(1 + \sqrt{3})^2}{81 \sqrt{3}} \cdot \pi^2 \].
Lemma 1: Every positive definite binary Hermitian form over $\mathbb{Q}(\sqrt{-3})$
is equivalent to a form:

$$uu + Buv + B\overline{uv} + zv\overline{v}$$

where $z \geq 1$, $B = x + \frac{iy}{\sqrt{3}}$ and $-1/2 \leq x \leq y \leq 0$.

Proof: Let

$$f = tuu + Buv + B\overline{uv} + zv\overline{v}$$

be a positive definite Hermitian form over $\mathbb{Q}(\sqrt{-3})$. As in lemma 1 of
Chapter I we can assume that $t$ and $z$ are the successive minima of $f$.
Dividing by $t$ we have

$$f = uu + Buv + B\overline{uv} + zv\overline{v}$$

where $z \geq 1$.

$$z = f(1, \pm 1) = 1 \pm (B + B) + z = 1 \pm 2x + z.$$ 

Hence $-1 \leq \pm 2x$ and $|x| \leq 1/2$.

If $xy < 0$ we apply the transformation $u \rightarrow \overline{u}$, $v \rightarrow \overline{v}$ to get
Thus we assume $xy \geq 0$.

If $x$ or $y$ is positive, we apply the transformation $u \rightarrow -u$, $v \rightarrow v$ to get

$$u^2 - Buv - Buv + zvv.$$ 

Thus we can assume $x \leq 0$ and $y \leq 0$.

If $y < x$ we consider two cases:

i) $3x \leq y < x \leq 0$

Apply the transformation $u \rightarrow \varepsilon u$, $v \rightarrow v$ where $\varepsilon = \frac{1 + \sqrt{3}i}{2}$.

$f$ becomes

$$u^2 + \varepsilon B uv + \overline{B}\varepsilon \overline{uv} + zvv$$

i.e. $B$ is replaced by $B_1 = \overline{B}\varepsilon$

$$B_1 = x_1 + iy_1 = \frac{xy}{2} + \frac{1}{2\sqrt{3}} (3x-y)$$

$$y_1 = \frac{3x-y}{2} \leq 0 \text{ since } 3x \leq y$$

$$x_1 = \frac{xy}{2} < \frac{3x-y}{2} = y_1 \text{ since } y < x$$

ii) $y \leq 3x$

Apply the transformation $u \rightarrow \varepsilon u$, $v \rightarrow v$ where $\varepsilon = \frac{1 + \sqrt{3}i}{2}$. 

\( f \) becomes
\[
uu + B\bar{\epsilon}uv + B\bar{\epsilon}uv + zvv
\]
e.g. \( B \) is replaced by \( B_2 = B\bar{\epsilon} \)

\[
B_2 = x_2 + iy_2 = \frac{x+iy}{2} + \frac{1}{2\sqrt{3}}(y - 3x)
\]

\[
y_2 = \frac{y-3x}{2} \leq 0 \quad \text{since} \quad y \leq 3x
\]

\[
x_2 = \frac{x+iy}{2} = \frac{y-3x}{2} = y_2 \quad \text{since} \quad x \leq 0.
\]

In cases i) and ii) the new forms obtained still have 1 and \( z \) for their successive minima therefore \( z < f(1,1) \) implies \( x \geq -1/2 \).

**Lemma 2:** All positive definite binary Hermitian forms over \( Q(\sqrt{-3}) \) are in a cone equivalent to \( \Delta'' \).

**Proof:** Let \( f \) be a positive definite binary Hermitian form over \( Q(\sqrt{-3}) \). By Lemma 1, we can write \( f \) as
\[
uu + B\bar{u}v + B\bar{u}v + zvv
\]
where \( z \geq 1, \quad B = x + \frac{iy}{\sqrt{3}} \) and \(-1/2 \leq x \leq y \leq 0\).

Letting \( u = u_1 + u_2 (\frac{1+\sqrt{3}i}{2}) \) and \( v = u_3 + u_4 (\frac{1+\sqrt{3}i}{2}) \),
\[
f = u_1^2 + u_2^2 + u_1u_2 + z(u_3^2 + u_4^2 + u_3u_4) + 2\text{Re}(Buv)
\]
\[
= u_1^2 + u_2^2 + u_1u_2 + z(u_3^2 + u_4^2 + u_3u_4)
\]
\[\quad + x(2u_1u_3 + u_2u_3 + u_1u_4 + 2u_2u_4) + y(u_2u_3 - u_1u_4) \].
Applying the transformation $T$: $u_1 \mapsto -u_3$, $u_2 \mapsto u_1$, $u_3 \mapsto -u_2$, $u_4 \mapsto u_4$, we have

$$T(f) = u_1^2 + u_3^2 - u_1 u_3 + z(u_2^2 + u_4^2 - u_2 u_4)$$

$$+ x(2u_2 u_3 + 2u_1 u_4 - u_3 u_4 - u_1 u_2) + y(u_3 u_4 - u_1 u_2)$$

$$= (x+1/2)(\sum_{i=1}^{4} u_i^2 + (u_1 - u_3)^2 + (u_2 - u_4)^2)$$

$$+ \frac{x-1}{2} (u_2^2 + u_4^2 + (u_2 - u_4)^2)$$

$$+ \frac{y-x}{2} (\sum_{i=1}^{4} u_i^2 + (u_1 - u_3)^2 + (u_1 - u_4)^2 + (u_2 - u_3)^2)$$

$$+ (u_2 - u_4)^2 + (u_1 + u_2 - u_3 - u_4)^2$$

$$- \frac{1}{2} y(2 \sum_{i=1}^{4} u_i^2 + 2u_1 u_2 - 2u_1 u_3 - 2u_1 u_4 - 2u_2 u_3 - 2u_2 u_4).$$

Since the coefficients $x + 1/2$, $\frac{x-1}{2}$, $\frac{y-x}{2}$ and $-y$ are non-negative, $T(f)$ is in $\triangle$.

**Lemma 3:** Let $v = (v_1, v_2, v_3, v_4)$ be a vertex of $\Pi(g)$ defined by the integer points $m_1, m_2, m_3, m_4$ where $g = 2T(f)$. Then $g(v) = w'A^{-1}w$ where $A$ is the matrix of $2f$, $w = (-t_3, t_1, -t_2, t_4)$ and $t$ is defined by $2m_1t = g(m_1)$.

**Proof:** The proof of this lemma is the same as that of lemma 3 of Chapter II. Here we have to take
the matrix of the transformation $T$ and $w = (S')^{-1}t$.

$$S = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$w = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} = (-t_3, t_1, -t_2, t_4)$$

Lemma 4: If $v$ is a type I vertex then $g(v) = \frac{2}{D} (z - x^2 + y^2)(1 + x + y + z)$

where $D = 3z - 3x^2 - y^2$.

Proof: Choose $v = [3, 4, 13, 14]$ of type I. The integer points defining $v$ are

$m_3 = (0, 0, 1, 0)$ \hspace{1cm} $m_4 = (0, 0, 0, 1)$

$m_{13} = (1, 0, 1, 1)$ \hspace{1cm} $m_{14} = (0, 1, 1, 1)$

$g(m_3) = 2$ \hspace{1cm} $g(m_4) = 2z$

$g(m_{13}) = 2 + 2x + 2y + 2z$ \hspace{1cm} $g(m_{14}) = 2 + 2x + 2y + 2z$

From $2m_1^t = g(m_1)$ we have

$2t_3 = 2$, \hspace{1cm} $2t_4 = 2z$
\[ 2(t_1 + t_3 + t_4) = 2 + 2x + 2y + 2z, \quad 2(t_2 + t_3 + t_4) = 2 + 2x + 2y + 2z \]

hence \( 2t = (2x + 2y, 2x + 2y, 2, 2z) \) and \( 2w = (-2, 2x + 2y, -2x - 2y, 2z). \)

The matrix of \( 2f \) is

\[
A = \begin{pmatrix}
2 & 1 & 2x & x-y \\
1 & 2 & x+y & 2x \\
2x & x+y & 2z & z \\
x-y & 2x & z & 2z
\end{pmatrix}
\]

\[
A^{-1} = \frac{1}{D} \begin{pmatrix}
2z & -z & -2x & 2x+y \\
-z & 2z & x-y & -2x \\
-2x & -x+y & 2 & -1 \\
x+y & -2x & -1 & 2
\end{pmatrix}
\]

where \( D = \sqrt{\det A} = 3z - 3x^2 - y^2. \)

Since \( h_g(v) = \frac{1}{D} (2w)'A^{-1}(2w), \)

\[
h_Dg(v) = 2z(4 + 4x^2 + 4y^2 + 8xy) + 2(4x^2 + 4y^2 + 8xy + 4z^2) - 2(-4x - 4y) - 4x(4x + 4y) + (2x + 2y)(-4z) + (2x - 2y)(-4x^2 - 4y^2 - 8xy) - 4x(4xz + 4yz) - 2(-4xz - 4yz) = z(8 + 8x + 8y + 8z) - x^2(8 + 8x + 8y + 8z) + y^2(8 + 8x + 8y + 8z) = 8(z - x^2 + y^2)(1 + x + y + z)\]
Define $\tilde{f}(x, y, z) = \frac{2}{D^{3/2}} (z - x^2 + y^2)(1 + x + y + z)$.
If vertices of type I are maximal then $\tilde{f}(x, y, z) = \tilde{f}(g)$. In any case $\tilde{f}(x, y, z) \leq \tilde{f}(g)$ for any $(x, y, z)$ subject to the restrictions $z \geq 1$, $-1/2 \leq x \leq y \leq 0$.

**Lemma 5:**

i) $\frac{\partial \tilde{f}(x, y, z)}{\partial x}$ and $\frac{\partial \tilde{f}(x, y, z)}{\partial z}$ do not vanish simultaneously if $x \neq -1/2$

ii) $\frac{\partial \tilde{f}(x, y, 1)}{\partial x}$ and $\frac{\partial \tilde{f}(x, y, 1)}{\partial y}$ do not vanish simultaneously if $y \neq 0$ and $x \neq -1/2$

iii) $\frac{\partial \tilde{f}(-1/2, y, z)}{\partial y}$ and $\frac{\partial \tilde{f}(-1/2, y, z)}{\partial z}$ do not vanish simultaneously if $y \neq 0$ and $z \neq 1$.

**Proof:**

i) $\frac{\partial \tilde{f}}{\partial x} = \frac{2}{D^{3}} \left\{ D^{3/2}((z-x^2+y^2) - 2x(1+x+y+z)) + 9xD^{1/2}(1+x+y+z)(z-x^2+y^2) \right\}$.

$\frac{\partial \tilde{f}}{\partial x} = 0$ implies

(1) $D(z-x^2+y^2) - 2xD(1+x+y+z) + 9x(z-x^2+y^2)(z-x^2+y^2) = 0$.

$\frac{\partial \tilde{f}}{\partial z} = \frac{2}{D^{3}} \left\{ D^{3/2}(1+x+y+z) + D^{3/2}(z-x^2+y^2) - \frac{2}{2} D^{1/2}(z-x^2+y^2)(1+x+y+z) \right\}$. 
\( \frac{\partial^2 \mu}{\partial z} = 0 \) implies

\[
2xD(1+x+y+z) + 2xD(z-x^2+y^2) = 9x(z-x^2+y^2)(1+x+y+z)
\]

Using (1),

\[
D(z-x^2+y^2) -2xD(1+x+y+z) + 2xD(1+x+y+z) \\
+ 2xD(z-x^2+y^2) = 0.
\]

Dividing by \( D \) and simplifying, we get

\[ 1 + 2x = 0 \quad \text{or} \quad x = -1/2. \]

ii) \( \mu(x, y, 1) = \frac{2}{D^{3/2}} (1-x^2+y^2)(2+x+y) \)

where

\[ D = 3 - 3x^2 - y^2 \]

\[
\frac{\partial^2 \mu}{\partial y} = \frac{2}{D^3} \left\{ D^{3/2}(1-x^2+y^2) + 2yD^{3/2}(2+x+y) \\
+ 3yD^{1/2}(1-x^2+y^2)(2+x+y) \right\}.
\]

\[
\frac{\partial^2 \mu}{\partial x} = 0 \quad \text{implies}
\]

\[
-D(1-x^2+y^2) -2yD(2+x+y) - 3y(1-x^2+y^2)(2+x+y) = 0.
\]

\[
\frac{\partial^2 \mu}{\partial x} = \frac{2}{D^3} \left\{ D^{3/2}(1-x^2+y^2) - 2xD^{3/2}(2+x+y) \\
+ 9xD^{1/2}(1-x^2+y^2)(2+x+y) \right\}.
\]

\[
\frac{\partial^2 \mu}{\partial x} = 0 \quad \text{implies}
\]
(3) \[ D(1-x^2+y^2) -2xD(2+x+y) + 9x(1-x^2+y^2)(2+x+y) = 0 \]

adding (2) and (3),

\[ -2(x+y) D(2+x+y) + (9x-3y)(1-x^2+y^2)(2+x+y) = 0 \]

hence

(4) \[ -2(x+y) D + (9x-3y)(1-x^2+y^2) = 0 \]

We multiply (2) by 3x and (3) by y and add to get

\[ (y-3x) D(1-x^2+y^2) - 8xyD(2+x+y) = 0 \]

or

\[ (y-3x)(1-x^2+y^2) - 8xy(2+x+y) = 0 \]

adding this to (4) we get

\[ 0 = (1-x^2+y^2)(6x-2y) + (x+y)(-2D - 8xy) - 16 xy \]

\[ = 6x-2y-6x+2x^2y + 6xy^2-2y^2 - 16xy-(x+y)(-6+6x^2+2y^2-8xy) \]

\[ = -8y-16xy = -2y(1+2x) . \]

iii) \[ \vec{\omega}(-1/2, y, z) = \frac{2}{p^{3/2}} (z-1/4+y^2)(1/2 + y + z) \]

where \( D = (3z - 3/4 - y^2) \).

Let \( A = z - 1/4 + y^2 \)

and

\( B = 1 + 2y + 2z . \)

Then

\[ \vec{\omega}(-1/2, y, z) = \frac{AB}{p^{3/2}} \]
\[ f^{(-1/2, y, z)} = \frac{1}{d^3} \left\{ 2yD^{3/2}_B + 2D^{3/2}_A + 3yB^{1/2}AB \right\} \]

implies

\[ f^{(-1/2, y, z)} = 0 \]

(6) \[ 2yDB + 2DA + 3yAB = 0 \]

\[ f^{(-1/2, t, z)} = \frac{1}{d^3} \left\{ D^{3/2}_B + 2D^{3/2}_A - \frac{9}{2} D^{1/2}AB \right\} \]

implies

\[ f^{(-1/2, y, z)} = 0 \]

(7) \[ DB + 2DA - \frac{9}{2} AB = 0 \]

Subtracting this from (6) we get

\[ 2yDB - DB + 3yAB + \frac{9}{2} AB = 0 \]

hence

(8) \[ 2yD - D + 3yA + \frac{9}{2} A = 0 \]

We multiply (6) by 3 and (7) by 2y and add to get

\[ 8yDB + (6 + 4y) DA = 0 \]

or

(9) \[ 8yB + (6 + 4y)A = 0 \]

We multiply (8) by 4 and subtract (9) from it to get

\[ (8y-4)D + (12y + 18)A - 8yB - (6 + 4y)A = 0 \]
0 = (8y-4)D + (8y + 12)A - 8yB

= (8y-4)(3z - \frac{3}{4} - y^2) + (8y + 12)(z - \frac{1}{4} + y^2) - 8y(1+2y+2z)

= (24yz - 12z - 6y + 3 - 8y^3 + 4y^2 + 8yz - 2y + 8y^3

+ 12yz - 3 + 12y^2 - 8y - 16y^2 - 16yz

+ 16yz - 16y = -16y(1-z)

Lemma 6: \( \bar{\mu} \) has a minimum of \( \frac{2(-17 + 5\sqrt{13})}{(-5 + 2\sqrt{13})^{3/2}} \) and this minimum is attained at \((-1/2, -4 + \sqrt{13}/2, 1)\).

Proof: From lemma 5 we see that the minimum of \( \bar{\mu} \) cannot be in the interior of the faces z=1, y=0 or x=-1/2. If x=y,

\[ \bar{\mu} = z(1 + 2x + z)/D^{3/2} \text{ where } D = 3z - 4x^2 \]

\[ \frac{\partial \bar{\mu}}{\partial x} = 0 \text{ implies } \]

\[ 0 = D + 6x(1 + 2x + z) \]

\[ = 3z + 6xz + 6x + 8x^2 \]

\[ \frac{\partial \bar{\mu}}{\partial z} = 0 \text{ implies } \]

\[ 0 = D(1 + 2x + 2z) - \frac{9}{2} z(1 + 2x + z) \]

\[ = -6x(1 + 2x + z)(1 + 2x + 2z) - \frac{9}{2} z(1 + 2x + z). \]

Hence

\[ 0 = -4x(1 + 2x + 2z) - 3z \]

\[ = -3z - 4x - 8x^2 - 8xz \]

Adding this to 0 = 3z + 6xz + 6x + 8x^2

we get

\[ 0 = -2xz + 2x = 2x(1 - z). \]
We can now conclude that the minimum of $\tilde{J}$ cannot be in the interior of a 2-dimensional face. By lemma 5 and the fact that $\tilde{J}$ is an increasing function of $z$ for sufficiently large $z$, the minimum of $\tilde{J}$ must be on an edge of: $z \geq 1, -1/2 \leq x \leq y \leq 0$.

i) The edge $z=1, y=0$

$$\tilde{J}(x) = \frac{2(1-x^2)(2+x)}{(3-3x^2)^{3/2}}$$

$$\tilde{J}'(x) \text{ is of the same sign as}$$

$$(1-x^2) + x(2+x) = 3 + 2x - 2x^2$$

$$\geq 2 - 2x^2 = 2(1-x^2) > 0.$$ 

Therefore the minimum on this edge is at $x = -1/2$ and

$$\tilde{J}(-1/2, 0, 1) = \frac{2(\frac{3}{2})}{3^{3/2}(\frac{3}{4})^{1/2}} = \frac{2}{3}$$

ii) The edge $z=1, x=y$.

$$\tilde{J} = \frac{2(2+2x)}{(3-4x^2)^{3/2}}$$

$\tilde{J}'$ is of the same sign as

$$(3-4x^2) + 12x(1+x) = 8x^2 + 12x + 3$$

$8x^2 + 12x + 3$ is negative if $x = -1/2$, it is equal to zero if $x = \frac{-3 + \sqrt{3}}{4}$ and it is positive if $x = 0$. Therefore $\tilde{J}$ has a minimum at $x = (-3 + \sqrt{3})/4$ on this edge.
\[
\tilde{\mu}(-\frac{3 + \sqrt{3}}{4}, -\frac{3 - \sqrt{3}}{4}, 1) = \frac{2^{3/2}(1 + \sqrt{3})}{3^{1/2}} = 0.652\ldots
\]

iii) The edge \( z = 1, x = -1/2 \)

\[
\tilde{\mu} = \frac{2(\frac{3}{4} + y^2)(\frac{3}{2} + y)}{(\frac{3}{4} - y^2)^{3/2}}
\]

\[
\tilde{\mu}'(y) = \frac{2}{(\frac{3}{4} - y^2)^{3/2}} \left\{ \frac{9}{4} - y^2 \right\}^{3/2} (2y)(3/2 + y) + \frac{\left( \frac{9}{4} - y^2 \right)^{3/2}}{(\frac{3}{4} - y^2)^{3/2}} \left( \frac{3}{4} + y^2 \right) + 3y \left( \frac{9}{4} - y^2 \right)^{1/2} \left( \frac{3}{4} + y^2 \right) \left( \frac{3}{2} + y \right)
\]

\( \tilde{\mu}' \) is of the same sign as

\[2y(\frac{9}{4} - y^2)(3/2 + y) + (\frac{9}{4} - y^2)(\frac{3}{4} + y^2) + 3y(\frac{3}{4} + y^2)(\frac{3}{2} + y)\]

which in turn is of the same sign as

\[2y(\frac{9}{4} - y^2)(\frac{3}{2} - y)(\frac{3}{4} + y^2) + 3y(\frac{3}{4} + y^2)\]

\[= \frac{9}{2}y - 2y^3 + \frac{9}{8} + \frac{3}{2}y^2 - \frac{3}{4}y - y^3 + \frac{9}{4}y + 3y^3 = \frac{3}{2}y^2 + 6y + \frac{9}{8}.
\]

This polynomial is negative if \( y = -1/2 \), it is positive if \( y = 0 \) and it equals zero if \( y = (-4 + \sqrt{13})/2 \). Therefore the minimum on this edge is at \( y = (-4 + \sqrt{13})/2 \).

\[
\tilde{\mu}(-1/2, -4 + \frac{\sqrt{13}}{2}, 1) = \frac{2(-17 + 5\sqrt{13})}{(-5 + 2\sqrt{13})^{3/2}} = 0.6251\ldots
\]

iv) The edge \( y = 0, x = -1/2 \)

\[
\tilde{\mu} = \frac{2z(z - \frac{1}{4})(z + \frac{1}{2})}{3^{3/2}(z - \frac{1}{4})^{3/2}} = \frac{2(z + \frac{1}{2})}{3^{3/2}(z - \frac{1}{4})^{1/2}}
\]
\[ J^{\mu'}(z) = \frac{2}{3^{3/2}} \left( z - \frac{1}{4} \right)^{1/2} - \frac{2(z + \frac{1}{2})}{2 \cdot 3^{3/2}(z - \frac{1}{4})^{1/2}} \]

\[ J^{\mu'} \text{ is of the same sign as} \]

\[ (z - \frac{1}{4}) - \frac{1}{2}(z + \frac{1}{2}) = 1/2(z - 1) \geq 0. \]

Therefore the minimum on this edge is at \( z = 1 \) and

\[ J^{\mu'}(-1/2, 0, 1) = \frac{2(3z)}{3^{3/2}(3z - 1)^{1/2}} = \frac{2}{3} \]

v) The edge \( x = y = 0 \).

\[ J^{\mu} = \frac{2z(1 + z)}{3z} = \frac{2(1 + z)}{3} \]

\[ J^{\mu'}(z) = 2/3 > 0. \]

Therefore the minimum on this edge is at \( z = 1 \) and

\[ J^{\mu}(0, 0, 1) = \frac{4}{3} \]

vi) The edge \( x = y = -1/2 \)

\[ J^{\mu} = \frac{2z^2}{(3z - 1)^{3/2}} \]

\[ J^{\mu'}(z) = \frac{1}{(3z - 1)^3} \left\{ (3z - 1)^{3/2}(4z) - 9z^2(3z - 1)^{1/2} \right\} \]

\[ J^{\mu'} \text{ is of the same sign as} \]

\[ 4(3z - 1) - 9z = 3z - 4. \]

Therefore the minimum on this edge is
In parts i) through vi) we have considered all edges of the region: $z \geq 1$, $-1/2 \leq x \leq y \leq 0$.

**Theorem 1:** The extreme positive definite Hermitian forms over $Q(\sqrt{3})$ are equivalent to

$$g_0 = uu + Buv + Buv + vv$$

where

$$B = \frac{1}{2} - \frac{1}{2} \left( \frac{-4 + \sqrt{13}}{2 \sqrt{3}} \right)$$

and

$$\int (-1/2, -1/2, 4/3) = \frac{2(16)}{3^{3/2}} = \frac{32 \sqrt{3}}{81} = 0.684...$$

**Proof:** We need to show that vertices of type I are maximal when $z = 1$, $x = -1/2$ and $y = \frac{-4 + \sqrt{13}}{2}$.

Putting $z = 1$ and $x = -1/2$ in the form obtained in lemma 2, we consider forms:

$$g = \left( y + \frac{1}{2} \right) \left( \sum_{i=1}^{4} u_i^2 + (u_1 - u_3)^2 + (u_1 - u_4)^2 + (u_2 - u_3)^2 + (u_2 - u_4)^2 + (u_1 + u_2 - u_3 - u_4)^2 \right)$$

$$-2y(2 \sum_{i=1}^{4} u_i^2 + 2u_1u_2 - 2u_1u_3 - 2u_1u_4 - 2u_2u_3 - 2u_2u_4)$$
m₁, the minima modulo 2 and g(m₁) are summarized in the following table.

<table>
<thead>
<tr>
<th>i</th>
<th>m₁</th>
<th>g(m₁)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1, 0, 0, 0)</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>(0, 1, 0, 0)</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>(0, 0, 1, 0)</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>(0, 0, 0, 1)</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>(1, -1, 0, 0)</td>
<td>3+2y</td>
</tr>
<tr>
<td>6</td>
<td>(1, 0, 1, 0)</td>
<td>2</td>
</tr>
<tr>
<td>7</td>
<td>(1, 0, 0, 1)</td>
<td>2</td>
</tr>
<tr>
<td>8</td>
<td>(0, 1, 1, 0)</td>
<td>2</td>
</tr>
<tr>
<td>9</td>
<td>(0, 1, 0, 1)</td>
<td>2</td>
</tr>
<tr>
<td>10</td>
<td>(0, 0, 1, -1)</td>
<td>3-2y</td>
</tr>
<tr>
<td>11</td>
<td>(1, 1, 1, 0)</td>
<td>3-2y</td>
</tr>
<tr>
<td>12</td>
<td>(1, 1, 0, 1)</td>
<td>3-2y</td>
</tr>
<tr>
<td>13</td>
<td>(1, 0, 1, 1)</td>
<td>3+2y</td>
</tr>
<tr>
<td>14</td>
<td>(0, 1, 1, 1)</td>
<td>3+2y</td>
</tr>
<tr>
<td>15</td>
<td>(1, 1, 1, 1)</td>
<td>2</td>
</tr>
</tbody>
</table>
1) Choose $v_1 = [3, 4, 13, 14]$ of type I. From lemma 4 we see that

$$4Dg(v_1) = (3 + 4y^2)(3 + 2y) = 8y^3 + 12y^2 + 6y + 9$$

ii) Choose $v_2 = [1, 2, 11, 15]$ of type II.

$$2m_1t = g(m_1) \text{ implies }$$

$$2t_1 = 2t_2 = 2$$
$$2(t_1 + t_2 + t_3) = 3 - 2y$$
$$2(t_1 + t_2 + t_3 + t_4) = 2$$
$$2t = (2, 2, -1 - 2y, -1 + 2y)$$
$$2w = (1 + 2y, 2, -2, -1 + 2y)$$

$$4Dg(v_2) = (2w)'(DA^{-1})(2w) = F(w)$$

$$= 2 \sum w_1^2 - 2w_1w_2 + 2w_1w_3 - w_1w_4 + 2yw_1w_4$$

$$- w_2w_3 - 2yw_2w_3 + 2w_2w_4 - 2w_3w_4$$

$$= 2 \sum w_1^2 + 2(w_1w_3 - w_1w_2 + w_2w_4 - w_3w_4)$$

$$- w_1w_4 - w_2w_3 + 2y(w_1w_4 - w_2w_3)$$

$$= 2(1 + 4y^2 + 4y + 8 + 1 + 4y^2 - 4y - 2 - 4y)$$

$$- 2 - 4y - 2 + 4y - 2 + 4y)$$

$$+ 1 - 4y^2 + 4 + 2y(-1 + 4y^2 + 4)$$

$$= 2(2 + 8y^2) + 5 - 4y^2 + 6y + 8y^3$$

$$= 8y^3 + 12y^2 + 6y + 9$$

iii) Choose $v_3 = [1, 6, 11, -4]$ of type III.
\[2t_1 = -2t_4 = 2\]
\[2t_1 + 2t_3 = 2\]
\[2(t_1 + t_2 + t_3) = 3 - 2y\]
\[2t = (2, 1-2y, 0, -2)\]
\[2w = (0, 2, -1 + 2y, -2)\]
\[lDG(v_3) = 2(8 + 1 + 4y^2 - 4y - 2 + 4y - 4)\]
\[+ 2 - 4y + 2y(2 - 4y)\]
\[= 2(3 + 4y^2) + 2 - 4y + 4y - 8y^2 = 8\]

iv) Choose \(v_4 = [4, 3, 13, -2]\) of type IV.

\[2t_3 = 2t_4 = -2t_2 = 2\]
\[2(t_1 + t_3 + t_4) = 3 + 2y\]
\[2t = (-1 + 2y, -2, 2, 2)\]
\[2w = (-2, -1 + 2y, 2, 2)\]
\[lDG(v_4) = 2(12 + 1 + 4y^2 - 4y - 4 - 2 + 4y - 2\]
\[+ 4y - 4) + 4 + 2 - 4y + 2y(-4+2-4y)\]
\[= 2(1 + 4y^2 + 4y) + 6 - 4y - 4y - 8y^2 = 8\]

v) Choose \(v_5 = [1, 2, -3, -4]\) of type V

\[2t = (2, 2, -2, -2)\]
\[2w = (2, 2, -2, -2)\]
\[lDG(v_5) = 2(16 - 4 - 4 - 4) + 4 + 4\]
\[+ 2y(-4 + 4) = 8\]

vi) Choose \(v_6 = [1, 2, 12, 15]\) of type VI

\[2t_1 = 2t_2 = 2\]
\[2(t_1 + t_2 + t_4) = 3 - 2y\]
\[ 2(t_1 + t_2 + t_3 + t_4) = 2 \]
\[ 2t = (2, 2, -1 + 2y, -1 - 2y) \]
\[ 2w = (1 - 2y, 2, -2, -1 - 2y) \]
\[ 4D_{\text{g}} (v_6) = 2(8 + 1 + 4y^2 - 4y + 1 + 4y^2 + 4y - 2 + 4y^2 + 4) \]
\[ + 2y(-1 + 4y^2 + 4) \]
\[ = 2(2 + 8y^2) + 5 - 4y^2 + 6y + 8y^3 \]
\[ = 8y^3 + 12y^2 + 6y + 9 \]

vii) Choose \( v_7 = [1, 7, 12, -3] \) of type VII

\[ 2t_1 = -2t_3 = 2 \]
\[ 2t_2 + 2t_4 = 2 \]
\[ 2(t_1 + t_2 + t_4) = 3 - 2y \]
\[ 2t = (2, 1-2y, -2, 0) \]
\[ 2w = (2, 2, -1 + 2y, 0) \]
\[ 4D_{\text{g}} (v_7) = 2(8 + 1 + 4y^2 - 4y - 2 + 4y - 4) + 2 - 4y \]
\[ + 2y(-1 - 4y) \]
\[ = 2(3 + 4y^2) + 2 - 8y^2 = 8 \]

viii) Choose \( v_8 = [-1, 3, 4, 14] \) of type VIII

\[ -2t_1 = 2t_3 = 2t_4 = 2 \]
\[ 2(t_2 + t_3 + t_4) = 3 + 2y \]
\[ 2t = (-2, -1 + 2y, 2, 2) \]
\[ 2w = (-2, -2, 1-2y, 2) \]
4Dg(v_8) = 2(12 + 1 + 4y^2 - 4y - 4 - 2 + 4y - 2 + 4y - 4) + 4 + 2 - 4y + 2y(-4 + 2 - 4y)
   = 2(1 + 4y + 4y^2) + 6 - 4y + 2y(-2 - 4y) = 8

ix) Choose \( v_9 = [1, 2, -3, 12] \) of type IX

\[
2t_1 = 2t_2 = -2t_3 = 2 \\
2(t_1 + t_2 + t_4) = 3 - 2y \\
2t = (2, 2, -2, -1 - 2y) \\
2\bar{w} = (2, 2, -2, -1 - 2y)
\]

\[4Dg(v_9) = 2(12 + 1 + 4y^2 + 4y - 4 - 4 - 2 - 4y - 2 - 4y) + 2 + 4y + 4 + 2y(-2 - 4y + 4) = 2(1 - 4y + 4y^2) + 6 + 4y + 2y(2 - 4y) = 8\]

x) Choose \( v_{10} = [1, 2, 11, -4] \) of type X

\[
2t_1 = 2t_2 = -2t_4 = 2 \\
2(t_1 + t_2 + t_3) = 3 - 2y \\
2t = (2, 2, -1 - 2y, -2) \\
2\bar{w} = (1 + 2y, 2, -2, -2)
\]

\[4Dg(v_{10}) = 2(12 + 1 + 4y^2 + 4y - 2 - 4y - 2 - 4y - 4 - 4) + 2 + 4y + 4 + 2y(-2 - 4y + 4) = 2(1 - 4y + 4y^2) + 6 + 4y + 2y(2 - 4y) = 8\]

xi) Choose \( v_{11} = [-1, 3, 8, 14] \) of type XI

\[
-2t_1 = 2t_3 = 2 \\
2t_2 + 2t_3 = 2
\]
\[ 2t_2 + 2t_3 + 2t_4 = 3 + 2y \]
\[ 2t = (-2, 0, 2, 1+2y) \]
\[ 2w = (-2, -2, 0, 1+2y) \]

\[ 4Dg(v_{11}) = 2(8 + 1 + 4y^2 + 4y - 4 - 2 - 4y) + 2 \]
\[ + 4y + 2y(-2 - 4y) \]
\[ = 2(3 + 4y^2) + 2 + 4y - 4y - 8y^2 = 8 \]

xii) Choose \( v_{12} = [-1, 4, 9, 14] \) of type XII

\[ -2t_1 = 2t_4 = 2 \]
\[ 2t_2 + 2t_4 = 2 \]
\[ 2(t_2 + t_3 + t_4) = 3 + 2y \]
\[ 2t = (-2, 0, 1 + 2y, 2) \]
\[ 2w = (-1 - 2y, -2, 0, 2) \]

\[ 4Dg(v_{12}) = 2(8 + 1 + 4y^2 + 4y - 2 - 4y - 4) + 2 + 4y \]
\[ + 2y(-2 - 4y) \]
\[ = 2(3 + 4y^2) + 2 - 8y^2 = 8 \]

We see from parts i) through xii) that

\[
4Dg(v) = \begin{cases} 
8y^3 + 12y^2 + 6y + 9 & \text{if } v \text{ is of type I, II, VI} \\
8 & \text{otherwise}
\end{cases}
\]

\[ 8y^3 + 12y^2 + 6y + 9 - 8 = (2y + 1)^3 \geq 0 \]

hence vertices of type I are maximal when \( z = 1 \) and \( x = -1/2 \).
Formulated in terms of lattices the above theorem becomes:

**Theorem 2:** The extreme Eisenstein lattices are equivalent to

\[
L = \left\{ (1, 0), (-1/2 + \frac{y}{\sqrt{3}}, 1/2 \sqrt{3} - 4y^2) \right\}
\]

where

\[
y = \frac{-4 + \sqrt{13}}{2}
\]

Furthermore

\[
\theta(S) = \frac{4(35 - 13\sqrt{13})}{243} \pi^2
\]

where \( S \) is the unit sphere in complex 2-space.

**Proof:** With \( x = -1/2 \) and \( z = 1 \) the matrix \( A \) of \( f \) can be expressed as \( P'P \) with

\[
P = \begin{pmatrix}
1 & 1/2 & -1/2 & \frac{-1-2y}{4} \\
0 & \frac{\sqrt{3}}{2} & \frac{y}{\sqrt{3}} & \frac{y}{2\sqrt{3}} - \frac{3}{4\sqrt{3}} \\
0 & 0 & \frac{1}{2}\sqrt{3-4y^2} & \frac{1}{2}\sqrt{3-4y^2} \\
0 & 0 & 0 & \frac{1}{2}\sqrt{1-2y^2}
\end{pmatrix}
\]

As a real 4-dimensional lattice, the extreme lattice is generated by the columns of \( P \). Hence it has a basis

\[
\left\{ (1,0), (-1/2 + \frac{y}{\sqrt{3}}, 1/2 \sqrt{3} - 4y^2) \right\}
\]

as a 2-dimensional complex lattice over the integers in \( \mathbb{Q}(\sqrt{-3}) \).
It has already been established that \( y = \frac{-4 + \sqrt{13}}{2} \)
if \( L \) is extreme.

The formula

\[
\theta(S) = \inf \int_n \mu(f)^2
\]

and

\[
\inf \mu(f) = \frac{2(-17 + 5\sqrt{13})}{(-5 + 2\sqrt{13})^{3/2}}
\]
yield

\[
\theta(S) = \frac{-4(35 - 13\sqrt{13})}{243}\int
\]


