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$L_p$ SPACES AND DECOMPOSITIONS IN BANACH SPACES

DISSERTATION

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By

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* * * * * *

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**INTRODUCTION**

Spaces were introduced by Lindenstrauss and Pełczyński in [13]. A Banach space $X$ is called a $S_{p,\lambda}$ space if given any finite dimensional subspace $B \subset X$, there exists a finite dimensional subspace $W$ such that $B \subset W \subset X$ and $d(W, \ell_p^{(n)}) \leq \lambda$, $n = \dim(W)$. The characterization and classification of a Banach space in terms of its finite dimensional subspaces is a technique that has proved very fruitful in recent years in solving problems in Banach spaces. In [11], Lindenstrauss used this technique for the extension of operators and other related problems. In [15], using this approach, Michael and Pełczyński attacked some problems about $c(k)$ spaces, and in [3], Davis, Dean and Singer employed this technique to study complemented subspaces of a Banach space. In [14], Lindenstrauss and Rosenthal established a very deep result viz. the principle of local reflexivity which states that every finite dimensional subspace of $X^{**}$ is 'close' to a finite dimensional subspace of $X$. Recently in [7], Johnson, Rosenthal and Zippin developed this technique in a very deep way with the work culminating in solving affirmatively the outstanding problem that if $X^*$ has a basis, then $X$ has a basis.

In [13], the authors proved the main representation theorem about $S_p$ spaces viz. every $S_p$ space is isomorphic to a complemented $1$
subspace of some $L_p(\mu)$ space, $1 < p < \infty$ and for $p = 1, \infty$, if $X$ is complemented in $X^{**}$. In this paper, they also studied $p$-absolutely summing operators originally introduced by Grothendieck for $p = 1$. Combining the finite dimensional subspace approach and some powerful results about $p$-absolutely summing operators, they proved that every bounded unconditional basis of a $\ell_1$ space ($\ell_\infty$ space) is equivalent to the usual basis of $\ell_1(c_0)$.

In 1969, Lindenstrauss and Rosenthal [14] solved some of the problems left open in [13]. For example, they proved that every complemented subspace of an $L_p(\mu)$ space is either a Hilbert space or a $\ell_p$ space, $1 < p < \infty$ and for $p = 1, \infty$, it is always a $\ell_p$ space. This result together with the main representation theorem about $\ell_p$ spaces implies that if $X$ is a $\ell_p$ space, then $X^*$ is a $\ell_q$ space, $\frac{1}{p} + \frac{1}{q} = 1$. Recently in [7], the authors have proved that every separable $\ell_p$ space has a basis.

There are still many open problems in the theory of $\ell_p$ spaces. For example, the number of different (upto isomorphism) infinite dimensional separable $\ell_p$ spaces is unknown $1 < p < \infty$. Only six different such $\ell_p$ spaces are known [18]. Bessage and Pelczynski (for $p = \infty$) and Lindenstrauss (for $p = 1$) showed that there are infinitely many different separable infinite dimensional $\ell_p$ spaces. We shall discuss some more unsolved problems in Chapter IV.

In this paper, $\ell_p$ spaces are studied. We also study some properties that one might naturally ask in terms of finite dimensional subspaces of a Banach space to study the structure of that
space. In Chapter I, the notations and definitions are given and some known theorems are stated which are used in the subsequent chapters. Also a new proof of a theorem of Lindenstrauss and Pelczynski [13] is given which says that every bounded unconditional basis of an $c_0$ space is equivalent to the usual basis of $c_0$.

In Chapter II, $l_p$ spaces are studied. Proposition 2.1 establishes a property of $l_p$ spaces that is generally known but which nowhere has a very explicit statement in the literature. The technique of proof is frequently used in this work. Theorem 2.4 gives a sufficient condition for a subspace of $l_p$ to be complemented in $l_p$ in terms of the 'support' of a basis of the subspace. Theorem 2.6 is the main result of this chapter which says that if a Banach space $X$ is non-$l_p$, then it has a subspace $Y$ which is non-$l_p$ and which has a Schauder decomposition into finite dimensional subspaces. This result sheds more light on Proposition 7.2 [13]. Lemma 2.10 supporting Theorem 2.6 is of independent interest. Theorem 2.11 gives a sufficient condition for a subspace of $l_p$ to be a $l_p$ space.

In Chapter III, we introduce two new properties in Banach spaces. A Banach space $X$ has the $\lambda$-finite dimensional subspace basis property ($\lambda$-f. d. s. b) if for any finite dimensional subspace $B \subseteq X$, there exists a finite dimensional subspace $W$ such that $B \subseteq W \subseteq X$ and $W$ has a basis with basis constant less than or equal to $\lambda$. The introduction of this property is further justified by the fact that every Banach space with a basis has f. d. s. b.

Using the principle of local reflexivity, it is proved in
Proposition 3.6 that if $X^{**}$ has $\lambda$-f. d. s. b., then $X$ has $(\lambda + \varepsilon)$-f. d. s. b. for every $\varepsilon > 0$. As a corollary to Theorem 3.18, we obtain that if $X^*$ has a basis, then $X$ has f. d. s. b. Theorem 3.12 states that if a Banach space $X$ does not have f. d. s. b., then it has a subspace $Y$ which does not have f. d. s. b. and which has a Schauder decomposition into finite dimensional subspaces. In this chapter, we also study $B_\lambda$ spaces [Definition 3.15]. The study of this property is motivated by the f. d. s. b. property and the $\pi_\lambda$ property. Every space with a basis is a $B_\lambda$ space. Using two lemmas of [7], it is proved that if $X^*$ is a separable $B_\lambda$ space, then $X$ is a $B_\mu$ space for some $\mu$ [Theorem 3.18] and if $X$ is a $B_\lambda$ space, $X^*$ has $\mu$ metric approximation property, then $X^*$ is a $B_\tau$ space for some $\tau$ [Theorem 3.21].

Finally in Chapter IV, we discuss unsolved problems which have arisen in the course of this study and whose solution would shed great light on the structure of Banach spaces in terms of finite dimensional subspaces.
CHAPTER I

NOTATION, DEFINITIONS AND KNOWN RESULTS

In this paper, we deal with real Banach spaces. If $X$ is a Banach space, then $U(X)$ denotes the unit ball of $X$. By a subspace of $X$, we always mean a closed linear subspace. By an operator, we understand a bounded linear operator. We denote the dual of $X$ by $X^*$ and, as usual, identify $X$ with a subspace of $X^{**}$.

Definition 1.1

By a projection $P$ on a Banach space $X$, we mean an operator on $X$ such that $P^2 = P$. A projection with a finite dimensional range is called a finite rank projection.

Definition 1.2

A subspace $Y$ of a Banach space $X$ is said to be complemented in $X$ if there exists a projection $P$ from $X$ onto $Y$.

Notation 1.3

If $(x_i)_{i=1}^\infty (\ell^1)$ is a sequence of elements in a Banach space $X$, the subspace of $X$ spanned by $(x_i)_{i=1}^\infty (\ell^1)$ will be denoted by $[x_i]_{i=1}^\infty ([\ell^1])$. $[x_i]_{i=1}^\infty$ will be written as $[x_i]$ if no confusion arises.

Definition 1.4

Two Banach spaces $X$ and $Y$ are said to be isomorphic if there
exists an invertible operator from X onto Y, then we write $X \sim Y$.

**Definition 1.5**

If Y is a subspace of a Banach space X, then $X/Y$ denotes the **quotient space**. Y has **finite codimension** in X if $X/Y$ is finite dimensional.

**Definition 1.6**

A Banach space X is called an **injective space** if there exists a real number $\lambda$ such that for any Banach space $Z$, $Z \supset X$, there is a projection $P$ from Z onto X with $\|P\| \leq \lambda$.

**Definition 1.7**

A sequence $(x_i)$ of a Banach space X is called a **basis** for X if for each $x \in X$, there exists a unique sequence $(a_i)$ of real numbers such that $x = \sum a_i x_i = \lim_{n \to \infty} \sum_{i=1}^{n} a_i x_i$. A sequence $(x_i)$ in a Banach space X is called **basic** if it is a basis for the subspace that it spans in X. A sequence $(x_i)$ is called an **unconditional basis** for X if $x = \sum a_i x_i$ converges unconditionally to $x$.

**Theorem 1.8**

A sequence $(x_i)$ in a Banach space X is a basic sequence if and only if there exists a real number $k$ such that

$$\| \sum_{i=1}^{r} a_i x_i \| \leq k \| \sum_{i=1}^{s} a_i x_i \| \quad (1)$$

for all sequences $(a_i)$, $r \leq s$. The infimum of $k$ appearing in (1) is called the **basis constant** for $(x_i)$.
Definition 1.9

If \((x_i)\) is a basic sequence in a Banach space \(X\), then the sequence \((f_i)\) in \([x_i]^*\) with \(f_i(x_j) = \delta_{ij}\), \(i \neq j, i,j = 1,2,3,...\) is called biorthogonal functionals sequence. It is well known and is, in fact, a simple consequence of Theorem 1.8 that they constitute a basic sequence.

Definition 1.10

Let \(X\) be a Banach space. A family \((B_n)_{n=1}^\infty\) of subspaces of \(X\) is called a Schauder decomposition for \(X\), written as \(X = \bigoplus X_n\) if every \(x \in X\) can be uniquely written as \(x = \sum x_n\), \(x_n \in X_n\). It is well known that the projections \(P_m\) such that \(P_m(\sum x_n) = x_m\) are uniformly bounded.

Definition 1.11

Let \((B_n)\) be a sequence of Banach spaces. We define for \(p \geq 1\),

\[
B = \left( \bigoplus_{n} B_n \right)_p (\text{resp. } c_0) = \{(b_n) : b_n \in B_n \text{ and } \sum ||b_n||^p < \infty \} \quad (\text{resp. } ||b_n|| \to 0)\}.
\]

In \(B\), we define

\[
||(b_n)|| = \left( \sum ||b_n||^p \right)^{1/p} \quad (\text{resp. } \max_n ||b_n||),
\]

then \(B\) is a Banach space [16].

Notation 1.18

By \(L^p(\mu) = L^p(\Omega, \sigma, \mu)\), \(1 \leq p \leq \infty\), we denote the Banach space of equivalence classes of measurable functions on the measure space \((\Omega, \sigma, \mu)\) whose \(p\)th power is integrable if \(1 \leq p < \infty\) and essentially
bounded if \( p = \infty \). If \((\Omega, \sigma, \mu)\) is the usual measure space on \([0,1]\) we denote \( L_p(\mu) \) by \( L_p \). If \( \Omega \) is discrete and \( \mu(\{\gamma\}) = 1 \) for \( \gamma \in \Omega \), we denote \( L_p(\mu) \) be \( L_p(\Omega) \) and if \( \Omega \) is countable, then by \( L_p \). If \( \Omega = \{1, 2, \ldots, n\} \) with counting measure, then \( L_p(\mu) \) is denoted by \( L_p^n \). The subspace of \( l_\infty \) spanned by sequences which vanish at \( \infty \) is denoted by \( c_0 \).

**Definition 1.13**

Let \( X \) and \( Y \) be Banach spaces. Define

\[
d(X,Y) = \begin{cases} 
\inf(\|T\|, \|T^{-1}\|): T = X \to Y \text{ is an isomorphism onto} \\
\infty \text{ if } T \text{ is not isomorphic to } Y.
\end{cases}
\]

Also \( X \) and \( Y \) are said to be \( \epsilon \)-isometric if there exists an onto isomorphism \( T: X \to Y \) such that

\[
(1 - \epsilon)\|x\| \leq \|Tx\| \leq (1 + \epsilon)\|x\|, \quad x \in X.
\]

If \( \epsilon = 0 \), then \( T \) is an isometry and \( X \) and \( Y \) are said to be isometric. We state, without proof, the following useful relation connecting \( \epsilon \)-isometry with \( d(X,Y) \).

**Lemma 1.14**

\[
d(X,Y) \leq \lambda \text{ if and only if } X \text{ and } Y \text{ are } \frac{\lambda - 1}{\lambda + 1} \text{ isometric.}
\]

We shall use this lemma in computations throughout this paper.

As was mentioned in the introduction, \( L_p \) spaces were introduced by Lindenstrauss and Pełczyński in [13] in 1968 and in [14]...
Lindenstrauss and Rosenthal solved some questions left open in [13]. We state (without proof) below the main theorems of [13] and [14] which are relevant to the contents of this paper.

**Definition 1.15**

A Banach space $X$ is called a $\mathfrak{F}_{p,\lambda}$ space, $1 \leq p \leq \infty$, if given any finite dimensional subspace $B \subset X$, there exists a finite dimensional subspace $W$ s.t. $B \subset W \subset X$ and $d(W, \ell_p^{(n)}) \leq \lambda$ where $n = \dim(W)$. $X$ is called a $\mathfrak{F}_p$ space if it is $\mathfrak{F}_{p,\lambda}$ for some $\lambda$.

**Examples 1.16**

1. Every $L_p(\mu)$ is a $\mathfrak{F}_p$ space. The verification of this fact is non-trivial. We would discuss this result in Proposition 2.1. Here $1 \leq p \leq \infty$.
2. $\ell_2 \oplus \ell_p$, $1 < p < \infty$, is a $\mathfrak{F}_p$ space [13].
3. $(\ell_2 \oplus \ell_2 \oplus \cdots )_p$ is a $\mathfrak{F}_p$ space $1 < p < \infty$ [13].
4. In [13], Rosenthal has given examples of $\mathfrak{F}_p$ spaces ($p > 2$) spanned by certain sequences of random variables on $[0, 1]$ which are isomorphically different from the above examples.

**Theorem 1.17 [13]**

(a) Let $X$ be a $\mathfrak{F}_p$ space, $1 < p < \infty$. Then $X$ is isomorphic to a complemented subspace of some $L_p(\mu)$. If $X$ is separable, then it is isomorphic to a complemented subspace of $L_p$.
(b) For $p = 1$, $\infty$, $X$ is isomorphic to a subspace of some $L_p(\mu)$.
(c) For $p = 1$, $\infty$, $X$ is isomorphic to a complemented subspace of $L_p(\mu)$ if $X$ is complemented in $X^{**}$.
For $p = 1$, it is not always true that if $X$ is $l_p$, then $X$ is isomorphic to a complemented subspace of some $L_p(\mu)$. In fact, $c_0$ is an $\ell_\infty$ space and if $c_0$ is complemented is some $L_\infty(\mu)$, then $c_0$ would be injective as every $L_\infty(\mu)$ space is injective. But this is known to be false as no separable space can be an injective space. For $p = 1$, Lindenstrauss [9] has given an example of a $\ell_1$ space which is a subspace of $\ell_1$ and which is not isomorphic to a complemented subspace of any $L_1(\mu)$.

**Theorem 1.19** [8]

Every infinite dimensional $\ell_p$ space, $1 \leq p < \infty$, has a complemented subspace isomorphic to $\ell_p$.

**Note 1.20**

Theorem 1.19 is not true for $p = \infty$. In fact, $C([0,1])$ is a $\ell_\infty$ space but it has no subspace isomorphic to $\ell_\infty$.

Now we state some theorems connecting the structure of $X$ and $X^*$ if $X$ is a $\ell_p$ space.

**Theorem 1.21** [14]

(a) Every complemented subspace of a $L_p(\mu)$ space is either a Hilbert space or a $\ell_p$ space, $1 < p < \infty$.

(b) For $p = 1$, $\infty$, every complemented subspace of a $L_p(\mu)$ is a $\ell_p$ space.

**Remark**

For $1 < p < \infty$, every $L_p(\mu)$ has a complemented subspace isomorphic to a Hilbert space. In fact, the space spanned by Rademacher functions...
in $L_p$ is a complemented subspace of $L_p$ [16]. Also using Theorem 1.17, Theorem 1.21 and the principle of local reflexivity, it follows that Theorem 1.21 is true if we replace $L_p(\mu)$ by any $L_p$ space.

As a simple corollary to Theorem 1.17, Theorem 1.21 and the principle of local reflexivity, it was obtained that

**Theorem 1.22 [14]**

$X$ is a $L_p$ space if and only if $X^*$ is a $L_p$ space, $1 \leq p \leq \infty$, 

$$\frac{1}{p} + \frac{1}{q} = 1.$$

The question of "nice" finite dimensional superspaces of finite dimensional subspaces in a $L_p$ space was also settled in [14] viz.

**Theorem 1.23 [14]**

Let $X$ be a $L_p$ space, $1 \leq p \leq \infty$. Then there exists a constant $\lambda$ such that for any finite dimensional subspace $B$ of $X$, there exists a finite dimensional subspace $W \ni B \subseteq W \subseteq X$, $d(W, L_p^n) \leq \lambda$, $n = \dim(W)$ and there exists a projection $P: X \to W$, $\|P\| \leq \lambda$.

One must mention the following:

**Theorem 1.24 [7]**

Every separable $L_p$ space has a basis, $1 \leq p \leq \infty$.

We close this chapter with a new proof of the following theorem of Lindenstrauss and Pełczynski [13].

**Theorem 1.25**

Let $(x_i)$ be a bounded unconditional basis of an $L_\infty$ space $X$. 
Then \((x^i)\) is equivalent to the usual basis of \(c_0\).

We need the following lemmas:

**Lemma 1.26** [12]

Let \((y^i)\) be a bounded unconditional basis of a \(\ell_1\) space. Then \((y^i)\) is equivalent to the usual basis of \(\ell_1\).

In [13], the authors proved this lemma using some strong results about \(p\)-absolutely summing operators.

Before stating the next lemma, we fix a notation and give a definition. If \((x^i)\) is a basic sequence and \((f^i)\) are the corresponding biorthogonal functionals, we shall write \((x^i; f^i)\).

Also a basic sequence \((x^i)\) is called **bounded** if there exists positive numbers \(m\) and \(M\) such that

\[
m \leq \|x^i\| \leq M \quad i = 1, 2, \ldots
\]

**Lemma 1.27**

Let \((x^i; f^i)\) and \((y^i; g^i)\) be basic sequences. If \((x^i)\) is equivalent to \((y^i)\), then \((f^i)\) is equivalent to \((g^i)\).

**PROOF:**

The lemma is known. We include its proof.

First of all, we say \((x^i)\) is equivalent to \((y^i)\) if \(T: [x^i] \to [y^i] \ni T(x^i) = y^i\) is an onto isomorphism.

Since \((x^i)\) is equivalent to \((y^i)\), there exists \(T: [x^i] \to [y^i] \ni T(x^i) = y^i\) and \(T\) is an onto isomorphism. Then \(T^*: [y^i]^* \to [x^i]^*\) is such that

\[
(T^*g^i)(x^j) = g^i(Tx^j)
\]

\[
= g^i(y^j) = \delta_{ij}, \quad i, j = 1, 2, \ldots
\]
Also \( f_j(x_j) = \delta_{ij} \)  \hspace{1cm} (2)

Since \((x_j)\) is a basis for \([x_j]\), from (1) and (2), we conclude that \( T^*(g_i) = f_i \). Since \( T \) is an isomorphism,

\[
T^* : [g_i] \to [f_i]
\]

is an onto isomorphism. Hence the lemma.

\textbf{Lemma 1.28 (James)}

Let \((x_i)\) be an unconditional basis for a Banach space \(X\). If \((x_i)\) is not shrinking, then \(X\) has a complemented subspace isomorphic to \(\ell_1\).

\textbf{Proof of Theorem 1.25}

Assert that \((x_i)\) is shrinking. If \((x_i)\) is not shrinking, then by Lemma 1.28, \(X\) has a complemented subspace \(Y\) isomorphic to \(\ell_1\). By Theorem 1.21, being complemented in \(X\), \(Y\) is a \(\ell_\infty\) space. Thus we get a contradiction implying that \((x_i)\) is shrinking. Therefore, \((f_i)\), the functionals biorthogonal to \((x_i)\), constitute a bounded unconditional basis for \(X^*\) \([19]\) and \(X^*\) is a \(\ell_1\) space \([\text{Theorem 1.22}]\). Therefore, by Theorem 1.26, \((f_i)\) is equivalent to the usual basis \((h_i)\) of \(\ell_1\). Let \((g_i)\), \((k_i)\) denote the functionals biorthogonal to \((f_i)\) and \((h_i)\) respectively. Therefore, by Lemma 1.27, \((g_i)\) is equivalent to \((k_i)\). But \((g_i)\) is equivalent to \((x_i)\) and \((k_i)\) is equivalent to \((e_i)\), the usual basis for \(c_0\). Therefore, \((x_i)\) is equivalent to \((e_i)\). \hspace{1cm} Q. E. D.
CHAPTER II

SOME RESULTS ABOUT $L_p$ SPACES

In this chapter, some results about $L_p$ spaces are proved.

The main result of this chapter is that if $X$ is not a $L_p$ space, then there exists a subspace $Y$ of $X$ which is not a $L_p$ space and which has a Schauder decomposition into finite dimensional subspaces.

Proposition 2.1

Let $X$ be a Banach space such that $X = \bigcup_{n} B_n$ where $(B_n)$ is an increasing sequence of finite dimensional subspaces of $X$ and there exists a constant $\lambda \geq 1$ such that

$$d(B_n, L_p^m) \leq \lambda, \quad m = \dim(B_n),$$

then $X$ is a $L_p$ space. Conversely if $X$ is a separable $L_p, \lambda$ space, then $X = \bigcup_{n} B_n$ for some $B_n$'s as above.

PROOF:

Let $B$ be a finite dimensional subspace of $X$ and let $B = [x_i]_{i=1}^n$ where $x_1, x_2, \ldots, x_n$ are linearly independent. Let $\epsilon > 0$ be arbitrary. There exists an integer $N$ and $(y_i)_{i=1}^n \subseteq B_N$ such that

$$\|x_i - y_i\| < \epsilon, \quad 1 \leq i \leq n.$$  (1)
Also there exists $\delta > 0$ such that
\[ \left\| \sum_{i=1}^{n} a_i x_i \right\| \geq \delta \sum_{i=1}^{n} |a_i| \]
(2)
for all finite sequences $(a_i)_{i=1}^{n}$ of scalars. We split the proof of the theorem into several steps.

**Step I**

Assert that $(y_i)_{i=1}^{n}$ is a linearly independent set for $\varepsilon$ small enough.

Consider
\[ \left\| \sum_{i=1}^{n} a_i x_i - \sum_{i=1}^{n} a_i y_i \right\| \]
\[ \leq \sum_{i=1}^{n} |a_i| \left\| x_i - y_i \right\| \]
\[ \leq \varepsilon \sum_{i=1}^{n} |a_i| \quad \text{by (1)} \]
\[ \leq \frac{\varepsilon}{\delta} \left\| \sum_{i=1}^{n} a_i x_i \right\| \quad \text{by (2)} \]
(3')

Thus
\[ \left\| \sum_{i=1}^{n} a_i y_i \right\| \geq (1 - \frac{\varepsilon}{\delta}) \left\| \sum_{i=1}^{n} a_i x_i \right\| \]
(3)

Since $\varepsilon$ is arbitrary, let $\varepsilon < \delta$.

Then
\[ \sum_{i=1}^{n} a_i y_i = 0 \Rightarrow \sum_{i=1}^{n} a_i x_i = 0 \quad \text{by (3)} \]
\[ \Rightarrow a_i = 0, \ 1 \leq i \leq n \text{ as } x_1, \ldots, x_n \]
are linearly independent $\Rightarrow y_1, \ldots, y_n$ are linearly independent.
Step II

Let $B_N = \{y_{1,1}^{n}\} \oplus E$ for some $E$. Assert that $E \cap \{x_{1,1}^{n}\} = \{0\}$.

Suppose there exists $x = \sum \alpha_i x_i$, $\|x\| = 1$ and $x \in E$. Let $P: B_N \to \{y_{1,1}^{n}\}$ be the projection such that $P(E) = \{0\}$. By hypothesis,

$$\sum \alpha_i x_i - \sum \alpha_i y_i \in B_N.$$ Therefore,

$$\|P(x) - (\sum \alpha_i x_i - \sum \alpha_i y_i)\| = \|\sum \alpha_i x_i\| = 1$$ (4)

Also

$$\|P(x) - (\sum \alpha_i x_i - \sum \alpha_i y_i)\| \leq \|P - \sum \alpha_i x_i - \sum \alpha_i y_i\| \leq \|P - \sum \alpha_i x_i\| \frac{\epsilon}{\delta} \text{ by (3)},$$ (5)

Thus from (4) and (5), we get $1 \leq \|P - \sum \alpha_i x_i\| \frac{\epsilon}{\delta}$ and this is false by choosing $\epsilon$ small enough. Thus $E \cap \{x_{1,1}^{n}\} = \{0\}$ and so $W = E \oplus \{x_{1,1}^{n}\}$ is well defined.

Step III

Assert that $d(W, B_N) \leq 1 + \frac{\eta}{\lambda}$ for every $\eta > 0$. Indeed define $T: W \to B_N$ as follows:
\[ T(\sum \alpha_i x_i + e) = \sum \alpha_i y_i + e \]

where
\[ \sum \alpha_i x_i \in [x_i]^n \quad \text{and} \quad e \in E. \]

Then
\[
\|T(\sum \alpha_i x_i + e) - (\sum \alpha_i x_i + e)\|
\]
\[
= \|\sum \alpha_i (x_i - y_i)\|
\]
\[
\leq \sum \alpha_i \|x_i - y_i\|
\]
\[
\leq \frac{\varepsilon}{\delta} \|\sum \alpha_i x_i\| \quad \text{by (1) and (2)}.\]
\[
\leq \frac{\varepsilon}{\delta} \|Q(\sum \alpha_i x_i + e)\|
\]

where \( Q: \mathbb{W} \to [x_i]_{i=1}^n \) is a projection with \( Q(E) = \{0\}. \)

\[
\leq \frac{\varepsilon}{\delta} \|Q\| \|\sum \alpha_i x_i + e\|
\]

Thus
\[
d(\mathbb{W}, B_N) \leq \frac{1 + \frac{\varepsilon}{\delta} \|Q\|}{1 - \frac{\varepsilon}{\delta} \|Q\|}.\]

Finally choose \( \varepsilon \) such that, in addition to satisfying the conditions of Step I and Step II,
\[
\frac{1 + \frac{\varepsilon}{\delta} \|a\|}{1 - \frac{\varepsilon}{\delta} \|a\|} \leq 1 + \frac{\eta}{\lambda}.
\]

Then \(d(W, \ell_p^{(m_n)}) \leq d(W, B_n)\ell(B_n, \ell_p^{(m_n)})\)

\[\leq (1 + \frac{\eta}{\lambda})\lambda = \lambda + \eta.\]

Thus \(X\) is a \(\ell_{p, \lambda + \eta}\) space for every \(\eta > 0\).

Conversely let \((x_i)\) be a dense set in \(X\). Since \(X\) is \(\ell_{p, \lambda'}\), there exists a finite dimensional subspace \(B_1\) such that \(x_1 \in B_1 \subset X\) and \(d(B_1, \ell_p^{(m_1)}) \leq \lambda\). Again there exists a finite dimensional subspace \(B_2\) such that \([B_1, x_2] \subset B_2 \subset X\) and \(d(B_2, \ell_p^{(m_2)}) \leq \lambda\). In general there exists a finite dimensional subspace \(B_n\) such that \([B_{n-1}, x_n] \subset B_n \subset X\) and \(d(B_n, \ell_p^{(m_n)}) \leq \lambda\). Thus from the above construction, it is clear that there exists a sequence \((B_n)\) of finite dimensional subspaces such that \(X = \bigcup_{n=1}^{\infty} B_n\), \(B_n \subset B_{n+1}\), \(n = 1, 2, \ldots\)

and \(d(B_n, \ell_p^{(m_n)}) \leq \lambda.\) Q.E.D.

We obtain the following corollaries:

**Corollary 2.2**

Let \((x_i)\) be a basis for a Banach space \(X\). Suppose for every \(n, [x_i]_{i=1}^n\) is contained in a finite dimensional subspace \(B_n\) of \(X\) such that \(d(B_n, \ell_p^{(m_n)}) \leq \lambda\) where \(\lambda\) is a constant, then \(X\) is a \(\ell_{p, \lambda}\) space.
PROOF:

Let $B$ be any finite dimensional subspace of $X$, $(e_i)_{i=1}^k$ be a basis of $B$ and $\epsilon > 0$. Then there exists an integer $N$ such that

$$\| e_i - \sum_{j=1}^N a_{ij} x_j \| < \epsilon, \ 1 \leq i \leq k$$

for some $(a_{ij})_{i=1}^k$. Now the proof is completed exactly as in Proposition 2.1.

The proof of the following result was suggested by and is an expansion of a remark in [14, page 326].

**Corollary 2.3**

$L^p(\mu)$ is a $\mathcal{L}_{p,1+\epsilon}$ space for every $\epsilon > 0$, $1 \leq p < \infty$.

**PROOF:**

Let $x_1, x_2, \ldots, x_n$ be a basis of an arbitrary finite dimensional subspace of $L_p(\mu)$. Since simple functions are dense in $L_p(\mu)$, for $\eta > 0$, there exist simple functions $y_1, y_2, \ldots, y_n$ such that

$$\| x_i - y_i \| < \eta, \ i = 1, 2, \ldots, n.$$ 

As in the proof of Proposition 2.1, we can show that $(y_i)_{i=1}^n$ are linearly independent. Let

$$y_i = \sum_{j=1}^m a_{ij} \chi_{A_{ij}}$$

where $(A_{ij})_{j=1}^m, 1 \leq i \leq n$ are measurable sets and $\chi_A$ denotes the characteristic function of a set $A$. Let $\Lambda$ denote the class of all
measurable sets obtained by taking all possible intersections
of the sets \(((A_{ij})_{j=1}^m)_{i=1}^n\). Let \(D\) denote the subspace of \(L^p(\mu)\)
spanned by the characteristic functions of the sets in \(\Lambda\). Then
\(D\) is isometric to \(\ell_p^{(r)}\) where \(r = \dim(D)\).

Write \(D = [y_i]_{i=1}^n \oplus E\) and
\[ W = [x_i]_{i=1}^n \oplus E. \]

As in the proof of the proposition 2.1, we can show that \(W\) is well
defined and \(d(W, \ell_p^{(r)}) < 1 + \varepsilon\) for every \(\varepsilon > 0\) by choosing \(\eta\)
suitably. Q. E. D.

Let \(1 \leq p < \infty\) and \((x_i)\) be a sequence in \(\ell_p\). Let \(x_i = \sum_{j=1}^\infty a_{ij} e_j\),
where \((e_j)\) denotes the usual basis of \(\ell_p\). Define \(N_i = \{j: a_{ij} \neq 0\}\).

\(N_i\) is called the support of \(x_i\). Pelczynski [16] proved that if
\(N_i \cap N_j = \emptyset, i \neq j, i, j = 1, 2, 3, \ldots\), then \([x_i]\) is complemented in
\(\ell_p\). Using his result, we prove the following:

**Theorem 2.4**

Let \(1 \leq p < \infty\) and \((x_i)\) be a sequence in \(\ell_p\) such that
\(K = \bigcup_{i,j} (N_i \cap N_j)\) is a finite set. Then \([x_i]\) is complemented in \(\ell_p\).

**Proof:**

Let \(y_n = \sum_{j \in K} a_{nj} e_j\) where \(a_{nj}\) is zero if \(j\) does not appear
in the support of \(x_n\) but belongs to \(K\). Let \(z_n = x_n - y_n = \sum_{j \notin K} a_{nj} e_j\)
$= \sum \beta_{nj} e_j$, say. Let $N' = \{j: \beta_{nj} \neq 0\}$. It is clear from the definition of $N_i$ that $N_i' \cap N_j' = \emptyset$ if $i \neq j$. Thus by [16], $[z_n]$ is complemented in $\ell_p$. Let $[y_n] = B'$. $B'$ is a finite dimensional subspace as $B' \subset \{e_i\}_{i=1}^k = B$, say.

Assert that $[x_n] + B = [z_n] \oplus B$. \hspace{1cm} (1)

Clearly $[z_n] \oplus B$ is well defined. Since $z_n = x_n - y_n \in [x_n] + B$, we conclude that $[z_n] \oplus B \subset [x_n] + B$. \hspace{1cm} (2)

Also $x_n \in [z_n] \oplus B$ for all $n$, we conclude that $[x_n] + B \subset [z_n] + B$ (3)

From (2) and (3), we obtain (1). Since $[z_n]$ is complemented in $\ell_p$, $[z_n] \oplus B$ is complemented in $\ell_p$. Since $[x_n]$ has finite codimension in $[x_n] + B$, we conclude that $[x_n]$ is complemented in $\ell_p$. Q.E.D.

Following [7], we give the following definition:

**Definition 2.5**

A Banach space $X$ has the **finite dimensional decomposition** property (written as f.d.d.) if it has a Schauder decomposition into finite dimensional subspaces. Now we prove the main theorem of this chapter.
Theorem 2.6

Let $1 < p < \infty$. If a Banach space $X$ is non-$L^p$ space, then it has a subspace $Y$ which is non-$L^p$ and which has f.d.d.

The following lemmas are needed frequently below and in Chapter III. The first and the third are known, [4], [21] and the second is generally known but is not anywhere explicitly stated. We include the proofs here. The fourth lemma is of independent interest.

Lemma 2.7 [4]

Let $B$ be a finite dimensional subspace of a Banach space $Z$. Then, for $\epsilon > 0$, there exists a subspace $W$ of $Z$ with the following properties:

(1) $W \cap B = \{0\}$ and the natural projection of $W \oplus B \to B$ has norm $\leq 1 + \epsilon$.

(2) $W$ has finite codimension in $Z$.

Proof of Lemma 2.7

Since $B$ is finite dimensional, $U(B^*)$ is compact. Let $\delta > 0$ be arbitrary. Let $(g_i)_{i=1}^r$ be a $\delta$-net on $U(B^*)$ and let $f_{i}$ denote the Hahn-Banach extension of $g_i$ to $Z$, $1 \leq i \leq r$. For each $b \in B$, define $|b| = \sup_{1 \leq i \leq r} |f_i(b)|$. Assert $|\cdot|$ defines a new norm on $B$ equivalent to the given norm $\|\cdot\|$. Clearly $|b| \leq \|b\|$ (1'). Since $B$ is finite dimensional, there exists $g_0 \in U(B^*)$ such that
\( g_0(b) = \|b\|, \|g_0\|_B = 1. \)

There exists some \( g_i, 1 \leq i \leq r, \) such that \( \|g_0 - g_i\|_B < \delta. \)

Therefore,
\[
\|b\| = g_0(b) \leq \|g_0(b) - g_i(b)\| + |g_i(b)| \\
\leq \|g_0 - g_i\|_B \|b\| + |b| \\
\leq \delta \|b\| + |b| \\
\text{i.e.} \ (1 - \delta)\|b\| \leq |b| \quad (2')
\]

From (1') and (2'), we get
\[
(1 - \delta)\|b\| \leq |b| \leq \|b\| \quad (3')
\]
which shows that \( \cdot \) is a norm on \( B \) equivalent to the given norm.

Let \( W = \bigcap_{i=1}^{r} f_i^{-1}(0). \) Clearly \( W \) has finite codimension in \( Z \) so that (2) is satisfied. Suppose there exists \( b \in B \cap W, b \neq 0. \)

\( f_i(b) = 0, 1 \leq i \leq r = |b| = 0 = b = 0 \) by (3') which is a contradiction that \( b \neq 0. \) Hence \( W \cap B = \{0\}. \) Also by (3'), we have
\[
\|b\| \leq \frac{|b|}{1 - \delta} = \frac{|f_i(b)|}{1 - \delta} \quad \text{for some} \ i \\
= \frac{|f_i(b + w)|}{1 - \delta}, \ w \in W \\
\leq \frac{\|f_i\|}{1 - \delta} \|b + w\| \\
\leq \frac{1}{1 - \delta} \|b + w\|
\]
\leq (1 + \varepsilon)\|b + w\| \\
\text{if} \\
\delta \leq \frac{\varepsilon}{1 + \varepsilon}.

Thus the natural projection of $W \oplus B \to B$ has norm $\leq 1 + \varepsilon$. Q.E.D.

**Lemma 2.8**

Let $W$ be a subspace of a Banach space $Z$ such that $W$ has finite codimension in $Z$. If $W$ is a $\ell_p$ space, so is $Z$.

**PROOF:**

Suppose $W$ has codimension $n$ in $Z$. Therefore there exists linearly independent vectors $x_1, \ldots, x_n$ in $Z$ such that $Z = W \oplus [x_i]_{i=1}^n$.

Since any two finite dimensional subspaces of the same dimension are isomorphic, $\exists \lambda > 1 \ni d([x_i]_{i=1}^n, \ell_p^{(n)}) \leq \lambda$. Direct sum of two $\ell_p$ spaces being a $\ell_p$ space [13], we conclude that $Z = W \oplus [x_i]_{i=1}^n$ is a $\ell_p$ space.

**Lemma 2.9** [21]

Let $E_1, E_2$ be $n$ dimensional subspaces of an $(n+1)$ dimensional Banach space $E$, then $d(E_1, E_2) \leq 9$.

**PROOF:**

By hypothesis, $E_1 \cap E_2$ is a subspace of $(n-1)$ dimensions. Therefore, there exists a projection $P_1: E_1 \to E_1 \cap E_2$ such that $\|I - P_1\| = 1$, $i = 1, 2$. We can write $E_1 = (E_1 \cap E_2) \oplus [x_i]$ where
\( P_i(x_i) = 0, \|x_1\| = 1, i = 1,2. \) Define \( T: E_1 \rightarrow E_2 \) as follows:

\[ T(e + ax_1) = e + ax_2, \quad e \in E_1 \cap E_2. \]

Then

\[
\|T(e + ax_1)\| \leq \|e\| + |a|\|x_2\|
\]

\[
= \|e\| + \|ax_1\|
\]

\[
= \|P_1(e + ax_1)\| + \|(I - P_1)(e + ax_1)\|
\]

\[
\leq (2 + 1)\|e + ax_1\|
\]

i.e. \( \|T\| \leq 3. \) Similarly \( \|T^{-1}\| \leq 3. \) Hence the lemma.

The following lemma says that if \( Y \) has finite codimension in \( Z \), and finite dimensional subspaces of \( Y \) have 'good' finite dimensional superspaces in \( Z \), then finite dimensional subspaces in \( Z \) have 'good' finite dimensional superspaces in \( Z \). The technique of the proof is used in Chapter III also.

**Lemma 2.10**

Let \( Y \) be a finite codimensional subspace of a Banach space \( Z \). Suppose there exists a constant \( \lambda \geq 1 \) such that for every finite dimensional subspace \( B \subset Y \), there exists a finite dimensional subspace \( W \) such that \( B \subset W \subset Z \) and \( d(W, \mathcal{L}^p) \leq \lambda \), then \( Z \) is a \( \mathcal{L}^p \) space.

**PROOF:**

We shall give the proof by induction on \( n \), the codimension of \( Y \) in \( Z \).

Let \( Z = Y \oplus [x_0] \) for some \( x_0 \in Z \) and \( P: Z \rightarrow Y \) be the corresponding
natural projection with \( \|P\| \leq 2 \). Let \( E \subset Z \) be any finite dimensional subspace. If \( E \subset Y \), then it has already a "nice" superspace in \( Z \).

If \( E \notin Y \), there exists a finite dimensional subspace \( E' \subset Y \) such that \( E \subset E' \oplus [x_o] \). By hypothesis of the lemma for \( n = 1 \), there exists a finite dimensional subspace \( F \), \( E' \subset F \subset Z \) and \( d(F, \ell_p^{(m)}) \leq \lambda \). If \( x_o \in F \), then \( E \subset E' \oplus [x_o] \subset F \subset Z \) and we have a "nice" superspace of \( E \). Suppose \( x_o \notin F \). Since \( d(F, \ell_p^{(m)}) \leq \lambda \), there exists an operator \( T: F \to \ell_p^{(m)} \) such that \( \|T\| \leq 1 \), \( \|T^{-1}\| \leq \lambda \). Define \( S: F \oplus [x_o] \to \ell_p^{(m+1)} \) as follows:

\[
S(f + \alpha x_o) = T(f) + \alpha e_{m+1}
\]

where \((e_i)_{i=1}^{m+1}\) denotes the usual basis of \( \ell_p^{(m+1)} \). Then

\[
\|S(f + \alpha x_o)\| \leq \|T\|\|f\| + \|\alpha e_{m+1}\|
\]

\[\leq \|f\| + \|\alpha x_o\|\]

\[= \|(T - R)(f + \alpha x_o)\| + \|R(f + \alpha x_o)\|\]

where \( R: F \oplus [x_o] \to [x_o] \) is a projection with \( R(F) = \{0\} \) and \( \|R\| \leq 1 \)

\[\leq (2 + 1)\|f + \alpha x_o\| = \|S\| \leq 3.\]

Also

\[
\|S^{-1}(\sum_{i=1}^{m+1} \alpha_i e_i)\| = \|T^{-1}(\sum_{i=1}^{m} \alpha_i e_i + \alpha_{m+1}x_o)\|
\]
Thus the theorem is proved for $n = 1$. Suppose the theorem is known for $n-1$. Finally let $Z = Y \ominus \bigoplus_{i=1}^{n-1} \mathbf{x}_{i}$. By hypothesis of the lemma, for $n$, there exists a finite dimensional subspace $W$ such that $E \subseteq W \subseteq Z$ and $d(W, \mathfrak{l}^{W}) \leq \lambda$.

If $W \subseteq Y \ominus \bigoplus_{i=1}^{n-1} \mathbf{x}_{i}$, we proceed to the next step, otherwise let $W' = [W, \mathbf{x}_{n}]$. If $W = W'$, we proceed to the next step. If $W \neq W'$, then we can show, as in the first part of the proof, that

$$d(W', \mathfrak{l}^{W'}) \leq 3(1 + \lambda).$$

Let $S: W' \rightarrow \mathfrak{l}^{W'}$ be such that $\|S\| \leq 1$, $\|S^{-1}\| \leq 3(1 + \lambda)$.

Write $W' = G \oplus \bigoplus_{i=1}^{n-1} \mathbf{x}_{i}$ for some $G \subseteq Y \oplus \bigoplus_{i=1}^{n-1} \mathbf{x}_{i}$. Then $A = S/G: G \rightarrow \mathfrak{L}^{W'}$ is an isomorphism of $G$ into $\mathfrak{L}^{W'}$ and its range $L$ is an $m$ dimensional subspace of $\mathfrak{L}^{W'}$. Thus $d(L, G) \leq 3(1 + \lambda)$. By Lemma 2.9, $d(L, \mathfrak{L}^{L}) \leq 9$. Therefore, $d(G, \mathfrak{L}^{G}) \leq 27(1 + \lambda)$. Thus $E \subseteq G \subseteq Y \ominus \bigoplus_{i=1}^{n-1} \mathbf{x}_{i}$.

The conditions of the induction hypothesis for $n-1$ are satisfied and thus $Y \ominus \bigoplus_{i=1}^{n-1} \mathbf{x}_{i}$ is a $\mathfrak{L}$ space. The fact that every one dimensional
space is a $\mathbb{L}_p$ space and the direct sum of two $\mathbb{L}_p$ spaces is a $\mathbb{L}_p$ space implies that $Z = (Y \oplus [x_i]_{i=1}^{n-1}) \oplus [x_n]$ is a $\mathbb{L}_p$ space. This completes the proof of the lemma.

Proof of Theorem 2.6

Let $(\lambda_n)$ be any increasing sequence of reals tending to infinity. Since $X$ is not a $\mathbb{L}_p$ space, given $\lambda_1$, there exists a finite dimensional subspace $V_1$ of $X$ such that for any finite dimensional subspace $B_1$, $W_1 \subset B_1 \subset X$, $d(B_1, \mathbb{L}_p^{(m_1)}) > \lambda_1$. Choose one such $B_1$. By Lemma 2.7, given $\epsilon > 0$, there exists a subspace $Y_1$ such that

1. $Y_1$ has finite codimension in $X$.
2. $Y_1 \cap B_1 = \{0\}$.
3. the natural projection of $Y_1 \oplus B_1 \to B_1$ has norm $\leq 1 + \epsilon$.

By Lemma 2.8, $Y_1$ is not a $\mathbb{L}_p$ space. Since $Y_1 \oplus B_1$ has finite codimension in $X$ and $X$ is not a $\mathbb{L}_p$ space, $Y_1 \oplus B_1$ is also not $\mathbb{L}_p$. Therefore, by Lemma 2.10, given $\lambda_2$, there exists a finite dimensional subspace $W_2 \subset Y_1$, such that for any finite dimensional subspace $B_2$, $W_2 \subset B_2 \subset Y_1 \oplus B_1$, $d(B_2, \mathbb{L}_p^{(m_2)}) > \lambda_2$. Choose one such $B_2 \subset Y_1$. Suppose $B_1, B_2, \ldots, B_n$ and $Y_1, Y_2, \ldots, Y_{n-1}$ have been chosen such that

1. $\dim(B_i) < \infty, 1 \leq i \leq n$
2. $Y_i \subset Y_{i-1}, Y_0 = X, 1 \leq i \leq n-1$
3. $B_{i+1} \subset Y_i, 1 \leq i \leq n-1$. 
(iv) The natural projection of \((B_1 \otimes B_2 \otimes \ldots \otimes B_i) \otimes Y_i \rightarrow B_1 \otimes \ldots \otimes B_i\) has norm \(\leq 1 + \varepsilon, 1 \leq i \leq n-1\).

(v) \(d(B_i, B_p^{(m_i)}) > \lambda_i, 1 \leq i \leq n\).

(vi) Any finite dimensional superspace of \(B_i\) in \(B \otimes B_2 \otimes \ldots \otimes B_{i-1} \otimes Y_{i-1}\) also satisfies (v), \(1 \leq i \leq n\).

By Lemma 2.7, there exists a subspace \(Y_n\) such that

(a) \(Y_n \subseteq Y_{n-1}\) and has finite codimension in \(Y_{n-1}\).

(b) \(Y_n \cap [B_1, B_2, \ldots, B_n] = \{0\}\).

(c) The natural projection of \(Y_n \otimes (B_1 \otimes \ldots \otimes B_n) \rightarrow B_1 \otimes \ldots \otimes B_n\) has norm \(\leq 1 + \varepsilon\).

Since \(Y_{n-1}\) is not a \(\ell_p\) space, by Lemma 2.8, \(Y_n\) is not a \(\ell_p\) space. Therefore, given \(\lambda_{n+1}\), by Lemma 2.10, there exists a finite dimensional subspace \(W_{n+1}\) of \(Y_n\) such that for any finite dimensional subspace \(B_{n+1}, W_{n+1} \subseteq B_{n+1} \subseteq Y_n \otimes B_1 \otimes B_2 \otimes \ldots \otimes B_n\), \(d(B_{n+1}, B_p^{(m_{n+1})}) > \lambda_{n+1}\).

We choose one such \(B_{n+1}\) contained in \(Y_n\). This, by induction, completes the choice of the sequence \((B_n)\) and \((Y_n)\). Let \(Y = \Sigma \otimes B_n\).

By standard arguments, we can show that \(Y\) is well defined and is a closed subspace of \(X\). Also the way we have constructed \(B_n's\), it is clear that if \(D_n\) is a finite dimensional subspace such that
$B_n \subset D_n \subset Y$, then $d(D_n, \ell_p^n) > \lambda_n$ where $s_n = \dim(D_n)$. Since $\lambda_n \to \infty$, we conclude that $Y$ can't be a $\ell_p$ space. Q.E.D.

**Remark**

From the proof of the above theorem, it follows that if every separable subspace $Y$ of a Banach space $X$ can be contained in a separable subspace $Z$ which is a $\ell_p$ space, then $X$ is a $\ell_p$ space. This is converse to [Proposition 7.2 [13]].

**Remark**

It is not known that if a Banach space has f.d.d., then it has a basis [3]. It is worth noting that in the proof of Theorem 2.6, we had a lot of freedom in the choice of $B_n$'s. In fact, any finite dimensional superspace $B_n$ of $W_n$ would have been good enough for our purposes. Thus if the space $X$ (as in Theorem 2.6) has the property that every finite dimensional subspace $E$ of $X$ can be contained in a finite dimensional subspace $B$ of $X$ such that $B$ has a basis with basis constant less than or equal to $\mu$ for some constant $\mu$, then we could construct a subspace $Y$ of $X$ such that $Y$ has a basis and $Y$ is non-$\ell_p$. This above mentioned property is of independent interest in general Banach spaces. We shall study this property and other related properties in detail in Chapter IV.

In the next theorem, we obtain a sufficient condition for a subspace of $\ell_p$ to be a $\ell_p$ space.
Theorem 2.11

Let $X$ be an infinite dimensional subspace of $\ell_p$, $1 \leq p < \infty$, or $c_0$. Suppose there exist real numbers $\lambda_1$, $\lambda_2$ such that for any finite dimensional subspace $B$ of $X$, there exists a finite dimensional subspace $E$ of $\ell_p$ (respectively $c_0$) with the following properties:

i) there exists a natural projection $P_E: E = B \oplus A \to B$ with norm $\leq \lambda_1$

ii) $d(E, \ell_p^{(n)}) \leq \lambda_2$, $\dim(E) = n$.

Then $X$ is a $\ell_p$ space.

PROOF:

By Lemma 2.7, there exists a subspace $Y \subset X$ such that $Y$ has finite codimension in $X$, $Y \cap B = \{0\}$ and the natural projection of $Y \oplus B \to B$ has norm $\leq 2$. Since every infinite dimensional subspace of $\ell_p$ has a subspace isometric to $\ell_p$, $1 \leq p < \infty$ [16], we see that there exists $Z \subset Y$ such that $d(Z, \ell_p) \leq 2$. Thus there exists $Z \subset X$ such that $Z \cap B = \{0\}$ and the natural projection $Q: Z \oplus B \to B$ has norm $\leq 2$ and $d(Z, \ell_p) \leq 2$. By hypothesis, there exists a finite dimensional subspace $E$ of $\ell_p \oplus E = B \oplus A$ and the natural projection $P: B \oplus A \to B$ has norm $\leq \lambda_1$, and $d(E, \ell_p^{(n)}) \leq \lambda_2$, $n = \dim(E)$. Let $A'$ be a finite dimensional subspace of $Z \oplus d(A',A)$ $\leq 2$. This is possible as $d(Z, \ell_p) \leq 2$. Let $R: A' \to A$ be the isomorphism such that $\|R\| \leq 1$, $\|R^{-1}\| \leq 2$. Let $W = B \oplus A'$. Clearly $W$ is well defined. Define $T: W \to E$ as follows:

$$T(b + a') = b + R(a'), \quad b \in B, \quad a' \in A'.$$

Then
\[ \|T(b + a')\| \leq \|b\| + \|R(a')\| \]
\[ \leq \|b\| + \|a'\| \]
\[ = \|Q(b + a')\| + \|(I - Q)(b + a')\| \]
\[ \leq \|(Q + (I - Q))\|b + a'\| \]
\[ \leq 5\|b + a'\| \]
\[ = \|x\| \leq 5 \]

Also
\[ \|T^{-1}(b + a)\| = \|b + R^{-1}(a)\| \]
\[ \leq \|b\| + \lambda\|a\| \]
\[ = \|P(b + a)\| + \|(I - P)(b + a)\| \]
\[ \leq (1 + 2\lambda_1)\|b + a\) \]
\[ = \|T^{-1}\| \leq (1 + 2 \lambda_1) \]

Hence \(d(W, \ell_p^{(n_1)}) \leq 5(1 + 2\lambda_1)\lambda_2 \)

The proof in case of \(c_0\) is the same.

Remark

The theorem would still be true if it was hypothesized that for any finite dimensional subspace \(B\) of \(X\), \(\exists\) a finite dimensional subspace \(B_1 \subset X\) \(\exists\ B \subset B_1 \subset X\) and \(B_1 \subset E \subset \ell_p\) satisfying (i), (ii) in the theorem.
Proposition 2.12

Let $X$ be a $L_p$ space, $1 < p < \infty$ and $Y$ be a Banach space. Then $X \subset Y$ is complemented in $Y$ if and only if there exists a constant $\lambda$ such that given any finite dimensional subspace $B$ of $X$, there exists a finite dimensional subspace $W \ni B \subset W \subset X$, $d(W, L_p^{(n)}) \leq \lambda$ and there exists a projection $P: Y \to W$, $\|P\| \leq \lambda$.

We need the following result.

Lemma 2.13 [10]

Let $Z$ be a reflexive Banach space and $Z \subset Y$, a Banach space. Then $Z$ is complemented in $Y$ if there exists a constant $\lambda$ such that given any finite dimensional subspace $B$ of $Z$, $\exists$ a finite dimensional subspace $W \ni B \subset W \subset Z$ and there exists a projection $P: Y \to W$, $\|P\| \leq \lambda$.

Proof of Proposition 2.12

Let $X$ be complemented in $Y$ and let $Q: Y \to X$ be a projection. By definition of $L_p$ and Theorem 1.23, there exists a constant $\mu$ such that given a finite dimensional subspace $B$ of $X$, there exists a finite dimensional subspace $W$, $B \subset W \subset X$, $d(W, L_p^{(n)}) \leq \mu$ and $P: X \to W$ is a projection, $\|P\| \leq \mu$. Then $PQ: Y \to W$ is a projection and $\|PQ\| \leq \mu\|Q\|$.

The converse is an immediate consequence of Lemma 2.13. Note that for this part the assumption that $X$ is $L_p$ is superfluous. Q.E.D.

Theorem 2.14

Let $X$ be a $L_p$ space with a bounded unconditional basis $(x_i)$ such that no subspace of $X$ is isomorphic to $L_2$, $1 \leq p \leq \infty$, $p \neq 2$. 


Then $X$ has a Schauder decomposition $\sum X_n$ where each $X_n$ is isomorphic to $\ell_p$.

We need the following lemma:

Lemma 2.15

Let $(x_i)$ be a bounded unconditional basic sequence in $L_p$ such that $[x_i]$ is complemented in $L_p$, $1 < p < \infty$. Then $(x_i)$ has a subsequence which is equivalent to the usual basis of $\ell_2$ or $\ell_p$.

Proof of Lemma 2.15

For $p > 2$, the lemma is known by a result of Kadec and Pelczynski [8]. Now consider $1 < p < 2$. Since $[x_i]$ is complemented in $L_p$, the functionals $(f_i)$ biorthogonal to $(x_i)$ can be, in a natural way, extended over $L_p$ in such a way that $[f_i]$ is isomorphic to a complemented subspace of $L_q$, $\frac{1}{p} + \frac{1}{q} = 1$. Since $q > 2$, by [8], $(f_i)$ has a subsequence, say $(f_{ij})$ equivalent to the usual basis of $\ell_2$ or $\ell_q$.

Let $(y_{ij})$ denote the functionals in $[f_{ij}]^*$ biorthogonal to $(f_{ij})$. Since $(f_i)$ is an unconditional basic sequence, by Lemma 2 [8], $(y_{ij})$ is equivalent to $(x_{ij})$ and hence $(x_{ij})$ is equivalent to the usual basis of $\ell_2$ or $\ell_p$ as $y_{ij}$ is equivalent to the usual basis of $\ell_2$ or $\ell_q$.

Hence the lemma.

Remark

In the above lemma, the condition that $[x_i]$ be complemented is essential. In fact, Kadec and Pelczynski [8] have pointed out
that for $1 \leq p \neq q < 2$, there is a subspace $X_q$ of $l_p$ which is isomorphic to $l_q$.

**Proof of Theorem 2.14**

**Case I**

Let $p = 1, \infty$. In this case, the hypothesis that no subspace of $X$ is isomorphic to $l_2$ is unnecessary. In fact, by Lemma 1.25 and Theorem 1.26, $(x_i)$ is equivalent to the usual basis of $l_1$ for $p = 1$ and $c_0$ for $p = \infty$.

**Case II**

First, without loss of generality, we can assume that $X \subset l_p$ and complemented in $l_p$ [Theorem 1.17]. By hypothesis and Lemma 2.15, $(x_i)$ has a subsequence $(x_{i_1}^{(1)})$ equivalent to the usual basis of $l_p$.

Let $x_1 = [x_{i_1}^{(1)}]$. Consider the sequence $\{((x_i) \backslash (x_{i_1}^{(1)}))\}$ where $\backslash$ denotes the set theoretic difference. Since $(x_i)$ is an unconditional basis and $[x_i]$ is complemented in $l_p$, $\{((x_i) \backslash (x_{i_1}^{(1)}))\}$ is also complemented in $l_p$. Therefore, by Lemma 2.15, $\{((x_i) \backslash (x_{i_1}^{(1)}))\}$ has a subsequence $(x_{i_2}^{(2)})$ equivalent to the usual basis of $l_p$. Let

$$y_{i_2}^{(2)} = x_{i_2}^{(2)}$$

for $j \geq 1$ and $y_{i_0}^{(2)} = x_1$ if $x_1 \notin [x_{i_2}^{(2)}]$. Then

$$x_2 = [x_{i_2}^{(2)}, x_1] = [y_{i_2}^{(2)}]_{j=0}$$

is also equivalent to the usual basis of $l_p$. Continuing inductively as above, we shall get a sequence $(X_n)$ of subspaces of $X$ such that
(1) $X_n$ is spanned by a subsequence of $(x_i)$.  

(2) $X_n$ is isomorphic to $\ell_p$.  

(3) $X_i \cap X_j = \{0\}$, $i \neq j$, and the natural projection of  

$$ \sum_{i=1}^{n} X_i \to \sum_{i=1}^{m} X_i, \ m \leq n \text{ has norm } \leq \lambda, \ 1 \leq i \leq n \text{ where} $ 

$\lambda$ denotes the unconditional basis constant of $(x_i)$.  

(4) $\bigcup_{n} X_n = X$.  

Clearly then $X = \sum X_n$ where each $X_n$ is isomorphic to $\ell_p$. Q.E.D.  

**Definition 2.16**  

Let $X$ be a Banach space and $A$ be a subspace of $X$. Then the 

projection constant of $A$ with respect to $X$ is the infimum of the norms of the projections from $X$ onto $A$ and is denoted by $\lambda_X(A)$.  

**Theorem 2.17**  

Let $1 < p < \infty$. Then the following conditions are equivalent:  

(1) There exists a sequence of subspaces and a sequence $(n_m)$ of integers such that  

$$ B_m \subseteq \ell_p^{(n_m)}, d(B_m, \ell_p^{(i_m)}) \leq \mu $$ 

for a fixed $\mu$ and $\lambda_p(B_m) \to \infty$ as $m \to \infty$.  

(2) $\exists$ an infinite dimensional subspace $X$ of $\ell_p$ such that $X \cong \ell_p$ but $X$ is uncomplemented in $\ell_p$.  


PROOF:

(1) \Rightarrow (2).

Let \( Z = (\Sigma \oplus \ell_p^m) \ell_p \). By [16], \( Z \) is isomorphic to \( \ell_p \). Let

\[ S \]

denote the isomorphism of \( Z \) onto \( \ell_p \). Clearly \( Y \subset Z \). Let

\[ X = S(Y) \subset \ell_p \]. Since \( Y \) is isomorphic to \( \ell_p \) and \( S \) is an isomorphism, we conclude that \( X \) is isomorphic to \( \ell_p \). Assert that \( X \) is uncomplemented in \( \ell_p \). For this, it is enough to show that \( Y \) is uncomplemented in \( Z \). Suppose \( Y \) is complemented in \( Z \) and \( P: Z \to Y \) is a projection.

Let \( P_m: Y \to B_m, m = 1, 2, \ldots \) be the natural projections and let

\[ Q_m = P/\ell_p^m, m = 1, 2, \ldots \]. Let \( R_m = P_m Q_m: \ell_p^m \to B_m, m = 1, 2, \ldots \).

It is a straightforward calculation that \( R_m \)'s are projections and are uniformly bounded. Thus

\[ \sup_m (\lambda_{\ell_p^m} B_m) < \infty \]

which is a contradiction that \( \lambda_{\ell_p^m} (B_m) \to \infty \) as \( m \to \infty \). This completes the proof that \( (1) \Rightarrow (2) \).

Next we prove \( (2) \Rightarrow (1) \).

Since \( X \) is isomorphic to \( \ell_p \), it is a \( \ell_p, \eta \) space for some \( \eta \).

Consider a finite dimensional subspace \( B_1' \subset X \) such that \( d(B_1', \ell_p^{(1)}) \leq \eta \). Since \( X \) is a \( \ell_p \) space, \( 1 < p < \infty \), there exists a finite dimensional subspace \( B_2' \subset B_1' \subset X, d(B_2', \ell_p^{(2)}) \leq \eta \) and \( \lambda_{\ell_p^2} (B_2') > 2 \) (otherwise, one can project onto every \( B_2', B_2' \supset B_1' \)).
with norm ≤ 2. But then Lemma 2.12 would imply that X is complemented in $\ell_p^n$. Inductively, we shall get a sequence $(B'_n)$ of finite dimensional subspaces of X such that

(a) $B'_n \subseteq B'_1 \quad \forall n$

(b) $d(B'_n, \ell_p^n) \leq \eta$

(c) $\lambda_p(B'_n) \to \infty$ as $n \to \infty$.

As a typical case, we show that we can get $B'_n$'s as promised in the theorem. Let $f_1, \ldots, f_n$ denote a basis for $B'$ and $\varepsilon > 0$. Then there exists an integer $m_n \ni \|f_r - \sum_{i=1}^{m_n} a_{ir} e_i\| < \varepsilon$, $1 \leq r \leq n$ where $(e_i)$ denotes the usual basis of $\ell_p$. Let $g_r = \sum_{i=1}^{m_n} a_{ir} e_i$. As in the proof of Proposition 2.1, we can show that $g_r$'s are linearly independent and, in fact,

$$d([f_r]^n, [g_r]^n) \leq 1 + \varepsilon$$

for a small $\varepsilon$.

Let $B_n = [g_r]^n$. Then

$$d(B_n, \ell_p^n) = d(B_n, B')d(B', \ell_p^n) \leq \eta(1 + \varepsilon)$$

$$= \mu, \text{ say.}$$
Also $B_n \subset \ell_p^n$ for $n$, since there is a norm 1 projection from
$\ell_p$ onto $\ell_p^n$ and $\lambda_p(B^n) \to \infty$, we conclude that $\lambda_p(B_p^n) \to \infty$.

Hence $(2) \Rightarrow (1)$. Q.E.D.
CHAPTER III

SOME FINITE DIMENSIONAL SUBSPACE PROPERTIES

In this chapter, a property (the finite dimensional subspace basis property) which arises very naturally in the study of $L_p$ spaces and in the Basis Theory is studied. Some other related properties are also studied.

Definition 3.1

A Banach space $X$ has the $\lambda$-finite dimensional subspace basis property (written as $\lambda$-f.d.s.b.) if there exists a constant $\lambda$ such that for any finite dimensional subspace $B$ of $X$, there exists a finite dimensional subspace $W$ such that $B \subset W \subset X$ and $\eta(W) \leq \lambda$ where $\eta(W)$ denotes the infimum of the set of basis constants taken over all bases of $W$. $X$ is said to have f.d.s.b. if it has $\lambda$-f.d.s.b. for some $\lambda$.

Definition 3.2

A Banach space $X$ is called a $\pi_\lambda$ space if there exists a set $(B_\alpha)$ of finite dimensional subspaces of $X$, directed by inclusion, such that $X = \bigcup_\alpha B_\alpha$ and there exists a projection $P_\alpha : X \to B_\alpha$, $\|P_\alpha\| \leq \lambda$ for all $\alpha$.

We have indirectly referred to the f.d.s.b. property in the
note at the end of Theorem 2.6. Since every Banach space with a basis has this property (Cor. 3.4) the property has interest from the Basis Theory point of view.

**Proposition 3.3**

Let $X$ be a Banach space such that $X = \bigcup_{n} B_{n}$ where $(B_{n})$ is an increasing sequence of finite dimensional subspaces of $X$. If there exists a constant $\lambda$ such that $\eta(B_{n}) \leq \lambda$ for every $n$, then $X$ has $(\lambda + \varepsilon)$-f.d.s.b. for every $\varepsilon > 0$. Conversely if $X$ is separable and has $\lambda$-f.d.s.b., then there exists an increasing sequence $(B_{n})$ of finite dimensional subspaces such that $X = \bigcup_{n} B_{n}$ and $\eta(B_{n}) \leq \lambda$.

**PROOF:**

Let $[x_{i}]_{i=1}^{n}$ be any finite dimensional subspace of $X$ and $\varepsilon > 0$. Then there exists an integer $N$ and $y_{1}, y_{2}, \ldots, y_{n} \in B_{N}$ such that $\|x_{i} - y_{i}\| < \delta$ where $\delta$ is arbitrary to be chosen suitably later. As in the proof of Proposition 2.1, we can show that $(y_{i})_{i=1}^{n}$ is a linearly independent set for a small $\delta$. If we write $B_{N} = [y_{i}]_{i=1}^{n} \oplus A$, then

$$W = [x_{i}]_{i=1}^{n} \oplus A$$

is well defined and, in fact, $d(B_{N}, W) \leq 1 + \frac{\varepsilon}{\lambda}$ by a suitably choice of $\delta$ (this can be achieved as in the proof of Proposition 2.1).

Since $\eta(B_{N}) \leq \lambda$, we see that $\eta(W) \leq \lambda(1 + \frac{\varepsilon}{\lambda}) = \lambda + \varepsilon$. This completes the first part of the proposition.
As for the second part, let \((z_n)\) be a dense set in \(X\). Since \(X\) has \(\lambda\)-f.d.s.b., there exists a finite dimensional subspace \(B_1\) such that \(z_1 \in B_1\) and \(\eta(B_1) \leq \lambda\). Similarly, there exists a finite dimensional subspace \(B_2\) such that \([B_1, z_2] \subseteq B_2\) and \(\eta(B_2) \leq \lambda\).

Proceeding inductively, we get a sequence \((B_n)\) of finite dimensional subspaces of \(X\) such that

(i) \(B_n \subseteq B_{n+1}\).

(ii) \(z_n \in B_n\).

(iii) \(\eta(B_n) \leq \lambda, \ n = 1, 2, 3, \ldots\)

Thus \(X = \bigcup_{n} B_n\). Q.E.D.

**Remark**

Let \(X = \bigcup_{n} B_n\) where \((B_n)\) is an increasing sequence of finite dimensional subspaces. Suppose there exists a projection \(P_n : X \to B_n\), \(\|P_n\| \leq \lambda, \ n = 1, 2, \ldots\) where \(\lambda\) is a constant. Then given \(B \subseteq X\), \(\dim(B) < \infty\), there exists a finite dimensional subspace \(W, B \subseteq W \subseteq X\) and a projection \(P : X \to W, \|P\| \leq \lambda + \epsilon\) for every \(\epsilon > 0\). This result, for example, is discussed in [7].

**Corollary 3.4**

Let \(X\) be a Banach space with a basis \((b_1)\). Then \(X\) has \((\lambda + \epsilon)\)-f.d.s.b. for every \(\epsilon > 0\) where \(\lambda\) is the basis constant for \((b_1)\).
PROOF:

Let \( B_n = \{ b_i \}_{i=1}^n \). Then \( X = \bigcup_{n} B_n \) and \( \eta(B_n) \leq \lambda \). Therefore, the corollary follows from Proposition 3.3.

It is well known that every Banach space contains an infinite dimensional subspace with a basis with basis constant less than or equal to \( 1 + \epsilon \) for every \( \epsilon > 0 \) [2]. The following proposition shows that this result can be strengthened if the space has f.d.s.b.

**Proposition 3.5**

Let \( X \) be a Banach space. Then the following conditions are equivalent:

(i) \( X \) has f.d.s.b.

(ii) there exists a constant \( \mu \) such that given any finite dimensional subspace \( B \subset X \), there exists an infinite dimensional subspace \( Y \) such that \( B \subset Y \subset X \), \( Y \) has a basis with basis constant less than or equal to \( \mu \).

PROOF:

(i) \(\Rightarrow\) (ii).

Let \( B \subset X \) be any finite dimensional subspace. Let \( X \) have \( \lambda \)-f.d.s.b. Therefore, \( X \) a finite dimensional subspace \( W \) such that \( B \subset W \subset X \) and \( \eta(W) \leq \lambda \). By Lemma 2.7, there exists an infinite dimensional subspace \( Z \subset X \) such that

(a) \( Z \cap W = \{0\} \).
(b) the natural projection $P: Z \oplus W \rightarrow W$ has norm less than or equal to $1 + \varepsilon$ for every $\varepsilon > 0$.

There exists an infinite dimensional subspace $Y'$ of $Z$ with a basis $(b_i)$ and basis constant of $(b_i)$ less than or equal to $1 + \varepsilon$ for every $\varepsilon > 0$. Let $Y = Y' \oplus W$. Then the natural basis for $Y$ has basis constant less than or equal to $(1 + \varepsilon)(\lambda + 2 + \varepsilon)$. Hence $(i) \Rightarrow (ii)$. That $(ii) \Rightarrow (i)$ is an immediate consequence of Corollary 3.4.

With the introduction of the f.d.s.b. property, quite a few natural questions arise e.g., if $X$ has f.d.s.b., does $X^*$ have f.d.s.b. and conversely? In this direction, we have some partial results.

**Proposition 3.6**

If $X^{**}$ has $\lambda$-f.d.s.b., then $X$ has $(\lambda + \varepsilon)$-f.d.s.b. for every $\varepsilon > 0$.

We need the following powerful result of Lindenstrauss and Rosenthal.

**Lemma 3.7** [14]

Let $X$ be a Banach space and $\varepsilon > 0$. Let $B \subset X^{**}$ be a finite dimensional subspace. Then there exists a 1-1 operator $T: B \rightarrow X$ such that $T$ is identify on $B \cap X$ and $\|T\| \|T^{-1}\| \leq 1 + \varepsilon$.

**Proof of Proposition 3.6**

Let $E \subset X$ be a finite dimensional subspace. Since $X^{**}$ has
there exists a finite dimensional subspace $B \ni E \subseteq B \subseteq X^{**}$ and $\eta(B) \leq \lambda$. By Lemma 3.7, there exists a 1-1 operator $T: B \to X$ such that $T$ is identity on $B \cap X$ and

$$\|T\|\|T^{-1}\| \leq 1 + \frac{\varepsilon}{\lambda}.$$  

Then $X \ni T(B) \supseteq E$ and

$$\eta(T(B)) \leq \lambda\|T\|\|T^{-1}\| \leq \lambda(1 + \frac{\varepsilon}{\lambda})$$

$$= \lambda + \varepsilon$$

Q.E.D.

**Proposition 3.8**

Let $X_i$ be a Banach space such that $X_i$ has $\lambda_i$-f.d.s.b., $i = 1,2$. Then $X_1 \oplus X_2$ has f.d.s.b.

**PROOF:**

Since f.d.s.b. is preserved under isomorphism, we can assume that

$$\|(x_1, x_2)\| = \|x_1\| + \|x_2\|, \quad x_i \in X_i, \quad i = 1,2.$$  

Let $B \subseteq X_1 \oplus X_2$ be any finite dimensional subspace. Choose finite dimensional subspaces $E_i \subseteq X_i, \quad i = 1,2$ such that $B \subseteq E_1 \oplus E_2$. Also choose $W_i \subseteq X_i, \quad \dim(W_i) < \infty$ such that $\eta(W_i) \leq \lambda_i, \quad i = 1,2$. Then $B \subseteq W_1 \oplus W_2 \subseteq X_1 \oplus X_2$ and $W_1 \oplus W_2$ has a basis with basis constant less than or equal to $\max(\lambda_1, \lambda_2)$. Q.E.D.
Proposition 3.9

Let a Banach space $X$ have $\lambda$-f.d.s.b. and $Y \subset X$ be a separable space. Then there exists a separable subspace $Z \subset Y \subset Z \subset X$ and $Z$ has f.d.s.b. .

PROOF:

Let $(y_n)_{n=1}^\infty$ be a dense set in $Y$. Since $X$ has $\lambda$-f.d.s.b., $\exists$ a subspace $B_1$ such that $y_1 \in B_1 \subset X$ and $\tau(B_1) \leq \lambda$. Inductively, we shall have a sequence $(B_n)$ of finite dimensional subspaces such that

(i) $B_n \subset B_{n+1} \subset X$, $i = 1,2,...$

(ii) $\tau(B_1) \leq \lambda$, $i = 1,2,...$

Then, clearly, $Z = \bigcup_{n} B_n$ is a separable subspace of $X$ containing $Y$. By Proposition 3.3, $Z$ has $(\lambda + \varepsilon)$-f.d.s.b. for every $\varepsilon > 0$. Q.E.D.

In the next proposition, we improve the above theorem if $X$ is reflexive.

Proposition 3.10

Let $X$ be a reflexive Banach space with f.d.s.b. property. Let $Y \subset X$ be a separable subspace. Then $\exists$ a separable space $Z \exists$

(1) $Y \subset Z \subset X$ and $Z$ has f.d.s.b.

(2) $\exists$ a norm 1 projection from $X$ onto $Z$.

We need the following result of Lindenstrauss.
Lemma 3.11 \[12\]

Let $X$ be a reflexive Banach space and $W \subset X$ be a separable subspace. Then $\exists$ a separable subspace $L \in$

(i) $W \subset L \subset X$.

(ii) $\exists$ a norm 1 projection from $X$ onto $L$.

Proof of Proposition 3.10

By Proposition 3.9, there exists a separable subspace $Y_1 \in$

(i) $Y \subset Y_1 \subset X$.

(ii) $Y_1$ has $(\lambda + \varepsilon)$-f.d.s.b. for every $\varepsilon > 0$.

By Lemma 3.11, there exists a separable subspace $Y_2 \in$

(iii) $Y_1 \subset Y_2 \subset X$.

(iv) There exists a norm 1 projection from $X$ onto $Y_2$.

Continuing like this inductively, we get a sequence $(Y_n)$ of separable subspaces of $X$ such that

(a) $Y_n \subset Y_{n+1}$, $n = 1, 2, \ldots$

(b) $Y_{2n-1}$ has $(\lambda + \varepsilon)$-f.d.s.b. for every $\varepsilon > 0$, $n = 1, 2, \ldots$

(c) $\exists$ a norm 1 projection $P_{2n}: X \to Y_{2n}$, $n = 1, 2, \ldots$
Let $Z = \bigcup_{n} Y_n$. Then $Y \subset Z \subset X$ and $Z$ is separable. Following the technique of Proposition 3.3, we can show that $Z$ has f.d.s.b and thus (1) is satisfied. Also it can be shown, as in [Lemma 2, [12]], that through the sequence of projections $(P_{2n})$, we get a projection $P$ from $X$ onto $Z$ with $\|P\| = 1$. This proves (2). Q.E.D.

**Theorem 3.12**

Let $X$ be a Banach space such that $X$ does not have f.d.s.b. Then there exists a separable subspace $Y \subset X$ such that

(i) $Y$ does not have f.d.s.b.

(ii) $Y$ has f.d.d.

We need the following lemmas:

**Lemma 3.13**

Let $W$ be any finite codimensional subspace of a Banach space $Z$. If $W$ has f.d.s.b., then $Z$ has f.d.s.b.

**PROOF:**

We can write $Z = W \oplus [x_1]_{i=1}^{n}$ for some linearly independent vectors, $x_1, x_2, \ldots, x_n$. If a real number $\lambda_1$ such that $\eta([x_1]_{i=1}^{n}) \leq \lambda_1$. Suppose $W$ has $\lambda_2$-f.d.s.b. Then by Proposition 3.8, $W \oplus [x_1]_{i=1}^{n} = Z$ has f.d.s.b.

The following lemma is similar to Lemma 2.10.
Lemma 3.14

Let $W$ be a finite codimensional subspace of a Banach space $Z$. Suppose there exists a constant $\lambda$ such that for any finite dimensional subspace $B \subseteq W$, there exists a finite dimensional subspace $C$ such that $B \subseteq C \subseteq Z$, $\eta(C) \leq \lambda$, then $Z$ has f.d.s.b.

PROOF:

The proof will be given by induction on $n$, the number of codimension of $W$ in $Z$.

Suppose $Z = W \oplus [x_1]$. Let $B$ be a finite dimensional subspace of $Z$. There exists a finite dimensional subspace $A \subseteq W \oplus A \oplus [x_1]$. By hypothesis of the lemma, $\exists$ a finite dimensional subspace $C \ni A \subseteq C \subseteq Z$ and $\eta(C) \leq \lambda$. If $x_1 \in C$, then $B \subseteq C \subseteq Z$ and we have already a 'nice' superspace of $B$. If $x_1 \notin C$, let $D = [C, x_1] = C \oplus [z_1]$ for some $z_1 \in Z$. Then the 'nice' basis of $C$ together with $\{z_1\}$ constitutes a basis of $D \ni \eta(D) \leq 1 + 2\lambda$. Thus the result is proved for $n = 1$.

Suppose the result is known for codimension less than or equal to $n-1$. For $n$, let $Z = W \oplus [x_1]^n$. Let $C$ be any finite dimensional subspace of $W$. By the hypothesis of the lemma for $n$, $\exists$ a finite dimensional subspace $D$ such that $C \subseteq D \subseteq Z$ and $\eta(D) \leq \lambda$. If $D \subseteq W \oplus [x_1]^{n-1}$, then we proceed to the next step, otherwise let $F = [D, x_n] = E \oplus [x_n]$ for some $E \subseteq W$. Then by Lemma 2.9, $d(E, D) \leq 9$. Since $\eta(D) \leq \lambda$, therefore, $\eta(E) \leq 9\lambda$. Clearly $C \subseteq E \subseteq W \oplus [x_1]^{n-1}$. Thus we obtain that given any finite dimensional subspace $C \subseteq W$, $\exists$ a finite dimensional subspace $E \ni C \subseteq E \subseteq W \oplus [x_1]^{n-1}$ and $\eta(E) \leq 9\lambda$. 


Thus the conditions for the validity for the induction hypothesis for $n-1$ are satisfied and thus $W \oplus [x_i]_{i=1}^{n-1}$ has f.d.s.b. Therefore, by Proposition 3.8,

$$Z = W \oplus [x_i]_{i=1}^{n} = (W \oplus [x_i]_{i=1}^{n-1}) \oplus [x_n]$$

has f.d.s.b. This completes the proof of Lemma 3.14.

Proof of Theorem 3.12

Let $(\lambda_n)$ be any increasing sequence of real numbers tending to infinity. Since $X$ does not have f.d.s.b., given $\lambda_1$, there exists a finite dimensional subspace $B_1 \subset X$ such that for every finite dimensional subspace $W_1 \ni B_1 \subset W_1 \subset X$, $\eta(W_1) > \lambda_1$. Choose one such $W_1$. By Lemma 2.7, given $\varepsilon > 0$, there exists a finite codimensional subspace $Y_1$ of $X$ such that

(i) $Y_1 \cap W_1 = \{0\}$.

(ii) the natural projection of $Y_1 \oplus W_1 \rightarrow W_1$ has norm less than or equal to $1 + \varepsilon$.

By Lemma 3.13, $Y_1$ does not have f.d.s.b. Therefore, given $\lambda_2$, $\exists$ a finite dimensional subspace $B_2$ of $Y_1$ such that for any finite dimensional subspace $W_2$ such that $B_2 \subset W_2 \subset Y_1$, $\eta(W_2) > \lambda_2$. By Lemma 3.14, we can choose one such that $W_2 \subset Y_1$, $\exists \eta(W_2) > \lambda_2$. By induction, using the technique of the proof of Theorem 2.6, we shall get the sequences $(Y_n)$ and $(W_n)$ such that
(i) \( Y_n \subset Y_{n-1} \), \( Y_n \) has finite codimension in \( Y_{n-1} \), \( Y_0 = X \)

(ii) \( Y_n \) does not have have f.d.s.b.

(iii) \( W_n \subset Y_{n-1} \), \( \dim(W_n) < \infty \) and \( \gamma(W_n) > \lambda_n \).

(iv) The natural projection of \( Y_n \oplus (W_1 \oplus \ldots \oplus W_n) \rightarrow W_1 \oplus \ldots \oplus W_n \)

has norm less than or equal to \( 1 + \varepsilon, n = 1, 2, \ldots \)

(v) For any finite dimensional superspace \( K_n \) of \( W_n \) in

\( Y_{n-1} \oplus W_{n-1} \oplus \ldots \oplus W_1 \), \( \gamma(K_n) > \lambda_n \), \( n = 1, 2, \ldots \).

Then, as in the proof of Theorem 2.6, \( Y = \Sigma \oplus W_n \) is a well defined closed subspace of \( X \). Clearly \( Y \) has f.d.d. so that (ii) is satisfied. Also, from the construction of \( W_n \)'s, it is clear that \( W_n \)'s can not be contained in finite dimensional subspaces of \( Y \) with 'nice' basis constant and thus (i) is proved. Q.E.D.

Now we introduce another property in Banach spaces viz. the \( B_\lambda \) property.

Definition 3.15

A Banach space \( X \) is called a \( B_\lambda \) space if given any finite dimensional subspace \( B \subset X \), there exists a finite dimensional subspace \( W \) such that \( B \subset W \subset X \), \( \gamma(W) \leq \lambda \) and there exists a projection \( P: X \rightarrow W \), \( \|P\| \leq \lambda \).

Remark

The study of \( B_\lambda \) spaces arises naturally from \( \ell_p \) spaces and also
the viewpoint of Basis Theory. Note that every $B_\lambda$ space has
$\lambda$-f.d.s.b. The converse is unknown.

**Example 3.16**

1. Every space with a basis is a $B_\lambda$ space. This follows from
   Proposition 3.3 and the remark at the end of Proposition 3.3.

2. Every $L_p$ space is a $B_\lambda$ space for some $\lambda$. This follows from
   Theorem 1.23.

In the following results, we study some natural questions
associated with $B_\lambda$ spaces such as if $X^*$ is a $B_\lambda$ space, is $X$ a $B_\mu$
space for some $\mu$? For these, we need two powerful lemmas proved
recently by Johnson, Rosenthal and Zippin in [7].

We shall start with a definition.

**Definition 3.17**

Two Banach spaces $X$ and $Y$ are said to be ecloose if there exists
a 1-1 onto operator $T : X \to Y$ such that $\|T(x) - x\| \leq \varepsilon \|x\|, x \in X$.

**Theorem 3.18**

If $X^*$ is a separable $B_\lambda$ space, then $X$ is a $B_\mu$ space for some $\mu$.

We need the following lemma:

**Lemma 3.19** [Lemma 4.5 [7]]

Let $X^*$ be a $\lambda_\alpha$ space represented as $X = \bigcup_\alpha E_\alpha$. Let $E \subset X$,
be a finite dimensional subspace and $\varepsilon > 0$. Then there exists a
finite rank projection \( Q \) on \( X \) \( \exists \)

(i) \( Q/E \) is identity.

(ii) \( \|q\| \leq 4\lambda + 4\lambda^2 \).

(iii) \( Q^*(X^*) \) is \( \varepsilon \)-close to some \( E_\alpha \).

Proof of Theorem 3.18

Since \( X^* \) is a separable \( B_\lambda \) space, there exists an increasing sequence \( (E_n) \) of finite dimensional subspaces of \( X^* \) such that

\[
X^* = \bigcup_n E_n, \quad \eta(E_n) \leq \lambda
\]

and there exists a projection \( P_n : X^* \to E_n, \quad \|P_n\| \leq \lambda, \; n = 1, 2, \ldots \).

This can be achieved by taking a dense set in \( X^* \) and then taking suitable finite dimensional superspaces as in the proof of Proposition 3.9. Let \( B \subset X \) be any finite dimensional subspace. By Lemma 3.19, there exists a finite rank projection \( Q \) on \( X \) such that \( Q/B = I_B \), \( Q^*(X^*) \) is \( \varepsilon \)-close to some \( E_\alpha \), and \( \|Q\| \leq 4\lambda + 4\lambda^2 \). Since \( \eta(E_o) \leq \lambda \), therefore,

\[
\eta(Q^{**}(X^{**})) \leq \lambda \frac{1+\varepsilon}{1-\varepsilon} \|Q^{**}\|
\]

\[
\leq \lambda \frac{1+\varepsilon}{1-\varepsilon} (4\lambda + 4\lambda^2).
\]

But \( Q^{**}(X^{**}) = Q(X) \supset B \).

Therefore,
\[ \tau(Q(X)) \leq (4\lambda^2 + 4\lambda^3) \left( \frac{1+\varepsilon}{1-\varepsilon} \right) \]
\[ \leq 8(\lambda^2 + \lambda^3) \text{ if } \varepsilon < \frac{1}{3}. \quad Q.E.D. \]

**Remark**

Since every space with a basis is a \( B_\mu \) space for some \( \mu \), it follows immediately from Theorem 3.18 that if \( X^* \) has a basis, then \( X \) is a \( B_\lambda \) space for some \( \lambda \). It should be pointed out that much stronger result is known viz. if \( X^* \) has a basis, then \( X \) has a basis, \([7]\].

**Definition 3.20**

A Banach space \( X \) has \( \mu \) metric approximation property (written as \( \mu\text{-m.a.p.} \)) if given any finite dimensional subspace \( B \subset X \) and \( \varepsilon > 0 \), there exists an operator \( T \) on \( X \), such that

\[ \|T\| \leq \mu \text{ and } \|T(x) - x\| < \varepsilon \|x\| \quad \forall x \in B. \]

**Theorem 3.21**

Let \( X \) be a separable \( B_\lambda \) space and \( X^* \) have \( \mu\text{-m.a.p.} \), then \( X^* \) is a \( B_\mu \) space for some \( \mu \).

We need the following lemma:

**Lemma 3.22 [Lemma 4.4 [7]]**

Let \( X \) be a \( \pi_\lambda \) space represented as \( X = \bigcup_{\alpha} E_\alpha \) and \( X^* \) have \( \mu\text{-m.a.p.} \). Let \( F \subset X^* \) be a finite dimensional subspace and \( \varepsilon > 0 \).
Then there exists a finite rank projection \( Q \) on \( X \) such that \( Q^*F = I_F \)

\[
\|Q\| \leq 2\mu + 2\lambda + 4\mu \lambda \text{ and } Q(X) \text{ is } \varepsilon\text{-close to some } E_\alpha.
\]

**Proof of Theorem 3.21**

There exists an increasing sequence of finite dimensional subspaces of \( X \) such that

\[
X = \bigcup_{n} E_n \quad \text{and} \quad \eta(E_n) \leq \lambda
\]

and there exists a projection \( P_n : X \rightarrow E_n \), \( \|P_n\| \leq \lambda \), \( n = 1, 2, \ldots \). Let \( B \subseteq X^* \) be any finite dimensional subspace. Then by Lemma 3.22, there exists a finite rank projection \( Q \) on \( X \) such that \( Q^*(X^*) \supseteq B \) and \( Q(X) \) is \( \varepsilon \)-close to some \( E_{n_0} \). Since \( \eta(E_{n_0}) \leq \lambda \), therefore,

\[
\eta(Q(X)) \leq \lambda \left( \frac{1 + \varepsilon}{1 - \varepsilon} \right).
\]

Therefore,

\[
\eta(Q^*(X^*)) \leq \lambda \left( \frac{1 + \varepsilon}{1 - \varepsilon} \right) \|Q^*\|
\]

\[
\leq \lambda \left( \frac{1 + \varepsilon}{1 - \varepsilon} \right) (2\mu + 2\lambda + 4\mu \lambda)
\]

\[
\leq 2\lambda(2\mu + 2\lambda + 4\mu \lambda) \quad \text{if } \varepsilon < \frac{1}{3}.
\]

Q.E.D.
Remark

Theorems 3.18 and 3.21 are true without the condition of separability. The proofs are almost the same.

The proof of [Theorem 2.1, [14]] can be modified to give the following theorem:

Theorem 3.22

Let $X$ be a $B_\Lambda$ space and $Y$ be a complemented subspace of $X$. Suppose there exists $\varepsilon > 0$ such that every infinite dimensional subspace of $X$ has a subspace $\varepsilon$-isometric to $X$. Then $Y$ has f.d.s.b.
CHAPTER IV

SOME UNSOLVED PROBLEMS

In this chapter, we wish to summarize in a systematic way some unsolved problems related to the material of this dissertation.

The first set of problems is related to $L_p$ spaces.

Problem 1

Is every $L_p$ subspace of $L_p$ isomorphic to $L_p$, $1 < p < \infty$?

This problem was mentioned both in [13] and [14]. Theorem 2.11 and Theorem 2.14 give some partial answers to this problem. As was pointed out before in Chapter II, the answer to Problem 1 is 'no' for $p = 1, \infty$.

Problem 2

Let $1 < p < \infty$, $X$ be a $L_p$ space and $X \subseteq L_p(\mu)$ for some measure $\mu$. Is it possible to find an embedding of $X$ in $L_p(\mu)$ such that the corresponding embedded space is complemented in $L_p(\mu)$?

Problem 2 is very naturally suggested by Problem 1. A positive answer to problem 2 would imply a positive answer to Problem 1, because if $X \subseteq L_p$ and $X$ is $L_p$, let $Y$ be an embedding of $X$ in $L_p$ on

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which we can project. Then by a result of Pelczynski [16], Y would be isomorphic to \( l_p \) and hence X would be isomorphic to \( l_p \). As was mentioned earlier in Chapter II, Problem 2 has a negative answer for \( p = 1, \infty \).

**Problem 3**

Let \( X, Y \subset l_p \) be isometric Banach spaces and each has infinite codimension in \( l_p \). If \( X \) is complemented in \( l_p \), is \( Y \) complemented in \( l_p \), \( 1 \leq p < \infty \)?

**Problem 4**

In Theorem 2.6, can we choose \( Y \) with a basis?

The next set of problems is related to f.d.s.b. and \( B_\lambda \) properties.

**Problem 5**

Does every Banach space have f.d.s.b.?

If there exists a Banach space \( X \) which does not have f.d.s.b., then by Theorem 3.12, \( X \supset Y \) such that \( Y \) does not have f.d.s.b. and \( Y \) has f.d.d. Therefore, by Proposition 3.3, \( Y \) can't have a basis. Thus if the answer to Problem 5 is 'no,' then the basis problem would be answered negatively.

**Problem 6**

If \( X \) has f.d.s.b., does \( X^* \) have f.d.s.b.?

**Problem 7**

If \( X^* \) has f.d.s.b., does \( X \) have f.d.s.b.?
If the answer to Problem 6 is yes, then by Proposition 3.6, the answer to Problem 7 is also yes.

It is an unsolved problem that if a Banach space $X$ has f.d.d., then it has a basis [3]. In connection with this we ask

**Problem 8**

If $X$ has f.d.d. and f.d.s.b., does it have a basis? More strongly, if $X$ has f.d.d. and it is a $B^*$ space, does it have a basis?

If the answer to the question that $X$ has f.d.d. $\implies$ $X$ has a basis is yes, then by Theorem 3.12, it follows that every Banach space has f.d.s.b.

It is unknown that if a Banach space $X$ has a basis, then every complemented subspace of $X$ has a basis. We ask a formally weaker question.

**Problem 9**

If $X$ has f.d.s.b., does every complemented subspace of $X$ have f.d.s.b.?
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