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OF CONTRACTION OPERATORS

DISSERTATION

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the Degree Doctor of Philosophy in the Graduate
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By

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The Ohio State University
1971

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ACKNOWLEDGMENT

I wish to express my deep gratitude towards my adviser, Professor Ulrich Krengel, for suggesting this problem and for his guidance and encouragement during my mathematical studies.
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INTRODUCTION

Throughout this paper \((X, \mathcal{M}, \mu)\) will represent a \(\sigma\)-finite measure space. \(L_1(X, \mathcal{M}, \mu)\) will represent the Banach space of equivalence classes of integrable functions with the norm, \(\|f\|_1 = \int_X |f| \, d\mu\). \(L_\infty(X, \mathcal{M}, \mu)\) will represent the Banach space of equivalence classes of essentially bounded functions with the norm, \(\|f\|_\infty = \text{ess. sup.} |f|\).

The results in this paper deal with semigroups of linear operators on the Banach space \(L_1(X, \mathcal{M}, \mu)\). A mapping \(T\) of \(L_1(X, \mathcal{M}, \mu)\) into \(L_1(X, \mathcal{M}, \mu)\) is a linear operator if for all \(f, g\) in \(L_1(X, \mathcal{M}, \mu)\) and all \(\alpha, \beta\) in \(\mathbb{R}\) (set of real numbers) we have that \(T(\alpha f + \beta g) = \alpha Tf + \beta Tg\). The norm of the operator \(T\) is defined by \(\|T\| = \sup_{f \in L_1(X, \mathcal{M}, \mu)} \|Tf\| / \|f\|_1\).

The operator \(T\) is called a contraction if \(\|T\| \leq 1\). \(T\) is called positive if for all \(f\) in \(L_1^+(X, \mathcal{M}, \mu) = \{f \in L_1(X, \mathcal{M}, \mu) : f \geq 0 \text{ a.e.}\}\) we have that \(Tf\) is in \(L_1^+(X, \mathcal{M}, \mu)\). The family of operators \(\{T(t) : t \geq 0\}\) is called a one-parameter semigroup of operators on \(L_1(X, \mathcal{M}, \mu)\) if the following two conditions hold,

(i) \(T(t)\) is a linear operator on \(L_1(X, \mathcal{M}, \mu)\) for each \(t \geq 0\)

(ii) \(T(t)T(s) = T(t+s)\) holds for all \(t, s \geq 0\).
The family of operators \( (T(t_1, \ldots, t_N); t_i \geq 0, i=1, \ldots, N) \) is called an \textbf{N-parameter semigroup} of operators on \( L_1(X, M, \lambda) \) if the following two conditions hold,

(i') \( T(t_1, \ldots, t_N) \) is a linear operator on \( L_1(X, M, \lambda) \) for all \( (t_1, \ldots, t_N), t_i \geq 0, i=1, \ldots, N \),

(ii') \( T(t_1, \ldots, t_N)T(s_1, \ldots, s_N) = T(t_1+s_1, \ldots, t_N+s_N) \) for all \( (t_1, \ldots, t_N), (s_1, \ldots, s_N), t_i, s_i \geq 0, i=1, \ldots, N \).

The \textbf{N-parameter semigroup} \( (T(t_1, \ldots, t_N); t_i \geq 0, i=1, \ldots, N) \) is said to be \textbf{strongly continuous} if

\[
\|T(t_1, \ldots, t_N)f - T(s_1, \ldots, s_N)f\|_1 \rightarrow 0 \text{ as } (t_1, \ldots, t_N) \rightarrow (s_1, \ldots, s_N) \text{ holds for all } (s_1, \ldots, s_N), s_i \geq 0, i=1, \ldots, N, \text{ and for all } f \text{ in } L_1(X, M, \lambda).
\]

A transformation, \( T \), of a measure space, \((X, M, \lambda)\), into itself is called \textbf{measurable} if for all sets \( B \) in \( M \), \( T^{-1}B \) is also in \( M \). Such a transformation is called \textbf{measure preserving} if for all sets \( B \) in \( M \), \( \lambda(T^{-1}B) = \lambda(B) \). A measure preserving transformation is said to be \textbf{invertible} if its inverse transformation, \( T^{-1} \), is also a measure preserving transformation of \((X, M, \lambda)\) into itself. A family of invertible measure preserving transformations on \((X, M, \lambda)\), \( (T(t_1, \ldots, t_N); -\infty < t_i < \infty, i=1, \ldots, N) \), is called a \textbf{group} if \( T(t_1, \ldots, t_N)T(s_1, \ldots, s_N) = T(t_1+s_1, \ldots, t_N+s_N) \) for all \( (t_1, \ldots, t_N), (s_1, \ldots, s_N), -\infty < t_i, s_i < \infty, i=1, \ldots, N \). A group of invertible measure preserving transformations on \((X, M, \lambda)\),
\((T(t_1, \ldots, t_N); -\infty < t_i < \infty, i=1, \ldots, N)\), is said to be strongly measurable if for each \(f\) in \(L_1(X, \mathcal{M}, \lambda)\) and for almost every \(w\) in \(X\), \(f(T(t_1, \ldots, t_N)w)\) is a Lebesgue measurable function of \((t_1, \ldots, t_N)\).

In 1939, Norbert Wiener [7], presented a paper in which there appeared the following two local limit theorems for groups of invertible measure preserving transformations of a finite measure space.

**Theorem 0.1 [7, Theorem III']** Let \((X, \mathcal{M}, \lambda)\) be a finite measure space, \((T(t); -\infty < t < \infty)\) a strongly measurable group of invertible measure preserving transformations on \((X, \mathcal{M}, \lambda)\). Then for all \(f\) in \(L_1(X, \mathcal{M}, \lambda)\)

\[
f(w) = \lim_{\alpha \to 0^+} \frac{1}{\alpha} \int_0^\alpha f(T(t)w) \, dt \quad \text{a.e. } [\lambda].
\]

**Theorem 0.2 [7, Theorem III''']** Let \((X, \mathcal{M}, \lambda)\) be a finite measure space, \((T(t_1, \ldots, t_N); -\infty < t_i < \infty, i=1, \ldots, N)\) a strongly measurable \(N\)-parameter group of invertible measure preserving transformations on \((X, \mathcal{M}, \lambda)\). Then for all \(f\) in \(L_1(X, \mathcal{M}, \lambda)\)

\[
f(w) = \lim_{\alpha \to 0^+} \frac{1}{V(\alpha)} \int_0^{\alpha^2} \cdots \int_0^{\alpha^2} f(T(t_1, \ldots, t_N)w) \, dt_1 \cdots dt_N \quad \text{a.e. } [\lambda],
\]

where \(V(\alpha)\) is the volume of a sphere in \(N\)-space of radius \(\alpha\).

In 1969, U. Krengel and D. Ornstein independently
produced an operator theoretic generalization of Theorem 0.1. Their theorem is as follows:

**Theorem 0.3** [4], [5, Theorem 4.1] If \((T(t); t \geq 0)\) is a strongly continuous one-parameter semigroup of positive contraction operators on \(L_1(X, \mathcal{M}, \lambda)\), then for all \(f\) in \(L_1(X, \mathcal{M}, \lambda)\)

\[
T(0)f = \lim_{\alpha \to 0^+} \frac{1}{\alpha} \int_0^{\alpha} T(t)f \, dt \quad \text{a.e.} \ [\lambda].
\]

The main objective of this paper is to present two operator theoretic generalizations of Theorem 0.2. One of the theorems will be the extension of Theorem 0.3 to the general \(N\)-parameter case.

In section 1 I will make some comments on the proof of Theorem 0.3 and discuss the difficulties involved in changing from the one-parameter case to the general \(N\)-parameter case. Also in section 1 I will present the first operator theoretic generalization of Theorem 0.2. This theorem will have some stronger restrictions than those of Theorem 0.3. However, it will also apply to certain types of semigroups of non-positive operators. In section 2 I will present another operator theoretic generalization of Theorem 0.2. This theorem will also be the extension of Theorem 0.3 to the case of \(N\)-parameter semigroups. Then finally in section 3 I will give a variation of the theorem appearing in section 2 using a more general form of averaging.

Throughout the paper all equations and inequalities
are assumed to hold only almost everywhere unless specifically indicated otherwise. The important steps and equations are numbered consecutively throughout the paper.
SECTION 1

Let \(( T(t_1, \ldots, t_N); t_i \geq 0, i=1, \ldots, N )\) be a semigroup of contraction operators on \(L_1(X, \mu, \lambda)\). Throughout this paper we will be concerned with operator averages of the form

\[
(1) \quad \mathcal{M}(\alpha)f(w) = \frac{1}{\alpha} \int_0^\alpha \int_0^\alpha \cdots \int_0^\alpha T(t_1, \ldots, t_N)f(w) \, dt_1 \cdots dt_N,
\]

where \(f\) is in \(L_1(X, \mu, \lambda)\). The first fact that must be justified is the validity of the integration in (1). It is the case that strong continuity of the semigroup is a sufficient condition for choosing \(T(t_1, \ldots, t_N)f\) from its equivalence classes in such a way that for almost every \(w\) in \(X\), \(T(t_1, \ldots, t_N)f(w)\) is Lebesgue measurable in \((t_1, \ldots, t_N)\). This then justifies the validity of the Lebesgue integrals in (1). We present this result in the following proposition.

**Proposition 1.1** Let \(( T(t_1, \ldots, t_N); t_i \geq 0, i=1, \ldots, N )\) be a strongly continuous semigroup of contraction operators on \(L_1(X, \mu, \lambda)\). Then given \(f\) in \(L_1(X, \mu, \lambda)\) we can change \(T(t_1, \ldots, t_N)f\), for each \((t_1, \ldots, t_N)\) in \(\mathbb{R}_N^+ = ( (t_1, \ldots, t_N) : t_i \geq 0, i=1, \ldots, N )\), on a set of measure zero in such a way that \(T(t_1, \ldots, t_N)f(w)\) is a measurable function on \(X \times \mathbb{R}_N^+\).
proof: (The proof given here is a modification of a proof given by Ornstein [5, Proposition 4.1])

Let \( F_n(t_1, \ldots, t_N, w) = T(\frac{i_1}{n}, \ldots, \frac{i_N}{n}) f(w) \) for \( 0 \leq \frac{i_j-1}{n} \leq \frac{i_j}{n} \), \( j = 1, \ldots, N \). Now for each \( (t_1, \ldots, t_N) \) in \( \mathbb{R}^N_+ \) fixed we have

\[
(2) \quad \| F_n(t_1, \ldots, t_N, w) - T(t_1, \ldots, t_N) f(w) \|_1 \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.
\]

Moreover, the convergence in (2) is uniform in \( (t_1, \ldots, t_N) \). Thus there exist a subsequence \( n^*_i \) such that

\[
(3) \quad F_{n^*_i}(t_1, \ldots, t_N, w) \longrightarrow T(t_1, \ldots, t_N) f(w) \quad \text{both a.e.} \quad [\lambda]
\]

and in \( L_1(X, \mathcal{M}, \lambda) \).

The uniformity of the convergence in (2) also gives us that

\[
(4) \quad F_n(t_1, \ldots, t_N, w) \longrightarrow T(t_1, \ldots, t_N) f(w) \quad \text{in} \quad L_1 \text{ of } [0,1]^N \times X.
\]

So the subsequence \( n^*_i \) can be chosen such that besides (3) we also have

\[
(5) \quad F_{n^*_i}(t_1, \ldots, t_N, w) \text{ converges in } L_1 \text{ of } [0,1]^N \text{ for a.e. } w.
\]

The proposition now follows by changing
\[
T(t_1, \ldots, t_N) f(w) \quad \text{to} \quad \lim_{i \to \infty} F_{n^*_i}(t_1, \ldots, t_N, w).
\]

We are now ready to consider the generalization of Theorem 0.2. What we want to do is find conditions on the semigroup \( \{ T(t_1, \ldots, t_N) : t_i \geq 0, i = 1, \ldots, N \} \) under which \( M(\alpha) f \) converges almost everywhere as \( \alpha \to 0^+ \) for all \( f \) in \( L_1(X, \mathcal{M}, \lambda) \). In the proof of Theorem 0.3, as in many other ergodic theorems, the technique of proof was to establish the result for a dense subclass of functions and then to
apply a maximal type lemma or inequality. The establishment of this maximal lemma is usually the most difficult part of the proof. For example, Krengel's proof of Theorem 0.3 was based on the following continuous time version of Hopf's Maximal Ergodic Theorem.

**Lemma [4]** If \(( T(t); t \geq 0 )\) is a strongly continuous one-parameter semigroup of positive contraction operators on \(L^1(X, \mathcal{M}, \mu)\), then for any \(f\) in \(L^1(X, \mathcal{M}, \mu)\)

\[
\int_A T(0)f d\mu \geq 0
\]

where \(A = \{ w : \sup_{\alpha > 0} \int_0^\alpha \int \cdots \int T(t_1, \ldots, t_N)f(w) dt_1 \cdots dt_N > 0 \}\).

One might expect that the methods used in the proof of the one-parameter case could be directly extended to the \(N\)-parameter case. However, this could not be done since this maximal ergodic lemma does not extend to the \(N\)-parameter case. To be more precise, to use Krengel's technique of proof we would need to have that if \(( T(t_1, \ldots, t_N); t_i \geq 0 )\) is a strongly continuous semigroup of positive contractions on \(L^1(X, \mathcal{M}, \mu)\), then for all \(f\) in \(L^1(X, \mathcal{M}, \mu)\)

\[
\int_A T(0, \ldots, 0)f d\mu \geq 0
\]

where \(A = \{ w : \sup_{\alpha > 0} \int_0^\alpha \cdots \int T(t_1, \ldots, t_N)f(w) dt_1 \cdots dt_N > 0 \}\).

However, we can easily show this to be a false statement in the case \(N > 1\). Consider the following simple counterexample. Let \(X = \mathbb{R}_2, \mathcal{M} = \text{Borel } \sigma\text{-field of } \mathbb{R}_2, \text{ and}

\( \lambda = \text{Lebesgue measure on } \mathbb{R}_2 \). Then let \(( T(t_1,t_2), t_i \geq 0, i=1,2 \) be the semigroup of two dimensional translations. Then define

\[
\begin{align*}
\lambda &= 3 \text{ on } [0,1) \times [0,1) \\
&= -2 \text{ on } [1,2) \times [0,1) \\
&= -2 \text{ on } [0,1) \times [1,2) \\
&= 0 \text{ otherwise}.
\end{align*}
\]

Then \( f \) is in \( L^1(\lambda, \mu \times \lambda) \). However, for all \((x,y)\) in

\(( [0,1) \times [1,2) ) \cup ( [1,2) \times [0,1) )

\[
\int_0^2 \int_0^2 T(t_1,t_2) f(x,y) \, dt_1 dt_2 > 0
\]

Thus,

\[
A = \left\{ (x,y) : \sup_{\alpha > 0} \int_0^\alpha \int_0^\alpha T(t_1,t_2) f(x,y) \, dt_1 dt_2 \right\}
\]

\[
\subseteq ( [0,1) \times [0,1) ) \cup ( [0,1) \times [1,2) ) \cup ( [1,2) \times [0,1) )
\]

and thus

\[
\int_A f \, d\lambda = \int_{A + (0,0)} f \, d\lambda = -1 < 0.
\]

Since Krengel's maximal ergodic lemma does not extend to the general \( N \)-parameter case, one might look for other maximal ergodic type inequalities which do hold for \( N \)-parameter semigroups. In Wiener's paper there appears such a lemma for point transformations \([7, \text{Theorem IV'}]\) which Dunford and Schwartz generalized to the operator case \([3, \text{lemma 11}]\). Their lemma is as follows:

**Lemma 1.1** \([3]\) Let \(( T(t_1,\ldots,t_N), t_i \geq 0, i=1,\ldots,N )\) be a strongly continuous \( N \)-parameter semigroup of contraction
operators on $L^1(X,M,\lambda)$ which also satisfy

\[(*) \|T(t \ldots , t) f\|_\infty \leq \|f\|_\infty \] for all $f$ in $L^1 \cap L^\infty$ and all $t_i \geq 0, \ i=1, \ldots , N$.

then there exist an absolute constant $c_N > 0$ such that for all $f$ in $L^1(X,M,\lambda)$ and all $\beta > 0$

\[
\lambda(w) \sup_{\alpha > 0} \left| \frac{1}{N^\alpha} \int_0^\alpha \cdots \int_0^\alpha T(t_1, \ldots , t_N) f(w) \ dt_1 \cdots dt_N \right| > \beta
\]

\[
\leq \frac{1}{\beta c_N} \left( \lambda(w) \|f\|_{\geq \beta c_N} \right).
\]

It is with this lemma that we are able to prove a generalization of Theorem 0.2. However, before presenting this theorem I will need to prove another lemma which is needed in both this section and later sections.

**Lemma 1.2:** Let $(T(t_1, \ldots , t_N); t_i \geq 0, \ i=1, \ldots , N)$ be a strongly continuous semigroup of contraction operators on $L^1(X,M,\lambda)$. For each $f$ in $L^1(X,M,\lambda)$ and each $0 < \alpha < 1$ define the function

\[
m(\alpha)f = \frac{1}{\alpha^N} \int_0^\alpha \cdots \int_0^\alpha T(t_1, \ldots , t_N) f \ dt_1 \cdots dt_N.
\]

Then each $m(\alpha)f$ has the property that

\[
\lim_{\beta \to 0+} \frac{1}{\beta^N} \int_0^\beta \cdots \int_0^\beta T(t_1, \ldots , t_N) m(\alpha)f \ dt_1 \cdots dt_N = T(0, \ldots , 0)m(\alpha)f
\]

almost everywhere $[\lambda]$. Furthermore, the set of functions $m^* = \{ m(\alpha)f : 0 < \alpha < 1, \ f \ in \ L^1(X,M,\lambda) \}$ is dense in $T(0, \ldots , 0)L^1(X,M,\lambda)$. 
Proof: Since each $M(\alpha)f$ can be written as the limit of Riemann sums in the strong operator topology we have that $T(t_1, \ldots, t_N)M(\alpha)f = M(\alpha)(T(t_1, \ldots, t_N)f)$. We also have that $M(\alpha)(t(0, \ldots, 0)f) = M(\alpha)f$. Therefore, it follows that $T(0, \ldots, 0)M(\alpha)f = M(\alpha)(T(0, \ldots, 0)f) = M(\alpha)f$. Thus all $M(\alpha)f$ are in $T(0, \ldots, 0)L_1(X, M, \lambda)$.

The denseness of $M^*$ in $T(0, \ldots, 0)L_1(X, M, \lambda)$ results from the strong continuity of the semigroup at zero. By an application of Fubini's Theorem we have

$$M(\alpha)f - T(0, \ldots, 0)f \leq \frac{1}{\alpha N} \int_0^\alpha \int_0^\alpha T(t_1, \ldots, t_N)f \quad \text{for} \quad \alpha \to 0^+,$$

which converges to zero as $\alpha \to 0^+$.

Finally we need to show each $M(\alpha)f$ satisfies the local limit property. This is established as follows:

$$\left| \frac{1}{\beta N} \int_0^\beta \int_0^\beta T(t_1, \ldots, t_N)M(\alpha)f \, dt_1 \ldots dt_N - M(\alpha)f \right|$$

$$= \left| \frac{1}{\beta N} \int_0^\beta \int_0^\beta \int_0^\beta \int_0^\beta T(t_1 + s_1, \ldots, t_N + s_N)f \, ds_1 \ldots ds_N \, dt_1 \ldots dt_N \right|$$

$$= \left| \frac{1}{\beta N} \int_0^\beta \int_0^\beta \int_0^\beta \int_0^\beta T(s_1, \ldots, s_N)f \, ds_1 \ldots ds_N \, dt_1 \ldots dt_N \right|$$

$$= \left| \frac{1}{\beta N} \int_0^\beta \int_0^\beta \int_0^\beta \int_0^\beta T(r_1, \ldots, r_N)f \, dr_1 \ldots dr_N \, dt_1 \ldots dt_N \right|$$

$$- \left| \frac{1}{\beta N} \int_0^\beta \int_0^\beta \int_0^\beta \int_0^\beta T(r_1, \ldots, r_N)f \, dr_1 \ldots dr_N \, dt_1 \ldots dt_N \right|$$
\[
\begin{aligned}
&= \frac{1}{\beta^N} \frac{1}{\alpha^N} \int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_N}^{\beta_N} \left[ \int_{0}^{t_1} \int_{0}^{t_2} \int_{0}^{t_N} \tau(r_1, \ldots, r_N) \, dr_1 \cdots dr_N \\ &+ \int_{t_1}^{t_2} \int_{0}^{t_3} \int_{0}^{t_4} \tau(r_1, \ldots, r_N) \, dr_1 \cdots dr_N + \cdots \right] dt_1 \cdots dt_N \\
&- \frac{1}{\beta^N} \frac{1}{\alpha^N} \int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_N}^{\beta_N} \left[ \int_{0}^{t_1} \int_{0}^{t_2} \int_{0}^{t_N} \tau(r_1, \ldots, r_N) \, dr_1 \cdots dr_N \\ &+ \int_{t_1}^{t_2} \int_{0}^{t_3} \int_{0}^{t_4} \tau(r_1, \ldots, r_N) \, dr_1 \cdots dr_N + \cdots \right] dt_1 \cdots dt_N \\
&\leq \frac{1}{\beta^N} \frac{1}{\alpha^N} \int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_N}^{\beta_N} \left[ \int_{0}^{t_1} \int_{0}^{t_2} \int_{0}^{t_N} \tau(r_1, \ldots, r_N) \, dr_1 \cdots dr_N \\ &+ \int_{t_1}^{t_2} \int_{0}^{t_3} \int_{0}^{t_4} \tau(r_1, \ldots, r_N) \, dr_1 \cdots dr_N + \cdots \right] dt_1 \cdots dt_N
\end{aligned}
\]
Now it is easy to show that the last expression converges to zero almost everywhere as \( \beta \to 0^+ \), since by the integrability of \( T(r_1, \ldots, r_N)f \) we have

\[
\begin{align*}
&\int_{t_1}^{\alpha+t_1} \int_{t_2}^{\alpha+t_2} \int_{t_3}^{\alpha+t_3} \cdots \int_{t_N}^{\alpha+t_N} |T(r_1, \ldots, r_N)f| \, dr_1 \cdots dr_N \to 0 \quad \text{a.e.} \\
&\int_{t_1}^{\alpha+t_1} \int_{t_2}^{\alpha+t_2} \int_{t_3}^{\alpha+t_3} \cdots \int_{t_N}^{\alpha+t_N} |T(r_1, \ldots, r_N)f| \, dr_1 \cdots dr_N \to 0 \quad \text{a.e.} \\
&\int_{t_1}^{\alpha+t_1} \int_{t_2}^{\alpha+t_2} \int_{t_3}^{\alpha+t_3} \cdots \int_{t_N}^{\alpha+t_N} |T(r_1, \ldots, r_N)f| \, dr_1 \cdots dr_N \to 0 \quad \text{a.e.} \\
\end{align*}
\]
\[ \int_0^{t_1} \cdots \int_0^{t_{N-1}} \int_0^{t_N} |T(r_1, \ldots, r_N)f| dr_1 \cdots dr_N \to 0 \quad \text{a.e.} \]

\[ \vdots \]

\[ \int_0^{t_1} \cdots \int_0^{t_{N-1}} \int_0^{t_N} |T(r_1, \ldots, r_N)f| dr_1 \cdots dr_N \to 0 \quad \text{a.e.} \]

\[ \text{as } t_1 \to 0^+, \ i=1, \ldots, N. \]

We now have all the tools necessary to prove our first operator theoretic generalization of Theorem 0.2.

**Theorem 1.1**: Let \((T(t_1, \ldots, t_N); t_i \geq 0, \ i=1, \ldots, N)\) be a strongly continuous semigroup of contraction operators on \(L^1(X, \mu, \lambda)\) which also satisfies

\((*)\) \[ \|T(t_1, \ldots, t_N)f\|_\infty \leq \|f\|_\infty \]

for all \(f\) in \(L^1(X, \mu, \lambda) \cap L^\infty(X, \mu, \lambda)\) and all \(t_i \geq 0, \ i=1, \ldots, N,\) then for all \(f\) in \(L^1(X, \mu, \lambda)\)

\[ \lim_{\alpha \to 0^+} \frac{1}{\alpha^N} \int_0^\alpha \cdots \int_0^\alpha T(t_1, \ldots, t_N)f \ dt_1 \cdots dt_N = T(0, \ldots, 0)f \]

almost everywhere.

**Proof**: Suppose the theorem would not hold. Then there would exist an \(f\) in \(L^1(X, \mu, \lambda)\) such that

\((6)\) \[ \lambda \{ \omega : \lim_{\alpha \to 0^+} \sup M(\alpha) f(\omega) > T(0, \ldots, 0)f(\omega) \} > 0. \]

Hence, there would also be some \(b > 0\) such that
(7) \( \lambda[w: \limsup_{\alpha \to 0+} M(\alpha)f(w) > b + T(0,\ldots,0)f(w)] = \delta > 0 \).

Then choose an \( f_0 \) from \( M^* \) such that

(8) \( \|T(0,\ldots,0)f - f_0\|_1 < \min(\frac{\delta b c_N}{8}, \frac{\delta b}{4}) \) where \( c_N \) is as in Lemma 1.1. From (8) it follows that

(9) \( \lambda[w: |T(0,\ldots,0)f(w) - f_0(w)| \geq \frac{b}{2}] < \frac{\delta}{2} \), and

(10) \( \|T(0,\ldots,0)f - f_0\|_1 < \frac{\delta b c_N}{8} \).

By Lemma 1.1 we have that

(11) \( \lambda[w: \limsup_{\alpha \to 0+} M(\alpha)(f - f_0)(w) > \frac{b}{2}] \)

\[ \leq \lambda[w: \sup_{\alpha > 0} M(\alpha)(f - f_0)(w) > \frac{b}{2}] \]

\[ \leq \frac{2}{bc_N} \|T(0,\ldots,0)f - f_0\|_1 < \frac{\delta}{4} \).

Now (9), (10) and the properties of \( f_0 \) yield

(12) \( \lambda[w: \limsup_{\alpha \to 0+} M(\alpha)(f - f_0)(w) > \frac{b}{2}] \)

\[ = \lambda[w: \limsup_{\alpha \to 0+} M(\alpha)f(w) > \frac{b}{2} + f_0] \]

\[ \geq \lambda[w: \limsup_{\alpha \to 0+} M(\alpha)f(w) > b + T(0,\ldots,0)f(w)] \]

\[ - \lambda[w: T(0,\ldots,0)f(w) - f_0(w) \geq \frac{b}{2}] \]

\[ > \delta - \frac{\delta}{2} = \frac{\delta}{2} \).

But (11) and (12) yield a contradiction, thus proving the theorem.
It has been claimed that Theorem 1.1 is an operator theoretic generalization of Theorem 0.2. The extent of the generalization can be seen by considering the relationship between measure preserving transformations and contraction operators. If \( (T(t_1, \ldots, t_N); t_i \geq 0, i=1, \ldots N) \) is a semigroup of measure preserving point transformations of \( (X, m, \lambda) \), then \( (U(t_1, \ldots, t_N); t_i \geq 0, i=1, \ldots N) \), defined by \( U(t_1, \ldots, t_N)f(w) = f(T(t_1, \ldots, t_N)w) \), is a semigroup of operators on \( L_1(X, m, \lambda) \). Since all the transformations are measure preserving, all the operators have norm one. Also for each \( f \) in \( L_1 \cap L_\infty \), we also have
\[
\left\| U(t_1, \ldots, t_N)f(w) \right\|_\infty = \left\| f(T(t_1, \ldots, t_N)w) \right\|_\infty = \left\| f(w) \right\|_\infty.
\]
So semigroups of measure preserving point transformations are just a special case of Theorem 1.1.

In Theorem 1.1 we do not require the operators to be positive. However, we do require the condition (*), which seems to be very restrictive. Theorem 0.3 might lead us to hope that by restricting semigroups to positive operators we could eliminate the condition (*). In section 2 we will prove this conjecture. However, it will require a completely different type of proof. The reason that condition (*) is necessary in the proof of Theorem 1.1 is for the validity of Lemma 1.1. It is the case that even under the assumption that the operators are positive Lemma 1.1 is false without condition (*). I will give a very simple example to illustrate this point. Let \( X = [0, \infty) \),
$\mathcal{M}$ is Borel sets on $[0,\infty)$ and $\lambda$ is the measure which is absolutely continuous with respect to Lebesgue measure and has density $d(x) = \frac{1}{n+1}$ on $[n,n+1)$. Then define $(T(t); t \geq 0)$ on $L^1(X,\mathcal{M},\lambda)$ by

$$T(t)f(x) = \begin{cases} 0 & \text{for } 0 \leq x < t \\ f(x-t) \frac{d(x-t)}{d(x)} & \text{for } x \geq t. \end{cases}$$

Then $(T(t); t \geq 0)$ is a strongly continuous semigroup of positive contractions on $L^1(X,\mathcal{M},\lambda)$. Let $f(x) = 1_{[0,1)}(x)$ (i.e. $f(x)$ is the indicator function of $[0,1)$). Then we have $\|f\|_1 = 1$ and for any $n \leq x < n+1$

$$\frac{1}{x} \int_0^x T(t)f(x) \, dt \geq \frac{1}{x} \int_0^1 \frac{1}{d(x)} = \frac{1}{x} (n+1) \geq 1.$$

Therefore, we have

$$\lambda([\omega; \sup_{\alpha>0} \frac{1}{\alpha} \int_0^\alpha T(t)f(x) \, dt \geq 1]) = \lambda([0,\infty)) = \infty \geq \frac{1}{c} \|f\|_1$$

for all finite values of $c$. 
In this section we will be concerned with semi-groups of positive operators. Our main objective is to generalize Theorem 0.3 to the $N$-parameter case. Our technique in this section will be much different from the technique used to prove Theorem 1.1. We will prove our theorem by induction starting from the known result for one-parameter semigroups, i.e., Theorem 0.3. The crucial step in our proof is the reduction of a $2N$-parameter semigroup to an $N$-parameter semigroup. The method appearing here is basically a modification of a method used by Dunford and Schwartz in the proof of their Lemma 11[3].

We first must prove some lemmas to set up the machinery for the induction argument.

**Lemma 2.1.** For each $u > 0$ define the function

$$
\theta_u(x) = \begin{cases} 
\frac{u}{2\sqrt{\pi}} e^{-\frac{x^2}{4u}} & \text{if } x > 0 \\
0 & \text{if } x \leq 0
\end{cases}
$$

Then $(\theta_u, u > 0)$ has the following properties

(i) $\theta_u(x) \geq 0$ for all $x$ and for all $u > 0$.

(ii) $\int_{-\infty}^{\infty} \theta_u(x) \, dx = 1$ for all $u > 0$. 

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(iii) \((\theta_u; u > 0)\) is a semigroup under convolution,
(iv) for all \(s > 0\) \(\lim_{u \to 0^+} \int_s^\infty \theta_u(x) \, dx = 0\).

**proof:** This is essentially Dunford and Schwartz's lemma 12 [3, p. 160]. In their proof they verify (i), (ii) and (iii), so here we will just prove condition (iv).

For \(s > 0\) fixed, if \(u < 1\) then

\[
0 \leq \int_s^\infty \theta_u(x) \, dx = \frac{u}{2\sqrt{\pi}} \int_s^\infty x^{-3/2} e^{-u^2/4x} \, dx \leq \frac{u}{2\sqrt{\pi}} \int_s^\infty x^{-3/2} \, dx
\]

But \(\int_s^\infty x^{-3/2} \, dx < \infty\) so \(\frac{u}{2\sqrt{\pi}} \int_s^\infty x^{-3/2} \, dx \to 0\)
as \(u \to 0\). 

For technical reasons arising later in this section we will revise the requirements on "strongly continuous semigroups".

**Definition 2.1:** \((T(t_1,\ldots,t_N), T(0); t_i > 0, i=1,\ldots,N)\)
will be called a strongly continuous \(N\)-parameter semigroup
of positive contractions on \(L_1(X,\mu,\lambda)\) if
(i) \((T(t_1,\ldots,t_N); t_i > 0, i=1,\ldots,N)\) is a strongly continuous semigroup of positive contractions,
(ii) \(T(0)\) is a positive contraction on \(L_1(X,\mu,\lambda)\) such that
\(T(0)T(t_1,\ldots,t_N) = T(t_1,\ldots,t_N)T(0) = T(t_1,\ldots,t_N)\)
for all \((t_1,\ldots,t_N), t_i > 0, i=1,\ldots,N\)
and
(iii) for all \(f\) in \(L_1(X,\mu,\lambda)\) \(\|T(t_1,\ldots,t_N)f - T(0)f\|_1 \to 0\)
as \((t_1,\ldots,t_N) \to (0,\ldots,0)^+\).
Clearly our previous definition of strongly continuous semigroups of positive contractions meets the requirements of definition 2.1 (in which case $T(0) = T(0,...,0)$).

Lemma 2.2: Let $(\theta_u, u > 0)$ be as in Lemma 2.1, and let $(T(t_1, ..., t_{2N}), T(0) ; t_i > 0, i = 1, ..., 2N)$ be a strongly continuous 2N-parameter semigroup of positive contractions on $L_1(X, M, \lambda)$. Then for $x_i > 0, i = 1, ..., N$, define the operators

$$S(x_1, ..., x_N)f = \int_0^\infty \int_0^\infty \cdots \int_0^\infty \theta(t_1)x_1 \theta(t_2)x_2 \cdots \theta(t_{2N})x_{2N} f \, dt_1 \cdots dt_{2N}.$$ 

Then $(S(x_1, ..., x_N), T(0) ; x_i > 0, i = 1, ..., N)$ is a strongly continuous N-parameter semigroup of positive contractions on $L_1(X, M, \lambda)$.

**Proof:** The operators are positive since the $T(t_1, ..., t_{2N})$ are positive and $\theta_x(t) > 0$ for all $t > 0$ and all $x > 0$.

The operators are contractions since for all $f$ in $L_1(X, M, \lambda)$

$$\|S(x_1, ..., x_N)f\|_1 = \left\|\int_0^\infty \int_0^\infty \cdots \int_0^\infty \theta(t_1)x_1 \theta(t_2)x_2 \cdots \theta(t_{2N})x_{2N} f \, dt_1 \cdots dt_{2N}\right\|_1$$

$$\leq \int_0^\infty \int_0^\infty \cdots \int_0^\infty \theta(t_1)x_1 \cdots \theta(t_{2N})x_{2N} \|T(t_1, ..., t_{2N})f\|_1 \, dt_1 \cdots dt_{2N}$$

$$\leq \|f\|_1 \left[ \int_0^\infty \theta_x(t) \, dt \right]^{2N} = \|f\|_1.$$ 

To show that the operators form a semigroup we first observe that since $S(x_1, ..., x_N)f$ can be written as the limit of Riemann sums in the norm topology it
follows that
\[ T(t_1, \ldots, t_{2N})S(x_1, \ldots, x_N)f = S(x_1, \ldots, x_N)(T(t_1, \ldots, t_{2N})f). \]
Using this we obtain
\[ S(x_1, \ldots, x_N)S(y_1, \ldots, y_N)f \]
\[ = \int_0^\infty \cdots \int_0^\infty \theta_x(t_1) \theta_x(t_{2N}) \theta_y(s_1) \theta_y(s_{2N}) \]
\[ \cdot T(t_1 + s_1, \ldots, t_{2N} + s_{2N})f \cdot ds_1 \cdots ds_{2N} dt_1 \cdots dt_{2N} \]
\[ = \int_0^\infty \cdots \int_0^\infty \theta_x(v_1 - w_1) \theta_y(w_1) \theta_x(v_2 - w_2) \theta_y(w_2) \cdots \theta_x(v_{2N} - w_{2N}) \]
\[ \cdot \theta_y(w_{2N}) T(v_1, \ldots, v_{2N})f \cdot dv_1 \cdots dv_{2N} dw_1 \cdots dw_{2N} \]
\[ = \int_0^\infty \cdots \int_0^\infty \theta_x(v_1) \theta_x(v_2) \cdots \theta_x(v_{2N-1}) \theta_x(v_{2N}) \]
\[ \cdot T(v_1, \ldots, v_{2N})f \cdot dv_1 \cdots dv_{2N} \]
\[ = S(x_1 + y_N, \ldots, x_1 + y_N)f \cdot \]

To prove the strong continuity of the semigroup we distinguish two cases. First let \( y_i > 0, i = 1, \ldots, N \). Then for \( f \) in \( L_1(X, \mathbb{M}, \lambda) \)
\[ \| S(x_1, \ldots, x_N)f - S(y_1, \ldots, y_N)f \|_1 \]
\[ = \| \int_0^\infty \cdots \int_0^\infty \left[ \theta_x(t_1) \cdots \theta_x(t_{2N}) - \theta_y(t_1) \cdots \theta_y(t_{2N}) \right] \]
\[ T(t_1, \ldots, t_{2N})f \cdot dt_1 \cdots dt_{2N} \|_1 \]
\[ \leq \| f \|_1 \int_0^\infty \cdots \int_0^\infty \left| \theta_x(t_1) \cdots \theta_x(t_{2N}) - \theta_y(t_1) \cdots \theta_y(t_{2N}) \right| dt_1 \cdots dt_{2N} \]
But since each \( y_i > 0 \), \( \theta_x(t) \rightarrow \theta_y(t) \) as \( x_i \rightarrow y_i \)
uniformly in \( t \), and moreover for any fixed \( y > 0 \) there
exists a $K_i > 0$ such that $\int_{K_i} y_i(t) \, dt < \gamma$. This is sufficient to give that

$$\int_0^\infty \int_0^\infty \left| \theta_{x_1}(t_1) \cdots \theta_{x_N}(t_{2N}) - \theta_{y_1}(t_1) \cdots \theta_{y_N}(t_{2N}) \right| \, dt_1 \cdots dt_{2N}$$

converges to zero as $(x_1, \ldots, x_N) \rightarrow (y_1, \ldots, y_N)$. Thus $\| S(x_1, \ldots, x_N)f - S(y_1, \ldots, y_N)f \|_1 \rightarrow 0$ as $(x_1, \ldots, x_N) \rightarrow (y_1, \ldots, y_N)$.

Finally, we must show strong continuity at zero.

So let $f$ be in $L_1(X, M, \lambda)$ and $\gamma > 0$ fixed. Then by the strong continuity of the semigroup $(T(t_1, \ldots, t_{2N}), T(0); t_i > 0, i=1, \ldots, 2N)$ there exists an $s > 0$ such that for $\gamma < t_i < s$, $i=1, \ldots, 2N$, we have

$$(13) \left\| T(t_1, \ldots, t_{2N})f - T(0)f \right\|_1 < \frac{\gamma}{2}.$$  

By Lemma 2.1 $\int_0^\infty \theta_{x_i}(t) \, dt \rightarrow 0$ as $x_i \rightarrow 0$, thus there exist an $r > 0$ such that if $x_i < r$ then

$$(14) \int_0^\infty \theta_{x_i}(t) \, dt < \frac{\gamma}{8N \| f \|_1}.$$  

Therefore, for $0 < x_i < r$, $i=1, \ldots, N$, we have

$$\left\| S(x_1, \ldots, x_N)f - T(0)f \right\|_1$$

$$\leq \left\| \int_0^\infty \int_0^\infty \left( \theta_{x_1}(t_1) \cdots \theta_{x_N}(t_{2N}) - \theta_{y_1}(t_1) \cdots \theta_{y_N}(t_{2N}) \right) T(t_1, \ldots, t_{2N})f \, dt_1 \cdots dt_{2N} \right\|_1$$

$$\leq \int_0^\infty \int_0^\infty \left( \theta_{x_1}(t_1) \cdots \theta_{x_N}(t_{2N}) \right) T(t_1, \ldots, t_{2N})f - T(0)f \, dt_1 \cdots dt_{2N}$$
The last inequalities follow from (13) and (14). So we now also have that \( \| S(x_1, \ldots, x_N)f - T(0)f \|_1 \rightarrow 0 \) as \( (x_1, \ldots, x_N) \rightarrow (0, \ldots, 0)^+ \).

The next lemma gives us the important relationship between the original semigroup and the derived semigroup. Although this lemma does not appear explicitly in Dunford and Schwartz [3], the idea was suggested by their proof of lemma 13 [3, p. 161].

**Lemma 2.3:** Let \( (T(t_1, \ldots, t_{2N}), \ T(0); t_i > 0, i=1, \ldots, 2N) \) be a strongly continuous semigroup of positive contractions on \( L_1(X, M, \lambda) \) and let \( (S(x_1, \ldots, x_N), \ T(0); x_i > 0, i=1, \ldots, 2N) \) be the semigroup derived in Lemma 2.2. Then there exists an absolute constant \( d > 0 \), which depends only on the size of \( N \), such that for all \( f \) in \( L_1^+(X, M, \lambda) \) and any \( \alpha > 0 \) we have that

\[
\frac{1}{(\sqrt{\alpha})^N} \int_0^{\sqrt{\alpha}} \cdots \int_0^{\sqrt{\alpha}} S(x_1, \ldots, x_N)f \ dx_1 \cdots dx_N \geq \frac{d}{\alpha^{2N}} \int_0^{\alpha} \cdots \int_0^{\alpha} T(t_1, \ldots, t_{2N})f \ dt_1 \cdots dt_{2N}. \]
proof:

\[
\frac{1}{(\sqrt{\alpha})^N} \int_0^{\sqrt{\alpha}} \cdots \int_0^{\sqrt{\alpha}} S(x_1, \ldots, x_N) \, dx_1 \cdots dx_N
\]

\[
= \frac{1}{(4\pi)^N} \int_0^\infty \cdots \int_0^\infty h(\sqrt{\alpha}, t_1, t_2) \cdots h(\sqrt{\alpha}, t_{2N-1}, t_{2N}) \, T(t_1, \ldots, t_{2N}) \, dt_1 \cdots dt_{2N}
\]

where

\[
h(\sqrt{\alpha}, t_1, t_{i+1}) = \frac{4\pi}{\sqrt{\alpha}} \int_0^{\sqrt{\alpha}} \theta(t_i) \theta(t_{i+1}) \, du
\]

\[
= \frac{1}{\sqrt{\alpha}} (t_i t_{i+1})^{-\frac{3}{2}} \int_0^{\sqrt{\alpha}} u e^{-u^2 \left( \frac{1}{4t_i} + \frac{1}{4t_{i+1}} \right)} \, du.
\]

I shall now show that there exists a \( \delta > 0 \) such that

\[(16) \quad h(\sqrt{\alpha}, t_1, t_{i+1}) > \frac{\delta}{\alpha^2} \quad \text{for all } 0 < t_1, t_{i+1} < \alpha.\]

It is clearly sufficient to work with the case \( t_1 = t_1, t_{i+1} = t_2 \). By making the change of variables \( u = v\sqrt{\alpha}, t_1 = s_1\sqrt{\alpha}, t_2 = s_2\sqrt{\alpha} \), (16) becomes equivalent to

\[(17) \quad (s_1 s_2)^{-\frac{3}{2}} \int_0^{\sqrt{s_1} + \sqrt{s_2}} \int_0^{1} v e^{-v^2 \left( \frac{1}{4s_1} + \frac{1}{4s_2} \right)} \, dv > \delta \quad \text{for } 0 < s_1, s_2 < 1.\]

Let \( H(a) = \int_0^{1} v e^{-v^2 a} \, dv \). Then what we are trying to show is that

\[(s_1 s_2)^{-\frac{3}{2}} H(\frac{1}{4s_1} + \frac{1}{4s_2}) > \delta \quad \text{for } 0 < s_1, s_2 < 1.\]

Now \( H(a) \) is positive and continuous, so if \( s_1 \) and \( s_2 \) are bounded away from zero then so is \( H(\frac{1}{4s_1} + \frac{1}{4s_2}) \). So our only problem is when \( s_1 \) or \( s_2 \) is close to zero.
But \( H(a) = \int_{0}^{a} v e^{-v} dv = \frac{a^{3/2}}{2} \int_{0}^{v^{2}} e^{-v} dv \).

We can choose \( a, b > 0 \) such that for all \( a > b \)

\[
0 < \frac{\Gamma(3/2)}{2} \leq \int_{0}^{a} v^{b} e^{-v} dv \leq \Gamma(3/2).
\]

Then for all \( a > b \) we have

\[
H(a) \geq M a^{-3/2}
\]

where \( M = \frac{\Gamma(3/2)}{4} \).

Therefore, for \( \frac{1}{4s_1} + \frac{1}{4s_2} > b \) we have

\[
(s_1, s_2)^{-3/2} H\left( \frac{1}{4s_1} + \frac{1}{4s_2} \right) \geq 8M(s_1 + s_2)^{-3/2}
\]

which is bounded away from zero if \( s_1 \) or \( s_2 \) is close to zero. So the existence of \( \delta \) is clear, in fact we could use

\[
\delta = \min\left\{ \frac{\Gamma(3/2)}{2}, \frac{1}{2b^{3}} \int_{0}^{1/2} v e^{-v} dv \right\}.
\]

Now we have that

\[
\frac{1}{(\sqrt{\alpha})^{N}} \int_{0}^{\alpha} \cdots \int_{0}^{\alpha} S(x_1, \ldots, x_N)f \, dx_1 \cdots dx_N
\]

\[
= \frac{1}{(4\pi)^N} \int_{0}^{\alpha} \cdots \int_{0}^{\alpha} h(\sqrt{\alpha}, t_1, t_2) \cdots h(\sqrt{\alpha}, t_{2N-1}, t_{2N})
\]

\[
T(t_1, \ldots, t_{2N})f \, dt_1 \cdots dt_{2N}
\]

\[
\geq \frac{1}{(4\pi)^N} \int_{0}^{\alpha} \cdots \int_{0}^{\alpha} h(\sqrt{\alpha}, t_1, t_2) \cdots h(\sqrt{\alpha}, t_{2N-1}, t_{2N})
\]

\[
T(t_1, \ldots, t_{2N})f \, dt_1 \cdots dt_{2N}
\]

\[
\geq \left( \frac{\delta}{4\pi} \right)^N \frac{1}{\alpha^{2N}} \int_{0}^{\alpha} \cdots \int_{0}^{\alpha} T(t_1, \ldots, t_{2N})f \, dt_1 \cdots dt_{2N}.
\]

Thus the lemma is proved with \( d = \left( \frac{\delta}{4\pi} \right)^N \).
We are now ready to prove our generalization of Theorem 0.3. In the following proof \( M^* \) will again refer to the class of functions defined in Lemma 1.2.

**Theorem 2.1:** Let \((T(t_1, \ldots, t_N), T(0); t_i > 0, i=1, \ldots, N)\) be a strongly continuous semigroup of positive contraction operators on \( L^1(X, M, \lambda) \). Then for all \( f \) in \( L^1(X, M, \lambda) \)

\[
\lim_{\alpha \to 0+} \frac{1}{\alpha^N} \int_0^\infty \cdots \int_0^\infty T(t_1, \ldots, t_N)f \, dt_1 \cdots dt_N = T(0)f
\]

almost everywhere.

**proof:** First let us notice that if \((T(t_1, \ldots, t_k), T(0); t_i > 0, i=1, \ldots, k)\) satisfies the hypotheses of the theorem and if \( 1 \leq k \leq m \) and \( P(t_1, \ldots, t_m) = T(t_1, \ldots, t_k) \) then \((P(t_1, \ldots, t_m), T(0); t_i > 0, i=1, \ldots, m)\) also satisfies the hypotheses of the theorem, and furthermore

\[
\frac{1}{\alpha^m} \int_0^\infty \cdots \int_0^\infty P(t_1, \ldots, t_m) \, dt_1 \cdots dt_m
\]

\[
= \frac{1}{\alpha^k} \int_0^\infty \cdots \int_0^\infty T(t_1, \ldots, t_k) \, dt_1 \cdots dt_k.
\]

So if the theorem is known to hold for an integer \( m \), then it also holds for all integers \( 1 \leq k \leq m \). Therefore, it will suffice to show the theorem to hold for all integer of the form \( N = 2^k, k \geq 0 \).

I now proceed by induction. The theorem is known to hold for \( N = 1 \) by virtue of Theorem 0.3. So suppose it holds for \( N = 2^k \). Then I shall show it holds for \( N \) replaced by \( 2N \).
Let \((T(t_1, \ldots, t_{2N}), T(0); t_i > 0, i=1, \ldots, 2N)\) be a 2N-parameter semigroup satisfying the hypotheses of the theorem. Then construct \((S(x_1, \ldots, x_N), T(0); x_i > 0, i=1, \ldots, N)\) as in Lemma 2.2. Then by Lemma 2.3 and the induction assumption we have that for all \(f \) in \(L^+_1(X, \mathcal{M}, \lambda)\)

\[
T(0)f = \lim_{\alpha \to 0^+} \frac{1}{\alpha^N} \int_0^\alpha \cdots \int_0^\alpha S(x_1, \ldots, x_N)f \, dx_1 \cdots dx_N
\]

\[
\geq d \limsup_{\alpha \to 0^+} \frac{1}{\alpha^{2N}} \int_0^\alpha \cdots \int_0^\alpha T(t_1, \ldots, t_{2N})f \, dt_1 \cdots dt_{2N}
\]

\[
= d \limsup_{\alpha \to 0^+} M(\alpha)f.
\]

Thus for all \(f \) in \(L^+_1(X, \mathcal{M}, \lambda)\) we must have

\[
\lambda \{w: \limsup_{\alpha \to 0^+} M(\alpha)f(w) > \frac{1}{d} T(0)f(w)\} = 0.
\]

Suppose now that the result does not hold. Then there would exist some \(f \) in \(L^+_1(X, \mathcal{M}, \lambda)\) such that

\[
\lambda \{w: \limsup_{\alpha \to 0^+} M(\alpha)f(w) > T(0)f(w)\} > 0.
\]

Hence, we could find a \(\beta > 0\) such that

\[
\lambda \{w: \limsup_{\alpha \to 0^+} M(\alpha)f(w) > \beta + T(0)f(w)\} = b > 0.
\]

Let \(\delta > 0\) and choose \(f_0\) in \(\mathcal{M}^*\) such that

\[
\|T(0)f - f_0\|_1 < \min \left(\frac{5b}{4}, \frac{b}{8}\right).
\]

Since \(T(0)\) is a contraction we would then also have

\[
\|T(0)f - f_0\|_1 < \frac{b}{4}.
\]
(20) and (21) imply that

\[ (22) \quad \lambda \{ w : \left| T(0) f - f_0 \right| > \delta \} < \frac{b}{4}, \text{ and} \]

\[ (23) \quad \lambda \{ w : \left| T(0) f - f_0 \right| > \frac{b}{2} \} < \frac{b}{4}. \]

Therefore, there must exist a set \( A \subset \{ w : \limsup_{\alpha \to 0^+} M(\alpha)f(w) > \beta + T(0)f(w) \} \) with \( \lambda[A] \geq \frac{b}{2} \) and such that on \( A \)

\[ (24) \quad \left| T(0) f - f_0 \right| < \delta, \text{ and} \]

\[ (25) \quad \left| T(0) f - f_0 \right| < \frac{\beta}{2}. \]

Now by the properties of \( f_0 \) we have for almost every \( w \) in \( A \)

\[
\limsup_{\alpha \to 0^+} M(\alpha)\left| T(0) f - f_0 \right| \geq \left| \limsup_{\alpha \to 0^+} M(\alpha)(T(0)f - f_0) \right|
\]

\[ = \left| \limsup_{\alpha \to 0^+} M(\alpha)f - f_0 \right| \]

\[ > (T(0)f + \beta) - (T(0)f + \frac{\beta}{2}) \]

\[ = \frac{\beta}{2}. \]

But if \( \delta \) had been chosen such that \( \frac{\beta}{2} > \frac{\delta}{d} \), then for almost every \( w \) in \( A \) we would have

\[
\limsup_{\alpha \to 0^+} M(\alpha)\left| T(0) f - f_0 \right| > \frac{\beta}{2} > \frac{\delta}{d} > \frac{1}{d} \left| T(0)f - f_0 \right|
\]

thus giving us a contradiction to (19). Hence our induction is completed and the theorem is proved. \( \Box \)
SECTION 3

In this section we will consider using a more general form of averaging. That is, instead of considering the averages as in Theorem 1.1 or Theorem 2.1 we might consider the following types of averages

\[
(26) \frac{1}{\alpha_1 \cdots \alpha_N} \int_0^{\alpha_1} \cdots \int_0^{\alpha_N} T(t_1, \ldots, t_N) f \, dt_1 \cdots dt_N.
\]

We cannot hope that the limit exists almost everywhere as the \( \alpha_i \to 0^+ \) independently, for it has long been known that if \((X, M, \lambda)\) is a two dimensional Lebesgue space and \((T(t_1, t_2); t_1, t_2 \geq 0)\) two dimensional translations then the convergence of (26) can fail if the \( \alpha_i \to 0^+ \) independently (see Busemann and Feller, [2]). However, it was shown by Busemann and Feller that if a slight regularity condition is placed on the convergence of the \( \alpha_i \) then convergence can be proved for the case of translation semigroups on \( \mathbb{R}^N \). I should remark that Busemann and Feller were not considering semigroups of translations, but rather the problem of the differentiability of \( N \)-dimensional indefinite Lebesgue integrals. However, the study of translation semigroups on \( \mathbb{R}^N \) encompasses part of their study.

The regularity condition required by Busemann and
Feller was that the $\alpha_i \to 0^+$ in such a way that the ratios of the $\alpha_i$ are bounded above and below. I shall show that this condition is sufficient for the convergence almost everywhere of the averages of type (26).

**Definition 3.1:** Let us say that $(\alpha_1, \ldots, \alpha_N) \sim (0, \ldots, 0)^+$ if $\alpha_i \to 0^+$ for $i=1, \ldots, N$ and we also always have

$$\frac{\alpha_i}{\alpha_j} < B \quad \text{for} \quad i=1, \ldots, N, \quad j=1, \ldots, N.$$

**Theorem 3.1:** Let $(T(t_1, \ldots, t_N), \lambda(t), \lambda(t) > 0, \ i=1, \ldots, N)$ be a strongly continuous $N$-parameter semigroup of positive contractions on $L^1(\mathbb{X}, \mathbb{M}, \lambda)$. Then for all $f$ in $L^1(\mathbb{X}, \mathbb{M}, \lambda)$

$$\lim_{(\alpha_1, \ldots, \alpha_N) \sim (0, \ldots, 0)^+} \frac{1}{\alpha_1 \cdots \alpha_N} \int_0^{\alpha_1} \cdots \int_0^{\alpha_N} T(t_1, \ldots, t_N)f \, dt_1 \cdots dt_N$$

$$= T(0)f \quad \text{almost everywhere.}$$

**proof:** I shall use the notation

$$m(\alpha_1, \ldots, \alpha_N)f = \frac{1}{\alpha_1 \cdots \alpha_N} \int_0^{\alpha_1} \cdots \int_0^{\alpha_N} T(t_1, \ldots, t_N)f \, dt_1 \cdots dt_N.$$

Let $\mathcal{M}^*$ be the class of functions defined in Lemma 1.2. The proof of Lemma 1.2 easily carries over to the type of convergence in this theorem. So each $f$ in $\mathcal{M}^*$ also satisfies the conclusion of this theorem.

For all $(\alpha_1, \ldots, \alpha_N)$ satisfying $\frac{\alpha_i}{\alpha_j} < B$ for $i=1, \ldots, N, \quad j=1, \ldots, N$ let $\alpha^* = \max (\alpha_1, \ldots, \alpha_N)$. Then we have
holds almost everywhere for all \( f \) in \( L^+(X, m, \lambda) \). Thus as a result of Theorem 2.1 we have

\[
\limsup_{(\alpha_1, \ldots, \alpha_N) \sim B(0, \ldots, 0) +} M(\alpha_1, \ldots, \alpha_N)f < B^{N-1} T(0)f
\]

almost everywhere.

The proof now proceeds as in Theorem 2.1. By virtue of (27) we have that

\[
(27) \quad \lambda\left[ w: \limsup_{(\alpha_1, \ldots, \alpha_N) \sim B(0, \ldots, 0) +} M(\alpha_1, \ldots, \alpha_N)f(w) > T(0)f(w) \right] = 0.
\]

Let us suppose that the theorem does not hold. Then there would exist some \( f \) in \( L^+(X, m, \lambda) \) such that

\[
\lambda\left[ w: \limsup_{(\alpha_1, \ldots, \alpha_N) \sim B(0, \ldots, 0) +} M(\alpha_1, \ldots, \alpha_N)f(w) > T(0)f(w) \right] > 0.
\]

Hence, we could find a \( \beta > 0 \) such that

\[
\lambda\left[ w: \limsup_{(\alpha_1, \ldots, \alpha_N) \sim B(0, \ldots, 0) +} M(\alpha_1, \ldots, \alpha_N)f(w) > \beta + T(0)f(w) \right] = b > 0.
\]

Let \( \delta > 0 \) and choose \( f_0 \) in \( M^* \) such that

\[
(29) \quad \| T(0)f - f_0 \|_1 < \min\left( \frac{\delta b}{4}, \frac{\beta b}{8} \right).
\]

Since \( T(0) \) is a contraction we also have

\[
(30) \quad \| T(0)T(0)f - f_0 \|_1 < \frac{\delta b}{4}.
\]
By (29) and (30) we have

\[ (31) \quad \lambda \{ \omega : |T(0)f - f_0| > \delta \} < \frac{b}{4}, \quad \text{and} \]

\[ (32) \quad \lambda \{ \omega : |T(0)f - f_0| > \frac{B}{2} \} < \frac{b}{4}. \]

Therefore, there must exist a set \( A \subset \{ \omega : \lim \sup_{(\alpha_1, \ldots, \alpha_N) \sim B(0, \ldots, 0)} m(\alpha_1, \ldots, \alpha_N)f(\omega) > \beta + T(0)f(\omega) \} \) with \( \lambda[A] > \frac{b}{2} \) such that for almost every \( \omega \) in \( A \)

\[ (33) \quad \limsup_{(\alpha_1, \ldots, \alpha_N) \sim B(0, \ldots, 0)} \|T(0)f - f_0\| < \delta, \quad \text{and} \]

\[ (34) \quad |T(0)f - f_0| < \frac{B}{2}. \]

Now by (33), (34) and the properties of \( f_0 \) we have for almost every \( \omega \) in \( A \)

\[
\limsup_{(\alpha_1, \ldots, \alpha_N) \sim B(0, \ldots, 0)} m(\alpha_1, \ldots, \alpha_N) |T(0)f - f_0|
\geq \limsup_{(\alpha_1, \ldots, \alpha_N) \sim B(0, \ldots, 0)} m(\alpha_1, \ldots, \alpha_N) (T(0)f - f_0)
= \limsup_{(\alpha_1, \ldots, \alpha_N) \sim B(0, \ldots, 0)} m(\alpha_1, \ldots, \alpha_N) |f - f_0|
\geq (T(0)f + \beta) - (T(0)f + \frac{B}{2})
= \frac{B}{2}.
\]

But if \( \delta \) had been chosen such that \( \frac{B}{2} > \delta B^{N-1} \), then we would have for almost every \( \omega \) in \( A \)

\[
\limsup_{(\alpha_1, \ldots, \alpha_N) \sim B(0, \ldots, 0)} m(\alpha_1, \ldots, \alpha_N) |T(0)f - f_0| > \frac{B}{2} \delta B^{N-1}
> B^{N-1} |T(0)f - f_0|.
\]
thereby giving us a contradiction to (28). Thus the theorem is proved.
BIBLIOGRAPHY


