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THE RADIATIVE PICTURE FOR THE NUCLEON ANOMALOUS MAGNETIC MOMENTS IN THE SIDEWISE DISPERSIVE FRAMEWORK

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

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The Ohio State University
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INTRODUCTION

One of the long standing problems of particle physics has been the calculation of the nucleon anomalous magnetic moments. Although we do not present in this thesis the long sought after definitive calculation, we do present an investigation of a dynamical method which we think merits consideration as a definitive approach. Confirmation of this approach waits upon the acquisition of further experimental information.

The method we are recommending is the sidewise dispersive approach. In this approach, dispersion takes place in the incoming nucleon channel instead of the photon channel considered in the usual dispersive approach. This means, in the sidewise dispersive approach, that the absorptive parts of the form factors develop from the product of a strong interaction vertex times a photo- or electro-production amplitude in contrast to the conventional dispersive approach where the absorptive parts develop from a nucleon-antinucleon annihilation amplitude times an electromagnetic vertex. The first advantage of being in the sidewise channel is that the electromagnetic interaction is experimentally accessible in terms of the photo- or electro-production amplitude. Another advantage may be faster convergence of the dispersion integrals. In the normal approach, vector meson dominance appears to be inadequate and
the inclusion of other higher intermediate states such as $\bar{N}N$ seems required. In the sidewise approach, while the inclusion of higher intermediate states other than the lowest (pion-nucleon) states likewise seems necessary, we find that these higher states have lower thresholds by comparison.

In the literature, applications of the sidewise dispersive technique can be found concerning the axial vector coupling $g_A$, the nucleon mass difference, the neutron electric dipole moment, the $NN\gamma$ vertices where $N^*$ is either the $N^*(1470)$ or the $N^*(1236)$, and the isovector nucleon radius. A common feature of these applications appears to be the dominance of the low energy contributions.

In this thesis, we illustrate this feature (following Drell and Pagels) by the sidewise dispersive calculation of the electron anomalous magnetic moment (Schwinger term and the two next highest order corrections) using a second-order perturbative Compton amplitude. We

find that such a calculation has its analog in the original quantum electrodynamics calculation where the same diagrams are employed and are interpreted as radiative corrections (self-energy and vertex corrections for second-order diagrams).

Thus, in turning to the same calculations for the nucleons, we interpret the internal pion as a radiative correction and expect the pion-nucleon intermediate state to give a quickly convergent and dominant contribution. Thus, using a simple perturbative photoproduction amplitude, we can account for the basic magnitude of the nucleon moment. Explicitly, for vanishing pion mass, we find that the proton has \[ \mu_p = \frac{1}{2\pi} \frac{9\pi N}{4\pi}, \]
result which is analogous to the Schwinger correction \[ \mu_e = \frac{\alpha}{2\pi} = \frac{1}{2\pi} \frac{e^2}{4\pi}. \]

We are misled, however, if we take this simple perturbative approach seriously in the complex realm of hadronic physics. Indeed, we show in this thesis, using a phenomenological model for the photoproduction amplitude (which gives a rapidly falling factor), that, upon demanding the pion-nucleon contribution to be dominant, the pion-nucleon vertex cannot be approximated by a coupling constant but must, in fact, display a peaked behavior as a function of the energy. Otherwise, the vertex factor does not compensate for the rapidly falling photoproduction factor and the total contribution is much too small. As discussed in the text, a sidewise dispersive treatment of the pion-nucleon vertex can give this peaked behavior. This corresponds to adding another internal pion (a pion-nucleon scattering amplitude is inserted into the original electromagnetic dispersive diagram) in continuation of the low energy radiative philosophy, thus preserving the low energy interpretation given the
magnetic moment. In this manner, we are also able to explain the moderately successful results obtained by Drell and Pagels in their approximation of the vertex by a coupling constant and the photoproduction amplitude by its threshold limit.

In this thesis, under the general theoretical considerations of the first three chapters, we discuss the structure of the electromagnetic vertex, the low energy radiative philosophy in connection with unsubtracted dispersion relations and the calculation of the Schwinger correction, and the development of the formalism of sidewise dispersion relations (following Bincer) for the electromagnetic and pion-nucleon vertices through the application of the reduction technique. In Chapter IV, we develop the radiative concept for the nucleon anomalous moments and discuss the critical role of the pion-nucleon form factor. Chapter V provides a review of the radiative concept for the anomalous moments. In the appendixes, we construct the projection operators for the anomalous form factors and give a proof of the analyticity of the form factors.

We conclude these introductory remarks with the following prologue to the calculations of the future. The notion that the anomalous moment is a low energy quantity will probably be preserved in the context of a properly conceived radiative philosophy. Thus, we should find in the sidewise dispersive approach the amplitudes of each intermediate state being naturally damped and falling so quickly at high energies as to remove the imposition of an arbitrary cutoff upon the dispersion integrals.


I. STRUCTURE OF THE ELECTROMAGNETIC VERTEX

The most general form the electromagnetic vertex \( \bar{u}(p)\Gamma_\mu(p,p+q)u(p+q) \) can have under the requirement of Lorentz covariance is given by

\[
\bar{u}(p)\gamma_\mu(p,p+q)u(p+q) = \bar{u}(p)
\left[
F_1(x^2) + i\frac{2\mu\mathbf{q}^2}{2x} F_3(x^2) - \frac{\mathbf{q}^2}{m^2} F_2(x^2) - \frac{\mathbf{q}^2}{x}\Gamma_\mu(p,p+q) G(x^2)\right]u(p+q)
\]  \( (1.1) \)

where the vertex function \( \Gamma_\mu(p,p+q) \) is defined by its connection to the electromagnetic current \( j_\mu(x) \) in the relation

\[
\langle p|j_\mu(x)|p+q\rangle = \left[\frac{1}{2\pi^2}\right]^2 \int \frac{d^4k}{p_0} \int \frac{d^4k'}{p_0'} \bar{u}(p)\Gamma_\mu(k,k')\gamma^\mu\bar{u}(k')j_\mu(k) \]  \( (1.2) \)

and where \( p + q, p, \) and \( q \) are, respectively, the momenta of the incoming nucleon, the outgoing nucleon, and the emitted photon\(^\text{10} \), and \( (p+q)^2 = W^2 \) and \( p^2 = W'^2 \). From the viewpoint of a minimal electromagnetic coupling, the Dirac term \( \bar{u}(x)\gamma_\mu\bar{u}(x)j_\mu(x) \) is fundamental and the second and third terms are induced by renormalization effects. Thus, the second form factor gives only the anomalous part of the magnetic moment, the normal Dirac moment being incorporated together with the charge in the first form factor. This can be seen by a Gordon decomposition of the Dirac term and comparison to the classical electric charge and magnetic dipole interactions. When both nucleons and the photon are on

\(^{10}\) In the literature, it is customary to consider the photon as being absorbed rather than emitted. Thus, we have introduced the negative signs before the induced form factors in order to retain the conventional definitions.
their mass shells, the sum of the first and second form factors is the total magnetic moment.

Under time-reversal invariance, the third form factor vanishes when both nucleons are on their mass shells. This corresponds to the experimental situation (as shown in Figure 1) in which elastic electron-nucleon scattering proceeding through the exchange of a single space-like photon \( q^3 < 0 \) is studied in order to determine \( F_1(q^3) \) and \( F_2(q^3) \). Thus, in the conventional approach\(^1\)(see Figure 2), dispersion relations have been written directly in the \( q^2 \) variable with the absorptive part being generated from the right-hand cut in the \( q^2 \) plane. The present such dispersion studies\(^2\) have introduced vector meson dominance of the form factors with only limited success, for this approach fails to reproduce the experimental dipole fit characterizing the Rosenbluth form factors\(^3\). Apparently, these dispersion representations converge slowly at high energies and the inclusion of higher intermediate states (such as \( N\bar{N} \)) is needed.

We elaborate no further upon this conventional approach for we present, in this thesis, an alternative dispersive approach. The pertinent observation to be made is that dispersion representations can be written


Figure 1. Electron-nucleon elastic scattering proceeding through the exchange of a single virtual photon.

\[ \text{Abs} \rightarrow \begin{array}{c} N \\ N \end{array} = \sum_n \frac{2q^2 < 0}{(p+q)^2 W^2 + p^2 m^2} \]

Figure 2. Normal dispersive approach for the electromagnetic form factors.

Figure 3. Electromagnetic vertex in the sidewise dispersive approach.
in the nucleon channel (as shown in Figure 3) as well as in the photon channel. While not suggested by the experimental situation (the nucleon being dispersed off its mass shell), these representations have the advantage that their absorptive parts are calculable from such experimentally accessible quantities as scattering and production amplitudes. In addition, these dispersion representations, known as sidewise dispersion relations, may afford several other advantages over the usual dispersion relations. In particular, they may possess faster convergence properties needed for emphasizing the low energy contributions\(^\text{14}\). Another advantage is that they may present the only calculable dispersion channel for some vertices. Lastly, another feature is that they can be rigorously proved.

In this thesis, we follow closely the developments of Bincer\(^\text{15}\), who first developed and proved the sidewise dispersive approach for the electromagnetic and pion-nucleon form factors, and Drell and Pagels\(^\text{16}\), who made the first calculations in the sidewise dispersive framework by the application of their threshold dominance idea. Since sidewise dispersion relations can be written in either the \(W\) or the \(W^2\) plane (Bincer used the first, Drell and Pagels the second), we present the general theoretical analysis concurrently in both planes. We also employ the metric and notational conventions of Bjorken and Drell\(^\text{17}\).


\(^{15}\) A.M. Bincer, Phys. Rev. 118, 855 (1960).


In the sidewise dispersive approach, $q^2$ is held fixed and understood to be zero or negative, the outgoing nucleon is put on the mass shell ($p^2 = m^2$), and the incoming nucleon energy $W^2 = (p+q)^2$ is varied. Since the incoming nucleon is off its mass shell, we are forced to admit negative energy states. Therefore, we rewrite the electromagnetic vertex with positive and negative energy projection operators in the following manner:

$$\bar{u}(p)F_{\mu} (p, p+q) = \bar{u}(p) \left\{ \left[ F_1(w) \gamma_{\mu} - \frac{i\gamma_{\nu} q_{\nu}}{2m} F_2(w) - g_{\mu} F_3(w) \right] \frac{p^2 + q^2 + W^2}{2W} 
+ \left[ F_1(-w) \gamma_{\mu} - \frac{i\gamma_{\nu} q_{\nu}}{2m} F_2(-w) - g_{\mu} F_3(-w) \right] \frac{-p^2 - q^2 + W^2}{2W} \right\}$$

(1.3)

and

$$\bar{u}(p)F_{\mu} (p, p+q) = \bar{u}(p) \left\{ \left[ F_1^+(w) \gamma_{\mu} - \frac{i\gamma_{\nu} q_{\nu}}{2m} F_2^+(w) - g_{\mu} F_3^+(w) \right] \frac{p^2 + q^2 + m^2}{2m} 
+ \left[ F_1^-(w) \gamma_{\mu} - \frac{i\gamma_{\nu} q_{\nu}}{2m} F_2^-(w) - g_{\mu} F_3^-(w) \right] \frac{-p^2 - q^2 + m^2}{2m} \right\}$$

(1.4)

By multiplying from the right with positive and negative energy spinors, it is easily seen that the form factors in the $W$ and $W^2$ planes are related as follows:

$$F_i(\pm W) = \frac{1}{2m} \left[ F_i^\pm(w^2)(m \pm W) + F_i^\pm(w^2)(m \mp W) \right]$$

(1.5)

We note that this simple transformation introduces no additional singularities into the mapping between the two planes. Thus, the right-hand cut in the $W^2$ plane maps directly onto a right-hand cut and invertedly onto a left-hand cut in the $W$ plane. Furthermore, there is only one function $F_i(W)$ in the entire $W$ plane since $F_i(W)$ has the

---

boundary values

\[ F_{e}^{*}(w) = F_{e}^{*}(w) \quad \text{for} \quad w > 0 \]

and

\[ F_{e}^{*}(w) = F_{e}^{*}(-w) = F_{e}^{*}(|w|) \quad \text{for} \quad w < 0 \]  \quad (1.6)

Thus, we may employ the Schwarz reflection principle for form factors analytic in both planes to write dispersion relations. For example, since we are interested in the anomalous magnetic moment (obtained from the second form factor by putting the incoming nucleon and the photon on their mass shells), we write for the second form factor, assuming no subtractions are necessary, the following sidewise dispersion relations:

\[ F_{2}(w) = \frac{1}{\pi} \int_{m+\mu}^{\infty} dw' \left[ \frac{\text{Im} F_{2}(w'+i\varepsilon)}{w' - w} + \frac{\text{Im} F_{2}(-w'-i\varepsilon)}{w' + w} \right] \]  \quad (1.7)

and

\[ F_{2}^{+}(w') = \frac{1}{\pi} \int_{m+\mu}^{\infty} dw'^{2} \frac{\text{Im} F_{2}^{+}(w'^{2})}{w'^{2} - w'^{2}} \]  \quad (1.8)

where the Schwarz reflection principle applied to the form factor on the left-hand cut is \( F_{2}(-w+i\varepsilon) = F_{2}^{*}(-w-i\varepsilon) \).

Gauge invariance may be imposed through the generalized Ward identity, stated in the present context in the form \(^{19}\)

\[ \bar{u}(p) \gamma_{\mu} (p,\gamma_{\mu} + q) q_{\mu} = \bar{u}(p) \not{\gamma} \]

(1.9)

to give the following relations between the charge form factors:

---

where $e_N$ is the nucleon charge. These relations are consistent with the transformation (1.5), and the last of these relations implies

$$F_3(\pm w) = \frac{1}{2m} (m \mp w) F_3^-(w^2)$$  \hspace{1cm} (1.12)

consistent with the time-reversal requirement

$$F_3(q^2, m, m) = 0$$  \hspace{1cm} (1.13)

From (1.10), we derive by differentiation the relation

$$F_1(q^2) = F_1(q^2, m, m) = e_N + q^2 F_3'(q^2, m, m)$$  \hspace{1cm} (1.14)

where the prime means differentiation with respect to $W$. Thus, gauge invariance has provided a subtraction constant for the charge form factor. Finally, we eliminate the first form factor by the above relations and rewrite the vertex in the forms

$$\tilde{u}(p) \Gamma_\mu (p, q+p) = \tilde{u}(p) \left\{ e_N \gamma_\mu - \left[ \frac{i q_\mu q_0}{2m} F_2(w) + (q_\mu \mp \frac{q^2}{m \mp w}) F_3(w) \right] \frac{q + q + W}{2W} \right. \right.$$  \hspace{1cm} (1.15)

$$- \left[ \frac{i q_\mu q_0}{2m} F_2(-w) + (q_\mu \mp \frac{q^2}{m \mp w}) F_3(-w) \right] \frac{-q - q + W}{2W} \}$$

and

$$\tilde{u}(p) \Gamma_\mu (p, q+p) = \tilde{u}(p) \left\{ e_N \gamma_\mu - \left[ \frac{i q_\mu q_0}{2m} F_2^+(w) + (q_\mu \mp \frac{q^2}{m \mp W}) F_3^-(w) \right] \frac{q + q + M}{2m} \right.$$  \hspace{1cm} (1.16)

$$- \left[ \frac{i q_\mu q_0}{2m} F_2^-(w) + (q_\mu \mp \frac{q^2}{m \mp W}) F_3^+(w) \right] \frac{-q - q + M}{2m} \}$$
In assuming the anomalous magnetic moment satisfies an unsubtracted dispersion relation, we are adhering to the notion of a fundamental minimal electromagnetic coupling, for had we subtracted the second form factor and identified the subtraction constant with the anomalous moment, we could admit the anomalous moment as a fundamental, noncalculable coupling. Furthermore, along these same lines, any other subtraction constant could be considered as a fundamental coupling. Proceeding further, we would be at a loss in trying to identify such a mysterious subtraction constant, for we are without an apparent fundamental constant, such as the charge, and we are also without a principle, such as gauge invariance, with which to find such a fundamental subtraction constant.

If we do not however, consider the subtraction constant as a fundamental coupling, we could straightforwardly try to identify the subtraction constant with high energy contributions. Harari, for example, has subtracted the $\Delta l = 1$ amplitude for the isospin one multiplet mass differences and identified the subtraction constant with a tadpole term. On the basis of our investigations, however, we do not think this straightforward approach can succeed either. It appears to us that the anomalous moment is entirely calculable in the unsubtracted form for the simple reason that it is mainly induced by low energy radiative corrections. The anomalous moment of the electron, for example, can be computed from this viewpoint.

Before proceeding with this calculation in the next chapter, we remark that once the absorptive part of the vertex is obtained from dynamical considerations, a projection operator is needed to extract the part pertaining to the absorptive part of the form factor. Explicit definition and construction of these projection operators is given in Appendix A.
II. CALCULATION OF THE ANOMALOUS MAGNETIC MOMENT OF THE ELECTRON

The notion of radiative corrections has traditionally been incorporated into perturbation theory and has already provided a successful calculation of the electron anomalous magnetic moment in quantum electrodynamics. We present here another calculation of the electron moment in the dispersion theoretic framework, using the perturbative technique to obtain an expression for the absorptive part. Since the Feynman propagators are analytic, we are assured of the analytic properties (to all finite orders of perturbation theory) required for the writing of dispersion relations in the $(\text{mass})^2$ of one of the external lines with the other two lines on their mass shells.

We begin with the construction of the $S$-matrix for the lowest order corrections for emission of a photon on the mass shell ($q^2 = 0$). We have

$$S = S = \frac{(-i)}{3!} \int d\chi_1 d\chi_2 d\chi_3 \overline{T} \left( H(\chi_1) H(\chi_2) H(\chi_3) \right)$$

(2.1)

where the electromagnetic interaction takes the standard form

$$\mathcal{H}(x) = -e \overline{\Psi}(x) \gamma_\mu \Psi(x) A_\mu(x)$$

(2.2)
Expanding the time-ordered product by the Wick expansion, we obtain

\[
\langle p, q | S(p+q) \rangle = -i e^3 \int \frac{d^4q}{(2\pi)^4} \frac{1}{(2\pi)^2} \frac{\sqrt{m}}{\sqrt{2q_0}} \frac{\sqrt{m}}{\sqrt{p_0}} \int \frac{m}{m} \frac{m}{m} \left[ \bar{u}(p) \gamma_\mu \frac{i}{q + q - m} \gamma_\beta \frac{i}{\epsilon - m} \gamma_\nu \frac{1}{k^2 + i\epsilon} \bar{u}(p+q) + \bar{u}(p) \gamma_\mu \frac{i}{q - K - m} \gamma_\beta \frac{i}{\epsilon - m} \gamma_\nu \frac{-i\gamma_5 \gamma_\beta}{k^2 + i\epsilon} \bar{u}(p+q) \right]
\]

where we define

\[
\bar{u}(p) \gamma_\mu \frac{1}{k^2 + i\epsilon} \bar{u}(p) = -e^2 \bar{u}(p) \left[ \gamma_\mu \frac{1}{q + q - m} \gamma_\nu + \gamma_\nu \frac{1}{q - K - m} \gamma_\mu \right] \bar{w}(q)
\]

(2.4)

To obtain the absorptive part of the S-matrix, we set the internal photon and electron on their positive energy mass shells by the replacement

\[
\frac{1}{\epsilon^2 + i\epsilon} \rightarrow 2\pi \delta(\epsilon^2 - m^2) \delta(k^2) \Theta(k_0) \Theta(k_0)
\]

(2.5)

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22. S. Mandelstam, Phys. Rev. 115, 1741 (1955); R. Cutkosky, J. Math. Phys. 1, 429 (1960). The above replacement is generally referred to as the Cutkosky rule. In the general case of an arbitrary Feynman amplitude, the discontinuity of the Feynman amplitude (essentially twice the imaginary part) is given by replacing each Feynman propagator function for each internal line which is cut by \(2\pi \delta(q^2 - m^2)\Theta(q_0)\). Thus, the singular parts of the propagator functions not only give the discontinuity but also put the internal particles on their positive energy mass shells as required by the
We thus obtain for the absorptive part

\[ \text{Abs} \langle p, q | S | p+q \rangle = -\frac{\hbar^2}{m^2} \int \frac{d^4l}{(2\pi)^4} \frac{\delta((\mathbf{u}_k - \mathbf{u}_0))}{(2\pi)^4} \delta((\mathbf{v}_l - \mathbf{v}_0)) \Theta(\mathbf{e}) \]

\[ \times \left[ \frac{1}{(2\pi)^3} \right]^{3/2} \sqrt{\frac{m}{p_0}} \sqrt{\frac{m}{p_0 + q}} \tilde{u}(\mathbf{p}) \Gamma_{\mu \nu} u(\mathbf{e}) \Gamma_{\nu \alpha} u(\mathbf{p} + q) \] (2.6)

Landau conditions. In the case at hand, the absorptive part can also be obtained by direct calculation from form factor unitarity. If we write the S-matrix as \( S = 1 - i(2m)^2 \delta(\mathbf{P}_f - \mathbf{P}_i) \), then the unitarity condition \( SS^\dagger = 1 \) gives the following expression for the imaginary part of the T-matrix:

\[ \text{Im} \langle p, q | T | p+q \rangle = -\frac{1}{2} \sum_n (2\pi)^4 \delta(n - p - q) \langle p, q | T | n \rangle \langle n \cdot m \cdot T^\dagger | p+q \rangle \]

\[ = -\frac{1}{2} \int \frac{d^4l}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} (2\pi)^4 \delta(k + l - p - q) \langle p, q | T_1 | k, l \rangle \langle k, l | T_2 | p+q \rangle \]

\[ = -2\hbar^2 \int \frac{d^4l}{(2\pi)^4} \delta((\mathbf{u}_k - \mathbf{u}_0)) \Theta(\mathbf{e}) \delta((\mathbf{v}_l - \mathbf{v}_0)) \langle p, q | T_1 | k, l \rangle \langle k, l | T_2 | p+q \rangle \]

where \( T = T_1 + T_2 \) is the sum of the first and second order terms,

\[ \langle k, l | T_1^\dagger | p+q \rangle = -\frac{1}{(2\pi)^3} \left[ \frac{m}{p_0 + q} \right] \frac{1}{\sqrt{2k_0}} \tilde{u}(\mathbf{p}) \Gamma_{\mu \nu} u(\mathbf{e}) \]

and

\[ \langle p, q | T_2 | k, l \rangle = -\frac{1}{(2\pi)^3} \left[ \frac{1}{(2\pi)^3} \right] \frac{m}{p_0} \frac{1}{\sqrt{2q_0}} \tilde{u}(\mathbf{p}) \Gamma_{\mu \nu} u(\mathbf{e}) \]
where $e_{\mu}(q)e_{\nu}(k)\bar{u}(p)T_{\mu\nu}(k)$ has become the second-order Compton amplitude and $\bar{u}(k)\gamma_{\nu}u(p+q)$ the off-shell, on-shell transition vertex. Thus, from the dispersive viewpoint, the self-energy and vertex corrections are linked, respectively, to the direct and crossed channel poles in the Compton amplitude (diagrams (a) and (b) in Figure 4).

The phase space can be integrated immediately except for an angular integration over $x = \cos \theta$, where $\cos \theta = \frac{k \cdot q}{|k| |q|}$ is related to the center of mass scattering angle for Compton scattering. The $d^4k$ differential can be reduced to $d\theta d^2k \pi dx$ and the delta functions can be rearranged as

$$\delta(k^2) = \frac{1}{2w} \delta\left(\frac{w^2+m^2}{2w} - k^0\right)$$

and

$$\delta(k^2-w^2) = \delta\left(\frac{(w^2-m^2)^2}{4w^2} - k^2\right)$$

(2.7)

![Figure 4. Radiative graphs for the Schwinger correction to the electron anomalous magnetic moment.](image)
in order to obtain

\[ \text{Abs } \langle p, q | S(p+q) \rangle = \frac{-i}{2} \frac{e^{-m \beta}}{8 \pi} \sigma(w^2) (\sigma(w) \gamma^0(\sigma)) \]

\[ \times \int_{-1}^{+1} \frac{e^{\mu(q)}}{\sqrt{2 q^0}} \left[ \frac{m}{p_0} \frac{m}{p_0 + q_0} \left( \frac{1}{e^2 \gamma \lambda} \right)^3 \bar{u}(p) \bar{\Gamma}^\nu u(c) \bar{\tilde{u}}(n) \gamma_\nu \bar{u}(p+q) \right] \]

(2.8)

where \( \rho(w^2) = \frac{w^2 - m^2}{w^2} \) is a phase space factor.

The reduction technique can be applied to the electromagnetic emission matrix element \( \langle p, q \text{ out} | p+q \text{ in} \rangle \) to relate the S-matrix to the electromagnetic vertex. This procedure gives

\[ \langle p, q | S(p+q) \rangle = -i e \left[ \frac{1}{(2\pi)^3} \right] \int_{-1}^{+1} \frac{e^{\mu(q)}}{\sqrt{2 q^0}} \left[ \frac{m}{p_0} \frac{m}{p_0 + q_0} \left( \frac{1}{e^2 \gamma \lambda} \right)^3 \bar{u}(p) \bar{\Gamma}^\nu u(c) \bar{\tilde{u}}(n) \gamma_\nu \bar{u}(p+q) \right] \]

(2.9)

Thus, directly from the above relations we obtain for the absorptive part of the vertex

\[ \bar{u}(p) \text{Abs } \bar{\Gamma}^\mu(q, p+q) \bar{u}(q) = \frac{-i}{2} \frac{m}{8 \pi} \rho(w) \int_{-1}^{+1} \bar{u}(p) \bar{\Gamma}^\nu u(c) \bar{\tilde{u}}(n) \gamma_\nu \bar{u}(p+q) \]

(2.10)

The imaginary part of the form factor is then obtained from the condition

\[ \text{Tr } \bar{u}(p) \text{Abs } \bar{\Gamma}^\mu(q, p+q) \sigma^{(a)}(w^2) \bar{u}(q) = \frac{-i}{2m} \text{Im } F_2(w^2) \]

(2.11)

where \( \sigma^{(a)}(w^2) \) is an appropriate projection operator. We thus find

\[ \text{Im } F_2(w^2) = \frac{m}{8 \pi} \rho(w) \int_{-1}^{+1} \bar{u}(p) \bar{\Gamma}^\nu u(c) \bar{\tilde{u}}(n) \gamma_\nu \gamma^{(a)}_\mu(w^2) \bar{u}(p+q) \]

(2.12)
The projection operator has the general form

\[
\mathcal{V}_\mu^{(2)}(w^2) = \frac{-\mathbf{q}^2}{(w^2 - m^2)^2} \left[ \frac{\mathbf{q}^2 + K + m}{2m} (-i\sigma_{\mu\nu} q_{\nu}) + 3q_{\mu} \frac{K + q - m}{2m} \right]
\]

(2.13)

which can be simplified since the Compton amplitude is gauge invariant, i.e., \(q_{\mu} \bar{u}(p) T_{\mu\nu} u(x) = 0\). Thus, noting that \(i\sigma_{\mu\nu} q_{\nu} = (\gamma_\mu - q_{\mu}) = (q_{\mu} - \gamma_\mu q_{\mu})\), we reduce the projection operator to the form

\[
\mathcal{V}_\mu^{(2)}(w^2) = \frac{-\mathbf{q}^2}{(w^2 - m^2)^2} \frac{\mathbf{q}^2 + K + m}{2m} \gamma_\mu \gamma_0
\]

(2.14)

With this expression and the one for the Compton amplitude, the absorptive part becomes

\[
\text{Im} F_\mu(w^2) = \frac{i}{\pi} \frac{m}{8(w^2 - m^2)^2} \phi(w^2)
\]

\[
\times \int \text{Tr}(p + m) \left[ \gamma_\mu \frac{p + K + m}{2(pq)} \gamma_\nu - \gamma_\nu \frac{p + K + m}{2(pq)} \gamma_\mu \right] (\gamma_\nu K + m) \gamma_\nu (p + m) \delta_{\mu\nu}
\]

(2.15)

where the trace calculation is best performed separately for each pole in the Compton amplitude.

The first trace calculation to be performed is

\[
I = \text{Tr}(p + m) \gamma_\mu (p + q + m) \gamma_\nu (K + m) \gamma_\nu (p + m) \gamma_\mu \gamma_0
\]

(2.16)

By using the anticommutation relations \(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu}\), one can easily show that

\[
\gamma_\nu (K + m) \gamma_\nu = -2K + 4m
\]

\[
\gamma_\mu \gamma_0 (p + m) \delta_{\mu\nu} = \nu(pq) - 2m \eta_{\mu\nu}
\]

(2.17)
Then $I = 4 \, \text{Tr} \left( (2pq) \cdot (p^2 + m^2) \right)$ which is easily reducible to

$$I = 16m \cdot 2(pq)(m^2 + (pq)) \quad (2.18)$$

The second trace calculation is

$$\Pi = \text{Tr} \left( (p+q)\nu \nu (p-q+m) \nu \nu (q+m) \nu \nu \right) \quad (2.19)$$

In similar fashion to the above, we show that

$$\nu \nu (p+q)\nu \nu (p+q) \nu \nu = 2 \left[ \nu \nu \nu - 2p^2 \nu - m^2 \nu + 2m(p\nu + q\nu) \right]$$

and

$$\nu \nu (p+q)\nu \nu (p-k+m) = 2 \left[ K \nu \nu + 2(p\nu - k\nu) \right] \quad (2.20)$$

Performing the contractions, we obtain

$$\nu \nu (p-k+m) \nu \nu (k+m) \nu \nu (p+m) \nu \nu \nu \nu \nu \nu \nu$$

$$= 4 \left[ -2m^2p - m^2g + 2(pq)k + mK(2p+q) + 2m(2m^2 - 2pq) \right] \quad (2.21)$$

After taking the trace and making some simplifications, we obtain

$$\Pi = 16m \cdot 2(pq)(m^2 + (pq)) \quad (2.22)$$

Thus, the entire trace is given by $16m(qk)$.

Finally, we perform the angular integration and obtain for the imaginary part

$$\Im F(t) = \frac{1}{2} \alpha \frac{w^2 - m^2}{w^2} \frac{m^2}{w^2} \quad (2.23)$$
The anomalous moment is then

\[ \mu_e = \frac{1}{\pi} \int_{m^2}^{\Lambda^2} dW_{12} \frac{\text{Im} \hat{F}_2(W_{12})}{W_{12}^2 - m^2} = \frac{\alpha}{2\pi} \left(1 - \frac{1}{\Lambda^2}\right) \]

(2.24)

where we have inserted an artificial cutoff \(\Lambda\) to demonstrate the convergence properties of the integral. We first note that without a cutoff the Schwinger correction\(^23\) is reproduced exactly (as expected).

Secondly, we note that the dispersion integral converges very quickly to the Schwinger correction as \(\Lambda\) increases from threshold. For example, for \(\Lambda = 5\) (roughly twice the one-electron threshold energy) already 80 per cent of the Schwinger correction is obtained. This means that the low energy contributions dominate and that in approximate calculations the high energy contributions can be neglected.

It is this rationale which has provided the basis for an estimation of the sixth-order correction\(^24\). For this estimation, the bare vertices \(\gamma_{\mu}^\dagger\) in the second-order Compton amplitude are replaced by \(\gamma_{\mu}^\dagger \frac{1}{\mu_m} F_2(m^2) [\gamma_{\mu}, q]\) in order not only to include the anomalous interaction but also to provide a method of iteration. Two and three photon intermediate states are neglected in this approach. Due to the presence of the momentum \(q\) in the anomalous interaction, this procedure amounts to a low energy expansion and not a perturbation series. The presence of \(q\) also means the dispersion integral loses its convergence properties and, in fact, diverges logarithmically, and hence a cutoff is introduced.

23. J. Schwinger, Phys. Rev. 73, 416 (1948).

The calculation proceeds first to the computation of the $\alpha^2$ correction by setting $F_2 (m^2) = \frac{1}{2} (g-2) = \frac{\alpha}{2\pi}$. From this term, the range of cutoff values giving reasonable approximations to the known $\alpha^2$ correction is determined. With the cutoff determined, one proceeds next to an estimation of the sixth-order term by introducing $\frac{1}{2} (g-2) = \frac{\alpha}{2\pi} - 0.328 \frac{\alpha^3}{\pi^3}$, the Petermann-Sommerfield result from quantum electrodynamics. Drell and Pagels have obtained

$$\frac{1}{2} (g-2) = \frac{\alpha}{2\pi} - 1.28 \frac{\alpha^2}{\pi^2} + 0.14 \frac{\alpha^3}{\pi^3} \text{ for } \Lambda \sim 5,$$

which is consistent with the accuracy of present experimental determinations of $\alpha$ and the anomalous moment.

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III. CALCULATION OF THE ABSORPTIVE PART FROM THE REDUCTION TECHNIQUE

We now give a general theoretical formulation of the absorptive part for the electromagnetic vertex using the reduction technique. We will show that the absorptive part of the anomalous moment form factor can be written as the product of a time-reversed electroproduction amplitude times a meson-baryon form factor which can be obtained from a dispersion integral over the $J = \frac{1}{2}, T = \frac{1}{2}$ S and P wave phase shifts. We also include, in preparation for the applications of the next chapter, expressions for the absorptive parts in the perturbative context and in the context of a nonrelativistic multiple expansion of the photoproduction amplitude.

We begin by taking the nucleon out of the in-state to give

$$\langle p|j_{\mu}(0)|p'q\rangle = \frac{-i}{(2\pi)^{3/2}} \int \frac{d^{3}p}{p_{0}+q_{0}} \int d^{3}x \langle p|T(J_{\mu}(0)\Psi(0))|0\rangle \tilde{T}_{x} e^{-i(p+q)x}$$

$$= \frac{-i}{(2\pi)^{3/2}} \int \frac{d^{3}p}{p_{0}+q_{0}} \int d^{3}x \langle p|T(J_{\mu}(0)\tilde{\eta}(x))|0\rangle \tilde{T}_{x} e^{-i(p+q)x}$$

where $\tilde{T}_{x} = (-i\gamma_{\mu}\gamma_{\nu}-m)$ is the Dirac operator and $\tilde{\eta}(x) = \tilde{\eta}(x)(-i\gamma_{\mu}\gamma_{\nu}-m)$ is the nucleon source current. In the second expression, an equal-time commutator term is omitted since it makes no contribution to the absorptive part. In order to obtain the absorptive part immediately, the time-ordered product can be replaced by the retarded commutator $	heta(-x_{0})[j_{\mu}(0),\tilde{\eta}(x)]$. Then the absorptive part is given by replacing
\[ i \theta(-x_0) \text{ by } \frac{i}{2}, \text{ as we now show by explicit calculation. We have the vertex expansion} \]
\[ \langle p | j_\mu(0) | p+q \rangle = \left[ \frac{1}{(2\pi)^\frac{3}{2}} \right] \sqrt{\frac{P^0}{p^0 + q^0}} \bar{u}(p) \left[ F_1(q^2, W) \gamma_\mu - \frac{i e g_\mu}{2m} F_2(q^2, W) \right] u(p+q) \]  
(3.2)

from which we can obtain under time-reversal and parity invariance of the form factors
\[ \langle p+q | j_\mu(0) | p \rangle^\dagger = \left[ \frac{1}{(2\pi)^\frac{3}{2}} \right] \sqrt{\frac{p^0}{p^0 + q^0}} \bar{u}(p) \left[ F_1^*(q^2, W) \gamma_\mu - \frac{i e g_\mu}{2m} F_2^*(q^2, W) \right] u(p+q) \]  
(3.3)

Thus, the imaginary part of the vertex is given by
\[ \tilde{\omega}(p) \text{Abs} \sum_{n} \langle n | p+q \rangle = \frac{1}{2i} \left[ \frac{1}{(2\pi)^\frac{3}{2}} \right] \sqrt{\frac{p^0}{p^0 + q^0}} \left[ \langle p | j_\mu(0) | p+q \rangle - \langle p+q | j_\mu(0) | p \rangle^\dagger \right] \]  
(3.4)

Then, by taking the nucleon out of the out-state, we obtain
\[ \langle p+q | j_\mu(0) | p \rangle^\dagger = \frac{i}{\sqrt{(2\pi)^3}} \int d^4x \langle p | \Theta(x) | j_\mu(0), \bar{\eta}(x) \rangle | 0 \rangle e^{-i(p+q)x} \]  
(3.5)

and finally we have for the absorptive part
\[ \tilde{\omega}(p) \text{Abs} \sum_{n} \langle p+q \rangle = -\frac{1}{2} \left[ \frac{1}{(2\pi)^\frac{3}{2}} \right] \sqrt{\frac{p^0}{p^0 + q^0}} \int d^4x \langle p | j_\mu(0), \bar{\eta}(x) \rangle | 0 \rangle e^{-i(p+q)x} \]  
(3.6)

Introducing complete sets of intermediate states \( n \) and \( n' \), we have
\[ \tilde{\omega}(p) \text{Abs} \sum_{n} \langle p+q \rangle = -\frac{1}{2} \left[ \frac{1}{(2\pi)^\frac{3}{2}} \right] \sqrt{\frac{p^0}{p^0 + q^0}} \int d^4x e^{-i(p+q)x} \left\{ \sum_{n} \left[ \frac{\delta_m^n}{(2\pi)^3} \langle j_\mu(0) | n \rangle \langle n | \bar{\eta}(x) \rangle | 0 \rangle e^{i\omega x} \right. \right. \]
\[ \left. \left. - \sum_{n'} \int \frac{d^4p'}{(2\pi)^3} \langle \bar{\eta}(0) | n' \rangle \langle n' | j_\mu(0) \rangle | 0 \rangle e^{i(p-x)} \right\} \]  
(3.7)

Provided the order of the \( d^4x \) integration can be interchanged with
the $d^3n$ and $d^3n'$ integrations, we obtain

$$
\tilde{u}(p) A_{\mu} \langle p, p' | q \rangle = -\frac{1}{2} (\pi m)^{3/2} \left\{ \sum_{n} \left[ \sum_{n'} \frac{d^3n'}{(2\pi)^3} \langle \psi_n | \psi_n' \rangle \langle \psi_n' | \psi_n \rangle \langle \psi_n | j_{\mu}(0) | \psi_n \rangle \right]
- \sum_{n'} \left[ \sum_{n} \frac{d^3n}{(2\pi)^3} \langle \psi_n | \psi_n' \rangle \langle \psi_n' | \psi_n \rangle \langle \psi_n | j_{\mu}(0) | \psi_n \rangle \right] \right\}
$$

Using the identities

$$
\delta(p_0 + q_0 - n_0) = \left[ \delta(p_0 + q_0 - n_0) + \delta(p_0 + q_0 + n_0) \right] \Theta(p_0 + g) 
= 2 \langle p_0 + q_0 \rangle \Theta(p_0 + g) \delta(w^2 - m_n^2)
$$

and

$$
\delta(q_0 + n_0) = 2 g_0 \Theta(-q_0) \delta(q^2 - m_{n'}^2)
$$

we are able to cast the vertex into suitable form for the examination of the intermediate states:

$$
\tilde{u}(p) A_{\mu} \langle p, p' | q \rangle = -\frac{1}{2} (\pi m)^{3/2} \left\{ \sum_{n} \left[ \sum_{n'} \frac{d^3n'}{(2\pi)^3} \langle \psi_n | \psi_n' \rangle \langle \psi_n | j_{\mu}(0) | \psi_n \rangle \right]
- \sum_{n'} \left[ \sum_{n} \frac{d^3n}{(2\pi)^3} \langle \psi_n | \psi_n' \rangle \langle \psi_n' | \psi_n \rangle \langle \psi_n | j_{\mu}(0) | \psi_n \rangle \right] \right\}
$$

Since the states $n'$ and $n$ couple to the photon through $j_{\mu}(0)$ in $\langle n' | j_{\mu}(0) | n \rangle$ and $\langle p | j_{\mu}(0) | n \rangle$, the states $n'$ have nucleon number zero whereas the states $n$ have nucleon number one. For the states $n'$, only the vacuum state can contribute because, for $q^2 < 0$, the argument of the delta function can vanish only for $m_{n'} = 0$. Furthermore, since

27. Sidewise dispersion relations can be proved for the form factors provided this interchange is permissible. Consult Appendix B for a proof of form factor analyticity.
\[ \langle \text{one nucleon} \mid \pi(0) \mid 0 \rangle = 0, \text{ even the vacuum state cannot contribute.} \]

 Had we allowed \( q^2 > 0 \), the situation would be reversed (then only the states \( n' \) would be nonvanishing) and would correspond to the \( t \)-channel dispersion approach.

 Also because \( \langle \text{one nucleon} \mid \pi(0) \mid 0 \rangle = 0 \), the state \( n \) cannot simply be a nucleon. As shown in Figure 5, the lowest intermediate states are various meson-baryon states and the \( J^P = \frac{1}{2}^\pm \) resonances. The most important of these intermediate states is the pion-nucleon state, and since it is the lowest mass state, the threshold of the unitarity cut occurs at \( W^2 = (m + \mu)^2 \).

 In general, the intermediate state \( n \) can consist of any number of particles, all on their physical mass shells, such that the total internal momentum is \( n = p + q = n_1 + n_2 + \ldots \) and such that the phase space appears as

\[
\left( \frac{1}{2\pi^3} \right)^N \left( \frac{m}{2\pi} \right)^2 \sum_{\eta_i} d^4 \eta_i \cdot \frac{d^4 \eta_i^{*}}{2\pi} \ldots \left[ \delta(n_i^* - m_i^*) \right]^N \left( \frac{2\pi}{g + h} \right)^N \langle p \mid \pi(0) \rangle \langle n \rangle \langle \pi(0) \mid 0 \rangle
\]

\[ (3.11) \]

 We proceed with our calculations by setting the internal baryon momentum equal to \( \Lambda \) and the internal meson momentum equal to \( k \) and write for the absorptive part

\[
\left( \frac{1}{2\pi^3} \right)^N \left( \frac{m}{2\pi} \right)^2 \sum_{\eta_i} d^4 \theta \cdot \frac{d^4 \theta^{*}}{2\pi} \left[ \delta(\xi - m_i^*) \right]^N \left( \frac{2\pi}{g + h} \right)^N \langle p \mid \pi(0) \rangle \langle n \rangle \langle \pi(0) \mid 0 \rangle
\]

\[ (3.12) \]
Figure 5. Intermediate states contributing to the nucleon anomalous moments.
Setting
\[ d^{4}l = d\xi d\eta d\xi' d\eta' = \pi d\chi \theta(d\xi), \]

\[ \delta(\chi^{2} - m_{1}^{2}) = \delta \left( \frac{(\chi^{2} + m_{1}^{2} - \mu^{2})^{2}}{4\chi^{2}} - m_{2}^{2} - \xi^{2} \right) \]

and
\[ \delta(\chi^{2} - \mu^{2}) = \frac{1}{2\chi} \delta \left( \frac{\chi^{2} + m_{1}^{2} - \mu^{2}}{2\chi} - l_{0} \right) \]

we obtain
\[ \overline{u}(p) A_{\mu}(p,q+q') = \frac{\pi^{2} A_{\mu}}{m^{2}} \delta \left( \frac{\chi^{2} + m_{1}^{2} - \mu^{2}}{2\chi} \right) \frac{1}{w} \frac{1}{l_{0}} \int d\chi \langle p|j_{\mu}(0)|n\rangle \langle n|\overline{u}(q')|0\rangle \] (3.14)

The matrix elements \( \langle p|j_{\mu}(0)|n\rangle \) and \( \langle n|\overline{u}(0)|0\rangle \) can be obtained from experimental observations, the first directly from the electroproduction \( (q^{2}<0) \) amplitude and the second indirectly from the meson-baryon scattering phase shifts. Since we are only interested in calculating the anomalous moment, we need to retain only the photoproduction \( (q^{2} = 0) \) amplitude.

For the photoproduction amplitude \( \langle n|\overline{v}_{p}\rangle \), the T-matrix and reduction formalism expressions can be equated to give

\[ \varepsilon_{\mu} \langle n|j_{\mu}(0)|p\rangle = \frac{1}{C_{F} A_{q}^{2} \sqrt{2} k_{e} \sqrt{2} \gamma_{\mu} \varepsilon_{\mu}} \frac{1}{l_{0}} \int d\chi \varepsilon_{\mu}(\omega) \varepsilon_{\mu} \overline{u}(q) \varepsilon_{\mu} T_{\mu} \psi(\omega) \] (3.15)

where the T-matrix has the expansion

\[ \varepsilon_{\mu} T_{\mu} = \xi_{i} A_{i} M_{i} \] (3.16)

in terms of invariant structure functions \( A_{i} \) and a gauge invariant
basis \( M_j \) given by

\[
\begin{align*}
M_1 &= \frac{1}{2} i \gamma_5 \{ \gamma, \gamma \} \\
M_2 &= 2 i \gamma_5 \{ P, \gamma \} \\
M_3 &= \gamma_5 \{ \gamma, \gamma \} \\
M_4 &= 2 \gamma_5 \left( \{ \gamma, P \}^2 - \frac{1}{2} i m \{ \gamma, \gamma \} \right)
\end{align*}
\]

(3.17)

where \( \{a,b\} = (ae)(bq)-(aq)(be) \) is a gauge invariant combination, \( P = \frac{1}{2}(p+\ell) \), and the \( M_j \) are identical with the standard CGLN\(^{28}\) basis for photoproduction.

Under the time reversal operation, \( <n|\gamma_0|p> \) becomes the time-reversed photoproduction amplitude \( <\gamma_0|n> \) and thus \( \epsilon_\mu <n|j_\mu(0)|p> \) becomes\( -\epsilon_\mu^* <|p|j_\mu(0)|n> \). This implies the time-reversed T-matrix \( \epsilon_\mu^* T\mu^* \), in which the order of the matrices is reversed, has a relative negative sign, that is,

\[
\epsilon_\mu^* <p|j_\mu(0)|n> = -\frac{1}{(2\pi)^{3/2}} \frac{1}{(2k_0)^{1/2}} \frac{1}{P^0} \frac{1}{E_0} \tilde{u}(p) \epsilon_\mu^* T\mu^* u(\ell)
\]

(3.18)

Assuming the structure functions are real in the physical region, we find that the time reversal operation changes the sign of all but the second basis function.

The pseudoscalar meson-baryon vertex \( <n|\bar{\eta}(0)|0> \) is obtained from the adjoint operation \( <0|\bar{\eta}(0)|n>^+\gamma_0 \) where under Lorentz invariance \( <0|\bar{\eta}(0)|n> \) has the form

\[
<0|\bar{\eta}(0)|n> = \epsilon_\delta \sqrt{\gamma_5} \frac{1}{\sqrt{2k_0}} \Gamma \gamma_5 u(\ell)
\]

(3.19)

---

where

\[ \Gamma(w) = \frac{\Phi'(-w)}{2w} K(w) + \frac{w - \frac{\Phi'(-w)}{2}}{2w} K'(w) \]  (3.20)

or

\[ \Gamma(w^2) = \frac{\Phi'(-w^2)}{2w^2} K'(w^2) + \frac{w - \frac{\Phi'(-w^2)}{2}}{2w^2} K''(w^2) \]  (3.21)

and \( g \) is the meson-baryon coupling constant.

We are now ready to extract the imaginary part of the second form factor from the absorptive part of the electromagnetic vertex through the projection operation

\[ \text{Im} F_2(w) = -2m \text{Tr} \bar{u}(p) \text{Abs} \Gamma_{\mu}(p, p + q) \nu^{(a)}(w) u(p) \]  (3.22)

or

\[ \text{Im} F_2(w^2) = -2m \text{Tr} \bar{u}(p) \text{Abs} \Gamma_{\mu}(p, p + q) \nu^{(a)}(w^2) u(p) \]  (3.23)

where the projection operators \( \nu^{(a)}(w) \) and \( \nu^{(a)}(w^2) \) are given in Appendix A. Thus, inserting the expressions developed above for the matrix elements into the expression for \( \bar{u}(p) \text{Abs} \Gamma_{\mu} \) in (3.14) and projecting, we find

\[ \text{Im} F_2(w) = i g F(w^2) \frac{m \cdot m}{8\pi} \int_{-1}^{+1} \text{Tr} \bar{u}(p) \Gamma_{\mu}(\omega) \bar{u}(\omega) \kappa_{s} K_{s}(w^2) \nu^{(a)}(w) u(p) \]  (3.24)
and

\[ \text{Im} F_2(w^2) = i g \rho(w^2) \frac{m_1 m_2}{\hat{s}} \int_{-1}^{1} d\chi \tilde{u}(\chi) \bar{T}_\mu u(\chi) \bar{r}_5 \gamma_\mu \gamma_5 \]

\[ \times \left\{ K_+^R(w^2) \left[ \frac{w^2 + m_1^2 - 2m_1(p+q)}{2m} A_\mu + \frac{w^2 - m_1^2}{2m} B_\mu \right] \right\} \]

\[ + K_-^R(w^2) \left[ \frac{w^2 - m_1^2}{2m} A_\mu + \frac{w^2 - m_1^2}{2m} (p+q) B_\mu \right] \]  

(3.25)

where

\[ \rho(w^2) = \left( \frac{w^2 + m_1^2 - \mu^2}{4w^2 m_1^2} \right) \frac{1}{w} = \frac{2I(w)}{w} \]  

(3.26)

is a phase space factor.

In the perturbative context, point couplings are used and the above expressions reduce to

\[ \text{Im} F_2(w) = i g \rho(w^2) \frac{m_1 m_2}{\hat{s}} \int_{-1}^{1} d\chi \tilde{u}(\chi) \bar{T}_\mu u(\chi) \bar{r}_5 \gamma_\mu \gamma_5 \]

(3.27)

and

\[ \text{Im} F_2(w^2) = i g \rho(w^2) \frac{m_1 m_2}{\hat{s}} \int_{-1}^{1} d\chi \tilde{u}(\chi) \bar{T}_\mu u(\chi) \bar{r}_5 \gamma_\mu \gamma_5 \gamma_5^{(c)}(w^2) \]

(3.28)

upon setting \( K(W) = K_+(W^2) = K_-(W^2) = 1 \).

In the next chapter, the calculation of the nucleon magnetic moments is presented in the perturbative context (see Figure 6) using the absorptive amplitude which we now calculate from the perturbative photopion amplitude \( (q^2 = 0) \). With the S-matrix expansion given by

\[ S = \frac{(\gamma_5 \rho)^L}{2!} \int_{x_1} dx_2 T \{ H(x_1) H(x_2) \} \]  

(3.29)
the electromagnetic interaction by

\[ H^{\text{em}}(\omega) = e g_{\mu \nu} A_{\mu}(\omega) + i e A_\mu(\omega) \left( \pi^+ \pi^- - \pi^0 \pi^0 \right) \]

for the nucleon and pion poles, and the strong interaction for the \( \eta^+ \) state by

\[ H^{\text{strong}}(\omega) = i \sqrt{2} g_{\pi N} \bar{\psi}(x) \gamma_5 \psi(x) \pi^-(x), \]

we obtain for the T-matrix, defined as

\[ S = 1 - i \frac{\pi m^4}{\left(2\pi^2\right)^2} \frac{\delta(p+q-k-e)}{\sqrt{2q_0} \sqrt{2k_0} \sqrt{p_0} \sqrt{e_0}} \tilde{u}(e) T u(c, p), \]

the expression

\[ T = \sqrt{2} e g_{\pi N} \left[ \gamma_5 \frac{i}{p+k-m} \not{\!q} + \gamma_5 \frac{i}{(q-k)^2 - \mu^2} \left(2k^2-q^2\right) \right] \]

Figure 6. Perturbative context for the nucleon moments
We simplify this expression to

\[ T_\mu = \sqrt{2} g_{\pi N} \left[ \frac{-i\gamma_5 \gamma_\mu \gamma_0}{2(q^2)} + \frac{i\gamma_5}{(p^2)(q^2)} (p_\mu(kq) - k_\mu(pq)) \right] \]  

(3.34)

and noting that \( p_\mu(kq) - k_\mu(pq) = p_\mu(kq) - k_\mu(pq) \), and that \( M_\mu^2 = i\gamma_5 \gamma_\mu \gamma_0 \) and \( M_\mu^2 = 2i\gamma_6(p_\mu(kq) - k_\mu(pq)) \), we can identify

\[ A_1 = -\frac{\sqrt{2} g_{\pi N}}{2(q^2)} = -\frac{\sqrt{2} g_{\pi N}}{(s-m^2)} \]

and

\[ A_2 = \frac{\sqrt{2} g_{\pi N}}{2(q^2)(kq)} = -\frac{2\sqrt{2} g_{\pi N}}{(s-m^2)(t-m^2)} \]  

(3.35)

Thus, the photopion amplitude has only two invariant amplitudes.

Working in the \( W^2 \) plane, we insert the expression for the projection operator \( \nu_{J}^{(2)}(q^2 = 0, W) \) into (3.28) to obtain

\[ \text{Im} F_2(q^2 = 0, W^2) = \frac{i\sqrt{2} g_{\pi N}}{8\pi} \rho(w^2) \frac{m}{q(w^2-w^3)^2} \int_{-\frac{1}{2}}^{+\frac{1}{2}} dx \text{Tr}(p+m) T_\mu(k+m) \gamma_5(qk) \gamma_\mu \gamma_0 \]  

(3.36)

We first show that the pion pole does not contribute to the anomalous moment. This is not unexpected since the projection operator essentially performs a partial wave analysis, keeping only the \( S \) and \( P \) wave contributions for the \( J = \frac{1}{2}, T = \frac{1}{2} \) intermediate state. The trace calculation for the pion pole is

\[ \text{Tr}(p+m) \gamma_5(qk) \gamma_5(qm) \gamma_\mu(qk) = \text{Tr}(-2m(pq) + 2m^*(pq)) \left[ q(qk) - k(pq) \right] = 0 \]  

(3.37)
For the contributing nucleon pole, we note that

\[(\not{\xi} + m)^2 \gamma_\mu \gamma_5 (\xi^2 + m^2) \gamma_5 (\not{\xi} + m) \gamma_\mu = (-2\xi^2 + 4m^2) \gamma_5 (-\xi^2 + m^2) \gamma_5 \]

in order to obtain for the trace calculation

\[Tr (-2\xi^2) \gamma_5 (-\xi^2 + m^2 - m^2 + m^2 \gamma_5) \gamma_5 = 8 - 2(pq) \left[ m(q\xi) - m_1(pq) \right] \]

The imaginary part then becomes

\[\text{Im} F_2(q^2, w^2) = \frac{g^2}{4\pi} \rho(w^2) m \frac{m}{(w^2 - m^2)^2} \int_0^\infty dx \left[ m_1(pq) - m(q\xi) \right] \]

\[= \frac{1}{2} \frac{g^2}{4\pi} \rho(w^2) E_p(w^2) \]

where

\[E_p(w^2) = \frac{2m}{(w^2 - m^2)} \left[ m_1 - m \frac{m_1 w}{2w_2} (w^2 + m_1^2 - \mu^2) \right] \]

For the neutron, we similarly find from the crossed channel pole in

\[\gamma_n \rightarrow \pi^+ n^-\]

\[\text{Im} F_2(w^2) = \frac{1}{2} \frac{g^2}{4\pi} \rho(w^2) E_n(w^2) \]

where

\[E_n(w^2) = \frac{2m}{(w^2 - m^2)} \left[ m - \frac{m_1 w^2}{(w^2 + m_1^2 - \mu^2)} \frac{Q_0}{L_1} \ln \left( \frac{1 + \frac{i\gamma}{L_0}}{1 - \frac{i\gamma}{L_0}} \right) \right] \]

with

\[L_0 = (w^2 + m_1^2 - \mu^2) / 2w \]
Also in the context of point couplings, we can introduce resonances into the photopion amplitude (see Figure 7) for the low energy region by working with the nonrelativistic CGLN amplitudes $\tilde{F}_L$. These amplitudes have multipole expansions in terms of derivatives of Legendre polynomials which permits us to perform the angular integration. We then need the nonrelativistic reductions

$$\tilde{u}(p)\tilde{T}_\mu u(p) = 4\pi \frac{W^L}{m} \chi^L \tilde{F}_L^k \chi$$

(3.44)

where $k = 1,2,3$ and where

$$\tilde{F}_L = i(\sigma \cdot \varepsilon^*) \tilde{F}_T - \frac{\sigma \cdot \varepsilon^* \sigma \cdot \varepsilon^*}{4\pi} \frac{i(\sigma \cdot \varepsilon^*) \sigma \cdot \varepsilon^*}{4\pi} \frac{i(\sigma \cdot \varepsilon^*) \sigma \cdot \varepsilon^*}{4\pi} \frac{i(\sigma \cdot \varepsilon^*) \sigma \cdot \varepsilon^*}{4\pi}$$

(3.45)

$$\tilde{u}(p)\tilde{Y}_S u(p) = \frac{1}{2m} \chi^L \sigma \cdot (\varepsilon^* - \varepsilon^*) \chi$$

(3.46)

$$\tilde{Y}_\mu^{(q)}(w) = \frac{m^2}{(w^2 - m^2)^2} \sum_{\text{spins}} u(q)\tilde{u}(p)i\sigma_\mu\sigma_q u(p)$$

$$= \frac{m^2}{(w^2 - m^2)^2} \tilde{u}(p)\chi^L \frac{(w^2 + m^2)}{2mw} i(\sigma \cdot \varepsilon^*)^k \chi$$

(3.47)

The absorptive part becomes in the nonrelativistic approximation

$$\text{Im} F_2(w) = -\frac{1}{2} \frac{g_{\pi N}^2}{4\pi} \rho(w) \frac{W^2}{(w^2 - m^2)^2} \int_{-\infty}^{\infty} \frac{dx_1}{2} \frac{4\pi}{4\pi} \sigma \cdot (\varepsilon^* - \varepsilon^*) \frac{(\sigma \cdot \varepsilon^*)}{(w^2 - m^2)} \tilde{F}_L$$

(3.48)

Upon performing the trace calculation, we find

$$F_{\infty} = \frac{1}{2} \text{Tr} (\sigma \cdot q \cdot \tilde{F}_L) (\sigma \cdot q \cdot q)$$

$$= 2 (\sigma \cdot (\sigma \cdot q \cdot q) \tilde{F}_L - (\sigma \cdot q \cdot q) \tilde{F}_L - (\sigma \cdot q \cdot q) \tilde{F}_L)$$

(3.49)

Figure 7. Approximation of the pion-nucleon contribution with a constant pion-nucleon form factor.

The $S_i$ have the above mentioned expansion

\[
S_i = \sum_{l=0}^{\infty} \left\{ \left[ (l+1) E_{l+} + E_{l+} \right] P_{l+}^l(x) + \left[ (l+1) M_{l+} + E_{l+} \right] P_{l+1}^l(x) \right\}
\]

\[
S_i^2 = \sum_{l=1}^{\infty} \left[ (l+1) M_{l+} + 2 M_{l-} \right] P_{l+}^l(x)
\]

\[
S_i^3 = \sum_{l=1}^{\infty} \left\{ \left[ E_{l+} - M_{l+} \right] P_{l+1}^l(x) + \left[ E_{l+} + M_{l-} \right] P_{l+1}^l(x) \right\}
\]

\[
S_i^4 = \sum_{l=1}^{\infty} \left[ M_{l+} - E_{l+} - M_{l-} - E_{l-} \right] P_{l+}^l(x)
\]

Retaining only the $S$ and $P$ wave multipoles, these become

\[
S_i = E_{l+} + 3 \lambda (M_{l+} + E_{l+})
\]

\[
S_i^2 = 2 M_{l+} + M_{l-}
\]

\[
S_i^3 = 3 (E_{l+} - M_{l+})
\]

\[
S_i^4 = 0
\]

Finally, we perform the angular integration to obtain an expression
The behavior of the pion-nucleon form factors may be obtained by application of the same procedures used for the electromagnetic form factors. We begin by taking the pion out of the in-state to obtain

\[ \langle 0 | \eta(\omega) | n \rangle = \frac{2}{(2\pi)^2 \sqrt{2E}} \int d^4k e^{-ikx} \langle 0 | \Theta(-x) [\eta(\omega), J(x)] | n \rangle \]

where the equal-time commutator term is omitted and \( J(x) = (\Box + \mu^2) \phi(x) \) is the meson source current. Introducing intermediate states, we find

\[ \langle 0 | \eta(\omega) | n \rangle = \frac{2}{(2\pi)^2 \sqrt{2E}} \int d^4k e^{-ikx} \Theta(-x) \left\{ \sum \frac{d^4k_1}{(2\pi)^4} \langle 0 | \eta(\omega) | n_1 \rangle \langle n_1 | J(x) | n \rangle \right\} \]

\[ - \sum \frac{d^4k_2}{(2\pi)^4} \langle 0 | J(x) | n_2 \rangle \langle n_2 | \eta(\omega) | 0 \rangle \]

The set of states \( n_2 \), analogously to the states \( n_1 \) in the electromagnetic vertex, do not contribute to the absorptive part, which we now write as

\[ \text{Abs} \ P(\omega) Y_\omega \mu = \frac{-i}{9} \frac{1}{(2\pi)^4 \sqrt{m_i}} \frac{1}{2} \int d^4N \sum d^4k \Theta(-x) S(ke-x-N) \]

\[ \times \left\{ \bar{\psi} \gamma_{\mu} \bar{\psi} \psi \Theta(\omega) \Theta(\eta) \langle n_2 | J(x) | n_1 \rangle \langle n_1 | J(x) | n_2 \rangle \right\} \]

assuming the pion-nucleon intermediate state dominates (\( \pi \) and \( N \) refer, respectively, to the pion and nucleon momenta; see Figure 8). Upon integrating the phase space, we obtain
The absorptive part for the form factor is next obtained by projection, as defined in the condition

$$\text{Im } K(\pm w) = \text{Tr } \tilde{u}(\vec{p}) U(\pm w) \text{Abs } F(w) Y_S u(q)$$

(3.57)

Figure 8. Elastic approximation for the pion-nucleon vertex in the sidewise dispersive approach.

where

$$V(\pm w) = -\frac{g}{\sqrt{2} m_N} \frac{1}{2\pi} \sqrt{\frac{m}{N_0}} \frac{1}{16\pi} \frac{\hat{W}}{(\hat{W}-m)^2 - \mu^2}$$

We thus obtain

$$\text{Im } K(w) = -\frac{1}{2\pi e^4} \frac{g^2}{m_N} \frac{1}{2\pi} \sqrt{\frac{m}{N_0}} \frac{1}{16\pi} \frac{\hat{W}}{(\hat{W}-m)^2 - \mu^2}$$

(3.58)

$$\times K(w) \int d\xi \text{Tr } \tilde{u}(\vec{p}) Y_S (\vec{p}+\vec{q}+\vec{w}) Y_S u(q) \langle m_1 J(0) | 1 \rangle$$
Since the state $n_1 = \pi + N$ in the matrix element $\langle n_1 | J(0) | \ell \rangle$ labeled by the "in" convention, we obtain $\langle n_1 | J(0) | \ell \rangle$ by taking the hermitian conjugate of the pion-nucleon scattering amplitude $\langle \ell | J(0) | n_1 \rangle$. Using the reduction technique, we obtain

$$\langle \ell | J(0) | m_i \rangle = \frac{-i}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{m_{\ell}}{2\lambda}} \sqrt{\frac{m_i}{2\lambda}} \tilde{u}(q) \tilde{T} u(N)$$

(3.59)

where the $T$-matrix is given by

$$T = -A + \frac{1}{2} (\kappa + \kappa') B$$

(3.60)

where $A$ and $B$ are two invariant functions.

Thus the absorptive part becomes

$$\text{Im} K(w) = \frac{|w|^2}{w} \left[ \frac{m^2}{(w - m)^2 - \mu^2} \right] K(w)$$

$$\times \int_{-1}^{1} d \cos \theta \, \text{Tr} \left[ \tilde{u}(\theta) (\frac{-w + q + \mu}{w + q + \mu}) \tilde{u}(\theta) \right] \left( -A + \frac{1}{2} (\kappa + \kappa') B \right) \kappa(q)$$

(3.61)

For the coefficient of $A^*$, the integrated trace is

$$\int_{-1}^{1} d \cos \theta \, \text{Tr} \left[ \tilde{u}(\theta) (\frac{-w + q + \mu}{w + q + \mu}) \tilde{u}(\theta) \right] = \frac{2}{w} \left[ (w - m)^2 - \mu^2 \right]$$

(3.62)

and similarly for the coefficient of $B^*$

$$\frac{1}{2} \int_{-1}^{1} d \cos \theta \, \text{Tr} \left[ \tilde{u}(\theta) (\frac{-w + q + \mu}{w + q + \mu}) \tilde{u}(\theta) \right] = \frac{-2(w + m)}{w} \left[ (w - m)^2 - \mu^2 \right]$$

(3.63)

We thus have

$$\text{Im} K(w) = \left[ \frac{2}{w} \int_{-1}^{1} d \cos \theta \, \kappa(q) \right] K(w) = \left[ e^{i \delta_{L}(\omega)} \sin \delta_{L}(\omega) \right] K(w)$$

(3.64)
Similarly, we find for negative energies

\[
\text{Im} K(c \omega) = \left( \pi i \int_{0+}^{c \omega} K(c \omega) \, d\omega \right) = \left[ e^{i \delta_{o+}(\omega)} \sin \delta_{o+}(\omega) \right] K(-\omega) \quad (3.65)
\]

Using the MacDowell\textsuperscript{30} symmetry relation \( \delta_{o+}(\omega) = \delta_{1-}(-\omega) \), we generalize this expression to the form

\[
\text{Im} K(\pm \omega) = \left[ e^{i \delta(\pm \omega)} \sin \delta(\pm \omega) \right] K(\pm \omega) \quad (3.66)
\]

where \( \delta(\omega) \) is a P wave phase shift and \( \delta(-\omega) \) is an S wave phase shift.

The reappearance of the form factor in the absorptive part leads to coupled integral equations of the Omnes\textsuperscript{31} type when unsubtracted dispersion relations are written over \( \text{Im} K(\pm \omega) \). We can omit this formal method of solution, however, by the following procedure. We note that the identity

\[
K(\pm \omega \pm i \varepsilon) = K(\pm \omega \pm i \varepsilon) = (1 - e^{-i \delta(\pm \omega)}) K(\pm \omega \pm i \varepsilon) \quad (3.67)
\]

reduces to

\[
\frac{K(\pm \omega \pm i \varepsilon)}{K(\pm \omega \mp i \varepsilon)} = e^{2i \delta(\pm \omega)} \quad (3.68)
\]

which shows that the ratio of the form factors above and below the cut


is an analytic function. Taking the logarithm of the above function and using the Schwarz reflection principle, we obtain

$$\text{Im} \ln K(\tau w) = \delta(\tau w)$$  \hspace{1cm} (3.69)

Thus, writing a subtracted dispersion relation for the logarithmic function and then exponentiating, we obtain for the pion-nucleon form factor

$$K(\tau w) = \exp \frac{\tau w - m}{\mu} \int_0^\infty dw' \left[ \frac{\delta(w')}{(w'-w)(\tau w'-m)} + \frac{\delta(-w')}{(w'+w)(\tau w'-m)} \right]$$  \hspace{1cm} (3.70)
IV. DEVELOPMENT OF THE RADIATIVE CONCEPT FOR THE NUCLEON ANOMALOUS MOMENTS IN THE SIDEWISE DISPERSSIVE FRAMEWORK

As was shown in the calculation of the Schwinger correction for the electron anomalous moment, the sidewise dispersive framework provides the natural dispersive setting for the calculation of radiative corrections. The radiative picture, for example, does not even develop in the photon channel dispersion relations. For the nucleon anomalous moments there are obvious parallels, and we thus develop in this chapter the radiative concept for the nucleon anomalous moments in the sidewise dispersive framework.

We begin, in analogy to the calculation of the Schwinger correction, with the investigation of the nucleon anomalous moments in the lowest order perturbative context. In a physical sense, we are simply replacing the internal, electromagnetically coupled photon by a strongly coupled pion. For a massless pion, in fact, we can simply replace $e^2/4\pi$ in the Schwinger correction by $g^2_{\pi N}/4\pi$ and obtain a good approximation to the proton anomalous moment.

We demonstrate this explicitly in the context of the perturbative calculations for the nucleon anomalous moments. The absorptive parts, calculated in Chapter III, are given by

$$\text{Im} \mathcal{F}_2(w^2) = \frac{1}{2} g^2_{\pi N}/\pi \rho(w^2) E(w^2)$$ (4.1)
For the nucleon moments, we make the approximations for vanishing pion mass ($\mu^2 = 0$)

\[ E_p(w^2) = \frac{m^2}{w^2} \]  \hspace{1cm} (4.5)

\[ E_n(w^2) = \frac{2m^2}{(w^2 - m^2)} \left[ 1 - \frac{w^2}{w^2 + m^2} \ln \left( \frac{1 + \frac{2w^2}{m^2}}{1 - \frac{2w^2}{m^2}} \right) \right] \]  \hspace{1cm} (4.6)

where \[ \frac{\ln \left( \frac{1 + \frac{2w^2}{m^2}}{1 - \frac{2w^2}{m^2}} \right)}{(\frac{2w^2}{m^2})} \geq 2 \quad \text{and} \quad \rho(w^2) = \frac{w^2 - m^2}{w^2}.

Inserting these approximations into the dispersion integral

\[ \mu = \frac{1}{\pi} \int \frac{d w^2}{w^2 - m^2} \frac{\text{Im} \, \Pi_2(w^2)}{(m^2 + m^2)^2} \]  \hspace{1cm} (4.7)

we obtain

\[ \mu_p = \frac{1}{2\pi} \frac{g^2}{\pi^2} \frac{1}{4\pi^2} \left( 1 - \frac{1}{\lambda} \right) \left( 1 + \frac{m^2}{M^2} \right)^2 \]  \hspace{1cm} (4.8)
and

$$\mu_n = -\frac{1}{2\pi} \frac{g_{nn}^2}{4\pi} \ln \left( \frac{\Lambda}{\mu_n} \right) \left( 1 + \frac{\mu_n}{\Lambda} \right)^2$$  \hspace{1cm} (4.9)$$

With the cutoff removed and the threshold relaxed to the Kroll-Ruderman point \((\frac{\mu}{m} = 0)\), we arrive at the Schwinger form for the proton anomalous moment

$$\mu_p = \frac{1}{2\pi} \frac{g_{nn}^2}{4\pi}$$ \hspace{1cm} (4.10)$$

If the physical threshold is retained, we obtain an even better approximation \((\mu_p = \frac{1}{2\pi} \frac{g_{nn}^2}{4\pi} 3 = 1.75)\) to the physical proton moment \((\mu_p = 1.79)\). For the neutron, however, the simple perturbative context for the radiative concept does not provide a good approximation to the physical neutron moment \((\mu_n = -1.91)\). It gives an overestimate \((\mu_n = -2.6)\) which leads us to suspect that the inclusion of resonances in the photoproduction amplitude will diminish the Born pole behavior of \(E_n(W^2)\). Likewise, we expect the inclusion of resonances to affect \(E_p(W^2)\) in the same fashion and thus open the way for contributions from higher intermediate states.

As was shown in Figure 5, the lowest lying pion-nucleon intermediate state is accompanied by other higher meson-baryon states and a string of \(J^P = \frac{1}{2}^\pm\) resonances. It is difficult at this time to estimate the contributions of these higher intermediate states since partial wave analyses are not currently available. One can, however, compare the

total experimental photoproduction cross sections\textsuperscript{33} and meson-baryon couplings at the vertices for the various meson-baryon states to obtain relative estimates.

When this is done, the pion-nucleon state is clearly seen to be dominant. The photoproduction cross sections for the higher meson-baryon states are several times smaller and peak at much higher energies than the pion-nucleon photoproduction cross section. In addition, these meson-baryon states generally have smaller coupling constants.

These considerations thus indicate that the pionic states, such as $\pi^0 N$ and $\pi N$, give the next most important contributions and that the heavier mesonic states, such as $\pi N$, $\rho N$, and $K^0 \Lambda$, give the least important contributions. To be specific, the pionic states have couplings on the same order as the pion-nucleon coupling and their photoproduction cross sections are down by a factor of about two. The heavier mesonic states have not only generally smaller couplings ($g_{\rho N} \sim \frac{1}{2} g_{\pi N}$, $g_{NN\pi} = \sqrt{3} (4\alpha - 3) g_{\pi N} \sim \frac{1}{2} g_{\pi NN}$, and $g_{N\Lambda K} = \frac{1}{\sqrt{3}} (3 - 2\alpha) g_{\pi N} \sim g_{\pi N}$) but also much smaller photoproduction cross sections.

Provided the $J^P = \frac{1}{2}^\pm$ resonances make only small contributions, the above pattern of indications can be considered as suggestive of a concept of low energy radiative mesonic corrections for the anomalous baryon moments. In regard to the $J^P = \frac{3}{2}^\pm$ resonances, it is possible that these resonances can be considered as chiral pairs giving contributions of opposite signs\textsuperscript{34}.

\textsuperscript{33} Consult, for example, "Proceedings of the 1967 International Symposium on Electron and Photon Interactions", op. cit.

Since experimental information seems to be available only for the pion-nucleon state, we turn now to a more thorough investigation of this dominant state in order to complete the development of the radiative concept. We begin by estimating the effect of resonances upon the absorptive amplitude \( E(w^2) \) in the low energy region in terms of phenomenological photoproduction multipoles. As was shown in Chapter III, the absorptive part is given by

\[
\text{Im} F_2(w) = \frac{1}{2} \frac{9 \pi}{4 \pi} q(w^2) E(w^2)
\]

(4.11)

where

\[
E(w^2) = 4 \pi \alpha \frac{W^2 + m^2}{(W^2 - m^2)^2} 4 \left( q^2 E_{\alpha+} - \frac{1}{2} \frac{|q|}{|r|} M_{j+} + \frac{1}{2} |q| |r| M_{j-} \right)
\]

(4.12)

Since we are interested only in a rough description, we let the pion mass vanish \((m^2 = 0)\), giving

\[
E(w^2) = 4 \pi \alpha \frac{W^2 + m^2}{W^2} \left( E_{\alpha+} - \frac{1}{2} M_{j+} + \frac{1}{2} M_{j-} \right)
\]

(4.13)

To obtain the behavior of \( E(w^2) \), we extract the multipoles from Walker's phenomenological photoproduction model\(^{35}\). In this model, partial wave helicity elements are constructed from nonrelativistic Born pole amplitudes and complicated Breit-Wigner resonance forms (see Figure 9). The helicity elements are linear combinations of partial wave helicities of opposite final state helicities. They have a definite parity and are related to the CGLN multipole coefficients \( E_{\alpha\pm} \) and \( M_{\pm} \), where \( j = \ell \pm \frac{1}{2} \). In particular, we need the relations

\[ E_{0^+} = A_{0^+}, \quad M_{1^-} = A_{1^-}, \quad \text{and} \quad M_{1^+} = \frac{1}{2}(A_{1^+} - \frac{3}{2}B_{1^+}), \] where \( E_{0^+}, \ M_{1^-}, \) and \( M_{1^+} \) are dominated by the \( S_{11}(1550), \ P_{11}(1470), \) and \( P_{33}(1238), \) respectively.

The Breit-Wigner resonant form used is

\[ A(w) = A(w_0) \frac{k_0 q_0^2}{|k_0 l|^2} \left( \frac{w_0^2}{s_0 - s - iW_0} \right) \]

where

\[ \Gamma = \Gamma_0 \left( \frac{k}{k_0} \right)^2 \left( \frac{k^2 + x^2}{k^2 + x^2} \right)^2 \]

and

\[ \Gamma_y = \Gamma_0 \left( \frac{q}{q_0} \right)^2 \left( \frac{q^2 + x^2}{q^2 + x^2} \right) \]

where \( W_0 \) is the resonance energy, \( q_0, \ k_0, \) and \( s_0 \) are the values of \( q, \ k, \) and \( s \) at the resonance energy \( W = W_0. \) In addition to the Born poles and resonances, small contributions having a smoothly varying behavior are added arbitrarily.

As shown in Figure 10, a plot of the helicity elements for both neutron and proton can be fitted by a \( 1/W^3 \) parametrization. Requiring the Kroll-Ruderman theorem \( (E(m^2) = 1) \) to be satisfied, we obtain

\[ E(w^2) = \frac{W^2 + m_\text{n}^2}{2W^2} \frac{m_\text{p}^3}{W^3} \]

This gives for the magnitude of the nucleon moment \( (\mu = \mu_p = |\mu_n|) \)

\[ \mu_\text{N}(\lambda) = \frac{1}{2\pi} \frac{q_n^2}{4^\pi} \int \frac{dW^2}{W^2} E(w^2) \]

\[ = \frac{1}{2\pi} \frac{q_n^2}{4^\pi} \left[ \left( \frac{2}{3} \left( -\frac{1}{\lambda^2} \right) \right) + \frac{v}{5} \left( -\frac{1}{\lambda^2} \right) \right] \]

\[ = \frac{1}{2} \left[ \left( \frac{1}{\lambda^2} \right) + \frac{9}{20} \left( \frac{1}{\lambda^2} \right) \right] \]
In Figure 11, the nucleon moment function $\mu(\Lambda)$ shows a saturation near the end of the low energy region (about twice the threshold energy). Choosing, for example, $\Lambda = 3$ ($W \sim 1.9$ Bev), we find $\mu = .65$. In comparison, the simple pole model gives $\mu_P = 1.2$ for the same cutoff, indicating that the resonance contributions subtract away a large part of the Born amplitude. In a similar calculation (which we discovered later), Love and Rankin$^{36}$ use the Walker multipoles and the $W$ plane dispersion relation to obtain $\mu_P = .6$ in agreement with our estimate.

Since these results are too small for the required dominance of the pion-nucleon intermediate state, we are forced to conclude that the

Figure 10. Multipole amplitude $E'(W)$ in terms of the Walker model multipoles.

Figure 11. Nucleon anomalous moment $\mu(\Lambda)$ as extracted from the Walker model.
pion-nucleon form factor must exhibit a peaked behavior in the low energy region which would compensate for the rapidly falling photoproduction factor. In this manner, we could explain the moderately successful results obtained by Drell and Pagels\textsuperscript{37} in the threshold dominance approach where the amplitude $E(W^2)$ is approximated by the coupling constant and the photoproduction amplitude by its threshold limit.

The above interpretation also accounts for the partial success obtained by Pagels\textsuperscript{38} in evaluating the octet baryon moments by the use of physical threshold amplitudes and SU(3) coupling constants for the various meson-baryon intermediate states. Although Pagels\textsuperscript{1} results are only in fair agreement with the SU(3) predictions, we note that the simple pole model gives even worse results since the amplitude $E_n(W^2)$ for the crossed channel poles is too large. In addition, if a low energy cutoff is employed, the results are not consistent for all the octet moments and a similar matching cutoff on $E_p(W^2)$ cannot be employed since $E_p(W^2)$ seems to give, at least in an average sense, a good approximation to the physical amplitude.

We turn now to a dispersive calculation of the pion-nucleon form factors which shows the required peaking in the form factors can be obtained. As shown in Chapter III, the behavior of these form factors can be obtained in the sidewise dispersive framework in the elastic approximation of the Omnes solution (an integral over the $S$ and $P$ wave pion-nucleon phase shifts). Lusanna\textsuperscript{39} has incorporated the inelasticity $\Pi_\Delta$

into this phase representation and evaluated the form factors in terms of the Donnachie phase shifts. The inelastic phase representation, similar to (3.70) and converging to (3.70) in the elastic approximation, is given by

\[
K(w) = \exp \left( \frac{w-m}{\pi} \right) \int dw' \left[ \frac{\psi(w')}{(w'-w)(w'-m)} - \frac{\psi(-w')}{(w'+w)(w'+m)} \right] \tag{4.18}
\]

where

\[
\psi(w) = \psi_{P_{11}}(w) = \tan^{-1} \frac{1 - \eta_{P_{11}}(w) \cos 2 \delta_{P_{11}}(w)}{\eta_{P_{11}}(w) \sin 2 \delta_{P_{11}}(w)} \tag{4.19}
\]

and

\[
\psi(-w) = \psi_{S_{11}}(w) = \tan^{-1} \frac{1 - \eta_{S_{11}}(w) \cos 2 \delta_{S_{11}}(w)}{\eta_{S_{11}}(w) \sin 2 \delta_{S_{11}}(w)} \tag{4.20}
\]

and where the partial wave amplitude \( f_{\pm} = \frac{1}{2i} (\eta_{\pm} e^{2i\delta_{\pm}} - 1) \) contains the inelasticity parameter \( \eta_{\pm} \). As can be seen in Figure 12, the phases are dominated by the \( S_{11}(1550) \) and \( P_{11}(1470) \) resonances and the form factors peak (at the resonance energies) as required by our considerations. Beyond the low energy region, these form factors fall off rapidly, making \( K(w) = 0 \) a plausible assumption.

Several estimations of the nucleon anomalous moments have already been made using pion-nucleon form factors of the Omnes type and realistic models for the photoproduction amplitude. Love and Rankin\(^{41} \) have used the Walker multipoles and the Donnachie phase shifts.

Figure 12. The inelastic phases and pion-nucleon form factors.
and have obtained $\mu_p = 1.05$. Also, from the Moorhouse and Rankin\(^{42}\) multipoles they have obtained $\mu_p < 1.50$. Ademollo, Gatto, and Longhi\(^{43}\) have, on the other hand, introduced scattering length approximations for the phase shifts in addition to the Donnachie phase shifts into three models. Model I contains the scattering length approximations

$$K^*(\omega) = \left(1 - a_\omega \ell_t^3 \right) / \left(1 - i a_\omega \ell_t \right)$$

and

$$K^*(\omega) = \left(1 + a_\omega \ell_t \right) / \left(1 - i a_\omega \ell_t \right)$$

where $\ell_t = \mu \left[1 - \frac{\mu^2}{4m^2} \right]^{\frac{1}{2}}$. Model II changes to a resonance form for the P wave phase shift so that

$$K^*(\omega) = \left(\omega_R - \omega - \gamma_R \ell_t^2 \right) / \left(\omega_R - \omega - i \gamma_R \ell_t \right)$$

where $\gamma_R = \frac{\Gamma_R}{2 \ell_R^3}$. Finally, Model III uses the experimental elastic phase shifts for both S and P waves. All these models employ a side-wise dispersive representation for the photoproduction multipoles and contain a sizable contribution from the $N^*(1238)$. In the following table we summarize these various calculations for the nucleon anomalous moments.

### TABLE 1

CONTRIBUTIONS TO THE NUCLEON ANOMALOUS MOMENTS FROM THE DOMINANT PION-NUCLEON INTERMEDIATE STATE

<table>
<thead>
<tr>
<th>Authors &amp; Model</th>
<th>( \mu_p )</th>
<th>( \mu_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Love and Rankin Walker multipoles</td>
<td>1.05</td>
<td>-1.2</td>
</tr>
<tr>
<td>Love and Rankin Moorhouse and Rankin multipoles</td>
<td>1.50</td>
<td></td>
</tr>
<tr>
<td>Ademollo et al. I</td>
<td>.9</td>
<td>-1.25</td>
</tr>
<tr>
<td>Ademollo et al. II</td>
<td>1.0</td>
<td>-1.4</td>
</tr>
<tr>
<td>Ademollo et al. III</td>
<td>.8</td>
<td>-1.1</td>
</tr>
<tr>
<td>Experimental Values</td>
<td>1.79</td>
<td>-1.91</td>
</tr>
</tbody>
</table>

With respect to the other meson-baryon intermediate states, we can expect their mesonic form factors to similarly exhibit peaking in the low energy region and vanishing behavior in the high energy region. The mesonic form factors would thus assume the critical role in the anomalous moment calculations. Together with the rapidly falling photoproduction factor, these sorts of form factors provide for a natural damping of the absorptive part and the elimination of a need for an arbitrary cutoff, in support of the low energy dominance viewpoint and in support of an enlarged radiative concept.

Such a radiative concept, for example, can be thought of as being applied by the insertion of a scattering amplitude into the dispersive diagram for the electromagnetic vertex (see Figure 13) to account for the behavior of the strong interaction vertex. It is a result of essentially the low energy physics involved and the mathematical nature of the Omnes solution which solves the vertex problem in terms of an observable scattering amplitude. Had the integral equation been of
Figure 13. Radiative or rescattering approximation for baryon anomalous magnetic moments.

another type, the solution may have involved a perturbation series in the scattering amplitude. Due to the Omnes solution, however, we need proceed no further than the insertion of one scattering amplitude.
V. SUMMARY

We have investigated the calculation of the anomalous magnetic moment in the sidewise dispersive framework by regarding the anomalous moment as being completely calculable from renormalization effects. Thus, in analogy with the situation for the electron anomalous moment, we have applied the concept of radiative corrections to the calculation of the nucleon anomalous moments by retaining the radiative meson-baryon states as the dominant intermediate states and by regarding the \( J^P = \frac{1}{2}^\pm \) resonances and other higher intermediate states as corrections to the radiative states.

When applied to the neutron anomalous moment in the perturbative context and to both nucleons in the context of a realistic photoproduction amplitude and a constant pion-nucleon form factor, the simple radiative concept fails. Its only success is the proton moment in the perturbative context.

The situation is saved, however, by the Omnes solution for the pion-nucleon form factor which corresponds to inserting a pion-nucleon scattering amplitude into the original dispersive diagram for the electromagnetic vertex in order to provide a radiative correction for the strong interaction vertex. The required peaking in the pion-nucleon form factor thus obtained retrieves the dominance of the pion-nucleon intermediate state by compensating for the rapidly falling photoproduction factor.
In this way, the moderate success achieved by Drell and Pagels in approximating the absorptive part by constant mesonic vertex and the threshold limit of the photoproduction amplitude can be understood.

The critical role of the mesonic form factor is further underscored by its probable rapid convergence to zero in the high energy region. We thus expect that an arbitrary, unphysical cutoff will not need to be introduced in future calculations which extend into the high energy region. In sum, the behavior of the mesonic form factor provides the ingredient necessary to interpret the anomalous moment as a low energy quantity.

In regard to future calculations of anomalous moments, the sidewise dispersive framework seems better suited than the normal dispersive framework because in the latter, the vector meson dominance picture is pursued and reliance must be placed upon knowledge of the vector meson-photon coupling and the vector meson-nucleon form factor, whereas in the radiative picture, we can rely directly upon experimental information concerning photoproduction and meson-baryon scattering amplitudes. The inclusion of higher intermediate states poses difficulties in either dispersive framework and we only mention here that we expect vanishing asymptotic behavior from their absorptive parts.

With the acquisition of further experimental information and/or theoretical insight for the various photoproduction and meson-baryon scattering amplitudes, we expect that an accurate and dynamical account can be given for the anomalous magnetic moments of the entire baryon octet in the sidewise dispersive framework. Such calculations would be interesting to compare with the internal symmetry predictions to see how
well hadronic dynamics is arranged by the symmetry principles in the low energy region.
APPENDIX A

CONSTRUCTION OF THE PROJECTION OPERATORS

As shown in Chapter I, the electromagnetic vertex function has either the form

\[ \bar{u}(p) P_\mu (p) u(q) = \bar{u}(p) \left\{ e_\mu \gamma_\mu - \left[ \frac{i q_\mu q_0}{2m} F_\mu (w) + \left( q_\mu + \frac{q^2}{m} \gamma_\mu \right) F_\mu (w) \right] \frac{w + Q + W}{2w} \right. \]

\[- \left. \left[ \frac{i q_\mu q_0}{2m} F_\mu (-w) + \left( q_\mu + \frac{q^2}{m} \gamma_\mu \right) F_\mu (w) \right] \frac{w - Q + W}{2w} \right\} \quad (A.1)\]

in the \( W \) plane or, alternatively, in the \( W^2 \) plane, the form

\[ \bar{u}(p) P_\mu (p) u(q) = \bar{u}(p) \left\{ e_\mu \gamma_\mu - \left[ \frac{i q_\mu q_0}{2m} F_\mu (w^2) + \frac{q^2}{2m} \gamma_\mu F_\mu (w^2) \right] \frac{w + Q + W}{2m} \right. \]

\[- \left. \left[ \frac{i q_\mu q_0}{2m} F_\mu (w^2) + \left( q_\mu + \frac{q^2}{2m} \gamma_\mu \right) F_\mu (w^2) \right] \frac{w - Q + W}{2m} \right\} \quad (A.2)\]

In order to obtain the absorptive part of the form factor, a projection operator is needed to extract the part corresponding to the form factor from the absorptive part of the general vertex. We construct in this appendix only the projection operators \( \psi_\mu (a) (w) \) and \( \psi_\mu (a) (w^2) \) for the second form factor where the projection operation reads either

\[ \text{Tr} \ \bar{u}(p) P_\mu (p) P_{\mu} (p) \psi_\mu (a) (w) u(q) = - \frac{1}{2m} F_\mu (w) \quad (A.3) \]

or

\[ \text{Tr} \ \bar{u}(p) P_\mu (p) P_{\mu} (p) \psi_\mu (a) (w^2) u(q) = - \frac{1}{2m} F_\mu (w^2) \quad (A.4) \]
In addition to providing an appropriate definition for projection operation, the first of these conditions also serves as the source of a sufficient number of relations to be used for the construction of the projection operator \( v_{\mu}^{(a)}(w) \). The \( W^a \) plane projection operators \( v_{\mu}^{\pm(a)}(W^a) \) can then be obtained from the \( W \) plane projection operator \( v_{\mu}^{(a)}(W) \) from the relations

\[
\begin{align*}
U_{\mu}^{+(a)}(W^2) &= \frac{m+W}{2W} U_{\mu}^{(a)}(W) + \frac{m-W}{2W} U_{\mu}^{(a)}(-W) \\
U_{\mu}^{-(a)}(W^2) &= \frac{m-W}{-2W} U_{\mu}^{(a)}(W) + \frac{m+W}{2W} U_{\mu}^{(a)}(-W)
\end{align*}
\]

which are derived by imposing the relations (1.5) upon (A.3) and (A.4).

The projection operators can assume the following general forms:

\[
U_{\mu}^{(a)}(q^2, W) = (\gamma + q + W) A_{\mu}(q^2, W)
\]

and

\[
U_{\mu}^{(a)}(q^2, W^2) = (\gamma + q + W) A_{\mu}(q^2, W^2) + (\gamma + q - W) B_{\mu}(q^2, W^2)
\]

where

\[
A_{\mu} = A_1 q_{\mu} + A_2 \gamma_{\mu} + A_3 \gamma_{\mu} q^2
\]

and

\[
B_{\mu} = B_1 q_{\mu} + B_2 \gamma_{\mu} + B_3 \gamma_{\mu} q^2
\]

We proceed with the determination of the coefficients \( A_i(q^2, W) \).
For the sake of clarity, we begin by displaying the general contraction of $\Gamma_{\mu\nu}(w)$ in the form

$$
\begin{align*}
\Gamma_{\mu\nu}(w) &= e_N \delta_{\mu\nu} (\varphi + \bar{\varphi} + w) A_\mu \\
&\quad - \frac{i \delta_{\mu\nu}}{2m} F_2(w) (\varphi + \bar{\varphi} + w) A_\mu \\
&\quad - \left( q_\mu + \frac{q_\nu}{m - w} \delta_{\mu\nu} \right) F_3(w) (\varphi + \bar{\varphi} + w) A_\mu
\end{align*}
$$

(A.11)

From the defining condition (A.3) we can obtain the following relations. First, from the term with $e_N$, we obtain

$$
\text{Tr} (\varphi + m) \delta_{\mu\nu} (\varphi + \bar{\varphi} + w) A_\mu = 0 \quad (A.12)
$$

Second, with this relation, we obtain from the term with $F_3(w)$

$$
\text{Tr} (\varphi + m) q_\mu (\varphi + \bar{\varphi} + w) A_\mu = 0 \quad (A.13)
$$

Each of these relations fixes the relationship between a different pair of the coefficients and thus in this fashion all pairwise relationships are determined. Finally, from the projection term, the condition

$$
\text{Tr} (\varphi + m) i \delta_{\mu\nu} (\varphi + \bar{\varphi} + w) A_\mu = 2m \quad (A.14)
$$

determines the functional form of one of the coefficients.
For convenience, we note the following useful identities:

\[ 2(q^2) = w^2 - m^2 - q^2 \]
\[ i \delta_{\mu \nu} q^\nu = q^\mu - q^\nu \rho^\nu = q^\mu - q^\mu \]
\[ q^\mu (q^2 + w) A_\mu = (q^2 + q^2 + w) A_1 + (q^2 + q^2 + w) A_2 \]
\[ q^\mu (q^2 + w) A_\mu = (q^2 + q^2 + w) A_1 + (-2q^2 - 2q^2 + 4w) A_2 \]
\[ i \delta_{\mu \nu} q^\nu (q^2 + w) A_\mu = (-3q^2 + 3w q^2 + q^2 - 4pq^2) A_2 + (3q^2 - 3w q^2 - q^2 + 4pq^2) A_3 \]

We are now ready to explicitly evaluate the coefficients. From (A.13), we find

\[ q^2 A_1 + (w - m) A_2 = 0 \]  \hspace{1cm} (A.16)

and from (A.12), we obtain, upon elimination of \( A_2 \) by the above relation,

\[ [(w + m)^2 + 2q^2] A_1 - 3(w^2 - m^2) A_3 = 0 \]  \hspace{1cm} (A.17)

Finally, combination of the above two relations fixes the third pairwise relationship as

\[ [(w + m)^2 + 2q^2] A_2 + 3q^2 (w + m) A_3 = 0 \]  \hspace{1cm} (A.18)

From this third pairwise relationship, we eliminate the appearance of \( A_2 \) in the projection condition (A.14) to obtain the following functional form for \( A_3(q^2, w) \):

\[ A_3(q^2, w) = \frac{m [(w + m)^2 + 2q^2]}{8w^2 - 8[(w + m)^2 - q^2]} \]  \hspace{1cm} (A.19)
where

\[ 4 w^2 q^2 = \left[ (w^2 - m^2)^2 - 2 q^2 (w^2 + m^2) + q^4 \right] \]  
(A.20)

For convenience, we list here the other two projection coefficients:

\[ A_2 (q^2, w) = - \frac{3 m q^2 (w^2 + m^2)}{8 w^2 q^2 [ (w^2 + m^2)^2 - q^4]} \]  
(A.21)

\[ A_1 (q^2, w) = \frac{3 m (w^2 - m^2)}{8 w^2 q^2 [ (w^2 + m^2)^2 - q^4]} \]  
(A.22)

To obtain \( \nu_{(3)}^\mu (-W) \), \( W \) is simply replaced by \(-W\) in the above relations.

We also list here the coefficients for the \( W^2 \) plane projection operators\(^{44}\).

\[ A_1 (q^2, w^2) = -3 m q^2 (w^2 - m^2)/2 (4 w^2 q^2)^2 \]
\[ A_2 (q^2, w^2) = \left[ 6 q^4 m^2 + \frac{3}{2} q^4 (w^2 - m^2) - \frac{3}{2} q^2 (w^2 - m^2)^2 \right]/2 (4 w^2 q^2)^2 \]
\[ A_3 (q^2, w^2) = -m \left[ q^4 + 4 q^2 w^2 + 2 q^2 (w^2 - m^2) - (w^2 - m^2)^2 \right]/2 (4 w^2 q^2)^2 \]  
(A.23)
\[ B_1 (q^2, w^2) = -3 m (w^2 - m^2)^2/2 (4 w^2 q^2)^2 \]
\[ B_2 (q^2, w^2) = (w^2 - m^2) \left[ \frac{3}{2} q^4 - \frac{3}{2} q^2 (w^2 - m^2) \right]/2 (4 w^2 q^2)^2 \]
\[ B_3 (q^2, w^2) = -3 m q^2 (w^2 - m^2)/2 (4 w^2 q^2)^2 \]

APPENDIX B

PROOF OF ANALYTICITY

From the reduction technique, we obtain

\[
\langle p | j_\mu (x) | p+q \rangle = \frac{i}{(2\pi)^{3/2}} \sqrt{\frac{m}{p_0+q_0}} \int d^4 x \langle p | T ( j_\mu (0) \overline{\psi} (x)) | 0 \rangle \left( i \overline{\psi} (x) \gamma_\mu - m \right) e^{i (p+q) x}
\]

\[
= \frac{i}{(2\pi)^{3/2}} \sqrt{\frac{m}{p_0+q_0}} \int d^4 x \langle p | \left\{ \Theta (x) \left[ j_\mu (0), \overline{\psi} (x) \right] - i \left[ j_\mu (0), \overline{\psi} (x) \right] \overline{\psi} (x) \gamma_\mu \Theta (x) \right\} | 0 \rangle e^{i (p+q) x}
\]

\[
= \frac{i}{(2\pi)^{3/2}} \sqrt{\frac{m}{p_0+q_0}} \int d^4 x \langle p | \left\{ \Theta (x) \left[ j_\mu (0), \overline{\psi} (x) \right] + i \left[ j_\mu (0), \overline{\psi} (x) \right] \overline{\psi} (x) \right\} | 0 \rangle e^{-i (p+q) x}
\]

(B.1)

The anticommutation relations

\[
\delta (x) \left\{ \psi_\mu (x), \psi^\dagger_\mu (x) \right\} = \delta_{\mu \nu} \delta^4 (x) \quad \text{and} \quad \left\{ \psi^\dagger_\mu (x), \psi^\dagger_\nu (x) \right\} = \delta_{\mu \nu} \delta^4 (x)
\]

(B.2)

can be used to show

\[
i \delta (x) \left[ j_\mu (0), \overline{\psi} (x) \right] \psi_\mu = i \delta^4 (x) \overline{\psi} (0) \gamma_\mu
\]

for a current of the basic form \( j_\mu (0) = \overline{\psi} (0) \gamma_\mu \psi (0) \). We thus obtain

\[
\langle p | j_\mu (0) | p+q \rangle = \frac{i}{(2\pi)^{3/2}} \sqrt{\frac{m}{p_0+q_0}} \int d^4 x \langle p | \left\{ \Theta (x) \left[ j_\mu (0), \overline{\psi} (x) \right] \right\} | 0 \rangle
\]

\[
\quad + \left[ \frac{1}{(2\pi)^{3/2}} \right] \sqrt{\frac{m}{p_0+q_0}} \sqrt{\frac{m}{p_0}} \overline{\psi} (0) \gamma_\mu
\]

(B.4)
Omitting the equal-time commutator term since it plays no role in the analysis of the analytic structure, we obtain in the center of mass system (where $\vec{p} + \vec{q} = 0$)

$$
\bar{u}(p) \gamma_\mu (\bar{q}, p+q) = -ie \gamma^0 \gamma^i \int \frac{d^4x}{m} e^{iWx} \phi(0(x)) \left[ \gamma^\mu(x), \n(x, -x) \right] \psi \tag{8.5}
$$

Due to causality, the commutator vanishes for space-like separations ($x_0^2 - x_i^2 < 0$) and due to $\theta(x_0)$, only contributions in the forward light cone $x_0 > 0$ are nonvanishing. Thus, in the upper half-plane ($ImW > 0$), we obtain an exponentially falling factor which makes the vertex an analytic function and permits us to write a Cauchy representation for the vertex by enclosing the upper half-plane in a semicircular contour.

To discuss the analyticity of the form factors requires an examination of the analytic properties as influenced by the presence of the projection operator. For this purpose, we find it useful to work in the rest frame of the outgoing nucleon $p$ and we will then be able to prove analyticity in the $W^2$ plane, and as a consequence, analyticity in the $W$ plane also. We denote in the rest frame of $p$

$$
p = (0, m)
$$

$$
P = p + q = (\hat{x}^R, P_0)
$$

$$
q = (\hat{x}^R, P_0 - m)
$$

and derive from momentum conservation

$$
P_0 = \left( W^2 + m^2 - q^2 \right) / 2m
$$

$$
q^2 - 2q^i (W^2 + m^2) + q^2 \left( W^2 - m^2 \right) + q^2 \right) / 4m^2 \tag{8.7}
$$
where $\hat{\xi}$ is an arbitrary unit vector and $P_0$ may be considered the invariant dispersion variable in the $W^2$ plane.

The second form factor can be written as (we omit a similar discussion for the third form factor)

$$F_2(\pm w) = \frac{i(2\pi)^2}{8W} \sum_{\text{spin}} \int d^4x \ e^{-iP_0x} \langle p|\Theta(x_0)[\eta_{\mu}(x_0)\eta(x)](0)|\eta(\pm w)\rangle U_\mu(x_0)$$

$$= \int_0^\infty \int_{-\infty}^{+\infty} dr \int d^4x \ e^{iP_0x} \ G_2(\pm w, x_0, r) \quad \text{where} \quad r = |x_0|$$

(8.8)

We need to show that $G_2(\pm W, x_0, r)$ as defined by

$$G_2(\pm w, x_0, r) = \frac{i(2\pi)^2}{8W} \sum_{\text{spin}} r^4 \int d^4x \ e^{-iP_0x} \langle \Theta(x_0)[\eta_{\mu}(x_0)\eta(x_0-\xi)](0)|\eta(\pm w)\rangle U_\mu(x_0)$$

(8.9)

has no singularities as a function of $W$, for it is already obvious by the above discussion that it has the required vanishing behavior outside the forward light cone. Since $\hat{\xi}$ is an arbitrary unit vector, $G_2$ cannot depend upon $\hat{\xi}$ and we thus integrate over the angles of $\hat{\xi}$. In this way we obtain

$$G_2(\pm w, x_0, r) = \frac{i(2\pi)^2}{8W} \sum_{\text{spin}} \int d^4x \ e^{-iP_0x} \langle \Theta(x_0)[\eta_{\mu}(x_0)\eta(x_0-\xi)](0)|\eta(\pm w)\rangle U_\mu(x_0)$$

(8.10)

where

$$\lambda_2(\pm w, x) = \frac{1}{4\pi} \int d^4\xi \ e^{i\xi \cdot x} \eta_{\mu} U_\mu(x_0)$$

(8.11)

and $j_\mu = j_\mu(0)\eta_\mu$ with $\eta_\mu$ a four vector in the direction of $j_\mu(0)$. When explicitly evaluated, the $\lambda_2(\pm W, x)$ are found to have no

singularity as a function of $W$ and $r$ and take the form

$$8m\lambda_2(tW, \vec{x}) = -2(\eta + \frac{1}{3} \vec{v} \cdot \vec{r})_0(r^2) - i\vec{v} \cdot \vec{x} \left[ \frac{(m \pm W)^2 - q^2}{m} \right]$$

$$- (m \pm W) i\vec{v} \cdot \vec{x} \left( J_1(r^2) - \frac{q^2 + W^2 + W^2 - 2m^2}{m} \right)$$

(B.12)

where the $j_\ell(r^2)$ are spherical Bessel functions of order $\ell$ and $(r^2)^{-\ell} j_\ell(r^2)$ is finite at $r^2 = 0$.

Thus, $G_2(\pm W, x_0, r)$ is an analytic function of $W$ and the linear combinations

$$\frac{1}{2} \left[ G_2(W, x_0, r) + G_2(-W, x_0, r) \right]$$

and

$$\frac{1}{2W} \left[ G_2(W, x_0, r) - G_2(-W, x_0, r) \right]$$

are analytic functions of $W^2$. The first combination singles out the even part of $G_2(\pm W, x_0, r)$ and the second the odd part. Both parts satisfy Hilbert relations which we write, as an assumption, in the unsubtracted form

$$Re H_2^\pm(P_0, r) = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{d\vec{P}_0}{\vec{P}_0 - P_0} Re H_2^\pm(P_0, r)$$

(B.14)

where

$$H_2^+(P_0, r) = \frac{1}{2} \int_{-\infty}^{+\infty} dx_0 e^{i\vec{P}_0 x_0} \left[ G_2(W, x_0, r) + G_2(-W, x_0, r) \right]$$

(B.15)

and

$$H_2^-(P_0, r) = \frac{1}{2W} \int_{-\infty}^{+\infty} dx_0 e^{i\vec{P}_0 x_0} \left[ G_2(W, x_0, r) - G_2(-W, x_0, r) \right]$$
Finally, we obtain dispersion relations for the even and odd parts of the form factors $F_2(\pm W)$ by interchanging the order of the $dP_0$ and $d\tau$ integrations

$$
R_{2} F_{2}^{\pm}(P_{0}) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{dP_{0}'}{P_{0}' - P_{0}} \ I_{\nu} F_{2}^{\pm}(P_{0}')
$$

The interchange of the order of integrations is permissible provided $h$ is never imaginary for $P_0$ in the range $-\infty < P_0 < \infty$. Since we are interested in the experimental region (space-like $q^2 < 0$), $h$ is always real. For the normal dispersion relations in $q_0$, however, $h = ((q_0 + m)^2 - p^2)^{1/2}$ can be imaginary in the range $-\infty < q_0 < \infty$ and thus this approach at a proof fails. As yet, no other approach has succeeded.
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