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A RELATED UNIQUENESS THEOREM.

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DISSERTATION

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the Requirement for the Degree
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By

Jessie Ann Nelson Engle, B.A., M.S.

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"Broad Band Antenna," (to the United States of America as represented by the Secretary of War), U. S. Patent 2,624,844, 6 January 1953. (Jessie A. Nelson and Darrel I. Wilhoit).


"Coaxial-Line Switch," (to the United States of America as represented by the Secretary of War), U. S. Patent 2,662,142, 8 December 1953. (Jessie A. Nelson).

FIELDS OF STUDY

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LIST OF SYMBOLS

\[ a, b, x \quad \text{elements of a set} \]
\[ A, B, X \quad \text{subsets of a set} \]
\[ A, T, F \quad \text{families of subsets of a set} \]
\[ \mathcal{U} \quad \text{a uniformity on a set} \]
\[ \mathcal{U} \quad \text{a set in a uniformity} \]
\[ A \setminus B \quad \{ab : a \in A, b \in B\} \]
\[ a \setminus B \quad \{ab : b \in B\} \]
\[ \text{cl} A \quad \text{the closure of } A \]
\[ \text{int} A \quad \text{the interior of } A \]
\[ v^{-1} \quad \{v^{-1} : v \in V\} \]
INTRODUCTION

The locally compact topological group is a structure which generalizes certain properties of the real line, among them the translation-invariance of Lebesgue measure and the continuity of translations with respect to the euclidean topology. The Haar measure which exists on any locally compact topological group is a generalization of Lebesgue measure on the line.

This dissertation is concerned with determining sufficient conditions for the existence of a non-trivial invariant measure on a structure which is weaker than a locally compact topological group. The weaker structure chosen for study is a group endowed with a locally compact Hausdorff topology with respect to which all left translations are continuous. Such a group is here called a "left-continuous group."

Only left-invariance and left translations are considered, since the theory for right-invariance and right translations is symmetrical.

In Chapter I it is shown that there exists a non-trivial left-translation-invariant measure on any left-continuous group which satisfies Condition (I), a condition on the open neighborhoods of the group identity.

A left-continuous group; a left-continuous group which satisfies
Condition (I); and a locally compact Hausdorff topological group—these are successively less general structures. The relationship among the three is made clear when a neighborhood characterization of a topological group is used, as in Bourbaki [3]. (Numbers in square brackets refer to numbered entries in the bibliography.) Common to all three is an underlying structure, a group G with a locally compact Hausdorff topology. We let U be the family of open neighborhoods of the identity in G, and consider the following conditions which may be imposed on the underlying structure.

i) For every x in G and every open set 0, the set x0 is an open set.

i') For every x in G and every open set 0, the sets x0 and Ox are open sets.

ii) For every U in U, there is a V in U with V^2 ⊂ U.

iii) For every U in U, there is a V in U with V^{-1} ⊂ U.

iii') For every U in U, U^{-1} is in U.

iv) For every x in G and every U in U, the set xUX^{-1} is in U.

If condition i) is satisfied, the structure is a left-continuous group; if conditions i), ii), and iii) are satisfied, the structure is a left-continuous group which satisfies Condition (I); and if conditions i'), ii), iii'), and iv) are satisfied, the structure is a locally compact Hausdorff topological group.

The main result in Chapter I is that Haar measure exists on a structure— that is substantially weaker than a topological group,
that is, that Haar measure exists on a left-continuous group which satisfies Condition (I).

Chapter II discusses equivalent conditions for the existence of a Haar measure on a left-continuous group.

A left-continuous group may be regarded as a locally compact Hausdorff topological space \((X, T)\) operated on by the family \(H\) of left translations, \(x \to ax\). The translation associated with each \(a\) in \(X\) is a homeomorphism from \(X\) onto \(X\); the family \(H\) is a group under composition; and \(H\) is transitive, that is, for each pair of elements \(x, y\) in \(G\), there is an element \(h_a\) in \(H\) associated with \(a = yx^{-1}\) such that \(y = h_a(x)\).

It is shown in Chapter II that imposing Condition (I) on a left-continuous group is equivalent to imposing certain conditions on the family \(H\). Equivalent conditions are 1) that the topology on \(X\) be a uniform topology and that the family \(H\) be equicontinuous with respect to a uniformity which induces the topology; 2) that the family \(H\) be strongly evenly continuous; and 3) that Strong Condition A hold: for every pair of disjoint sets, one closed and one compact, there is an open set such that no left translate of the open set intersects both the closed set and the compact set.

In Chapter III a method is given for constructing left-continuous groups from topological groups. A topological group may be embedded in a left-continuous group, and under certain conditions the embedding group will not be a topological group. For instance, if for some \(x\) in \(G\), the set \(\{a: axa^{-1} = x\}\) is not an open set, then
there is an embedding left-continuous group which is not a topological group. In this case the embedding group is a product space with a non-normal subgroup which is isomorphic to the embedded topological group. Left cosets are copies of the embedded group, and are endowed with a copy of the topology on the embedded group. Since right cosets are not unions of open sets, right translations are not continuous, and the embedding group is not a topological group.

Chapter IV contains a related uniqueness theorem--a 5-r uniqueness theorem for invariant measures on a separable metric space. The theorem is a sharpening of the 5-r uniqueness theorem of Mickle and Rado [8] which was proved by them to show the uniqueness of a translation-invariant measure on the space of oriented lines in three-space. The measure under consideration is Haar with respect to a family of homeomorphisms which satisfy Condition (II), a condition guaranteeing that translates of closed spheres of small radius are bounded within and without by closed spheres of related radius. It is shown in Chapter IV that a Haar measure is unique, up to a multiplicative constant, when it satisfies the 5-r condition at one point. The 5-r condition is a condition on the rate of increase of measure of a closed sphere of arbitrarily small radius as the radius increases.

The final chapter, Chapter V, contains an example of a left-continuous group with its associated left-translation-invariant measure. The group is the open right half-plane, that is, the product space \{(x,y): x > 0, y \text{ is real}\}. The group operation is
defined by \((x,y)(a,b) = (xa, xb + y)\). Open sets are unions of sets which are open intervals on horizontal lines; e.g., \([(x,y_0): a < x < b}\) is an open set. With this topology, left translates of open sets are open, and right translates of open sets are open if and only if the translating element is of the form \((a,0)\). This example illustrates the embedding theorem: the subgroup \([(x,0): x > 0}\) is a topological group with the induced subspace topology, and is an open neighborhood of \((1,0)\), the identity in \(G\). Hence Condition (I) is satisfied, and there is a Haar measure on the left-continuous group, a measure \(\Lambda\) which on open neighborhoods of \((x_0, y_0)\) takes the value

\[
\Lambda[\{(x, y_0): x_0 \in (a,b), a < x < b\}] = \ln \frac{a}{b},
\]

where \(\ln\) is the natural logarithm. The measure \(\Lambda\) is clearly non-trivial and left-translation-invariant.
CHAPTER I

THE EXISTENCE OF A LEFT-INVARIANT MEASURE
ON A LEFT-CONTINUOUS GROUP

This chapter is concerned with defining a non-trivial left-
translation-invariant measure on a left-continuous group. A left-
continuous group is a group, with the group operation here called
multiplication, which is endowed with a locally compact Hausdorff
topology with the property that all left translations are continuous,
where left translation by an element $x$ in $G$ is multiplication on
the left by $x$.

Clearly every locally compact Hausdorff topological group is
a left-continuous group, but there are left-continuous groups which
are not topological groups, and an example will be given below in
Chapter V. It was proved by Ellis [4] in 1957 that the continuity
of both left and right translations, or continuity of either the
left or right translations and of the inverse operation, guarantee
that a group with a locally compact topology be a topological group.
Thus locally compact semi-topological groups, para-topological groups,
and quasi-topological groups, to use Bourbaki ([3]) nomenclature,
are all topological groups, whereas a left-continuous group is
locally compact, but not necessarily a topological group.
Definition 1.1.

A left-continuous group is a structure \((G, \circ, T)\) where \((G, \circ)\) is a group, \((G, T)\) is a locally compact Hausdorff topological space, and for every element \(g\) in \(G\) and every open set \(O\) in \(T\) the set \(gO = \{gx : x \in O\}\) is an open set.

Definition 1.2.

A left-continuous group \((G, \circ, T)\) is said to satisfy Condition (I) if for every \(U\), with \(e \in U \subseteq T\) (where \(e\) is the group identity in \(G\)), there is a \(V\), with \(e \in V \subseteq T\) such that \(VV^{-1} \subseteq U\).

Note that the set \(V^{-1}\) is not necessarily an open set.

Definition 1.3.

A non-negative real-valued set function \(\Lambda\) defined on all subsets of a set \(G\) will be said to be an outer measure on \(G\) if

1) \(\Lambda(\emptyset) = 0\), where \(\emptyset\) is the empty set,

2) for \(A \subseteq B\), \(\Lambda(A) \leq \Lambda(B)\), and

3) for \(A_n, n = 1, 2, \cdots\), a sequence of sets in \(G\),

\[\Lambda(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \Lambda(A_n)\].

Definition 1.4.

An outer measure \(\Lambda\) is said to be Borel-regular if for every set \(E\) in \(G\) there is a Borel set \(B\) such that \(E \subseteq B\) and \(\Lambda(E) = \Lambda(B)\).

Theorem 1.5. (Existence Theorem).

If \((G, \circ, T)\) is a left-continuous group satisfying Condition (I),
then there exists an outer measure \( \Lambda \) on \( G \) such that

1) \( \Lambda \) is Borel-regular,
2) if \( E \subset G \), \( x \in G \), then \( \Lambda(xE) = \Lambda(E) \),
3) if \( 0 \in T \), \( 0 \neq \emptyset \), then \( \Lambda(\emptyset) > 0 \),
4) if \( K \) is compact, then \( \Lambda(K) < \infty \), and
5) Borel sets are \( \Lambda \)-measurable.

The proof of Theorem 1.5 will follow after a series of lemmas.

For the remainder of Chapter I, \( (G, \circ , T) \) is a left-continuous group which satisfies Condition (I). Several consequences of assuming Condition (I) are given in Lemmas 1.6 through 1.11. Since the topology on \( G \) is locally compact and Hausdorff, it is also regular, and each point in \( G \) has a closed neighborhood base. Below, \( U_e \) will denote the family of open neighborhoods of the identity element \( e \) in \( G \), and \( C_e \) will denote the family of closed neighborhoods of \( e \).

Lemma 1.6.

For every \( U \) in \( U_e \) there is a \( V \) in \( U_e \) such that \( V^2 \subset U \).

Proof:

Fix \( U \in U_e \). Then there is \( W \in U_e \) with \( WW^{-1} \subset U \), and \( V \in U_e \) with \( VV^{-1} \subset W \), and \( V^2 \subset VV^{-1}VV^{-1} = VV^{-1}(VV^{-1})^{-1} \subset WW^{-1} \subset U \).

Lemma 1.7.

For every \( C \) in \( C_e \) there is a \( D \) in \( C_e \) such that \( DD^{-1} \subset C \).

Proof:

Fix \( C \in C_e \). Then there is \( U \in U_e \) with \( U \subset C \), \( V \in U_e \) with
Lemma 1.8.

For every compact set $K$ and open set $O$ that contains $K$, there is a $U$ in $U_e$ such that $KU \subseteq O$.

Proof:

For each $x \in K \subseteq O$, there is a $W_x \in U_e$ with $xW_x \subseteq O$, and there is a $V_x \in U_e$ with $V_x^2 \subseteq W_x$. Since $K$ is compact, and $K = \bigcup \{xV_x : x \in K\}$, there is a finite set, $\{x_1, x_2, \ldots, x_n\}$, such that $K \subseteq \bigcup_{i=1}^n x_i V_{x_i}$.

Letting $V = \bigcap_{i=1}^n V_{x_i}$, we have $V \in U_e$ and

$$KV \subseteq \bigcup_{i=1}^n x_i V_{x_i} \subseteq \bigcup_{i=1}^n x_i V_{x_i} V_{x_i} \subseteq \bigcup_{i=1}^n x_i W_{x_i} \subseteq O.$$  

Lemma 1.9.

For every subset $A$ of $G$, and every open set $O$, the set $AO$ is an open set.

Proof:

If $x \in A$, then $xO \in T$, and

$$AO = \bigcup \{xO : x \in A\} \in T.$$  

Lemma 1.10.

For every compact set $K$ and closed set $F$ where $K$ and $F$ are disjoint, there is a $U$ in $U_e$ such that $KU$ and $FU$ are disjoint.

Proof:

It suffices to show there is a $U \in U_e$ such that $KU^{-1} \cap F = \emptyset$,
since if $KU \cap FU \neq \emptyset$, we have $k_1u_1 = f_2u_2$ for some $u_1, u_2$ in $U$ and
$k_1$ in $K$, $f_2$ in $F$. Then $f_2 = k_1u_1u_2^{-1} \in KU^{-1} \cap F \neq \emptyset$.

Since $K \cap F = \emptyset$, $K \subset (G - F) \in T$, and by Lemma 1.8 there is
a $V \in \mathcal{U}_e$ such that $KV \subset (G - F)$. Then there is a $U$ in $\mathcal{U}_e$ with
$UU^{-1} \subset V$, and since $KU^{-1} \subset KV \subset (G - F)$, we have $KU^{-1} \cap F = \emptyset$,
and $KU \cap FU = \emptyset$.

Lemma 1.11.

For every compact set $K$ and closed set $F$ with $K$ and $F$ disjoint,
there is a $U$ in $\mathcal{U}_e$ such that no left translate of $U$ has non-empty
intersections with both $K$ and $F$.

Proof:

Let $K$ and $F$ be disjoint, and compact and closed respectively.

Then, by Lemma 1.10, there is a $V \in \mathcal{U}_e$ such that $KV \cap FV = \emptyset$, and
by Condition (I) and Lemma 1.6, there is $U \in \mathcal{U}_e$ such that
$(UU^{-1})^2 \subset V$. Assume $xU \cap K \neq \emptyset$. Then
$xu_1 = k_1$ for some $u_1 \in U$
and $k_1 \in K$. Assume $xu_2 = f_2 \in xU \cap F$, for some $u_2 \in U$ and $f_2 \in F$.
Then $f_2 = xu_2 = k_1u_1u_2^{-1} \subset k_1U^{-1}U \subset k_1(UU^{-1})^2 \subset KV$, and
$f_2 \in KV \cap F \subset KV \cap FV = \emptyset$. Contradiction.

Hence $xU \cap F = \emptyset$.

Below, in Chapter II, there is a discussion of various conditions
which are equivalent to Condition (I).

The following three lemmas lead directly to the proof of
Theorem 1.5, the existence theorem. In what follows, $(G, \circ, T)$
is a left-continuous group which satisfies Condition (I), $\mathcal{U}_e$ is the
family of open neighborhoods of the identity in $G$, and $F$ will denote
the class of compact sets in $G$ with non-empty interiors. Since $(G, \circ, \mathcal{T})$ is locally compact, every element in $G$ has a neighborhood base of sets in $\mathcal{F}$. In Lemma 1.12, a left-invariant real-valued set function $\lambda$ will be defined on sets in $\mathcal{F}$.

**Lemma 1.12.**

There is a real-valued set function $\lambda$ defined on the family $\mathcal{F}$ of compact sets with non-empty interiors such that for all $A, B$ in $\mathcal{F}$ and all $x \in G$:

1) $0 < \lambda(A) < \infty$,

2) if $A \subset B$, then $\lambda(A) \leq \lambda(B)$,

3) $\lambda(A \cup B) \leq \lambda(A) + \lambda(B)$,

4) $\lambda(A) = \lambda(xA)$, and

5) if $A \cap B = \emptyset$, then $\lambda(A \cup B) = \lambda(A) + \lambda(B)$.

**Proof:**

For $A, B \in \mathcal{F}$, let

$$(A:B) = \inf\{n: A \subset \bigcup_{i=1}^{n} x_i \text{ int}B\},$$

where int$B$ is the interior of $B$. Then for $A, B, C \in \mathcal{F}$ and for $x \in G$, we have:

i) $1 \leq (A:B) < \infty$,

ii) $(xA:B) = (A:B) = (A:xB)$,

iii) if $A \subset B$, then $(A:C) \leq (B:C)$,

iv) $(A:C) \leq (A:B)(B:C)$,

v) $(A \cup B:C) \leq (A:C) + (B:C)$, and

vi) if $A \cap B = \emptyset$, then for some $C$ in $\mathcal{F}$, $(A \cup B:C) = (A:C) + (B:C)$. 
Conclusions i) through v) are obvious, and vi) follows from Lemma 1.11. For $A, B \in \mathcal{F}$, $A \cap B = \emptyset$, there is $U \in \mathcal{U}_e$ such that for any $x \in G$, if $xU \cap A \neq \emptyset$, then $xU \cap B = \emptyset$. There is a compact set $K$ that is a neighborhood of $e$ with $K \subset U$, and $K \in \mathcal{F}$. If $A \cup B \subset \bigcup_{i=1}^{n} x_i \text{int} K$, then $(A:K) + (B:K) \leq n$, since none of the $n$ sets $x_i \text{int} K$ intersects both $A$ and $B$. Hence $(A \cup B:K) \leq (A:K) + (B:K) \leq (A \cup B:K)$, and equality follows.

Fix $F_0 \in \mathcal{F}$, and define, for each $A, B \in \mathcal{F}$,

$$\lambda_B(A) = \frac{(A:B)}{(F_0:B)}.$$ 

Then for all $A, B, C \in \mathcal{F}$, and $x \in G$, we have:

(i) $\lambda_B(A) = \lambda_B(xA)$,

(ii) if $A \subset C$, then $\lambda_B(A) \leq \lambda_B(C)$,

(iii) $\lambda_B(A \cup C) \leq \lambda_B(A) + \lambda_B(C)$,

(iv) if $A \cap C = \emptyset$, then there is $B \in \mathcal{F}$ with $\lambda_B(A \cup C) = \lambda_B(A) + \lambda_B(C)$, and

(v) $0 < \frac{1}{(F_0:A)} \leq \lambda_B(A) \leq (A:F_0) < \infty$.

Conclusions (i) through (iv) are obvious, and (v) follows from iv) above:

$$0 < \frac{1}{(F_0:A)} = \frac{(A:B)}{(F_0:A)(A:B)} \leq \frac{(A:B)}{(F_0:B)} = \lambda_B(A)$$

$$\leq \frac{(A:F_0)(F_0:B)}{(F_0:B)} = (A:F_0) < \infty$$
From (v), it can be seen that for each $A$ in $F$, the set 
\{\lambda_B(A): B \in F\} is a subset of the closed interval \([1/(F_o:A), (A:F_o)]\),
which is a subset of \((0, \infty)\).

Let $P$ be the product space, indexed by \{A: A \in F\}, of the closed
intervals \([1/(F_o:A), (A:F_o)]\), with the product topology induced by
the usual real-line topology on the closed intervals:

$$P = \prod_{A \in F} \left[ \frac{1}{(F_o:A)}, (A:F_o) \right].$$

Since each interval is compact, the product space $P$ is compact
(by the Tychonoff product theorem; see Kelley [7]), and for each
$B$ in $F$, $\lambda_B$ is an element of the product space $P$.

For each compact neighborhood $K$ of the identity element in $G$,
that is, for $K \subseteq C_e \cap F$, let

$$S(K) = \{\lambda_B: B \subseteq C_e \cap F, B \subseteq K\}.$$ 

Then $S(K) \subseteq P$ for each $K$, and $cl S(K)$, the closure of $S(K)$, is a
closed subset of the compact space $P$, hence is compact. The family
of $cl S(K)$ for $K \subseteq C_e \cap F$ has the finite intersection property, since

$$\bigcap_{i=1}^{n} cl S(K_i) \supset cl \bigcap_{i=1}^{n} S(K_i) = cl \bigcap_{i=1}^{n} \{\lambda_B: B \subseteq C_e \cap F, B \subseteq K_i\}$$

$$= cl \{\lambda_B: B \subseteq C_e \cap F, B \subseteq \bigcap_{i=1}^{n} K_i \subseteq F \cap C_e \} \neq \emptyset.$$ 

Thus, $\bigcap \{cl S(K): K \subseteq F \cap C_e \} \neq \emptyset$.

Let $\lambda \in \bigcap \{cl S(K): K \subseteq F \cap C_e \}$. Then $\lambda$ is a real-valued
function on $F$ that satisfies (1) - (5) of the lemma.
It is clear that (1) holds, since \( \lambda \in \mathbb{P} \) and for every \( A \) in \( \mathbb{F} \),
\[
0 < 1/(\mathbb{F}_0:A) \leq \lambda(A) \leq (A: \mathbb{F}_0) < \infty.
\]
For an arbitrary element \( q \) in \( \mathbb{P} \) and set \( A \) in \( \mathbb{F} \), let \( p_A(q) \) be the projection of \( q \) at \( A \). Then \( p_A \) is continuous for all \( A \) in \( \mathbb{F} \). The projections \( p_A \) are used in the proofs of (2) - (5).

To show that (2) holds, let \( A \subset B \) and let
\[
Q_2 = \{ q: p_A(q) \leq p_B(q) \}.
\]
Since \( p_A \) and \( p_B \) are continuous, \( Q_2 \) is a closed set, and for
\( K \in C_e \cap \mathbb{F} \), \( \lambda \in Q_2 \) since \( \lambda(K) \leq \lambda(K) \). Hence \( \text{cl}S(K) \subset Q_2 \)
and \( \lambda \in Q_2 \). Then if \( A \subset B \), \( \lambda(A) \leq \lambda(B) \), and (2) holds.

Similarly, to show that (3) holds, for \( A, B \in \mathbb{F} \), let
\[
Q_3 = \{ q: p(A \cup B)(q) \leq p_A(q) + p_B(q) \}.
\]
Then \( Q_3 \) is closed, \( \text{cl}S(K) \subset Q_3 \) and \( \lambda \in Q_3 \), hence \( \lambda(A \cup B) \leq \lambda(A) + \lambda(B) \).

For arbitrary \( A \in \mathbb{F} \) and \( x \in G \), let
\[
Q_4 = \{ q: p_A(q) = p_{xA}(q) \}.
\]
Again \( \text{cl}S(K) \subset Q_4 \), \( \lambda \in Q_4 \), \( \lambda(A) = \lambda(xA) \), and (4) holds.

To establish (5), let \( A, B \) be disjoint sets in \( \mathbb{F} \). Let
\[
Q_5 = \{ q: p_A \cup B(q) = p_A(q) + p_B(q) \}.
\]
Then \( Q_5 \) is a closed set. Now there is a \( K \) in \( \mathbb{F} \) such that
\( \lambda_K(A \cup B) = \lambda_K(A) + \lambda_K(B) \) (by (iv) above), and by ii), for \( x \in \text{int}K \),
\( \lambda^{-1}_x(A \cup B) = \lambda^{-1}_x(A) + \lambda^{-1}_x(B) \), and \( x^{-1}_K \in C_e \). Then for
\( D \subset x^{-1}_K \), \( D \in C_e \), we have \( \lambda_D(A \cup B) = \lambda_D(A) + \lambda_D(B) \), and \( S(x^{-1}_K) \subset Q_5 \),
\( \text{cl}S(x^{-1}_K) \subset Q_5 \), and \( \lambda \in Q_5 \). Hence \( \lambda(A \cup B) = \lambda(A) + \lambda(B) \) when
\( A \cap B = \emptyset \), and the proof of the lemma is complete.
The real-valued set function $\lambda$, defined on the family $\mathcal{F}$ of compact sets with non-empty interiors, will be used to define a set function $\rho$ on all open sets in $G$.

**Lemma 1.13.**

For all $0$ in $\mathcal{T}$, the family of open sets in $G$, let

\[
\rho(0) = \sup\{\lambda(F) : F \in \mathcal{F}, F \subset 0\} \text{ for } 0 \neq \emptyset, \text{ and}
\]

\[
\rho(\emptyset) = 0.
\]

Then for $A,B$ in $\mathcal{T}$, $x$ in $G$:

1. if $A \subset B$, then $\rho(A) \leq \rho(B)$,
2. $\rho(A \cup B) \leq \rho(A) + \rho(B)$,
3. $\rho(A) = \rho(xA)$,
4. if $A \cap B = \emptyset$, then $\rho(A \cup B) = \rho(A) + \rho(B)$, and
5. $\rho\left(\bigcup_{i=1}^{\infty} O_i\right) \leq \sum_{i=1}^{\infty} \rho(O_i)$ for all $(O_i)_{i=1}^{\infty}$ in $G$.

**Proof:**

For every $0 \neq \emptyset$ and for every $x$ in $0$, there is a compact neighborhood of $x$ which is a subset of $0$. Hence $\rho$ is defined for every open set in $G$.

Conclusions (1), (3), and (4) are obvious from the definition of $\rho$ and the properties of the set function $\lambda$.

To prove (2), let $F \subset A \cup B$, $F \in \mathcal{F}$. Then

\[
F \subset \bigcup_{x \in A, x \in O_x \in \mathcal{T}, \operatorname{cl} O_x \subset A} O_x \cup \bigcup_{y \in B, y \in O_y \in \mathcal{T}, \operatorname{cl} O_y \subset B} O_y.
\]

Since $F$ is compact, $F \subset \bigcup_{i=1}^{n} O_{x_i} \cup \bigcup_{i=1}^{m} O_{y_i}$ for some $(x_i), i=1,2,\ldots,n$, and $(y_i), i=1,2,\ldots,m$.
and \((y_i), i=1,2,\ldots,m\), and letting

\[
F_1 = F \cap \bigcup_{i=1}^{n} cl(x_i), \quad F_2 = F \cap \bigcup_{i=1}^{m} cl(y_i),
\]

we have \(F_1, F_2 \in F\), \(F_1 \subseteq A\), \(F_2 \subseteq B\), and \(F_1 \cup F_2 = F\). Then

\[
\lambda(F) \leq \lambda(F_1) + \lambda(F_2) \leq \rho(A) + \rho(B),
\]

and since \(F\) was an arbitrary subset of \(A \cup B\), we have \(\rho(A \cup B) \leq \rho(A) + \rho(B)\).

To prove (5), first assume \(\rho(\bigcup_{i=1}^{\infty} O_i) < \infty\). Then for every \(\epsilon > 0\),

there is an \(F\) in \(F\) with \(F \subseteq \bigcup_{i=1}^{\infty} O_i, \lambda(F) > \rho(\bigcup_{i=1}^{\infty} O_i) - \epsilon\). Since \(F\)

is compact, there are a finite number of the \(O_i\)'s that cover \(F\):

\[
F \subseteq \bigcup_{i=1}^{n} O_i. \quad \text{Then}
\]

\[
\rho(\bigcup_{i=1}^{\infty} O_i) - \epsilon < \lambda(F) \leq \rho(\bigcup_{i=1}^{n} O_i) \leq \sum_{i=1}^{n} \rho(O_i) \leq \sum_{i=1}^{\infty} \rho(O_i),
\]

where the second inequality follows from (2). Since \(\epsilon\) is arbitrary,

\[
\rho(\bigcup_{i=1}^{\infty} O_i) \leq \sum_{i=1}^{\infty} \rho(O_i).
\]

when \(\rho(\bigcup_{i=1}^{\infty} O_i) < \infty\).

Assuming \(\rho(\bigcup_{i=1}^{\infty} O_i) = \infty\), we have for every \(n\) an \(F_n\) in \(F\) with

\[
n < \lambda(F_n) \text{ and } F_n \subseteq \bigcup_{i=1}^{\infty} O_i. \quad \text{Since } F_n \text{ is compact, there are a finite number of the } O_i\text{'s that cover } F_n: F_n \subseteq \bigcup_{i=1}^{m} O_i, \text{ and}
\]

\[
n < \lambda(F_n) \leq \rho(\bigcup_{i=1}^{m} O_i) \leq \sum_{i=1}^{m} \rho(O_i) \leq \sum_{i=1}^{\infty} \rho(O_i).
\]
Since this is true for arbitrary \( n \), \( \sum_{i=1}^{\infty} \rho(0_i) = \infty \), and (5) holds.

**Definition 1.14.**

For each subset \( E \) of \( G \), define

\[
\Lambda(E) = \inf\{\rho(0): 0 \in T, E \subset 0\}.
\]

**Lemma 1.15.**

For every open set \( O \) in \( T \), \( \Lambda(O) = \rho(O) \).

**Proof:**

Let \( O \in T \). Then since \( O \subset 0 \), \( \Lambda(O) \leq \rho(O) \). For every \( Q \) in \( T \) with \( O \subset Q \), we have \( \rho(O) \leq \rho(Q) \), so \( \rho(O) \leq \inf\{\rho(Q): O \subset Q, Q \in T\} = \Lambda(O) \), and \( \Lambda(O) = \rho(O) \).

The existence theorem, Theorem 1.5, will be restated and proved, using the preceding lemmas.

**Existence Theorem.**

If \((G, \circ, T)\) is a left-continuous group satisfying Condition (I), then there exists an outer measure \( \Lambda \) on \( G \) such that:

1) \( \Lambda \) is Borel-regular,
2) if \( E \subset G \), \( x \in G \), then \( \Lambda(xE) = \Lambda(E) \),
3) if \( 0 \in T \) and \( 0 \neq \emptyset \), then \( \Lambda(0) > 0 \),
4) if \( K \) is compact, then \( \Lambda(K) < \infty \), and
5) Borel sets are \( \Lambda \)-measurable.

**Proof:**

Let \( \Lambda \) be the set function defined in Definition 1.14. Then \( \Lambda \)
is an outer measure if \( \Lambda(\emptyset) = 0 \), \( \Lambda \) is monotone, and \( \Lambda \) is countably subadditive. Since \( \rho(\emptyset) = 0 \), \( \Lambda(\emptyset) = 0 \). Clearly \( \Lambda \) is monotone.

If \( (A_i)_{i=1}^\infty \) is a sequence of subsets of \( G \), and \( \sum_{i=1}^\infty \Lambda(A_i) = \infty \), then \( \Lambda(\bigcup A_i) \leq \sum_{i=1}^\infty \Lambda(A_i) \). So assume \( \sum_{i=1}^\infty \Lambda(A_i) < \infty \). Then \( \Lambda(A_i) < \infty \) for each \( i \). Fix \( \epsilon > 0 \). For each \( A_i \) there is an \( O_i \) with \( A_i \subset O_i \) and \( \rho(O_i) < \Lambda(A_i) + \epsilon/2^i \), and \( \Lambda(\bigcup A_i) \leq \rho(\bigcup O_i) \leq \sum_{i=1}^\infty \rho(O_i) < \sum_{i=1}^\infty \Lambda(A_i) + \epsilon \). Hence \( \Lambda \) is countably subadditive, and \( \Lambda \) is an outer measure.

The fact that \( \Lambda \) is Borel-regular follows from the definition of \( \Lambda \). For arbitrary \( E \subset G \), if \( \Lambda(E) = \infty \), then \( E \subset G \), \( \Lambda(G) = \infty \), and \( G \) is in \( T \), hence a Borel set. If \( \Lambda(E) < \infty \), for each \( n \) there is an \( O_n \) containing \( E \) with \( \Lambda(E) + 1/n > \rho(O_n) \). Then \( \cap_{n=1}^\infty O_n \) is a Borel set which contains \( E \), and \( \Lambda(\cap_{n=1}^\infty O_n) \leq \rho(O_n) < \Lambda(E) + 1/n \) for every \( n \), so \( \Lambda(\cap_{n=1}^\infty O_n) \leq \Lambda(E) \). Since any open set containing \( \cap_{n=1}^\infty O_n \) also contains \( E \), \( \Lambda(E) \leq \Lambda(\cap_{n=1}^\infty O_n) \), and \( \Lambda(E) = \Lambda(\cap_{n=1}^\infty O_n) \).

Conclusion 2) follows directly from Lemma 1.13 (3).

To prove conclusion 3), let \( O \) be an arbitrary non-empty open set, and \( x \) a point in \( O \). There is a closed neighborhood \( C \) of \( x \) such that \( C \subset O \), and a compact neighborhood \( K \) of \( x \), since the topology is regular and locally compact. Then \( C \cap K \) in \( F \), and \( \lambda(C \cap K) > 0 \), by Lemma 1.12, 1), so \( \Lambda(O) = \rho(O) \geq \lambda(C \cap K) > 0 \), and \( \Lambda(O) > 0 \).
Conclusion 4), that for every compact set $K$, $\Lambda(K) < \infty$, follows from the compactness of $K$ and the left-translation invariance of $\Lambda$ if it can be shown that there is one non-empty open set with finite $\Lambda$-measure. Fix $x$, and let $C$ be a compact neighborhood of $x$. Then $\text{int} C$ is an open neighborhood of $x$, hence non-empty, and $0 < \Lambda(\text{int} C) = \sup \{ \lambda(F) : F \subseteq \text{int} C, F \in F \} \leq \lambda(C) < \infty$. Any compact set $K$ may be covered by a finite number of translates of $\text{int} C$, each of finite $\Lambda$-measure; hence $\Lambda(K) < \infty$.

To prove that all Borel sets are $\Lambda$-measurable, it suffices to prove that all open sets are measurable, and to prove all open sets are measurable, it suffices to show, for arbitrary $O$ in $\mathcal{T}$ and $E \subseteq G$ with $\Lambda(E) < \infty$, that

$$\Lambda(E) \geq \Lambda(E \cap O) + \Lambda[E \cap (G - O)].$$

The notation of Lemmas 1.12 and 1.13 is used below. Let $Q$ be in $\mathcal{T}$ with $E \subseteq Q$ and $\Lambda(Q) < \infty$. Then

$$O \cap E \subseteq (O \cap Q) \in \mathcal{T},$$

and

$$E \cap (G - O) \subseteq Q \cap (G - O).$$

For every $n$ there is an $F_n$ in $F$ such that

$$F_n \subseteq O \cap Q, \quad \Lambda(O \cap Q) = \rho(O \cap Q) < \lambda(F_n) + 1/n,$$

and there is an $O_n$ in $\mathcal{T}$ and a $K_n$ in $F$ such that

$$E \cap (G - O) \subseteq O_n \subseteq Q \cap (G - F_n), K_n \subseteq O_n,$$

and

$$\Lambda(O_n) = \rho(O_n) < \lambda(K_n) + 1/n.$$
Then $\lambda(E \cap (G - 0)) + \lambda(E \cap 0) \leq \lambda(0_n) + \lambda(0 \cap q)$

$< \lambda(K_n) + 1/n + \lambda(F_n) + 1/n = \lambda(K_n \cup F_n) + 2/n \leq \rho(q) + 2/n$

$= \lambda(q) + 2/n.$

The first equality holds since $K_n \cap F_n = \emptyset$, and $\lambda$ is additive on disjoint sets in $\mathcal{F}$. Since

$\lambda(E \cap (G - 0)) + \lambda(E \cap 0) < \lambda(q) + 2/n$

holds for every $n$, we have $\lambda(E \cap (G - 0)) + \lambda(E \cap 0) \leq \lambda(q)$ for every open set $Q$ containing $E$, hence

$\lambda(E \cap (G - 0)) + \lambda(E \cap 0) \leq \lambda(E).$

The proof of the theorem is complete.

An example of a left-continuous group which satisfies Condition (1), and the left-invariant measure that exists on it, will be given below in Chapter V.
CHAPTER II

CONDITIONS EQUIVALENT TO CONDITION (I)

In Chapter I it was shown that if a left-continuous group G satisfies Condition (I) there exists a non-trivial measure on G that is invariant with respect to left translations. In this chapter some conditions equivalent to Condition (I) will be given.

Definition 2.1.

A left-continuous group \((G, \circ, T)\) will be said to satisfy Condition (Ia) if for every \(U\) in \(\mathcal{U}\) (where \(\mathcal{U}\) is the family of open neighborhoods of the identity in \(G\)) there is a \(V\) in \(\mathcal{U}\) such that \(V^2 \subset U\).

Definition 2.2.

A left-continuous group \((G, \circ, T)\) will be said to satisfy Condition (Ib) if for every \(U\) in \(\mathcal{U}\) there is a \(V\) in \(\mathcal{U}\) such that \(V^{-1} \subset U\).

Lemma 2.3.

A left-continuous group \((G, \circ, T)\) satisfies Condition (I) if and only if it satisfies Conditions (Ia) and (Ib).

Proof:

Assume Condition (I) holds, and let \(U\) be an open set in \(\mathcal{U}\).
Then there is an open set $W$ in $U_e$ with

$$W^{-1} \subset U$$

and there is an open set $V$ in $U_e$ with

$$V^{-1} \subset W.$$ 

Now

$$VV = V e V e \subset V V^{-1} V V^{-1} = (V V^{-1})(V V^{-1})^{-1} \subset W^{-1} \subset U,$$

and

$$V^{-1} = e V^{-1} \subset V V^{-1} \subset W \subset W^{-1} \subset U.$$ 

So if Condition (I) holds, Conditions (Ia) and (Ib) hold.

Assume Conditions (Ia) and (Ib) hold, and let $U$ be an open set in $U_e$. Then there are sets $W$ and $R$ in $U_e$ with

$$W^2 \subset U, \ R^{-1} \subset W.$$ 

Letting $V = R \cap W$, we have $V \in U_e$ and

$$V V^{-1} \subset W R^{-1} \subset W^2 \subset U,$$

and the lemma is proved.

A left-continuous group $(G, o, T)$ may be regarded as a locally compact Hausdorff topological space $(G, T)$ and a family $H$ of homeomorphisms of $G$ onto $G$ with

(1) \quad $H = \{h_x : x \in G$, and for $y \in G, h_x(y) = xy\}.$

Thus $H$ is the family of left translates from $G$ onto $G$. Each $h_x$
in \( H \) is both continuous and open, and clearly \( H \) is a group with composition as its binary operation.

In what follows, a left-continuous group \((G, \circ, T)\) and a locally compact Hausdorff topological space \((G, T)\) with a family \( H \) of homeomorphisms as at (1) will be used interchangeably as convenient.

Requiring that \((G, \circ, T)\) satisfy Conditions (1), (1a), or (1b) is equivalent to imposing certain conditions on the family of homeomorphisms \( H \). Steinlage [15] has defined Strong Condition A on a group of homeomorphisms from a topological space onto itself. Using the notation of this dissertation, Strong Conditions A may be defined as follows.

Definition 2.4. (Steinlage)

A group \( H \) of homeomorphisms of a topological space \( G \) onto itself is said to satisfy Strong Condition A if for each pair of disjoint sets \( B \) and \( C \), one of which is compact and one of which is closed, there exists a non-empty open set \( O \) such that \( h(O) \cap B = \emptyset \) or \( h(O) \cap C = \emptyset \) for all \( h \) in \( H \).

Several other conditions are also equivalent; after defining them proof of the equivalences will be given.

Definition 2.5.

A group \( H \) of homeomorphisms of a topological space \( G \) is said to be evenly continuous if, for each pair of elements \( x, y \) in \( G \), and for each open neighborhood \( O_y \) of \( y \), there are open neighborhoods \( O_1 \) and \( O_2 \) of \( x \) and \( y \) respectively such that for every \( h \in H \),
if \( h(x) \in O_x \), then \( h(O_x) \subseteq O_y \).

**Definition 2.6.**

A group \( H \) of homeomorphisms of a topological space \( G \) will be said to be strongly evenly continuous if, for every pair of elements \( x, y \) in \( G \), and every open neighborhood \( O_y \) of \( y \), there are open neighborhoods \( O_x \) and \( O_2 \) of \( x \) and \( y \) respectively such that for every \( h \in H \) if \( h(O_x) \cap O_2 \neq \emptyset \), then \( h(O_x) \subseteq O_y \).

One equivalent condition concerns uniformities: Condition (I) holds on the left-continuous group \( (G, \circ, T) \) if and only if there is a uniformity on \( G \) such that the topology on \( G \) is generated by that uniformity, and the family \( H \) is equicontinuous with respect to the uniformity. For definitions, notation, and discussion of uniformities, the reader is referred to Kelley [7].

**Definition 2.7.**

A family \( H \) of transformations of a uniform space \( (G, U) \) is said to be equicontinuous if for every \( x \) in \( G \) and every \( U \in U \) there is an open neighborhood \( O_x \) of \( x \) such that for every \( h \) in \( H \), \( h(O_x) \subseteq U[h(x)] \).

**Theorem 2.8.**

The left-continuous group \( (G, \circ, T) \) satisfies Condition (Ia) if and only if the family \( H \) is evenly continuous.

**Proof:**

When \( H \) is a family of left translates of a left-continuous group, even continuity of \( H \) means that for every \( x, y \) in \( G \) and
U in $U_e$, there are $R, S$ in $U_e$ such that if $zx \in yR$, then $zxS \subseteq yU$.

Assume that $H$ is evenly continuous. Let $x = y = e$ and fix $U \in U_e$. Then there are $R, S$ in $U_e$ such that

$$z \in R \Rightarrow zS \subseteq U.$$

Let $V = R \cap S$. Then $V \in U_e$, $V^2 \subseteq \bigcup_{z \in R} zS \subseteq U$, $e z \in r$ and Condition (Ia) is satisfied.

Assume Condition (Ia) is satisfied. Fix $x, y, e$ and $U$. There is a $V$ in $U_e$ such that $V^2 \subseteq U$. Let $R = S = V$. Then

$$zx \in yR = yV \Rightarrow zxS \subseteq yVV \subseteq yU,$$

and the theorem is proved.

**Theorem 2.9.**

In a left-continuous group $(G, \circ, T)$ where $H$ is the family of left translations, the following conditions are equivalent:

a) Condition (I) is satisfied.

b) $H$ is strongly evenly continuous.

c) $H$ satisfies Strong Condition A.

d) There is a uniformity $\mathcal{U}$ on $G$ which generates the topology $T$, and $H$ is equicontinuous with respect to $\mathcal{U}$.

**Proof:**

The equivalence will be shown by proving that $a) \Rightarrow d) \Rightarrow b) \Rightarrow c) \Rightarrow a)$.

First assume Condition (I) holds. Let $\mathcal{B}$ be the family of sets
of the form

\[ \mathcal{U} = \{ (x,y) : x^{-1}y \in U, U \in \mathcal{U}_e \} . \]

The family \( \mathcal{B} \) is a base for a uniformity if it satisfies four conditions: (See Kelley \([7]\).)

1) For every \( x \) in \( G \) and every \( U \) in \( \mathcal{B} \), \( (x,x) \in U \).

2) For every \( U \) in \( \mathcal{B} \), there is a \( V \) in \( \mathcal{B} \) such that \( V \subseteq U^{-1} \).

3) For every \( U \) in \( \mathcal{B} \), there is a \( W \) in \( \mathcal{B} \) such that \( W \circ V \subseteq U \).

4) For every \( U \) and \( V \) in \( \mathcal{B} \), there is a \( W \) in \( \mathcal{B} \) such that \( W \subseteq U \cap V \).

Clearly condition 1) is satisfied. To prove condition 2), let \( \mathcal{U} = \{ (x,y) : x^{-1}y \in U, U \in \mathcal{U}_e \} \). Then there is a \( V \) in \( \mathcal{U}_e \) such that \( V \subseteq U \). Let \( \mathcal{V} = \{ (t,w) : t^{-1}w \in V \} = \{ (t,w) : w^{-1}t \in V^{-1} \subseteq V^{-1} \subseteq U \} \subseteq \mathcal{U}^{-1} \).

For 3), let \( \mathcal{U} \) be given with the associated \( U \in \mathcal{U}_e \). There is a \( V \) in \( \mathcal{U}_e \) such that \( V^2 \subseteq U \). Then \( \mathcal{V} \circ \mathcal{V} = \{ (x,y) : (x,z),(z,y) \in \mathcal{V} \} = \{ (x,y) : x^{-1}z, z^{-1}y \in V^2 \subseteq V \} = \{ (x,y) : x^{-1}y = x^{-1}zz^{-1}y \in V \subseteq U \} \subseteq \mathcal{U} \).

To prove 4), fix \( \mathcal{U} \) and \( \mathcal{V} \). Then

\[ \mathcal{U} \cap \mathcal{V} = \{ (x,y) : x^{-1}y \in U, x^{-1}y \in V \} = \{ (x,y) : x^{-1}y \in U \cap V \} \]

\[ = (U \cap V) \in \mathcal{B} . \]

Thus \( \mathcal{B} \) is a base for a uniformity \( \mathcal{U} \).

The uniformity \( \mathcal{U} \) generates the topology \( T \) if each set \( \mathcal{U}[x] \) contains an open neighborhood of \( x \), and each open neighborhood of \( x \) contains a set \( \mathcal{U}[x] \) for some \( \mathcal{U} \in \mathcal{B} \). This is easily shown:
$U(x) = \{y: (x,y) \in U\} = \{y: x^{-1}y \in U\} = \{y: y \in xU\} = xU.$

It remains to show that the family of homeomorphisms $H$ is equicontinuous with respect to $\mathcal{U}$, that is, that for each $x$ in $G$, each $U$ in $\mathcal{U}$ there is a $V$ in $\mathcal{U}_e$ such that for every $z$ in $G$, $zxV \subseteq U[zx]$. This is trivially true, since $U[zx] = zxU$.

Second, to show that d) implies b), assume there is a uniformity $\mathcal{U}$ on $G$ which generates the topology $T$, and assume $H$ is equicontinuous with respect to $\mathcal{U}$. For fixed $x,y$, and $U$ in $\mathcal{U}_e$ we need to show there are $W$ and $T$ in $\mathcal{U}_e$ such that for all $z$ in $G$, if $zxW \cap yT \neq \emptyset$, then $zxW \subseteq yU$.

Since $yU$ is an open neighborhood of $y$, there is a $U$ in $\mathcal{U}$ such that $U[y] \subseteq yU$, and there is a $V$ in $\mathcal{U}$ such that $V = V^{-1}$ and $V \circ V \circ V \subseteq U$. Then $V \circ V \circ V[y] \subseteq yU$, and by equicontinuity there is a $W$ in $\mathcal{T}$ such that for every $z$ in $G$, $zxW \subseteq V[zx]$. Assume $t \in zxW \cap V[y]$, and let $p \in xW$. Then, since $V = V^{-1}$, $(y,t) \in V$, $(zx,zp) \in V$, and $(zx,t) \in V$, we have $(y,zp) \in V \circ V \circ V$, and $zp \in V \circ V \circ V[y] \subseteq yU$ for every $p$ in $xW$, hence $zxW \subseteq yU$.

Third, to show that b) implies c), fix $C$ compact and $A$ closed with $C \cap A = \emptyset$. For $x = e$, and any $y \in C \subseteq (G - A) \in \mathcal{T}$, by strong even continuity there is a $W_y$ such that for all $z$ in $G$, if $zW_y \cap yW_y \neq \emptyset$, then $zW_y \subseteq (G - A)$. Since $C$ is compact and $C \subseteq \bigcup \{yW_y: y \in C\}$, there is a finite subfamily, $y_1W_{y_1}, y_2W_{y_2}, \ldots, y_nW_{y_n}$.
with $C \subseteq \bigcup_{i=1}^{n} y_i W_i$. Let $W = \bigcap_{i=1}^{n} W_i \in T$. Then if $zW \cap C \neq \emptyset$, $z W_i \cap y_i W_i \neq \emptyset$ for some $i$ and $zW \subseteq (G - A)$, or $zW \cap A = \emptyset$, and Strong Condition A is satisfied.

It remains to show that c) implies a), that is, that if $H$ satisfies Strong Condition A, $(G, \circ, T)$ satisfies Condition (I).

Assume $H$ satisfies Strong Condition A, and let $U \in U_e$. We may assume that $U$ has compact closure, since $G$ has a locally compact topology. There is a $C$ in $C_e$ with $C \subseteq U$. Then $C$ is compact, $B = G - U$ is closed, and $C \cap B = \emptyset$. Hence by Strong Condition A there is a non-empty open set $O$ such that $xO \cap C = \emptyset$ or $xO \cap B = \emptyset$ for every left translate in $H$. Choose $x$ in $O$. Then $x^{-1}O \in U_e$.

Let $V = x^{-1}O \cap \text{int}C$. Then $V \in U_e$, $V \neq \emptyset$, and for $v \in V$, we have $vV \cap C \neq \emptyset$, hence $vV \cap B = \emptyset$, and

$$V^2 = \bigcup_{v \in V} vV \subseteq U.$$

Now for $v \in V$, $e = v^{-1}v \in v^{-1}V$, and $v^{-1}V \cap C \neq \emptyset$, hence $v^{-1}V \cap B = \emptyset$, and $V^{-1} \subseteq v^{-1}V = \bigcup_{v \in V} v^{-1}V \subseteq U$.

Thus if Strong Condition A holds, then both Condition (Ia) and Condition (Ib) hold, hence by Lemma 2.3, Condition (I) holds, and the theorem is proved.
CHAPTER III
AN EMBEDDING THEOREM

Under some conditions it is possible to embed a topological group in a left-continuous group which is not a topological group. In this chapter an embedding theorem will be stated and proved.

Definition 3.1.
A topological group \((H, \cdot, S)\) is said to be embedded in a left-continuous group \((G, \circ, T)\) if there is a group \(A\) which is a subgroup of \(G\), and a map from \(A\) onto \(H\) which is a group isomorphism and is also a topological homeomorphism with respect to the induced subspace topology on \(A\).

Theorem 3.2.
A topological group \((H, \cdot, S)\) may be embedded as a proper subgroup in a left-continuous group \((G, \circ, T)\) if conditions 1) and 2) are satisfied, and \((G, \circ, T)\) will not be a topological group if and only if condition 3) holds.

1) \((H, S)\) is locally compact and Hausdorff.
2) There is a group \((K, \cdot)\) and a homomorphism, \(h \rightarrow t_h\), from \(H\) into the automorphism group of \(K\).
3) For some \(k\) in \(K\), the set \(\{h: t_h(x) = k\}\) is not an open set in \((H, S)\).
**Corollary 3.3.**

If \((H, \cdot, S)\) is a locally compact Hausdorff group such that for at least one element \(x\) in \(H\) the set \(\{h : hxh^{-1} = x\}\) is not an open set, then \((H, \cdot, S)\) may be embedded in a left-continuous group which is not a topological group.

**Proof:**

The corollary follows from the theorem by letting \((K, \cdot) = (H, \cdot)\) and \(t_h(x) = hxh^{-1}\).

The proof of the theorem will follow from a series of lemmas. In what follows, \((H, \cdot, S)\) is a topological group, \((K, \cdot)\) is a group, and there is a map \(h \rightarrow t_h\) which is a homomorphism from \(H\) into the automorphism group of \((K, \cdot)\). Thus we have

\[
\begin{align*}
\left[t_{h_1}\right] t_{h_2}(k) &= t_{h_1h_2}(k), \\
t_h(k_1k_2) &= t_h(k_1)t_h(k_2), \\
t_h(e_H) &= k, \\
t_h(e_K) &= e_K
\end{align*}
\]

for all \(h\) in \(H\), and \(k\) in \(K\), where \(e_H\) and \(e_K\) are the identity elements of \((H, \cdot)\) and \((K, \cdot)\) respectively, and the group operation in the automorphism group of \((K, \cdot)\) is composition. It should also be noted that for each \(h\) in \(H\), \(t_h\) is one-to-one and onto.

**Lemma 3.4.**

Let \(G = H \times K = \{(h, k) : h \in H, k \in K\}\), and let an operation
be defined on $G \times G$ such that

$$(h_1, k_1)(h_2, k_2) = (h_1 h_2, k_1 t_{h_1} k_2).$$

Then $G$ is a group with respect to this operation.

**Proof:**

Since $(h_1 h_2, k_1 t_{h_1} k_2) \in G$, the defined operation is a binary operation on $G$.

It is clear that $(e_H, e_K)$ is the identity in $G$, and that $(h, k)^{-1} = (h^{-1}, t_{h^{-1}}(k^{-1}))$. To show $G$ is a group it remains to prove that the binary operation is associative. Now

$$(a, b)[(x, y)(w, z)] = (a, b)(xw, yt_x(z)) = (axw, bt_a[yt_x(z)])
= (axw, bt_a(y)t_{ax}(z)),
$$

and

$$[(a, b)(x, y)](w, z) = (ax, bt_a(y))(w, z) = (axw, bt_a(y)t_{ax}(z));$$

hence the operation is associative.

A topology $T$ will be defined on $G$ by defining a family of neighborhoods at each point, and showing that these neighborhoods form a base for a topology.

Let neighborhoods of $(e_H, e_K)$ be sets of the form

$$N(e_H, e_K) = \{(h, e_K): h \in N_{e_H} e_H\}$$

where $N_{e_H}$ is an open neighborhood of $e_H$ in $(H, S)$. Let neighborhoods of an arbitrary point $(h_o, k_o)$ in $G$ be left translates of neighborhoods
of \((e_H,e_K)\), that is, sets of the form
\[ N(h_0,k_0) = (h_0,k_0)N(e_H,e_K) = \{(h_0h,k_0h) (e_K) : h \in N_{e_H}\} \]
\[ = \{(h_0h,k_0e_K) : h_0h \in h_0N_{e_H}\} \]
\[ = \{(h,k_0) : h \in h_0N_{e_H}\}. \]

Note that \(h_0N_{e_H}\) is an open neighborhood of \(h_0\) in \((H,S)\). Hence the sets \(N_{h_0} \times \{k_0\}\) are neighborhoods of \((h_0,k_0)\). Such sets form a base for a topology (see Kelley [7]), since if
\[(h_1,k_1) \in \{(h,k_0) : h \in N_{h_0}\} \cap \{(h,k_2) : h \in N_{h_2}\}, \]
then \(k_1 = k_0 = k_2\), and \(h_1 \in N_{h_0} \cap N_{h_2} = N_{h_1}\), which is an open neighborhood of \(h_1\).

Then \(\{(h,k_1) : h \in N_{h_1}\}\) is an open neighborhood of \((h_1,k_1)\) and
\[ \{(h,k_1) : h \in N_{h_1}\} \subseteq \{(h,k_0) : h \in N_{h_0}\} \cap \{(h,k_2) : h \in N_{h_2}\}. \]

Let \(T\) be the topology induced by this base.

**Lemma 3.5.**

Left translations in \((G, \circ, T)\) are homeomorphisms, hence continuous.

**Proof:**

The left translation \((x,y) \mapsto (a,b)(x,y)\) is a one-to-one onto mapping for each \((a,b)\) and the topology is so defined that it is an
open mapping. The inverse map \((x, y) \rightarrow (a, b)^{-1}(x, y) = (a^{-1}, t_{a^{-1}}(b^{-1}))(x, y)\) is also an open map, hence each left translate and its inverse are continuous, and left translations are homeomorphisms of \(G\) onto \(G\).

**Lemma 3.6.**

\((G, T)\) is Hausdorff if and only if \((H, S)\) is Hausdorff.

**Proof:**

Assume \((H, S)\) is Hausdorff. Then if \((h_1, k_1) \neq (h_2, k_2)\) and \(k_1 = k_2\), we have \(h_1 \neq h_2\), and there are disjoint open neighborhoods \(O_1\) and \(O_2\) in \(S\) of \(h_1\) and \(h_2\) respectively. Then \((h_1, k_1) \in O_1 \times \{k_2\}\), \((h_2, k_2) \in O_2 \times \{k_2\}\), and \(O_1 \times \{k_1\} \cap O_2 \times \{k_2\} = \emptyset\). If \(k_1 \neq k_2\), then \((h_1, k_1) \in H \times \{k_1\}\), \((h_2, k_2) \in H \times \{k_2\}\), and \(H \times \{k_1\} \cap H \times \{k_2\} = \emptyset\). Hence if \((H, S)\) is Hausdorff, then \((G, T)\) is Hausdorff. In a similar manner it can be shown that \((H, S)\) is Hausdorff whenever \((G, T)\) is Hausdorff.

**Lemma 3.7.**

\((G, T)\) is locally compact if and only if \((H, S)\) is locally compact.

**Proof:**

Assume \((H, S)\) is locally compact, and let \((h_0, k_0)\) be an element of \(G\). Then there is a compact neighborhood \(C\) of \(h_0\) in \((H, S)\).

It will be shown that \(C \times \{k_0\}\) is a compact neighborhood of \((h_0, k_0)\) in \((G, T)\). If \((O_\alpha)\) is any open cover of \(C \times \{k_0\}\), then \((U_\alpha)\), where \(U_\alpha = O_\alpha \cap (H \times \{k_0\})\), is also an open cover of \(C \times \{k_0\}\). Each \(U_\alpha\) is of the form \(\{(h, k_0): h \in Q_\alpha \subseteq S\}\) for some \(Q_\alpha\) in \(S\), and \((Q_\alpha)\) is an open cover of \(C\). Since \(C\) is compact, there is a finite subset,
which covers \( C \), and the corresponding finite set 
\[ 0_{a_1}, 0_{a_2}, \ldots, 0_{a_n} \]

is a finite subset of the \((0_\alpha)\) which covers 
\( C \times \{k_0\} \). Hence \( C \times \{k_0\} \) is compact in \((G,\tau)\), and \((G,\tau)\) is locally compact. In a similar manner it can be shown that if \((G,\tau)\) is locally compact, so is \((H,\mathfrak{s})\).

**Lemma 3.8.**

Right translations in \((G,\circ,\tau)\) are continuous if and only if for every element \( k \) of \( K \), the subgroup \( \{ h: h \in H, t_h(k) = k \} \) is an open set in \((H,\mathfrak{s})\).

**Proof:**

Fix \( k_0 \in K \). Consider the set 
\[ P_{k_0} = \{ h: t_h(k_0) = k_0 \}. \]

The set \( P_{k_0} \) is a subgroup of \( H \), since if \( h_1, h_2 \in P_{k_0} \), we have 
\[ t_{h_1 h_2}^{-1}(k_0) = t_{h_1} [t_{h_2}^{-1}(k_0)] = t_{h_1} (k_0) = k_0, \]
and \( h_1 h_2^{-1} \in P_{k_0} \).

Then for \( h' \) in \( H \), \( h' P_{k_0} \) is a left coset of \( P_{k_0} \), and 
\[ h' P_{k_0} = \{ h' h: t_h(k_0) = k_0 \} = \{ h' h: t_h(t_h(k_0)) = t_h(k_0) \} \]
\[ = \{ h: t_h(k_0) = t_h(k_0) \}. \]

Now let \( O \) be an open set in \((G,\circ,\tau)\). We may assume \( O \) is a
The set $0(h_o, k_o)$ is open in $(G, T)$ if and only if for each second component, $bt_h(k_o)$, that is, for each $h'$ in $O'$, the set

$$Q' = \{hh_o: bt_h(k_o) = bt_{h'}(k_o)\} = \{hh_o: t_h(k_o) = t_{h'}(k_o)\}$$

is an open set in $(H, S)$, and $Q'$ is open just when $Q = Q'_{h_o^{-1}} = \{h: t_h(k_o) = t_{h'}(k_o)\}$ is an open set, since in the topological group $(H, \ast, S)$, right translates of open sets are open.

Now $t_h(k_o) = t_{h'}(k_o)$ just when $h \in h'P_{k_o}$, so $Q = \{h: t_h(k_o) = t_{h'}(k_o)\} = P_{k_o}$, and $Q$ is open just when $P_{k_o}$ is an open set.

Therefore, all right translates are continuous transformations just when $P_k = \{h: t_h(k) = k\}$ is an open set in $(H, S)$ for every $k$ in $K$.

Theorem 3.2 will now be restated and proved.

**Theorem 3.2.**

A topological group $(H, \ast, S)$ may be embedded as a proper subgroup in a left-continuous group $(G, \circ, T)$ if conditions 1) and 2) are satisfied, and $(G, \circ, T)$ will not be a topological group if and
only if condition 3) holds.

1) \((H, S)\) is locally compact and Hausdorff.

2) There is a group \((K, \ast)\) and a homomorphism, \(h \mapsto t_h\), from \(H\) into the automorphism group of \((K, \ast)\).

3) For some \(k\) in \(K\), the set \(\{h : t_h(k) = k\}\) is not an open set in \((H, S)\).

Proof:

Let \((H, \ast, S)\) and \((K, \ast)\) be a topological group and a group respectively that satisfy condition 1) and 2), let \((G, \circ, T)\) be the product space \(H \times K\) with group operation and topology as defined above in this chapter. Then, since \((H, S)\) is locally compact and Hausdorff, by Lemmas 3.6 and 3.7, \((G, T)\) is also locally compact and Hausdorff. By Lemma 3.5, left translations in \((G, \circ, T)\) are continuous. Thus \((G, \circ, T)\) is a left-continuous group. By Lemma 3.8, \((G, \circ, T)\) is a topological group just when condition 3) does not hold.

The set

\[ H \times \{e_K\} = \{(h, e_K) : h \in H\} \]

is a subgroup of \(G\), since

\[ (h_1, e_K)(h_2, e_K)^{-1} = (h_1 h_2^{-1}, e_K t_{h_1}(t_{h_2^{-1}}(e_K))) = (h_1 h_2^{-1}, e_K) \]

is in \(H \times \{e_K\}\), and this subgroup is clearly isomorphic to \((H, \ast)\) under the map \(h \mapsto (h, e_K)\), and homeomorphic to \(H \times \{e_K\}\) under the subspace topology. Hence \((H, \ast, S)\) is embedded in \((G, \circ, T)\), and the theorem is proved.
An example of a left-continuous group that is not a topological group and that has a topological group as a subgroup is given in Chapter V.
A FIVE-R UNIQUENESS THEOREM

During an investigation into the possibility that every left-continuous group satisfies Condition (I), an attempt was made to show that every left-continuous group with a metric topology satisfies Condition (I). Although neither a counterexample to nor confirmation of this possibility was obtained, the investigation led to consideration of the Mickle-Rado 5-r condition for uniqueness \[8\], and a sharpening of the original Mickle-Rado uniqueness theorem was obtained.

Several definitions will be needed for the development of the theory. In what follows, \( c(x,r) \) is the closed sphere of center \( x \) and radius \( r \).

Definition 4.1

A real-valued set function \( \Lambda \), defined on all subsets of a separable metric space \((X,d)\), is said to be a Carathéodory outer measure if

\[
\text{i)} \quad \text{for } E \subset X, \quad 0 \leq \Lambda(E) \leq \infty, \\
\text{ii)} \quad \Lambda(\emptyset) = 0, \\
\text{iii)} \quad \text{if } E_1 \subset E_2, \text{ then } \Lambda(E_1) \leq \Lambda(E_2), \\
\text{iv)} \quad \text{if } E_1, E_2, \ldots \text{ is any sequence of subsets of } X,
\]

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then $\Lambda \left( \bigcup_{n=1}^{\infty} E_n \right) \leq \sum_{n=1}^{\infty} \Lambda(E_n)$, and

\( \nu \) if $E_1$ and $E_2$ are a positive distance apart, then

$$\Lambda(E_1 \cup E_2) = \Lambda(E_1) + \Lambda(E_2).$$

**Definition 4.2.**

Let $(X,d)$ be a separable metric space, and let $H$ be a family of homeomorphisms from $X$ onto $X$. A Borel-regular Caratheodory outer measure $\Lambda$ on $X$ will be called $H$-invariant, or Haar with respect to $H$, if $\Lambda[h(E)] = \Lambda(E)$ for every $E \subseteq X$ and every $h$ in $H$, and if there is a non-empty set $0$ for which $0 < \Lambda(0) < \infty$.

**Definition 4.3.**

Let $(X,d)$ be a separable metric space, and let $H$ be a family of homeomorphisms from $X$ onto $X$. The family $H$ is said to satisfy Condition (II) if for every pair of elements $x, y$ in $X$ there are homeomorphisms $h_1$ and $h_2$ in $H$ with $h_1(x) = y$ and $h_2(y) = x$, and positive real numbers $\lambda(x,y)$ and $L(x,y)$ such that if $0 < r < \lambda$, then

$$c(x,r) \subseteq h_2[c(y,rL)] \subseteq c(x,rL^2),$$

and

$$c(y,r) \subseteq h_1[c(x,rL)] \subseteq c(y,rL^2).$$

**Definition 4.4.**

A Caratheodory outer measure $\sigma$ on a separable metric space $(X,d)$ is said to satisfy the strong $5$-$r$ condition if for every bounded set $E$ there are positive real numbers $k(E)$ and $K(E)$ such that for every $x$ in $E$ if $0 < r < k(E)$, then
\[ \sigma[c(x,5r)] < K(E)\sigma[c(x,r)]. \]

**Definition U.5.**

A Caratheodory outer measure \( \sigma \) on a separable metric space \((X,d)\) is said to satisfy the 5-r condition at \( x \) if there are positive constants \( k(x) \) and \( K(x) \) such that for \( 0 < r < k(x) \),

\[ \sigma[c(x,5r)] < K(x)\sigma[c(x,r)]. \]

If \( \sigma \) satisfies the 5-r condition at every \( x \) in \( X \), it is said to satisfy the 5-r condition on \( X \), and it will be called a 5-r measure.

**Definition U.6.**

A Borel-regular Caratheodory outer measure \( \Lambda \) on \( X \) will be called locally finite if for each \( x \) in \( X \) there is an open set \( O_x \) such that \( x \in O_x \) and \( \Lambda(O_x) < \infty \).

If \( \Lambda \) is locally finite, then, since \( X \) is separable, there is a sequence of open sets \( O_1, O_2, \ldots \) such that

\[ (1) \; X = \bigcup_{n=1}^{\infty} O_n \quad \text{and} \quad \Lambda(O_n) < \infty, \; n=1,2,\ldots. \]

It is clear that if a measure satisfies the strong 5-r condition, it satisfies the 5-r condition. Note also that if \( \Lambda \) satisfies the 5-r condition at \( x \), there is an open set containing \( x \) that has finite \( \Lambda \)-measure, hence if \( \Lambda \) satisfies the 5-r condition on \( X \), then \( \Lambda \) is locally finite.

The main uniqueness theorem which will be proved in this chapter can now be stated.
Theorem U.7. (Uniqueness theorem)

Let \((X,d)\) be a separable metric space, let \(H\) be a family of homeomorphisms from \(X\) onto \(X\) which satisfies Condition (II), and let \(\Lambda_1\) and \(\Lambda_2\) be Borel-regular Caratheodory outer measures on \(X\) that are Haar with respect to \(H\). Then if \(\Lambda_1\) satisfies the 5-r condition at one point in \(X\), there is a positive real number \(c\) such that for every \(E \subseteq X\), \(\Lambda_2(E) = c\Lambda_1(E)\).

Theorem U.7 is a sharpening of the 5-r uniqueness theorem proved by Mickle and Rado [8] in 1959. Their theorem established the uniqueness of an invariant measure on the space of oriented lines in three-space, a result which does not follow from the standard uniqueness theorems for invariant measures. In their original theorem, one of the invariant measures is required to satisfy the strong 5-r condition. In the present theorem, this condition is weakened to the requirement that one of the invariant measures satisfy the 5-r condition at one point. First it will be shown that if a Haar measure satisfies the 5-r condition at one point, it satisfies the 5-r condition at every point in \(X\), and then that if it satisfies the 5-r condition on \(X\), it is unique.

Theorem 4.8.

Let \((X,d)\) be a separable metric space, let \(H\) be a family of homeomorphisms from \(X\) onto \(X\) satisfying Condition (II), and let \(\Lambda\) be an \(H\)-invariant outer measure on \(X\) that satisfies the 5-r condition at one point. Then \(\Lambda\) satisfies the 5-r condition everywhere on \(X\).
Proof:

Assume for $0 < r < k$, $\Lambda[c(x_o, 5r)] < K \Lambda[c(x_o, r)]$, and let $x$ be an arbitrary element in $X$. Then, since $H$ satisfies Condition (II), there are positive constants $\ell(x_o, x)$, $L(x_o, x)$, and $h$, $h_o$ in $H$ with $h(x) = x_o$ and $h_o(x_o) = x$ such that if $0 < r < \ell$,

$$c(x, r) \subseteq h_o[c(x_o, r_L)] \subseteq c(x, r_L^2)$$

and

$$c(x_o, r) \subseteq h[c(x, r_L)] \subseteq c(x_o, r_L^2).$$

Then for $r < \frac{1}{25L} \min (k, \ell)$, we have

$$\Lambda[c(x, 5r)] = \Lambda[h(c(x, 5r))] \leq \Lambda[c(x_o, 5rL)],$$

and, letting $n$ be the positive integer with $5^{n-1} \leq L < 5^n$,

$$\Lambda[c(x_o, 5rL)] \leq \Lambda[c(x_o, r_5^{n+1})] < K^{2n+1} \Lambda[c(x_o, r_5^{-n})]$$

$$\leq K^{2n+1} \Lambda[c(x_o, rL^{-1})] = K^{2n+1} \Lambda[h_o(c(x_o, rL^{-1}))]$$

and $\Lambda$ satisfies the 5-r condition at an arbitrary $x$ in $X$, hence on $X$.

There are several different conditions that are sufficient for a Haar measure to satisfy the 5-r condition at a point, and they all concern an underlying Caratheodory outer measure which satisfies the 5-r condition, but which is not necessarily Haar itself. In what follows, this underlying measure will usually be denoted $\sigma$. 
and use will be made of the upper and lower densities of a Haar measure $\Lambda$ with respect to the underlying $5\cdot r$ measure $\sigma$.

In Lemmas 4.9 through 4.13 below, $(X, d)$ is a separable metric space, and $\sigma$ is a Carathéodory outer measure which satisfies the $5\cdot r$ condition on $X$. Letting

$$(2) \quad A_n = \{ x : x \in X, \ 0 < r < 1/n \Rightarrow \sigma[c(x, 5r)] < n\sigma[c(x, r)] \},$$

we have

$$X = \bigcup_{n=1}^{\infty} A_n \text{ and } A_n \subseteq A_{n+1}, \ n=1, 2, \ldots.$$ 

**Lemma 4.9**

Let $\Lambda$ be a Borel-regular Carathéodory outer measure on $X$, let $t$ be a positive finite number, and let $E$ be a subset of $X$. Set

$$G_t(E) = \{ x : x \in X, \ \lim_{r \to 0} \sup_{r \cdot t} \frac{\Lambda[c(x, r) \cap E]}{\sigma[c(x, r)]} > t \}.$$ 

Then for the sets $A_n$ in (2),

$$\sigma[A_n \cap G_t(E)] < \frac{t}{n} \Lambda(E).$$

**Proof:**

Since

$$(3) \quad A_n \cap G_t(E) \subseteq \bigcup \{ c(x, r) : x \in A_n \cap G_t(E), \ r < \frac{1}{n},$$

$$\Lambda[c(x, r) \cap E] > t\sigma[c(x, r)] \},$$

by a covering theorem of Mickle and Rado [9], there exists a pairwise
disjoint sequence of these spheres, \( c(x_1, r_1), c(x_2, r_2), \ldots \), such that

\[(4) \quad A_n \cap G_t(E) \subset \bigcup \{c(x_i, 5r_i): i=1,2,\ldots\} \).

From (2), (3), and (4) it follows that

\[
\sigma[A_n \cap G_t(E)] \leq \sum_{i=1}^{\infty} \sigma[c(x_i, 5r_i)] \leq n \sum_{i=1}^{\infty} \sigma[c(x_i, r_i)] < \frac{n}{t} \sum_{i=1}^{\infty} \Lambda[c(x_i, r_i) \cap E] \leq \frac{n}{t} \Lambda(E).
\]

Lemma 4.10.

Let \( \Lambda \) be a locally finite Borel-regular Caratheodory outer measure in \( X \) and let \( E \) be a \( \Lambda \)-measurable subset of \( X \). Then for \( x \in X - E \),

\[
(5) \quad \lim_{r \to 0} \frac{\Lambda[c(x,r) \cap E]}{\sigma[c(x,r)]} = 0 \quad \text{a.e.} \ (\sigma).
\]

Proof:

It suffices to show that (5) holds for \( \Lambda(E) < \infty \), since letting \( E_n = E \cap O_n \), with \( O_n \) as in (1), we have

\[
\{x: x \in X - E, \lim sup_{r \to 0} \frac{\Lambda[c(x,r) \cap E_n]}{\sigma[c(x,r)]} > 0\} 
\subset \bigcup_{n=1}^{\infty} \{x: x \in X - E_n, \lim sup_{r \to 0} \frac{\Lambda[c(x,r) \cap E_n]}{\sigma[c(x,r)]} > 0\},
\]
and it suffices to show $\sigma(E_n) = 0$, $n = 1,2,...$

Assume $\Lambda(E) < \infty$. For $0 < t < \infty$, let

$$H_t(E) = \{x: x \in X - E, \limsup_{r \to 0} \frac{\Lambda[c(x,r) \cap E]}{\sigma[c(x,r)]} > t\}.$$ 

For arbitrary $\varepsilon > 0$, there is a closed set $C$ such that $C \subset E$ and $\Lambda(C - E) < \varepsilon$. (See Morse and Randolph [10]). Since $C$ is a closed subset of $E$, for $r$ sufficiently small and for $x \in X - E$, we have $c(x,r) \cap E = c(x,r) \cap (E - C)$, and for $x \in H_t(E)$,

$$\limsup_{r \to 0} \frac{\Lambda[c(x,r) \cap (E - C)]}{\sigma[c(x,r)]} > t.$$ 

Hence $H_t(E) \subset G_t(E - C)$ and by Lemma 4.9 for each positive integer $n$,

$$\sigma[A_n \cap H_t(E)] < \varepsilon \frac{n}{t}.$$ 

Since $\varepsilon > 0$ is arbitrary, for each positive integer $n$ and $t > 0$,

$$\sigma[A_n \cap H_t(E)] = 0,$$ 

and by (2), $\sigma[H_t(E)] = 0$. Then since

$$\{x: x \in X - E, \limsup_{r \to 0} \frac{\Lambda[c(x,r) \cap E]}{\sigma[c(x,r)]} > 0\} = \bigcup_{m=1}^{\infty} H_{1/m}(E),$$

(5) follows, and the lemma is proved.

**Lemma 4.11.**

If $\Lambda$ is a Borel-regular Caratheodory outer measure and $E$ is a subset of $X$ such that for $x \in E$

$$\liminf_{r \to 0} \frac{\Lambda[c(x,r) \cap E]}{\sigma[c(x,r)]} = 0,$$
then $\Lambda(E) = 0$.

**Proof:**

It suffices to show that $\Lambda(A_n \cap E) = 0$ for $A_n$ as in (2). Since there is a sequence $(O_m)$ of open sets such that

$$X = \bigcup_{m=1}^{\infty} O_m, \quad \sigma(O_m) < \infty, \quad m = 1, 2, \ldots,$$

it suffices to show that $\Lambda(A_n \cap O_m \cap E) = 0$. For $x \in E_{nm} = A_n \cap O_m \cap E$, we have

$$\lim_{r \to 0} \inf \frac{\Lambda[c(x, r) \cap E_{nm}]}{\sigma[c(x, r)]} = 0.$$

Let $\epsilon > 0$ be given. The family of closed spheres $c(x, r/5)$ for which $x \in E_{nm}$, $r < 1/n$, $c(x, r) \subset O_m$, and $\Lambda[c(x, r) \cap E_{nm}] < \epsilon \sigma[c(x, r)]$ covers $E_{nm}$. Accordingly (by [9]), there is a pairwise disjoint sequence of closed spheres $c(x_1, r_1/5)$, $c(x_2, r_2/5)$, ... such that

$$E_{nm} \subset \bigcup_{i=1}^{\infty} c(x_i, r_i) \text{ and } \Lambda(E_{nm}) \leq \bigcup_{i=1}^{\infty} \Lambda[c(x_i, r_i) \cap E_{nm}] \leq \Sigma_{i=1}^{\infty} \Lambda[c(x_i, r_i)] < \epsilon \text{ n } \Sigma_{i=1}^{\infty} \sigma[c(x_i, r_i/5)]$$

$$= \epsilon \text{ n } \sigma[\bigcup_{i=1}^{\infty} c(x_i, r_i/5)] \leq \epsilon \text{ n } \sigma(O_m).$$

Since $\epsilon > 0$ is arbitrary, $n$ and $m$ are fixed, and $\sigma(O_m) < \infty$, we have $\Lambda(E_{nm}) = 0$.

**Lemma 4.12.**

Let $\Lambda$ be a locally finite Borel-regular Caratheodory outer measure and let
Then \( E = \{ x : x \in X, \limsup_{r \to 0} \frac{\Lambda(c(x,r))}{\sigma(c(x,r))} = \infty \} \).

Proof:

It suffices to show \( \sigma(E_{nm}) = 0 \), where \( E_{nm} \) is defined as in the proof of the previous lemma.

Let \( t > 0 \) be given. By Lemma 4.9,

\[
\sigma(E_{nm}) \leq \sigma[A_n \cap G_t(O_m)] < \frac{\Lambda(O_m)}{t},
\]

and since \( t \) is arbitrary, \( \sigma(E_{nm}) = 0 \).

**Lemma 4.13.**

Let \( \Lambda \) be a locally finite Borel-regular Caratheodory outer measure on \( X \) and let

\[
E = \{ x : x \in X, \liminf_{r \to 0} \frac{\Lambda(c(x,r))}{\sigma(c(x,r))} < \infty, \limsup_{r \to 0} \frac{\Lambda(c(x,r))}{\sigma(c(x,r))} = \infty \}.
\]

Then \( \Lambda(E) = 0 \).

Proof:

By the previous lemma, we have \( \sigma(E) = 0 \). Let

\[
E_j = \{ x : x \in E, \liminf_{r \to 0} \frac{\Lambda(c(x,r))}{\sigma(c(x,r))} < j \}, j = 1, 2, ...
\]

To show that \( \Lambda(E) = 0 \), it suffices to show \( \Lambda(E_j) = 0 \), for \( j = 1, 2, ... \), and to show that \( \Lambda(E_j) = 0 \), it suffices to show that \( \Lambda(E_{jn}) = 0 \), where \( E_{jn} = E_j \cap A_n \), with \( A_n \) as in (2). Let \( \epsilon > 0 \) be given. There
is an open set \( O \) such that \( E_{jn} \subset O \) and \( c(0) < \varepsilon \). (See Mickle and Rado [9]). The set \( E_{jn} \) is covered by the family of closed spheres \( c(x, r/5) \) such that \( x \in E_{jn}, r < 1/n, c(x, r) \subset O, \) and \( \Lambda[c(x, r)] < j\sigma[c(x, r)] \). Hence, by [9] there is a pairwise disjoint sequence of the spheres, \( c(x_1, r_1/5), c(x_2, r_2/5), \ldots, \) such that

\[
E_{jn} \subset \bigcup_{i=1}^{\infty} c(x_i, r_i).
\]

Then

\[
\Lambda(E_{jn}) \leq \sum_{i=1}^{\infty} \Lambda[c(x_i, r_i)] < j \sum_{i=1}^{\infty} \sigma[c(x_i, r_i)] < jn \sum_{i=1}^{\infty} \sigma[c(x_i, r_i/5)] = jn \sigma[\bigcup_{i=1}^{\infty} c(x_i, r_i/5)] \leq jn \sigma(0) < jn \varepsilon.
\]

Since \( \varepsilon > 0 \) is arbitrary, \( \Lambda(E_{jn}) = 0. \)

It should be noted that Lemmas 4.9 through 4.13 are concerned only with the relationship between two measures on \( X \), and that the family \( H \) of homeomorphisms is not involved. The following two lemmas concern properties of covers of \( X \), and are also independent of the family \( H \).

**Lemma 4.14.**

Let \( (X, d) \) be a separable metric space, \( \Lambda \) a locally finite Borel-regular Caratheodory outer measure on \( X \), and \( \sigma \) a Borel-regular Caratheodory outer measure on \( X \) that satisfies the \( 5-r \) condition. If \( B \) is a Borel set with \( \Lambda(B) < \infty, \sigma(B) > 0 \), and if \( S \) is a family
of Borel sets covering $B$ with the property that for every $x$ in $B$ there is an $S_x$ in $\mathcal{S}$ such that $x \in S_x$ and

$$\limsup_{r \to 0} \frac{\Lambda[c(x, r) \cap S_x]}{\sigma(c(x, r))} > 0,$$

then there is an $S$ in $\mathcal{S}$ with $\Lambda(B \cap S) > 0$.

Proof:

By Lemma 4.10, for a.e. $x$ in $B$,

$$\lim_{r \to 0} \frac{\Lambda[c(x, r) \cap (X - B)]}{\sigma(c(x, r))} = 0,$$

and since $\sigma(B) > 0$, there is such an $x$. For that $x$, its associated $S_x = S$, and any radius $r$, we have

$$\frac{\Lambda[c(x, r) \cap S]}{\sigma(c(x, r))} \leq \frac{\Lambda[c(x, r) \cap S \cap B]}{\sigma(c(x, r))} + \frac{\Lambda[c(x, r) \cap S \cap (X - B)]}{\sigma(c(x, r))}.$$

Hence

$$\limsup_{r \to 0} \frac{\Lambda[c(x, r) \cap S \cap B]}{\sigma(c(x, r))} \geq \limsup_{r \to 0} \frac{\Lambda[c(x, r) \cap S]}{\sigma(c(x, r))} > 0,$$

and $\Lambda(S \cap B) > 0$.

Lemma 4.15.

Let $(X, d)$ be a separable metric space, $\Lambda$ a locally-finite
Borel-regular Caratheodory outer measure, and \( \sigma \) a Borel-regular Caratheodory outer measure that satisfies the \( 5-r \) condition. If \( B \) is a Borel set, and \( S \) is a family of Borel sets such that for each \( x \) in \( B \), there is an \( S_x \) in \( S \) with \( x \in S_x \) and

\[
\limsup_{r \to 0} \frac{\lambda(c(x,r) \cap S_x)}{\sigma(c(x,r))} > 0,
\]

then there is a sequence \( (S_1^i)_{i=1}^\infty \) of sets in \( S \) with

\[
\sigma(B - \bigcup_{i=1}^\infty S_i^i) = 0.
\]

**Proof:**

We may assume \( \sigma(B) > 0 \), since if not the lemma is trivially true. First assume \( \lambda(B) < \infty \), and let

\[
(6) \quad a = \inf \{ \lambda(B - \bigcup_{i=1}^\infty S_i^i) : (S_i^i) \text{ is a sequence of sets in } S \}.
\]

Since \( \lambda(B) < \infty \), \( a < \infty \). Note that there is a sequence \( (S_i^*)_{i=1}^\infty \) with

\[
a = \lambda(B - \bigcup_{i=1}^\infty S_i^*).
\]

Take

\[
(S_i^*) = \bigcup_{n=1}^\infty \{ (S_i,n) : \lambda(B - \bigcup_{i=1}^\infty S_i,n) < a + 1/n \}.
\]

Assume \( \sigma(B - \bigcup_{i=1}^\infty S_i^*) > 0 \). Then \( B - \bigcup_{i=1}^\infty S_i^* \) and the family \( S \) satisfy the conditions for Lemma 4.14, and there is a set \( S \) in \( S \) with \( \lambda[(B - \bigcup_{i=1}^\infty S_i^*) \cap S] > 0 \), and \( \lambda[B - (S \cup \bigcup_{i=1}^\infty S_i^*)] < a \), contradicting (6).
Hence $\sigma(B - \bigcup_{i=1}^{\infty} S_i) = 0$.

For arbitrary $B$, let $B_n = B \cap 0_n$, with $0_n$ as in (1). For each $n, n=1,2,\ldots$, we have $(S_i^n), i=1,2,\ldots$, such that $\sigma(B_n - \bigcup_{i=1}^{\infty} S_i^n) = 0$. Then

$$\sigma(B - \bigcup_{i=1}^{\infty} S_i^n) \leq \sigma(\bigcup_{n=1}^{\infty} (B_n - \bigcup_{i=1}^{\infty} S_i^n)) \leq \sum_{n=1}^{\infty} \sigma(B_n - \bigcup_{i=1}^{\infty} S_i^n) = 0.$$  

In Lemmas 4.16 through 4.24 $(X,d)$ is a separable metric space, $H$ is a family of homeomorphisms from $X$ onto $X$ that satisfies Condition (II), $\Lambda$ and $\sigma$ are Borel-regular Caratheodory outer measures that are Haar with respect to $H$, and $\sigma$ satisfies the 5-r condition.

**Lemma 4.16.**

Let $E \subset X$ and let $x \in E$ such that

$$\lim_{r \to 0} \sup \frac{\Lambda[c(x,r) \cap E]}{\sigma[c(x,r))}>0,$$

and let $y \in X$. Then there is an $h \in H$ such that $h(x) = y$ and

$$\lim_{r \to 0} \sup \frac{\Lambda[c(y,r) \cap h(E)]}{\sigma[c(y,r)]}>0.$$

**Proof:**

Since $H$ satisfies Condition (II), we have $h \in H$, $f(x,y)$ and $L(x,y)$ such that $h(x) = y$ and if $0 < r < f(x,y)$, $c(y,r) \subset h[c(x,rL)] \subset c(y,rL^2)$. We may assume $L(x,y) > 1$. Since $\sigma$ satisfies the 5-r condition, there is an $n$ such that $x \in A_n$ and $y \in A_n$, with $A_n$ as in
(2).

Also there is a positive integer $q$ such that $5 \leq L(x,y) < 5^q$.

Note that $\sigma[c(x,r)] \leq \sigma[c(x,5^q r)] < n^{2q} \sigma[c(x,5^{-q} r)] \leq n^{2q} \sigma[c(x,r/L)]$.

Then, for $0 < r < L(x,y)/nL$, we have

$$\frac{\Lambda[c(y,r) \cap h(E)]}{\sigma[c(y,r)]} \geq \frac{\Lambda[h[c(x,r/L)] \cap h(E)]}{\sigma[h[c(x,r/L)]]} = \frac{\Lambda[c(x,r/L) \cap E]}{\sigma[c(x,rL)]}$$

$$= \frac{\Lambda[c(x,r/L) \cap E]}{\sigma[c(x,r/L)]} \frac{\sigma[c(x,r/L)]}{\sigma[c(x,rL)]}$$

$$> \frac{\Lambda[c(x,r/L) \cap E]}{\sigma[c(x,r/L)]} \frac{1}{n^{2q}}$$

and

$$\limsup_{r \to 0} \frac{\Lambda[c(y,r) \cap h(E)]}{\sigma[c(y,r)]} > \frac{1}{n^{2q}} \limsup_{r \to 0} \frac{\Lambda[c(x,r) \cap E]}{\sigma[c(x,r)]} > 0.$$

Lemma 4.17

Let $B$ be a Borel set in $X$ such that $\Lambda(B) > 0$. Then there exists a sequence $(h_i)$, $i=1,2,\ldots$, of homeomorphisms in $H$ such that

$$\sigma[X - \bigcup_{i=1}^{\infty} h_i(B)] = 0.$$

Proof:

By Lemma 4.11, since $\Lambda(B) > 0$, there is a point $x_0 \in B$ with

$$\limsup_{r \to 0} \frac{\Lambda[c(x_0,r) \cap B]}{\sigma[c(x_0,r)]]} > 0.$$

Then, by Lemma 4.16, for any $x \in X$ there is an $h_x \in H$ such that
\[ h(x_0) = x \text{ and } \limsup_{r \to 0} \frac{\Lambda[c(x,r) \cap h(B)]}{\sigma(c(x,r))} > 0. \]

Thus \( X \) is covered by the Borel sets \( h_x(B) \) for \( x \in X \) in a manner to satisfy Lemma 4.15, and there is a sequence \( (h_i), i=1,2,\ldots \), in \( H \) such that

\[ \sigma[X - \bigcup_{i=1}^{\infty} h_i(B)] = 0. \]

**Lemma 4.18.**

Let \( B \) be a Borel set in \( X \) such that \( \sigma(B) > 0 \). Then there exists a sequence \( (h_i), i=1,2,\ldots \), of homeomorphisms in \( H \) such that

\[ \sigma[X - \bigcup_{i=1}^{\infty} h_i(B)] = 0. \]

**Proof:**

Since \( \sigma \) satisfies all the conditions placed upon \( \Lambda \), Lemma 4.17 holds with \( \Lambda = \sigma \).

**Lemma 4.19.**

Let \( B \) be a Borel set in \( X \). Then \( \sigma(B) = 0 \) if and only if \( \Lambda(B) = 0 \).

**Proof:**

Assume \( \sigma(B) = 0 \) and \( \Lambda(B) > 0 \). Then, by Lemma 4.17, there is a sequence \( (h_i), i=1,2,\ldots \), in \( H \) such that \( \sigma[X - \bigcup_{i=1}^{\infty} h_i(B)] = 0 \). Since \( \sigma(B) = 0 \), \( \sigma[h_i(B)] = 0 \), \( \sigma[\bigcup_{i=1}^{\infty} h_i(B)] = 0 \) and \( \sigma(X) = 0 \). But
σ is Haar with respect to H, hence σ(X) > 0. Thus if σ(B) = 0, A(B) = 0.

Assume A(B) = 0 and σ(B) > 0. Then, by Lemma 4.18, there is a sequence (h_n), n = 1, 2, ..., in H such that σ(X - ∪ h_n(B)) = 0, hence A(X - ∪ h_n(B)) = 0. But since A is Haar with respect to H, A(X) > 0, and since A[h_n(B)] = A(B), we have A(B) > 0, contradicting the assumption. Hence if A(B) = 0, then σ(B) = 0.

The separable metric space (X, d), the sigma-algebra of Borel sets B, with either A or σ restricted to the Borel sets, form sigma-finite measure spaces, and by Lemma 4.19, A is absolutely continuous with respect to σ. Hence the Radon-Nikodym Theorem may be applied (see Rudin [13]), and there is a Radon-Nikodym derivative, a real-valued Borel-measurable function f on X, such that for every Borel set B

\[ A(B) = \int f \, d\sigma. \]

In what follows, a Radon-Nikodym derivative, f, is fixed.

Lemma 4.20.

For fixed t, 0 < t < ∞, let \( L_t = \{x: f(x) > t\} \), and let \( S_t = \{x: f(x) < t\} \). Then at most one of \( L_t \) and \( S_t \) has positive σ-measure.

Proof:

Assume σ(L_t) > 0. Then by Lemma 4.18 there is a sequence
(h_i), i=1,2,..., in H such that \( \sigma[X - \bigcup_{i=1}^{\infty} h_i(L_t)] = 0 \). For each \( i, i=1,2,... \), if \( \sigma[S_t \cap h_i(L_t)] > 0 \), we have

\[
\Lambda[S_t \cap h_i(L_t)] = \int_{S_t \cap h_i(L_t)} f \, d\sigma < \sigma[S_t \cap h_i(L_t)],
\]

and

\[
\Lambda[h_i^{-1}(S_t) \cap L_t] = \int_{h_i^{-1}(S_t) \cap L_t} f \, d\sigma > \sigma[h_i^{-1}(S_t) \cap L_t],
\]

contradicting the invariance of \( \Lambda \) and \( \sigma \) under \( H \). Hence, if \( \sigma(L_t) > 0 \), then \( \sigma(S_t) = 0 \). In a similar fashion it can be shown that if \( \sigma(S_t) > 0 \), then \( \sigma(L_t) = 0 \).

**Lemma 4.21.**

Let

\[(7) \ c = \sup \{t: \sigma[x: f(x) \geq t] > 0\}.
\]

Then \( 0 < c < \infty \).

**Proof:**

If \( c = 0 \), then \( \Lambda(X) = \int_X f \, d\sigma = 0 \), which contradicts the fact that \( \Lambda \) is Haar with respect to \( H \).

Assume \( c = \infty \). Then for any Borel set \( B \) with \( \Lambda(B) > 0 \), we have \( \sigma(B) > 0 \), and by Lemma 4.20, for any \( t \), \( \sigma[B \cap \{x: f(x) \geq t\}] = \sigma(B) \). Hence \( \Lambda(B) > \sigma(B) \) for any \( t \); thus for every Borel set \( B \), \( \Lambda(B) = \infty \). But since \( \Lambda \) is Haar, there is a Borel set of positive
finite $\Lambda$-measure. Hence $0 < c < \infty$.

**Lemma 4.22.**

For a.e. $(\sigma) x$ in $X$, $f(x) = c$.

**Proof:**

The set

$$\{x: f(x) > c\} = \bigcup_{n=1}^{\infty} \{x: f(x) > c + \frac{1}{n}\},$$

and

$$\sigma\{x: f(x) > c + \frac{1}{n}\} = 0,$$

hence $\sigma\{x: f(x) > c\} = 0$.

The set

$$\{x: f(x) < c\} = \bigcup_{n=1}^{\infty} \{x: f(x) < c - \frac{1}{n}\},$$

By the definition of $c$, $\sigma\{x: f(x) > c - \frac{1}{n}\} > 0$, for $n=1,2,\ldots$, hence by Lemma 4.20, $\sigma\{x: f(x) < c - \frac{1}{n}\} = 0$ for every $n$, and $\sigma\{x: f(x) \neq c\} = 0$.

**Lemma 4.23.**

For every Borel set $B$, $\Lambda(B) = c\sigma(B)$.

**Proof:**

$$\Lambda(B) = \int_B f \, d\sigma = \int_B c \, d\sigma = c\sigma(B).$$

**Theorem 4.24.**

Let $(X,d)$ be a separable metric space, let $H$ be a family of homeomorphisms from $X$ onto $X$ which satisfies Condition (II), let
A and \( \sigma \) be Borel-regular Caratheodory outer measures on \( X \) that are Haar with respect to \( H \), and let \( \sigma \) satisfy the 5-r condition. Then there is a positive real number \( c \) such that for every \( E \subseteq X \), \( \Lambda(E) = c \sigma(E) \).

**Proof:**

Fix \( E \subseteq X \). Then since \( \Lambda \) and \( \sigma \) are Borel-regular, there are Borel sets \( B_1 \) and \( B_2 \) such that \( E \subseteq B_1 \), \( E \subseteq B_2 \), \( \Lambda(E) = \Lambda(B_1) \), and \( \sigma(E) = \sigma(B_2) \). Hence, with \( B = B_1 \cap B_2 \), \( B \) is a Borel set, \( \Lambda(E) = \Lambda(B) \), and \( \sigma(E) = \sigma(B) \).

By the previous lemma, with \( c \) defined as at (7), we have

\[
\Lambda(E) = \Lambda(B) = c \sigma(B) = c \sigma(E).
\]

**Theorem 4.7. (Uniqueness Theorem)**

Let \((X, d)\) be a separable metric space, let \( H \) be a family of homeomorphisms from \( X \) onto \( X \) which satisfies Condition (II), and let \( \Lambda_1 \) and \( \Lambda_2 \) be Borel-regular Caratheodory outer measures on \( X \) that are Haar with respect to \( H \). Then if \( \Lambda_1 \) satisfies the 5-r condition at one point in \( X \), there is a positive real number \( c \) such that for every \( E \subseteq X \), \( \Lambda_2(E) = c \Lambda_1(E) \).

**Proof:**

By Theorem 4.8, if \( \Lambda_1 \) satisfies the 5-r condition at one point in \( X \), then \( \Lambda_1 \) satisfies the 5-r condition on \( X \). Letting \( \Lambda_1 \) be the \( \sigma \) of Theorem 4.24, with \( c \) defined as at (7), we have \( \Lambda_2(E) = c \Lambda_1(E) \) for every \( E \subseteq X \).
The density properties of a Haar measure $\Lambda$ with respect to a 5-\(r\) measure $\sigma$ may be used to decompose the space $X$ into disjoint subsets. Letting

$$X_0 = \{x: \liminf_{r \to 0} \frac{\Lambda c(x,r)}{\sigma c(x,r)} = 0\}.$$ 

$$X_1 = \{x: 0 < \liminf_{r \to 0} \frac{\Lambda c(x,r)}{\sigma c(x,r)} \leq \limsup_{r \to 0} \frac{\Lambda c(x,r)}{\sigma c(x,r)} < \infty\},$$

(8) $$X_2 = \{x: \liminf_{r \to 0} \frac{\Lambda c(x,r)}{\sigma c(x,r)} < \limsup_{r \to 0} \frac{\Lambda c(x,r)}{\sigma c(x,r)} = \infty\},$$

$$X_\infty = \{x: \liminf_{r \to 0} \frac{\Lambda c(x,r)}{\sigma c(x,r)} = \infty\},$$

we have

$$X = X_0 \cup X_1 \cup X_2 \cup X_\infty$$

and the sets are pairwise disjoint.

From Lemmas 4.11, 4.12, and 4.13, we have $\Lambda(X_0 \cup X_2) = 0$ and $\sigma(X_2 \cup X_\infty) = 0$.

If $X_1 = \emptyset$, or if $X_1$ is either $\Lambda$-null or $\sigma$-null, then there are disjoint sets $A$ and $B$ such that for every subset $E$ of $X$,

$$\Lambda(E) = \Lambda(E \cap A), \text{ and } \sigma(E) = \sigma(E \cap B),$$

where $A = X_2 \cup X_\infty$ and $B = X_0$ if $X_1 = \emptyset$, or $B = X_0 \cup X_1$ if $X_1$ is $\Lambda$-null, and similarly if $X_1$ is $\sigma$-null. If this situation occurs, then both $A$ and $B$ are dense in $X$. This will be proved for the case $X_1 = \emptyset$; extension to the other cases is obvious.
Theorem 4.25.

Assume \( X = X^\circ \cup X_2 \cup X_\infty \) as in (8). Then \( X = \text{cl}X^\circ = \text{cl}X_\infty \), where \( \text{cl}E \) means the closure of \( E \).

Proof:

Assume \( X \neq \text{cl}X_\infty \). Then there is an \( x \in X \), and an open neighborhood \( O_x \) of \( x \), such that \( O_x \cap X_\infty = \emptyset \). Now \( X = \bigcup_{h \in H} h(O_x) \), and since \( X \) is separable, \( X = \bigcup_{i=1}^\infty h_i(O_x) \) for some sequence \( (h_i) \), \( i=1,2,\ldots \), of sets in \( H \), and \( \Lambda(X) \leq \sum_{i=1}^\infty \Lambda[h_i(O_x)] \). Since \( \Lambda[h_i(O_x)] = \Lambda(O_x) = \Lambda(O_x \cap X_\infty) = 0 \), we have \( \Lambda(X) = 0 \), which is a contradiction, since \( \Lambda \) is Haar. Hence \( X = \text{cl}X_\infty \). Similarly \( X = \text{cl}X^\circ \), since \( \sigma \) satisfies the 5-r condition, hence is positive on every open set.

If the set \( X_1 \neq \emptyset \), then there is a point \( x \) in \( X \) at which the Haar measure \( \Lambda \) satisfies the 5-r condition, as is shown in the following theorem.


Let \( (X,d) \) be a separable metric space, let \( \sigma \) be a Borel-regular Caratheodory outer measure in \( X \) which satisfies the 5-r condition, let \( \Lambda \) be a locally-finite Borel-regular Caratheodory outer measure in \( X \), and let \( x \) be a point in \( X \) such that

\[
0 < \liminf_{r \to 0} \frac{\Lambda[c(x,r)]}{\sigma[c(x,r)]} \leq \limsup_{r \to 0} \frac{\Lambda[c(x,r)]}{\sigma[c(x,r)]} < \infty.
\]

Then \( \Lambda \) satisfies the 5-r condition at \( x \).
Proof:

There are real numbers $a$ and $b$ such that

$$0 < a = \lim_{r \to 0} \inf \frac{\Lambda[c(x,r)]}{\sigma[c(x,r)]} \leq \lim_{r \to 0} \sup \frac{\Lambda[c(x,r)]}{\sigma[c(x,r)]} = b < \infty.$$  

Fix $\epsilon > 0$ with $\epsilon < a$. Then there is an $r_0$ such that for $0 < r < r_0$,

$$a - \epsilon < \frac{\Lambda[c(x,r)]}{\sigma[c(x,r)]} < b + \epsilon.$$  

Since $\sigma$ satisfies the $5$-$r$ condition, there are $k$ and $K$ such that

$$\sigma[c(x,5r)] < K\sigma[c(x,r)]$$

for $0 < r < k$. Then for $0 < r < \frac{r_0}{5}$, $r < k$, we have

$$\Lambda[c(x,5r)] < (b + \epsilon)\sigma[c(x,5r)] < (b + \epsilon)K\sigma[c(x,r)]$$

$$< \frac{b + \epsilon}{a - \epsilon} K\Lambda[c(x,r)],$$

and $\Lambda$ satisfies the $5$-$r$ condition at $x$.

There is also a condition on a point in $X_\infty$ which ensures that $\Lambda$ satisfy the $5$-$r$ condition.

Theorem 4.27.

Let $(X,d)$ be a separable metric space, let $\sigma$ be a Borel-regular Caratheodory outer measure in $X$ which satisfies a $5$-$r$ condition, let $\Lambda$ be a locally-finite Borel-regular Caratheodory outer measure in $X$, and let $x$ be a point in $X$ such that
1) \( \lim \inf_{r \to 0} \frac{\Lambda[c(x,r)]}{\sigma(c(x,r))] = \infty, \) and

2) there are positive constants \( r_0, m < 1 \) such that for 
\( 0 < r < r_0 \) and for \( 0 < t < mr, \)

\[
\frac{\Lambda[c(x,t)]}{\sigma(c(x,t))] \geq \frac{\Lambda[c(x,r)]}{\sigma(c(x,r))}.
\]

Then \( \Lambda \) satisfies the 5-r condition at \( x. \)

**Proof:**

Fix \( x, \) where \( x \) satisfies 1) and 2). Then \( x \in A_n \) for some \( n, \)
where \( A_n \) is defined as in (2), and for \( 0 < r < 1/n, \sigma[c(x,5r)] < \)
\( n\sigma[c(x,r)]. \) For the \( m \) in 2), there is a positive integer \( t \) with
\( 5^{-t} < m. \) Then for \( 0 < r < 1/n, r < r_0/5, \) we have

\[
\frac{\Lambda[c(x,5r)]}{\sigma(c(x,5r))] \leq \frac{\Lambda[c(x,rm)]}{\sigma(c(x,rm))] \leq \frac{\Lambda[c(x,r)]}{\sigma(c(x,r))} < n^{t+1}\frac{\Lambda[c(x,r)]}{\sigma(c(x,5r))}.
\]

and \( \Lambda \) satisfies a 5-r condition at \( x. \)

An additional condition imposed on the 5-r measure \( \sigma \) will ensure that a Haar measure \( \Lambda \) satisfies the 5-r condition.

**Lemma 4.28.**

Let \( (X,d) \) be a separable metric space, let \( H \) be a family of
homeomorphisms from \( X \) onto \( X \) which satisfies Condition (II), let
\( \Lambda \) be a Borel-regular Caratheodory outer measure that is Haar with
respect to \( H, \) and let \( \sigma \) be a Borel-regular Caratheodory outer measure
that satisfies the 5-r condition and has the property that for
x, y in X there are positive real numbers m and M such that for
0 < r < m,
\[ \sigma[c(x,r)] < M\sigma[c(y,r)], \text{ and} \]
\[ \sigma[c(y,r)] < M\sigma[c(x,r)]. \]

Then there is a point x in X such that \( A \) satisfies the 5-r condition
at x.

**Proof:**

If \( X_1 \neq \emptyset \), with \( X_1 \) defined as in (8), then by Lemma 4.26,
\( A \) satisfies the 5-r condition on X.

Assume \( X_1 = \emptyset \). Then \( X_\infty \neq \emptyset \) and \( X_0 \neq \emptyset \), by Theorem 4.25.

For \( x \in X_\infty \), \( y \in X_0 \), we have m and M as above. Note that for
0 < r < m,
\[ \frac{1}{M}\sigma[c(x,r)] < \sigma[c(y,r)] < M\sigma[c(x,r)], \text{ and} \]
\[ \frac{1}{M}\sigma[c(y,r)] < \sigma[c(x,r)] < M\sigma[c(y,r)]. \]

Since \( \sigma \) satisfies the 5-r condition, there is an n such that
if 0 < r < 1/n, \( \sigma[c(y,5r)] < n\sigma[c(y,r)] \).

Since \( H \) satisfies Condition (II), there are \( \ell(x,y), L(x,y) \)
and an h in \( H \) such that for 0 < r < \( \ell(x,y) \), we have h(y) = x and
\( c(x,r) \subseteq h[c(y,rL)] \).

There is a positive integer q such that \( 5^{q-1} \leq L(x,y) < 5^q \).
Since \( \lim_{r \to 0} \inf_{i=1,2,\ldots} \frac{\Lambda[c(y,r)]}{\sigma[c(y,r)]} = 0 \), there is a sequence \( (r_i) \), \( i=1,2,\ldots \), \( r_i \to 0 \), \( r_i < r_{i+1} \), and \( \lim_{i \to \infty} \frac{\Lambda[c(y,r_i)]}{\sigma[c(y,r_i)]} = 0 \). We may assume \( \frac{\Lambda[c(y,r_i)]}{\sigma[c(y,r_i)]} < 1 \) for \( i = 1,2,\ldots \).

Let \( t > M \). Then, since \( \lim_{r \to 0} \frac{\Lambda[c(x,r)]}{\sigma[c(x,r)]} = \infty \), there is an \( r_0 \) such that for \( 0 < r < r_0 \),

\[
\frac{\Lambda[c(x,r)]}{\sigma[c(x,r)]} > t > M.
\]

Fix \( r \) with \( 0 < r < r_0 \), \( r < 1/n \), \( r < l(x,y) \), \( r < m \), and

\[
\frac{\Lambda[c(y,r)]}{\sigma[c(y,r)]} < 1.
\]

Then

\[
\Lambda[c(x,r^*-)] > t \sigma[c(x,r^*-)] > M \sigma[c(x,r^*-)] > M \sigma[c(x,r)]
\]

\[
> \sigma[c(y,r)] > \Lambda[c(y,r)] = \Lambda[h[c(y,r)]] \geq \Lambda[c(x,r/L)] \geq \Lambda[c(x,r^*-)];
\]

a contradiction. Hence, if \( X_\delta \neq \emptyset \), then \( X_\delta = \emptyset \), and since \( \sigma \) is positive on \( X \) and null on \( X_\infty \cup X_2 \), we have \( X_1 \neq \emptyset \), contradicting the assumption that \( X_1 = \emptyset \). Since \( X_1 \neq \emptyset \), there is a point \( x \in X_1 \), and by Lemma 4.26, \( \Lambda \) satisfies the 5-r condition at \( x \).

The preceding lemmas are summarized in the following theorem.
Theorem 4.29.

Let \((X,d)\) be a separable metric space, let \(H\) be a family of homeomorphisms from \(X\) onto \(X\) which satisfies Condition (II), let \(\Lambda_1\) and \(\Lambda_2\) be Borel-regular Caratheodory outer measures on \(X\) that are Haar with respect to \(H\), and let \(\sigma\) be a Borel-regular Caratheodory outer measure on \(X\) which satisfies the 5-r condition. Then if i), ii), or iii) holds, there is a positive number \(c\) such that for every \(E \subset X\), \(\Lambda_1(E) = c\Lambda_2(E)\).

i) There is a point \(x\) in \(X\) such that

\[
0 < \liminf_{r \to 0} \frac{\Lambda[c(x,r)]}{\sigma[c(x,r)]} \leq \limsup_{r \to 0} \frac{\Lambda[c(x,r)]}{\sigma[c(x,r)]} < \infty.
\]

ii) There is a point \(x\) in \(X\) such that

\[
\liminf_{r \to 0} \frac{\Lambda[c(x,r)]}{\sigma[c(x,r)]} = \infty,
\]
and there are positive constants \(r_0\) and \(m \leq 1\) such that for \(0 < r \leq r_0\) and \(0 < t < rm\),

\[
\frac{\Lambda[c(x,t)]}{\sigma[c(x,t)]} \geq \frac{\Lambda[c(x,r)]}{\sigma[c(x,r)]}.
\]

iii) For every pair of points \(x,y\) in \(X\) there are positive real numbers \(m\) and \(M\) such that for \(0 < r < m\), \(\sigma[c(x,r)] < M\sigma[c(y,r)]\), and \(\sigma[c(y,r)] < M\sigma[c(x,r)]\).
CHAPTER V
AN EXAMPLE

In this chapter is an example which illustrates several of the theorems presented in the previous chapters, and demonstrates that left-continuous groups that are not topological groups do indeed exist.

Following the example is a theorem on the decomposition of a measure into the sum of two measures, one trivially infinite or zero, the other unique, and infinite only on sets with subsets of finite positive measure.

Example 5.1.

Let \( G = \{ (x,y) : x,y \in \mathbb{R}_+, x > 0 \} \). For \((a,b),(x,y) \in G\), let \((a,b)(x,y) = (ax,ay+b)\). Clearly \((G, \circ )\) is a group, with \((1,0)\) the identity, and for each \((x,y)\) in \(G\), \((x,y)^{-1} = (1/x,-y/x)\). If \(G\) is mapped onto the set of all \(2 \times 2\) matrices of the form \(\begin{pmatrix} a & b \\ a & b \end{pmatrix}\), \(a > 0\), with the mapping \((a,b) \to \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\), then the group operation \(\circ \) corresponds to matrix multiplication, and left translates of horizontal lines are horizontal lines, but right translates of horizontal lines are lines which are not necessarily horizontal.

Let the topology \(T\) on \(G\) be the topology with neighborhood base
at \((x_0, y_0)\) consisting of sets of the form
\[
\{(x, y_0) : x \in (a, b), \ 0 < a < x_0 < b < \infty\}.
\]
Open sets are then unions of open intervals on horizontal lines.

With the group operation and topology as defined above, \((G, \circ, T)\) is a left-continuous group which is not a topological group.

To show that left translations are continuous, it suffices to show left translates of open base sets are open base sets, since for any open set \(O\), \((x, y)^{-1}O = (1/x, -y/x)O\). Then, since
\[
(t, w)[(x, y_0) : x \in (a, b)] = [(tx, ty_0 + w) : x \in (a, b)]
= [(x, y_1) : x \in (ta, tb), y_1 = ty_0 + w],
\]
left translates of open base sets are open, and left translations are continuous.

It is clear that \((G, T)\) is locally compact and Hausdorff, so \((G, \circ, T)\) is a left-continuous group.

To show that \((G, \circ, T)\) is not a topological group, we need to show that not all right translations are continuous. Consider the open set \(O = \{(x, 1) : 1 < x < 3\}\) and right translation by \((1, 2)\). Then \(O(1, 2) = \{(x, 2x+1) : 1 < x < 3\}\) is an open segment of the line \(y = 2x+1\), of slope 2, and contains no open sets. Hence right translations are not all continuous and \((G, \circ, T)\) is not a topological group.

The left-continuous group \((G, \circ, T)\) satisfies Condition (I), since if \(O\) is an open set containing \((1,0)\), the identity in \(G\), there is a set \(U \subset O\), \(U = \{(x, 0) : a < x < b\}\) for some \(a\) and \(b\) with \(0 < a < 1 < b < \infty\), and letting \(V = \{(x, 0) : 1-\varepsilon < x < 1+\varepsilon\}\) where
\[
\varepsilon < \frac{1 - a}{1 + a}, \quad \varepsilon < \frac{b - 1}{b + 1}, \text{ we have } \mathcal{V}^{-1} = \{(x,0): \frac{1}{1 + \varepsilon} < x < \frac{1}{1 - \varepsilon}\}, \text{ and } \mathcal{W}^{-1} = \{(x,0): \frac{1 - \varepsilon}{1 + \varepsilon} < x < \frac{1 + \varepsilon}{1 - \varepsilon}\} \subset U \subset \mathcal{O}.
\]

Since \((G, \cdot, \tau)\) satisfies Condition (I), there is a left-translation-invariant Borel-regular outer measure on \(G\) that is positive on open sets and finite on compact sets. For open base sets, let

\[
\Lambda[\{(x,y_0): a < x < b\}] = \ln \frac{b}{a}, \quad \Lambda(\emptyset) = 0,
\]

where \(\ln\) is the natural logarithm, and for arbitrary \(E \subset G\), letting \(B_i, i=1,2,\ldots\), be an open base set, let

\[
\Lambda(E) = \inf \left\{ \sum_{i=1}^{\infty} \Lambda(B_i): E \subset \bigcup_{i=1}^{\infty} B_i \right\} \text{ if there is a sequence of open base sets that covers } E, \text{ and } \Lambda(E) = \infty \text{ otherwise.}
\]

Clearly \(\Lambda\) is a Borel-regular outer measure that is positive on non-empty open sets and finite on compact sets, and it is invariant under left translation, since

\[
\Lambda[(t,w)\{(x,y_0): a < x < b\}] = \Lambda[(x,y_1): y_1 = ty_0 + b, ta < x < tb] = \ln \frac{ta}{tb} = \ln \frac{a}{b} = \Lambda[(x,y_0): a < x < b].
\]

This example serves as an illustration of the embedding theorem.

Let \((H, \cdot, S)\) be the group of positive real numbers under multiplication, with the subspace topology induced by the usual topology on the reals. Then \((H, \cdot, S)\) is a locally compact Hausdorff topological group. Let \((K, +)\) be the group of real numbers under addition. Then \(G = H \times K\).

For \(h \in H\), define the map \(h \rightarrow t_h\) by \(t_h(k) = hk\). For each \(h\), \(t_h\) is
an automorphism of $K$, since $t_h(x+y) = h(x+y) = hx+hy = t_h(x)+t_h(y)$, and $t_h$ is a one-to-one onto map. If $t_h(x) = t_h(y)$, then $hx=hy$, $x=y$, and $t_h$ is one-to-one. For $y \in K$, $y \neq 0$, we have $y = t_h(y/h)$, and for $y = 0$, $y = t_h(0)$ for any $h$. Since $t_{h_1h_2}(x) = h_1h_2x = t_{h_1}(h_2(x)) = t_{h_1}[t_{h_2}(x)]$, the map $h \mapsto t_h$ is a homomorphism from $H$ into the automorphism group of $K$.

The group operation on $G$, $(a,b)(x,y) = (ax,ay+b) = (ax,b+t_a(y))$, is as defined in the proof of the embedding theorem in Chapter III.

It is clear that for $k \neq 0$, the set $\{h: t_h(k) = k\} = \{1\}$ is not an open set in $H$, so the embedding left-continuous group is not a topological group.

Now the set $\{(x,0): x \in H\}$ is a subgroup of $G$ which is isomorphic to $H$, and the induced subspace topology is homeomorphic to the topology on $H$, both under the obvious map $x \mapsto (x,0)$, hence $(H,\cdot,\mathcal{S})$ is embedded in $(G,\circ,T)$.

One non-trivial left-translation-invariant outer measure, $\Lambda$, has been defined on $G$. Another non-trivial left-translation-invariant outer measure also exists: for $E \subset G$, let

$$\sigma(E) = \sup\{\Lambda(F): F \subset E, \Lambda(F) < \infty\}.$$ 

Then $\sigma$ is clearly an outer measure on $G$, since $\sigma(\emptyset) = 0$, $\sigma$ is monotone, and for any sequence of sets $E_n$ in $G$, and any $F \subset \bigcup_{n=1}^{\infty} E_n$ with $\Lambda(F) < \infty$, we have $\Lambda(F) = \Lambda(F \cap \bigcup_{n=1}^{\infty} E_n) = \Lambda(\bigcup_{n=1}^{\infty} F \cap E_n) \leq \sum_{n=1}^{\infty} \Lambda(F \cap E_n) \leq \sum_{n=1}^{\infty} \sigma(E_n)$, and since $F$ was arbitrary, $\sigma(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \sigma(E_n)$.
\[ \sum_{n=1}^{\infty} \sigma(E_n), \text{ and } \sigma \text{ is countably sub-additive. It is clear from the definition that } \sigma \text{ is left-translation-invariant. There are sets on which } \Lambda \text{ is infinite and } \sigma \text{ is not; for instance, any vertical line segment has infinite } \Lambda \text{-measure and zero } \sigma \text{-measure.} \]

The relationship between measures such as \( \Lambda \) and \( \sigma \) above has been discussed by Berberian [1], and a theorem concerning this relationship is given below.

In what follows, \((X, \mathcal{M}, \Lambda)\) represents a measure space, where \(\mathcal{M}\) is a sigma-algebra of subsets of \(X\) and \(\Lambda\) is a measure on \(\mathcal{M}\).

**Definition 5.2.**

A set \(S\) in \(\mathcal{M}\) is said to have the finite-subset property if there is a set \(A\) in \(\mathcal{M}\) with \(A \subseteq S\) and \(0 < \Lambda(A) < \infty\).

**Definition 5.3.**

A set \(S\) in \(\mathcal{M}\) is said to be locally-null if for every set \(F\) in \(\mathcal{M}\) with \(\Lambda(F) < \infty\), \(\Lambda(S \cap F) = 0\).

**Theorem 5.4.**

Let \((X, \mathcal{M}, \Lambda)\) be a measure space. Then there are measures \(\Lambda_c\) and \(\Lambda_\infty\) on \(\mathcal{M}\) such that

i) \(\Lambda = \Lambda_c + \Lambda_\infty\),

ii) for \(E \in \mathcal{M}\), \(\Lambda_\infty(E) = 0 \text{ or } \Lambda_\infty(E) = \infty\),

iii) for \(E \in \mathcal{M}\), \(\Lambda_c(E) = \infty\) if and only if \(\sup\{\Lambda(F) : F \in \mathcal{M}, F \subseteq E, \Lambda(F) < \infty\} = \infty\), and
iv) $\Lambda_c$ is unique.

Proof:

Let $\Lambda_c$ and $\Lambda_\infty$ be defined on $\mathcal{M}$ as follows: For $E \in \mathcal{M}$, let

$$\Lambda_c(E) = \sup\{\Lambda(F): F \in \mathcal{M}, F \subseteq E, \Lambda(F) < \infty\},$$

$$\Lambda_\infty(E) = \begin{cases} \infty & \text{if } E \text{ contains a non-null locally-null set,} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\Lambda_c(\emptyset) = 0$ and $\Lambda_\infty(\emptyset) = 0$, so to show that $\Lambda_c$ and $\Lambda_\infty$ are measures it suffices to show that they are countably additive. Note that $\Lambda_c(E) \leq \Lambda(E)$ for $E \in \mathcal{M}$, and that if $\Lambda(E) < \infty$, then $\Lambda(E) = \Lambda_c(E)$, and $\Lambda_\infty(E) = 0$.

Let $(E_n)_{n=1}^\infty$ be a sequence of pairwise disjoint sets in $\mathcal{M}$. First assume $\Lambda(E_n) < \infty$ for all $n$. Then $\sum_{n=1}^\infty \Lambda_c(E_n) = \sum_{n=1}^\infty \Lambda(E_n)$.

If $\sum_{n=1}^\infty \Lambda(E_n) < \infty$, we have $\sum_{n=1}^\infty \Lambda_c(E_n) = \sum_{n=1}^\infty \Lambda(E_n) = \Lambda(\bigcup_{n=1}^\infty E_n) = \Lambda(\bigcup_{n=1}^\infty E_n)$.

If $\sum_{n=1}^\infty \Lambda(E_n) = \infty$, then for all $m$, $\sum_{n=1}^m \Lambda(E_n) = \sum_{n=1}^m \Lambda_c(E_n) = \Lambda_c(\bigcup_{n=1}^m E_n) < \infty$.

and $\lim_{m \to \infty} \Lambda_c(\bigcup_{n=1}^m E_n) = \lim_{m \to \infty} \sum_{n=1}^m \Lambda_c(E_n) = \Lambda_c(\bigcup_{n=1}^\infty E_n) = \infty$. Since $\Lambda_c(\bigcup_{n=1}^m E_n) \geq \Lambda_c(\bigcup_{n=1}^m E_n)$ for all $m$, we have $\sum_{n=1}^\infty \Lambda_c(E_n) = \sum_{n=1}^\infty \Lambda_c(E_n)$, for the case where $\Lambda(E_n) < \infty$ for all $n$.

Next assume for some $n$, $\Lambda(E_n) = \infty$. Let $F \subseteq \bigcup_{n=1}^\infty E_n$, $F \in \mathcal{M}$, $\Lambda(F) < \infty$. Then $\Lambda(F \cap E_n) < \infty$ for all $n$, and

$$\Lambda(F) = \sum_{n=1}^\infty \Lambda(F \cap E_n) \leq \sum_{n=1}^\infty \Lambda_c(E_n),$$

so, since $F$ is arbitrary, $\sum_{n=1}^\infty \Lambda_c(E_n) \leq \sum_{n=1}^\infty \Lambda_c(E_n)$. Let $F_n \subseteq E_n$, $F_n \in \mathcal{M}$,
$\Lambda(F_n) < \infty$. Then $\sum_{n=1}^{m} \Lambda(F_n) = \Lambda(\bigcup F_n) \leq \Lambda_c(\bigcup F_n)$. Since this holds for arbitrary sets $F_n$ and for arbitrary positive integer $m$, we have

$$\sum_{n=1}^{\infty} \Lambda_c(E_n) \leq \sum_{n=1}^{\infty} \Lambda_c(E_n) = \Lambda_c(\bigcup E_n).$$

Hence $\Lambda_c$ is countably additive, and is a measure on $\mathcal{M}$. It is clear from the definition of $\Lambda_c$ that every set in $\mathcal{M}$ that has positive $\Lambda_c$-measure has the finite-subset property.

Now $\Lambda_\infty$ is countably additive, since, letting $(E_n)_{n=1}^{\infty}$ be a sequence of pairwise disjoint sets in $\mathcal{M}$, we have, first, if $\Lambda_\infty(\bigcup E_n) = 0$, there is no non-null set in $\bigcup E_n$, hence none in $E_n$ for any $n$, hence $\Lambda_\infty(E_n) = 0$ for every $n$, and $\sum_{n=1}^{\infty} \Lambda_\infty(E_n) = 0$. Second, if $\Lambda_\infty(\bigcup E_n) = \infty$, there is a locally-null non-null set $L \subset \bigcup E_n$. Then $L \cap E_n$ is locally-null for each $n$. If $L \cap E_n$ is null for each $n$, then $0 = \sum_{n=1}^{\infty} \Lambda(L \cap E_n) = \Lambda(\bigcup L \cap E_n) = \Lambda(L)$, contradicting the fact that $L$ is non-null. Hence for some $n$, $L \cap E_n$ is locally-null and non-null, and $\sum_{n=1}^{\infty} \Lambda_\infty(E_n) = \infty$. Hence $\Lambda_\infty$ is countably additive and is a measure.

For every $E \in \mathcal{M}$, $\Lambda(E) = \Lambda_c(E) + \Lambda_\infty(E)$, since first, if $\Lambda(E) < \infty$, we have $\Lambda_c(E) = \Lambda(E)$ and $\Lambda_\infty(E) = 0$. Second, let $\Lambda(E) = \infty$. If $\Lambda_\infty(E) = \infty$, then clearly $\Lambda(E) = \Lambda_c(E) + \Lambda_\infty(E)$. So assume $\Lambda_\infty(E) = 0$ and $\Lambda_c(E) = R < \infty$. If $R = 0$, then $E$ is locally-null and non-null, and
\( \Lambda_\alpha(E) = \infty \), a contradiction. For \( R > 0 \), there is a sequence of sets \( F_n \in M, F_n \subset E, \Lambda(F_n) > R - 1/n \). Then \( \Lambda(\bigcup F_n) > R - 1/m \) for all \( m \).

Now \( \Lambda(\bigcup F_n) \leq \Lambda_c(E) \) for all \( m \), hence \( \lim_{m \to \infty} \Lambda(\bigcup F_n) = \Lambda(\bigcup F_n) \leq \Lambda_c(E) \). Then \( \Lambda(E - \bigcup F_n) = \infty \). If for some \( F_0 \subset E - \bigcup F_n, 0 < \Lambda(F_0) / \infty < \infty, \) then \( \Lambda_c(E) \geq \Lambda(F_0) + \Lambda(E - \bigcup F_n) > R \), contradicting \( \Lambda_c(E) = R \).

So \( (E - \bigcup F_n) \) is a locally-null non-null subset of \( E \), and \( \Lambda_\infty(E) = \infty \), contradicting \( \Lambda_\alpha(E) = 0 \). Therefore we have \( \Lambda_c(E) = \infty \), and it has been shown that \( \Lambda = \Lambda_c + \Lambda_\infty \).

It remains to show that \( \Lambda_c \) is unique. Assume \( \Lambda_c + \Lambda_\infty = \Lambda = \sigma_c + \sigma_\infty \), where \( \Lambda_c \) and \( \sigma_c \) satisfy iii), and \( \Lambda_\infty \) and \( \sigma_\infty \) satisfy ii) in the statement of the theorem.

Assume \( \Lambda_c \neq \sigma_c \). Then there is a set \( E \in M \) such that \( \Lambda_c(E) \neq \sigma_c(E) \), and we may assume \( \Lambda_c(E) < \infty \).

If \( \Lambda_\infty(E) = 0 \), then \( \Lambda(E) = \Lambda_c(E) < \infty \), and \( \sigma_\infty(E) = 0 \), so \( \sigma_c(E) = \Lambda_c(E) \).

If \( \Lambda_\infty(E) = \infty \), then \( \Lambda_c(E) = \Lambda(E) = \Lambda(F_o), \) where \( F_o = \bigcup (F_n: \Lambda_c(E) < \Lambda(F_n) + 1/n, F_n \in M, F_n \subset E), \) and \( \Lambda_\infty(F_o) = 0, \sigma_\infty(F_o) = 0, \) and \( \sigma_c(F_o) = \Lambda_c(F_o) \). Since \( \Lambda_c(E - F_o) = \infty \), and \( \Lambda_c(E - F_o) = 0, E - F_o \) is locally-null, non-null, hence \( \sigma_c(E - F_o) = 0 \), and \( \sigma_c(E) = \Lambda_c(E) \), contradicting the assumption that \( \Lambda_c \neq \sigma_c \).

The measure \( \Lambda_\infty \) is not necessarily unique; consider the following case. Let \((X,M,\Lambda)\) be a non-sigma-finite measure space, assume \( \Lambda = \Lambda_c \),
and define $\Lambda_1$ as follows:

$$\Lambda_1(E) = \infty \text{ if } E \text{ is not sigma-finite, and}$$

$$\Lambda_1(E) = 0 \text{ if } E \text{ is sigma-finite.}$$

Then $\Lambda_1$ is clearly a measure, which assumes only the values 0 and $\infty$, and $\Lambda = \Lambda_c + \Lambda_1$. But also if $\Lambda_2(E) = 0$ for all $E \in \mathcal{M}$, then $\Lambda = \Lambda_c + \Lambda_2$, and $\Lambda_2 \neq \Lambda_1$.

It is of course only in non-sigma-finite spaces that Theorem 5.4 has any application.

* * * *

The following questions have arisen in the course of the research for this dissertation:

1) Does every left-continuous group satisfy Condition (I)?

2) Does every left-continuous group have an open subgroup which is a topological group?

3) If a left-continuous group satisfies Condition (I), does it then have an open subgroup that is a topological group?

4) Does there exist a separable metric space endowed with two Borel-regular Caratheodory outer measures, $\Lambda$ and $\sigma$, where $\Lambda$ is locally-finite, and $\sigma$ satisfies the 5-r condition, and for no point $x$ in $X$ is

$$0 < \liminf_{r \to 0} \frac{\Lambda(c(x,r))}{\sigma(c(x,r))} \leq \limsup_{r \to 0} \frac{\Lambda(c(x,r))}{\sigma(c(x,r))} < \infty ?$$
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