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AN IMPROVED APPROACH TO CRYSTAL SYMMETRY AND
THE DERIVATION AND DESCRIPTION OF THE THIRTY-
TWO CRYSTAL CLASSES BY MEANS OF THE STEREOGRAPHIC
PROJECTION AND GROUP THEORY.

The Ohio State University, Ph.D., 1971
Mineralogy

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AN IMPROVED APPROACH TO CRYSTAL SYMMETRY AND THE DERIVATION
AND DESCRIPTION OF THE THIRTY-TWO CRYSTAL CLASSES
BY MEANS OF THE STEREORAPHIC PROJECTION
AND GROUP THEORY

DISSERTATION
Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By
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* * * * *

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PLEASE NOTE:

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UNIVERSITY MICROFILMS.
PREFACE

The subject of order in the natural world is a favorite topic of scientists and philosophers alike. If there is any order in the universe, the crystalline state seems to be an excellent place to find it. That which is orderly should be comprehensible. There is no reason why a subject as rich in educational value, as rewarding in intellectual content, and as full of aesthetic satisfaction as crystallography should remain behind a veil of obscurity, the property of a few specialists.

As a teacher of crystallography and mineralogy for many years, it has become apparent to me that existing textbooks often do not serve the needs of either students or instructors. The material generally available on elementary crystallography seems to vary between excessively abstruse and involved on the one hand, or too brief on the other. Not only are the discussions often incomplete and insufficiently detailed in their explanations of the facts of crystallography and the concepts that enter into it, but often they are unnecessarily complex and oblique in their approach, using complicated techniques without explanation, and scientific shorthand notation without clarification.

This is not to say that it is possible to present the basics of so fundamental a discipline in a fashion that can be read as one reads light fiction. Many of the concepts are simple, but
often are deceptively so, and require considerable thought and insight if one is to understand them. Many of the proofs and explanations to be presented here will need very careful attention and must be followed step by step. However, I hope that by proceeding from the known to the unknown; by attempting to keep in mind the background and abilities of the average student, so that the remark "it is now obvious that" does not mean "an experienced mathematician will see that"; and by extensive use of examples, to provide a text that can be followed by any reader with the usual background in high-school mathematics.

It is my firm conviction that the study of crystallography should proceed from the outside in. This seems to me not only the logical, as well as the historical approach, but also to have much to be desired from the standpoint of pedagogy. Crystallography began as a branch of mineralogy, and the subjects of study were real crystals which could be handled by students. Interfacial angles could be measured and plotted on projections, and symmetry elements could not only be determined, but could be seen to exist in actual objects.

Since the development of x-ray diffraction, beginning in 1912 the emphasis quite properly has shifted to studies of internal structure, and certainly an ultimate goal of the crystallographer is the complete elucidation of structure; but this is no reason to ignore morphological studies to the extent that one finds in most textbooks which are aimed primarily at training workers in
structural crystallography. Whatever the ultimate fate of the student, whether he becomes a researcher in the area of x-ray crystallography or allied fields, or perhaps never deals in a professional way with crystallography at all, I feel that he should begin his training with a morphological approach. This has a number of advantages.

In the first place, the student is at the outset faced with things that he can see, handle, and measure for himself. Symmetry is real and observable, and not merely something to be extracted from pictures taken with a magic machine. Many current treatments, based from the start on lattice theory, open the discussion with a comma, or some similar object. They soon populate the universe with commas, and real crystals (and real people) may never make an appearance. The student is being asked to accept the fundamental concepts on faith, because they are being derived from things that he cannot see for himself. If he starts with a consideration of the shapes of real crystals, with help from wooden and plastic models, he can proceed logically from the directly observable to the inferred. Furthermore, such studies can be made the subject of a laboratory course for considerable numbers of students. Most schools do not have the facilities to accommodate many in x-ray courses involving laboratory.
In the second place, though one might gather the impression that morphological studies are no longer of significance in view of the numerous methods of gathering internal data, this is not so. Such methods often require that the material to be studied be oriented; in many cases the object may be a crystal with excellent faces, yet an investigator untrained in morphological studies will not be able to orient it by optical goniometry. In addition, the work of Donnay and others shows very clearly that one of the roads to the knowledge of the internal organization of crystals lies through the study of morphology.

Another one of my convictions is that crystallography, like every other subject, may be brightened and made more interesting by including some knowledge of the people who have created it. Crystallography without real people is as bad as crystallography without real crystals. I intend to include brief remarks about important contributors in the body of the text; but, in addition, there will be an appendix containing biographies from a few paragraphs to a few pages in length. Some of this material is from standard sources, but in other cases there are new facts obtained from more fundamental documents.

Also included will be quotations from original papers and books, both as translations (some of which have not, to my knowledge, appeared elsewhere), and as reproductions by xerox.
I am writing a textbook which I hope will remedy, at least in part, some of the deficiencies mentioned. Even though it is limited in scope to the externals of crystals, such a work is of necessity very lengthy. Therefore, this dissertation will contain excerpts from portions of the book, together with author's comments, set apart by double lines, as to why things are being done as they are, together with suggestions concerning techniques of teaching. It will be limited to a consideration of point-group symmetry, together with such other topics as seem necessary for logical discussion. The emphasis on symmetry is justified for the following reasons:

(1) The possession of symmetry is the fundamental property of crystalline matter. It might be said that crystallographic axes are imaginary, but symmetry axes really exist. The classification of crystals into 32 classes and 230 space groups really rests upon symmetry, even though historically the systems did not first arise in this manner.

(2) A thorough knowledge of morphological symmetry is by far the best preparation for an understanding of space-group symmetry.

(3) The methods of study of crystal properties, whether optical crystallography, x-ray or other forms of diffraction, or the determination of simple physical attributes such as hardness, are
founded on symmetry. Experience indicates, for instance, that it is far easier for a student to learn the use of the polarizing microscope if he has a good background in the symmetry of crystals.

(4) There has been a rapidly increasing use by chemists, both organic and inorganic, of the symmetry concept. Not only are they paying attention, as they have not for a century, to the solid state, but the symmetry of molecules is receiving even greater attention. Even though molecules are not subject to the crystallographic restriction, many chemists dealing with these subjects would have profited from prior knowledge of point and space-group symmetry.

(5) Symmetry is a subject that is inadequately handled in the available textbooks. The crystallographic sections of mineralogy books give it very little attention, and instructors could use an auxiliary source. Buerger (1956) gives a thorough treatment, but one which is too abstract and mathematical in my opinion. Books by Philips (1963) and by Bishop (1967) are incomplete on this subject.

(6) The method here used in the derivation of the thirty-two crystal classes, combining the use of the stereographic projection and group theory, does not exist in published form in English, or in any language to the best of my knowledge. It seems to me to be the most useful approach. If one learns to think of symmetry classes in terms of the stereographic projections of the
forms and the symmetry, he is possessed of an invaluable tool in
doing optics or x-ray crystallography.

(7) In addition to the derivation of the classes, certain
other aspects of the treatment of symmetry are unique.

The reader is reminded again of the format. The section
appearing between double lines are addressed to persons with a
knowledge of the subject, and in a sense, explain the text. The
text consists of excerpts and samples of a few of the possible
topics that will appear in the complete book.

I wish to acknowledge the consideration given to me by all
the members of the faculty of the Department of Mineralogy of
The Ohio State University. Especially, I wish to thank Dr. Henry
E. Wenden, without whose help, guidance, and patience, this work
would not have been possible.
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FIELDS OF STUDY

Major Field: Mineralogy

Studies in Crystallography and Crystal Chemistry. Professor Henry Wenden

Studies in Thermochemical Mineralogy. Professor W. R. Foster

Studies in Optical Mineralogy. Professor Ernest Ehlers
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<td>Class ( \frac{1}{m} )</td>
<td>235</td>
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<td>151</td>
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<td>152</td>
<td>Class ( \frac{4}{m} )</td>
<td>238</td>
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INTRODUCTION

Quartz, one of the most common minerals in the crust of the earth, has attracted the attention of men since the earliest times. Often it is as transparent as water, and indeed it was considered by the ancients to be a special form of ice. The Greek word *krystallos* is a derivative of *kystainein* meaning "to freeze", and originally was applied to quartz because of the idea that extreme cold was involved in its formation. Thus it is that a very poor conjecture on the part of the Greeks is the basis for our modern word "crystal", now come to have a much broader meaning.

Rocks and most of the minerals that make them up are usually dark, opaque, and weathered. It is no wonder that people have always been fascinated by the lovely color, the clarity and lustre, the transparency and sheen of crystals such as quartz, beryl, or topaz, when such are found amidst the dullness of the usual earth material. However, there is another aspect to crystals which adds much to their attractiveness. It was well expressed by the great English crystallographer and mineralogist, A. E. H. Tutton (1924, p. 5), when he remarked in a lecture at the Royal Institution,

"The beauty of crystals lies in the planeness of their faces."
When formed under proper conditions, whether in a crevice in a mass of granite, in a vesicle in a lava flow, or in a laboratory beaker, crystals are bounded by plane faces, arranged in a symmetrical manner, and meeting at precise angles.

Pliny the Elder (23-79 A.D.) gives perhaps the first suggestion concerning the relationships between the faces of crystals. In Book XXXVII of his Natural History (Eichholz, p. 183) appears the following statement concerning rock crystal, or clear quartz:

Why it is formed with hexagonal faces cannot readily be explained; and any explanation is complicated by the fact that, on the one hand, its terminal points are not symmetrical and that, on the other, its faces are so perfectly smooth that no craftsmanship could achieve the same effect.

This is an astute observation, as it is quite true that many quartz crystals are shaped as shown in Fig. 1, which fits Pliny's description very well.

![Fig. 1 Quartz crystal](image-url)
On the other hand, a few pages later (Eichholz, p. 235), Pliny remarks about beryl:

Beryls are produced in India and are rarely found elsewhere. All of them are cut by skilled craftsmen to a smooth hexagonal shape, since their colour, which is deadened by the dullness of an unbroken surface, is enhanced by the reflection from the facets.

Here Pliny is completely off the track, as beryl, like quartz, is formed by nature into hexagonal crystals. It is obvious that Pliny never collected a crystal of beryl. If so, he would have known that it does not have to be cut in order to be six-sided.

There perhaps is an important lesson here. The ancients knew and admired crystals, but the writers and philosophers never really looked at them. Crystals are found in places such as quarries and mines which were worked by lackeys and slaves. No Greek or Roman writer would have thought of visiting such a place. A craftsman might obtain a beautiful crystals which he would carve into an ornamental piece, but even if he observed the faces very carefully before cutting, the fact would not have been recorded; such people did not write books any more than did the miners.

It is remarkable that a man of the erudition and broad general knowledge of Albertus Magnus has nothing to say about crystals in his De Mineralibus, written about 1260, (Wyckoff, 1967). Albertus did visit mines, but apparently to study the metals. It appears that almost nothing of substance was written about crystals until the middle of the seventeenth century.
It was not until 1669 that the fundamental law of crystallography was first stated. Nicolaus Steno (the Latinized form of Niels Steensen) was born January 10, 1638, in Copenhagen, and studied mathematics, languages, and medicine in Copenhagen and Amsterdam. While attached to the court of Grand Duke Ferdinand II in Florence, he published his famous Prodromus (Steno, 1669), an introduction to a much larger work which unfortunately was never completed. The title in English would be Concerning A Solid Body Enclosed By Process Of Nature Within A Solid, and in it are many remarkable observations on geology, mineralogy, and crystallography. His observations on the nature of crystal growth (Winter, p. 237-243) are truly amazing when one considers that he had no previous work upon which to build.

It is in connection with the description of some of his drawings of cross-sections of quartz that there occurs his most astute observation. Note Fig. 2, which shows figures 5 and 6 of Steno.

![Fig. 2 Steno's figures](image)
He writes (Winter, p. 272)

Figures 5 and 6 belong to the class of those which I could present in countless numbers to prove that in the plane of the axis both the number and the length of the sides are changed in various ways without changing the angle.

Steno does not say how he produced the figures. It may have been shadow casting, or perhaps the crystals were cut and traced, or perhaps he had some form of contact goniometer; in any case, he gave the first statement of the fundamental law of crystallography, the Law of Constancy of Interfacial Angles, now usually stated as follows: (Hurlbut, p. 12)

The angles between equivalent faces of crystals of the same substance, measured at the same temperature, are constant.

Steno was operating on very slim evidence. Although he talks briefly about pyrite and hematite, his statement about constancy applies only to a few specimens of one mineral, quartz; but his observation was correct and basic. However, he and his successors, during the following century, although they made contributions which demand admiration, did very little to affect the thinking of the times, and crystallography could in no way be considered a science.

The next real step forward was taken by Jean Baptiste Louis Romé de l'Isle (1736-1790), an ex-army officer who added to his pension by giving lectures on mineralogy, spending the latter portion of his life in Paris. His 1772 work does not consider interfacial angles, but the greatly expanded version published in 1783 gives many such values and uses them as a means of identifying individual minerals. As is often the case, the advance took
place because of a technological development.

Arnould Carangeot (1742-1802) was a controller by profession and a student of Rome's and he was asked by the latter to produce clay models of some crystals. This led him to the development of the instrument known as the contact goniometer, the first device by means of which accurate measurements of interfacial angles can be made. It is a semi-circle to the center of the diameter of which is attached two bars which pivot together and which can be clamped in any position. Simple instruments have only one arm. The angle to be measured is fitted against the two arms (or one arm and the body) in a plane normal to the line of intersection of the two faces, and the angle read on the graduated side. See Fig. 3.

![Fig. 3 Contact Goniometer](image)
With this device it became possible to measure many angles quickly and accurately. Thus it was that Romé was able to state the Law of Constancy with considerable evidence to back it up. However, Carangeot should be given almost equal credit, as he not only developed the goniometer, but called to the attention of Romé the nature of his results. The story is told in Histoire de la Science by Daumas, and the following is a translation by Geisler (1966).

Several authors have observed the angles of certain isolated faces of different crystals and we have been given the measure of isosceles triangles of rock crystal or vitriolic tarter, of rhombus of Islandic Spar, of crystal rhomboids of selenite, of garnet, etc. But one had not made use of this primary data for measuring the solids themselves in order to know the geometrical rapport of the faces among them. This observation, which measures the identity of substances through the identity of forms, and simplify the study of mineralogy by calling attention to a primitive measure and constant quantity of varieties which may have many shapes, must be regarded as absolutely new and due to chance just as many other discoveries.

The author, a novice at crystallography but very desireous of perfecting himself by a profound study under the eye and through the instructions of the creator of this science, worked at cutting and modelling in clay, after nature, crystals which M. de l'Isle wished executed in baked clay to accompany his learned work. Desperate after several futile attempts to render a truly bizarre form of rock crystal, the author took it into his head to cut, proceeding cautiously, the angle which was formed from the two faces. These two faces cut, he was surprised to find, the same angle in the two opposite faces. He repeated the experiment on all the rock crystals which he had available. He recognized with satisfaction that the angles were constant and produced 104° for the junction of bases of each pyramid and consequently 76° for their summit; 142° for
the angle of incidence of the pyramidal faces on those of the prisms and, $120^\circ$ for each of the six angles of the prisms whatever form the crystal had. He hastened to inform M. de l'Isle of this experiment. This scholar, who was conscious of the usefulness of it, had it repeated for him and recognized, with the greatest pleasure, that it had taken place constantly on crystals of different mineral substances.

Nothing better proves the justice and profoundness of his theory than the constancy of angles in crystalline forms of the mineral substances, and which exist in varieties which seem to be the farthest removed.

In fact it is geometrically demonstrated that every salty, rocky, or metallic substance has a constant and determined polyhedral form, which is only varied by truncations more or less strong, more or less multiplied, but whose angles are constantly the same in all crystals of the same type."

Thus it was that the first law of crystallography was placed on a firm foundation of experimental evidence. With the Law of Constancy no more would result from studying the angles between crystal faces than from a statistical analysis of the facets on a chandelier ornament. The difference lies in the fact that the plane surfaces on crystals are an outward manifestation of a regular internal order. This is the distinction between crystalline and amorphous solids, and it is the fundamental difference that sets crystalline matter apart from liquids and gases. Robert Hooke (1665), and Christian Huyghens (1690) were among those who speculated on this matter of internal structure, but it was René Just Hauy who first stated the idea with precision.
Hauy was born at Saint Just, France, on February 28, 1743, the son of a poor weaver. However, he received a scholarship at the University of Paris from which institution he obtained the Master of Arts degree. He also studied theology and was ordained to the priesthood. Later he became an honorary canon of Notre Dame, and in most works is referred to as the Abbe Hauy. Excellent comments on his work are found in Burke (1966), Tutton (1924), Le Corbeiller (1953), and in the volume Rene-Just Hauy, published in 1945 by the Societe Francaise de Mineralogie. The latter includes a very complete bibliography.

The essence of Hauy's system is the idea that crystalline substances are built of units of the form enclosed by their prominent cleavages. He is supposed to have dropped a calcite crystal of the shape of a hexagonal prism, and upon finding it shattered into rhombohedrons, exclaimed "Tout est trouve". Further experimentation with calcite and other minerals convinced him that the basic unit, or molecule integrante, could be determined for any substance, and that the entire crystal is built by repetition of these fundamental building blocks. By the omission in regular fashion of rows of the blocks, faces with slopes other than that of the basic unit can be produced. He supposed that the units are of submicroscopic size, so that the "steps" are not visible to the eye, and the faces appear plane. The following figures, taken from the publication Rene-Just Hauy, illustrate this idea.
Hauy's idea was basically sound. If the modern concept of the unit cell is substituted for his small, solid cleavage blocks, his view of the nature of crystalline material is in accord with the present one. The difference between the crystalline state and the amorphous state is a matter of internal structure.

Crystalline substances have their constituent atoms arranged in such a way as to have three-dimensional internal order, with a periodic repetition of a basic pattern. When the regular internal arrangement results in a body with symmetrically arranged plane faces, which are an outward manifestation of the inner order, the body is termed a crystal.

Implicit in the work of Hauy is the second great law of crystallography, the Law of Rational Indices, which may be stated as follows (Tutton, 1972):

The intercepts on the crystal axes made by any face of the crystal, are such as can be expressed as rational multiples of the parametral unit lengths of those axes, as determined by the intersection of those axes with the primary face which has been chosen as the parametral plane.

When the notation of William Miller is used, three small whole numbers form the indices of most faces to be found on crystals.

Hauy was aware of symmetry, but though he gives the outlines of the symmetry of five of the seven crystal classes, he failed to develop the concept. However, he did erect a complete system of crystallography, with a notation for crystal faces and ways of making calculations that could still be used. Furthermore, he very nearly anticipated Dalton's atomic theory, and expressed
Fig. 4  Rhombic Dodecahedron Built of Cubes
Fig. 3 Pyritohedron Built of Cubes
Fig. 6  Scalenochedron Built of Rhombohedrons
ideas concerning the orientation of atoms in a crystalline structure. His most fundamental contribution is that he not only viewed matter as particulate, but that he erected a quantitative system to give substance to that concept. A charming compliment is paid to Hauy in the little essay written by Le Corbeiller (1953), which is written as a mock-Socratic dialogue.

This, Empeiros, was a momentous event in the history of science. For the good Abbe Hauy thus became the first experimental atomist. His successors were John Dalton, Gregor Johann Mendel, J. J. Thompson, Max Planck, Albert Einstein -- each discovering a new type of 'atom'. Rene Just Hauy thoroughly deserves the appellation usually accorded him -- "Father of Crystallography".

Although the contact goniometer of Carangeot is still useful for the study of large crystals, it is limited in accuracy at best, and small crystals can not be measured with it. For most purposes it has been replaced by the reflecting goniometer, first described by William Hyde Wollaston in 1809. Wollaston was a remarkably versatile medical doctor who made contributions to chemistry, physics, and astronomy, as well as to crystallography.

The principle of the reflecting, or optical, goniometer, as described originally by Wollaston (1809), is as follows:

The instrument which I use, consists of a circle graduated on its edge, and mounted on a horizontal axle, supported by an upright pillar (Plate XI). This axle being perforated, admits the passage of a smaller axle through it, to which any crystal of moderate size may be attached by a piece of wax, with its edge, or inter­section of the surfaces, horizontal and parallel to the axis of motion.

This position of the crystal is first adjusted, so
Fig. 7 Wollaston's Goniometer
that by turning the smaller axle, each of the two surfaces, whose inclination is to be measured, will reflect the same light to the eye.

The circle is then set to zero, or 180°, by an index attached to the pillar that supports it.

The small axle is then turned till the further surface reflects the light of a candle, or other definite object to the eye; and, lastly, (the eye being kept steadily in the same place) the circle is turned by its larger axle, till the second surface reflects the same light. This second surface is thus ascertained to be in the same position as the former surface had been. The angle through which the circle has moved, is in fact the supplement to the inclination of the surfaces; but as the graduations on its margin are numbered accordingly in an inverted order, the angle is correctly shown by the index, without need of any computation.

Wollaston's original instrument is pictured in Fig. 7. Note that in order to bring the two images one after the other to the reference spot, the crystal is rotated through the angle between the normals to the faces, as shown in Fig. 8. It is necessary to turn the crystal through the angle $\alpha$ in order to cause face B to occupy the position now occupied by face A. It is this angle $\alpha$.

Fig. 8 Angle Measured by Optical Goniometer
rather than the internal dihedral angle $\Theta$ between the faces which is used in crystallography.

Modern instruments are supplied with a collimator for the light source and a telescope for observation, and angles can be determined to minutes of arc. The most used type permits the crystal to be rotated around two mutually perpendicular axes, and thus determine directly the position of a crystal face in spherical coordinates. These are called "two-circle" goniometers. One and two-circle instruments are illustrated in Fig. 9.

Hauy's view of crystal structure involved a regular stacking of identical units which he considered to be extremely small solid blocks similar in shape to the cleavage fragments of the substance concerned.

The modern idea is essentially the same, but replaces the cleavage blocks with groups of atoms, ions, or molecules; but a further simplification is possible. The identical groups of atoms, ions, or molecules may be represented by the points of a "lattice". A lattice is an array of points arranged in such a manner that the environment of every point is precisely the same as that of any other point in the array.

Auguste Bravais showed in 1850 that there are exactly fourteen ways of placing identical points in space in a manner such that each point is repeated at regular intervals in all directions; that is, every point is exactly like every other point. All crystal lattices may be referred to one of the fourteen types as developed by Bravais. Such an array of points may be termed
Fig. 9 One-circle (above) and Two-circle Goniometers
a "regular arrangement" in the crystallographic sense, and all "regular arrangements" will meet the following test, as suggested in an unpublished work by O. C. Von Schlichten:

1. Take any two nearest identical atoms, and through them pass a straight line. On that line will be found identical atoms at the same interval as between the first two chosen, with no other identical atoms in between.

2. Take the next nearest (or any other) identical atom off that line, and through it pass a straight line parallel to the first line. On this second line will be found the same interval as on the first line.

Any crystalline substance may be referred to one of the seven crystal systems, based on the symmetry of the shape of its Bravais lattice. This symmetry may be expressed in terms of axes, planes, or a center of symmetry. If crystals are considered as finite bodies, so that all symmetry operations leave the center of the crystal unmoved, then there are just thirty-two possible combinations of symmetry elements, that is, thirty-two crystal classes.

The fact that there are more crystal classes than there are types of symmetry in the Bravais lattices (six) is easily explained. The space lattice is simply a collection of points, and a point itself has no symmetry. If the points are all located at the centers of one kind of atom or ion, the symmetry of the crystal is the symmetry of the lattice; but this is true only rarely in real crystals. Usually the points represent the centers of groups of atoms, ions, or molecules. These groups themselves have symmetry, and this means that, for instance, a hexagonal lattice might have tetrahedral groups of ions within it, thus leading to crystal
classes having other than simple hexagonal symmetry. The mineral quartz is such a substance.

The fact that there are just thirty-two symmetry combinations possible in a finite body built upon a regular interval arrangement was first discovered by Johann Friedrich Christian Hessel (1796-1872). Hessel was active as a teacher, researcher, and author for fifty years as a professor at the University of Marburg. His concept of the crystal classes was first published as an article called "Krystall" in Gehler's Physikalisches Wörterbuch in 1830. It was repeated in 1831 under the title "Krystallometrie oder Krystallographie", which was reprinted in Ostwald's Klassiker, number 88, 1897.

Hessel's method was to determine first all possible symmetrical finite bodies, and then eliminate those which cannot satisfy the law of rationality of indices. His work was a real tour de force, but lay forgotten for over half a century, until it was resurrected by L. Sohncke (1891). Because he was so far ahead of his time, Hessel was forced to invent a completely new and complicated terminology, and it seems clear that this was the main reason his great work remained obscure. For instance, William Whewell wrote in 1830:

Mr. Hessel himself (Professor at Marburg,) in his work entitled "Crystallometry" (Leipzig 1831), has adopted several other new denominations and modes of considering forms. Thus, by way of example, he states concerning the rhombic dodecahedron, that its twelve faces "are perpendicular to doubly-two-membered normals, (the edges are doubly-one-membered, like-sided, unlike-ended,) which are perpendicular to
doubly-one-membered four-and-three-spaced rays," etc.

The principle of this and similar methods of treating this subject consists in the permutations and combinations of various kinds of symmetry in lines, surfaces, and solids. One kind of symmetry, which occurs frequently in crystals, is not easily described by any common expression; and Mr. Hessel, who justly attaches much importance to the consideration of it, has introduced a peculiar term to designate it. The symmetry here spoken of is that which is seen in comparing the two ends of an oblique prism; and they are called by him "gerenstellig" gore-wise-placed, in opposition to "gleichstellig" alike-placed. One or two new phrases in such cases may perhaps be introduced with advantage; but the systems to which I here refer are so far laden with new phrasology and new views of the relations of space, that they will probably not be found by many a convenient mode of mineralogical study.

A more satisfying derivation was published by Alexis Gadolin in 1871. His method was complete and elegant, and for a time the entire concept of the thirty-two classes was attributed to him rather than to Hessel. Gadolin devoted his attention to the development of only those classes possible in crystals, and his treatment is far more brief and understandable than is Hessel's. Other discussions of the derivation of the thirty-two classes include those of Curie (1884), Schoenflies (1891), Federov (1891), Hilton (1903), Donnay (1942), and Buerger (1956). A good resume of the early literature is given by Swartz (1909).

The thirty-two classes represent all of the possible combinations of symmetry about a fixed point, and thus are termed "point-groups". If the crystal lattice is considered infinite in extent, so that translational symmetry elements are allowed, 230 "space-groups" are possible. This fact was discovered independently in the last decade of the nineteenth century by Federov (1891), the
Russian crystallographer, Schoenflies (1891), a German mathematician, and Barlow (1897), an English businessman and amateur scientist.

This work will be limited to a consideration of the thirty-two point groups only, and will not be concerned with space-groups. By the use of a combination of group theory and the stereographic projection, the crystal classes will be derived and their symmetry described.
Crystals are three-dimensional objects and to reproduce them on a two-dimensional surface requires the use of a projection. The problem is similar to that of a cartographer, who faces the necessity of depicting the globe, or a portion thereof, on a plane.

The situation is not exactly analogous, but since most people have at least some slight acquaintance with maps and the earth grid, the projections most used in crystallography will be introduced through the medium of mapping applications. It happens that the two projections widely applied to crystallographic problems, the stereographic and the gnomonic, are among the oldest known, and have been used by map-makers since before the time of Christ.

The gnomonic is said to be the first of all projections, and Fisher (1952) quotes A. Germain as follows:

The origin of this (gnomonic) projection, the oldest known, seems to go back before Thales of Miletus, the predictor of eclipses, who died in 548 B.C. Called successively by the names of 'horoscope' and 'analemme', it was much used in astronomy under the name of 'gnomonic' to trace the course of celestial phenomena from the surface of the earth.

Maps made on the gnomonic (and on the stereographic as well) are often called "eye perspective" maps, since the principle of projection is that seen in Fig. 10. In the gnomonic each point on the globe is projected on the tangent plane in the position it
would occupy if an observer were at the center of the earth looking out.

Fig. 10 Principles of the Gnomonic Projection

In Fig. 11 a portion of the western hemisphere as shown on a gnomonic projection is depicted.

Fig. 11 Western Hemisphere on the Gnomonic Projection

Any plane passing through the center of a sphere intersects the surface of that sphere in what is termed a great circle. On
the globe, the lines of longitude, or meridians, are all great circles. That is, if the earth were a true sphere, they would represent the circumference. Of the parallels, or lines of latitude, only the equator is a great circle.

In crystallographic studies, great circles have a special significance, and the most important property of the gnomonic projection is that all great circles are shown as straight lines. Note in the two preceding figures that this is true for all lines of longitude, and the equator.

A glance at Fig.10 will make apparent one of the chief drawbacks of the projection. It is easily seen that the portion mapped must be less than a hemisphere. This difficulty will appear in another form in the crystallographic applications.

Franz Ernst Neumann (1798-1895), one of the great names in early 19th century crystallography, used both the gnomonic and stereographic projections in his book *Beiträge zur Krystallonomie*, (1823). A more extensive use of the gnomonic is by François Mallard (1833-1894), Professor of Mineralogy at the School of Mines in Paris, in his *Traité de cristallographie géométrique et physique* (1879, 1884). However, it was the development of two-circle goniometry by Goldschmidt, Fedorov and others in the latter part of the 19th century and the early 20th century, and the widespread use of the reciprocal lattice concept following the development of x-ray crystallography that brought the gnomonic projection to its present importance.
The stereographic projection is usually said to have been invented by Hipparchus, the Greek astronomer, in about 150 B.C. Brown (1949) states that Claudius Ptolemy (90-168 A.D.) described in his work *Planisphaerium* a projection of the globe on the equatorial plane, with the eye located at the pole, and such projections have been used for maps since that time.

Fig. 12 shows the stereographic projection in a manner analogous to the illustration of the gnomonic.

Fig. 13 is a map made on the stereographic projection. A bit of trial and error testing by the reader will demonstrate one of the important qualities of this projection: that all circles or parts of circles on the sphere are projected as circles or parts of circles. All of the lines of latitude and longitude are thus arcs of circles (the equator and the center line connecting the two poles are straight lines -- arcs of circles of infinite radius).
Fig. 13 Map on the Stereographic Projection

On the stereographic projection a great circle is either a straight line through the center, the primitive circle, (the circle which outlines the projection), or any circular arc that has a diameter for a chord. Remember that on a gnomonic projection, all great circles are straight lines. Each of these qualities is useful in crystallographic studies. Great circles represent planes which contain the poles of faces which lie in zones. Zones are girdles of faces on crystals which have parallel edges of intersection, and the concept of zones is one of the most useful ideas in crystallography.

As noted earlier, Neumann introduced the stereographic projection into crystallography as early as 1823, but in his early works made little use of it. W. H. Miller (1839, 1860) probably provided the greatest impetus to its employment in crystallographic studies.
Projections are essential tools with which to attack many sorts of crystallographic problems, ranging from presentation of data obtained with the contact goniometer to the determination of unit cell and space groups in x-ray crystallography. There also appear to be secondary benefits derived from emphasizing projections in an elementary mineralogy course.

In most American colleges and universities such courses are taken in large part by students majoring in geology, and are taught almost exclusively in geology departments. The ability to think in three dimensions is vital to geologists in almost any branch of that science, and as any teacher knows, some students seem to have it, or acquire it, very readily, while with others such visualization is a painful, and often unsuccessful, process. Whatever the truth is concerning the efficiency of carry-over from one branch of knowledge to others, experience indicates to me that students who have had considerable work with projections in crystallography and mineralogy are able to perform the greater ease in areas such as structural geology. Of course, the stereographic projection is widely used by some structural geologists, and obviously prior knowledge is helpful; but even with the methods which are essentially those of descriptive geometry there seems to be a somewhat greater facility on the part of those who have worked with crystallographic projections.

In approaching the problem of teaching students to think in
three dimensions, and especially to work three-dimensional problems on a plane surface, often the best avenue is through maps. Most people have daily contact with maps, whether or not they make professional use of them. Indeed, it is hard to go through a twenty-four hour period without seeing a map in a newspaper, a magazine, or on one of the map types of television news shows. The student is made aware of maps in many of his courses, and it is a rare person of college age, who, in this era of the gasoline engine, does not use road maps. Therefore, on the general principle that there are pedagogic advantages in proceeding from the known to the unknown, it seems worthwhile to introduce projection through the cartographic application.

I find especially helpful in the early stages a large styrofoam ball marked with lines of latitude and longitude, and the angles phi and rho indicated on the surface and on a section cut out after the manner of Fig. 14.

If projection methods are taught at all in American mineralogy courses, it is the stereographic projection which receives almost exclusive treatment. The gnomonic projection receives far too little attention in such courses and in elementary crystallography books, and even less space in the sections devoted to crystals in mineralogy texts.

There are at least two good reasons for paying a bit more attention to the gnomonic. In the first place, any student doing further work in mineralogy will use it in x-ray interpretation, and possibly in two-circle goniometry or in crystal drawing.
Secondly, it provides a background for the reciprocal notation of Miller symbols which is encountered in the most elementary of courses. Indeed, later in this work I will use it in just this way.

It was stated earlier that the projection of the faces of a crystal is not exactly analogous to the projection of the features found on maps. The cartographer is interested in both shape and areal size, and his greatest problem lies in the fact that no projection can preserve both true shape and true area. The crystallographer has a great advantage; the shapes and sizes of crystal faces are accidents of growth, and for most purposes quite unimportant in crystallographic studies. The constant features of crystals are the angles between faces (Steno's principle of the constancy of interfacial angles). It is clear that if interfacial angles are constant, so also are the angles between face normals. That is to say, if one imagines lines drawn from the center of the crystal perpendicular to each face (extended if necessary), the angles between these normals preserve the essential angular relationships, Johann Jakob Bernhardi (1774-1850), Professor Medicine at Erfurt, stated this quite clearly (1808, p. 378):

One makes an incorrect approach to crystallography if one believes that its nature consists of the determination of primary and secondary forms. Let one rather conceive of lines perpendicular to each crystal face. Let these lines intersect on a common point. The relationships of these lines may be defined trigometrically....
By using the face normals as the features to be plotted, rather than the faces themselves, the difficulties of size and shape have been done away with. No longer are the essentials two-dimensional (plane faces); they are now one-dimensional (lines), which have no areal extent. It should be noted at this point that the optical goniometer measures the angles between face-normals, not those between faces; so the original data are in a form to be used without conversion.

Bearing in mind that the elements to be plotted are perpendiculars, the mathematical relationships found in the stereographic projection are best seen by using the concept of the spherical projection as an intermediate step. Fig. 14 depicts a crystal with its center coincident with the center of a sphere, and with various faces designated as (100), (101), etc. (this nomenclature will be explained in a later section). If the center

![Fig. 14 Spherical Projection](image)
is called 0, then lines such as OA, OC, and OD, for example, are normals to those faces. The points on the sphere (A, C, D, etc.) at which the normals intersect the sphere, are termed poles. That is, D is the pole on the spherical projection of the face (011). Since the plane of a crystal face has now been reduced to a point on a sphere, it is possible to designate uniquely that point, and thus the plane, by means of spherical coordinates. On the earth grid, as in Fig. 15, it would be said that Columbus, Ohio has a latitude of 40° N, because it is located 40° "up" from the equatorial plane. It has a longitude of 83° W, because the great circle of longitude on which it lies makes an angle in the equatorial plane of 83° with the meridian of the arbitrary zero starting point, Greenwich England.

Fig. 15 Spherical Coordinates
To see the manner in which a pole on the spherical projection is converted to a pole on the stereographic projection, again observe Fig 14. Let the plane of the equator be the plane of the stereographic projection, and draw a line from D to the S, the "south pole" of the sphere. The point D', where DS pierces the equatorial plane, is the pole of face (011) on the stereographic projection.

The poles of crystal faces are designated in much the same manner. The arbitrary starting point for "longitude" is the meridian passing through the pole of the face on the right side. The angle between this meridian and the one passing through the pole of the face in question is termed \( \phi \). In the previous figure, the \( \phi \) of the face "C" is \( 83^\circ \). This is exactly analogous to longitude.

The other coordinate is termed \( \rho \), and is measured from the north pole of the sphere. That is, from the upper end of the principle axis of the crystal. In terms of the earth grid, this is the colatitude. In our example, face C has a \( \phi \) of \( 83^\circ \), and a \( \rho \) of \( 50^\circ \).

In the usual orientation of a crystal, the \( 0^\circ \) \( \phi \) is placed at the right of the observer, and \( \phi \) angles are positive from \( 0^\circ \) to \( 180^\circ \) in the hemisphere toward the observer, and \( 0^\circ \) to \( -180^\circ \) in the rear hemisphere. \( \rho \) angles range from \( 0^\circ \) for a horizontal face, that is a face parallel to the plane of the equator (note that such a face does not have a \( \phi \) angle), to \( 90^\circ \) for a vertical face.
Fig. 16 represents a vertical section through Fig. 14, and will make clear the fundamental mathematical relationships in the stereographic projection. The angle \( \rho \) is the angle between the face normal and the vertical line \( N-O-S \). If \( r \) is the radius of the sphere, \( O S \) and \( O D \) are both equal to \( r \), and the triangle \( DOS \) is isosceles.

Therefore,

\[
\angle OSD = \angle ODS
\]

but,

\[
\rho = \angle OSD + \angle ODS \quad \text{(opposite exterior angle)}
\]

and thus

\[
\angle OSD = \frac{\rho}{2}
\]

since \( \frac{OD'}{OS} = \tan \frac{\rho}{2} \) and

\[
OD' = OS \tan \frac{\rho}{2} \quad \text{and}
\]

\[
OD' = r \tan \frac{\rho}{2}
\]
The plane of the equator of the spherical projection is the plane of the stereographic projection. It appears as a great circle having as its radius the radius of the sphere, and is termed the "primitive". To plot a crystal face having a phi of $83^\circ$ and a rho of $40^\circ$, proceed as follows:

1. Draw a circle representing the plane of projection. Any radius may be used, but if space permits 10 cm. is a convenient value. Fig. 17 has a 5 cm radius.

2. Mark $0^\circ \varphi$ on the right side, and measure with a protractor an angle of $83^\circ$ clockwise around the circle, as positive $\varphi$ angles are measured in that direction. It often is helpful to sketch in the lines from the center of the projection to the points on the primitive.

3. The line from 0 to the point marked $83^\circ \varphi$ is the locus of the poles of all faces having a phi of $83^\circ$. The particular face having a rho of $40^\circ$ must now be located.

4. The distance from the center to the stereographic pole of a face is equal to $r \tan \rho/2$, or in the example, $5 \times \tan 20^\circ$

$$5 \times 0.364 = 1.820 \text{ cm.}$$

5. Using a scale, the pole of face C is located as shown, measuring out from 0 a distance of 1.82 cm.

Take note of the fact that the distance from the center to the stereographic pole is proportional to a tangent function, and not directly proportional to the value of the rho angle in degrees. A rho value of $45^\circ$ does not fall halfway between the center and the edge, but rather at $0.414 \times (\tan 22.5^\circ)$ times that distance.
To plot a face in the lower hemisphere, that is, one which lies below the plane of the projection, it is imagined that the pole of the face is found on the spherical projection as before; but from this pole a line is drawn to the "North" pole of the projection, as in Fig. 18. $M'$ represents the stereographic pole of the face illustrated. A moment's consideration of the geometry will show that if the face in question has a phi of $83^\circ$, and a rho of $40^\circ$ (as shown by the dashed lines), but lies below the projection plane, the stereographic pole will be exactly at the same
spot as in the previous example. The most widely used convention is that of plotting faces in the upper hemisphere as X or +, and those in the lower hemisphere as small circles. Thus, both are present would appear on the projection. This convention is adopted here.

Fig. 18 Plotting a Lower Hemisphere Face
Many of the readily available texts, such as Bishop (1967), Phillips (1963), and Terpstra-Codd (1961), give rather complete instructions on the methods of making stereographic constructions using the compass, protractor, and straight-edge. In a beginning course in crystallography it is worth paying some attention to these fundamental constructions. To do so certainly provides the student with a better understanding of the nature of the projection. However, since most practical work will be done on the Wulff net, it is my judgment that practice in making basic constructions by compass, straight-edge, and protractor, can be kept to a minimum.

The problem that is faced most often which requires the use of the compass is that of constructing a small circle around a given point. Except in the special cases in which the point lies at the center of projection, or on the primitive, it can not be done on the net, so this seems to be the best example to use.

Any plane that intersects a sphere but does not go through the center intersects that sphere in a small circle. The projection of a small circle, or any part of one, results in a circle,
or an arc which is a part of a circle, on the stereographic projection. For a proof, see Terpstra-Codd (1961, p. 12).

Observe Fig. 19. If point A is the center of the small circle lying on the surface of the sphere, that circle is the locus of all points lying at a particular angular distance from A. If A represents the pole of a crystal face, A, and if (for instance) the angular radius of the circle is 20°, then the poles of all faces making an angle of 20° with face A will lie on that circle. We shall use this property in later problems involving the plotting of crystal faces, and it is necessary to be able to plot circles of any radius about a given point.

Fig. 19 Small Circle on a Sphere
Consider first the general case, in which the pole of the center of the required circle lies within the primitive circle. Our problem will be as follows: construct a circle of $20^\circ$ radius about a point which represents the pole of a face having a phi of $26^\circ$ and a rho of $66^\circ$. Note Fig. 20.

1. Draw a primitive 7 cm in radius, and locate the required pole, P. Draw the radius from O through P.

2. Construct $N' - S'$ perpendicular to the line $O P$.

3. Draw a line from $S'$ through P to $P'$, and with a protractor lay off arc $P' A'$ and $P' B'$, each of $20^\circ$. This represents the limits of the circle of the required radius, $20^\circ$.

4. Draw $A' S'$ and $B' S'$, locating $A$ and $B$, the limits of the required circle on the stereographic plane.

5. Bisect line segment $A B$, finding C, the center of the required circle.

6. Using a compass centered on C, draw the circle with $C B$ (or $C A$) as the radius.

Note that C is displaced toward the edge of the primitive from P. This will not be surprising if you remember that distance from the center is related to the tangent of the angle, and is not a linear function.

Virtually all modern textbooks dealing with morphological symmetry use the stereographic projection to show symmetry ele-
Fig. 20 Stereographic Projection of a Small Circle
ments, to illustrate the distribution of faces in a form, the
repetition of a face by the operations of the symmetry elements,
etc. However, they do not make clear to the student what is being
done.

The most obvious deficiency is lack of explanation of the
fact that while faces (which are planes) are projected as points,
planes of symmetry are shown as lines; furthermore, axes (which
are lines) are projected as lines! Of course, the answer is that
the stereographic projection is being used as a reciprocal projec-
tion on the one hand (poles of faces), and as a direct projection
on the other. Fisher (1952) was so taken with this problem that
he suggested the term verstereographic be used for the reciprocal
type of projection. He seems to overlook the fact that commonly
both types are often used on the same stereogram. Textbooks often
show the direct projection of a plane, but fail to connect this
with the symmetry planes on the dozens of stereograms illustrating
crystal classes.

At this point the student has not been introduced to the con-
cept of symmetry elements; but the difference between direct and
reciprocal projection can be presented and the terms "plane of
symmetry" and "axis of symmetry", used in the examples, even though
the terms will not be fully explained until later.
DIRECT AND RECIPROCAL STEREORAPHIC PROJECTION

The stereographic method may be used for direct and for reciprocal projection. Indeed, we have been using it in both manners. In the opening sections dealing with map projections, the earth grid is projected in a direct manner. That is, any plane, such as that of the equator, is extended until it strikes the sphere of the spherical projection, producing a line. An infinite number of points on that line are projected stereographically, producing another line. Any plane may be treated in this manner, as illustrated in Fig. 21.

Fig. 21. Direct Projection of a Plane

In this illustration, a plane passing through the center of the sphere is shown. Its intersection with the sphere is a great circle, ABC, and the projection of an infinite number of points
on the plane of the equator in the customary manner produces a line, ADC. This line is an arc of a circle, and represents a great circle in the projection. Later planes of symmetry will be projected on stereograms in the direct manner.

Axes of symmetry, which are imaginary lines in crystals, will also be projected. These lines will be projected directly, in the manner of Fig. 22, where line OB is projected as line OB'.

Fig. 22. Direct Projection of a Line

If it is considered that an infinite number of points on line OB have been projected stereographically in the usual manner, OB' results. This is a direct projection of line OB.
As noted earlier, in spite of the fact that most actual work with the stereographic projection is done on a Wulff net, there is surprisingly little about its use in the literature commonly available to the average student. Probably the best source is in the laboratory exercise by Ehlers and Tettenhorst (1968).

An introductory text in crystallography should contain instructions for performing all of the usual operations on the Wulff net. An excellent method of teaching the Wulff net is the method used in *Angular Relations of Lines and Planes* by Higgs and Tunell (1959). No crystallography or mineralogy text uses their approach. In Higgs and Tunell there are numerous problems of a geologic nature which they proceed to explain and solve by the stereographic net. Included with the text are plastic sheets with the problems worked in the usual manner. The reader can place the solutions over the net and follow the instructions given step by step. Even a complicated procedure can rather readily be learned in this manner. The only drawback obviously is the possible cost to the publisher.

For demonstrating in the classroom, an excellent device is a Wulff net printed on transparent plastic. When used with an overhead projector the image size is large enough to be seen readily by students seated in the rear. Furthermore, quite good accuracy can be obtained and the overlay sheets can be retained and used again.
The problems worked should not be simply exercises on the projection. They should be real problems having crystallographic significance, even though early in the game they must of necessity be straightforward and easy to follow. If the course involved is in mineralogy, such problems may be used to bring to the students' attention facts concerning common minerals, or information on minerals he is unlikely to encounter elsewhere. Furthermore, problems given early in the course can be enlarged later when additional concepts have been covered. For instance, the illustrative examples given next can later be expanded to include determination of symmetry, designation of the class by the International System and the faces by Miller symbols, determination of zonal relationships, etc.
STEREOGRAPHIC PROJECTION PROBLEMS

In each case plot the faces on the stereographic projection.

1. Where the waves of the Aegean wash the ancient lead slags of Laurion, from which came much of the wealth of Athens in the Golden Age of Greece, crystals of basic lead chloride Pb(OH)Cl, the mineral laurionite, have formed over the centuries. These crystals yield angular measurements as follows:

<table>
<thead>
<tr>
<th>Face</th>
<th>$\phi$</th>
<th>$\rho$</th>
<th>Face</th>
<th>$\phi$</th>
<th>$\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>90°00'</td>
<td>90°00'</td>
<td>7</td>
<td>15°27'</td>
<td>90°00'</td>
</tr>
<tr>
<td>2</td>
<td>22°33'</td>
<td>90°00'</td>
<td>8</td>
<td>-51°15'</td>
<td>70°30'</td>
</tr>
<tr>
<td>3</td>
<td>90°00'</td>
<td>36°16'</td>
<td>9</td>
<td>-157°27'</td>
<td>90°00'</td>
</tr>
<tr>
<td>4</td>
<td>-90°00'</td>
<td>36°16'</td>
<td>10</td>
<td>128°45'</td>
<td>70°30'</td>
</tr>
<tr>
<td>5</td>
<td>51°15'</td>
<td>70°30'</td>
<td>11</td>
<td>-128°45'</td>
<td>70°30'</td>
</tr>
<tr>
<td>6</td>
<td>-22°33'</td>
<td>90°00'</td>
<td>12</td>
<td>-90°00'</td>
<td>90°00'</td>
</tr>
</tbody>
</table>

2. Of the three natural polymorphs of TiO$_2$, Rutile is the most common. Excellent crystals have been found near Ambatofinandrahana in Madagascar and at Magnet Cove, Arkansas. The following forms have been measured:

<table>
<thead>
<tr>
<th>Face</th>
<th>$\phi$</th>
<th>$\rho$</th>
<th>Face</th>
<th>$\phi$</th>
<th>$\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0°</td>
<td>90°00'</td>
<td>8</td>
<td>135°00'</td>
<td>90°00'</td>
</tr>
<tr>
<td>2</td>
<td>-</td>
<td>0°00'</td>
<td>9</td>
<td>-45°00'</td>
<td>90°00'</td>
</tr>
<tr>
<td>3</td>
<td>90°00'</td>
<td>90°00'</td>
<td>10</td>
<td>133°00'</td>
<td>42°20'</td>
</tr>
<tr>
<td>4</td>
<td>45°00'</td>
<td>90°00'</td>
<td>11</td>
<td>-135°00'</td>
<td>90°00'</td>
</tr>
<tr>
<td>5</td>
<td>45°00'</td>
<td>42°20'</td>
<td>12</td>
<td>-45°00'</td>
<td>42°20'</td>
</tr>
<tr>
<td>6</td>
<td>180°00'</td>
<td>90°00'</td>
<td>13</td>
<td>-135°00'</td>
<td>42°20'</td>
</tr>
<tr>
<td>7</td>
<td>-90°00'</td>
<td>90°00'</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Fig. 23 illustrates the principle of the gnomonic projection as seen in vertical cross section. $A'$ is the spherical pole and $A''$ the gnomonic pole of face A. If $ON$ is the radius of the sphere of projection, then

$$\frac{NA''}{ON} = \tan \rho$$

and

$$NA'' = ON \tan \rho$$

Thus, if the plane of the paper is the plane of projection, the distance from the center of the gnomonic projection to the pole of a face is readily determined by the formula distance = radius $\times$ $\tan \rho$. A sphere of any radius can be used, with 5 cm or 10 cm
being easily calculated, and therefore convenient. If the angle rho is above $60^\circ$, the projection becomes very large, as the tangent is increasing rapidly in magnitude. A face with a rho of $90^\circ$ cannot be plotted directly, which is related to the fact noted earlier that in cartographic work only areas less than a hemisphere in extent can be mapped on the gnomonic projection.

The angle phi is plotted in exactly the same way as on the stereographic, being measured clockwise or counterclockwise from the $0^\circ$ starting point.

On Fig. 24 the following faces have been plotted on a gnomonic projection using a 5 cm radius.

1. $\phi = 83^\circ$, $\rho = 40^\circ$
2. $\phi = -30^\circ$, $\rho = 28^\circ$
3. $\phi = 120^\circ$, $\rho = 90^\circ$

Note the convention adopted to handle face 3. Since tan $90^\circ$ is equal to infinity, the proper phi is plotted, and an arrow placed on the end of the line, indicating the direction in which face 3 lies.
Fig. 24  Faces Plotted on the Gnomonic Projection
Symmetry is a familiar concept and everyone "knows" what is meant by symmetrical arrangement. The word suggests a systematic or periodic repetition of the elements of a whole. The markings of a piece of wallpaper, the petals of a flower, the patterns found in a rug, the arms of a starfish, the spokes of a wheel, or the sides of a polygon may be symmetrically distributed. However, in spite of the familiarity of the concept, it is a difficult word to define, and the usual dictionary definitions are not satisfactory. For instance, the following is from The American Heritage Dictionary (1969):

"A relationship of characteristic correspondence, equivalence, or identity among constituents of a system or between different systems."

For crystallographic purposes, such a definition is neither sufficiently broad nor precise. It is much more satisfactory to describe symmetry in an operational sense.

The characteristic feature of a symmetrical body or arrangement lies in the fact of periodic repetition. If one can describe the way in which the periodic repetition takes place, he is describing the symmetry. This may be done by referring the repetition to rotation about an axis, reflection across a plane, or inversion through a point. In the following sections, we will discuss each of these elements of symmetry, the axis, the plane, and the point.
Axis of Symmetry

If all parts and properties of a crystal can be brought into complete congruence periodically by a simple rotation about a line, that line is an axis of symmetry.

By "all parts and properties" we mean all crystal space; that is, all of the atoms, molecules, or ions that make up the internal structure; all physical properties such as hardness, index of refraction, electrical conductivity, or any others that are vectorial in nature; and especially important to our discussion, the geometric shape of any perfectly formed crystal.

By "congruence" we mean absolute sameness or complete identity. A rotation has brought the crystal into congruence when there is a one to one correspondence between all points and properties before and after the rotation. A simple example will illustrate the geometric meaning of a rotation axis. See Fig. 25.

![Fig. 25 Rotation Axis]
If one imagined that there is an axis perpendicular to the plane of the paper and passing through the center of the square, he might rotate the square around the axis $90^\circ$. It would occupy exactly the same position as before the rotation, and would appear identical in every respect. Indeed, an observer who looked away while the rotation was being made would be unaware that anything has happened. The rotation has achieved or produced congruence.

Axes are designated according to the number of times identity is achieved in a $360^\circ$ rotation. That is, a 3-fold axis brings about complete congruence three times in a $360^\circ$ revolution. The period of the axis is the smallest angular interval at which complete congruence occurs. Thus, the period of a 3-fold axis is $120^\circ$, of a 5-fold axis, $72^\circ$, and of a 6-fold axis $60^\circ$. To put it another way, $360^\circ$ divided by the number of times the axis produces congruence during a complete rotation is equal to the period of the axis. By convention, the period of an axis is always stated as the smallest angle at which repetition takes place. For example, a 6-fold axis repeats at $60^\circ$, $120^\circ$, $180^\circ$, $240^\circ$, $300^\circ$, and $360^\circ$, but its period is the smallest interval or $60^\circ$. In general terms, when the period of a symmetry axis is the angle alpha, repetition will occur $n$ times in $360^\circ$, where $n$ is an integer such that $n = \frac{360^\circ}{\alpha}$.

Fig. 26 shows one of the symmetry axes in a body of "shoe-box" shape. If such a body is rotated about the axis, it will come
Fig. 26 2-fold Axis

Fig. 27 3-fold Axis

Fig. 28 4-fold Axis

Fig. 29 6-fold Axis
into congruence twice, at $180^\circ$ and $360^\circ$. Therefore

$$n = \frac{360^\circ}{180^\circ} = 2$$

and the axis is called a 2-fold axis.

Or, $\alpha = \frac{360^\circ}{2} = 180^\circ$.

and the period of the axis, $\alpha$, is $180^\circ$.

In Fig. 27,

$$n = \frac{360^\circ}{120^\circ} = 3$$

or

$$\alpha = \frac{360^\circ}{3} = 120^\circ$$

and illustrated is a 3-fold axis of a period of $120^\circ$.

In the same manner Fig. 28 shows a 4-fold axis having a period of $90^\circ$, and in Fig. 29, there is a 6-fold axis with a period of $60^\circ$.

The action that brings about repetition is termed an operation, and the total number of times that repetition is achieved is the number of operations of a symmetry element. That is, a 3-fold axis which achieves congruence at $120^\circ$, $240^\circ$, and $360^\circ$ by successive rotations of $120^\circ$ is said to have three operations.

Note that a rotation of $360^\circ$ is considered to be an operation. This returns the rotated object to its original position and the end result is the same as if there had been no rotation at all.

So far in our definitions we have said nothing to preclude the possibility of axes of any period such that $\frac{360^\circ}{n}$.
only restriction being that \( n \) must be an integer. A 15-fold axis, repeating identical parts every \( 24^\circ \) is a geometrical possibility. Certainly many are familiar with the highly symmetrical pentagonal dodecahedron, one of the five regular Platonic solids of classical geometry. The pentagonal dodecahedron has several axes of 5-fold symmetry. However, it has long been a matter of observed fact that only 1-fold, 2-fold, 3-fold, 4-fold and 6-fold axes appear in crystals, and the Platonic dodecahedron does not exist in crystalline matter. Since obviously it would be an easy matter to construct by carving or other means a body having an axis or axes of symmetry of any period that is an integral part of \( 360^\circ \), it would seem that the existence of only five types of axes in crystals must be related to the internal nature of these bodies; and indeed, this is so.

The limitation exists because of the nature of a regular arrangement, as of the atoms in a crystal. We will now examine several of the proofs of the fact that only the types of axes mentioned occur in crystalline bodies. Each proof presents a different approach to the question.

Perhaps the proof most widely used in crystallography texts is that by Niggli (1919, p. 33). He stated that he believed it to be original at that time. The method of Niggli requires only plane geometry and trigonometry, but as usually presented, even in the original text, seems at several points to leave questions in the minds of students. In the following
presentation we will attempt to clarify these points:

Observe Fig. 30. Let a and $a_1$ be two nearest nodes in a regular arrangement, the distance between them being $b$. Assume that axes of symmetry pass through a and $a_1$ perpendicular to the plane of the paper, and let the period of the axes be any angle $\alpha$. Rotate about a and $a_1$ through $\alpha$ producing $a_2$ and $a_3$.

The question often asked at this point is "How can you rotate about a in a counterclockwise direction and about $a_1$ in a clockwise direction?" The answer can be put in at least two ways:

(1) If $\alpha$ is a symmetry operation, so also is $-\alpha$. Referring back to the square in Fig. , if we achieve congruence by rotating $90^\circ$ clockwise, we can also achieve it by rotating $90^\circ$ counterclockwise and returning to the original position.

(2) Since $\alpha$ is an integral part of $360^\circ$, a cannot only be carried into $a_3$ by a clockwise rotation, it can also reach the same point by one or more counterclockwise movements. Note Fig. 31.
Fig. 30 Niggli Proof
Fig. 31  6-fold Axis

Here is a figure with a 6-fold axis in the center and points located on the periphery 60° apart. Point $a_1$ can be carried into point $a_2$ by a 60° counterclockwise rotation, or by a 300° counterclockwise rotation.

Let us now return to the Niggli proof and Fig. 30. Draw line $c$ connecting $a_2$ and $a_3$. From $a$ and $a_1$ draw perpendiculars to the line $a_2a_3$. Then:

1. $c = n \cdot b$, where $n$ is a whole number or zero.

By construction the lines $a-a_1$, and $a_2a_2$ are parallel. From our earlier definition of a regular arrangement, parallel lines contain equally spaced nodes. At this point, it is easy to see intuitively the essence of this proof. It is clear that $c = n \cdot b$ only for certain angles. Since $a$ and $a_1$ are nearest nodes in a regular arrangement, the distance between $a_2$ and $a_3$ must be some integral multiple of the distance between $a$ and $a_1$. For rotations of 60°, 90°, 120°, and 180° the integers are respectively
0, 1, 2, and 3. For all other angles the distance $a-a_1$, $a_2-a_3$
are not related by an integer, that is, $c \neq n \cdot b$, and so the
resulting net is not a regular arrangement.

(2) $c = b \pm 2X$ (if $\alpha > 90^\circ$, it is $+2X$,
    if $\alpha < 90^\circ$, it is $-2X$)

(3) $X = \frac{b}{b} \cos \alpha$ (the cos of 180 - $\alpha = \cos \alpha$)

(4) $X = b \cos \alpha$

(5) $c = b \pm 2b \cdot \cos \alpha$ (from (2) and (4))

(6) $n \cdot b = b \pm 2b \cdot \cos \alpha$ (from (1) and (5))

(7) $n = 1 \pm 2 \cos \alpha$ (divide by b)

(8) $\cos \alpha = \pm \frac{n-1}{2}$

but if $n$ is an integer, so also is $n-1$. Let $n-1 = N$, which
is an integer. Then,

(9) $\cos \alpha = \pm \frac{N}{2}$, where $N$ is a whole number or 0.

Now a cosine value greater than 1 is not possible, so the
only values of $N$ are 0, 1, and 2. Note table 1.
<table>
<thead>
<tr>
<th>N</th>
<th>( \frac{N}{2} )</th>
<th>( + \cos \alpha )</th>
<th>( \alpha )</th>
<th>( - \cos \alpha )</th>
<th>( \alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>90°</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>1</td>
<td>.5</td>
<td>+ .5</td>
<td>60°, 360°</td>
<td>- .5</td>
<td>120°</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>+ 1</td>
<td>0°, 360°</td>
<td>- 1</td>
<td>180°</td>
</tr>
<tr>
<td>3</td>
<td>1.5</td>
<td>impossible</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>

Since the only possible values of \( \alpha \) are 0°, 60°, 90°, 120°, 180°, and 360°, the only possible axes are 1-fold, 2-fold, 3-fold, 4-fold, and 6-fold.
A somewhat different proof, in that it involves use of the sine rather than the cosine, is presented by Buttgenbach (1953, p. 32). He did not state that it is original, but neither is there a reference.

In Fig. 32, let 0 be points or atoms in a regular arrangement and let there be axis A, of any period d, passing through 0 perpendicular to the plane of the paper. Choose point N such that the distance a is as small as the distance between any such nodes.

![Fig. 32 Buttgenbach Proof](image)

Rotate N counterclockwise about 0 through the angle \( \alpha \) producing N', and, by a similar rotation in the opposite sense, produce N''.

It will prove instructive to examine again the rationale which permits the two rotations in opposite directions. Observe Fig. 33, and consider the numbers as a part of a regular arrangement with a 4-fold axis through the center of each number. Note that 24 and 25 represent two nearest points or atoms. If 24 is rotated clockwise around 25, it is repeated at 18, 26, and 32.
<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
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<td>49</td>
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</tbody>
</table>

*Fig. 33 Regular Arrangement*
If 24 is rotated counterclockwise around 25, it is repeated at 32, 26, and 18. The repetition at 18 is there, regardless of which way the rotation takes place. Indeed, a moment's examination shows that the entire array can be produced by continued rotations. If 25 is rotated about 26, the nodes at 19, 27, and 33 result. A rotation of 25 around 18 produces 17, 11, and 19, and so on for the entire lattice.

Now, draw lines $N N'$ and $N''$, and construct $N'M$ parallel to $N N''$, and $N'M$ parallel to $N N'$. Since $N'M$ is parallel to $N N''$, and there is a node at $N''$, so also there must be a point or atom at $M$. By construction it lies on the line of which $ON$ is a segment.

Draw $N'N''$, cutting $OM$ at $R$. Note that $\angle ORN'$ and $\angle ORN''$ are right angles.

From the figure,

1. $NM = 2 \ NR = 2 \ (a + OR)$

and $\frac{OR}{a} = - \cos \alpha$

2. $OR = - \ a \ \cos \alpha$

3. $NM = 2 \ (a - a \ \cos \alpha)$ from (1) and (2)

4. $= 2 \ a \ (1 - \cos \alpha)$

5. $1 - \cos \alpha = 2 \ \sin^2 \alpha$ (identity)

6. $NM = 2 \ a \ (2 \ \sin^2 \frac{\alpha}{2})$ (from (4) and (5))

7. $= 4 \ a \ \sin^2 \frac{\alpha}{2}$

By the nature of a regular arrangement,

8. $NM = ka$ (k is any integer)

9. $ka = 4 \ a \ \sin^2 \frac{\alpha}{2}$ (from (7) and (8))
Since \( k \) is an integer, and a sine value greater than one is impossible, the only possibilities are those shown in the following table:

**TABLE 2**

Possible Symmetry Axes

<table>
<thead>
<tr>
<th>( k )</th>
<th>( \frac{\chi}{2} )</th>
<th>( \frac{\alpha}{2} )</th>
<th>( \alpha )</th>
<th>( n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0°</td>
<td>0°</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{1}{2} )</td>
<td>30°</td>
<td>60°</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{1}{\sqrt{2}} )</td>
<td>45°</td>
<td>90°</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>60°</td>
<td>120°</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>90°</td>
<td>180°</td>
<td>2</td>
</tr>
</tbody>
</table>
One of the most widely quoted proofs is that of Barlow (1901). Not only is it used by crystallographers but it seems to be also a favorite of mathematicians. For instance, see Coxeter (1961). Barlow appears to have been the first to use the idea of a regular arrangement as a basis for his proof. In view of the date, a decade before the Laue experiment, Barlow's statement concerning structure is significant.

FUNDAMENTAL DEFINITION.--Homogeneity of structure consists in a likeness of the ultimate parts or molecules of a body, both as to their nature and their relative arrangement, of the following kind:--Corresponding to every mathematical point in the mass are found evenly distributed at finite intervals a number of points whose relation to the ultimate structure, regarded as of unlimited extent, is the same as that of the point selected; so that the aspect of this structure viewed successively from all such corresponding points is identically the same, although the actual orientations of the similar aspects may be different.

Barlow's proof, based upon his concept of structure, follows:

In Fig. 34, $A$ and $A'$ are identical axes in a homogeneous structure, and are perpendicular to the plane of paper. The distance between these two axes, $a$, is "as small as that separating any of such axes."

![Fig. 34 Axes Greater than 6-fold Impossible](image)

Rotate $A'$ about $A$ through the period of the axis $A$, thus producing $A''$ a third identical axis. The distance from $A'$ to $A''$ cannot be
less than $A A'$, already specified as the minimum separation of two such axes; therefore the triangle $A A' A''$ has an angle at $A$ which must be at least $60^\circ$, therefore an axis of greater than 6-fold is impossible.

It remains only to show that a 5-fold axis is not possible. Consider Fig. 35.

![Fig. 35 5-fold Axis Impossible](image)

Let $A$ and $A'$ be two nearest identical 5-fold axes. Locate $A''$ by rotation of $A'$ around $A$ for $360^\circ / 5 = 72^\circ$, and locate $A'''$ in the same manner by rotation in the opposite sense around $A''$. Note that $A A' = A A'' = A'' A'''$, and that angles $A A'' A'''$ and $A'' A A'$ both equal $72^\circ$. But since $A' A'''$ cannot be smaller than $A A''$, both of the angles $A' A A''$ and $A A'' A'''$ cannot be less than $90^\circ$; so 5-fold axes are impossible.

Of all of the available proofs, Barlow's seems to the author to be the most satisfactory in teaching beginning students. It is short, simple, and appeals especially to those who find most mathematical proofs unappealing.
The following proof is adapted from Wood (1964), who credits it to H. L. Frisch. It is very satisfying for those who understand the elementary use of vectors.

![Diagram showing a portion of a two-dimensional lattice](image)

Fig. 36 Frisch's Proof

Fig. 36 shows a portion of a two-dimensional lattice generated by the translations \( a \) and \( b \), so that

\[
T n_1 n_2 = n_1 a + n_2 b \quad \text{where } n_1 \text{ and } n_2 \text{ are any integers.}
\]

Allow \( a \) to be parallel to the \( x \) axis of a rectangular coordinate system. Let there be an axis of symmetry through the origin \( O \), perpendicular to the plane of the drawing, and having the angle \( \alpha \) as its period. Since \( \alpha \) is a symmetry operation, so also is \( -\alpha \).
If the vector $a$ is rotated about through the angle $\alpha$, the vector $a'$ results.

$$\frac{a' x}{2} = \cos \alpha$$

and $\frac{a' y}{a} = \sin \alpha$

So,

1. $a' x = a \cos \alpha$
2. $a' y = a \sin \alpha$

Since $a'$ is a point in the lattice, there must exist a lattice translation which will carry point $a$ into point $a'$. Such a translation, $T_{n_1' n_2'}$, is equal to $n_1' a + n_2' b$ where $n_1'$ and $n_2'$ are integers. So,

$$T_{n_1' n_2'} = n_1' a + n_2' b$$

In a similar manner, a rotation through $-\alpha$ moves point $a$ into lattice point $a''$.

$$\frac{a'' x}{a} = \cos -\alpha$$

and

$$\frac{a'' y}{a} = \sin -\alpha$$

and

1. $a'' x = a \cos -\alpha = a \cos \alpha$
2. $a'' y = a \sin -\alpha = -a \sin \alpha$

There must be also a lattice translation which will carry $a$ into $a''$. This translation is $T_{n_1'' n_2''}$ and equal to $n_1'' a + n_2'' b$
where \( n_1'' \) and \( n_2'' \) are integers.

(6) \( T_{n_1''} n_2'' = n_1'' a + n_2'' b \)

The sum of vectors \( a' \) and \( a'' \), that is, \( a' + a'' \), is equal to \( T_{n_1''} n_2'' + a + T_{n_1''} n_2'' + a \), since the movement from the origin to point \( a \) is the vector \( a \), and the movement from \( a \) to \( a' \) or \( a'' \) is \( T_{n_1''} n_2'' \) or \( T_{n_1''} n_2'' \) as the case may be.

(7) \( a' + a'' = T_{n_1''} n_2'' + T_{n_1''} n_2'' + 2 a \)

(8) \( a' + a'' = n_1 a + n_2 b + n_1'' a + n_2'' b + 2 a \)

(from (3), (6), and (7))

(9) \( a' + a'' = (n_1' + n_1'') a + (n_2' + n_2'') b \)

but

(10) \( a' x + a'' x = 2 \cos \alpha \) (from (1) and (4))

(11) \( a'y + a''y = 0 \) (from (2) and (5))

But since the translations in the \( a \) direction are along the \( x \) axis

(12) \( a' x + a'' x = (n_1' + n_1'' + 2) a \)

(13) \( (n_1' + n_1'' + 2) a = 2 a \cos \alpha \) (from (10) and (12))

(14) \( n_1' + n_1'' + 2 = 2 \cos \alpha \)

Now \( n_1' \), \( n_1'' \), and \( 2 \) are all integers, so \( 2 \cos \alpha \) must be an integer; but \( \cos \alpha \leq 1 \), so the only possible values are \( 0, \pm \frac{1}{2}, \pm 1 \), and the only possible axes are 1-fold, 2-fold, 3-fold, 4-fold, and 6-fold.
A proof such as this one is extremely difficult for most liberal arts students, but may prove very satisfying to engineers or others with a more complete background in mathematics. Most students have no intuitive feeling for the results of such an approach, and the less mathematically inclined may learn it, but won't really believe it!
Several proofs of the fact that only axes of 1-fold, 2-fold, 3-fold, 4-fold, and 6-fold symmetry can exist in crystals have been presented. These have all been predicated upon the lattice concept; but it is possible to develop this fact using only the "second law" of crystallography, the idea of rational indices. For such proofs see Gadolin (1867), Hilton (1903), and De Jong (1959). To my knowledge, it has not heretofore been demonstrated in exactly the following fashion:

In Fig. 37 let there be an axis of symmetry, 0, having a period of any angle , passing through 0 and perpendicular to the plane of the paper. Let OA be a line parallel to a possible edge of the crystal. Act upon line OA with the axis 0, thus repeating OA as OB, OC, OD, OE.....

Choose Ob, OC, and axis b as the crystallographic axes, which is permissible since these are all parallel to possible edge of the crystal. Let the face CB, which cuts OB and OC equally be the parametral face. We need not be concerned with the intercept on 0.

There is a possible face in the plane determined by OD and axis 0. Let this face be located in a parallel position so that it passes through point C and cuts axis OB at OH. Now, following the "Law of Rational Indices", the relationship OB is a rational one.

But,

(1) OC = OB

(2) Therefore, OC is rational also

Draw HK perpendicular to OC. Note
Fig. 37 Development of Possible Axes of Symmetry
Using Law of Rational Indices
(3) \( \angle KCH = \alpha \) (KCH and KOD are alternate interior angles)
(4) \( OK = KC \) (since OHC is an isosceles triangle)
(5) \( 2 \frac{OK}{OH} = OC \)
(6) \( \frac{OK}{OH} = \cos \alpha \)
(7) \( 2 \frac{OK}{OH} = 2 \cos \alpha \)
(8) \( \frac{OC}{OH} = 2 \cos \alpha \) (from (5) and (7))

But, from (2), \( OC \) is rational, and therefore \( 2 \cos \alpha \) is rational. If \( 2 \cos \alpha \) is rational, so also is \( \cos \alpha \) rational. The only angles which are integral parts of \( 360^\circ \) that have rational cosines are those shown in the following table:

**TABLE 3**

Rational Cosines

<table>
<thead>
<tr>
<th>Angle</th>
<th>Cosine</th>
<th>Axis</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>1</td>
<td>1-fold</td>
</tr>
<tr>
<td>60°</td>
<td>( \frac{1}{2} )</td>
<td>6-fold</td>
</tr>
<tr>
<td>90°</td>
<td>0</td>
<td>4-fold</td>
</tr>
<tr>
<td>120°</td>
<td>-( \frac{1}{2} )</td>
<td>3-fold</td>
</tr>
<tr>
<td>180°</td>
<td>-1</td>
<td>2-fold</td>
</tr>
</tbody>
</table>

It is well to remember that the original derivations of the 32 classes of necessity had to rest upon the principal of rational indices. Workers such as Hessel (1830) and Gadolin (1871) had no clean-cut theory of regular internal arrangement with which to work.
Coxeter (1961, p. 52, p. 61) defines a tessellation as "any arrangement of polygons fitting together so as to cover the whole plane without overlapping", and a regular tessellation as being one with "equal regular polygons". The problem of the limitations on possible regular tessellations is analogous to that of the question of possible symmetry axes in a regular arrangement. It does not speak to the question of 1 and 2-fold axes, but is a nice way of demonstrating the non-existence of 5-fold and greater than 6-fold axes in such situations.

Let us assume a regular polygon of \( K \) sides. Such a polygon can be divided into \( K \) triangles, one for each side.

![Fig. 38 Regular Polygon of K Sides and Angles of A°](image)

\( K \) triangles will have a total of \( K \times 180° \) of internal angles. Of this total, \( 360° \) is in the angles surrounding the center, and therefore the other angles equal \((K \times 180°) - 360°\).

Now \( K \times 180° - 360° = 180° (K-2) = (K-2) 180° \)
Let each angle of the polygon be \( A^\circ \), as in Fig. 38. There are \( K \) such angles, so

\[
A = \frac{(K - 2) 180^\circ}{K}
\]

If the polygons are to fill space, let \( N \) polygons fit so that their angles add up to \( 360^\circ \).

Thus \( N \times A = 360^\circ \), or \( A = \frac{360^\circ}{N} \)

Equate the two values for \( A \)

\[
\frac{360^\circ}{N} = \frac{(K - 2) 180^\circ}{K}
\]

\[
\frac{2}{N} = \frac{K - 2}{K}
\]

(divide by \( 180^\circ \))

\[
\frac{N}{2} = \frac{K}{K - 2}
\]

(invert)

\[
N = \frac{2K}{K - 2}
\]

\[
N = \frac{K - 4 + 4}{K - 2}
\]

(\( \text{add 4, subtract 4} \))

Remembering that \( N \) equals the total number of polygons meeting at a point, and \( K \) is the number of sides of each polygon, the relationship \( N = 2 + \frac{4}{K-2} \) can be used to test each possibility. If

\[
K = 1, \quad N = 2 + \frac{4}{1-2} = -2 \text{ which is negative and therefore impossible.}
\]

If \( K = 2 \), \( N = 2 + \frac{4}{2-2} = 2 + \frac{4}{0} \) and therefore indeterminate. If

\[
K = 3, \quad N = 2 + \frac{4}{3-2} = 6, \text{ and it is seen in Fig. 39 that 6 regular}
\]

polygons of three sides can meet to fill space.
If \( K = 4 \), \( N = \frac{2 + 4}{4 - 2} = 4 \), as in Fig. 40 and four sided polygons meet the requirements.

If \( K = 5 \), \( N = \frac{2 + 4}{5 - 2} = 3 \frac{1}{3} \), an impossibility.

When \( K = 6 \), \( N = \frac{2 + 4}{6 - 2} = 3 \), and 3 hexagons can meet to fill space, as in Fig. 41.

If \( K \geq 7 \), \( N \) would be a fraction, and therefore no regular polygons of greater than six sides can form a tessellation.
Consider some other facts about the nature of symmetry axes. So far nothing has been required of an axis than that it repeat or bring into congruence identical parts and properties at certain permissible periods. If a polyhedron with a square base and four sides, each of which is an equilateral triangle (Fig. 42), is rotated about the axis OA, it will assume an identical appearance four times in a rotation of 360°. Thus OA answered to the definition of an axis of symmetry. Note, however, that the two ends of the axis OA are not the same. The end at O is in the center of a square face. The end at A is at a corner formed by the intersection of four faces. That is, the axis presents a completely different aspect when viewed from opposite ends. Such a symmetry axis may be termed a polar axis. Polar axes are consistent with the definition of a regular arrangement and polar axes of symmetry occur often in crystals, the 3-fold axis of tourmaline, the 2-fold axis of quartz, and the 6-fold axis of zincite being common examples. The various possible relationships between the opposite ends of the same axis of symmetry will be discussed in more detail in a later section.

If such a pyramid is combined with an identical pyramid by joining the bases, the result is a dipyramid (Fig. 42). Such a figure has a 4-fold axis of symmetry, A' - A, but a view along this axis from either end presents the same aspect. Both ends terminate at a corner point determined by the intersection of four faces.

A symmetry axis of this type may be termed a bipolar axis and regarded as two identical polar 4-fold axes standing at 180° to.
Fig. 42 Bipolar Axes
each other. That is, the line OA makes an angle of $180^\circ$ with the line OA', producing a single bipolar 4-fold axis.

The question of the various possible relationships between opposite ends of a single symmetry axis is one that has not been adequately discussed. After the thirty-two point groups have been derived and therefore logically are available as a basis of discussion, we will consider these relationships. For the present, I am using the term "bi-polar" to mean an axis having opposite ends which are identical. That is, opposite ends can be brought into congruence by rotation around another axis.
In the discussions of the various symmetry elements and the
derivation of the thirty-two crystal classes, extensive use will
be made of the stereographic projection. The following conven­
tions are adopted:

1. Faces will be projected indirectly. That is, they will
be represented by the poles of the faces, with X indicating the
upper and 0 the lower hemispheres.

2. Symmetry elements will be projected directly.

3. Axes of symmetry will be marked by small geometric figures
indicating the nature of the axes

- 2-fold axis
- 3-fold axis
- 4-fold axis
- 6-fold axis

4. Vertical axes will be indicated by one of the above
symbols since both ends of the axis plot stereographically at the
same point.

5. Horizontal axes will be shown by dashed lines, with the
axial symbol at each end.

6. The primitive circle will be a dashed line. However,

7. Planes of symmetry will be solid lines, so that a solid­
lined primitive indicates a symmetry plane in the plane of the
paper.

8. A solid line with axial symbols depicts both a plane and
an axis contained in it.
Some other conventions will be introduced later, but the following illustrations should clarify the above.

Fig. 43 Three 2-fold Axes, Mutually Perpendicular, But No Planes.

Fig. 44 A Plane of Symmetry in the Plane of the Paper.
Fig. 45 Two Vertical Planes of Symmetry Intersecting in a 2-fold Axis.

Fig. 46 Three Mutually Perpendicular Planes of Symmetry, and Three 2-fold Axes, Each Lying in Two of the Planes
Note, however, that the dipyramid has additional symmetry. To say that it has a bipolar 4-fold axis alone does not express completely the symmetry of such a figure. It has been stated that the opposite ends of the 4-fold axis are identical. In symmetry terms this means that they are capable of being brought into congruence. The symmetry element which brings things into congruence — that is, produces identity — is the axis. Therefore, there must be a minimum of additional symmetry to accomplish the needed operation of bringing the opposite ends of the 4-fold axis into identity. This minimum might be a rotation of $180^\circ$ about another axis EE' which stands perpendicular to the original 4-fold axis. EE' in turn will be repeated every $90^\circ$ by the 4-fold axis thus creating another 2-fold axis DD'. Since opposite ends of EE' and DD' are identical these axes are also bipolar. Furthermore two EE' additional 2-fold axis, MM' and NN' also mutually perpendicular, and also with opposite identical ends and standing at right angles to each other, arise as a result of the initial symmetry. MM' and NN' are identical because of 4-fold axis; DD' and EE' are identical for the same reason; however, MM' is not identical to DD' or EE'. If this were so, — that is, if all four of the bipolar 2-fold axis were identical, then the vertical axis would be an 8-fold axis, which has been shown to be impossible.

The way in which the additional symmetry arises as a result of the presence of a bipolar 4-fold axis may be illustrated on the stereographic projection. In Fig. 47 the bipolar 4-fold axis will
Fig. 47 Stereogram of One 2-fold and One 4-fold Axis

Fig. 48 One 4-fold and Two 2-fold Axes

Fig. 49 One 4-fold and Three 2-fold Axes

Fig. 50 One 4-fold and Four 2-fold Axes
be brought into congruence end for end by a 2-fold axis which lies at right angles to it as at position M. The 4-fold repeats this 2-fold at N, N', and N'', and thus there are two identical 2-fold axes, MM' and NN', as in Fig. 48.

If a face is placed in a general position, X, the 4-fold axis will repeat it every $90^\circ$, at X', X'', and X'''. Furthermore the 2-fold axes will repeat the face in the lower hemisphere, at O, O', O'', and O''', in each instance $180^\circ$ from an X position.

Note in Fig. 49 that the presence of a 2-fold axis is now required at E, because face X is $180^\circ$ from identical face O', face C is $180^\circ$ from identical face X', face X''' is $180^\circ$ from identical face O'', and O'' is $180^\circ$ from identical face X''. Axis E performs exactly these operations. If it is considered polar as shown in Fig. 47, it is repeated by the 4-fold axis at D, E', and D', thus creating two bipolar 2-fold axes. Hence from a single 2-fold perpendicular and a 4-fold the total complement of four 2-fold axes has arisen of necessity.

Even though the discussion of symmetry elements is far from complete, it is obvious at this point simply from the discussion of bipolar axes that there is an interdependence among symmetry elements. The presence of certain symmetry elements will necessitate the presence of others.

This is related to the fact that since a crystal can be brought into congruence with itself only in a finite number of ways, the symmetry operations of a crystal form a finite group in the mathematical sense. In a later section some of the elementary
aspects of group theory will be discussed. At this point it is enough to state that in group theory, postulate number one states that the product, or result, of any number of operations of a group is also a member of the group. That is, if rotation about symmetry axis $A$ through its period $\alpha$ is followed by rotation around symmetry axis $B$ through its period $\alpha''$, the result, or product of the two operations must be the same as the result of a single operation, such as rotation about axis $C$ through its period, $\alpha''$. That is, whenever two rotations about intersecting axes of symmetry bring a face or other crystallographic feature from an initial point to a new identical position, there exists a third axis which will accomplish the same result in one operation. Thus if only one symmetry axis is present, no other axis arises; if there are two axes of symmetry, at least one other additional axis must be present which may by one operation produce the same final result as consecutive operations about the two other axes.

Some examples will clarify the above points. In Fig. 50, a rotation of $180^\circ$ around $MM'$ takes $X$ into $O'$; a rotation of $90^\circ$ around the central 4-fold axis takes $O$ into $O'$; but note that rotation of $180^\circ$ around $EE'$ also takes $X$ into $O'$. That is, the single operation of rotation around axis $EE'$ is exactly equivalent to the double operation of rotation around axis $MM'$, followed by rotation around the central axis.

A more concrete example may prove useful. Fig. 51 shows a crystal having three 2-fold axes of symmetry, $A$, $B$, and $C$. The face on the upper right is lettered $X$. 
Fig. 51 Three 2-fold Axes

If the crystal is rotated $180^\circ$ around Axis B, it becomes the face in the lower right rear, as in Fig. 52.

Fig. 52 Rotation About Axis B
If now it is rotated $180^\circ$ around axis $C$, it comes to the left lower front, as in Fig. 53.

Fig. 53 Rotation About Axis C

But, referring back to Fig. 51, notice that a rotation of $180^\circ$ around axis $A$ produces the same result as successive rotations around $B$ and $C$. This can be stated as

$$B_{\alpha'} \cdot C_{\alpha''} = A_{\alpha'}$$

where $\cdot$ is read as "followed by".

It follows that some of the limitations on the possible symmetry combinations of crystals are independent of the internal regular arrangement of crystalline matter. That is to say, there are limitations which are a result of the nature of finite groups. These limitations apply equally to carved or artificially faceted bodies as well as to crystals. One can readily perceive intuitively that it is impossible to produce by any means a body having
three 6-fold axes of symmetry. Each element of symmetry must repeat all other elements and most students feel instinctively that there must be limits.

There is some value at this point in taking a look at the body with the ultimate in symmetry, the sphere. The sphere has an infinite number of symmetry axes of any period one wishes to choose, as well as an infinite number of planes, plus of course a center of symmetry, and is group \([\infty\infty m]\). It can be pointed out that the seemingly endless repetition produced by three 6-fold axes would approach the sphere in shape, and thus ultimately destroy the 6-fold symmetry.

The most elegant approach to the relationship between multiple symmetry axes makes use of a construction credited to Leonard Euler, the great Swiss mathematician. Euler's construction has been used by many crystallographers; see for instance Hilton (1903, p. 24), Niggli (1919, p. 47), Donnay (1942, p. 44), and Buerger (1956, p. 36). The version presented here follows most closely that of Shubnikov (1964, p. 19) and Buerger.
In Fig. 54 let OA and OC be two axes of symmetry, with 0 the center of a sphere and A and C being the points at which the axes intersect the surface of the sphere. Let $\alpha$ be the period of axis OA and $\gamma$ the period of axis OC. Draw the arc of the great circle, $R$, connecting A and C. Draw $S$, the arc of a great circle making the angle $\frac{\alpha}{2}$ with arc $R$, and $T$, the arc of a great circle making the angle $\frac{\gamma}{2}$ with arc $R$. Arches $S$ and $T$ intersect at point $B$.

Now, obtain $R'$ by rotating $R$ about $A$ through the angle $\alpha$. Note that the angle between arcs $S$ and $R'$ is $\frac{\alpha}{2}$, since $R \land R'$ is $\alpha$ and $R \land S$ is $\frac{\alpha}{2}$. Obtain $R''$ in the same manner by rotation through the angle $\gamma$ around $C$. The angle between $T$ and $R''$ is $\frac{\gamma}{2}$.

In a similar fashion, arc $T'$ making the angle $\frac{\beta}{2}$ with $R$, and arc $S'$, making the angle $\frac{\alpha}{2}$ with $R$ can be constructed.

Now let $B$ become the point of emergence of an axis $OB$, having as its period twice the angle between arcs $T$ and $S$; that is, the period of $OB$ is $\beta$, and the angle between $T$ and $S$ is $\frac{\beta}{2}$.

The spherical triangle $ACB'$, $ABC'$, and $BCA'$ are now congruent, having all sides and all angles equal. Call them respectively triangles I, II, and III.

Rotate triangle I counterclockwise $\alpha$ degrees about axis $A$, and it becomes triangle II. Rotate triangle II counter-
Fig. 54 Euler Proof
clockwise $\beta$ degrees about axis B, and it becomes triangle III; but if triangle I is rotated clockwise $\gamma$ degrees about axis C, it also becomes triangle III.

Therefore, the result of rotation through $\alpha$ about A, followed by rotation about B through $\beta$, is equivalent to rotation in the opposite sense about C through $\gamma$.

Or $\alpha \beta = C - \gamma$

From the Euler construction it is possible to calculate the relationship between arcs R, S, and T, and the angles between these arcs, $\frac{\alpha}{2}$, $\frac{\beta}{2}$, and $\frac{\gamma}{2}$. It is clear that ABC is the spherical triangle shown in Fig. 55.

![Fig. 55 Triangle from Euler Proof](image)

Note that the three arcs represent the angles between the three axes, and $\frac{\alpha}{2}$, $\frac{\beta}{2}$, and $\frac{\gamma}{2}$ represent 1/2 the periods of each of the axes. Therefore, if we solve the spherical triangle of Fig. 55, we have all of the relationships possible among multiple symmetry axes. These relationships are given in
texts in spherical trigonometry, for instance see *Engineering Trigonometry*. (Pease and Wadsworth, 1946, p. 282). According to the Law of Cosines for angles,

\[ \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \cos S - \cos \frac{\alpha}{2} \cos \frac{\beta}{2} = \cos \frac{\gamma}{2} \]

\[ \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \cos S = \cos \frac{\gamma}{2} + \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \]

\[ \cos S = \cos \frac{\gamma}{2} + \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \]

In the same manner, the equations for arcs \( R \) and \( T \) can be written

\[ \cos R = \frac{\cos \frac{\beta}{2} + \cos \frac{\alpha}{2} \cos \frac{\gamma}{2}}{\sin \frac{\alpha}{2} \sin \frac{\gamma}{2}} \]

\[ \cos T = \frac{\cos \frac{\alpha}{2} + \cos \frac{\beta}{2} \cos \frac{\gamma}{2}}{\sin \frac{\beta}{2} \sin \frac{\gamma}{2}} \]

\( \alpha, \beta, \) and \( \gamma \) represent periods of possible axes, and therefore are limited to angles of \( 60^\circ, 90^\circ, 120^\circ, 180^\circ, \) and \( 360^\circ \). Furthermore, one should suspect at this point that arcs \( R, S, \) and \( T \) will be limited to certain values, since the equations produce cosine values for these arcs, and a cosine cannot be greater than one.
Indeed this proves to be the case, as the following examples readily show:

First examine the possibility of having the combination of three axes be 234; that is, the first rotation is about a 2-fold axis (period 180°), the second about a 3-fold axis (period 120°), and the third about a 4-fold axis (period 90°). When the proper values are substituted in equation (3), it is seen that

\[
\cos S = \frac{.707 + 0 \times .5}{1} = .866
\]

\[
= \frac{.707}{.866}
\]

\[
\cos S = .816
\]

\[
S = 35° 16'
\]

Thus, 234 is a viable combination and the angle between the 2-fold axis and the 3-fold axis is 35° 16'.

To continue with the combination 234, set up equation (4) to find the angle between the 2-fold and the 4-fold axis.

\[
\cos R = \frac{.5 + 0 \times .707}{1 \times .707}
\]

\[
= \frac{.5}{.707}
\]

\[
\cos R = .707
\]

\[
R = 45°, \text{ the angle between the 2-fold and the 4-fold axes.}
\]

The angle between the 2-fold and the 3-fold axes can be obtained from equation (5).
Thus it is that 234 is a possible combination of symmetry axes, and the angles involved are:

- 2-fold ∧ 3-fold = 35° 15' 52"
- 2-fold ∧ 4-fold = 45°
- 3-fold ∧ 4-fold = 54° 44' 08"

The first look at these angles always seems to startle students. A useful exercise is to have them calculate the angles between the space diagonals of a cube using trigonometry. They seem to associate only right angles with a cube and usually are surprised by the 35°+ and 54°+ values. For that matter, most are not aware of the 3-fold axes in a cube, and may discover them at this point.

Now try the combination 246. That is, examine the possibility that a rotation around a 2-fold axis followed by a rotation around a 3-fold axis is the equivalent of a rotation around a 6-fold axis.
Using equation (3)

\[ \cos S = \frac{.866 + 0 \times .5}{1 \times .707} \]

\[ = \frac{.866}{.707} \]

\[ \cos S = >1 \]

Cosine values greater than one do not exist, and so the combination 246 is impossible.

Note that equations (3), (4), and (5) are not predicated upon the Law of Rational Indices or regular internal arrangement. It is impossible to have the combination 246 in a body produced by any means. By the same token, it may be that 235 is a legitimate combination. To examine that possibility, note that the period of a 5-fold axis \( = \frac{360^\circ}{5} = 72^\circ \).

Using equation (3)

\[ \cos S = \frac{.809 + 0 \times .5}{1 \times .866} \]

\[ = .934 \]

\( S = 20^\circ 54' \), the angle between the 2-fold and the 3-fold axes.

Now, taking equation (4)

\[ \cos R = \frac{.5 + 0 \times .809}{1 \times .588} \]

\[ = .850 \]

\( R = 31^\circ 47' \), the angle between the 2-fold and the 5-fold axes.

Setting up equation (5)
\[
\cos T = 0 + \frac{.5 \times .809}{.866 \times .588}
\]

\[= .794\]

\[T = 37^\circ 24',\] the angle between the 3-fold and the 5-fold axes.

It has been demonstrated that 235 is a legitimate combination of symmetry axes in a polyhedron, but at the same time it is clear that such a body could not exist as a crystal because of the presence of the 5-fold axis; this combination of symmetry axes is, however, an extremely interesting one, representing the symmetry of two of the five regular or Platonic solids of classical geometry. These are the pentagonal dodecahedron and the icosahedron, pictured in Fig. 56. It should be noted, however, that these bodies have other symmetry in addition to the axes.

There is a real question here as to how much time one can afford to spend on the non-crystallographic applications of symmetry theory. The Platonic solids are not only a lot of fun but they do have important applications in biology. Furthermore, pentagonal symmetry plays an essential role in the symmetry of molecules, if not in crystals.

I find it useful to give the following simple proof of the existence of only five regular polyhedra:

At least three faces must meet at each apex, and the sum of the face angles forming the apex must be less than 360°. Therefore since the internal angles of a regular hexagon are 120°, and any
Pentagonal Dodecahedron

Icosahedron

Fig. 56
regular polygon of more than six sides has even larger angles (for instance, 135° in the case of an octagon), it is not possible to have a regular polyhedron with faces having more than five sides.

The internal angles of a pentagon are 108°, so a convex polyhedral angle of less than 360° can be formed from three regular pentagons, but four times 108° = 432°, so there is only one possibility, which is, of course, the pentagonal dodecahedron.

A square has internal face angles of 90°, so three square faces meet the requirement of being less than 360°, but four do not. So, there is only the cube.

Now an equilateral triangle has angles of 60°, so convex polyhedral angles may be formed by combining three, four, or five such figures, but not six. This produces the tetrahedron, octahedron, and icosahedron.

The pentagonal dodecahedron and the icosahedron are class $\overline{5} 3 2$, having six 5-fold, ten 3-fold, and 15 2-fold axes, plus fifteen planes and of course a center, as shown in Fig. 57. The dodecahedral faces are located on the ends of the 5-fold axes, and

![Fig. 57 Class $\overline{5} 3 2$](image-url)
the icosahedral faces on the ends of the 3-folds. There are a
total of 120 operations, which would be the number of faces on the
general form of this symmetry class.

By following the procedure of substituting all possible
crystallographic values in equations (3), (4), and (5), one can
find all of the combinations of symmetry axes possible in
crystals. (Much of this is available in Buerger, 1956, p. 39-45.)
It is found that the only possible combinations are 222, 223, 224,
226, 233, 234, 236, 244, 333, of which the latter three are
trivial. In each of these three cases, the angle between the
axes is zero. That is to say, they are coincident.

In 236, a rotation of 180° clockwise, followed by a rotation
of 120° clockwise, gives a result of 300° clockwise. A counter-

clockwise rotation of 60° is equivalent.

In 244, a rotation of 180° followed by one of 90° is the
same as 90° rotation in the opposite sense. The combination 333
simply means that 120° plus 120° in one direction equals 120° in
the opposite direction. These situations all fit the requirement
that the third rotation be the equivalent of a combination of
the first two rotations, but have no crystallographic significance.
The relationships that are significant are shown in Table 4.

<table>
<thead>
<tr>
<th>Combinations</th>
<th>Angle between 1st and 2nd axes</th>
<th>Angle between 2nd and 3rd axes</th>
<th>Angle between 1st and 3rd axes</th>
</tr>
</thead>
<tbody>
<tr>
<td>222</td>
<td>90°</td>
<td>90°</td>
<td>90°</td>
</tr>
<tr>
<td>223</td>
<td>60°</td>
<td>90°</td>
<td>90°</td>
</tr>
<tr>
<td>224</td>
<td>45°</td>
<td>90°</td>
<td>90°</td>
</tr>
<tr>
<td>226</td>
<td>30°</td>
<td>90°</td>
<td>90°</td>
</tr>
<tr>
<td>233</td>
<td>54°44'</td>
<td>70°32'</td>
<td>54°44'</td>
</tr>
<tr>
<td>234</td>
<td>35°16'</td>
<td>54°44'</td>
<td>45°</td>
</tr>
</tbody>
</table>
Fig. 58 shows the axial combinations and Fig. 59 the essential angles.

Fig. 58 Possible Axial Combinations
Fig. 59 Angles Between Possible Axes
Now consider the following question:

Can identical axes of symmetry stand to each other in any angle, or are such relationships limited? The answer is that they are limited, and it is possible to determine the relationships. The development that follows is after Niggli (1919, p. 48).

In Fig. 60, let $OJ$ and $OJ_1$ be identical axes of symmetry passing through two nearest identical points ($J$ and $J_1$) in a regular arrangement, the angle between the two axes being $\Theta$.

Make $OJ$ and $OJ_1$ equal and the distance between two identical points $J$ and $J_1$.

Let the period axes of $OJ$ and $OJ_1$ be $\alpha$.

Rotate $OJ$ about $OJ_1$, and $OJ_1$ about $OJ$, through angle $\alpha$ producing $OJ_2$ and $OJ_3$, axes which are identical to $OJ$ and $OJ_1$. $J_2$ and $J_3$ are points in the regular arrangement identical to points $J$ and $J_1$. Connect $J_2$ and $J_3$.

Note.

Triangles $OJ_2$ and $OJ_1J_3$ are isosceles and congruent with triangle $OJ_1J_3$.

$J_2J_3$ is parallel to $JJ_1$ by construction, therefore $J, J_1, J_2$ and $J_3$ are coplanar.

Therefore, from an earlier proof, $\gamma$ is limited to $60^\circ$, $90^\circ$, $120^\circ$, $180^\circ$, and $360^\circ$, the permissible periods of symmetry axes.

Through any point on $OJ$, $P$, pass a plane $RPR_1$ perpendicular to axis $OJ$ and intersecting plane $JJ_1J_2J_3$ along the line $RR_1$.

$OJ$ is now the edge of a dihedral angle $J_1JOJ_2$ formed by
Fig. 60 Relationships Between Identical Symmetry Axes
rotating the triangle $OJ_1J$ about $OJ$ through the angle (plane angle $RPR_1$).

Now, pass a plane through $OJ$, bisecting the dihedral angle $J_1JO-J_2$ and cutting the plane $J_1J_2J_3$ along the line $JO$.

Bisect angle $\Theta$ of triangle $J_0J_1$

Then, angle $RJP = 90^\circ - \frac{\Theta}{2}$

angle $QJR = \frac{\gamma}{2}$

angle $QPR = \frac{\alpha}{2}$

Thus, $JPQR$ is a right angled tetrahedron, with angles $JPQ$, $PQR$, $JQR$, and $JPR$ being $90^\circ$

and

$$\sin \frac{\gamma}{2} = \frac{QR}{JR}$$

(1) $\sin \frac{\gamma}{2} \cdot JR = QR$

$$\sin \frac{\alpha}{2} = \frac{QR}{PR}$$

(2) $\sin \frac{\alpha}{2} \cdot PR = QR$

$$\cos \frac{\Theta}{2} = \frac{PR}{JR}$$

(3) $\cos \frac{\Theta}{2} \cdot JR = PR$

(4) $\sin \frac{\gamma}{2} \cdot JR = \sin \frac{\alpha}{2} \cdot PR$ (from (1) and (2))

(5) $\sin \frac{\gamma}{2} \cdot JR = \sin \frac{\alpha}{2} \cdot \cos \frac{\Theta}{2} \cdot JR$ (from (3) and (4))
(6) \( \sin \frac{\gamma}{2} = \sin \frac{\alpha}{2} \cdot \cos \frac{\theta}{2} \) (divide by JR)

(7) \( \cos \frac{\theta}{2} = \sin \frac{\gamma}{2} \cdot \sin \frac{\alpha}{2} \)

But \( \frac{\gamma}{2} \) and \( \frac{\alpha}{2} \) are limited to one-half the permissible periods of axes of symmetry, \(30^\circ, 45^\circ, 60^\circ, 90^\circ, \) and \(180^\circ\).

In Table 5, all permitted values of \( \sin \frac{\alpha}{2} \) and \( \sin \frac{\gamma}{2} \) are related according to equation (7), producing all permitted values of \( \cos \frac{\theta}{2} \).

<table>
<thead>
<tr>
<th>Possible values of sin ( \frac{\alpha}{2} )</th>
<th>Possible Values of sin ( \frac{\gamma}{2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha = 360^\circ )</td>
<td>0            0            0            0            0</td>
</tr>
<tr>
<td>( \alpha = 180^\circ )</td>
<td>1            0            .5           .707           .866           1</td>
</tr>
<tr>
<td>( \alpha = 120^\circ )</td>
<td>.866         0            .577         .816           1</td>
</tr>
<tr>
<td>( \alpha = 90^\circ )</td>
<td>.707         0            .707         1</td>
</tr>
<tr>
<td>( \alpha = 60^\circ )</td>
<td>.5           0            1</td>
</tr>
</tbody>
</table>
TABLE 6  Values of $\frac{\Theta}{2}$

<table>
<thead>
<tr>
<th>When $\Theta$ is as below</th>
<th>The corresponding values of $\frac{\Theta}{2}$ are</th>
</tr>
</thead>
<tbody>
<tr>
<td>$360^\circ$</td>
<td>$90^\circ$ $90^\circ$ $90^\circ$ $90^\circ$ $90^\circ$ $90^\circ$</td>
</tr>
<tr>
<td>$180^\circ$</td>
<td>$90^\circ$ $60^\circ$ $45^\circ$ $30^\circ$ $0^\circ$ $0^\circ$</td>
</tr>
<tr>
<td>$120^\circ$</td>
<td>$90^\circ$ $54^\circ$ $44^\circ$ $8^\circ$ $35^\circ$ $15^\circ$ $52^\circ$ $0^\circ$ $0^\circ$</td>
</tr>
<tr>
<td>$90^\circ$</td>
<td>$90^\circ$ $45^\circ$ $0^\circ$ $0^\circ$ $0^\circ$ $0^\circ$ $0^\circ$ $0^\circ$ $0^\circ$</td>
</tr>
<tr>
<td>$60^\circ$</td>
<td>$90^\circ$ $0^\circ$ $0^\circ$ $0^\circ$ $0^\circ$ $0^\circ$ $0^\circ$ $0^\circ$ $0^\circ$</td>
</tr>
</tbody>
</table>

However, we are not concerned with $\frac{\Theta}{2}$, but with $\Theta$, the permissible angles of intersection between identical axes.

TABLE 7  Angles at Which Identical Axes Can Intersect

<table>
<thead>
<tr>
<th>Periods</th>
<th>Angles at which identical axes can intersect</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two fold (180°)</td>
<td>$0^\circ$ $180^\circ$ $120^\circ$ $--$ $--$ $90^\circ$ $60^\circ$</td>
</tr>
<tr>
<td>Three-fold (120°)</td>
<td>$0^\circ$ $180^\circ$ $--$ $109^\circ$ $28^\prime$ $16^\prime$ $70^\circ$ $31^\prime$ $44^\prime$ $--$ $--$</td>
</tr>
<tr>
<td>Four-fold (90°)</td>
<td>$0^\circ$ $180^\circ$ $--$ $--$ $--$ $90^\circ$ $--$</td>
</tr>
<tr>
<td>Six-fold (60°)</td>
<td>$0^\circ$ $180^\circ$ $--$ $--$ $--$ $--$ $--$</td>
</tr>
</tbody>
</table>
If there are identical axes of symmetry in a crystal, there must be another axis about which they can be rotated into congruence. This is the crystallographic interpretation of identity. Now, this axis which produces the congruence must make some angle with the congruent axes (and the same angle, otherwise they would not be identical), and it must have one of the permissible periods. Let us now address ourselves to the following question:

Is there a relationship between the angle between identical axes and the other axis which makes them identical?

The following is adapted from Niggli (1919, p. 47).

In Fig. 61, let OA and OB be identical axes of symmetry intersecting at angle $\theta$. If these axes are identical, they can be brought into congruence by rotation about a third axis of symmetry. Let this axis be OC having as a period $\alpha$.

The angle between OC and OB is $\beta$. Therefore, since OA and OB are identical, the angle between OC and OB is $\beta$.

Now, pass a plane ABC perpendicular to OC, and a plane OCP through the axis OC and bisecting the dihedral angle $A - CO - B$. Note the following:

AOB and ACB are isosceles triangles, and

Each face of the tetrahedron OCPB (or OCPA) is a right triangle, angles OPB, CPB, OCB, and OCP being 90°.
Fig. 61 Relationships Between Like and Unlike Axes
Therefore

\[ \sin \frac{\theta}{2} = \frac{BP}{OB} \]

(1) \[ \sin \frac{\theta}{2} \cdot OB = BP \]

\[ \sin \frac{\alpha}{2} = \frac{BP}{CB} \]

(2) \[ \sin \frac{\alpha}{2} \cdot CB = BP \]

\[ \sin \beta = \frac{CB}{OB} \]

(3) \[ \sin \beta \cdot OB = CB \]

Combining (1), (2), and (3)

\[ \sin \frac{\theta}{2} \cdot OB = \sin \frac{\alpha}{2} \cdot \sin \beta \cdot OB \]

(4) \[ \sin \frac{\theta}{2} = \sin \frac{\alpha}{2} \cdot \sin \beta \] (dividing by OB)

or

(5) \[ \sin \beta = \sin \frac{\theta}{2} \]

\[ \frac{\sin \frac{\alpha}{2}}{\sin \frac{\alpha}{2}} \]

This is a perfectly general relationship and would apply to any such right-angled tetrahedron. However, in the case at hand, \( \alpha \) represents the possible values of periods of symmetry axes, and therefore is limited; furthermore, we have at hand the possible values of \( \theta \), the angles between like axes. Therefore, the possible values of \( \beta \), the angle between like and unlike axes, can readily be calculated.
The following statements concerning axes of symmetry in crystals can now be made:

1. When rotated through $360^\circ$ about an axis of symmetry, all parts and properties of a crystal come into complete congruence periodically.

2. If $n$ is an integer representing the number of times congruence is achieved in a complete rotation, $\frac{360^\circ}{n}$ is the period of the axis; and axes are named by the value of $n$.

3. Due to internal regular arrangement, crystals have axes only of 1-fold, 2-fold, 3-fold, 4-fold, and 6-fold symmetry. Or, the permissible periods are $360^\circ$, $180^\circ$, $120^\circ$, $90^\circ$, and $60^\circ$.

4. A polar axis is one having opposite ends unrelated by any sort of symmetry operation; therefore such an axis can exist alone in a crystal.

5. A bipolar axis has opposite ends which are identical. Therefore, such axes cannot exist alone but require the presence of additional symmetry.

6. While one axis of symmetry may exist alone, if two are present a third arises of necessity. The relationships among these three axes can be shown by formulae such as:

$$\cos S = \frac{\cos \frac{\gamma}{2} + \cos \frac{\alpha}{2} \cos \frac{\beta}{2}}{\sin \frac{\alpha}{2} \sin \frac{\beta}{2}}$$

7. Using these formulae it has been determined that the only non-trivial axial combinations are 222, 223, 224, 226, 233, and 234.
Furthermore, the necessary angular relationships have been calculated. (Table 4)

8. Identical axes of symmetry can intersect only at certain angles. (Table 8)

9. When two axes of symmetry are identical in the crystallographic sense, there must be present a third axis about which the first two can be rotated into congruence. The following formulae shows the relationship:

$$\sin \beta = \frac{\sin \Theta}{\sin \frac{\alpha}{2}}$$

$\Theta$ = the angle between the identical axes

$\alpha$ = the period of the other axis which brings the identical axes into congruence.

$\beta$ = the angle between the identical axes and the other axis.
Planes of Symmetry

The type of symmetry previously considered consists of the periodic repetition of a decorative motif, a crystal face, or an arrangement of atoms or ions, in the course of rotation through $360^\circ$ about a line or axis. Complete congruence was obtained at each position of repetition. An axis of symmetry relates objects or pattern units which are absolutely identical; but there is another sort of symmetry which we will now consider. A right hand and a left hand are symmetrical to each other, as are a right foot and a left foot. However, no amount of rotation will make a right hand and a left hand congruent. Such objects are said to be enantiomorphous, or in more common terms, they are mirror images of each other. Using this concept we may define a plane of symmetry as an imaginary plane which divides the crystal into two halves, each of which is a mirror image of the other.

Symmetry by "reflection" through an imaginary plane is the most familiar type of symmetry. The external shape of the human body (if one excludes more minor details) many wallpaper and necktie patterns, the majority of common household objects and furniture all provide example of symmetry by "reflection" in a plane; but one must keep in mind that the "reflection" is simply a way of stating the nature of the repetition.
The optics of mirror reflection are such that the mirror seems to repeat each point of an object on a line perpendicular to the mirror and as far behind the mirror as the original point is in front. Some examples in two dimensions follow:

![Fig. 62 Repetition by Symmetry Planes](image)

In Fig. 62, lines A A', B B', C C', and D D' are all the traces of planes of symmetry. Note, for instance, that point a has its counterpart at a', which is as far behind "mirror" D D' as a is in front; at a", which represents a "reflection" in A A', at a''' representing "reflection" by B B', and a'''' upon repetition by C C'.

In Fig. 63, there is a plane of symmetry A A', but B B' does not represent the trace of such a plane. Points a, b, and c have their "reflections" at a', b', and c', but not at a", b", or c". 
Fig. 63 Points Repeated by a Symmetry Plane

Fig. 64 Planes Which Are Not Symmetry Planes
Note that in Fig. 64, A A' and B B' are not planes of symmetry, though students often interpret them as such. For instance, point "a" does not have its counterpart in the proper position across either A A' or B B'.

Two dimensional figures, or perspective drawings of three dimensional objects, are not a completely satisfactory substitute for actual objects. Much of the preceding discussion can be made clear by the use of right and left gloves stuffed with cotton or other material, a pair of scissors, a geologist's hammer, mirrors, (about which more will follow), paper clips - indeed, almost anything that is at hand. Above all, crystal models should be introduced, not only wooden, plaster, or plastic representations of forms, but styrofoam ball and "exploded" wooden ball models of structures. The student need not be told at this point about symmetry elements that have not been discussed - but if even one discovers the rotary inversion relationship, or possibly a glide plane, real progress is being made! Wooden models of right and left-handed quartz crystals will quickly bring home the point that they are similar, but not identical.
While an object is not necessarily congruent with its mirror image, it may be. For example, the rear wheels of a child's tricycle are mirror images of each other. That is, the tricycle has a vertical plane of symmetry which can divide it into two halves; but the rear wheels are also inter-changeable. That is, they are congruent. This is not true of a pair of gloves, or the hands that occupy them.

These various relationships may be shown on the stereographic projection as in Fig. 65. The symmetry represented here is that of two planes of symmetry at 90° to each other with their line of inter-section being a 2-fold axis. Let us consider face R representing a right hand as the starting point. The 2-fold axis repeats R as an identical right hand at R'. The symmetry axis which runs east-west repeats R at L' as a left hand, that is, a mirror image of a right hand. L' might also be said to represent a mirror image of R' as repeated across the north-south symmetry plane. By the same token, L is the mirror image of R' across the east-west plane. Note face X, however; it has a mirror image X' repeated across the east-west plane; but X' is also an identical face, due to the presence of the vertical 2-fold axis. Therefore, X' is a mirror image of X but it is also congruent with X.

When two planes of symmetry intersect their line of intersection is an axis of symmetry whose period is twice the angle between the planes. This is related to the fact that a mirror
image of a mirror image reproduces the original object. That is, reflect a right hand once and a left hand is obtained. Reflect the left hand and a right hand is the result. As an example, reconsider Fig. 65. The right hand at R is reflected by the east-west plane as left hand L' and is reflected by the north-south plane as left hand L. Left hand L is reflected by the east-west plane into right hand R' but right hand R' is also the result of reflection through the north-south plane of left hand L'. Note that the two planes do not express entirely the symmetry. The line of intersections of the planes is obviously a 2-fold axis as rotation through 180° repeats the right hand R at R' and the left hand L at L'.

Fig. 65 Stereogram of Intersecting Planes
Further examples may prove instructive. Examine the case of two planes intersecting at 60° as shown in Fig. 66. Consider face R as the starting point, the plane running northeast-southwest repeats R as a left hand at L. The east-west plane repeats R at L' and repeats L at R'. It is now clear that another plane has arisen. This plane running northwest-southeast could be considered the reflection of the northeast-southwest plane in the east-west plane or the reflection of the east-west plane in the northeast-southwest plane. Or, it may be stated that the plane must be there since L' and R' are mirror images of one another, thus necessitating a plane in between. The northwest-southeast plane reflects L as L'' and R as R'' which in turn are mirror images of each other across the east-west plane. A moment's examination makes it clear that there is a 3-fold axis at the line of intersection of three planes.

If the angle of intersection of the two initially assumed planes is thirty degrees, the result, which follows much the same pattern as in the previous example, is that of a total of six planes all intersecting at angles of 30°, and their line of intersection is a 6-fold axis (Fig. 67). The 45° angle initial intersection case is shown in Fig. 68.

In each situation it will be noted that the period of the axis which lies in the intersection of the planes is exactly twice the angle between the planes. It is easy to demonstrate geometrically the reason for this relationship.
Fig. 66 Symmetry Planes Intersecting at 60°

Fig. 67 Symmetry Planes Intersecting at 30°
Fig. 68 Symmetry Planes Intersecting at 45°
In Fig. 69 let OA and OB be two planes of symmetry standing perpendicular to the plane of the paper and intersecting at any angle. Let R represent a right hand and construct the line RO. Let R be repeated by the plane of symmetry OB as a left hand at L. L will appear to be located on the perpendicular and as far behind the plane OB as R is in front of OB. Draw line LO. The angle B thus equals the angle B'. Now let L be repeated by the symmetry plane OA as a right hand at R'. R' will appear to be located on the perpendicular and as far behind the plane OA as OL is in front of OA. Construct the line OR'. The angle Y thus equals the angle Y'. Since R and R' are congruent (a mirror image of a mirror image is the original), the symmetry operation which repeats R at R' can also be described as an axis of symmetry perpendicular to the paper and passing through point O. The period of this axis is the angle $\alpha$, but $\alpha = B + B' + Y + Y'$, or $2B + 2Y$, or $2(B + Y)$. Since the angle between the planes OA and OB is $B + Y$, the period of the axis, $\alpha$, is twice the angle between the planes, as noted above.

At about this point it is not only instructive, but enjoyable to pose the question "Why does a mirror seem to reverse right and left, but not up and down?" If the student looks in a mirror and scratches his right ear, the image scratches its left; but if he touches his hair, the image doesn't touch its chin. Even if the
Fig. 69 Intersecting Symmetry Planes Produce an Axis
mirror is rotated or the subject bends to a horizontal position, 
the same is true. This question is discussed at length in the 
wonderful "The Ambidextrous Universe" by Martin Gardner (1964), 
a book strongly suggested as collateral reading. A bit later 
in this work a "crystallographic" explanation will be given.

Another relationship may be seen as follows:

Let the number of intersecting planes be any number \( n \), the 
angle between them \( \beta \), and let \( \alpha \) represent the period of the 
resulting symmetry axis.

Then \( 2 \times n \times \beta = 360^\circ \)

\[
\frac{360^\circ}{2 \times \beta} = n
\]

but since \( \alpha = 2 \times \beta \)

\[
\frac{360^\circ}{\alpha}
\]

That is, if a plane of symmetry contains an axis of period 
\( \alpha \), \( \frac{360^\circ}{\alpha} \) planes will intersect in that axis, making an angle 
with each other of \( \frac{\alpha}{2} \).

It now should be obvious that the angles at which symmetry 
planes can intersect are limited to certain values. When symmetry 
planes intersect, their line of intersection is an axis of 
symmetry whose period is twice the angle between the planes. 
That is,

\[
\alpha = 2 \beta \quad \text{or} \quad \beta = \frac{\alpha}{2}
\]
But $\alpha$ can only have values of $180^\circ$, $120^\circ$, $90^\circ$, or $60^\circ$, and therefore, $\beta$ is limited to $90^\circ$, $60^\circ$, $45^\circ$, and $30^\circ$.

Furthermore, since $30^\circ$ is the smallest permissible angle of intersection, and

$$n = \frac{360^\circ}{2 \times \beta}$$

Also, it might be noted that 5 planes cannot intersect in crystals, since the intersection of 5 planes would produce a 5-fold axis of symmetry with a period of $72^\circ$, which is not possible in a regular arrangement.

$$n = \frac{360^\circ}{5}$$

$$\alpha = \frac{360^\circ}{n} = \frac{360^\circ}{5} = 72^\circ$$

It has been shown that there are certain limitations to the number, kind and relationships between symmetry axes. Furthermore, it has now been demonstrated that there are limitations on the relationships between the planes of symmetry. Thus the question arises, are there limitations on the positions which planes may occupy with respect to axes? The answer to this question is yes, and the following sections will discuss these limited relationships.

Like axes of symmetry may intersect each other only at certain angles. Thus, planes of symmetry can stand in relation to such axes only in those positions that will not demand a repetition of like axes at forbidden angles. For examples, note
Fig. 70 Axes at Impermissible Angles

Fig. 71 Axes at Permissible Angles
Fig. 70. OA and OA' are 4-fold axes of symmetry making with each other the angle \(90^\circ\), which is one of the permissible angles. If a plane OP were placed in the position shown and if this plane were a symmetry plane, then OA for instance would be repeated by the symmetry plane in the position OA''. This, it will be noted, produces two 4-fold axes OA and OA' which intersect at an angle of \(30^\circ\), an impossible situation. Further inspection will show that there will be additional reflections of 4-fold axes in the plane OP and these also will be in impossible positions.

However, if a symmetry plane bisects a permissible angle between two like axes, the axes will then be repeated only as themselves and thus no new axes arise in impossible positions. Note Fig. 71. OA and OA' are two axes of symmetry intersecting at some permissible angle theta. The symmetry plane OP bisects the angle theta producing a reflection of OA' in OA and a reflection of OA in OA'. Thus nothing impermissible is introduced.

It further should be noted that the angle between unlike ends of two identical polar axes may be bisected by a symmetry plane. Remember that a plane of symmetry does not produce identity, merely enantiomorphism, so that the plane of symmetry does not make the two unlike ends of the identical axes congruent; it simply makes them alike by reflection.

Such a situation may be nicely shown by starting with a consideration of the symmetry of the common mineral quartz. (Fig. 72) Quartz has a bipolar 3-fold axis at right angles to
Fig. 72 Polar Axes at 120°

Fig. 73 Addition of a Plane of Symmetry to Polar Axes
three polar 2-fold axes. The latter axes intersect each other at angles of 120° (so if we ignore the fact that the axes are polar, they could be said to intersect at 60°; but if this were true, the central axis would be 6-fold, not 3-fold. It is the unlike ends that intersect at 60°. The identical ends intersect at 120°). Any face R representing a right hand is repeated by the symmetry in the positions shown. All of the faces can be represented by right hands as there is no enantiomorphic symmetry produced, only the axes being present. The circles around three of the R's show that these faces are in the lower hemisphere, that is, below the plane of the paper in the stereographic projection.

Now is a plane of symmetry is added to the existing symmetry in the position PP' which bisects one of the 60° angles between unlike ends of the polar 2-fold axes, the symmetry shown in Fig. 73 arises as a result. The resulting symmetry may be derived in several different ways: for instance, by simply repeating the plane by means of the 3-fold axis every 120°, or by noting from the previous section that if a plane of symmetry contains a 3-fold axis, that axis will be the line of intersection of three planes intersecting at angles of 60°.

Once again select a random face R for repetition. Note that an assemblage of faces results which is shown in Fig. 73. Some faces are represented by right hands and are identical with the original face R. Others are represented by left hands and are related to the original face R only by reflection. Incidentally,
In general, an axis of symmetry may stand perpendicular to a plane of symmetry. If the axis of symmetry under consideration is a bipolar axis, then the symmetry plane might be considered as bisecting the angle of $180^\circ$ between two identical bipolar axes. However, the presence of the plane of symmetry does not make the axis bipolar. That is, it does not make the two ends identical, it makes them mirror images. An example might be the type of symmetry shown by the important mineral scheelite which is illustrated in Fig. 74. Here there is a 4-fold axis standing perpendicular to the plane of symmetry. However, there are no other axes present to make the two ends of the 4-fold axis identical. Therefore a face such as R is repeated at four positions in the upper hemisphere as right hands and immediately underneath as four left hands. There is no way to bring about congruency between the top four faces and the bottom four faces.

The question of the various possible relationships between opposite ends of a single symmetry axis is one that is not adequately discussed. After the thirty-two point groups have been derived and therefore logically are available as a basis of discussion, we will consider these relationships. For the present, I am using the term "bipolar" to mean an axis having opposite ends which are identical. That is, capable of being brought into congruence by
Fig. 74 Symmetry of Scheelite

Fig. 75 Symmetry of Class $\frac{2}{m}$

Fig. 76 Relationship between 3-fold Axes and Symmetry Plane
rotation around another axis.

Planes may also contain or be parallel to a symmetry axis and perpendicular to another axis at the same time for neither situation demands the repetition of the axis in any new or impossible position. Note Fig. 75 which is one of the most common of all the symmetry classes.

In general, planes may not occur in positions that would demand the intersection of axes at impermissible angles or which would produce axes of impermissible periods. If no impossible demands are made at the same time, planes may (1) intersect at $30^\circ$, $45^\circ$, $60^\circ$, or $90^\circ$ with the line of intersection being a symmetry axis, (2) bisect the angles between like axes, (3) stand perpendicular to a symmetry axis, (4) stand parallel to a symmetry axis, that is, contain a symmetry axes, (5) stand both parallel and perpendicular to a symmetry axis.

If any impossible demands are made by the addition of symmetry planes, such planes cannot be present. Three-fold axes making an angle of $109^\circ 28$ minutes and 16 seconds with each other are a possible case. Such a situation exists between axes OA and OA' in Fig. 76. Now, if a plane of symmetry were to be placed in a position such as to make an angle of $90^\circ$ with axis OA, it would make an angle of $19^\circ 28$ minutes and 16 seconds with axis OA'. The plane of symmetry would repeat OA' at OA'', thus creating the impossible situation of having two 3-fold axes of symmetry making
angles of 38° 56 minutes and 32 seconds with one another.

The discussion of planes of symmetry may be summarized as follows:

1. A plane of symmetry is an imaginary plane that divides a crystal into two halves, each of which is a mirror image of the other. The repetition is of objects which are enantiomorphs, and are not necessarily identical. However, it is possible for an object to be congruent with its mirror image.

2. When two symmetry planes intersect, their line of intersection is an axis of symmetry whose period is twice the angle between the planes.

That is, $\alpha = 2\theta$ where $\alpha$ is the period of the axis and $\theta$ is the angle between the planes. Or, in operations algebra

$$M_1 \cdot M_2 = A\alpha,$$

indicating that if mirrors $M_1$ and $M_2$ intersect at angle $\alpha$, a reflection in $M_1$ followed by a reflection in $M_2$ is the equivalent of a rotation about an axis having a period $\alpha$.

3. Since $\theta = \alpha/2$, the angles at which symmetry planes can intersect can have only those values which are 1/2 of the possible periods of axes.

4. The permissible positions of symmetry planes with respect to axes are as follows:
(1) planes may intersect (in an axis of symmetry at 
   $30^\circ$, $45^\circ$, $60^\circ$ and $90^\circ$).

(2) planes may bisect the angles between identical axes.

(3) planes may stand perpendicular to a symmetry axis.

(4) planes may stand parallel to a symmetry axis.

(5) planes may stand both parallel and perpendicular 
   to a symmetry axis.

5. When a plane contains an axis of period $\alpha$ that axis will 
   be the line of intersection of $\frac{360^\circ}{\alpha}$ planes which intersect at an 
   angle of $\frac{\alpha^\circ}{2}$.
THE CENTER OF SYMMETRY

Another way of expressing enantiomorphic symmetry is by means of the center of symmetry, or symmetry by inversion. This is symmetry of the type produced by the lens in a pinhole camera, as in Fig. 77. A crystal may be said to have a center of symmetry if an imaginary line drawn from some point on the surface of the crystal through the crystal encounters a similar point at an equal distance beyond the center.

Fig. 77 Repetition by a Center

The tetrahedron shown in Fig. 78 is a highly symmetrical body, but does not contain a symmetry center. If a line is drawn from corner X through the center it comes out in the middle of the front face, not at another corner.

The crystal of axinite shown in Fig. 79 does have a center of symmetry. A line drawn from face X through the center encounters a like point on face X'.
Fig. 78 Tetrahedron

Fig. 79 Axinite

Fig. 80 Plane Perpendicular to a 2-fold Axis
Later on it will be seen that eleven of the possible thirty-two classes of crystals have a center of symmetry. Of these eleven, all but two contain one or more planes of symmetry, and all but one contain at least one axis of symmetry.

A center of symmetry produces an enantiomorph, or mirror image, just as does a plane of symmetry; but unlike the repetition produced by a plane, the image repeated by a center is "upside down". That is to say, it has been rotated as well as reflected. This indicates the possibility that a plane plus an axis might produce the same results as a center; and in some cases it is true.

Refer to Fig. 80, which shows in stereographic projection a two-fold symmetry axis perpendicular to a plane of symmetry, with a face R, representing a right hand, in the general position. The two-fold will repeat R at R', and the plane of symmetry will repeat both as left-hands, L and L', in the lower hemisphere, as seen in Fig. 80. Face L' is the enantiomorph of R, and is the same distance from the center of the crystal as R, but directly opposite. That is to say, the relationship is that produced by a center of symmetry. Now, since a 4-fold axis repeats at 180° as well as at 90°, and a 6-fold axis repeats at 180° as well as at 60°, it can be seen that when an axis of symmetry with n as an even number stands perpendicular to a plane of symmetry, a center of symmetry must of necessity be present.

Axes of 1-fold and 3-fold symmetry perpendicular to a plane
of symmetry do not produce centers. A 1-fold axis is, in one sense, no symmetry at all, so a 1-fold axis, plus a plane, is simply a plane.

A 3-fold axis perpendicular to a plane does not give rise to a center. Fig. 81 shows the situation on a projection. Since $180^\circ$ is not one of the repeat positions, no center can be present.

Fig. 81 Plane Perpendicular to a 3-fold Axis
COMPOUND SYMMETRY ELEMENTS

There are crystals which have faces repeated in a symmetrical manner but whose symmetrical distribution cannot be described in terms of an axis, a plane, a center, either singly or combined in any manner previously considered. In Fig. 82 there is seen a perspective drawing and in Fig. 83 stereographic projection showing such a distribution. This symmetry may be related to a type of symmetry element called an axis of rotary inversion. Such axes consist of rotation through the period of the axis, and a simultaneous inversion through a center. It is not a rotary axis plus a center of symmetry; it is one operation.

The symbols for the five types of inversion axes are \( \bar{1} \), \( \bar{2} \), \( \bar{3} \), \( \bar{4} \), and \( \bar{6} \). These are read as "bar one", "bar two", etc. On the stereographic projection axes of rotary inversion will be designated as follows:

\[
\begin{align*}
\bar{1} & = \circ \\
\bar{2} & = \bigcirc \\
\bar{3} & = \triangle \\
\bar{4} & = \Box \\
\bar{6} & = \bigotimes
\end{align*}
\]
Fig. 82 Tetragonal Sphenoid

Fig. 83 Stereogram of Tetragonal Sphenoid
The pattern of faces generated by each of the inversion axes is illustrated in Fig. 84.

![Diagram of Inversion Axes]

Fig. 84 Stereograms of Inversion Axes
Other combinations of symmetry elements may be substituted for all of the types of rotary inversion axes except in the case of 4. This element is unique and there is one symmetry class which consists only of a 4-fold inversion axis. Therefore, although the concept of rotary inversion is useful in expressing several symmetry combinations, it is only 4 which is essential. The symmetries equivalent to the rotary inversion axes are as follows:

\[ \bar{1} = \text{a center of symmetry} \]
\[ \bar{2} = \text{a mirror plane} \]
\[ \bar{3} = \text{a 3-fold axis plus a center} \]
\[ \bar{4} \text{ is unique} \]
\[ \bar{6} = \text{a 3-fold axis normal to a mirror plane} \]
Many current texts use terms such as "point-groups" and "space groups" with no indication of the meaning of the term "group". One gets the impression that "group" is being employed in the general sense, and simply means a collection or set, the members of which have some sort of relationship, one to the other. That this relationship is rigidly defined in a mathematical sense is often not stated. It seems worthwhile to give the student the briefest of explanations of the theory of groups, so that he gets some feel for the closely knit relationships between symmetry elements, and the limitations upon the various possible combinations of those elements as expressed in group theory. Furthermore, some knowledge of this approach to symmetry may prove extremely valuable to those who move on to more advanced work.
The theory of groups is an abstract branch of mathematics which has wide application and many important concrete interpretations, including symmetry studies. Consider some of the fundamentals of group theory and crystallographic applications. A group is defined by assuming the following postulates.

The Group Postulates (after Richardson, 1958)

Let $G$ be a set of undefined objects, called elements of $G$, and denoted by $a, b, c, \ldots$, and let there be an undefined operation denoted by $\circ$, which will be read as "followed by."

In specific examples, $\circ$ will be defined.

**Postulate I.** (Law of closure) To every pair of elements $a$ and $b$ of $G$, given in the stated order, there corresponds a definite element of $G$, denoted by $a \circ b$. That is, $a \circ b \in G$, where $\in$ means "is an element of".

That is to say that when any two elements of a group are combined, the result is also a member of the group, and is often called "the group property".

**Postulate II.** (Associative law) If $a, b, c$, are any elements of $G$, then $a \circ (b \circ c) = (a \circ b) \circ c$.

**Postulate III.** There exists a unique element of $G$, denoted by $I$, having the property that, if $a$ is any element of $G$, then $a \circ I = a$.

The element $I$ is called the identity element.
Postulate IV. To each element \( a \) of \( G \) there corresponds a unique element of \( G \), denoted by \( a^{-1} \) having the property that 
\[
    a \circ a^{-1} = a^{-1} \circ a = I.
\]

The element \( a^{-1} \) is called the inverse of \( a \).

A set \( G \) of elements with an operation satisfying the above postulates is called a group.

Let us examine some concrete examples of sets which are groups.

1. Let the set be that of all the integers, \( 0, \pm 1, \pm 2 \ldots \) and the operation \( \circ \) be addition.

Then,

Postulate I. When any two integers are added, the result is an integer, and therefore in the group.

Postulate II. As an example, \( 6 \circ (3 + 2) = 11 \), and \( (6 + 3) \circ 2 = 11 \), so the associative law is satisfied.

Postulate III. Any integer plus zero is equal to that integer, therefore zero is the identity element, \( I \)

Postulate IV. If \( a \) is any integer, then 
\[
    a \circ (-a) = 0
\]

Thus, two integers of opposite sign combine to form the identity element.

The reader might test for himself that the following are groups:

2. Let the set be all positive rational numbers and let the operation be multiplication.

3. Let the set be \( \pm 1, \pm 1, \pm 1, \pm 1 \), and the operation
be multiplication.

4. Let the set be +1, -1, and the operation be multiplication. Note that set 3 is contained within set 2, and therefore is a subgroup of the larger group.

Note that the following is not a group:

Let the set be all positive integers and the operation be multiplication. This cannot satisfy postulate IV, because, for instance, there is no positive integer such that

\[ 9 \cdot a = a \cdot 9 = 1 \]

The next interpretation of a group, after Richardson (1958), has a geometric basis which is close to the nature of symmetry axes. See Fig. 85.

Fig. 85 Interpretation of a Group
Let the elements of $G$ be the numbers on the dial, $0$, $1$, $2$, $3$, $4$. Let the operation $\circ$ be "addition" in the following sense:

the "sum" $2 + 1$ shall mean the number to which the hand points after rotating in the indicated direction two spaces starting from $\circ$ and then through one space. Thus, $2 + 1 = 3$. Zero is interpreted to mean no rotation.

$$
\begin{array}{c|ccccc}
0 & 0 & 1 & 2 & 3 & 4 \\
1 & 1 & 2 & 3 & 4 & 0 \\
2 & 2 & 3 & 4 & 0 & 1 \\
3 & 3 & 4 & 0 & 1 & 2 \\
4 & 4 & 0 & 1 & 2 & 3 \\
\end{array}
$$

A much more succinct way of presenting the same information, and which will be used in this text, is that of an "operations table," or "multiplication table". Each operation is listed both horizontally and vertically, so that all of the combinations are present.

By examination of this table, it is seen, for instance, that a rotation of $4$ spaces followed by a rotation of $3$ spaces results
in the dial hand pointing at 2, as noted previously.

The operations of axes of symmetry, either simple rotation axes or axes of rotary inversion, form a group. Rotations about such axes satisfy:

**Postulate I.** They can be combined algebraically by addition.

**Postulate II.** They can be combined (rotation + rotation) + rotation, or, rotation + (rotation + rotation)

**Postulate III.** By including $0^\circ$ rotation (or $360^\circ$) the identity element, designated $I$, is present.

**Postulate IV.** By including both positive and negative rotations, the inverse of any operation is present.

Examine the operations of an axis of 4-fold symmetry such as that in the center of the square in Fig. 86.

![Fig. 86 Axis of 4-fold Symmetry](image)
The operations of the axis A will be designated as follows:

I - the identity operation, that is, a rotation of $0^\circ$ or $(360^\circ)$

$A_{90}$ - a rotation of $90^\circ$ counter clockwise

$A_{180}$ - a rotation of $180^\circ$ counter clockwise

$A_{270}$ - a rotation of $270^\circ$ counter clockwise

The table of operations of a group is termed the "multiplication table" of the group, regardless of the nature of the operation. Hereafter, such tables will be referred to in that manner.

The multiplication table of the above group is as follows:

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>$A_{90}$</th>
<th>$A_{180}$</th>
<th>$A_{270}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>I</td>
<td>$A_{90}$</td>
<td>$A_{180}$</td>
<td>$A_{270}$</td>
</tr>
<tr>
<td>$A_{90}$</td>
<td>$A_{90}$</td>
<td>$A_{180}$</td>
<td>$A_{270}$</td>
<td>I</td>
</tr>
<tr>
<td>$A_{180}$</td>
<td>$A_{180}$</td>
<td>$A_{270}$</td>
<td>I</td>
<td>$A_{90}$</td>
</tr>
<tr>
<td>$A_{270}$</td>
<td>$A_{270}$</td>
<td>I</td>
<td>$A_{90}$</td>
<td>$A_{180}$</td>
</tr>
</tbody>
</table>

Virtually all discussions of the application of group theory to crystal symmetry use symbols for the various operations which are derived from the Schoenflies notation. Buerger (1956) does not, but his symbols designate angles by radian measure. It is my belief that the student will follow with greater ease if the symbols are indicative of degrees and are related more closely to the International notation.
In the section developing and describing the thirty-two crystal classes the operations of each of the symmetry elements will be designated as follows:

<table>
<thead>
<tr>
<th>Name</th>
<th>Symbol</th>
<th>Operations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-fold axis</td>
<td>1</td>
<td>I</td>
</tr>
<tr>
<td>2-fold axis</td>
<td>2</td>
<td>I, $A_{180}$</td>
</tr>
<tr>
<td>3-fold axis</td>
<td>3</td>
<td>I, $A_{120}$, $A_{240}$</td>
</tr>
<tr>
<td>4-fold axis</td>
<td>4</td>
<td>I, $A_{90}$, $A_{180}$, $A_{270}$</td>
</tr>
<tr>
<td>6-fold axis</td>
<td>6</td>
<td>I, $A_{60}$, $A_{120}$, $A_{180}$, $A_{240}$, $A_{300}$</td>
</tr>
<tr>
<td>Center of symmetry</td>
<td>i</td>
<td>I, i</td>
</tr>
<tr>
<td>Mirror plane</td>
<td>m</td>
<td>I, m</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Name</th>
<th>Symbol</th>
<th>Operations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-fold inversion axis</td>
<td>1</td>
<td>I, i</td>
</tr>
<tr>
<td>2-fold inversion axis</td>
<td>2</td>
<td>I, m</td>
</tr>
<tr>
<td>3-fold inversion axis</td>
<td>3</td>
<td>I, $A_{120}$, $A_{120}$, $A_{240}$, $A_{240}$, i</td>
</tr>
<tr>
<td>4-fold inversion axis</td>
<td>4</td>
<td>I, $A_{90}$, $A_{180}$, $A_{240}$</td>
</tr>
<tr>
<td>6-fold inversion axis</td>
<td>6</td>
<td>I, $A_{60}$, $A_{120}$, $A_{180}$, m, $A_{270}$, $A_{300}$</td>
</tr>
</tbody>
</table>
Note, for example, the operations of a 4-fold axis of rotary inversion. These are as follows:

I, the identity, which is present as an operation for all symmetry elements.  
$\overline{A}_{90}$, which represents a rotation of 90° together with a simultaneous inversion.

$A_{180}$, a rotation of 180°  
$A_{270}$, a rotation of 270° accompanied by a simultaneous inversion through a center.

In the next section, in which the thirty-two classes are derived, a multiplication table for each class will be given.
DERIVATION OF THE CRYSTAL CLASSES

The foundation for the determination of all of the combinations of symmetry elements which are possible in crystalline bodies has been laid. The types of axes that can exist, the relationships between such axes, the nature of planes of symmetry, and the limitations which arise of necessity when more than one plane is present have all been discussed. Furthermore, it has been shown that combination of axes and planes are limited, and that centers of symmetry and axes of rotary inversion are possible in crystals.

By combining the elements of symmetry in all of the permissible ways, the possible crystal classes, of which there are thirty-two only, will be deduced. These "classes of crystals" are the thirty-two point groups of the mathematician, that is, all of the possible combinations of symmetry elements about a fixed point. Such a combination is a group in the mathematical sense. The method used here is to show the "multiplication tables" of the groups, but to combine these with visual representations of the operations by means of the stereographic projection. In each case an "initial symmetry" will be assumed and the additional symmetry, if any, which arises will be deduced.
Many methods of deriving the thirty-two crystal classes have appeared in print. The combination of the group theoretical approach plus visual representation by the stereographic projection has not been used. It seems to have the advantages of the elegance of group theory plus the concreteness of direct visualization on the projection.

The derivation of the possible crystal classes will be in the following order, duplications being omitted;

1. The classes containing only one axis of symmetry.
2. The classes containing more than one axis of symmetry.
3. The classes obtained by adding planes of symmetry in all permissible positions to each of the axial classes.
4. The classes containing only axes of rotary inversion.

The use of the stereographic projection to determine any symmetry that arises as a result of the operation of one symmetry on another may be illustrated as follows:

Assume an initial symmetry of one 2-fold axis. Add a plane containing that axis. To the projection of that initial symmetry add a face in the general position, that is, located on none of the symmetry elements. The situation is depicted in Fig. 87.
Fig. 87 Derivation of m\textsubscript{m}2

The symmetry present will repeat the face as shown in Fig. 88.

Fig. 88 Derivation of m\textsubscript{m}2

Fig. 89 Derivation of m\textsubscript{m}2
The thirty-two crystal classes can be grouped into seven crystal systems based on symmetry elements. These are as follows:

**Isometric**  Four 3-fold axes of symmetry

**Tetragonal**  One, and one only, 4-fold axis

**Hexagonal**  One 6-fold axis

**Trigonal**  One, and one only, 3-fold axis

**Orthorhombic**  Three mutually perpendicular 2-fold axes, or 2-fold axis at the intersection of two mirror planes

**Monoclinic**  One, and one only, 2-fold axis. Note that $\overline{2} = m$

**Triclinic**  No symmetry higher than $\overline{1}$

The system of international symbols will be used to designate each crystal class. A simplified explanation of the system follows:

1. Rotation axes are indicated by 1, 2, 3, 4, and 6. Rotary inversion axes are designated by $\overline{1}$, $\overline{3}$, $\overline{4}$, and $\overline{6}$. The symbol $\overline{2}$ is not used, as it is the same as $m$, a mirror plane. If a plane is perpendicular to an axis, the axis symbol is written as a number over $m$, as in $\frac{3}{m}$.

2. In the isometric, tetragonal, hexagonal, and monoclinic systems, the first part of the symbol refers to the principal axis. An example is the 6 in 6 mm.

3. In the isometric system the second part of the symbol refers to the 3-fold symmetry elements, and third part to the
2-fold symmetry elements.

4. In the tetragonal system the second symbol refers to the axial elements and the third symbol to the diagonal elements of symmetry. Thus, in 4 2 2 the principal axis is a 4-fold and these are two sets of 2-fold axes.

5. In the hexagonal system the second symbol refers to the axial elements and the third symbol to the alternate elements of symmetry.

6. In the orthorhombic system the three symbols refer to the three principal crystal directions, which are mutually perpendicular.

The symbols of Schoenflies are given for possible future reference, but are not used in this work and will not be explained.
Derivation of Class 1

**Initial Symmetry**

One one-fold rotary axis

Fig. 90 Class 1

Fig. 90 shows the repetition of a face in the general position. Obviously no new symmetry arises, as shown by the projection or by the multiplication table. The total symmetry is the same as the initial symmetry. To state that there is such a class of symmetry is simply to state that it is possible to have a regular arrangement which is not repetitive except at 360° rotation.

---

Symmetry of the class - 1 $A_1$

Name of the class - Pedial

International symbol - 1

Schoenflies symbol - $C_1$

**Multiplication Table**

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<tbody>
<tr>
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</table>
Derivation of Class 2

**Initial Symmetry**

One two-fold rotary axis

Fig. 91 shows the repetition of a face in the general position. No new symmetry arises. The total symmetry is the same as the initial symmetry.

**Symmetry of the Class** - 1A\(_2\)

**Name of the Class** - Sphenoidal

**International Symbol** - 2

**Schoenflies Symbol** - C\(_2\)

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</tbody>
</table>

Do not deduce a possible symmetry plane. The two faces are identical, not mirror images.
Derivation of Class 3

Initial Symmetry

One three-fold rotary axis

Fig. 92 Class 3

Once again it is clear that no additional symmetry is indicated, as shown in Fig. 92.

Symmetry of the Class - 1 \( A_3 \)
Name of the Class - Trigonal-pyramidal
International Symbol - 3
Schoenflies Symbol - \( C_3 \)

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Derivation of Class 4

Initial Symmetry

One four-fold rotary axis

Fig. 92 Class 4

The symmetry and the repetition of a general face are shown in Fig. 93.

Symmetry of the Class - 1 $A_4$
Name of the Class - Tetragonal-pyramidal
International Symbol - 4
Schoenflies Symbol - $C_4$

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Derivation of Class 6

**Initial Symmetry**

One six-fold rotary axis

The symmetry and repetition of a general face are shown in Fig. 94.

**Symmetry of the class** - 1 $A_6$

**Name of the class** - Hexagonal-pyramidal

**International Symbol** - 6

**Schoenflies Symbol** - $C_6$

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Derivation of Class 222

Initial Symmetry
Three two-fold axes at 90° to each other

Fig. 95 shows the symmetry and the repetition of a general face. No new symmetry arises.

Symmetry of the class - 3 A₂
Name of the class - Rhombic-disphenoidal
International symbol - 222
Schoenflies symbol - D₂

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Derivation of Class 32

Initial Symmetry

One three-fold axis and three two-fold axes, each at $90^\circ$ to the three-fold. It is especially important to note here that the identical ends of the two-fold axes make an angle of $120^\circ$ with each other. It is the unlike ends that stand at $60^\circ$ (otherwise the central axis would be six-fold, not three-fold).

Fig. 95 Class 32

Fig. 96 shows the symmetry, and the repetition of a general face.

Symmetry of the class - $1A_3, 3A_2$

Name of the class - Trigonal-trapezohedral

International Symbol - 32

Schoenflies Symbol - $D_3$
### Multiplication Table

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Derivation of Class 422

Initial Symmetry

One four-fold, four two-fold axes. Note that there are two sets of two-fold axes. That is, axes B and \( ^2B \) are identical, and axes C and \( ^2C \) are identical; but the "B" axes are not identical with the "C" axes (this would create an eight-fold axis at the center, which is impossible).

Fig. 97 shows the symmetry and the repetition of a general face.

Symmetry of the class - \( 1 A_4, 4 A_2 \)

Name of the class - Tetragonal-trapezohedral

International symbol - 422

Schoenflies symbol - D_4
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</table>
Derivation of Class 622

**Initial Symmetry**

One six-fold, six two-folds. Note again that there are two sets of two-fold axes. The identical axes make angles of $60^\circ$ with each other.

Fig. 98 shows the symmetry and the repetition of a general face.

Symmetry of the class - $1 A_6, 6 A_2$

Name of the class - Hexagonal-trapezohedral

International symbol - 622

Schoenflies symbol - $D_6$
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Derivation of Class 23

Initial Symmetry

The group which contains a two-fold axis and four three-fold axes. The three-fold axes each make an angle of $54^\circ 44'$ with the two-folds. Note that identical ends of the three-fold axis stand at $109^\circ 28'$ to each other, but that an identical ends stand at the angle of $54^\circ 44'$, as in Fig. 99. If identical ends made the $54^\circ 44'$ angle with each other, the present two-fold axes would each be four-fold axes. This is a possible case, but represents a different group. When plotted on the stereographic projection, this symmetry appears as shown in Fig. 100.

![Axes of 23](image-url)
The ends of the axes $C$ and $^3C$, which appear in the upper hemisphere are identical, as are the upper ends of $^2C$ and $^4C$; but it is the lower hemisphere ends of $^2C$ and $^4C$ which are identical to the upper ends of $C$ and $^2C$; and vice versa.
Symmetry of the Class - $3A_2$, $4A_3$

Name of the class - Tetartoidal

International symbol - $23$

Schoenflies symbol - $T$

Number of operations - 12

The operations of this class are shown below, together with the point at which an initial face 1 is repeated by each operation. Those numbers which have a circle around them are faces in the lower hemisphere. This repetition is illustrated in Fig.

$$
\begin{align*}
  I &= 1 \\
  A_{180} &= 19 \\
  2A_{180} &= 11 \\
  3A_{180} &= 14 \\
  C_{120} &= 3 \\
  C_{240} &= 5 \\
  2C_{120} &= 7 \\
  2C_{240} &= 16 \\
  3C_{120} &= 9 \\
  3C_{240} &= 21 \\
  4C_{120} &= 18 \\
  4C_{240} &= 23
\end{align*}
$$
Derivation of Class 432

Initial Symmetry

The group which contains a two-fold axis, a three-fold axis, and a four-fold axis. The angular relationships are shown in Fig. 101.

Fig. 101 Angles of 432

When plotted on the stereographic projection, these elements of symmetry may be placed as illustrated in Fig. 102.

Fig. 102 Initial Symmetry
When the elements assumed act upon one another, the total resulting symmetry is that of Fig. 103.

On the stereographic projection, it appears as in Fig. 104.
Symmetry of the Class - $3A_4, 4A_3, 6A_2$

Name of the Class - Gyroidal

International Symbol - 432

Schoenflies Symbol - 0

Number of operations - 24

The operations of this class are shown below, together with the point at which an initial face 1 is repeated by each operation. Those numbers with a circle around them indicate faces in the lower hemisphere. This repetition is illustrated in Fig. 105.

\[
\begin{align*}
I &= 1 \\
A_{90} &= 24 \\
A_{180} &= 13 \\
A_{270} &= 12 \\
A_{90} &= 22 \\
A_{180} &= 14 \\
A_{270} &= 8 \\
B_{180} &= 9 \\
B_{180} &= 17 \\
B_{180} &= 20
\end{align*}
\]
Derivation of Class m

Initial Symmetry

The axial class containing one one-fold axis, plus a plane. Since a one-fold axis is really no symmetry, the total symmetry consists of a plane of symmetry, as indicated in Fig. 106 and in the multiplication table.

Fig. 106 Class m

The symmetry and repetition of a general face are shown in Fig. 106.

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</table>
Derivation of Class $\frac{2}{m}$

**Initial Symmetry**

The symmetry group consisting of one two-fold axis, plus a plane of symmetry perpendicular to that axis. This is one of the possible relationships between planes and axes, as stated on page 134. The multiplication table, and the stereographic program show that the complete symmetry is indicated by the two initial symmetry elements. However, it is clear that a center of symmetry also is present, having arisen because of the presence of an even-fold axis perpendicular to a plane of symmetry. The symmetry and the repetition of a general face are shown in Fig. 107.

![Fig. 107 Class $\frac{2}{m}$](image)

- **Symmetry of the class**: $1 A_2, 1 P$
- **Name of the class**: Prismatic
- **International symbol**: $\frac{2}{m}$
- **Schoenflies symbol**: $C_{2h}$
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<tr>
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<th>I</th>
<th>A(_{180})</th>
<th>M</th>
<th>i</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>I</td>
<td>A(_{180})</td>
<td>M</td>
<td>i</td>
</tr>
<tr>
<td>A(_{180})</td>
<td>A(_{180})</td>
<td>I</td>
<td>i</td>
<td>M</td>
</tr>
<tr>
<td>M</td>
<td>M</td>
<td>i</td>
<td>I</td>
<td>A(_{180})</td>
</tr>
<tr>
<td>i</td>
<td>i</td>
<td>M</td>
<td>A(_{180})</td>
<td>I</td>
</tr>
</tbody>
</table>
Derivation of Class $\bar{6}$

**Initial Symmetry**

The group which has one three-fold axis, with a plane perpendicular to that axis. From the projection in Fig. 108, and the multiplication table, it is seen that no center of symmetry is present, because three-fold axis does not repeat at $180^\circ$. The symmetry of a three-fold, perpendicular to a plane expresses the total symmetry, but it also can be expressed by a six-fold axis of rotary inversion.

![Fig. 108 Class $\bar{6}$ or $\overline{3}$](image)

Symmetry of the class - $1 A_3, 1 P$ (or $1 \overline{6}$)

Name of the class - Trigonal-dipyramidal

International symbol - $\bar{6}$ or $\overline{3}$

Schoenflies symbol - $C_{3h}$
<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>$\overline{A}_{60}$</th>
<th>$A_{120}$</th>
<th>$\overline{A}_{180}$</th>
<th>$A_{240}$</th>
<th>$\overline{A}_{300}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I$</td>
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<td>$A_{120}$</td>
<td>$\overline{A}_{180}$</td>
<td>$A_{240}$</td>
<td>$\overline{A}_{300}$</td>
</tr>
<tr>
<td>$\overline{A}_{60}$</td>
<td>$\overline{A}_{60}$</td>
<td>$A_{120}$</td>
<td>$\overline{A}_{180}$</td>
<td>$A_{240}$</td>
<td>$\overline{A}_{300}$</td>
<td>$I$</td>
</tr>
<tr>
<td>$A_{120}$</td>
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<td>$A_{240}$</td>
<td>$\overline{A}_{300}$</td>
<td>$I$</td>
<td>$\overline{A}_{60}$</td>
</tr>
<tr>
<td>$\overline{A}_{180}$</td>
<td>$\overline{A}_{180}$</td>
<td>$A_{240}$</td>
<td>$\overline{A}_{300}$</td>
<td>$I$</td>
<td>$\overline{A}_{60}$</td>
<td>$A_{120}$</td>
</tr>
<tr>
<td>$A_{240}$</td>
<td>$A_{240}$</td>
<td>$\overline{A}_{300}$</td>
<td>$I$</td>
<td>$\overline{A}_{60}$</td>
<td>$A_{120}$</td>
<td>$\overline{A}_{180}$</td>
</tr>
<tr>
<td>$\overline{A}_{300}$</td>
<td>$\overline{A}_{300}$</td>
<td>$I$</td>
<td>$\overline{A}_{60}$</td>
<td>$A_{120}$</td>
<td>$\overline{A}_{180}$</td>
<td>$A_{240}$</td>
</tr>
</tbody>
</table>
Derivation of Class m

Initial Symmetry

One four-fold axis plus a plane perpendicular to the axis. Once again it can be seen that the axis and the plane completely express the symmetry, but that a center is also present, due to the fact that the four-fold axis does cause a repetition at 180°. Fig. 109 shows the symmetry and the repetition of a general face.

Fig. 109 Class m

Symmetry of the class - \(1 A_4, 1 P\)

Name of the class - Tetragonal-dipyramidal

International symbol - \(\frac{4}{m}\)

Schoenflies symbol - \(C_{4h}\)
<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>A_{90}</th>
<th>\bar{A}_{90}</th>
<th>A_{180}</th>
<th>i</th>
<th>A_{270}</th>
<th>\bar{A}_{270}</th>
<th>M</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>I</td>
<td>A_{90}</td>
<td>\bar{A}_{90}</td>
<td>A_{180}</td>
<td>i</td>
<td>A_{270}</td>
<td>\bar{A}_{270}</td>
<td>M</td>
</tr>
<tr>
<td>A_{90}</td>
<td>A_{90}</td>
<td>A_{180}</td>
<td>M</td>
<td>A_{270}</td>
<td>\bar{A}_{90}</td>
<td>i</td>
<td>i</td>
<td>\bar{A}_{270}</td>
</tr>
<tr>
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<td>\bar{A}_{90}</td>
<td>M</td>
<td>A_{180}</td>
<td>\bar{A}_{270}</td>
<td>A_{90}</td>
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<td>i</td>
<td>\bar{A}_{270}</td>
</tr>
<tr>
<td>A_{180}</td>
<td>A_{180}</td>
<td>A_{270}</td>
<td>\bar{A}_{270}</td>
<td>i</td>
<td>M</td>
<td>A_{90}</td>
<td>\bar{A}_{90}</td>
<td>i</td>
</tr>
<tr>
<td>i</td>
<td>i</td>
<td>\bar{A}_{90}</td>
<td>A_{90}</td>
<td>M</td>
<td>i</td>
<td>\bar{A}_{270}</td>
<td>A_{270}</td>
<td>A_{180}</td>
</tr>
<tr>
<td>A_{270}</td>
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<td>i</td>
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<td>\bar{A}_{90}</td>
</tr>
<tr>
<td>\bar{A}_{270}</td>
<td>\bar{A}_{270}</td>
<td>i</td>
<td>i</td>
<td>\bar{A}_{90}</td>
<td>A_{270}</td>
<td>M</td>
<td>A_{180}</td>
<td>A_{90}</td>
</tr>
<tr>
<td>M</td>
<td>M</td>
<td>\bar{A}_{270}</td>
<td>A_{270}</td>
<td>i</td>
<td>A_{180}</td>
<td>\bar{A}_{90}</td>
<td>A_{90}</td>
<td>I</td>
</tr>
</tbody>
</table>
**Initial Symmetry**

The group containing one six-fold axis, plus a plane perpendicular to that axis. The axis and the plane completely define the symmetry, but since a six-fold axis causes a repetition at 180°, there is a center. The symmetry and the repetition of a general face are shown in Fig. 110.

![Fig. 110 Class m](image)

**Symmetry of the class** - 1 A₆, 1 P

**Name of the class** - Hexagonal-dipyramidal

**International symbol** - \( \frac{6}{m} \)

**Schoenflies symbol** - C₆h
Derivation of Class \textit{mm2}

\textbf{Initial Symmetry}

The class having one two-fold axis, plus a plane containing that axis, as shown in Fig. 111.

\begin{center}
\includegraphics[width=0.5\textwidth]{fig111.png}
\end{center}

\textit{Fig. 111} Initial Symmetry for Class \textit{mm2}

When a face is placed in a general position, it is repeated as shown in Fig. 112.

\begin{center}
\includegraphics[width=0.5\textwidth]{fig112.png}
\end{center}

\textit{Fig. 112} Face Repeated by Initial Symmetry

It is seen that another plane is present, and the total symmetry is that of Fig. 113.

\begin{center}
\includegraphics[width=0.5\textwidth]{fig113.png}
\end{center}

\textit{Fig. 113} Class \textit{mm2}
Symmetry of the class - $1 \text{A}_2, 2 \text{P}$

Name of the class - Rhombic-pyramidal

International symbol - mm2

Schoenflies symbol - C$_2$V

### Multiplication Table

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>A180</th>
<th>M</th>
<th>$^2_M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>I</td>
<td>A180</td>
<td>M</td>
<td>$^2_M$</td>
</tr>
<tr>
<td>A180</td>
<td>A180</td>
<td>I</td>
<td>$^2_M$</td>
<td>M</td>
</tr>
<tr>
<td>M</td>
<td>M</td>
<td>$^2_M$</td>
<td>I</td>
<td>A180</td>
</tr>
<tr>
<td>$^2_M$</td>
<td>$^2_M$</td>
<td>M</td>
<td>A180</td>
<td>I</td>
</tr>
</tbody>
</table>
Derivation of Class 3m

**Initial Symmetry**

The class having one three-fold axis, plus a plane containing that axis. The axis repeats the plane at $120^\circ$ and $240^\circ$.

The initial symmetry is shown in Fig. 114.

The final symmetry with the repetition of a face in the general position is illustrated in Fig. 115.

---

Fig. 114  Initial Symmetry for Class 3m

Fig. 115  Class 3m
The products of the symmetry operations of \( A_3 \) and the operations of a mirror plane are as follows:

<table>
<thead>
<tr>
<th>Operations of ( A_3 )</th>
<th>Operations of ( m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I )</td>
<td>( m = M )</td>
</tr>
<tr>
<td>( A_{120} )</td>
<td>( m = 2M )</td>
</tr>
<tr>
<td>( A_{240} )</td>
<td>( m = 3M )</td>
</tr>
</tbody>
</table>

Symmetry of the Class - 1 \( A_3 \), 3 \( \text{P} \)

Name of the class - Ditrigonal-pyramidal

International Symbol - 3 \( m \)

Schoenflies Symbol - \( C_{3v} \)

The resulting multiplication table is:

<table>
<thead>
<tr>
<th></th>
<th>( I )</th>
<th>( A_{120} )</th>
<th>( A_{240} )</th>
<th>( M )</th>
<th>( 3M )</th>
<th>( 3M )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I )</td>
<td>( I )</td>
<td>( A_{120} )</td>
<td>( A_{240} )</td>
<td>( M )</td>
<td>( 3M )</td>
<td>( 3M )</td>
</tr>
<tr>
<td>( A_{120} )</td>
<td>( A_{120} )</td>
<td>( A_{240} )</td>
<td>( I )</td>
<td>( 2M )</td>
<td>( 3M )</td>
<td>( M )</td>
</tr>
<tr>
<td>( A_{240} )</td>
<td>( A_{240} )</td>
<td>( I )</td>
<td>( A_{120} )</td>
<td>( 3M )</td>
<td>( M )</td>
<td>( 2M )</td>
</tr>
<tr>
<td>( M )</td>
<td>( M )</td>
<td>( 3M )</td>
<td>( 2M )</td>
<td>( I )</td>
<td>( A_{240} )</td>
<td>( A_{120} )</td>
</tr>
<tr>
<td>( 2M )</td>
<td>( 2M )</td>
<td>( M )</td>
<td>( 3M )</td>
<td>( A_{120} )</td>
<td>( I )</td>
<td>( A_{240} )</td>
</tr>
<tr>
<td>( 3M )</td>
<td>( 3M )</td>
<td>( 2M )</td>
<td>( M )</td>
<td>( A_{240} )</td>
<td>( A_{120} )</td>
<td>( I )</td>
</tr>
</tbody>
</table>
Initital Symmetry

The group having one four-fold axis, plus a plane containing that axis. This symmetry is shown in Fig. 116. The four axis repeats the plane every $90^\circ$, as indicated in Fig. 117.

Fig. 116 Initial Symmetry of 4 mm

Fig. 117 Derivation of 4 mm
However, when one observes Fig. 117 with the repetition of a general face shown, it is seen that two additional planes of symmetry are present. Thus the final symmetry is that indicated in Fig. 118.

Of course, the resulting symmetry can be arrived at by taking the product of the operations of four-fold axis and a plane containing that axis, as follows:

Operations of fold axis $\times$ Operations of plane = result

$1 \times M = M$

$A_{90} \times M = 4_M$

$A_{180} \times M = 2_M$

$A_{270} \times M = 3_M$
It is seen that the two "extra" planes are necessary and are a part of the symmetry group.

Symmetry of the Class - 1 $A_4$, $4\ P$

Name of the Class - Ditetragonal-pyramidal

International Symbol - 4 mm

Schoenflies Symbol - $C_{4v}$

Multiplication Table:

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>$A_{90}$</th>
<th>$A_{180}$</th>
<th>$A_{270}$</th>
<th>M</th>
<th>$2M$</th>
<th>$3M$</th>
<th>$4M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>I</td>
<td>$A_{90}$</td>
<td>$A_{180}$</td>
<td>$A_{270}$</td>
<td>M</td>
<td>$2M$</td>
<td>$3M$</td>
<td>$4M$</td>
</tr>
<tr>
<td>$A_{90}$</td>
<td>$A_{90}$</td>
<td>$A_{180}$</td>
<td>$A_{270}$</td>
<td>I</td>
<td>$3M$</td>
<td>$4M$</td>
<td>$2M$</td>
<td>$M$</td>
</tr>
<tr>
<td>$A_{180}$</td>
<td>$A_{180}$</td>
<td>$A_{270}$</td>
<td>I</td>
<td>$A_{90}$</td>
<td>$2M$</td>
<td>M</td>
<td>$4M$</td>
<td>$3M$</td>
</tr>
<tr>
<td>$A_{270}$</td>
<td>$A_{270}$</td>
<td>I</td>
<td>$A_{90}$</td>
<td>$A_{180}$</td>
<td>$4M$</td>
<td>$3M$</td>
<td>M</td>
<td>$2M$</td>
</tr>
<tr>
<td>M</td>
<td>M</td>
<td>$4M$</td>
<td>$2M$</td>
<td>$3M$</td>
<td>I</td>
<td>$A_{180}$</td>
<td>$A_{270}$</td>
<td>$A_{90}$</td>
</tr>
<tr>
<td>$2M$</td>
<td>$2M$</td>
<td>$3M$</td>
<td>M</td>
<td>$4M$</td>
<td>$A_{180}$</td>
<td>I</td>
<td>$A_{90}$</td>
<td>$A_{270}$</td>
</tr>
<tr>
<td>$3M$</td>
<td>$3M$</td>
<td>M</td>
<td>$4M$</td>
<td>$2M$</td>
<td>$A_{90}$</td>
<td>$A_{270}$</td>
<td>I</td>
<td>$A_{180}$</td>
</tr>
<tr>
<td>$4M$</td>
<td>$4M$</td>
<td>$2M$</td>
<td>$3M$</td>
<td>M</td>
<td>$A_{270}$</td>
<td>$A_{90}$</td>
<td>$A_{180}$</td>
<td>I</td>
</tr>
</tbody>
</table>
Initial Symmetry

The group containing one six-fold axis, plus a plane containing that axis, as shown in Fig. 119.

Fig. 119 Initial Symmetry for Class 6 mm

The six-fold axis repeats the plane as planes $2M$ and $3M$, Fig. 120.

Fig. 120 Class 6 mm
Quite analogous to the previous case, the repetition of a face in general position, as in Fig. 120, shows that there is present a second set of planes $4M, 5M, \text{and } 6M$. The latter set are not identical with planes $M, 2M, \text{and } 3M$. If they were all identical, the axis at the center would be twelve-fold, which is not possible in crystals.

Again, by taking the products of the operations, we can see the necessity of the second set of planes.

<table>
<thead>
<tr>
<th>Initial Symmetry</th>
<th>x mirror plane</th>
<th>= result</th>
</tr>
</thead>
<tbody>
<tr>
<td>I $\times$ M</td>
<td>M</td>
<td>$M$</td>
</tr>
<tr>
<td>$A_{60}$ $\times$ M</td>
<td>$M$</td>
<td>$6M$</td>
</tr>
<tr>
<td>$A_{120}$ $\times$ M</td>
<td>$M$</td>
<td>$3M$</td>
</tr>
<tr>
<td>$A_{180}$ $\times$ M</td>
<td>$M$</td>
<td>$5M$</td>
</tr>
<tr>
<td>$A_{240}$ $\times$ M</td>
<td>$M$</td>
<td>$2M$</td>
</tr>
<tr>
<td>$A_{300}$ $\times$ M</td>
<td>$M$</td>
<td>$4M$</td>
</tr>
</tbody>
</table>

Symmetry of the group - $1A_6, 6P$

Name of the group - Dihexagonal-pyramidal

International symbol - $6\text{mm}$

Schoenflies symbol - $C_{6v}$

Number of operations - 12
Derivation of Class $\text{m m m} \quad \frac{2}{2} \frac{2}{2}$

Initial Symmetry

The class having three mutually perpendicular two-fold axes, plus a plane which contains any two of them, (and therefore is perpendicular to the third). It makes no difference which two are selected, a fact which the reader can readily verify by trial and error. Fig. 121 shows the initial symmetry.

If $x$ at position 1 represents a face in a general position, it is repeated in the lower hemisphere, as the circle at position 2, by the two-fold axis labeled $A$. Plane $M$ repeats the circle at position 2 as a circle at position 1, and the $X$ of position 1 as an $X$ at position 2. Either axis $B$ or axis $C$ will then repeat the faces as the $X$'s and circles at positions 3 and four. It is now obvious that two additional planes of symmetry are present,
designated as $^2\!M$ and $^3\!M$ in Fig. 122.

The products of the operations concerned are as follows:

<table>
<thead>
<tr>
<th>Axial operations</th>
<th>Plane</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$M$</td>
<td>$M$</td>
</tr>
<tr>
<td>A$_{180}$</td>
<td>$M$</td>
<td>$^3!M$</td>
</tr>
<tr>
<td>B$_{180}$</td>
<td>$M$</td>
<td>$1$</td>
</tr>
<tr>
<td>C$_{180}$</td>
<td>$M$</td>
<td>$^2!M$</td>
</tr>
</tbody>
</table>

As always, when an evenfold axis is perpendicular to a plane, a center is present also.
Symmetry of the Class - 3 A₂, 3 P, C

Name of the Class - Rhombic-dipyramidal

International Symbol - \( \frac{2 2 2}{m m m} \)

Schoenflies Symbol - \( D_{2h} \)

### Multiplication Table

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>A₁⁸₀</th>
<th>B₁⁸₀</th>
<th>C₁⁸₀</th>
<th>M</th>
<th>²M</th>
<th>³M</th>
<th>i</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>I</td>
<td>A₁⁸₀</td>
<td>B₁⁸₀</td>
<td>C₁⁸₀</td>
<td>M</td>
<td>²M</td>
<td>³M</td>
<td>i</td>
</tr>
<tr>
<td>A₁⁸₀</td>
<td>A₁⁸₀</td>
<td>I</td>
<td>C₁⁸₀</td>
<td>B₁⁸₀</td>
<td>³M</td>
<td>i</td>
<td>M</td>
<td>²M</td>
</tr>
<tr>
<td>B₁⁸₀</td>
<td>B₁⁸₀</td>
<td>C₁⁸₀</td>
<td>I</td>
<td>A₁⁸₀</td>
<td>i</td>
<td>³M</td>
<td>²M</td>
<td>M</td>
</tr>
<tr>
<td>C₁⁸₀</td>
<td>C₁⁸₀</td>
<td>B₁⁸₀</td>
<td>A₁⁸₀</td>
<td>I</td>
<td>²M</td>
<td>M</td>
<td>i</td>
<td>³M</td>
</tr>
<tr>
<td>²M</td>
<td>²M</td>
<td>i</td>
<td>³M</td>
<td>M</td>
<td>C₁⁸₀</td>
<td>I</td>
<td>B₁⁸₀</td>
<td>A₁⁸₀</td>
</tr>
<tr>
<td>³M</td>
<td>³M</td>
<td>M</td>
<td>²M</td>
<td>i</td>
<td>A₁⁸₀</td>
<td>B₁⁸₀</td>
<td>I</td>
<td>C₁⁸₀</td>
</tr>
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<td>M</td>
<td>³M</td>
<td>B₁⁸₀</td>
<td>A₁⁸₀</td>
<td>C₁⁸₀</td>
<td>I</td>
</tr>
</tbody>
</table>
Derivation of Class \( \bar{6} \text{ m} 2 \)

**Initial Symmetry**

The class having one three-fold axis and three two-fold axes all at 90° to the three-fold, plus a plane containing the three-fold and one of the two-fold axes, as shown in Fig. 123.

![Fig. 123 Initial Symmetry for Class \( \bar{6} \text{ m} 2 \)](image)

Note that the plane bisects the angle between the two two-fold axes which it does not contain. This, of course, is a permissible relationship.

The three-fold axis (or the two-fold axes labeled C and D) repeat the initial plane in the positions indicated in Fig. 124.
A face in the general position of the X at position 1 is repeated at X, position 2, by the plane M. The two-fold axis B would reproduce these faces at position 1 and 2 in the lower hemisphere, the circles. Then, the three-fold would repeat them as upper and lower faces in positions 3, 4, 5, and 6. It now becomes clear that the plane of the paper is a symmetry plane, and the complete symmetry is illustrated in Fig. 125.
Referring back to the discussion of axes of rotary conversion, we note that the symmetry of a three-fold axis perpendicular to a plane is equivalent to a six-fold axis of rotary inversion.

The products of the operations concerned are:

<table>
<thead>
<tr>
<th>Initial symmetry</th>
<th>operations of a plane</th>
<th>result</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I$</td>
<td>$x$</td>
<td>$M$</td>
</tr>
<tr>
<td>$A_{120}$</td>
<td>$x$</td>
<td>$3M$</td>
</tr>
<tr>
<td>$A_{240}$</td>
<td>$x$</td>
<td>$2M$</td>
</tr>
<tr>
<td>$B_{180}$</td>
<td>$x$</td>
<td>$\overline{A}_{180}$</td>
</tr>
<tr>
<td>$C_{180}$</td>
<td>$x$</td>
<td>$\overline{A}_{60}$</td>
</tr>
<tr>
<td>$D_{180}$</td>
<td>$x$</td>
<td>$\overline{A}_{300}$</td>
</tr>
</tbody>
</table>

Thus we see that the addition of a plane does require the
presence of two additional planes plus an axis of six-fold rotary inversion. The operation of the latter are I, $A_{60}^\prime$, $A_{120}^\prime$, $A_{180}^\prime$, $A_{240}^\prime$, $A_{300}^\prime$.

<table>
<thead>
<tr>
<th>Symmetry of the class -</th>
<th>1 $A_3$, 3 $A_2$, 4 $P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Name of the class -</td>
<td>Ditrigonal-dipyramidal</td>
</tr>
<tr>
<td>International symbol -</td>
<td>$\bar{6} m 2$</td>
</tr>
<tr>
<td>Schoenflies symbol -</td>
<td>$D_{3h}$</td>
</tr>
<tr>
<td>Number of operations -</td>
<td>12</td>
</tr>
</tbody>
</table>
Initial Symmetry

The class having one four-fold axis plus two sets of two-fold axes at 90° to the four-fold, with each two-fold standing at 90° to an identical two-fold axis and at 45° to a two-fold of the other set; plus a plane in any permissible position. The permissible positions for planes are shown in Figs. 126, 127, and 128, and trial will convince the reader that it does not matter which initial plane is chosen.

Fig. 126 Axial Plane

Fig. 127 Diagonal Plane

Fig. 128 Horizontal Plane
We will choose the initial plane in the position shown in Fig. 126 as plane \( M \), and use the symmetry to operate on a face in the general position of the \( X \) at location 1.

Observing Fig. 129 we see that the face \( X \) at position 1 is repeated by the four-fold axis as other faces designated \( X \) at positions 2, 3, and 4. The two-fold axis labeled \( B \) will repeat the upper hemisphere faces at positions 1, 2, 3, and 4 at lower hemisphere positions 5, 6, 7, and 8, marked by circles. Plane \( M \) would repeat circle 8 at position 1 in the lower hemisphere, and \( X \) number 1 as an \( X \) at position number 8 in the upper hemisphere. Corresponding repetition will take place between position 5 and 4, 2 and 7, 6 and 3. Furthermore, plane \( M \) will be repeated by the four-fold axis as plane \( \overline{2}M \), so that the symmetry and faces
appear as in Fig. 130.

An examination of the collection of faces makes it apparent that additional planes of symmetry are present, and that the total symmetry is that illustrated in Fig. 131.
The fact that planes $2M$, $3M$, $4M$, and $5M$ do arise is illustrated below.

1 \times M = M

A_{90} \times M = 4M

A_{180} \times M = 2M

A_{270} \times M = 3M

b_{180} \times M = 5M \ (\overline{A}_{180})

C_{180} \times M = \overline{A}_{270}

2b_{180} \times M = \overline{1}

2C_{180} \times M = \overline{A}_{90}

Symmetry of the class - 1 A_4, 4 A_2, 4 P, C

Name of the class - Ditetragonal-dipyramidal

International symbol - 4 \frac{2}{m} \frac{2}{m} m

Schoenflies symbol - D_{4h}

Number of operations - 16

A multiplication table 16 items square is too large to fit on a typed page. It is hoped that the point thus illustrated has been made a sufficient number of times that such tables can be omitted when the number of operations is greater than twelve.
Derivation of Class m m m

Initial Symmetry

The group containing one axis of six-fold symmetry plus two sets of two-fold axes making \(90^\circ\) with the former. The identical two-folds are at \(60^\circ\) to each other, the unlike two-folds at \(30^\circ\). Now add a plane in any permissible position. The situation is quite analogous to the previous case. It makes no difference which is the initial plane. Fig. 132 shows the initial symmetry.

Fig. 132 Initial Symmetry for Class m m m
A general face, selected (for instance) in the position shown in the previous figure, will be repeated as in Fig. 133.

It is readily apparent that the complete symmetry involves additional planes $4M, 5M, 6M,$ and $7M$, and is as shown in Fig. 134.
Symmetry of the group - 1 $A_6$, 6 $A_2$, 7 $P$, $C$

Name of the group - Dihexagonal-dipyramidal

International symbol - $\frac{6 2 2}{m m m}$

Schoenflies symbol - $D_{6h}$

Number of operations - 24

The multiplication table would contain the following operations:

<table>
<thead>
<tr>
<th>Operation</th>
<th>$B_{180}$</th>
<th>$2B_{180}$</th>
<th>$3B_{180}$</th>
<th>$C_{180}$</th>
<th>$2C_{180}$</th>
<th>$3C_{180}$</th>
<th>$M$</th>
<th>$2M$</th>
<th>$3M$</th>
<th>$4M$</th>
<th>$5M$</th>
<th>$6M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{60}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_{120}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_{180}$</td>
<td></td>
<td></td>
<td></td>
<td>$C_{180}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_{240}$</td>
<td></td>
<td></td>
<td></td>
<td>$2C_{180}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_{300}$</td>
<td></td>
<td></td>
<td></td>
<td>$3C_{180}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_{120}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$2M$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_{180} (7M)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$3M$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_{240}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$4M$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_{300}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$5M$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$i$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$6M$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Derivation of Class $3\bar{m}$

Initial Symmetry

The class containing one three-fold axis and three two-fold axes at $90^\circ$ to the three-fold, plus a plane containing the three-fold and bisecting the angle between any two of the two-fold axes. Trial will show that it makes no difference which of the latter are selected. This symmetry is shown in Fig. 135.

![Diagram of Initial Symmetry for Class $3\bar{m}$](image)

Fig. 135 Initial Symmetry for Class $3\bar{m}$

The three-fold axis (or the various two-fold axes) will repeat the plane as shown in Fig. 136. A face in the general position is reproduced as shown.
It will be noted that each two-fold axis is perpendicular to a plane of symmetry. This means that a center of symmetry is present, and a center of symmetry combined with a three-fold axis is a three-fold axis of rotary inversion. Therefore, the axis designated A is such an axis.

Again this can be shown by taking the products of the initial symmetry and the added plane of symmetry.

<table>
<thead>
<tr>
<th>Initial Symmetry</th>
<th>X Plane</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>X M</td>
<td>M</td>
</tr>
<tr>
<td>A_{120}</td>
<td>X M</td>
<td>3M</td>
</tr>
<tr>
<td>A_{240}</td>
<td>X M</td>
<td>2M</td>
</tr>
<tr>
<td>B_{180}</td>
<td>X M</td>
<td>A_{120}</td>
</tr>
<tr>
<td>2B_{180}</td>
<td>X M</td>
<td>1</td>
</tr>
<tr>
<td>3B_{180}</td>
<td>X M</td>
<td>A_{240}</td>
</tr>
</tbody>
</table>

Fig. 136 Class $\overline{3}\overline{2}M$
Thus the symmetry of a three-fold axis of rotary inversion is seen to be present.

Symmetry of the class - $1 A_3, 3A_2, 3P, C$

Name of the class - Hexagonal-scalenohedral

International Symbol - $\overline{3}m$

Schoenflies symbol - $D_{3d}$

Number of operations - 12
Derivation of Class $\overline{4} 2 m$

Initial Symmetry

The group having three mutually perpendicular two-fold axes, plus a plane containing one of them and bisecting the angle between the other two, a legitimate position for a plane. Trial will show that it makes no difference which axis is chosen to contain the plane. The initial symmetry is that illustrated in Fig. 137.

![Fig. 137 Initial Symmetry for $\overline{4} 2 m$](image)

A rotation of plane $M$ about axis $C$, or about axis $B$, will repeat the plane at $^2M$, as in Fig. 138.

![Fig. 138 Derivation of $\overline{4} 2 m$](image)
If a face is placed in a general position, as is the X in location 1, Fig. 139, the repetition by the symmetry is as follows:

Plane M repeats $1X$ at $2X$

Two-fold axis A repeats $1X$ and $2X$ at $5X$ and $6X$

Axis C repeats $1X$ at $30$ and $2X$ at $40$

Two-fold axis A repeats $30$ at $70$ and $40$ at $80$

Of course, other sequences of operations may be used, but the reader can easily satisfy himself that the results are the same.

Fig. 139 Class $42m$

It should be noted that although the two-fold axis at A (Fig. 138) is sufficient to designate the symmetry about that axis, the repetition pattern is such that A is actually a four-fold axis of rotary inversion, as indicated in Fig. 139.
Taking the product of the operations of the three two-fold axes and the mirror plane also shows the presence of the four-fold axis of rotary inversion.

<table>
<thead>
<tr>
<th>Operations of the three two-fold</th>
<th>x</th>
<th>operations of M</th>
<th>=</th>
<th>result</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td></td>
<td>M</td>
<td></td>
<td>M</td>
</tr>
<tr>
<td>$A_{180}$</td>
<td></td>
<td>M</td>
<td></td>
<td>$2M$</td>
</tr>
<tr>
<td>$B_{180}$</td>
<td></td>
<td>M</td>
<td></td>
<td>$\bar{A}_{270}$</td>
</tr>
<tr>
<td>$C_{180}$</td>
<td></td>
<td>M</td>
<td></td>
<td>$\bar{A}_{90}$</td>
</tr>
</tbody>
</table>

$I$, $A_{90}$, $A_{180}$, and $A_{270}$ are the operations of a four-fold axis of rotary inversion.

Symmetry of the class - $3A_{2}, 2P$

Name of the class - Tetragonal-scalenohedral

International symbol - $\overline{4}2m$

Schoenflies symbol - $D_{2d}$

Number of operations - 8
<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>A_{180}</th>
<th>B_{180}</th>
<th>C_{180}</th>
<th>M</th>
<th>2M</th>
<th>\overline{A}_{90}</th>
<th>\overline{A}_{270}</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>I</td>
<td>A_{180}</td>
<td>B_{180}</td>
<td>C_{180}</td>
<td>M</td>
<td>2M</td>
<td>\overline{A}_{90}</td>
<td>\overline{A}_{270}</td>
</tr>
<tr>
<td>A_{180}</td>
<td>A_{180}</td>
<td>I</td>
<td>C_{180}</td>
<td>B_{180}</td>
<td>2M</td>
<td>M</td>
<td>\overline{A}_{270}</td>
<td>\overline{A}_{90}</td>
</tr>
<tr>
<td>B_{180}</td>
<td>B_{180}</td>
<td>C_{180}</td>
<td>I</td>
<td>A_{180}</td>
<td>\overline{A}_{90}</td>
<td>\overline{A}_{270}</td>
<td>M</td>
<td>2M</td>
</tr>
<tr>
<td>C_{180}</td>
<td>C_{180}</td>
<td>B_{180}</td>
<td>A_{180}</td>
<td>I</td>
<td>\overline{A}_{270}</td>
<td>\overline{A}_{90}</td>
<td>2M</td>
<td>M</td>
</tr>
<tr>
<td>M</td>
<td>M</td>
<td>2M</td>
<td>A_{270}</td>
<td>\overline{A}</td>
<td>I</td>
<td>A_{180}</td>
<td>C_{180}</td>
<td>B_{180}</td>
</tr>
<tr>
<td>2M</td>
<td>2M</td>
<td>M</td>
<td>\overline{A}_{90}</td>
<td>\overline{A}_{270}</td>
<td>A_{180}</td>
<td>I</td>
<td>B_{180}</td>
<td>C_{180}</td>
</tr>
<tr>
<td>\overline{A}_{90}</td>
<td>\overline{A}_{90}</td>
<td>\overline{A}_{270}</td>
<td>2M</td>
<td>M</td>
<td>B_{180}</td>
<td>C_{180}</td>
<td>A_{180}</td>
<td>I</td>
</tr>
<tr>
<td>\overline{A}_{270}</td>
<td>\overline{A}_{270}</td>
<td>\overline{A}_{90}</td>
<td>M</td>
<td>2M</td>
<td>C_{180}</td>
<td>B_{180}</td>
<td>I</td>
<td>A_{180}</td>
</tr>
</tbody>
</table>
Derivation of Class $\frac{2}{m \ 3}$

**Initial Symmetry**

The class containing two-fold and three-fold axes as shown in Fig. , page , plus a plane perpendicular to either of the two-fold axes. Again, it makes no difference which position is chosen. Fig. 140 illustrates this symmetry with plane M the initial plane of symmetry.

![Diagram of Initial Symmetry for Class $\frac{2}{m \ 3}$](image)
This should not be a surprise. The plane initially assumed contained the two-fold axes $A$ and $3A$; but $A$, $2A$, and $3A$ are identical axes, made so by symmetry elements already present. Therefore, if there is a plane containing $A$ and $3A$, there must be a plane containing $A$ and $2A$, and one containing $2A$ and $3A$.

The complete symmetry is that of Fig. 141.

The operations of this class and the repetition of an initial face at position 1 in the upper hemisphere are: (refer to Fig. 141.)

$$
\begin{align*}
I &= 1 \\
A_{180} &= 19 \\
2A_{180} &= 11 \\
3A_{180} &= 14 \\
C_{120} &= 3 \\
\bar{C}_{120} &= 16 \\
C_{240} &= 5 \\
\bar{C}_{240} &= 18 \\
2C_{120} &= 7 \\
2\bar{C}_{120} &= 21 \\
2C_{240} &= 16 \\
2\bar{C}_{240} &= 3 \\
3C_{120} &= 9 \\
3\bar{C}_{120} &= 23 \\
3C_{240} &= 21 \\
3\bar{C}_{240} &= 7 \\
4C_{120} &= 18 \\
4\bar{C}_{120} &= 5 \\
4C_{240} &= 23 \\
4\bar{C}_{240} &= 9 \\
M &= 19 \\
2M &= 11 \\
9M &= 1 \\
i &= 14
\end{align*}
$$
When the elements of symmetry act upon each other, and upon a face in the general position 1, the pattern shown in Fig. in stereographic projection is the result. It is clear that additional planes have resulted from the initial symmetry.

Fig. 141 Class $\frac{2}{m \bar{3}}$
Symmetry of the class - $3 A_2, 4 A_3, 3 P, C$

Name of the class - Diploidal

International symbol - $\frac{2}{m} \bar{3}$

Schoenflies symbol - Th

Number of operations - 24
### Derivation of Class $\overline{4} 3 \text{m}$

**Initial Symmetry**

The class used in the previous example plus a plane containing one of the two-fold axes and one of the three-fold axes, and bisecting the angle between the other two-fold axes. This symmetry is as illustrated in Fig. 142. The plane is designated $4_M$.
When the symmetry elements act upon one another, and upon a face in position 1, the results are as illustrated in Fig. 143.

As might be suspected, planes are seen to be present in all positions determined by the intersection of the initial two-fold and three-fold axes. Trial will quickly convince the
reader that if any one of the six planes is introduced originally the end result is the same.

Symmetry of the Class - 3 \( A_2 \), 4 \( A_3 \), 6 \( P \)

Name of the Class - Hextetrahedral

International Symbol - \( \bar{4} \ 3 \ m \)

Schoenflies Symbol - \( Td \)

Number of Operations - 24

The results of adding a plane to the original symmetry are shown below.

\[
\begin{align*}
1 \times 4M & = 4M \\
A_{180} \times 4M & = 3\overline{A}_{90} \\
2A_{180} \times 4M & = 3\overline{A}_{270} \\
3A_{180} \times 4M & = 3M \\
C_{120} \times 4M & = \overline{A}_{90} \\
C_{240} \times 4M & = 2\overline{A}_{270} \\
2C_{120} \times 4M & = 7M \\
2C_{240} \times 4M & = 5M \\
3C_{120} \times 4M & = 2\overline{A}_{90} \\
3C_{240} \times 4M & = 6M \\
4C_{120} \times 4M & = 8M \\
4C_{240} \times 4M & = 8M
\end{align*}
\]
It is seen that the original two-fold axes are now four-fold axes of rotary inversion, as is indicated on the stereographic projection.

The positions resulting from the repetition of an original face "1" are shown below.

\[
\begin{align*}
I &= 1 \\
\bar{A}_{90} &= 15 \\
A_{180} &= 19 \\
\bar{A}_{270} &= 10 \\
2A_{90} &= 20 \\
2A_{180} &= 11 \\
3\bar{A}_{90} &= 22 \\
3A_{180} &= 14 \\
3\bar{A}_{270} &= 8 \\
C_{120} &= 3 \\
C_{240} &= 5 \\
2C_{120} &= 7 \\
2C_{240} &= 16 \\
3C_{120} &= 9 \\
3C_{240} &= 21 \\
4C_{120} &= 18 \\
4C_{240} &= 23 \\
3\bar{M} &= 4 \\
4\bar{M} &= 17 \\
5\bar{M} &= 2 \\
6\bar{M} &= 6 \\
7\bar{M} &= 12 \\
8\bar{M} &= 24
\end{align*}
\]
Derivation of Class $\frac{4}{m} \frac{2}{3} m$

Initial Symmetry

The class containing three four-fold axes, four three-fold axes, and six two-fold axes, as illustrated in Fig. 144.

Fig. 144 Initial Axes

The angular relationships are:

- two-folds $\land$ three-folds $= 35^{\circ}15'52''$
- two-folds $\land$ four-folds $= 45^{\circ}$
- three-folds $\land$ four-folds $= 54^{\circ}44'08''$
- two-folds $\land$ two-folds $= 90^{\circ}$
- three-folds $\land$ three-folds $= 70^{\circ}31'44''$ and $109^{\circ}28'16''$
- four-folds $\land$ four-folds $= 90^{\circ}$
This symmetry is plotted on Fig. 145.

Now, there are two possible ways to add a plane in a permissible position. These are:

Fig. 146, in which the plane contains only two-fold and four-fold axes (and equivalent positions)
Fig. 147, in which the plane contains two, three, and fourfold axes (and equivalent positions).

Fig. 147  432 Plus a Diagonal Plane
Trial will quickly show that it makes no difference which plane is added. The high degree of axial symmetry results in the development of all nine possible planes, regardless of which is chosen initially.

If plane $9M$ (the horizontal plane containing only two-fold and four-fold axes is chosen), the result of combining the operations of the plane and the initial symmetry is as follows:

$$432 \times 9M$$

<table>
<thead>
<tr>
<th>Operation</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I \times 9M$</td>
<td>$9M$</td>
</tr>
<tr>
<td>$A_{90} \times 9M$</td>
<td>$8M$</td>
</tr>
<tr>
<td>$A_{180} \times 9M$</td>
<td>$M$</td>
</tr>
<tr>
<td>$A_{270} \times 9M$</td>
<td>$5M$</td>
</tr>
<tr>
<td>$2A_{90} \times 9M$</td>
<td>$6M$</td>
</tr>
<tr>
<td>$2A_{180} \times 9M$</td>
<td>$2M$</td>
</tr>
<tr>
<td>$2A_{270} \times 9M$</td>
<td>$M$</td>
</tr>
<tr>
<td>$3A_{90} \times 9M$</td>
<td>$3A_{90}$</td>
</tr>
<tr>
<td>$3A_{180} \times 9M$</td>
<td>$1$</td>
</tr>
<tr>
<td>$3A_{270} \times 9M$</td>
<td>$3A_{270}$</td>
</tr>
<tr>
<td>$B_{180} \times 9M$</td>
<td>$3M$</td>
</tr>
<tr>
<td>$2B_{180} \times 9M$</td>
<td>$4M$</td>
</tr>
<tr>
<td>$3B_{180} \times 9M$</td>
<td>$2\overline{A}_{90}$</td>
</tr>
<tr>
<td>$4B_{180} \times 9M$</td>
<td>$\overline{A}_{270}$</td>
</tr>
<tr>
<td>$5B_{180} \times 9M$</td>
<td>$2\overline{A}_{270}$</td>
</tr>
<tr>
<td>$6B_{180} \times 9M$</td>
<td>$\overline{A}_{90}$</td>
</tr>
<tr>
<td>$C_{120} \times 9M$</td>
<td>$2C_{240}$</td>
</tr>
<tr>
<td>$C_{240} \times 9M$</td>
<td>$4\overline{C}_{120}$</td>
</tr>
<tr>
<td>$2C_{240} \times 9M$</td>
<td>$3\overline{C}_{240}$</td>
</tr>
<tr>
<td>$3C_{120} \times 9M$</td>
<td>$4\overline{C}_{240}$</td>
</tr>
<tr>
<td>$3C_{240} \times 9M$</td>
<td>$2\overline{C}_{120}$</td>
</tr>
<tr>
<td>$4C_{120} \times 9M$</td>
<td>$\overline{C}_{240}$</td>
</tr>
<tr>
<td>$4C_{240} \times 9M$</td>
<td>$3\overline{C}_{120}$</td>
</tr>
</tbody>
</table>
This results in 48 operations and the total symmetry shown in Fig. 148.

Note that a center of symmetry is present. That is, there is an even-fold axis perpendicular to a plane of symmetry (several, as a matter of fact!). Further, the center means that the three-fold axes are axes of rotary inversion.
Symmetry of the Class - $3A_4$, $4A_3$, $6A_2$, $9P$, $C$

Name of the Class - Hextetrahedral

International Symbol - $\frac{4}{m} \overline{2}m$

Schoenflies Symbol - $O_h$

Number of Operations - 48

The repetition by each operation is shown on Fig. 148.

and indicated below by number

<table>
<thead>
<tr>
<th>Operation</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I$</td>
<td>1</td>
</tr>
<tr>
<td>$A_{90}$</td>
<td>24</td>
</tr>
<tr>
<td>$A_{180}$</td>
<td>19</td>
</tr>
<tr>
<td>$A_{270}$</td>
<td>10</td>
</tr>
<tr>
<td>$A_{90}'$</td>
<td>6</td>
</tr>
<tr>
<td>$A_{180}'$</td>
<td>11</td>
</tr>
<tr>
<td>$A_{270}'$</td>
<td>13</td>
</tr>
<tr>
<td>$3A_{90}$</td>
<td>22</td>
</tr>
<tr>
<td>$3A_{180}$</td>
<td>14</td>
</tr>
<tr>
<td>$3A_{270}$</td>
<td>8</td>
</tr>
<tr>
<td>$3A_{90}'$</td>
<td>22</td>
</tr>
<tr>
<td>$3A_{270}'$</td>
<td>8</td>
</tr>
<tr>
<td>$B_{180}$</td>
<td>4</td>
</tr>
<tr>
<td>$2B_{180}$</td>
<td>17</td>
</tr>
<tr>
<td>$3B_{180}$</td>
<td>20</td>
</tr>
<tr>
<td>$4B_{180}$</td>
<td>10</td>
</tr>
<tr>
<td>$5B_{180}$</td>
<td>13</td>
</tr>
<tr>
<td>$6B_{180}$</td>
<td>15</td>
</tr>
<tr>
<td>$C_{120}$</td>
<td>3</td>
</tr>
<tr>
<td>$2C_{120}$</td>
<td>18</td>
</tr>
<tr>
<td>$C_{240}$</td>
<td>16</td>
</tr>
<tr>
<td>$2C_{240}$</td>
<td>7</td>
</tr>
<tr>
<td>$C_{240}'$</td>
<td>21</td>
</tr>
<tr>
<td>$4C_{120}$</td>
<td>18</td>
</tr>
<tr>
<td>$4C_{240}$</td>
<td>23</td>
</tr>
<tr>
<td>$4C_{240}'$</td>
<td>9</td>
</tr>
<tr>
<td>$i$</td>
<td>14</td>
</tr>
<tr>
<td>$M$</td>
<td>19</td>
</tr>
<tr>
<td>$2M$</td>
<td>11</td>
</tr>
<tr>
<td>$3M$</td>
<td>4</td>
</tr>
<tr>
<td>$4M$</td>
<td>17</td>
</tr>
<tr>
<td>$5M$</td>
<td>2</td>
</tr>
<tr>
<td>$6M$</td>
<td>6</td>
</tr>
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<td>$7M$</td>
<td>12</td>
</tr>
<tr>
<td>$8M$</td>
<td>24</td>
</tr>
<tr>
<td>$9M$</td>
<td>1</td>
</tr>
</tbody>
</table>
Note that in each position there are two faces present—one above the equatorial plane, and one below that plane.

That is, the symbol on the projection $24^0$ refers to one face, 24, in the upper hemisphere, and another directly below it in the lower hemisphere. In the preceding table, it is indicated that $A_{90}$ repeats face 1 at position 24 in the upper hemisphere, and plane $_{8}$ repeats face 1 at position 24 in the lower hemisphere.
Derivation of Class \( \bar{1} \)

Initial Symmetry

The group containing one one-fold axis of rotating inversion. As demonstrated earlier, this is the equivalent of a center of symmetry. The symmetry and repetition of a general face are as in Fig. 149.

![Diagram of Class 1 symmetry](image)

---

Symmetry of the Class - C

Name of the class - Finacoidal

International symbol - \( \bar{1} \)

Schoenflies symbol - \( \bar{C} \)

Number of operations - 2

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>i</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>I</td>
<td>i</td>
</tr>
<tr>
<td>i</td>
<td>i</td>
<td>I</td>
</tr>
</tbody>
</table>
Derivation of Class 3

Initial Symmetry

The class containing one three-fold axis of rotary inversion. This, of course, is the equivalent of a three-fold rotary axis plus a center, as shown in Fig. 150. The repetition of a general face also is indicated.

![Diagram of Class 3](image_url)

Fig. 151 Class 3

The operations resulting from the addition of a center to a three-fold rotary axis can be shown as follows:

<table>
<thead>
<tr>
<th>Operation of a three-fold</th>
<th>x</th>
<th>center</th>
<th>=</th>
<th>result</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>x</td>
<td>i</td>
<td></td>
<td>i</td>
</tr>
<tr>
<td>A_{120}</td>
<td>x</td>
<td>i</td>
<td></td>
<td>\overline{A}_{120}</td>
</tr>
<tr>
<td>A_{240}</td>
<td>x</td>
<td>i</td>
<td></td>
<td>\overline{A}_{240}</td>
</tr>
</tbody>
</table>
Symmetry of the class - 1 $A_3, C$

Name of the class - Rhombohedral

International symbol - $\bar{3}$

Schoenflies symbol - $C_3i$

Number of operations - 6

Multiplication table

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>$A_{120}$</th>
<th>$\bar{A}_{120}$</th>
<th>$A_{240}$</th>
<th>$\bar{A}_{240}$</th>
<th>I</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>I</td>
<td>I</td>
<td>I</td>
<td>I</td>
<td>I</td>
<td>I</td>
</tr>
<tr>
<td>$A_{120}$</td>
<td>$A_{120}$</td>
<td>$A_{240}$</td>
<td>$\bar{A}_{240}$</td>
<td>I</td>
<td>I</td>
<td>$A_{120}$</td>
</tr>
<tr>
<td>$\bar{A}_{120}$</td>
<td>$A_{120}$</td>
<td>$\bar{A}_{240}$</td>
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<tr>
<td>$A_{240}$</td>
<td>$\bar{A}_{240}$</td>
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<td>I</td>
<td>$A_{120}$</td>
<td>$\bar{A}_{120}$</td>
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<tr>
<td>$\bar{A}_{240}$</td>
<td>$\bar{A}_{240}$</td>
<td>I</td>
<td>I</td>
<td>$A_{120}$</td>
<td>$\bar{A}_{120}$</td>
<td>$A_{240}$</td>
</tr>
<tr>
<td>$I$</td>
<td>$I$</td>
<td>$\bar{A}_{120}$</td>
<td>$A_{120}$</td>
<td>$\bar{A}_{240}$</td>
<td>$A_{240}$</td>
<td>I</td>
</tr>
</tbody>
</table>
Derivation of Class $\bar{4}$

**Initial Symmetry**

The group having one four-fold axis of rotary inversion.

This is the one essential axis of rotary inversion, inasmuch as this symmetry can be expressed in no other way. The symmetry is as shown in Fig. 151.

![Fig. 152 Class $\bar{4}$](image)

Symmetry of the class - $\bar{1}\bar{A}_4$

Name of the class - Tetragonal-disphenoidal

International symbol - $\bar{4}$

Schoenflies symbol - $S_4$

Number of operations - 4

<table>
<thead>
<tr>
<th></th>
<th>$\bar{1}$</th>
<th>$\bar{A}_{90}$</th>
<th>$\bar{A}_{180}$</th>
<th>$\bar{A}_{270}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{1}$</td>
<td>$\bar{1}$</td>
<td>$\bar{A}_{90}$</td>
<td>$\bar{A}_{180}$</td>
<td>$\bar{A}_{270}$</td>
</tr>
<tr>
<td>$\bar{A}_{90}$</td>
<td>$\bar{A}_{90}$</td>
<td>$\bar{A}_{180}$</td>
<td>$\bar{A}_{270}$</td>
<td>$\bar{1}$</td>
</tr>
<tr>
<td>$\bar{A}_{180}$</td>
<td>$\bar{A}_{180}$</td>
<td>$\bar{A}_{270}$</td>
<td>$\bar{1}$</td>
<td>$\bar{A}_{90}$</td>
</tr>
<tr>
<td>$\bar{A}_{270}$</td>
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<td>$\bar{1}$</td>
<td>$\bar{A}_{90}$</td>
<td>$\bar{A}_{180}$</td>
</tr>
</tbody>
</table>
This completes the derivation of the thirty-two crystal classes. Any other permissable combination of symmetry elements will lead to one of the classes previously derived. The thirty-two classes are shown grouped into systems in Table 9.

TABLE 8
Crystal Systems and Classes

<table>
<thead>
<tr>
<th>Crystal system</th>
<th>International symbol</th>
<th>Crystal system</th>
<th>International symbol</th>
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</thead>
<tbody>
<tr>
<td>Triclinic</td>
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<td></td>
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<td></td>
<td>3</td>
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</tr>
<tr>
<td></td>
<td>m</td>
<td></td>
<td>3mm</td>
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<tr>
<td></td>
<td>2</td>
<td></td>
<td>6</td>
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<tr>
<td></td>
<td>m</td>
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<td>6m2</td>
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<td>Orthorhombic</td>
<td>222</td>
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<td></td>
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<td></td>
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<td>m m m</td>
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<tr>
<td>Isometric</td>
<td></td>
<td></td>
<td>23</td>
</tr>
</tbody>
</table>
REFERENCES


Guglielmini, D. (1705) *De salibus dissertatio epistolaris physico-medico-mechanica.* Venetiis.


Hooke, Robert (1665) *Micrographia*. London


