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1971

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INTRODUCTION

W. T. Tutte [1] generalized some well-known coloring problems of graph theory into a single geometric problem. This geometric problem is about the existence of a certain kind of subsets in the finite projective geometries over the Galois field of order 2. These subsets are called tangential 2-blocks. Now the question arises whether n-dimensional tangential 2-block exists in PG(n, 2), where PG(n, 2) denotes the finite projective geometry of dimension n, over the Galois field GF(2). W. T. Tutte [1] has answered to this question for n \leq 5. In fact, the answer is in the affirmative for n = 2, 3 and 5, but not for n = 4. Then Tutte [1] conjectured that n-dimensional tangential 2-block does not exist in PG(n, 2), for n > 5. An affirmative solution of the geometric problem, posed by Tutte [1], will imply an affirmative solution of the 4-Color Problem [8]; but the converse is not necessarily true. In this dissertation, it has been proved that n-dimensional tangential 2-block does not exist for n = 6. Furthermore, some general results have been developed to attack the general problem.
In Chapter I, we have tried to give a synopsis of Tutte's paper [1]. Here we have discussed some well-known concepts of graph theory and various results in [1], without giving any proof. Then we have stated a few conjectures connected with graph colorings, including Tutte's conjecture [1] about the existence of tangential 2-blocks in finite projective geometries over the field of order 2. In order to indicate how Tutte [1] has geometrized some graph coloring problems, we have also shown that the problem, posed by Tutte's conjecture [1], is a generalization of the problems posed by all the other conjectures.

In Chapter II, we have developed some new ideas and general results connected with tangential 2-blocks in PG(n, 2). For example, we introduced the notion of projection in PG(n, 2), attenuation space, polarising sets, \((r_1, r_2, r_3)\)-tangential stigm systems, non-tangential subspace in PG(n, 2); furthermore, some results connected with the notion of generators and axis of a cone [1] have been developed. These ideas and results are found to be very useful for the investigation into the existence of tangential 2-blocks in PG(n, 2). We concluded this chapter by using these ideas and results to give a different proof for the non-existence of 4-dimensional tangential 2-blocks in PG(4, 2).
The remaining chapters are all devoted to the investigation into the existence of a 6-dimensional tangential 2-block. From the definition of \((r_1, r_2, r_3)\)-tangential stigm systems and some results connected with them, it turns out that for every point belonging to a 6-dimensional tangential 2-block, there must correspond an \((r_1, r_2, r_3)\)-tangential stigm system, where \(r_1, r_2\) and \(r_3\) are three positive odd integers such that \(3 < r_i < 7\), \(i = 1, 2, 3\). In Chapter III, we have shown that none of these \(r_i\)'s is 7.

In Chapter IV, we have proved the non-existence of a \((5, 5, 5)\)-tangential stigm system for 6-dimensional tangential 2-block.

In Chapter V, it has been established that there cannot exist any \((5, 5, 3)\)-tangential stigm system for a 6-dimensional tangential 2-block.

In Chapter VI, we have proved the non-existence of \((5, 3, 3)\)-tangential stigm system for a 6-dimensional tangential 2-block.

In Chapter VII, we have made use of this information regarding \((r_1, r_2, r_3)\)-tangential stigm systems and proved the non-existence of 6-dimensional tangential 2-block in \(\text{PG}(n, 2)\).
CHAPTER I
Preliminaries.

In this chapter we are going to introduce various well known concepts of graph theory and various results of Tutte [1] without proof.

§1. Graphs and Subgraphs.

A graph $G$ is an ordered triplet $(V, E, I)$, where $V$ and $E$ are two disjoint sets and $I$ is an incidence map from the Cartesian product $V \times E$ into the set $\{0, 1, 2\}$ of integers such that $\sum_{v \in V} I(v, e) = 2, \forall e \in E$. $V$ is called the set of vertices and $E$ the set of edges of $G$. If $I(v, e) > 0$, then the edge $e$ is said to be incident with the vertex $v$ and $v$ is said to be an end point of $e$. Also $\sum_{v \in V} I(v, e) = 2$ implies that an edge can be incident with just two vertices, coincident or distinct. If an edge $e$ is incident with two distinct vertices, then $e$ is said to be a link. If an edge $e$ is incident with two coincident vertices, then $e$ is said to be a loop. If $I(v, e) = 2$, then $e$ is a loop and if $I(v, e) = 1$, then $e$ is a link. The two endpoints of an edge are said to be adjacent in $G$ and two edges having a vertex in common are said to be
An edge $e$ is called an **isthmus** if the vertex set $V$ can be partitioned into two sets $U_1$ and $U_2$ such that $e$ is the only edge with one end point in $U_1$ and the other in $U_2$. A graph $G = (V, E, I)$ is said to be finite iff $|V|$ and $|E|$ are both finite. As it will be necessary for our latter discussion, let us introduce three well known graphs by means of the following diagrams.

![Figures](a)(b)(c)

**Figure 1.**

Figure 1(a) is the complete bipartite graph on six vertices and it is denoted by $K_{3,3}$. Figure 1(b) is the complete graph on five vertices and it is denoted by $K_5$. Figure 1(c) is the Petersen graph [4], which is constructed from two disjoint pentagons $A_1A_2A_3A_4A_5$ and $B_1B_2B_3B_4B_5$ by making the five joins $A_iB_i$, $i = 1, 2, 3, 4, 5$.

**Subgraph of a graph.** Let $G = (V, E, I)$ and $H = (V', E', I')$ be two graphs. $H$ is called a subgraph of $G$ iff $V' \subseteq V$, $E' \subseteq E$ and $I'(v', e') = I(v, e), \forall (v', e') \in V' \times E'$ i.e. $I' = I|_{V' \times E'}$. 
Let \( G = (V, E, I) \) be a graph and let \( S \subseteq E \) and 
\[ V_S = \{ x \mid x \in V \text{ and } \exists e \in S \text{ such that } I(x, e) > 0 \} \]. Then 
\( G:S \) and \( G*S \) denote the following two subgraphs of \( G \).

\[ G:S = (V, S, I') \], where \( I' = I|_{V \times S} \)

\[ G*S = (V_S, S, I'') \), where \( I'' = I|_{V_S \times S} \).

\( G*S \) is called the reduction of \( G \) to \( S \). Again, we construct 
a graph, denoted by \( G \times S \), as follows. Its edges are the 
members of \( S \). Its vertices are the components of \( G:(E - S) \). 
An edge is incident with a vertex \( x \) in \( G \times S \) iff \( e \) is 
incident as an edge of \( G \), with a vertex of the component 
of \( G:(E - S) \) corresponding to \( x \). We say \( G \times S \) is derived 
from \( G \) by contracting the components of \( G:(E - S) \) to 
single vertices. \( G \times S \) is called the contraction of \( G \) to \( S \).

Two graphs are said to be isomorphic if there is a one-
to-one correspondence between their vertices and between 
their edges such that the incidences are preserved. In 
other words, if there is an edge between two vertices 
in one graph, there is a corresponding edge between the 
corresponding vertices in the other graph.

A graph \( G \) is said to be a planar graph if it is 
isomorphic to a graph \( G(\mathcal{P}) \), whose vertex set \( V \) is a point 
set in a plane \( \mathcal{P} \), while the edges are Jordan Curves in \( \mathcal{P} \) 
such that two different edges have at most endpoints in 
common. The following is the characterization of finite 
planar graphs.
Theorem 1.1. (Kuratowski) A finite graph is planar iff it does not contain any subgraph which is isomorphic, to within vertices of degree 2, to either $K_{3,3}$ or $K_5$.

Planarity of a graph $G$ remains preserved under reduction of a contraction of $G$.

§2. Chain-group.

Let $E$ be any finite set. A chain on $E$ is defined to be a mapping from $E$ into $GF(2)$. Let $C(E) = \{f | f$ is a chain on $E\}$. Let $f_1, f_2 \in C(E)$. We define $f_1 + f_2$ as follows.

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) \quad \forall \ x \in E.$$ 

Then clearly $f_1 + f_2 \in C(E)$. Let $0$ denote the map, called the zero chain, from $E$ into $GF(2)$ such that it takes every element $x$ of $E$ into the zero element of $GF(2)$. Then $C(E)$ is a group under the addition defined above. Any subgroup of $C(E)$ is called a chain-group on $E$. A chain-group $N$ on $E$ is called full iff $\forall x \in E, \exists f \in N$, such that $f(x) = 1$.

A coloring of a chain group $N$ on $E$ is defined to be a pair of chains $\{f, g\}$ of $N$ such that $\forall x \in E$, either $f(x) = 1$ or $g(x) = 1$. A chain group $N$ on $E$ is said to be chromatic iff it has a coloring and it is said to be achromatic iff it has no coloring.

Minors of chain-groups. Let $N$ be a chain group on a set $E$ and let $S \subseteq E$. Let $f \in N$. Then the restriction of $f$ to
S, denoted by $f_{S}$, is defined to be the chain on S such that $f_{S}(x) = f(x)$, $\forall x \in S$.

Let $N.S = \{f_{S} | f \in N\}$. Then it is easy to check that $N.S$ is a chain-group on S. $N.S$ is called the reduction of N to S.

Let $N \times S = \{f_{S} | f \in N$ and $f(x) = 0, \forall x \in E - S\}$. Then clearly $N \times S$ is a chain group on S and it is called the contraction of N to S.

A minor of N is a chain-group of the form $(N \times S).T$, where $T \subseteq S \subseteq E$. The reductions and contractions of N, including N itself are minors of N, since $N.S = (N \times E).S$, $N \times S = (N \times S).S$ and $N = (N \times E).E$.

An irreducible chain-group N is defined to be a full achromatic chain group which has no full achromatic minor other than itself.

Let f and g be two chains on a set E. Let $f \cdot g = \sum_{x \in E} f(x)g(x)$. Then f and g are called orthogonal iff $f \cdot g = 0$.

Let N be a chain group on E. Let $N^{*} = \{f | f \text{ is a chain on E and } f \cdot g = 0, \forall g \in N^{3}\}$. Then $N^{*}$ is a chain group on E, called the dual chain group of N.

§ 3. Cycles and Coboundaries of a graphs.

Let $G = (V, E, I)$ be a finite graph. Let $I': V \times E \rightarrow GF(2)$ be defined by $I'(x, e) = I(x, e) \pmod{2}$ for every $(x, e) \in V \times E$. Chains on V and E are called 0-chains and 1-chains of G respectively. Let f be
A 1-chain of $G$, Then the boundary of $f$ is a 0-chain, denoted by $\partial f$ and defined by

$$\partial f(x) = \sum_{e \in E} I'(x, e) f(e) \text{ for all } x \in V.$$  

A 1-chain $f$ of $G$ is a cycle iff $\partial f = 0$. It can easily be verified that

$$\partial (f + g) = \partial f + \partial g \text{ for any two 1-chains } f \text{ and } g.$$  

Hence the cycles of $G$ form a chain-group $\Gamma(G)$ on $E$. $\Gamma(G)$ is called the cycle-group of $G$.

Let $g$ be a 0-chain of $G$. The coboundary $\delta g$ of $g$ is a 1-chain defined by

$$\delta g(e) = \sum_{x \in V} I'(x, e) g(x), \quad \forall \ e \in E.$$  

Clearly, $\delta (g_1 + g_2) = \delta g_1 + \delta g_2$. Thus the coboundaries of the 0-chains of $G$ form a chain group $\Delta(G)$ on $E$. $\Delta(G)$ is called the coboundary group of $G$. Now we have the following result.

**Proposition 1.1.** Let $G = (V, E, I)$ be a finite graph and let $S \subseteq E$. Then

(i) $\Delta(G \cdot S) = \Delta(G) \cdot S$,  
(ii) $\Delta(G \times S) = \Delta(G) \times S$,  
(iii) $\Gamma(G \cdot S) = \Gamma(G) \times S$ and
(iv) $\Gamma(G \times S) = \Gamma(G) \cdot S$.

A chain group is defined to be graphic or cographic if it can be represented as the coboundary group or cycle group respectively, of some graph.
§4. Colorings of a graph.

Let $k$ be a positive integer. A $k$-coloring of a graph $G$ is a coloring if its vertices in $k$-colors, in such a way that the two ends of any edge have different colors. Thus if $G$ has a loop, it cannot have any $k$-coloring, for any positive integer $k$. Here we are interested in a 4-coloring of a loopless graph $G$. As our four colors, we take the four 2-vectors $(1, 1), (1, 0), (0, 1), (0, 0)$ with components in $GF(2)$. Then a 4-coloring of $G$ can be described as an ordered pair $(\mathbf{f}_1, \mathbf{f}_2)$ of 0-chains of $G$, the color assigned to a vertex $x$ being $(\mathbf{f}_1(x), \mathbf{f}_2(x))$. The condition for a given pair $(\mathbf{f}_1, \mathbf{f}_2)$ of 0-chains to be a 4-coloring of $G$ is that for the two end points $x_1$ and $x_2$ of each edge $e$, $\mathbf{f}_1(x_1)\neq \mathbf{f}_1(x_2)$, $i=1$ or 2, i.e., either $\mathbf{f}_1(e)=1$ or $\mathbf{f}_2(e)=1$. Now we have the proposition:

Proposition 1.2. Let $(\mathbf{f}_1, \mathbf{f}_2)$ is a pair of 0-chains of a graph $G$. Then $(\mathbf{f}_1, \mathbf{f}_2)$ is a 4-coloring of $G$ iff $(\mathbf{f}_1, \mathbf{f}_2)$ is a coloring of $\Delta(G)$.

A graph is trivalent iff each vertex is incident with just three edges, loops being counted twice. It is clear that if $G$ has a loop, then $G$ must have an isthmus.

A Tait coloring of a trivalent graph is a coloring of the edges in three colors so that each vertex is incident with one edge of each color. The following is a characterisation of Tait coloring of a graph.
Proposition 1.3. Let $G$ be a trivalent graph. Then $G$ has a Tait coloring iff $\Gamma(G)$ has a coloring.

§5. Embedding in a Projective Geometry.

In this section we deal with sets of points of a finite projective space $\text{PG}(n, 2)$ [2]. We admit the degenerate cases $n = -1, 0$ and $1$ of this geometry. We suppose a system of homogeneous coordinates in $\text{PG}(n, 2)$ is given and we identify each point with its coordinate vector. The zero $(n + 1)$-vector, which represents no point of $\text{PG}(n, 2)$ is represented in formulae by the symbol $0$. We assume that $\text{PG}(-1, 2) = \{0\}$. If $A$ and $B$ are two points of $\text{PG}(n, 2)$, then by $A + B$ we mean the point corresponding to the vector obtained from the vector sum of the two vectors corresponding to $A$ and $B$. We take a point as a space of dimension zero, a line of dimension $1$; and recursively, if $\sum_{n-1}$ is a space of dimension $n - 1$, $P$ a point not belonging to $\sum_{n-1}$, the set of all points in all lines $PB$, $B$ a point of $\sum_{n-1}$, is a space $\sum_n$ of dimension $n$. Let us now introduce the following notations. Let $\Delta$ be a set of points of $\text{PG}(n, 2)$. Then the subspace of $\text{PG}(n, 2)$ generated by the points of $\Delta$ is denoted by $\langle \Delta \rangle$ and the dimension of $\langle \Delta \rangle$ is denoted by $\dim \langle \Delta \rangle$. If $\Delta_1$ and $\Delta_2$ are two sets of points of $\text{PG}(n, 2)$, then $\langle \Delta_1, \Delta_2 \rangle$ denotes the subspace generated
by the points of $\triangle_1 \cup \triangle_2$. If $A_1, A_2, \ldots, A_m$ be a set of points of $PG(n, 2)$, then the space generated by $A_1, A_2, \ldots, A_m$ is denoted by $<A_1, A_2, \ldots, A_m>$ and its dimension by $\dim <A_1, A_2, \ldots, A_m>$. Again $\sum_k$ will always denote a subspace of $PG(n, 2)$, whose dimension is $k$.

Let $N$ be a chain group on a set $E$. Let $F$ be a mapping of $E$ onto a set of points of $PG(n, 2)$. We call $F$ an embedding of $N$ in $PG(n, 2)$ iff for every non-zero chain $f$ on $E$ the following holds:

$$\sum_{x \in E} f(x) F(x) = 0 \iff f \in N^*,$$

where $N^*$ is the dual chain-group of $N$.

Now we have the following proposition.

**Proposition 1.4.** A chain group $N$ on $E$ has an embedding in some $PG(n, 2)$ iff $N$ is full.

§6. $k$-Blocks.

Let $k$ be a positive integer. A non-empty set $\mathcal{B}$ of points in $PG(n, 2)$ is called a $k$-block iff its dimension is at least $k$ and $\mathcal{B} \cap \sum_{n-k} \neq \emptyset$, for every $(n-k)$-dimensional subspace $\sum_{n-k}$ in $PG(n, 2)$.

A $k$-block is minimal iff no proper subset of $\mathcal{B}$ is a $k$-block. Let $\triangle$ be a non null subset of a $k$-block $\mathcal{B}$ in $PG(n, 2)$ and $\dim <\triangle> \leq n - k$. We define tangent of $\triangle$, denoted by $t(\triangle)$, in $\mathcal{B}$ as any $(n-k)$-space $\sum_{n-k}$ in...
PG(N, 2) such that $\triangle \subseteq \sum_{n-k}$ and $A \in \bigcap \sum_{n-k}$ iff $A \in \langle \triangle \rangle$. A nonnull subset $\mathcal{B}$ in PG(n, 2) is said to be a 
tangential k-block iff $\mathcal{B}$ is a k-block and every non-null subset of $\mathcal{B}$, of dimension $\leq n - k$, has a tangent in $\mathcal{B}$.

If $\triangle = \{A_1, A_2, \ldots, A_v\}$ then for $t(\triangle)$ we may also use the notation $t(A_1, A_2, \ldots, A_v)$. Let us now state the following results.

**Proposition 1.5.** Every tangential k-block is minimal.

**Proposition 1.6.** Let $F$ be an embedding in PG(n, 2) of an achromatic chain group $N$ on $E$. Then $N$ is irreducible iff the following conditions hold:

(i) $F$ is a 1-1 mapping of $E$ onto $F(E)$.

(ii) $F(E)$ is a tangential 2-block in PG(n, 2).

We say that the irreducible chain group $N$ corresponds to the tangential 2-block $F(E)$ in the proposition 1.6.

§7. q-stigm.

Let $q$ be a positive integer. A q-stigm in PG(n, 2) is defined to be a set of $q$ points in an $(n - 2)$-space of PG(n, 2) such that each $q - 1$ of them are linearly independent.

An odd stigm, denoted by $St$, is a q-stigm for which $q$ is an odd integer greater than or equal to 3. If $\mathcal{B}$ is a k-block in PG(n, 2) and an odd stigm $St$ is contained in $\mathcal{B}$, then we say that $St$ is an odd stigm of $\mathcal{B}$. We now state the following results.
Proposition 1.7. Every k-block contains an odd stigm.

Proposition 1.8. The odd stigms are the minimal 1-blocks.

Proposition 1.9. Let $\mathcal{B}$ be a k-block of dimension $d$ in $\text{PG}(n, 2)$ and let $\sum_{d-k+1} \mathcal{B}$ be a subspace of dimension $d - k + 1$. Then $\mathcal{B} \cap \sum_{d-k+1}$ contains an odd stigm.

From proposition 1.9, we immediately deduce the following proposition.

Proposition 1.10. If $\mathcal{B}$ be an n-dimensional tangential 2-block in $\text{PG}(n, 2)$, then every $(n - 1)$-space of $\text{PG}(n, 2)$ contains an odd stigm of $\mathcal{B}$.

§8. Three important tangential 2-blocks.

We now describe three tangential 2-blocks, which play the most important part in the geometrization of some well known graph coloring problems.

Fano Block. In the Fano plane $\text{PG}(2, 2)$, the only 2-dimensional tangential 2-block is the one which contains all the seven points of the plane. We refer to this tangential 2-block as the Fano block.

We now state the following result on Fano block.

Proposition 1.11. The irreducible chain group corresponding to the Fano block is neither graphic nor cographic.

Desargues Block. In $\text{PG}(3, 2)$, the only 3-dimensional tangential 2-block is the one which is the complement in $\text{PG}(3, 2)$ of a 5-stigm. This 2-block is called the
Desargues block, since the points and lines in this 2-block constitute a Desargues configuration. We have the following result on Desargues block.

**Proposition 1.12.** The irreducible chain-group $N$ corresponding to Desargues block is the coboundary group of the complete 5-graph $K_5$ and $N$ is graphic.

**Petersen Block.** In $PG(5, 2)$, the only 5-dimensional tangential 2-block is the one which corresponds to an embedding of the cycle-group of the Petersen graph.

We state the following result on the Petersen graph.

**Proposition 1.13.** The irreducible chain-group $N$ corresponding to the Petersen block is the cycle-group of the Petersen graph and $N$ is cographic.


Here we discuss about W. T. Tutte's generalization of some well known coloring problems of graph theory into a single geometric problem about the existence of tangential 2-block in $PG(n, 2)$. First, we state the following coloring problems of graph theory.

**Statement 1** (4-color conjecture) If $G$ is a planar and loopless graph, then $G$ is 4-colorable.

**Statement 2.** If $G$ is a planar and trivalent graph without isthmuses, then $G$ has a Tait coloring.
Statement 3. (Special case of Hadwiger's conjecture). If $G$ is a loopless and not 4-colorable graph, then some reduction of a contraction of $G$ is a complete 5-graph $K_5[3]$.

Let us now make another statement which represents a generalization of a graph coloring problem into an algebraic problem about chain-groups.

Statement 4. (Tutte's conjecture on Tait coloring). The only irreducible chain-group which is co-graphic is the cycle group of the Petersen graph [1].

The following is the conjecture of W. T. Tutte [1], which is a generalisation of all the four problems stated above into a single problem about the finite projective geometries over $GF(2)$.

Statement 5. (Tutte's Conjecture). The only tangential 2-blocks in $PG(n, 2)$ are the Fano, Desargues and Petersen blocks [1].

The following theorem shows why Statement 5 (Tutte's conjecture) represents a generalization of all the four problems, posed by the first four statements.

Theorem 1.2.

(i) Statement 1 $\iff$ Statement 2

(ii) Statement 4 $\implies$ Statement 2

(iii) Statement 3 $\implies$ Statement 1

(iv) Statement 5 $\implies$ Statement 3

(v) Statement 5 $\implies$ Statement 4.
Proof: (i) Proof of (i) follows from Theorem 4, page 216 in [7].

(ii) Let us suppose that Statement 4 holds. If possible, let the Statement 2 not hold. Then there exists a planar trivalent graph \( G \) without isthmuses such that \( G \) has no Tait coloring. So, by proposition 1.3, \( \Gamma(G) \) does not have a coloring. Let \( N \) be an achromatic minor of \( \Gamma(G) \) such that no minor of \( N \), except \( N \) itself, is achromatic. It is easy to check that \( N \) is full. Therefore, \( N \) is irreducible. Now, we can write \( N = (\Gamma(G) \times S) \cdot T \), for some \( S \) and \( T \) where \( T \subseteq S \subseteq E \), \( E \) being the edge set of \( G \).

\[ \therefore N = \Gamma((G \cdot S) \times T). \therefore N \text{ is cographic}. \] By Statement 4, \( N \) is the cycle-group of the Petersen graph. \( \therefore (G \cdot S) \times T \) is not planar, which is not possible [ \( \because G \) is planar]. This completes the proof of (ii).

(iii) We assume that Statement 3 holds. Now, if possible, let Statement 1 not hold. Then there exists a graph \( G \) which is planar and loopless, but not 4-colorable. Since \( G \) is loopless and not 4-colorable, some reduction of a contraction of \( G \) contains \( K_5 \) [by Statement 3]. Since \( G \) is planar, we arrive at a contradiction [by theorem 1.1]. This completes the proof of (iii).

(iv) We assume that Statement 5 holds. Now, suppose that \( G \) is a loopless graph which is not 4-colorable. Then by proposition 1.2, \( \Delta(G) \) is achromatic. Let \( N \) be an
achromatic minor of \( \triangle(G) \) such that no minor of \( N \), other than \( N \) itself, is achromatic. Now it is easy to check that \( N \) is full. Therefore, \( N \) is irreducible chain-group on \( E \), where \( E \) is the edge set of \( G \). By proposition 1.4, there exists an embedding \( F \) of \( N \) in \( PG(n, 2) \) and by proposition 1.6, \( F(E) \) is a tangential 2-block. So, by Statement 5, \( F(E) \) is either Fano, Desargues or Petersen block. Now, we have \( N = (\triangle(G) \times S) \cdot T \), for some \( S \) and \( T \) such that \( T \subseteq S \subseteq E \). \( \therefore \) \( N = \triangle((G \times S) \cdot T) \). So, \( N \) is graphic. By propositions 1.11, 1.12, and 1.13, we conclude that \( (G \times S) \cdot T \) is \( K_5 \). This completes the proof.

(v) We assume that Statement 5 holds. We now want to show that Statement 4 holds. Let \( N_1 \) and \( N_2 \) be two irreducible chain-groups which are cographic. So, there exist graphs \( G_i, i = 1, 2 \) such that \( N_i = \Gamma(G_i), i = 1, 2 \). Since \( N_i, i = 1, 2 \), is full, there exists embedding \( F_i \) of \( N_i, i = 1, 2 \), in \( PG(n, 2) \) on \( E_i \), where \( E_i \) is the edge set of \( G_i \), for \( i = 1, 2 \). (by proposition 1.4). By proposition 1.6, it follows that \( F_i(E_i) \) is a tangential 2-block in \( PG(n, 2) \), for \( i = 1, 2 \). By Statement 5, \( F_i(E_i) \) is either Fano, Desargues or Petersen block, for \( i = 1, 2 \). Since \( N_1 \) and \( N_2 \) are both cographic, we get by propositions 1.11, 1.12 and 1.13, that \( F_i(E_i) \) is a Petersen block, for \( i = 1, 2 \). Thus both \( N_1 \) and \( N_2 \) correspond to Petersen block. So, by proposition 1.13,
1.13, \( N_i = \Gamma(G') \), \( i = 1, 2 \), where \( G' \) is the Petersen graph. So, \( N_1 = N_2 \). This completes the proof of the theorem.
A Few Concepts and General Results about Tangential 2-block in PG(n, 2).

In this chapter we shall always assume that in PG(n, 2) n is \( \geq 3 \), unless it is mentioned otherwise.

§1. Projection in PG(n, 2).

Let \( \mathcal{B} \) be a non-empty set of points in PG(n, 2) and \( \mathcal{P}(\mathcal{B}) \) the power set of \( \mathcal{B} \). Let "\( \overline{\cdot} \)" be a mapping from \( \mathcal{P}(\mathcal{B}) \rightarrow \mathcal{P}(\mathcal{B}) \), defined by

\[
\overline{\mathcal{A}} = \langle \mathcal{A} \rangle \cap \mathcal{B}, \forall \mathcal{A} \subseteq \mathcal{B}.
\]

Then we know that the mapping "\( \overline{\cdot} \)" is a closure operation, i.e. it satisfies the following three properties:

(i) \( \overline{\mathcal{A}} \supseteq \mathcal{A}, \forall \mathcal{A} \subseteq \mathcal{B} \).

(ii) \( \overline{\overline{\mathcal{A}}} = \overline{\mathcal{A}}, \forall \mathcal{A} \subseteq \mathcal{B} \).

(iii) \( \overline{\mathcal{A}_1} \supseteq \overline{\mathcal{A}_2} \Rightarrow \overline{\mathcal{A}_1} \supseteq \overline{\mathcal{A}_2}, \forall \mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{B} \).

Definition. Let \( \mathcal{A} \) and \( \mathcal{B} \) be two non-empty sets of points in PG(n, 2). Then \( \mathcal{A} \) is said to be closed subset of \( \mathcal{B} \) iff \( \mathcal{A} \subseteq \mathcal{B} \) and \( \mathcal{A} = \langle \mathcal{A} \rangle \cap \mathcal{B} \).

Proposition 2.1. Let \( \mathcal{B} \) be a 2-block in PG(n, 2). Then \( \mathcal{B} \) is tangential iff every non-empty closed subset of \( \mathcal{B} \), of dimension \( \leq n - 2 \), has a tangent.

Sufficiency. Let \( \emptyset \neq \sigma \subseteq R \), \( \dim <\sigma> \leq n - 2 \). Then \( \overline{\sigma} = <\sigma> \cap R \) is a closed subset of \( R \). Therefore, tangent of \( \overline{\sigma} \), i.e. \( t(\overline{\sigma}) \) exists. We claim that \( t(\overline{\sigma}) \) is a tangent of \( \overline{\sigma} \). For, \( \sigma \subseteq <\sigma> \cap R = \overline{\sigma} \subseteq t(\overline{\sigma}) \); also \( A \in R \cap t(\overline{\sigma}) \Rightarrow A \in <\sigma> \Rightarrow A \in <\sigma> \cap R \). Hence the proof.

Definition. Let \( \Sigma_k \) and \( \Sigma_{n-k-1} \) be two subspaces in \( PG(n, 2) \) such that \( 0 \leq k \leq n - 3 \) and \( \Sigma_k \cap \Sigma_{n-k-1} = \emptyset \). Let \( R \) be a non-empty subset of points in \( PG(n, 2) \) such that \( \Sigma_k \cap R = \emptyset \). Then the projection of \( R \) through \( \Sigma_k \) and \( \Sigma_{n-k-1} \) will be denoted by \( P(R, \Sigma_k, \Sigma_{n-k-1}) \) and is defined by

\[
P(R, \Sigma_k, \Sigma_{n-k-1}) = \{A | A \in \Sigma_{n-k-1} \text{ and } \exists S \in R \text{ such that } <\Sigma_k, \{S\}> \cap \Sigma_{n-k-1} = \{A, S\} \}.
\]

\( \Sigma_k \) will be called the vertex and \( \Sigma_{n-k-1} \) the base of the projection. Whenever we use the notation \( P(R, \Sigma_k, \Sigma_{n-k-1}) \), it should always be understood that \( k, R, \Sigma_k \) and \( \Sigma_{n-k-1} \) satisfy all the conditions required in the definition of projection.

Proposition 2.2. (i) \( R \cap \Sigma_{n-k-1} \subseteq P(R, \Sigma_k, \Sigma_{n-k-1}) \).
(ii) If \( \dim <R> = n \), then \( \dim <P(R, \Sigma_k, \Sigma_{n-k-1})> = n - k - 1 \).

Proof: (i) Proof is obvious.
(ii) For brevity, let \( R' = P(R, \Sigma_k, \Sigma_{n-k-1}) \). Clearly,
\[ \text{dim } < \beta' > \leq n - k - 1. \] If possible, let \( \text{dim } < \beta' > < n-k-1 \). Now \( \beta \subseteq < \Sigma_k, \beta' > \); so, \( n = \text{dim } < \beta > \leq \text{dim } < \Sigma_k, \beta' > = \text{dim } \beta' + k + 1 \). [Since \( \beta' \subseteq \Sigma_{n-k-1} \) and \( \Sigma_k \cap \Sigma_{n-k-1} = \emptyset \)]. Thus it follows that \( n < (n - k - 1) + k + 1 = n \). This is a contradiction. Hence the proposition.

**Proposition 2.3.** If \( \beta \) is a 2-block in \( PG(n, 2) \), then 
\[ \mathcal{P}(\beta, \Sigma_k, \Sigma_{n-k-1}) \] is a 2-block in \( \Sigma_{n-k-1} \).

**Proof:** For brevity, let \( \beta' = \mathcal{P}(\beta, \Sigma_k, \Sigma_{n-k-1}) \). Let \( \Sigma_{n-k-3} \) be any \((n - k - 3)\) space in \( \Sigma_{n-k-1} \). It is enough to show that \( \Sigma_{n-k-3} \cap \beta' \neq \emptyset \). Since \( \Sigma_k \cap \Sigma_{n-k-3} \subseteq \Sigma_k \cap \Sigma_{n-k-1} = \emptyset \), \( \langle \Sigma_k, \Sigma_{n-k-3} \rangle \) is an \((n - 2)\)-space, say \( \Sigma_{n-2} \), in \( PG(n, 2) \). Since \( \beta \) is a 2-block in \( PG(n, 2) \), \( \Sigma_{n-2} \cap \beta \neq \emptyset \). Let \( A \in \Sigma_{n-2} \cap \beta \). If \( A \in \Sigma_{n-k-3} \), then \( A \in \Sigma_{n-k-3} \cap \beta' \) [by proposition 2.1] and we are done. So, we assume that \( A \notin \Sigma_{n-k-3} \). Then \( \exists \ B \in \Sigma_k \) and \( C \in \Sigma_{n-k-3} \) such that \( A = B + C \) [since \( \Sigma_k \cap \beta = \emptyset \) and \( A \in \beta \)]. Therefore, by definition of projection, \( C \in \beta' \) i.e. \( C \in \beta' \cap \Sigma_{n-k-3} \). Thus \( \beta' \cap \Sigma_{n-k-3} \neq \emptyset \). Hence the proof.

Let us now state and prove the main result on projection \( \mathcal{P}(\beta, \Sigma_k, \Sigma_{n-k-1}) \).

**Theorem 2.1.** Let \( \beta \) be an \( n \)-dimensional tangential 2-block in \( PG(n, 2) \). Then 
\[ \mathcal{P}(\beta, \Sigma_k, \Sigma_{n-k-1}) \] is an \((n-k-1)\)-dimensional tangential 2-block in \( \Sigma_{n-k-1} \) provided there exists a \((k + 2)\)-space on \( \Sigma_k \) in \( PG(n, 2) \) such that
\[ (i) \quad \Sigma_k \subseteq \Sigma_{k+2} \quad \text{and} \quad \Sigma_k \neq \Sigma_k \Rightarrow \Sigma_k \cap \beta \neq \emptyset, \]
(ii) if \( \sum_{k+1}^i, i = 1, 2, 3, \) be the three \((k + 1)\)-spaces in \( \sum_{k+2} \) and on \( \sum_k \), then \( \dim \langle \beta \cap \sum_{k+1}^i \rangle = k + 1 \), for \( i = 1, 2, 3 \).

**Proof:** For brevity, let \( \beta' = P(\beta, \sum_k, \sum_{n-k-1}) \). Since \( \beta \) is an \( n \)-dimensional tangential 2-block in \( \text{PG}(n, 2) \), we get, by propositions 2.2 and 2.3, that \( \beta' \) is an \((n-k-1)\)-dimensional 2-block in \( \sum_{n-k-1} \). So, it remains to show that \( \beta' \) is tangential. By virtue of proposition 2.1, it is enough to show that every non-empty closed subset of \( \beta' \) has a tangent in \( \sum_{n-k-1} \). Let \( \beta' \cap \Delta' \subseteq \sum_{n-k-1} \), \( \dim \langle \Delta' \rangle \leq n - k - 3 \) and \( \beta' \cap \Delta' = \Delta' \). We need to show that tangent of \( \Delta' \) exists in \( \sum_{n-k-1} \). Let \( \Delta = \langle \sum_{k+1}', \Delta' \rangle \cap \beta \). Since \( \dim \langle \Delta' \rangle \leq n - k - 3 \), \( \dim \langle \Delta \rangle \leq n - 2 \). Since \( \beta \) is tangential, tangent of \( \Delta \), i.e. \( t(\Delta) \) exists in \( \text{PG}(n, 2) \). Let \( \Theta = t(\Delta) \cap \sum_{n-k-1} \). We claim that \( \Theta \) is a tangent of \( \Delta' \) in \( \sum_{n-k-1} \). First, we show that \( \sum_k \subseteq t(\Delta) \). Let \( \{ S_i \} \subseteq \sum_{k+1}^i \cap \sum_{n-k-1}, i = 1, 2, 3 \). Then \( S_i \in \beta', i = 1, 2, 3 \) [by condition (ii) of the hypothesis]. We now consider the following two cases.

**Case 1.** Let \( S_i \in \Delta' \), for some \( i \in \{ 1, 2, 3 \} \). We have \( \langle \sum_k, S_i \rangle = \sum_{k+1}^i \) and \( S_i \in \Delta' \); so, \( \langle \sum_k, S_i \rangle \cap \beta = \sum_{k+1}^i \cap \beta \subseteq \Delta \). Therefore, by condition (ii) of our hypothesis, we get \( \sum_k \subseteq \sum_{k+1}^i \subseteq \langle \sum_{k+1} \cap \beta \rangle \subseteq \langle \Delta \rangle \subseteq t(\Delta) \).

**Case 2.** Let \( S_i \not\in \Delta' \), for \( i = 1, 2, 3 \). Now \( t(\Delta) \) must meet \( \sum_{k+2} \) in at least a \( k \)-space, say \( \sum_k \), in \( \sum_{k+2} \). If \( \sum_k = \sum_k \), we are done. So, if possible, we assume that
\[ \Sigma_k \neq \Sigma_k. \] Then by condition (i) in our hypothesis, 
\[ \Sigma_k \cap \beta \neq \emptyset. \] Let \( \delta \in \Sigma_k \cap \beta \). Then \( \delta \in t(\Delta) \); therefore, \( \delta \in \langle \Delta \rangle = \langle \langle \Sigma_k, \Delta' \rangle \cap \beta \rangle \leq \langle \Sigma_k, \Delta' \rangle \), whence it follows that \( \delta \in \langle \Sigma_k, \Delta' \rangle \cap \beta = \Delta \). Now \( \delta \in \Delta \) and \( \delta \in \Sigma_k \subseteq \Sigma_{k+2} \implies \langle \Sigma_k, \delta \rangle = \sum_i \Sigma_{k+1} \) for some \( i \in \{1, 2, 3\} \) and \( \{\delta_i\} = \sum_i \Sigma_{k+1} \cap \Sigma_{n-k-1} \subseteq \Delta' \).

But this contradicts the assumption under case 2. Thus in both cases we get \( \Sigma_k \subseteq t(\Delta) \).

Let us now establish our claim that \( \theta = t(\Delta) \cap \Sigma_{n-k-1} \) is a tangent of \( \Delta' \) in \( \Sigma_{n-k-1} \). First, we show that \( \dim \langle \theta \rangle = n - k - 3 \). It is clear that \( \dim \langle \theta \rangle \geq n - k - 3 \). Now \( \langle \theta, \Sigma_k \rangle \subseteq t(\Delta) \implies \dim \langle \theta, \Sigma_k \rangle \leq n - 2 \). Since \( \Sigma_k \cap \theta \subseteq \Sigma_k \cap \Sigma_{n-k-1} = \emptyset, \dim \langle \theta, \Sigma_k \rangle \geq n - k - 3 + k + 1 \) i.e. \( \dim (\theta, \Sigma_k) \geq n - 2 \). Thus it follows that \( \dim \langle \theta, \Sigma_k \rangle = n - 2 \). Therefore, \( \dim \theta = n - k - 3 \). Again, let \( \delta' \in \Delta' \).

Then \( \exists \delta \in \Delta \) such that \( \langle \Sigma_k, \delta \rangle \subseteq t(\Delta) \); therefore, \( \delta' \in t(\Delta) \). Hence \( \Delta' \subseteq \theta \).

Finally, we assume that \( \delta' \in \beta' \cap \theta \). We need to show that \( \delta' \in \langle \Delta' \rangle \) i.e. \( \delta' \in \beta' \cap \langle \Delta' \rangle = \Delta' \) \( \Delta' \) is a closed subset of \( \beta' \). If \( \delta' \in \beta \), then \( \delta' \in \langle \Delta \rangle \subseteq \langle \Sigma_k, \Delta' \rangle \); but \( \delta' \in \Sigma_{n-k-1} \) and \( \langle \Sigma_k, \Delta' \rangle \cap \Sigma_{n-k-1} = \langle \Delta' \rangle \); so \( \delta' \in \langle \Delta' \rangle \cap \beta' = \Delta' \). If \( \delta' \notin \beta \), then \( \exists x \in \Sigma_k \) such that \( \delta' + x \in \beta \). Now \( \delta' \in \theta \) and \( x \in \Sigma_k \subseteq t(\Delta) \) \( \Rightarrow \delta' + x \in t(\Delta) \Rightarrow \delta' + x \in \langle \Delta \rangle \subseteq \langle \Sigma_k \rangle \cap \beta = \Delta' \). Since
Thus \( \Theta \) is a tangent of \( \triangle' \) in \( \Sigma_{n-k-1} \). Hence the theorem.

**Corollary 2.1.** Let \( \mathcal{B} \) be an \( n \)-dimensional tangential 2-block in \( \text{PG}(n, 2) \). Let \( \Sigma_2 \) be a plane in \( \text{PG}(n, 2) \) containing six points of \( \mathcal{B} \). Let \( Z \in \Sigma_2 \setminus \mathcal{B} \) and \( \Sigma_{n-1} \) be any \((n-1)\)-space in \( \text{PG}(n, 2) \) such that \( Z \not\in \Sigma_{n-1} \). Then \( \mathcal{P}(\mathcal{B}, \{Z\}, \Sigma_{n-1}) \) is an \((n-1)\)-dimensional tangential 2-block in \( \Sigma_{n-1} \).

**Proof:** In the theorem 2.1, take \( k = 0 \) and \( \Sigma_0 = \{Z\} \).

Then \( Z \in \Sigma_2 \) and \( Z \) and \( \Sigma_2 \) satisfy the conditions (i) and (ii) of the theorem 2.1. Therefore, it follows from theorem 2.1 that \( \mathcal{P}(\mathcal{B}, \{Z\}, \Sigma_{n-1}) \) is a tangential 2-block.

**Corollary 2.2.** Let the positive integer \( n \) be such that \((n-1)\)-dimensional tangential 2-block does not exist in \( \text{PG}(n-1, 2) \). If \( \mathcal{B} \) is an \( n \)-dimensional tangential 2-block in \( \text{PG}(n, 2) \), then there does not exist any plane in \( \text{PG}(n, 2) \), which may contain more than five points of \( \mathcal{B} \).

**Proof:** By virtue of Corollary 2.1, proof is immediate.

**Corollary 2.3.** Let the positive integer \( n \) be such that \((n-2)\)-dimensional tangential 2-block does not exist in \( \text{PG}(n-2, 2) \). If \( \mathcal{B} \) is an \( n \)-dimensional tangential 2-block in \( \text{PG}(n, 2) \), then there can be at most one plane in \( \text{PG}(n, 2) \), which may contain six points of \( \mathcal{B} \).

**Proof:** If possible, let \( \Sigma_2^1 \) and \( \Sigma_2^2 \) be two distinct planes in \( \text{PG}(n, 2) \), each having six points of \( \mathcal{B} \). Let \( Z_i \in \Sigma_2^i \setminus \mathcal{B} \), \( i = 1, 2 \). Now we consider the following two cases.
Case 1. Let $Z_1 \neq Z_2$. Then $Z_1 \notin \Sigma_2^2$. Let $\Sigma_2^1, X_2, Y_2$ be a basis of $\Sigma_2^2$. Then $\{Z_1, Z_2, X_2, Y_2\}$ form a set of independent points. We extend this set to a basis $\mathcal{O}'$ of $\text{PG}(n, 2)$. Then $\mathcal{O}' - \{Z_1\}$ generates an $(n - 1)$-space, say $\Sigma_{n-1}$, such that $Z_1 \notin \Sigma_{n-1}$ and $\Sigma_2^2 \subset \Sigma_{n-1}$. By Corollary 2.1, it follows that $\mathcal{P}(\mathcal{B}, \{Z_1\}, \Sigma_{n-1})$ is a tangential 2-block of dimension $n - 1$. For brevity, let $\mathcal{B}' = \mathcal{P}(\mathcal{B}, \{Z_1\}, \Sigma_{n-1})$. Now $\mathcal{B}'$ is a $(n-1)$-dimensional tangential 2-block in $\Sigma_{n-1} = \text{PG}(n - 1, 2)$ and $\Sigma_2^2$ is a plane in $\Sigma_{n-1}$ which contains six points of $\mathcal{B}'$. But there does not exist any $(n - 2)$-dimensional tangential 2-block. Thus we arrive at a contradiction. Therefore, case 1 is not possible.

Case 2. Let $Z_1 = Z_2 = Z$ say. In this case, we can always find a line $\{Z, X_2, Y_2\}$ such that $X_2, Y_2 \in \Sigma_2^2 \cap \mathcal{B}$, but $X_2, Y_2 \notin \Sigma_2^1$. [ $\therefore \Sigma_2^1 \neq \Sigma_2^2$]. Then clearly $t(X_2) \cap \Sigma_2^1 = \{Z\}$.

Therefore, $Z + X_2 = Y_2 \in t(X_2) \implies Y_2 \not\in <X_2> \implies Y_2 = X_2$.

But this is not possible. Thus case 2 cannot happen.

Hence the corollary is proved.

§2. Attenuation Space.

Definition. Let $\mathcal{B}$ be an $n$-dimensional tangential 2-block in $\text{PG}(n, 2)$. Let $k$ and $\ell$ be two non-negative integers such that $0 \leq k \leq n - 3$ and $k \leq \ell < n - 1$. A subspace $\Sigma_k$ is called an attenuation space for $\mathcal{B}$ with respect to $\Sigma_\ell$ iff
(i) $\Sigma_k \subset \Sigma_{k+1}$, 
(ii) $\dim <\Sigma_k \cap \mathcal{B}> < k$, and 
(iii) $\forall \Sigma_{k+1}$ in $PG(n, 2)$, $\Sigma_{k+1} \cap \Sigma_{k+1} = \Sigma_k \Rightarrow |(\Sigma_{k+1} - \Sigma_k) \cap \mathcal{B}| \leq 1$.

Note that in the above definition of attenuation space, condition (iii) implies that $\Sigma_1 + \Sigma_2 \notin \Sigma_k$, for any two points $\Sigma_1$ and $\Sigma_2$ of $\mathcal{B} - \Sigma_{k+1}$. In this sense $\Sigma_k$ actually attenuates the set $\mathcal{B} - \Sigma_{k+1}$. Let us now state and prove the following theorem on attenuation space which gives a sufficient condition for a subspace to be an attenuation space.

**Theorem 2.2.** Let $\mathcal{B}$ be an $n$-dimensional tangential $2$-block in $PG(n, 2)$. Let $\Sigma_k$ and $\Sigma_{k+1}$ be two subspaces in $PG(n, 2)$ such that $0 \leq k \leq n - 3$, $k + 2 \leq \ell \leq n - 1$ and $\Sigma_k \subset \Sigma_{k+1}$ and $\dim <\mathcal{B} \cap \Sigma_k> < k$. Then $\Sigma_k$ is an attenuation space for $\mathcal{B}$ with respect to $\Sigma_{k+1}$ provided there exists a $(k + 2)$-space $\Sigma_{k+2}$ and a non-empty subset $\Delta$ of $\mathcal{B}$ such that

(i) $\dim <\Delta> \leq n - 3$, $<\Delta> \subset \Sigma_{k+1}$ and $<\Delta> \cap \Sigma_k = \emptyset$, 

(ii) $\Sigma_k \subset \Sigma_{k+2} \subset \Sigma_{k+1}$ and 

(iii) $\forall \Sigma_{k+1} \subset \Sigma_{k+2}$, $\Sigma_{k+2} \neq \Sigma_k \Rightarrow \{<\Sigma_{k+1}, \Delta> - <\Delta>\} \cap \mathcal{B} \neq \emptyset$.

**Proof:** If possible, let $\Sigma_k$ be not an attenuation space. Then there must exist a $(k + 1)$-space, say $\Sigma_{k+1}$, such that $\Sigma_{k+1} \cap \Sigma_{k+1} = \Sigma_k$ and $|(\Sigma_{k+1} - \Sigma_k) \cap \mathcal{B}| > 1$. Let
Since \( \dim \triangle \leq n-3 \), \( t(\triangle, \{S_i\}) \) exists. \( S \in \mathcal{B} - \Sigma_2 \). We claim that 
\[ \forall S \in \mathcal{B} - \Sigma_2, \quad \dim \triangle \leq n-3 \]
\[ t(\triangle, \{S_i\}) \cap \Sigma_{k+2} \supseteq \Sigma_k, \quad \forall S \in \mathcal{B} - \Sigma_2. \]
For, \( \triangle \subseteq \Sigma_k \). If \( \Sigma_k \not\subseteq \Sigma_k \), then by the condition (iii), there exists a point \( Q \in \{\Sigma_k, \triangle\} \cap \mathcal{B} \). Therefore, 
\( Q \not\in \triangle \) and \( Q \in \Sigma_k \cap \mathcal{B} \). But \( Q \in t(\triangle, \{S_i\}) \), so that 
\( Q \in \triangle \), since \( Q \in \Sigma_k \cap \mathcal{B} \) and \( \triangle, \{S_i\} \cap \Sigma_{k+2} = \triangle \). Thus we arrive at a contradiction. Therefore, 
\[ t(\triangle, \{S_i\}) \cap \Sigma_{k+2} \supseteq \Sigma_k, \quad \forall S \in \mathcal{B} - \Sigma_2. \]
In particular, 
\[ t(\triangle, \{S_1\}) \supseteq \Sigma_k. \]
But \( S_1 + S_2 \in \Sigma_k \); so \( S_2 \in t(\triangle, \{S_1\}) \).
Therefore, it follows that \( S_1 + S_2 \in \triangle \), i.e. \( \triangle \cap \Sigma_k \neq \emptyset \), which contradicts (i) in the hypothesis. Hence the proof is completed.

When \( k = 0 \), it follows from the definition of attenuation space \( \Sigma_k \) for \( \mathcal{B} \) with respect to \( \Sigma_1 \), that \( \Sigma_k \) is a point, called attenuation point, and it does not belong to \( \mathcal{B} \).

Because of our special interest in the concept of attenuation points, let us rephrase its definition in the following way.

**Definition.** Let \( \mathcal{B} \) be a \( n \)-dimensional tangential 2-block in \( \text{PG}(n, 2) \). Let \( \Sigma_m \) be an \( m \)-dimensional subspace in \( \text{PG}(n, 2) \), where \( 0 \leq m \leq n - 1 \). A point \( X \) of \( \Sigma_m \) is said to be an attenuation point for \( \mathcal{B} \) with respect to \( \Sigma_m \) iff

(i) \( X \not\in \mathcal{B} \) and (ii) \( X \not\in S_1 + S_2 \), for every pair of points \( S_1 \) and \( S_2 \) belonging to \( \mathcal{B} - \Sigma_m \). Let us now state and prove
a theorem which establishes a set of sufficient conditions for a point being an attenuation point.

Theorem 2.3. Let \( \mathcal{B} \) be an \( n \)-dimensional tangential 2-block in \( \text{PG}(n, 2) \). Let \( \Sigma_m \) be an \( m \)-dimensional subspace in \( \text{PG}(n, 2) \), where \( 1 \leq m \leq n - 1 \), and let \( X \in \Sigma_m \). Then \( X \) is an attenuation point for \( \mathcal{B} \) with respect to \( \Sigma_m \) provided there exists a non-empty subset \( \triangle \) of \( \mathcal{B} \cap \Sigma_m \) such that

(i) \( \text{dim } \langle \triangle \rangle \leq n - 3 \),

(ii) \( X \in t(\triangle) \), but \( X \notin \triangle \), and

(iii) \( X \in t(\triangle, \{S\}), \forall S \in \mathcal{B} - \Sigma_m \).

Proof: Condition (i) implies \( t(\triangle, \{S\}) \) exists for every \( S \in \mathcal{B} - \Sigma_m \) and condition (ii) shows that \( X \notin \mathcal{B} \). If possible, let \( X \) be not an attenuation point. Then \( X = S_1 + S_2 \) for some \( S_1 \) and \( S_2 \) belonging to \( \mathcal{B} - \Sigma_m \). By condition (iii), we get \( S_1 + S_2 = X \in t(\triangle, \{S_1\}) \). Therefore, \( S_2 \in t(\triangle, \{S_1\}) \) and hence \( X = S_1 + S_2 \notin \langle \triangle \rangle \), which contradicts (ii). Hence the proof is completed.

Since we shall use theorem 2.2 quite often in the special case \( k = 0 \), let us restate the theorem 2.2 in the case of attenuation point.

Theorem 2.4. Let \( \mathcal{B} \) be an \( n \)-dimensional tangential 2-block and \( \Sigma_m \) an \( m \)-space in \( \text{PG}(n, 2) \), where \( 2 \leq m \leq n - 1 \). A point \( X \) of \( \Sigma_m \) is an attenuation point for \( \mathcal{B} \) with respect to \( \Sigma_m \) if there exists a plane \( \Sigma_2 \) and non-empty subset \( \triangle \) of \( \mathcal{B} \) such that
(i) \( \dim \langle \Delta \rangle \leq n - 3 \), \( \langle \Delta \rangle \subseteq \Sigma_m \) and \( X \notin \langle \Delta \rangle \),
(ii) \( X \in \Sigma_2 \subseteq \Sigma_m \), and
(iii) \( \forall x' \in \Sigma_2, x' \neq x \implies \{ \langle x', \Delta \rangle, \Delta \} \cap \Sigma \neq \emptyset \).

It is to be noted that the proof of theorem 2.4 also follows immediately from theorem 2.3. Let us now introduce the following definition, which will be utilised in the succeeding section.

**Definition.** Let \( \Sigma_m \) be a subspace, where \( 1 \leq m \leq n - 1 \), and \( \Omega \) be an \( n \)-dimensional tangential \( 2 \)-block in \( PG(n, 2) \). Let \( X \) be an attenuation point in \( \Sigma_m \) and \( \Delta \) be a non-empty subset of \( \Omega \cap \Sigma_m \). Then \( \Delta \) is said to induce the attenuation point \( X \) with respect to \( \Sigma_m \) iff \( \Delta \) and \( X \) satisfy the three conditions (i), (ii) and (iii) in theorem 2.3.

§3. Polarising set.

**Definition.** Let \( \Sigma_{n-1} \) be an \( n \)-dimensional tangential \( 2 \)-block and \( \Sigma_{n-1} \) be an \( (n - 1) \)-space in \( PG(n, 2) \). Let \( \emptyset \neq \Delta \subseteq \Omega \cap \Sigma_{n-1} \), and \( \dim \langle \Delta \rangle \leq n - 3 \). Then \( \Delta \) is said to be a weak polarising set for \( \Omega \) with respect to \( \Sigma_{n-1} \) iff \( \dim \langle \Omega - \Sigma_{n-1} \cup \Delta \rangle \leq \dim \langle \Delta \rangle + 3 \).

If \( \Delta \) be any non-empty subset of \( \Omega \), of dimension \( \leq n - 3 \), then \( \dim \langle \Omega - \Sigma_{n-1} \cup \Delta \rangle \) could be \( n \) or smaller. When \( \Delta \) is a weak polarising set of dimension less than \( n - 3 \), \( \Delta \) polarises all points of \( \langle \Omega - \Sigma_{n-1} \cup \Delta \rangle \) into a space of dimension smaller than \( n \). In the definition
we take the dimension of \( \langle (B - \sum_{n-1}) \cup \Delta \rangle \) to be at most \( \dim \langle \Delta \rangle + 3 \), as this seems to be the most important type of polarisation for our purposes.

**Proposition 2.4.** Let \( B \) be an \( n \)-dimensional tangential 2-block and \( \Delta \) be a weak polarising set for \( B \) with respect to \( \sum_{n-1} \) and \( \dim \langle \Delta \rangle = d \). Then there exist four \((d + 1)\)-spaces \( \rho_i^{d+1}, i = 1, 2, 3, 4 \) such that \( (B - \sum_{n-1}) \cup \Delta \subseteq \bigcup_{i=1}^{4} \rho_i^{d+1}, \) and \( \sum_{n-1} \supseteq \langle \Delta \rangle, \forall i \in \{1, 2, 3, 4\} \).

**Proof.** Let \( \langle \Delta \rangle = \sum_d \). Since \( \Delta \) is a weak polarising set, there exists a \((d + 3)\)-space, say \( \sum_{d+3} \), such that \( (B - \sum_{n-1}) \cup \Delta \subseteq \sum_{d+3} \). Let \( \sum_{d+3} = \langle \sum_d, \{A_1, A_2, A_3\} \rangle \), where \( A_i \in \text{PG}(n, 2) - \sum_{n-1}, \) \( i = 1, 2, 3 \). Then we must have \( B - \sum_{n-1} \subseteq \sum_{d+3} - \sum_{n-1} \subseteq \bigcup_{i=1}^{4} \rho_i^{d+1}, \) where \( \rho_i^{d+1} = \langle \sum_d, \{A_i^3\} \rangle, i = 1, 2, 3 \) and \( \rho_4^{d+1} = \langle \sum_d, \{A_1 + A_2 + A_3\} \rangle \). Hence the proposition is proved.

**Definition.** Let \( B \) be an \( n \)-dimensional tangential 2-block and \( \sum_{n-1} \) be an \((n - 1)\)-space in \( \text{PG}(n, 2) \). Let \( B \neq \Delta \subseteq B \cap \sum_{n-1} \) and \( \dim \langle \Delta \rangle = d \leq n - 3 \). Then \( \Delta \) is called a polarising set for \( B \) with respect to \( \sum_{n-1} \) iff there exist three \((d + 1)\)-spaces \( \rho_i^{d+1}, i = 1, 2, 3 \), such that

(i) \( \rho_i^{d+1} \supseteq \Delta \), \( \forall i \in \{1, 2, 3\} \) and

(ii) \( B - \sum_{n-1} \subseteq \bigcup_{i=1}^{3} \rho_i^{d+1} \).

Let us make the following remarks.

**Remarks** (i) A polarising set for \( B \) with respect to \( \sum_{n-1} \) is necessarily a weak polarising set for \( B \) with respect to \( \sum_{n-1} \).
(ii) If \( \emptyset \neq \triangle \subseteq \mathcal{P} \cap \Sigma_{n-1} \) and \( \dim \langle \triangle \rangle = n - 3 \), then clearly \( \triangle \) is a weak polarising set for \( \mathcal{P} \) with respect to \( \Sigma_{n-1} \).

(iii) If \( \emptyset \neq \triangle \subseteq \mathcal{P} \cap \Sigma_{n-1} \), \( \dim \langle \triangle \rangle = n - 3 \) and \( t(\triangle) \notin \Sigma_{n-1} \), then \( \triangle \) is a polarising set for \( \mathcal{P} \) with respect to \( \Sigma_{n-1} \); for, one of the four \((n - 2)\)-spaces, which are on \( \langle \triangle \rangle \), but not in \( \Sigma_{n-1} \), is \( t(\triangle) \), so that \( \mathcal{P} - \Sigma_{n-1} \) is contained in the remaining three \((n - 2)\)-spaces on \( \langle \triangle \rangle \).

(iv) The condition (ii) in the definition of polarizing set is obviously satisfied even when \( \emptyset \neq \mathcal{P} \subseteq \Sigma_{n-1} \) if

\[
\langle \triangle \rangle = \bigcup_{i \in J} \mathcal{P}^i \cup \mathcal{P}^{d+1},
\]

where \( \emptyset \neq J \subseteq \{1, 2, 3\} \).

Now we have the following theorem which gives a set of sufficient conditions in order that a set is a polarising set.

**Theorem 2.5.** Let \( \mathcal{P} \) be an \( n \)-dimensional tangential 2-block and \( \Sigma_{n-1} \) be an \((n - 1)\)-space in \( \text{PG}(n, 2) \). Let \( \emptyset \neq \triangle \subseteq \mathcal{P} \cap \Sigma_{n-1} \). Then \( \triangle \) is a polarising set for \( \mathcal{P} \) with respect to \( \Sigma_{n-1} \) provided

(i) \( t(\triangle) \notin \Sigma_{n-1} \) and

(ii) \( \triangle \) induces a set \( \mathcal{O} \) of attenuation points with respect to \( \Sigma_{n-1} \) such that \( \dim \langle \triangle, \mathcal{O} \rangle = n - 3 \).

**Proof.** Condition (ii) \( \implies \) \( \dim \langle \triangle \rangle \leq n - 3 \). If \( \mathcal{O} = \emptyset \), then \( \dim \langle \triangle \rangle = n - 3 \), so that by condition (i) we can conclude that \( \triangle \) is a polarising set [See Remark (iii) preceding theorem 2.5]. \( \therefore \) assume \( \mathcal{O} \neq \emptyset \). Let \( \langle \triangle, \mathcal{O} \rangle = \Sigma_{n-3} \) and \( \langle \triangle \rangle = \Sigma_d \), where \( d = \dim \langle \triangle \rangle \). Let
$\Sigma_{n-2}$, $i = 0, 1, 2, 3$, be all the four $(n - 2)$-spaces, which are on $\Sigma_{n-3}$, but not in $\Sigma_{n-1}$. Since $t(\Delta) \notin \Sigma_{n-1}$ (by (i)), we can assume (w.l.o.g.) that $t(\Delta) = \sum_{i=0}^{n-2}$.

We have

$$(1) \quad \mathcal{B} - \Sigma_{n-1} \subseteq \bigcup_{i=1}^{3} \Sigma_{n-2}$$

Let $s_i \in (\mathcal{B} - \Sigma_{n-1}) \cap \Sigma_{n-2}^i$, $i \in \{1, 2, 3\}$. Assert that

$\Sigma_{n-2} = t(\Delta, \{s_i\})$. Let $x \in \sigma\gamma$. Since $\Delta$ induces the attenuation point $x$, $x \in t(\Delta, \{s_i\})$. $\therefore \sigma\gamma \subseteq t(\Delta, \{s_i\})$.

Thus $t(\Delta, \{s_i\}) \supseteq \triangle, \sigma\gamma = \Sigma_{n-3} \Rightarrow t(\Delta, \{s_i\}) = \langle \Sigma_{n-3}, s_i \rangle = \Sigma_{n-2}$. [Since $\Sigma_{n-3} \subseteq \Sigma_{n-2}$ and $s_i \notin \Sigma_{n-3}$].

Assert that $(\mathcal{B} - \Sigma_{n-1}) \cap \Sigma_{n-2} \subseteq \triangle, s_i \rangle$.

Let $s_i^j \in (\mathcal{B} - \Sigma_{n-1}) \cap \Sigma_{n-2}^i$, $s_i^j \neq s_i$. Then from above, we get $s_i^j \in t(\Delta, \{s_i\})$. So $s_i^j \in \langle \Delta, s_i \rangle$.

Hence $(\mathcal{B} - \Sigma_{n-1}) \cap \Sigma_{n-2} \subseteq \langle \Delta, s_i \rangle$.

Let $J = \{i \mid (\mathcal{B} - \Sigma_{n-1}) \cap \Sigma_{n-2}^i \neq \phi\}$. Since (1) holds and $(\mathcal{B} - \Sigma_{n-1}) \neq \phi$, we conclude that $J \neq \phi$ and $|J| \leq 3$. Let $s_i^j \in (\mathcal{B} - \Sigma_{n-1}) \cap \Sigma_{n-2}^i$, for $i \in J$ and let $\langle \Delta, s_i^j \rangle = \rho_{d+1}^i$. Then from what we have gotten earlier, we get $\mathcal{B} - \Sigma_{n-1} \subseteq \bigcup_{i \in J} \rho_{d+1}^i$. Therefore $\Delta$ is a polarising set. Hence the proof is completed.

In connection with polarising set, let us introduce the following notations which we shall use very often.

If $\Delta$ is a polarising set for $\mathcal{B}$ with respect to $\Sigma_{n-1}$, there always exist three $(d + 1)$-spaces on $\langle \Delta \rangle$, where $\dim \langle \Delta \rangle = d$, such that the points of $\mathcal{B} - \Sigma_{n-1}$ are
distributed among these three \((d + 1)\)-spaces. Whenever \(\Delta\) is a polarising set, the three \((d + 1)\)-spaces are mentioned above, will always be denoted by \(\rho^i_{d+1}(\Delta)\), \(i = 1, 2, 3\). If \(\Delta = \{A_1, A_2, \ldots, A_k\}\), then the corresponding \((d + 1)\)-spaces will also be denoted by \(\rho^i_{d+1}(A_1, A_2, \ldots, A_k)\), \(i = 1, 2, 3\). Note that if \(S, S' \in \rho^i_{d+1}(\Delta)\), where \(S, S' \in \mathbb{P} - \sum_{n-1}\), then \(S + S' \in \langle \Delta \rangle\).

§4. Generators and Axis of a Cone.

In this section \(\mathbb{P}\) will always denote an \(n\)-dimensional tangential 2-block in \(\text{PG}(n, 2)\).

Let \(g_i(P) = \{P, A_i, B_i\}\), \(i = 1, 2, 3\), be any three lines of \(\mathbb{P}\). Then these three concurrent lines must generate a 3-space, say \(C_3(P)\), for \(n \geq 3\) and three concurrent and coplanar lines of \(\mathbb{P}\) would form a Fano block contained in \(\mathbb{P}\) and lead to a contradiction. Also note that the plane \(\langle P, A_1 + A_2, A_1 + A_3 \rangle\), denoted by \(V_2(P)\), is the only plane in \(C_3(P)\), which meets the lines \(g_1(P)\), \(g_2(P)\) and \(g_3(P)\) only in \(P\). Let the line \(\{P, A_1 + A_2 + A_3, A_1 + A_2 + B_3\}\) in \(C_3(P)\) be denoted by \(Q(P)\).

Definition. If \(g_i(P) = \{P, A_i, B_i\}\) be three concurrent lines of \(\mathbb{P}\), then \(g_1(P)\), \(g_2(P)\) and \(g_3(P)\) are called generators of a cone in \(C_3(P)\) and the line \(Q(P) = \{P, A_1 + A_2 + A_3, A_1 + A_2 + B_3\}\) is called the axis of the cone; \(P\) is called the vertex of the cone.
Theorem 2.6. Let \( g_1(P) = \{P, A_i, B_i\}, i = 1, 2, 3, \) be three concurrent lines of \( \mathcal{B} \).

(i) If \( \mathcal{L} \) be a line of \( C_3(P) \) through a point of \( \mathcal{A}(P) - \{P\} \), then \( \mathcal{L} \) must meet one of the generators \( g_1(P), g_2(P), \) and \( g_3(P) \).

(ii) \( \mathcal{A}(P) \cap \mathcal{B} = \{P\} \)

(iii) If \( n \geq 4 \) and \( \emptyset \neq \mathcal{O} \subseteq \mathcal{B} - C_3(P) \) such that \( \dim <\mathcal{O}> \leq n - 2 \) and \( <\mathcal{O}> \cap C_3(P) = \emptyset \), then \( t(\mathcal{O}) \cap \mathcal{A}(P) = \emptyset \).

(iv) If \( n \geq 4 \) and \( \emptyset \neq \mathcal{O} \subseteq \mathcal{B} - C_3(P) \) such that \( \dim <\mathcal{O}> \leq n - 2 \), then \( <\mathcal{O}> \cap C_3(P) \neq \mathcal{A}(P) - \{P\} \), unless \( <\mathcal{O}> \cap C_3(P) = \emptyset \).

(v) \( \nu_2(P) \) does not contain any odd stigm of \( \mathcal{B} \).

(vi) If \( n \geq 4 \) and \( X \in \mathcal{A}(P) - \{P\} \), then \( X \) is an attenuation point for \( \mathcal{B} \) with respect to \( C_3(P) \).

Proof: (i) Let \( X \in \mathcal{A}(P) - \{P\} \). It is easy to check that all the seven lines thru \( X \) in \( C_3(P) \) are obtained by joining \( X \) to the seven points on the generators \( g_1(P), g_2(P) \) and \( g_3(P) \). From this fact, proof of (i) follows immediately.

(ii) If possible, let \( \mathcal{A}(P) \cap \mathcal{B} - \{P\} \neq \emptyset \).

WLOG assume \( A_1 + A_2 + A_3 \in \mathcal{B} \). Then \( t(A_1 + A_2 + A_3) \) must contain a line of \( C_3(P) \) through \( A_1 + A_2 + A_3 \). So by (i), \( t(A_1 + A_2 + A_3) \) contains a point of the generators, which is not possible. So \( \mathcal{A}(P) \cap \mathcal{B} = \{P\} \).
(iii) If possible, let \( t(\Omega) \cap \Omega(p) \neq \emptyset \). Since \( \langle \Omega \rangle \cap C_3(p) = \emptyset \), \( p \in t(\Omega) \). WLOG assume \( A_1 + A_2 + A_3 \in t(\Omega) \). So \( t(\Omega) \) contains a line of \( C_3(p) \) through \( A_1 + A_2 + A_3 \). Now using (i) and the fact that \( \langle \Omega \rangle \cap C_3(p) = \emptyset \), we arrive at a contradiction.

(iv) If possible, let \( \langle \Omega \rangle \cap C_3(p) \neq \emptyset \) and \( \langle \Omega \rangle \cap C_3(p) \subset \Omega(p) - \{p\} \). This shows that \( \langle \Omega \rangle \) contains exactly one point \( X \) of \( C_3(p) \) such that \( X \in \Omega(p) - \{p\} \). WLOG assume \( X = A_1 + A_2 + A_3 \). Then \( t(\Omega) \) must contain a line of \( C_3(p) \) through \( A_1 + A_2 + A_3 \). Since \( \langle \Omega \rangle \cap C_3(p) = \{A_1 + A_2 + A_3\} \), we arrive at a contradiction by virtue of (i).

(v) If possible, let \( \nu_2(p) \) contain an odd stigm of \( \mathcal{B} \), say \( St \). Then \( St \) is a line of \( \mathcal{B} \). If \( p \in St \), then WLOG assume \( St = \langle p, A_1 + A_2 \rangle \). This implies \( \langle p, A_1, A_2 \rangle \) is a Fano block in \( \mathcal{B} \), which is a contradiction. If \( p \notin St \), then WLOG assume \( St = \{A_1 + A_2, A_1 + A_3, A_2 + A_3\} \). Since \( n \geq 4 \), \( \exists S \in \mathcal{B} - C_3(p) \). It is clear that \( t(S) \cap \langle A_1, A_2, A_3 \rangle = \{A_1 + A_2 + A_3\} \). But \( \langle S \rangle \cap C_3(p) = \emptyset \); so by (iii), we arrive at a contradiction.

(vi) If possible, let \( X \) be not an attenuation point. \( X \notin \mathcal{B} \); so, \( \exists S_1, S_2 \in \mathcal{B} - C_3(p) \) such that \( S_1 + S_2 = X \). This shows that \( \langle S_1, S_2 \rangle \cap C_3(p) \subset \Omega(p) - \{p\} \), which contradicts (iv). Hence the theorem.
§5. $(r_1, r_2, r_3)$-tangential stigm system.

In this section we assume $\mathcal{B}$ to be an $n$-dimensional tangential 2-block. In our investigation of a hypothetical tangential 2-block $\mathcal{B}$, we use the following notation.

Let $P \in \mathcal{B}$ and $\pi(P)$ denote a tangent of $P$. Then $\pi(P)$ is an $(n - 2)$-space of $\mathbb{P}G(n, 2)$, such that $\pi(P) \cap \mathcal{B} = \{P\}$. There are exactly three $(n - 1)$-spaces on $\pi(P)$ and we denote them by $\tau_i(P)$, $i = 1, 2, 3$. Then each $\tau_i(P)$ has an odd stigm [by prop. 1.10.] containing $P$; we denote this odd stigm by $St_i(P)$, $i = 1, 2, 3$. We denote the space $\langle St_i(P) \rangle \cap \pi(P)$ by $\lambda_i(P)$, $i = 1, 2, 3$.

Definition. Let $P \in \mathcal{B}$ and $St_i(P)$ be an $r_i$-stigm, where $r_i$ is an odd integer and $r_i \geq 3$. Then $St_1(P), St_2(P)$ and $St_3(P)$ are said to constitute a $(r_1, r_2, r_3)$-tangential stigm system with respect to $P$.

Proposition 2.5. (i) If $P \in \mathcal{B}$, then $P$ has a $(r_1, r_2, r_3)$-tangential stigm system for some odd integers $r_1, r_2, r_3$, where $r_i \geq 3$, $i = 1, 2, 3$.

(ii) Let $\sigma$ be a permutation on $\{1, 2, 3\}$ and $r_i$ is an odd integer $\geq 3$, for $i = 1, 2, 3$. Then existence (or non-existence) of $(r_1, r_2, r_3)$-tangential stigm system with respect to a point $P$ of $\mathcal{B}$ implies existence (or non-existence) of $(r_{\sigma(1)}, r_{\sigma(2)}, r_{\sigma(3)})$-tangential stigm system with respect to the point $P$. 
Proof: (i) Proof follows immediately from the discussion made in the beginning of this section.

(ii) Proof is obvious, since we can rearrange the suffixes of $\tau_i(p), i = 1, 2, 3$.

§ 6. Non-tangential Subspace in $PG(n, 2)$.

Let $\mathcal{B}$ denote an $n$-dimensional tangential $2$-block in $PG(n, 2)$.

Definition. Let $m$ be a non-negative integer such that $0 \leq m \leq n - 3$. An $m$-space $\Sigma_m$ in $PG(n, 2)$ is said to be non-tangential if and only if for every $(n - 2)$-space $\Sigma_{n-2}$, $\Sigma_{n-2} \cap \Sigma_m \Rightarrow \Sigma_{n-2} \cap \mathcal{B} \not\subset \Sigma_m \cap \mathcal{B}$.

Thus $\Sigma_m$ is non-tangential iff $\Sigma_m$ cannot be extended into a tangent of $\Sigma_m \cap \mathcal{B}$; in other words, any tangent $t(\Sigma_m \cap \mathcal{B})$ of $\Sigma_m \cap \mathcal{B}$ does not contain $\Sigma_m$.

Proposition 2.6. Let $\Sigma_k$ and $\Sigma_{n-k-1}$ be two subspaces in $PG(n, 2)$, where $0 \leq k \leq n - 3$, $\Sigma_k \cap \mathcal{B} = \emptyset$ and $\Sigma_k \cap \Sigma_{n-k-1} = \emptyset$. Then $\mathcal{P}(\mathcal{B}, \Sigma_k, \Sigma_{n-k-1})$ is an $(n - k - 1)$-dimensional tangential $2$-block provided there does not exist any non-tangential $m$-space $\Sigma_m$ such that $\Sigma_k \subset \Sigma_m$.

Proof: For brevity, let $\mathcal{B}' = \mathcal{P}(\mathcal{B}', \Sigma_k', \Sigma_{n-k-1})$. Then by proposition 2.2, $\mathcal{B}'$ is an $(n - k - 1)$-dimensional $2$-block. If possible, let $\mathcal{B}'$ be not tangential. Then by proposition 2.1, there exists a non-empty closed subset
\( \Delta' \) of \( \mathcal{B}' \) such that \( \dim <\Delta'> \leq n - k - 4 \) and \( \Delta' \) does not have a tangent in \( \mathcal{B}' \). Let \( \sum_m = <\Sigma_k, \Delta'> \). We claim that \( \sum_m \) is non-tangential. First of all, \( m = \dim <\Sigma_k, \Delta'> = \dim \Delta' + k + 1 \leq n - k - 4 + k + 1 = n - 3 < n - 2 \). Let \( \Delta = \sum_m \cap \mathcal{B} \). Clearly, \( \Delta \neq \emptyset \), since \( \Delta' \neq \emptyset \). If possible, let \( \sum_m \) be not non-tangential. Then there must exist an \((n - 2)\)-space \( \sum_{n-2} \) such that \( \sum_{n-2} \supset \sum_m \) and \( \sum_{n-2} = t(\sum_m \cap \mathcal{B}) = t(\Delta) \). Let \( \emptyset = \sum_{n-2} \cap \sum_{n-k-1} \). Since \( \sum_{n-2} \supset \Sigma_k, <\Delta'> \subseteq \emptyset \). It is also clear that \( \dim <\emptyset> = n - k - 3 \). Now, let \( A' \in \mathcal{B}' \) and \( A' \in \mathcal{B} \). If \( A' \in \mathcal{B} \), then \( A' \in \sum_{n-2} \) i.e. \( A' \in t(\Delta) \). Therefore, \( A' \in <\Delta> = <\Sigma_k, \Delta'> \); so, \( A' \in <\Delta> \), since \( <\Sigma_k, \Delta'> \cap \sum_{n-k-1} = <\Delta'> \). So we assume that \( A' \notin \mathcal{B} \). Then \( \exists \ S \in \mathcal{B} \) and \( C \in \sum_k \) such that \( A' = S + C \). Since \( A' \in \emptyset \) and \( \sum_k \subseteq t(\Delta) \), it follows that \( S = A' + C \in t(\Delta) \), i.e. \( A' + C \in <\Delta> \subseteq <\Sigma_k, \Delta'> \). But \( C \in \sum_k \); so \( A' \in <\Sigma_k, \Delta'> \), whence we get \( A' \in <\Delta'> \). Thus \( \emptyset \) is a tangent of \( \Delta' \) in \( \mathcal{B}' \). But this is a contradiction. Hence the proposition is proved.

\( \S 7 \). A few results on tangential 2-block.

**Theorem 2.7.** Let \( \mathcal{B} \) be an \( n \)-dimensional tangential 2-block in \( \text{PG}(n, 2) \), where \( n \geq 4 \). Let \( \sum_{n-1} \) be any \((n - 1)\)-space in \( \text{PG}(n, 2) \). Then \( |\mathcal{B} - \sum_{n-1}| \geq 5 \).
Proof: We prove proposition 3 with the help of the following two lemmas.

**Lemma 1.** \(|\mathcal{B} - \Sigma_{n-1}| \geq 4\), notations being those of proposition 3.

Proof: Clearly, \(\mathcal{B} - \Sigma_{n-1} \neq \emptyset\). Let \(S \in \mathcal{B} - \Sigma_{n-1}\). Then \(t(S) \cap \Sigma_{n-1}\) is an \((n - 3)\)-space, say \(\Sigma_{n-3}\). Now the number of \((n - 2)\)-spaces of \(\Sigma_{n-3}\), but not in \(\Sigma_{n-1}\) is four; let these four \((n - 2)\)-spaces be denoted by \(\Sigma_{n-2}^i, i = 0, 1, 2, 3\). Then \(t(S) = \Sigma_{n-2}^i\) for some \(i \in \{1, 2, 3, 4\}\). WLOG assume \(t(S) = \Sigma_{n-2}^0\). Since \(t(S) \cap \Sigma_{n-1} = \Sigma_{n-3}\) cannot contain any point of \(\mathcal{B} \cap \Sigma_{n-1}\) and \(\mathcal{B}\) is a 2-block, it follows that each of \(\Sigma_{n-2}^1, \Sigma_{n-2}^2, \Sigma_{n-2}^3\) must contain at least one point of \(\mathcal{B} - \Sigma_{n-1}\). Hence \(|\mathcal{B} - \Sigma_{n-1}| \geq 4\).

**Lemma 2.** If \(|\mathcal{B} - \Sigma_{n-1}| = 4\), then \(S + S' \in \mathcal{B} \cap \Sigma_{n-1}\), whenever \(S \neq S'\) and \(S, S' \in \mathcal{B} - \Sigma_{n-1}\); notations are those of proposition 3.

Proof: If possible, let \(S, S' \in \mathcal{B} - \Sigma_{n-1}\) such that \(S \neq S'\) and \(S + S' \notin \mathcal{B} \cap \Sigma_{n-1}\). Then \(t(S, S') \cap \Sigma_{n-1}\) is an \((n - 3)\)-space, say \(\Sigma_{n-3}\), and since \(S, S' \notin \mathcal{B} \cap \Sigma_{n-1}\) \(\mathcal{B} \cap \Sigma_{n-1}, \Sigma_{n-3} \cap \mathcal{B} = \emptyset\). Let \(\Sigma_{n-2}^i, i = 0, 1, 2, 3\), be the four \((n - 2)\)-spaces, which are in \(\Sigma_{n-3}\), but not in \(\Sigma_{n-1}\). WLOG assume \(\Sigma_{n-2}^0 = t(S, S')\). Then each of \(\Sigma_{n-2}^1, \Sigma_{n-2}^2, \Sigma_{n-2}^3\) must contain at least one point from \(\mathcal{B} - \Sigma_{n-1}\). Thus \(|\mathcal{B} - \Sigma_{n-1}| \geq 5\), which contradicts our hypothesis. Hence our lemma is proved.
Proof of proposition 2.7. We want to show that

$$|\mathcal{B} - \Sigma_{n-1}| \geq 5.$$  

If not, we must have $$|\mathcal{B} - \Sigma_{n-1}| = 4$$, by lemma 1. Let

$$S_1, S_2, S_3, \text{ and } S_4$$

be the four distinct points in $$\mathcal{B} - \Sigma_{n-1}$$. Then $$S_1, S_2, S_3, \text{ and } S_4$$ are linearly independent; for

$$S_4 \in \langle S_1, S_2, S_3 \rangle \implies S_4 = S_1 + S_2 + S_3 \implies S_1 + S_2 +$$

$$S_3 \in \mathcal{B} \iff \langle S_1, S_2, S_3 \rangle \text{ is a Fano block contained in } \mathcal{B} \text{ [by lemma 2]}. \text{ But this is not possible, since } \mathcal{B},$$

being a tangential 2-block, is a minimal 2-block and

$$\dim \langle \mathcal{B} \rangle = n \geq 4. \quad \therefore \quad \langle S_1, S_2, S_3, S_4 \rangle \text{ is a 3-space, say } \Sigma_3.$$  

Assert that every line in $$\Sigma_3$$ must contain a point of $$\Sigma_3 \cap \mathcal{B}$$.

By virtue of lemma 2, we can say that the only points that are probably not in $$\mathcal{B} \cap \Sigma_3$$ are points of the set

$$\Delta, \text{ where } \Delta = \{S_1 + S_2 + S_3, S_1 + S_2 + S_4, S_1 + S_3 + S_4,$$

$$S_2 + S_3 + S_4, S_1 + S_2 + S_3 + S_4\}.$$  

It is very easy to see that if $$l$$ is any line in $$\Sigma_3$$, then

$$l \not\subset \Delta. \text{ Also } \Sigma_3 - \Delta \subset \mathcal{B}. \text{ Therefore, } l \text{ must contain a point of } \Sigma_3 \cap \mathcal{B}.$$  

Let $$S \subset \mathcal{B} - \Sigma_3$$. [Note that $$\mathcal{B} - \Sigma_3 \neq 0, \text{ since } n \geq 4$$. Then $$t(S)$$ must contain a line, say $$l$$, of $$\Sigma_3$$. But $$l$$ contains a point of $$\Sigma_3 \cap \mathcal{B}$$; so $$t(S)$$ must contain a point of $$\Sigma_3 \cap \mathcal{B}$$. But this is not possible, since $$S \not\subset \Sigma_3$$. Hence the theorem is proved.
Theorem 2.8. Let $\mathcal{B}$ be an $n$-dimensional tangential 2-block, where $n \geq 3$. Let $P \in \mathcal{B}$ and let $St$ be any $(n+1)$-stigm such that $P \in St$. If $l_1$ and $l_2$ be any two lines of $\mathcal{B}$ through $P$, then at least one of $l_1$ and $l_2$ must lie in $<St>$.

Proof: Let $St = \{P, Q_1, Q_2, \ldots, Q_n\}$. Since $Q_1, Q_2, \ldots, Q_n$ are linearly independent, it follows immediately that $<Q_1 + Q_2, Q_1 + Q_3, \ldots, Q_1 + Q_{n-1}, Q_1 + Q_n>$ is an $(n-2)$-space, say $\sum_{n-2}$.

Let $\tau_1, \tau_2$ and $\tau_3$ be the three $(n-1)$-spaces on $\sum_{n-2}$. Then $<St> = \tau_i$, for some $i \in \{1, 2, 3\}$. WLOG assume that $\tau_1 = <St>$. We are now to show that at least one of $l_1$ and $l_2$ is in $\tau_1$.

If not, then both $l_1 - \{P\}$ and $l_2 - \{P\}$ does not intersect $\tau_1$.

Let $l_1 = \{P, L_1, L_2\}$ and $l_2 = \{P, M_1, M_2\}$. Then $L_i, M_i \in \mathcal{B} - \tau_1, i = 1, 2$. Therefore, $L_1 + M_j \in \tau_1 = <Q_1, Q_2, \ldots, Q_n>, i, j \in \{1, 2\}$. Let $L_1 + M_1 = \sum_{i=1}^{n} k_i Q_i$, $k_i \in \{0, 1\} \forall i$, all $k_i$'s are not zero. $L_1 + M_2 = \sum_{i=1}^{n} q_i Q_i$, $q_i \in \{0, 1\} \forall i$, all $q_i$'s are not zero. Let $\mu = \{k_1, k_2, \ldots, k_n\} \text{ and } \mu' = \{q_1, q_2, \ldots, q_n\}$. Let $J = \{i \mid k_i \neq 0, k_i \in \mu\}$ and $I = \{i \mid q_i \neq 0, q_i \in \mu'\}$. Since $L_1 + M_1 + L_2 + M_2 = P = \sum_{i=1}^{n} Q_i$, it follows that $J \cap I = \emptyset$. Also $|J| \geq 1$, $|I| \geq 1$ and $|J| + |I| = n \geq 3$.

Let $j \in J$ and $i \in I$. Then $J - \{j\}$ and $I - \{i\}$ are not both empty. Let $\Delta = \{Q_k \mid k \in (J - \{j\}) \cup (I - \{i\})\}$. Clearly, $\Delta \neq \emptyset$, since both $J - \{j\}$ and $I - \{i\}$ are not
empty. Also \( Q_i, Q_j \notin \triangle \), and \(|\triangle| = n - 2\). So tangent of 
\( \triangle \) exists. Let \( \sum_2 = \langle P, L_1, M_1 \rangle = 
\{P, L_1, L_2, M_1, M_2, L_1 + M_2, L_1 + M_1 + M_2 \}. \) Since \( \langle \triangle \rangle \subset \tau_1 \)
and \( L_1, M_i \not\in \tau_i \), \( i = 1, 2 \), it follows that \( L_1, M_i \not\in t(\triangle) \), for \( i = 1, 2 \). Clearly, \( P \not\in \triangle \), so that \( P \not\in t(\triangle) \). Now \( L_1 + M_1 \in t(\triangle) \Rightarrow \sum_{t \in J} Q_t \in t(\triangle) \Rightarrow 
Q_j \in \langle \triangle \rangle \) \( \because Q_t \in \triangle \), \( \forall t \in J - \{j\} \). But this is not possible, since \( Q_j \notin \triangle \) and \( Q_i \)'s are all linearly independent. Thus \( L_1 + M_1 \notin t(\triangle) \). Similarly, \( L_1 + M_2 \notin t(\triangle) \). \( \therefore t(\triangle) \cap \sum_2 = \emptyset \), which is a contradiction.

Hence the theorem is proved.

Theorem 2.9. Let \( \mathcal{B} \) be an \( n \)-dimensional tangential 2-block, where \( n \geq 3 \). Let \( \sum_{n-2} \) be an \((n - 2)\)-space and let \( \tau_1, \tau_2 \) and \( \tau_3 \) be the three \((n - 1)\)-spaces on \( \sum_{n-2} \) in \( \text{PG}(n, 2) \). Let \( \text{St}_i = \{P, A_i, B_i, C_i, D_i\} \) be a 5-stigm of \( \mathcal{B} \) in \( \tau_i \) such that \( \text{St}_i \cap \sum_{n-2} = \{P\} \), for \( i = 1, 2 \). Let \( \langle \text{St}_1 \rangle \cup \sum_{n-2} = \lambda_i \), \( i = 1, 2 \). Then \( \lambda_1 \neq \lambda_2 \).

**Proof:** If possible, let \( \lambda_1 = \lambda_2 \). Now \( \lambda_1 = 
\{P, A_1 + B_1, C_1 + D_1, A_1 + C_1, A_1 + D_1, B_1 + C_1, B_1 + D_1\} \).

Any point of \( \lambda_2 - \{P\} = \lambda_1 - \{P\} \) is the sum of two points of \( \{A_2, B_2, C_2, D_2\} \). \( \text{WLOG} \) we can assume that \( A_1 + B_1 = A_2 + B_2 \). So, \( C_1 + D_1 = C_2 + D_2 \). Therefore, \( A_1 + C_1 = X_2 + Y_2 \), where \( X_2 \in \{A_2, B_2\} \) and \( Y_2 \in \{C_2, D_2\} \).

But we can interchange \( A_2 \) and \( B_2 \) as well as \( C_2 \) and \( D_2 \) without affecting anything. So we can assume \( \text{WLOG} \) that
Thus we have the following relations:

\[ A_1 + B_1 = A_2 + B_2, \quad C_1 + D_1 = C_2 + D_2, \quad A_1 + C_1 = A_2 + C_2, \quad \]
\[ A_1 + D_1 = A_2 + D_2, \quad B_1 + C_1 = B_2 + C_2, \quad B_1 + D_1 = B_2 + D_2. \]

Let \( \sum_2 = \langle A_1, A_2, B_2 \rangle \) and \( \Theta = \tau( \mathcal{P}, D_2 ) \). Clearly, \( A_1, A_2, B_2 \) and \( A + B_1 = A_2 + B_2 \) are not in \( \Theta \). \( A_1 + A_2 \in \Theta \implies A_1 + A_2 + D_2 = A_1 + A_1 + D_1 = D_1 \in \Theta \implies D_1 \in \langle \mathcal{P}, D_2 \rangle \), which is a contradiction. \( A_1 + B_2 \in \Theta \implies A_1 + B_2 + D_2 + P = A_1 + B_1 + D_1 + P = C_1 \in \Theta \implies C_1 \in \langle \mathcal{P}, D_2 \rangle \), which is a contradiction. \( \therefore \Theta \cap \sum_2 = \emptyset \), which is a contradiction. Hence the theorem is proved.


**Theorem 2.10.** There does not exist any 4-dimensional tangential 2-block in \( \text{PG}(n, 2) \).

**Proof:** If possible, let \( \mathcal{B} \) be a 4-dimensional tangential 2-block in \( \text{PG}(4, 2) \). Let \( \mathcal{P} \in \mathcal{B} \). Then by Theorem 2.8 and theorem 2.9, we can conclude that none of \( \text{St}_i(\mathcal{P}) \), \( i = 1, 2, 3 \), is a 5-stigm. Let \( \text{St}_i(\mathcal{P}) = \{ \mathcal{P}, A_i, B_i \} \), \( i = 1, 2, 3 \), and \( \mathcal{U}_2(\mathcal{P}) = \langle \mathcal{P}, A_1 + A_2, A_1 + A_3 \rangle \). Let \( \tau_1, \tau_2, \text{ and } \tau_3 \) be the three 3-spaces on \( \mathcal{U}_2(\mathcal{P}) \), one of which is \( \mathcal{C}_3(\mathcal{P}) = \langle \text{St}_1(\mathcal{P}), \text{St}_2(\mathcal{P}), \text{St}_3(\mathcal{P}) \rangle \). WLOG assume \( \tau_1 = \mathcal{C}_3(\mathcal{P}) \). First we show that \( \mathcal{U}_2(\mathcal{P}) \cap \mathcal{B} = \{ \mathcal{P} \} \neq \emptyset \). If not, then \( \mathcal{U}_2(\mathcal{P}) \) is a tangent of \( \mathcal{P} \) and \( \tau_1 \) must contain a 3-stigm, say \( \text{St}_i = \{ \mathcal{P}, S_i, S_i^\perp \} \), for \( i = 1, 2 \). Then \( \langle \mathcal{P}, S_1, S_2 \rangle \) is a Fano block in \( \mathcal{B} \), since by theorem 2.6(vi), both
$S_1 + S_2$ and $S_1 + S_2$ are in $\mathcal{C}_3(P) - \mathcal{V}_2(P) \cup \mathcal{A}(P)$ and also
$\mathcal{C}_3(P) - \mathcal{V}_2(P) \cup \mathcal{A}(P) \subseteq \mathcal{B}$ [\mathcal{A}(P) = \text{axis of the cone}
\text{generated by } St_i(P), i = 1, 2, 3.] \text{ Thus we arrive at a}
\text{contradiction. So, } \mathcal{V}_2(P) \cap \mathcal{B} = \{P\} \neq \emptyset. \text{ WLOG assume}
A_1 + A_2 \in \mathcal{B}. \text{ As in case of } P, \text{ St}_i(A_1 + A_2), i = 1, 2, 3,
\text{are all lines of } \mathcal{B}. \text{ By theorem 2.6(v), none of } St_i
\text{St}_i(A_1 + A_2), i = 1, 2, 3, \text{ is in } \mathcal{V}_2(P). \text{ The four lines}
\text{through } A_1 + A_2, \text{ which are in } \mathcal{C}_3(P) = \mathcal{T}'_1, \text{ but not in}
\mathcal{V}_2(P), \text{ are } \{A_1 + A_2, A_1, A_2, A_1 + A_2, B_1, B_2\},
\{A_1 + A_2, A_3, A_1 + A_2 + A_3\} \text{ and } \{A_1 + A_2, B_3, A_1 + A_2 + B_3\},
two \text{of which are not lines of } \mathcal{B} [\text{by theorem 2.6(ii)}].
\text{So, one of the lines } St_i(A_1 + A_2), i = 1, 2, 3, \text{ is in } \mathcal{T}'_1,
\text{for some } j \in \{2, 3\}. \text{ WLOG assume } j = 2. \text{ Then there}
exist S_1, S_2 \in \mathcal{B} \cap \mathcal{T}'_2 - \mathcal{V}_2(P) \text{ such that } S_1 + S_2 =
A_1 + A_2. \text{ Again } \mathcal{T}'_3 \text{ must contain an odd stigm of } \mathcal{B}
\text{ and}
\mathcal{V}_2(P) \text{ does not contain any odd stigm of } \mathcal{B}; \text{ so there}
exist two points } S'_1, S'_2 \in \mathcal{B} \cap \mathcal{T}'_3 - \mathcal{V}_2(P). \text{ Again}
A_1 + A_2 \in \mathcal{B} \implies t(A_3) \cap \langle P, A_1, A_2 \rangle = \{A_1 + B_2\}.
\therefore \text{by theorem 2.4, } A_1 + B_2 \text{ is an attenuation point}
\text{with respect to } \mathcal{C}_3(P) \text{ and it is induced by } A_3. \text{ Also}
t(A_3) \not\subseteq \mathcal{C}_3(P), \text{ for } t(A_3) \text{ does not meet the line}
\{P, A_1, B_1\} \text{ of } \mathcal{C}_3(P). \text{ By theorem 2.5, } \{A_3\} \text{ is thus a polarising set with respect to } \mathcal{C}_3(P).
\therefore \exists \mathcal{P}_2^i(A_3), i = 1, 2, 3, \text{ such that } \mathcal{B} - \mathcal{C}_3(P) \subseteq \bigcup_{i=1}^{i=3} \mathcal{P}_2^i(A_3). \text{ Since}
S_1 + S_2 = A_1 + A_2 \text{ and } S'_1 + S'_2 \in \mathcal{V}_2(P), \text{ none of the pairs of}
\text{points } (S_1, S_2) \text{ and } (S'_1, S'_2) \text{ can be in the same } \mathcal{P}_2^i(A_3),
for some $i \in \{1, 2, 3\}$. By rearranging the suffixes of $S$'s and $S'$'s, we can assume (WLOG) that $S_i S'_i \subseteq \mathcal{P}_i(A_3)$, for some $i \in \{1, 2, 3\}$. \[ : S_1 + S' = A_3, \text{ but } S_1 + S_2 = A_1 + A_2. \] \[ : S_2 + S' = A_1 + A_2 + A_3, \text{ which violates theorem } 2.6 \text{ (vi). Hence the theorem is proved.} \]
CHAPTER III

Non-existence of \((7, r_2, r_3)\)-tangential stigm system for 6-dimensional tangential 2-block.

In this chapter as well as in the successive chapters, \(\mathcal{S}\) will always denote a 6-dimensional tangential 2-block in \(\text{PG}(6, 2)\); also the notations, used without explanation, can be found explained in Chapters I and II.

Let \(P \in \mathcal{S}\) and let \(P\) have an \((r_1, r_2, r_3)\)-tangential stigm system in \(\mathcal{S}\). We want to show that \(r_i \neq 7, i=1,2,3\).

\(\S 1\). Non-existence of \((7, 7, r_3)\)-tangential stigm system.

First of all, we shall show that no two of \(\text{St}_1(P), \text{St}_2(P), \text{St}_3(P)\), \(i = 1, 2, 3\), are 7-stigms.

Lemma 3.1. Let \(\text{St}_1(P) = \{P, A_1, A_2, A_3, A_4, A_5, A_6\}\) and \(X_1, X_2, X_3\) be three distinct points of \(\text{St}_1(P) - \{P\}\). Then \(t(P, X_1, X_2, X_3) \cap \Pi(P)\) is a 2-space.

Proof: WLOG we can take \(A_1, A_2, A_3\) for \(X_1, X_2, X_3\). Clearly, \(2 \leq \dim t(P, A_1, A_2, A_3) \cap \Pi(P) \leq 3\). If possible, let \(\dim t(P, A_1, A_2, A_3) \cap \Pi(P) = 3\). Now \(t(P, A_1, A_2, A_3) \cap \Pi(P) \supset \langle P, A_1 + A_2, A_1 + A_3 \rangle = \Sigma_2\), say. Also \(\Pi(P) = \langle P, A_1 + A_2, A_1 + A_3, A_1 + A_4, A_1 + A_5 \rangle\). \(t(P, A_1, A_2, A_3) \cap \Pi(P) = \langle \Sigma_2, \{X\} \rangle\), where \(X \in \Pi(P) - \Sigma_2\). WLOG we can assume that \(X \in\)
\[ <A_1 + A_4, A_1 + A_5> \cdot X = A_1 + A_4 \implies A_1 + A_4 \in t(P, A_1, A_2, A_3) \implies A_4 \in t(P, A_1, A_2, A_3) \implies A_4 \in <P, A_1, A_2, A_3>, \] which is a contradiction. Therefore, 

\[ X \neq A_1 + A_4. \] Similarly, \[ X \neq A_1 + A_5. \] Again \[ X = A_4 + A_5 \implies A_6 \in <P, A_1, A_2, A_3>, \] which is not possible. Thus we arrive at a contradiction. This completes the proof.

Lemma 3.2. Let \( St_1(P) = \{P, A_1, A_2, A_3, A_4, A_6\} \).

\( St_2(P) = \{P, B_1, B_2, B_3, B_4, B_5, B_6\} \) and \( X_1, X_2 \) and \( X_3 \) be any three distinct points in \( St_1(P) - \{P\} \). Then

(i) \( \{P, X_1, X_2, X_3\} \) form a polarising set for \( \mathcal{B} \) with respect to the 5-space \( \mathcal{T}_1(P) \);

(ii) None of \( \rho_i^4(P, X_1, X_2, X_3), i = 1, 2, 3 \) can contain more than two points of \( St_2(P) - \{P\} \).

Proof: (i) By theorem 2.5, proof is immediate.

(ii) If possible, let there exist \( X_1, X_2, X_3 \) and \( i \) such that \( \rho_i^4(P, X_1, X_2, X_3) \) contains at least three points of \( St_2(P) - \{P\} \). \( \text{WLOG} \) we assume \( X_j = A_j, j = 1, 2, 3 \) and \( i = 1, 2, 3 \). Therefore, \( B_i + B_j \in \rho_i^4(P, A_1, A_2, A_3), u = 1, 2, 3 \). Therefore, \( B_i + B_j \in <P, A_1 + A_2, A_1 + A_3> \cap \mathcal{T}_2(P) = <P, A_1 + A_2, A_1 + A_3> \), say, for \( i \neq j \), and \( i, j = 1, 2, 3 \). So, \( <\sum_2, \{B_i\}> = <\sum_2, \{B_j\}> \) for all \( i, j \in \{1, 2, 3\} \). But \( <\sum_2, \{B_i\}>, i = 1, 4, 5, 6 \) are all distinct, since \( \sum_2 = <P, B_1 + B_2, B_1 + B_3> \) and none of \( B_i + B_j, i \neq j, i = 1, 4, 5, 6 \) and \( j = 4, 5, 6 \), is in \( \sum_2 \). From lemma 3.1,
it follows that \( t(P, A_1, A_2, A_3) \) meets \( \tau_2(P) \) in a 3-space, say \( \Sigma_3 \), which is on \( \Sigma_2 \), but not in \( \tau(P) \). The number of 3-spaces in \( \tau_2(P) \), which are on \( \Sigma_2 \), but not in \( \tau \), is four. Thus from above we can conclude that

\[
\Sigma_3 = \langle \Sigma_2, B_i \rangle,
\]

for some \( i \in \{1, 4, 5, 6\} \). So, \( B_i \in t(P, A_1, A_2, A_3) \) i.e. \( B_i \in \langle P, A_1, A_2, A_3 \rangle \), which is not possible. This completes the proof.

**Remark:** Since \( |St_2(P) - \{P\}| = 6 \), it follows from lemma 3.2 that each of \( \rho_i^1(P, X_1, X_2, X_3) \) \( i=1,2,3 \) contains exactly two points of \( St_2(P) - \{P\} \), where \( X_1, X_2 \) and \( X_3 \) are any three distinct points of \( St_2(P) - \{P\} \).

Let \( St_1(P) = \{P, A_1, A_2, A_3, A_4, A_5, A_6\} \) and \( St_2(P) = \{P, B_1, B_2, B_3, B_4, B_5, B_6\} \). Let us consider \( t(P, A_i, A_j, A_k) \) where \( i, j, k \) are three distinct elements in \( I_6 = \{1, 2, 3, 4, 5, 6\} \). By the above remark there exist three pairs \( (n_1, n_2) \), \( (n_3, n_4) \), and \( (n_5, n_6) \), where \( n_u \in I_6 \) for all \( u \in I_6 \), and \( n_u \neq n_t \) if \( u \neq t \), such that no two of the three pairs of points \( \{B_{n_1}, B_{n_2}\} \), \( \{B_{n_3}, B_{n_4}\} \) and \( \{B_{n_5}, B_{n_6}\} \) are in the same \( \rho_i^1(P, A_i, A_j, A_k) \), \( \forall i \in \{1, 2, 3\} \), but each of these three pairs is in \( \rho_i^1(P, A_i, A_j, A_k) \), for some \( i \in \{1, 2, 3\} \). Thus the tangent \( t(P, A_i, A_j, A_k) \) induces a partition \( M(i, j, k) \) on \( I_6 \) into the three pairs \( (n_1, n_2) \), \( (n_3, n_4) \) and \( (n_5, n_6) \), so that \( I_6 = \{n_1, n_2\} \cup \{n_3, n_4\} \cup \{n_5, n_6\} \).

Each pair \( (n_u, n_t) \), \( u \neq t, u, t \in I_6 \), is called a block.

We shall also use the notation \( M(i, j, k) \) for the set of
blocks \{(n_1, n_2), (n_3, n_4), (n_5, n_6)\} induced by the tangent \(t(P, A_i, A_j, A_k)\).

Observe that for every triplet \((i, j, k)\), \(i, j, k \in I_6\) and \(i, j, k\) are all distinct, \(M(i, j, k)\) consists of three mutually disjoint blocks, each block consists of two distinct elements of \(I_6\) and union of the three blocks is \(I_6\). Also note that \((u, t)\) is a block of \(M(i, j, k)\) iff \(B_u + B_t \in \langle \langle P, A_i, A_j, A_k \rangle \cap \cap T(P)\).

**Lemma 3.3.** Let \(St_1(P) = \{P, A_1, A_2, A_3, A_4, A_5, A_6\}\) and \(St_2(P) = \{P, B_1, B_2, B_3, B_4, B_5, B_6\}\). Let \(i, j, k, \ell, m, n\) be the six distinct elements of \(I_6 = \{1, 2, 3, 4, 5, 6\}\). Neither \(M(i, j, k)\) and \(M(\ell, m, n)\) nor \(M(i, j, k)\) and \(M(i, \ell, m)\) can have any block in common.

**Proof:** If possible, let \(M(i, j, k)\) and \(M(\ell, m, n)\) have a block, say \((u, t)\), in common. This implies that \(B_u + B_t \in \langle \langle P, A_i, A_j, A_k \rangle \cap \langle P, A_\ell, A_m, A_n \rangle \cap \cap T(P) = \{P\}\), which is a contradiction.

Again \(\langle \langle P, A_i, A_j, A_k \rangle \cap \langle P, A_\ell, A_m \rangle \cap \cap T(P) = \{P\}\); so we can show similarly as above that \(M(i, j, k)\) and \(M(i, \ell, m)\) cannot have any block in common.

**Lemma 3.4.** Let \(St_1(P) = \{P, A_1, A_2, A_3, A_4, A_5, A_6\}\) and \(St_2(P) = \{P, B_1, B_2, B_3, B_4, B_5, B_6\}\). Let \(i, j, k, \ell\) be four distinct elements in \(I_6 = \{1, 2, 3, 4, 5, 6\}\). Then \(M(i, j, k)\) and \(M(i, j, \ell)\) have one and exactly one block in common.
Proof: Suppose the lemma is not true. WLOG we can assume that either (i) \( M(1, 2, 3) \) and \( M(1, 2, 4) \) have more than one block in common or (ii) \( M(1, 2, 3) \) and \( M(1, 2, 4) \) have no block in common. WLOG we assume that the blocks of \( M(1, 2, 3) \) are \((1, 2), (3, 4), \) and \((5, 6)\).

Case (i). Let \( M(1, 2, 3) \) and \( M(1, 2, 4) \) have two blocks in common and assume (WLOG) that \((1, 2)\) and \((3, 4)\) are their two common blocks. So, we conclude that

\[
\langle p, a_1 + a_2 \rangle = \langle p, a_1, a_2, a_3 \rangle \cap \langle p, a_1, a_2, a_4 \rangle \cap \pi(p)
\]

\[
\geq \langle p, b_1 + b_2, b_3 + b_4 \rangle = \text{a plane. But this is a contradiction. Thus case (i) is not possible.}
\]

Case (ii). Suppose \( M(1, 2, 3) \) and \( M(1, 2, 4) \) have no block in common. Since blocks of \( M(1, 2, 3) \) are \((1, 2), (3, 4)\) and \((5, 6)\), \((1, 2)\) is not a block of \( \Delta(1, 2, 4) \). This means that \( B_1 \) and \( B_2 \) are not in the same \( \rho_4^i(p, a_1, a_2, a_4) \), for any \( i \in \{1, 2, 3\} \). WLOG we assume that \( B_1 \in \rho_4^i(p, a_1, a_2, a_4), i = 1, 2 \). Using the same argument for block \((3, 4)\), and rearranging the suffixes and superscripts, we can assume (WLOG) that \( B_3 \in \rho_4^1(p, a_1, a_2, a_4) \). Then \( B_4 \notin \rho_4^2(p, a_1, a_2, a_4) \), for, otherwise, \( B_5, B_6 \in \rho_4^3(p, a_1, a_2, a_4) \) [by lemma 3.2], so that \((5, 6)\) would become a common block of \( M(1, 2, 3) \) and \( M(1, 2, 4) \). Again \((5, 6)\) is not a block of \( M(1, 2, 4); \) so, by using lemma 3.2 and rearranging the suffixes of \( B_5 \) and \( B_6 \) we assume (WLOG) that \( B_5 \in \rho_4^2(p, a_1, a_2, a_4) \). Therefore, \( B_6 \notin \rho_4^2(p, a_1, a_2, a_4) \).
$\rho_4^{3}(P, A_1, A_2, A_4)$ [by lemma 3.2]. Hence the blocks of $M(1, 2, 4)$ are $(1, 3), (2, 5)$ and $(4, 6)$. Let $\Sigma_2^1 = \langle B_1 + B_2, B_3 + B_4, B_5 + B_6 \rangle$ and $\Sigma_2^2 = \langle B_1 + B_3, B_2 + B_5, B_4 + B_6 \rangle$. We claim that $\Sigma_2^1 \cap \Sigma_2^2 = \{P\}$. If not, these two planes will generate a space of dimension less than or equal to 3. But it is easy to check that $\langle \Sigma_2^1, \Sigma_2^2 \rangle = \langle B_1 + B_2, B_1 + B_3, B_1 + B_4, B_1 + B_5, B_1 + B_6 \rangle$ which is a 4-space, since $St_2(P)$ is a 7-stigm. Therefore, $\Sigma_2^1 \cap \Sigma_2^2 = \{P\}$. But $\Sigma_2 = t(P, A_1, A_2, A_3) \cap \cap(P)$ and $\Sigma_2 = t(P, A_1, A_2, A_4) \cap \cap(P)$; so, $\langle P, A_1 + A_2 \rangle = \Sigma_2^1 \cap \Sigma_2^2 = \{P\}$, which is a contradiction. Thus case (ii) cannot happen. This completes the proof.

**Proposition 3.1.** There cannot be more than one 7-stigm among $St_1(P), St_2(P)$ and $St_3(P)$.

**Proof:** If possible, let $St_1(P)$ and $St_2(P)$ be both 7-sitgms. Let $St_1(P) = \{P, A_1, A_2, A_3, A_4, A_5, A_6 \}$ and $St_2(P) = \{P, B_1, B_2, B_3, B_4, B_5, B_6 \}$. Let us consider $M(1, 2, 3)$ and $M(1, 4, 5)$. Then by lemma 3.3, they cannot have any block in common. WLOG we assume that $M(1, 2, 3) = \{(1, 2), (3, 4), (5, 6)\}$. Now using the arguments made in the case (ii) in the proof of lemma 3.4, we can assume (WLOG) that $M(1, 4, 5) = \{(1, 3), (2, 5), (4, 6)\}$. Let $(i, j, k) = (1, 2, 4)$ or $(1, 2, 5)$. Then $M(i, j, k)$ must have exactly one block in common with $M(1, 2, 3)$ and exactly one with $M(1, 4, 5)$. Now $(1, 2) \subset M(i, j, k)$
Again, $(3, 4) \in M(i, j, k) \implies (2, 5) \in M(i, j, k) \implies M(i, j, k) = \{(3, 4), (2, 5), (1, 6)\}$. Finally, $(5, 6) \in M(i, j, k) \implies (1, 3) \in M(i, j, k) \implies M(i, j, k) = \{(5, 6), (1, 3), (2, 4)\}$. Hence $M(1, 2, 4)$ as well as $M(1, 2, 5)$ must be one of the following three sets of blocks: (i) $\{(1, 2), (4, 6), (3, 5)\}$; (ii) $\{(3, 4), (2, 5), (1, 6)\}$; (iii) $\{(5, 6), (1, 3), (2, 4)\}$.

But $M(1, 2, 4)$ and $M(1, 2, 5)$ have exactly one block in common (by lemma 3.4), whereas the three sets of blocks (i), (ii) and (iii) are mutually disjoint. Thus we arrive at a contradiction. Hence the proof is completed.

§2. **Non-existence of $(7, 5, r_3)$-tangential stigm system.**

Now we shall show that if one of $St_i(p), i=1,2,3$ be a 7-stigm, then none of the remaining two stigms is a 5-stigm.

**Proposition 3.2.** If $St_1(p)$ is a 7-stigm, then neither $St_2(p)$ nor $St_3(p)$ is a 5-stigm.

**Proof:** Suppose the proposition is not true. WLOG we assume that $St_2(p)$ is a 5-stigm. Let $St_1(p) = \{P, A_1, A_2, A_3, A_4, A_5, A_6\}$ and $St_2(p) = \{P, B_1, B_2, B_3, B_4\}$. We observe that $\langle St_1(p) \rangle = \mathcal{T}_1(p) \supseteq \mathcal{P}(p)$. Therefore, any point in $\mathcal{P}(p)$ can be expressed in the form $P + A_i + A_j$ or $P + A_i + A_j$, for some $i, j \in I_6 = \{1, 2, 3, 4, 5, 6\}, i \neq j$. Let $\lambda_i(p) = \langle St_i(p) \rangle \cap \mathcal{P}(p), i=1,2,3$. Any point of $\lambda_2(p)$ can be expressed in the form $P$ or $B_i + B_j$, for some $i, j \in \{1, 2, 3, 4\}, i \neq j$. Therefore, by rearranging the
suffixes of A's and B's, we can assume that $B_1 + B_2 = A_1 + A_2$. $\therefore$ $B_3 + B_4 = P + A_1 + A_2$. Again by making admissible rearrangement of the suffixes of B's we can assume (w.l.o.g.) that $B_1 + B_3 = A_1 + A_m$, for some $l, m \in I, l \neq m$.

**Case (i).** Let $\{l, m\} \cap \{1, 2\} = \emptyset$. By rearranging the suffixes of A_3, A_4, A_5, A_6, we can assume that $B_1 + B_3 = A_3 + A_4$. Thus we get that $B_1 + B_2 = A_1 + A_2$, $B_3 + B_4 = P + A_1 + A_2$, $B_2 + B_4 = P + A_3 + A_4$, $B_1 + B_4 = A_5 + A_6$ and $B_2 + B_3 = P + A_5 + A_6$. Note that $B_1 + B_j \notin \langle P, A_1, A_3, A_5 \rangle$, for all $i, j \in \{1, 2, 3, 4\}$, $i \neq j$.

But $\{B_1, B_2, B_3, B_4\} \subseteq \bigcup_{i=1}^{3} \mathcal{P}_i(P, A_1, A_3, A_5)$, since $\{P, A_1, A_3, A_5\}$ is a polarising set for $\mathcal{B}$ with respect to $\mathcal{T}_1(P)$ [by lemma 3.2]. So, there exists $i, j \in \{1, 2, 3, 4\}$, $i \neq j$, such that $B_i, B_j \in \mathcal{P}_i^k(P, A_1, A_3, A_5)$, for some $k \in \{1, 2, 3\}$. $\therefore$ $B_i + B_j \notin \langle P, A_1, A_3, A_5 \rangle$, which is a contradiction. $\therefore$ case (i) is not possible.

**Case (ii).** Let $\{l, m\} \cap \{1, 2\} \neq \emptyset$. It is clear that $\{l, m\} \neq \{1, 2\}$. $\therefore$ By adjusting the suffixes of A's, we can assume (w.l.o.g.) that $B_1 + B_3 = A_1 + A_3$. Thus we get $B_1 + B_2 = A_1 + A_2$, $B_3 + B_4 = P + A_1 + A_2$, $B_1 + B_3 = A_1 + A_3$, $B_2 + B_4 = P + A_4 + A_3$, $B_2 + B_3 = A_2 + A_3$ and $B_1 + B_4 = P + A_2 + A_3$. We note that $B_i + B_j \notin \langle P, A_1, A_4, A_5 \rangle$, for all $i, j \in \{1, 2, 3, 4\}$, $i \neq j$ and $\{P, A_1, A_4, A_5\}$ is a polarising set for $\mathcal{B}$ with respect to $\mathcal{T}_1(P)$. Now by arguments used in case (i), we arrive at a contradiction. Thus case (ii) is not possible. This completes the proof.
§3. Non-existence of \((7, r_2, r_3)\)-tangential stigm system.

Let us now state and prove the main result of this chapter.

**Theorem 3.1.** Let \(\mathcal{B}\) be a 6-dimensional tangential 2-block in \(\text{PG}(6, 2)\) and let \(P \in \mathcal{B}\). If \(P\) has a \((r_1, r_2, r_3)\)-tangential stigm system in \(\mathcal{B}\), then \(r_i \neq 7\), \(i = 1, 2, 3\).

**Proof.** If possible, let \(r_i = 7\), for some \(i \in \{1, 2, 3\}\). WLOG we assume \(i = 1\). This means \(\text{St}_1(P)\) is a 7-stigm. Then by propositions 3.1 and 3.2, \(\text{St}_2(P)\) and \(\text{St}_3(P)\) are both lines. But this contradicts theorem 2.8. Hence the theorem is proved.
CHAPTER IV.

Non-existence of (5, 5, 5)-tangential stigm system for a 6-dimensional tangential 2-block.

Let \( \mathcal{B} \) be a 6-diemsnional tangential 2-block in \( \text{PG}(6, 2) \). Let \( P \in \mathcal{B} \). In this chapter we want to show that \( P \) cannot have a (5, 5, 5)-tangential stigm system of \( \mathcal{B} \).

§1. Dimensions involved with \( \lambda_i(P), i = 1, 2, 3 \).

For brevity, we write \( \lambda_i \) for \( \lambda_i(P) \), \( i = 1, 2, 3 \), throughout this chapter.

Lemma 4.1. Let \( \text{St}_i(P) = \{P, A_i, B_i, C_i, D_i\} \), \( i = 1, 2, 3 \) and let \( A_1 + B_1 = A_2 + B_2 = A_3 + B_3 \). Let \( X_i \in \{A_i, B_i\} \) and \( Y_i \in \{C_i, D_i\}, i = 1, 2, 3 \). Then (i) \( X_3 \neq X_1 + X_2 \), for any \( X_i \in \{A_i, B_i\}, i = 1, 2, 3 \) and \( Y_3 \neq Y_1 + Y_2 \), for any \( Y_i \in \{C_i, D_i\}, i = 1, 2, 3 \); and (ii) at least one of the two sets \( \{X_1 + X_2 + X_3 \mid X_i \in \{A_i, B_i\}, i = 1, 2, 3 \} \) and \( \{Y_1 + Y_2 + Y_3 \mid Y_i \in \{C_i, D_i\}, i = 1, 2, 3 \} \) is contained in \( \langle \lambda_1, \lambda_2, \lambda_3 \rangle \).

Proof: (i) If possible, let \( X_1 + X_2 = X_3 \), for some \( X_i \in \{A_i, B_i\}, i = 1, 2, 3 \). WLOG we assume \( A_1 + A_2 = A_3 \). But \( A_1 + A_2 = A_3 \implies A_1 + B_2 = B_3 \) \( \therefore A_1 + B_1 = A_2 + B_2 = A_3 + B_3 \) \( \implies t(C_1, D_1) \cap \langle A_1, A_2, B_2 \rangle = \emptyset \), which is a

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contradiction. Similarly, we can show that $Y_3 \neq Y_1 + Y_2$, for any $Y_i \in \{C_i, D_i\}$, $i = 1, 2, 3$.

(ii) The result is obviously true if $<\lambda_1, \lambda_2, \lambda_3> = 4$-space $= \mathcal{N}(p)$. So, let us assume that $<\lambda_1, \lambda_2, \lambda_3> = 3$-space. Let $\sum_5 = <A_1, B_1, C_1, D_1, A_2, C_2>$. We note that if $A_3 \in \sum_5$, then $B_3 \in \sum_5$ and
\[ \{x_1 + x_2 + x_3 \mid x_i \in \{A_i, B_i\}, i = 1, 2, 3\} \subset <\lambda_1, \lambda_2, \lambda_3> \]
also if $C_3 \in \sum_5$, then $D_3 \in \sum_5$ and
\[ \{y_1 + y_2 + y_3 \mid y_i \in \{C_i, D_i\}, i = 1, 2, 3\} \subset <\lambda_1, \lambda_2, \lambda_3> \]
So, it is enough to show that at least one of the points $A_3, B_3, C_3, D_3$ is in $\sum_5$. If possible, let none of the points $A_3, B_3, C_3, D_3$ is in $\sum_5$. Let $m = \{p, A_1 + B_1, C_1 + D_1\}$. Since $\lambda_i \neq \lambda_j, i \neq j, i, j \in \{1, 2, 3\}$ (by theorem 2.9) and $<\lambda_1, \lambda_2, \lambda_3> = 3$-space, it follows that $\lambda_3 \subset <\lambda_1, \lambda_2>$ and if $Z \in \lambda_3 - m$, then $Z = U_1 + V_1 + U_2 + V_2$, for some $U_i, V_i \in St_i(p) - \{p\}$, $i = 1, 2$. Clearly $U_1 + V_1 \notin m$. WLOG we can assume that $A_3 + C_3 = A_1 + C_1 + U_2 + V_2$, where $U_2 \in \{A_2, B_2\}$ and $V_2 \in \{C_2, D_2\}$. Since $C_1 + D_1 = C_3 + D_3$, we get that $A_3 + D_3 = A_1 + D_1 + U_2 + V_2$. Let $\sum_2 = <A_3, C_3, D_3>$; since none of $A_3, B_3, C_3$ and $D_3$ is in $\sum_5$ and $\lambda_1 \neq \lambda_2$, it is easy to check that $\sum_2 \cap t(p, A_1, U_2, V_2) = \emptyset$; but this is a contradiction. This completes the proof.

Remarks. (i) The above lemma ensures that if $X_i \in \{A_i, B_i\}$ and $Y_i \in \{C_i, D_i\}, i = 1, 2, 3$, then both
\[X_1 + X_2 + X_3 \text{ and } Y_1 + Y_2 + Y_3 \text{ are points in the geometry.}\]

**Lemma 4.2.** Let \(\text{St}(P) = \{P, A_i, B_i, C_i, D_i\}, i = 1, 2, 3\) and let \(A_1 + B_1 = A_2 + B_2 = A_3 + B_3\). Let \(X_i \in \{A_i, B_i\}\) and \(Y_i \in \{C_i, D_i\}, i = 1, 2, 3\). Then neither \(X_1 + X_2 + X_3\) nor \(Y_1 + Y_2 + Y_3\) is a member of the line \(m = \{P, A_1 + B_1, C_1 + D_1\}\).

**Proof.** First we show that neither \(X_1 + X_2 + X_3\) nor \(Y_1 + Y_2 + Y_3\) is equal to \(P\). If possible, let \(P = X_1 + X_2 + X_3\), for some \(X_i \in \{A_i, B_i\}, i = 1, 2, 3\). WLOG assume that \(X_i = A_i\). So, \(P = A_1 + A_2 + A_3\). Now \(\langle A_1, B_1, A_2 \rangle = \{A_1, B_1, A_2, B_2, A_1 + A_2, A_1 + B_2, A_1 + B_1\}. A_1 + A_2 \in t(P, C_1) \implies P + A_1 + A_2 = A_3 \in t(P, C_1)\) which is not possible. \(A_1 + B_2 \in t(P, C_1) \implies P + A_1 + B_2 = A_1 + A_2 + A_3 + A_1 + B_2 = A_2 + B_2 + A_3 = A_3 + B_3 + A_3 = B_3 \in t(P, C_1)\), which is not possible. Thus it follows easily that \(t(P, C_1) \cap \langle A_1, B_1, A_2 \rangle = \emptyset\), which is a contradiction. Similarly, if \(P = Y_1 + Y_2 + Y_3\), we take the plane \(\langle C_1, D_1, C_2 \rangle\) and the tangent \(t(P, A_1)\) and arrive at a contradiction as before. Therefore, \(P \neq X_1 + X_2 + X_3\) and \(P \neq Y_1 + Y_2 + Y_3\). Next, we show that \(X_1 + X_2 + X_3 \notin \{A_1 + B_1, C_1 + D_1\}\) and \(Y_1 + Y_2 + Y_3 \notin \{A_1 + B_1, C_1 + D_1\}.\) If possible, let \(X_1 + X_2 + X_3 \in \{A_1 + B_1, C_1 + D_1\}\), for some \(X_i \in \{A_i, B_i\}, i = 1, 2, 3\). WLOG we assume \(X_i = A_i, i = 1, 2, 3\). \(\therefore A_1 + A_2 + A_3 \in \{A_1 + B_1, C_1 + D_1\}.\) But \(A_1 + A_2 + A_3 = C_1 + D_1 \implies P = B_1 + A_2 + A_3,\) which is not possible \([\because P \neq X_1 + X_2 + X_3]\)
A_1 + A_2 + A_3 = A_1 + B_1 \implies B_1 = A_2 + A_3 and A_1 = B_2 + A_3
\implies \langle B_1, A_2, B_2 \rangle \cap t(C_1, D_1) = \emptyset, which is a contradiction. Similarly, we can show that Y_1 + Y_2 + Y_3 \notin \{ A_1 + B_1, C_1 + D_1 \}. This completes the proof of the lemma.

Lemma 4.3. Let St_i(P) = \{P, A_i, B_i, C_i, D_i\}, i = 1, 2, 3 and let A_1 + B_1 = A_2 + B_2 = A_3 + B_3. Then there exists i, i \in \{1, 2, 3\}, such that either A_1 + A_2 + A_3 or C_1 + C_2 + C_3 belongs to \lambda_i - m, where m = \{P, A_1 + B_1, C_1 + D_1\}.

Proof: Case 1. Let \langle \lambda_1, \lambda_2, \lambda_3 \rangle = 3-space. Then either A_1 + A_2 + A_3 or C_1 + C_2 + C_3 is in \langle \lambda_1, \lambda_2, \lambda_3 \rangle (by lemma 4.1). But \lambda_1, \lambda_2, and \lambda_3 are all the three distinct planes on the common line m. So, there exist i, i \in \{1, 2, 3\}, such that either A_1 + A_2 + A_3 or C_1 + C_2 + C_3 belongs to \lambda_i - m (by lemma 4.2).

Case 2. Let \langle \lambda_1, \lambda_2, \lambda_3 \rangle = 4-space = \pi(P).
Since \lambda_i \neq \lambda_j (by theorem 2.9), \lambda_i and \lambda_j have a line in common, \forall i, j \in \{1, 2, 3\}, i \neq j, it follows that \langle \lambda_i, \lambda_j \rangle is a 3-space for every i, j \in \{1, 2, 3\}, i \neq j. We claim that either A_1 + A_2 + A_3 or C_1 + C_2 + C_3 belongs to \langle \lambda_i, \lambda_j \rangle, \forall i, j, i \neq j, i, j \in \{1, 2, 3\}. Suppose it is not true. Then there exist i, j, i \neq j, i, j \in \{1, 2, 3\} such that neither A_1 + A_2 + A_3 nor C_1 + C_2 + C_3 belongs to \langle \lambda_i, \lambda_j \rangle. WLOG we assume \{i, j\} = \{1, 2\}. But A_1 + A_2 + A_3, C_1 + C_2 + C_3 \notin \langle \lambda_1, \lambda_2, \lambda_3 \rangle - \langle \lambda_1, \lambda_2 \rangle \implies (A_1 + A_2 + A_3) +
(C_1 + C_2 + C_3) \in \langle \lambda_1, \lambda_2 \rangle \implies (A_1 + C_1 + A_2 + C_2) +
(A_3 + C_3) \in \langle \lambda_1, \lambda_2 \rangle \implies A_3 + C_3 \in \langle \lambda_1, \lambda_2 \rangle \implies
\lambda_3 \subset \langle \lambda_1, \lambda_2 \rangle \implies \langle \lambda_1, \lambda_2, \lambda_3 \rangle \text{ is a 3-space. But}
this is a contradiction. Thus our claim is justified.

WLOG we assume that \( A_1 + A_2 + A_3 \in \langle \lambda_1, \lambda_2 \rangle \). If
\( A_1 + A_2 + A_3 \in \langle \lambda_j, \lambda_3 \rangle \), \( j = 1 \) or \( 2 \), then \( A_1 + A_2 + A_3 \in
\langle \lambda_1, \lambda_2 \rangle \cap \langle \lambda_j, \lambda_3 \rangle = \lambda_j \), \( j = 1 \) or \( 2 \). So,
\( A_1 + A_2 + A_3 \in \lambda_j - m \), \( j = 1 \) or \( 2 \) (by lemma 4.2) and
we are done. If \( A_1 + A_2 + A_3 \not\in \langle \lambda_j, \lambda_3 \rangle \), \( j = 1, 2 \), then
\( C_1 + C_2 + C_3 \in \langle \lambda_j, \lambda_3 \rangle \), \( j = 1, 2 \). So, \( C_1 + C_2 + C_3 \in
\langle \lambda_1, \lambda_3 \rangle \cap \langle \lambda_2, \lambda_3 \rangle = \lambda_3 \), which implies that
\( C_1 + C_2 + C_3 \in \lambda_3 - m \) (by lemma 4.2). This completes
the proof.

Proposition 4.1. If \( St_1(P), St_2(P) \) and \( St_3(P) \) are all
5-stigms, then \( \lambda_1 \cap \lambda_2 \cap \lambda_3 = \{P\} \).

Proof: If possible, let \( \lambda_1 \cap \lambda_2 \cap \lambda_3 \not= \{P\} \). Then
by theorem 2.9, \( \lambda_1 \cap \lambda_2 \cap \lambda_3 \) must be a line through \( P \).
WLOG we assume that \( m = \{P, A_1 + B_1, C_1 + D_1\} \) and \( A_1 + B_1 =
A_2 + B_2 = A_3 + B_3 \). By lemma 4.3, we can assume (WLOG) that
\( C_1 + C_2 + C_3 \in \lambda_3 - m \). But \( \lambda_3 - m =
\{A_3 + C_3, B_3 + C_3, A_3 + D_3, B_3 + D_3\} \). Now \( C_1 + C_2 + C_3 =
A_3 + C_3 \implies C_1 + C_2 = D_1 + D_2 = A_3 \implies
t(B_3, C_3) \cap \langle A_3, C_1, D_1 \rangle = \emptyset \), which is a contradiction.
\( C_1 + C_2 + C_3 = A_3 + D_3 \implies C_1 + D_2 + D_3 = A_3 + D_3 \implies
C_1 + D_2 = D_1 + C_2 = A_3 \implies t(B_3, C_3) \cap \langle A_3, C_1, D_1 \rangle = \emptyset \),
which is a contradiction. Similarly, we can show that
\[ C_1 + C_2 + C_3 \neq B_3 + C_3, \quad B_3 + D_3. \] Therefore, \( C_1 + C_2 + C_3 \not\in \lambda_3 - m \); but this is a contradiction. This completes the proof.

Now we want to show that \( \langle \lambda_1, \lambda_2, \lambda_3 \rangle \) is a 3-space. For this purpose we need the following lemma.

**Lemma 4.4.** If \( i \in \{1, 2, 3\} \), then \( \exists \ j \in \{1, 2, 3\} - \{i\} \) such that \( \lambda_i \cap \lambda_j \) contains a line through \( P \).

**Proof.** The lemma is obviously true if \( \langle \lambda_1, \lambda_2, \lambda_3 \rangle \) is a 3-space. So we assume \( \langle \lambda_1, \lambda_2, \lambda_3 \rangle \) is a 4-space. If possible, let the lemma be not true. Then \( \exists \ i \in \{1, 2, 3\} \) such that \( \lambda_i \cap \lambda_j = \{p\}, \forall j \in \{1, 2, 3\} - \{i\} \). WLOG assume \( i = 1 \). Then \( \lambda_1 \cap \lambda_2 = \lambda_1 \cap \lambda_3 = \{p\} \). Let \( \Theta(x_2) = t(A_1, B_1, C_1, D_1, x_2) \), \( x_2 \in St_2(P) - \{p\} \). Since \( \Theta(x_2) \cap \Pi(P) = \lambda_4 \) and \( \lambda_1 \cap \lambda_3 = \{p\}, \Theta(x_2) \cap \Pi_3(P) = \langle \lambda_1, x_3 \rangle \), for some \( x_3 \in St_3(P) - \{p\} \). So, \( x_3 \in \Theta(x_2) \) and hence \( kP + X_1 + X_2 = X_3 \), for some \( X_1 \in St_1(P) - \{p\} \), and \( k = 0 \) or \( 1 \) (mod 2). WLOG we assume \( k_P + A_1 + A_2 = A_3 \), \( k = 0 \) or \( 1 \) (mod 2).

Taking \( X_2 = B_2 \), we get \( k_2P + X_1 + B_2 = X_3 \), for some \( X_1 \in St_i(P) - \{p\}, i = 1, 3 \), and \( k_2 = 0 \) or \( 1 \) (mod 2). Since \( \lambda_1 \cap \lambda_2 = \{p\} \), it follows that \( X_3 \not\in \lambda_3 - A_3 \). WLOG assume \( X_3 = B_3 \). \( \therefore k_2P + X_1 + B_2 = B_3 \). We now claim that \( X_1 \not\in A_1 \). If possible, let \( X_1 = A_1 \). Then we get \( k_1P + A_1 + A_2 = A_3 \) and \( k_2P + A_1 + B_2 = B_3 \). Assert that \( k_1 \not= k_2 \). For, if \( k_1 = k_2 \), then it follows that
\(<A_1, A_3, B_3> \cap t(C_2, D_2, B_1, C_1) = \emptyset \) \[\therefore \lambda_1 \cap \lambda_j = \{p\}, \quad j = 2, 3\]; but this is a contradiction. \(\therefore k_1 \neq k_2\). WLOG assume that \(P + A_1 + A_2 = A_3\) and \(A_1 + B_2 = B_3\). So, \(A_3 + B_3 = C_2 + D_2\) and \(\lambda_2 \cap \lambda_3 = \{p, A_2 + B_2, C_2 + D_2\}\). Again, let us take \(X_2 = C_2\). As before, we can write \(k_3^P + Y_1 + C_2 = Y_3\), for some \(Y_1 \in St_1(p) - \{p\}, i = 1, 3, \) and \(k_3 = 0 \) or \(1 \) (mod 2). Since \(\lambda_1 \cap \lambda_2 = \{p\}\), it follows that \(Y_3 \neq A_3, B_3\). WLOG assume \(Y_3 = C_3\). \(\therefore k_3^P + Y_1 + C_2 = C_3\). Since \(\lambda_2 \neq \lambda_3\) [by theorem 2.9], \(A_3 + C_3, B_3 + C_3 \not\in \lambda_2\), and hence \(Y_1 \neq A_1\). WLOG assume \(Y_1 = C_1\). Since \(C_3 + D_3 = A_2 + B_2\), WLOG we can write as before that \(P + C_1 + C_2 = C_3\) and \(C_1 + D_2 = D_3\). Let us consider the plane \(<A_1, A_3, B_3> =\{A_1, P + A_2, B_2, A_3, B_3, C_2 + D_2, A_1 + C_2 + D_2\}\). \(A_1 + C_2 + D_2 \in t(D_1, A_2, D_2, C_3) \implies A_1 + C_2 + D_2 + C_3 = A_1 + C_2 + D_2 + P + C_1 + C_2 = P + A_1 + C_1 + D_2 \in t(D_1, A_2, D_2, C_3) \implies P + A_1 + C_1 + D_1 = B_1 \in t(D_1, A_2, D_2, C_3) \implies B_1 \in \langle D_1, A_2, D_2, C_3\rangle\), which is not possible. Now it follows easily that \(t(D_1, A_2, D_2, C_3) \cap \langle A_1, A_3, B_3\rangle = \emptyset\), which is a contradiction. Thus we prove that \(X_1 \neq A_1\).

Also taking \(X_2 = C_2, D_2\), we can write down (WLOG) the following relations

\[
\begin{align*}
&k_1^P + A_1 + A_2 = A_3 \\
&k_2^P + B_1 + B_2 = B_3 \\
&k_3^P + C_1 + C_2 = C_3 \\
&k_4^P + D_1 + D_2 = D_3
\end{align*}
\]

\[\rightarrow (1)\]
where \( k_i = 0 \) or \( 1 \) (mod 2), \( i = 1, 2, 3, 4 \). Summing up the relations in (1) columnwise, we deduce that \( \exists \ i, j, i \neq j \), \( i, j \in \{1, 2, 3, 4\} \), such that \( k_i = 0 \) and \( k_j = 1 \). WLOG assume \( k_1 = 0 \) and \( k_2 = 1 \). From (1), it also follows that 
\[ \lambda_i \cap \lambda_j = \{p_i\}, \ i \neq j, \ i, j = 1, 2, 3. \]
Relations (1) now reduce to
\[
\begin{align*}
A_1 + A_2 &= A_3 \\
P + B_1 + B_2 &= B_3 \\
k_3P + C_1 + C_2 &= C_3 \\
k_4P + D_1 + D_2 &= D_3
\end{align*}
\]
Let \( \Sigma_2 = \langle A_1, B_1, A_2 \rangle = \{A_1, B_1, A_2, A_3, A_1 + B_1, B_1 + A_2, B_1 + A_3\} \) and \( \Theta = t(C_1, D_1, C_3, D_3) \). Since \( \lambda_i \cap \lambda_j = \{p_i\}, \ i \neq j \), \( i, j = 1, 2, 3 \), it follows that \( A_1, B_1, A_3, A_2 \) and \( A_1 + B_1 \) are not in \( \Theta \). \( A_2 + B_1 \in \Theta \implies A_2 + B_1 + C_1 + D_1 = A_2 + P + A_1 \in \Theta \implies P + A_3 \in \Theta \implies P + A_3 + C_3 + D_3 = B_3 \in \Theta \implies B_3 = C_1 + D_1 + Y_3 \), for some \( Y_3 \in \{C_3, D_3\} \implies B_3 + Y_3 = C_1 + D_1 \implies \lambda_1 \cap \lambda_3 \neq \{p_i\} \), which is a contradiction.
\( B_1 + A_3 \in \Theta \implies B_1 + A_3 + C_3 + D_3 = B_1 + P + B_3 = B_2 \in \Theta \implies B_2 \in \{C_1 + C_3, C_1 + D_3, D_1 + C_3, D_1 + D_3\} \). But \( B_2 = P + B_1 + B_3 \). Therefore, either \( B_3 + C_3 \) or \( B_3 + D_3 \) is in \( \lambda_1 \), which contradicts the assumption that \( \lambda_1 \cap \lambda_3 = \{p_i\} \). \( \therefore \Theta \cap \Sigma_2 = \emptyset \), which is a contradiction. This completes the proof of the lemma.

**Proposition 4.2.** \( \langle \lambda_1, \lambda_2, \lambda_3 \rangle \) is a 3-space.
Proof. It is enough to show that $\lambda_i \cap \lambda_j \neq \{p\}$, $\forall i, j \in \{1, 2, 3\}, i \neq j$. If possible, let there exist $i, j, i \neq j, i, j \in \{1, 2, 3\}$, such that $\lambda_i \cap \lambda_j = \{p\}$.

WLOG assume $i = 2, j = 3$ i.e. $\lambda_2 \cap \lambda_3 = \{p\}$. By the lemma 4.4, we must have $\lambda_1 \cap \lambda_j \neq \{p\}, j = 2, 3$. By theorem 2.9, $\lambda_1 \neq \lambda_j, j = 2, 3$ and by proposition 4.1, $\lambda_1, \lambda_2$ and $\lambda_3$ cannot have a common line. Therefore, by rearranging the letters $A, B, C$ and $D$, we can assume (WLOG) that $A_1 + B_1 = A_2 + B_2$ and $B_1 + C_1 = B_3 + C_3$. Now $t(A_2, B_2, C_2, D_2, A_1) \cap \Pi(P) = \lambda_2$; so $t(A_2, B_2, C_2, D_2, A_1) \cap \Pi_3(P) = <\lambda_2, X_3>$, for some $X_3 \in St_3(P) - \{p\}$, $\lambda_2 \cap \lambda_3 = \{p\}$; $kP + X_2 + A_1 = X_3$, for some $X_1 \in St_1(P) - \{p\}$, $i = 2, 3$ and $k = 0$ or $1$ (mod 2). We can interchange $A_2$ and $B_2$ as well as $C_2$ and $D_2$; also we interchange $A_3$ and $D_3$ as well as $B_3$ and $C_3$. So, it is enough to consider the following possibilities for $X_2$ and $X_3$.

$X_2 \in \{A_2, C_2\}$ and $X_3 \in \{A_3, B_3\}$.

Let us suppose that $X_2 = A_2$ and $X_3 = A_3$. Then $kP + A_1 + A_2 = A_3$. Let $\Sigma_2 = <A_3, A_1, B_1>$ =

$\{A_1, B_1, kP + A_2, kP + B_2, A_3, A_1 + B_1, A_1 + B_1 + A_3\}$ and 

$G = t(C_2, D_2, B_3, C_3)$. We note that 

$<C_2, D_2, B_3, C_3> \cap \Pi(P) = \{C_2 + D_2, B_1 + D_1, B_3 + C_3\}$,

$<C_2, D_2, B_3, C_3> \cap \Pi_1(P) = \Pi(P) = \{C_2 + B_3, C_2 + C_3, D_2 + B_3, D_2 + C_3\}$,
\[ \langle C_2, D_2, B_3, C_3 \rangle \cap \tau_2(p) - \tau(p) = \{ C_2, D_2, C_2 + B_3 + C_3, D_2 + B_3 + C_3 \} \]

\[ \langle C_2, D_2, B_3, C_3 \rangle \cap \tau_3(p) - \tau(p) = \{ B_3, C_3, B_3 + C_2 + D_2, C_3 + C_2 + D_2 \} \]

Since \( \cal{A}_2 \cap \lambda_3 = \{ \lambda \} \), it is easy to check that \( \sum_2 \cap \lambda = \emptyset \). This is a contradiction. Again, \( X_2 = C_2 \Rightarrow A_3 = kP + A_1 + C_2 \Rightarrow D_3 = kP + D_1 + C_2 [ \because A_1 + D_1 = A_3 + D_3 ] \Rightarrow D_3 = kP + C_1 + D_2 [ \because C_1 + D_1 = C_2 + D_2 ] \).

In this way we can now form the following table.

<table>
<thead>
<tr>
<th>( X_3 )</th>
<th>( X_2 )</th>
<th>( \sum_2 )</th>
<th>( \lambda )</th>
<th>( \sum_2 \cap \lambda )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_3 )</td>
<td>( A_2 )</td>
<td>( \langle A_3, A_1, B_1 \rangle )</td>
<td>( t(C_2, D_2, B_3, C_3) )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( A_3 )</td>
<td>( C_2 )</td>
<td>( \langle D_3, C_1, D_1 \rangle )</td>
<td>( t(A_2, B_2, B_3, C_3) )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( B_3 )</td>
<td>( A_2 )</td>
<td>( \langle B_3, A_1, B_1 \rangle )</td>
<td>( t(C_2, D_2, A_3, C_3) )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( B_3 )</td>
<td>( C_2 )</td>
<td>( \langle C_3, C_1, D_1 \rangle )</td>
<td>( t(A_2, B_2, A_3, B_3) )</td>
<td>( \emptyset )</td>
</tr>
</tbody>
</table>

The above table leads to a contradiction. This completes the proof.

\[ \text{§2. Non-existence of } (5, 5, 5)-\text{tangential stigm system.} \]

We consider the following proposition before we state and prove the main result of this chapter.

**Proposition 4.3.** \( \text{St}_1(p), \text{St}_2(p) \) and \( \text{St}_3(p) \) cannot be all 5-stigms.

**Proof.** If possible, let \( \text{St}_i(p) = \{ P, A_i, B_i, C_i, D_i \}, i = 1,2,3 \). Then \( \langle \lambda_1, \lambda_2, \lambda_3 \rangle \) is a 3-space (by proposition 4.2).

Now by using theorem 2.9, the proposition 4.1 and then adjusting the letters \( A, B, C, D \) we can assume WLOG that
\[ A_1 + B_1 = A_2 + B_2, \quad C_1 + D_1 = C_2 + D_2 \]
\[ A_1 + D_1 = A_3 + D_3, \quad B_1 + C_1 = B_3 + C_3 \]
\[ A_2 + C_2 = A_3 + C_3, \quad B_2 + D_2 = B_3 + D_3 \]

From the relations (1) it follows that \( A_3 + B_3 = B_1 + D_1 + A_2 + C_2 \).

Let \( \sum_3 = \langle A_3, B_3, C_3, D_3, A_2, D_2 \rangle \). We claim that \( \langle St_1(P), St_2(P), St_3(P) \rangle = \sum_5 \). Because of relations (1) it is enough to show that there exists \( X_1 \in St_1(P) - \{P\} \) such that \( X_1 \in \sum_5 \). If possible, let \( X_1 \notin \sum_5, \forall X_1 \in St_1(P) - \{P\} \). Let \( \sum_2 = \langle A_1, B_1, C_1 \rangle \)

\[ \langle A_1, B_1, C_1, P + D_1, A_2 + B_2, B_3 + C_3, A_1 + C_1 \rangle \]

and let \( \Theta = t(P, A_2, A_3, B_3) \). Since \( X_1 \notin \sum_5, \forall X_1 \in St_1(P) - \{P\} \), it is clear that \( \Theta \cap \sum_2 = \{A_1 + C_1\} \). But \( A_1 + C_1 \notin \Theta \) if \( A_1 + C_1 + B_3 = A_1 + C_1 + B_1 + D_1 + A_2 + C_2 = P + A_3 + C_3 = B_3 + D_3 \in \Theta \) if \( D_3 \in \Theta \) if \( D_3 \notin \Theta \), which is not possible. Thus we arrive at a contradiction. So, \( \langle St_1(P), St_2(P), St_3(P) \rangle = \sum_5 \).

Hence

\[ A_1 \in \langle A_3, B_3, C_3, D_3, A_2, D_2 \rangle \]

\[ \therefore A_1 = kP + X_2 + X_3, \] for some \( X_2 \in \{A_2, D_2\} \) and some \( X_3 \in \{A_3, B_3, C_3, D_3\} \) and \( k = 0 \) or 1 (mod 2). Let us consider the following cases.

**Case 1.** Let \( X_2 = D_2, X_3 = A_3 \) and \( k = 0 \). From relations (1) we get

\[ A_1 = D_2 + A_3, \quad P + A_1 = B_2 + C_3, \quad P + B_1 = C_2 + A_3, \]
\[ P + B_1 = A_2 + C_3, \quad P + C_1 = A_2 + B_3, \quad C_1 = C_2 + D_3, \quad D_1 = ... \]
$D_2 + D_3$ and $D_1 = B_2 + B_3$. Now it is easy to check that $t(A_1, C_1) \cap <B_2, B_3, C_3> = \emptyset$. Thus we get a contradiction. So, case 1 cannot happen. In case 1 we observe that there are exactly four relations of the type $P + Y_1 = Y_2 + Y_3$, $Y_1 \in \text{St}_i(p) - \{p\}$.

**Case 2.** Let $X_2 = D_2$, $X_3 = C_3$ and $k = 0 \pmod{2}$. Then $A_1 = D_2 + C_3$, $P + A_1 = B_2 + A_3$, $P + B_1 = A_2 + A_3$, $P + B_1 = C_2 + C_3$, $C_1 = A_2 + D_3$, $P + C_1 = C_2 + B_3$, $P + \check{B}_1 = D_2 + B_3$, and $P + D_1 = B_2 + D_3$. Now, $t(C_1, A_2) \cap <A_1, D_2, A_3> = \{D_2 + A_3\}$ and $t(C_1, A_2) \cap <A_1, D_2, B_3> = \{A_1 + B_3\}$. Therefore, $D_2 + A_3 + A_1 + B_3 + A_2 + C_1 \in t(C_1, A_2)$, i.e. $(D_2 + B_3) + (A_2 + A_3) + (A_1 + C_1) = (P + D_1) + (P + B_1) + (A_1 + C_1) = P \in t(C_1, A_2)$, which is a contradiction. So, case 2 cannot occur. In this case we observe that there are exactly six different relations of the type $P + Y_1 = Y_2 + Y_3$, $Y_1 \in \text{St}_i(p) - \{p\}$.

**Case 3.** Let $X_2 = D_2$, $X_3 = C_3$, $k = 1 \pmod{2}$. Then $P + A_1 = D_2 + C_3$, $A_1 = B_2 + A_3$, $B_1 = A_2 + A_3$, $B_1 = C_2 + C_3$, $C_1 = C_2 + B_3$, $P + C_1 = A_2 + D_3$, $D_1 = D_2 + B_3$, $D_1 = B_2 + D_3$.

We have $\Sigma_5 = \langle \text{St}_1(p), \text{St}_2(p), \text{St}_3(p) \rangle$.

t($A_1, C_1$) does not meet the line $\{B_1, A_2, A_3\}$ of $\mathcal{B}$ and $\{B_1, A_2, A_3\} \subset \Sigma_5$. So, $t(A_1, C_1) \not\subset \Sigma_5$. Further, $t(A_1, C_1) \cap <A_2, A_3, D_3> = \{B_1 + D_3\}$ and $t(A_1, C_1) \cap <C_3, C_2, D_2> = \{B_1 + D_2\}$. Thus it follows from theorem 2.4 that $B_1 + D_3$ and $B_1 + D_2$ are attenuation
points for $\mathcal{P}$ with respect to $\sum_5$ and $t(A_1, C_1) \cap \sum_5 =
\langle A_1, C_1, B_1 + D_3, B_1 + D_2 \rangle$, which is a 3-space. Hence by the theorem 2.5, $\{A_1, C_1\}$ is a polarising set of $\mathcal{P}$ with respect to $\sum_5$. Again, $t(A_2, D_2) \cap \langle D_3, C_1, D_1 \rangle$ $= \{C_1 + B_2, \}$ and $t(A_2, D_2) \cap \langle A_1, A_3, C_3 \rangle = \{D_2 + C_3\}$; also $t(C_3, D_3) \cap \langle C_1, C_2, A_2 \rangle = \{A_2 + D_3\}$ and $t(C_3, D_3) \cap \langle D_2, A_1, D_1 \rangle = \{A_1 + B_3\}$.

So, we can easily check as before that $\{A_2, D_2\}$ and $\{C_3, D_3\}$ are both polarising sets of $\mathcal{P}$ with respect to $\sum_5$. From theorem 2.7 we have $|\mathcal{P} - \sum_5| \geq 5$. Let $S_1, S_2, S_3, S_4$ and $S_5$ be five distinct points of $\mathcal{P} - \sum_5$.

We claim that $S_i + S_j \notin \{A_1 + C_1, A_2 + D_2, C_3 + D_3\}$, $i \neq j$, $i, j = 1, 2, 3, 4, 5$. $t(P, B_2, B_3) \cap \langle A_1, B_1, C_1 \rangle = \{A_1 + C_1\}$, $t(P, D_1, A_3) \cap \langle A_2, D_2, C_2 \rangle = \{A_2 + D_2\}$ and $t(P, B_1, D_2) \cap \langle C_3, D_3, B_3 \rangle = \{C_3 + D_3\}$. Using the theorem 2.4 we conclude that $A_1 + C_1, A_2 + D_2$ and $C_3 + D_3$ are all attenuation points. Thus our claim is justified.

Let $\{X, Y\} = \{A_1, C_1\}, \{A_2, D_2\}$ or $\{C_3, D_3\}$. Since $S_i + S_j \notin \{A_1 + C_1, A_2 + D_2, C_3 + D_3\}$, $i \neq j$, $i, j = 1, 2, 3, 4, 5$, it follows that none of $\rho_2^i(X, Y), i = 1, 2, 3$, can contain more than two points of $S_1, S_2, S_3, S_4$ and $S_5$.

WLOG we assume that each of $\rho_2^1(X, Y)$ and $\rho_2^2(X, Y)$ contains exactly two of the points $S_1, S_2, S_3, S_4, S_5$. We now claim that if $S_i + S_j \in \rho_2(X, Y)$ and $S_k, S_t \in \rho_2^2(X, Y)$, then $S_i + S_j \neq S_k + S_t$, $i \neq j, k \neq t, i, j, k, t$ are in
\{1, 2, 3, 4, 5\}. Let us assume that \((X, Y) = \{A_1, C_1\}\).

If possible, let us suppose that \(S_1, S_2 \in \rho^1_2(A_1, C_1)\) and \(S_3, S_4 \in \rho^2_2(A_1, C_1)\) such that \(S_1 + S_2 = S_3 + S_4\). Then \(S_1 + S_2 = S_3 + S_4 \in \langle A_1, C_1 \rangle\). But we have already seen that \(A_1 + C_1\) is an attenuation point for \(\mathcal{B}\) with respect to \(\Sigma_5\). So \(S_1 + S_2 = S_3 + S_4 = A_1\) or \(C_1\). Let us consider the distribution of \(S\)'s among \(\rho^i_2(A_2, D_2), i = 1, 2, 3\).

Since \(S_1 + S_2 \in \{A_1, C_1\}\) and \(\langle A_1, C_1 \rangle \cap \langle A_2, D_2 \rangle = \emptyset\), \(S_1\) and \(S_2\) cannot be in the same \(\rho^i_2(A_2, D_2), i \in \{1, 2, 3\}\).

WLOG we assume that \(S_1 \in \rho^1_2(A_2, D_2), i = 1, 2\). Since \(S_3 + S_4 \in \{A_1, C_1\}\), both \(S_3\) and \(S_4\) cannot be in \(\rho^3_2(A_2, D_2)\). Adjusting suffixes of \(S\)'s and superscripts of \(\rho\)'s, we assume (WLOG) that \(S_3 \in \rho^1_2(A_2, D_2)\). \(\therefore S_1 + S_3 \in \langle A_2, D_2 \rangle\) i.e. \(S_1 + S_3 \in \{A_2, D_2\}\), since \(A_2 + D_2\) is an attenuation point with respect to \(\Sigma_5\).

Since \(S_1 + S_3 = S_2 + S_4\) and the relations (1) hold, we can write down the following table.

<table>
<thead>
<tr>
<th>(S_1 + S_2)</th>
<th>(S_1 + S_3)</th>
<th>(\Sigma_2)</th>
<th>(\theta)</th>
<th>(\Sigma_2 \cap \theta)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A_1)</td>
<td>(A_2)</td>
<td>(\langle S_1, S_2, S_3 \rangle)</td>
<td>(t(B_2))</td>
<td>(\emptyset)</td>
</tr>
<tr>
<td>(A_1)</td>
<td>(D_2)</td>
<td>(\langle S_1, S_2, S_3 \rangle)</td>
<td>(t(P, C_2))</td>
<td>(\emptyset)</td>
</tr>
<tr>
<td>(C_1)</td>
<td>(A_2)</td>
<td>(\langle S_1, S_2, S_3 \rangle)</td>
<td>(t(P, B_2))</td>
<td>(\emptyset)</td>
</tr>
<tr>
<td>(C_1)</td>
<td>(D_2)</td>
<td>(\langle S_1, S_2, S_3 \rangle)</td>
<td>(t(C_2))</td>
<td>(\emptyset)</td>
</tr>
</tbody>
</table>

The above table leads to a contradiction. So \(S_1 + S_2 \neq S_3 + S_4\). Thus our claim is justified when \(\{X, Y\} = \{A_1, C_1\}\). Similarly we can justify our claim when \(\{X, Y\} = \{A_2, D_2\}\) or \(\{C_3, D_3\}\).
Now considering the distribution of $S_i$'s among $\rho_j^i(A_1, C_1)$, $j = 1, 2, 3$, we assume (WLOG) that $S_1, S_2 \in \rho_j^i(A_1, C_1)$ and $S_1 + S_2 = A_1$, $S_3 + S_4 = C_1$. Since $<A_1, C_1> \cap <A_2, D_2> = \emptyset$, none of the pairs $\{S_1, S_3\}$ and $\{S_2, S_4\}$ is in the same $\rho_j^i(A_2, D_2)$, $i \in \{1, 2, 3\}$. So, adjusting the suffixes of $S$'s and the superscripts of $\rho$'s, we assume (WLOG) that $S_1, S_3 \in \rho_j^i(A_2, D_2)$ and $S_2 \in \rho_j^i(A_2, D_2)$. Now $S_4 \not\in \Sigma_j^2(A_2, D_2)$, for otherwise, we would have $S_1 + S_3 + S_2 + S_4 = A_1 + C_1 = A_2 + D_2$, which would be a contradiction. \[\therefore S_4 \in \rho_j^i(A_2, D_2).\]

Let us now consider the following two cases:

**Case (a).** Let $S_5 \in \rho_j^i(A_2, D_2)$. Also we have $S_4 \in \rho_j^i(A_2, D_2)$. \[\therefore S_4 + S_5 \in \{A_2, D_2\}.\] Again $S_1 + S_3 \in \{A_2, D_2\}$ and $S_1 + S_3 \not\in S_4 + S_5$. So, $S_1 + S_3 + S_4 + S_5 = A_2 + D_2$. Also $S_1 + S_2 + S_3 + S_4 = A_1 + C_1$. \[\therefore S_2 + S_5 = A_1 + C_1 + A_2 + D_2.\] \[\Rightarrow S_2 + S_3 \in \{A_1 + A_2, A_1 + D_2, C_1 + A_2, C_1 + D_2\}, S_1 + S_5 \in \{A_1 + A_2 + D_2, C_1 + A_2 + D_2, S_3 + S_5 \in \{C_1 + D_2, C_1 + A_2, A_1 + D_2, A_1 + A_2\}.\] Now using the relations (1), we conclude that $\{S_1, S_2, S_3, S_5\} \subset \bigcup_{i=1}^3 \rho_j^i(C_3, D_3)$, but $S_1 + S_j \not\in \{C_3, D_3\}$, i.e. $S_1 + S_j \not\in <C_3, D_3>$, $\forall i, j \in \{1, 2, 3, 5\}, i \neq j$. But this is not possible, since $\{C_3, D_3\}$ is a polarising set of $\mathcal{P}$ with respect to $\Sigma_5$. \[\therefore S_5 \not\in \rho_j^i(C_3, D_3).\]
Case (b). Let $S_5 \subset P^2(A_2, D_2)$. Since $S_2 \subset P^2(A_2, D_2)$, $S_2 + S_5 \subset \{A_2, D_2\}$. Again $S_1 + S_2 \subset \{A_1, C_1\}$, $S_1 + S_3 \subset \{A_1, D_2\}$; so, $S_1 + S_5$, $S_2 + S_3$ \[\{A_1 + A_2, A_1 + D_2, C_1 + A_2, C_1 + D_2\}\] and $S_3 + S_5 \subset \{A_1, C_1, A_1 + A_2 + D_2, C_1 + A_2 + D_2\}$. By virtue of relations (1), we can conclude that \[\{S_1 + S_3, S_3 + S_5 \subset \bigcup_{i=1}^{3} P^{i}(C_3, D_3)\]
and $S_i + S_j \notin \{C_3, D_3\}$ i.e. $S_i + S_j \notin \langle C_3, D_3 \rangle$, $i, j \in \{1, 2, 3, 5\}$, $i \neq j$. But this is a contradiction. \[\therefore S_5 \notin \rho^2(C_3, D_3)\].

Since $\rho^1(A_2, D_2)$ cannot contain more than two points of $S_1$, $S_2$, $S_3$, $S_4$ and $S_5$, we get $S_5 \notin \rho^2(A_2, D_2)$, $\forall i \in \{1, 2, 3, 5\}$; but this is a contradiction. Hence $A_1 \notin P + D_2 + C_3$.

In case 3 we observe that there are exactly two relations of the form $P + Y_1 = Y_2 + Y_3$, $Y_i \in St_1(P) - \{P\}$, viz, $P + A_1 = D_2 + C_3$ and $P + C_1 = A_2 + D_3$. Also we observe that the polarising sets are $\{A_1, C_1\}$, $\{A_2, D_2\}$ and $\{C_3, D_3\}$.

We have $A_1 = kP + X_2 + X_3$, where $k = 0$ or $1$ (mod 2), $X_2 \in \{A_2, D_2\}$ and $X_3 \in \{A_3, B_3, C_3, D_3\}$. In the cases 1, 2, and 3, we have considered three values for the triple, $\{k, X_2, X_3\}$ and in each case we have arrived at a contradiction. Now it can be easily seen that for each admissible value of the triple $\{k, X_2, X_3\}$, the relation $A_1 = kP + X_2 + X_3$, together with the relations (1) gives rise to eight relations of the form $k_1P + Y_1 = Y_2 + Y_3$, $k_1 = 0$ or $1$ (mod 2), $Y_i \in St_i(P) - \{P\}$, $i = 1, 2, 3$. Further
it can be easily seen that these eight relations involve either 2, 4, or 6 relations of the form $P + Y_1 = Y_2 + Y_3$, $Y_i \in St_i(P) - \{P\}$, $i = 1, 2, 3$. If for a given value of the triple $\{k, X_2, X_3\}$ we get 4 or 6 relations of the form $P + Y_1 = Y_2 + Y_3$, we use the technique displayed in cases 1 and 2; i.e. we take a suitable plane and a suitable tangent and show that their intersection is empty. If for a given value of the triple $\{k, X_2, X_3\}$ we get exactly 2 relations of the form $P + Y_1 = Y_2 + Y_3$, we take three polarising sets and use the method displayed in case 3 [The way to choose the three polarising sets has been indicated at the end of case 3.] Thus it can be shown that for each value of the triple $\{k, X_2, X_3\}$, we arrive at a contradiction. Hence the proposition is proved.

Let us now state and prove the main result in this chapter.

**Theorem 4.1.** Let $\mathcal{B}$ be a 6-dimensional tangential 2-block in $\text{PG}(6, 2)$ and let $P \in \mathcal{B}$. Then $P$ cannot have a $(5, 5, 5)$-tangential stigm system.

**Proof.** The proof follows immediately from proposition 4.3.
CHAPTER V

Non-existence of $(5, 5, 3)$-tangential stigm system for a 6-dimensional tangential 2-block.

Let $\mathcal{B}$ be a 6-dimensional tangential 2-block in $\text{PG}(6,3)$ and let $P \in \mathcal{B}$. In this chapter we shall show that no two of $\text{St}_1(P)$, $\text{St}_2(P)$ and $\text{St}_3(P)$ are 5-stigs.

§1. Three odd stigs and the dimensions involved with them.

If $\text{St}_1(P)$ and $\text{St}_2(P)$ be two 5-stigs and $\text{St}_3(P)$ a 3-stigm, then we are concerned with the dimension of the space generated by the two 5-stigs and also we want to know whether $\text{St}_3(P)$ is contained in $\langle \text{St}_1(P), \text{St}_2(P) \rangle$ or not. In this connection we have the following proposition and remark.

**Proposition 5.1.** Let $\text{St}_1(P) = \{P, A_1, B_1, C_1, D_1 \}$, $i = 1, 2$, and $\text{St}_3(P) = \{P, A_3, B_3 \}$. Then $\langle \text{St}_1(P), \text{St}_2(P) \rangle$ is a 5-space.

**Proof.** Clearly $5 \leq \dim \langle \text{St}_1(P), \text{St}_2(P) \rangle \leq 6$ [\(\because \lambda_1(P) \neq \lambda_2(P)\)]. Suppose $\dim \langle \text{St}_1(P), \text{St}_2(P) \rangle = 6$. Then

\[
\lambda_1(P) \cap \lambda_2(P) = \{P\}. \text{ Since } t(A_1, B_1, C_1, D_1, A_3) \cap \tau(P) = \lambda_1(P), t(A_1, B_1, C_1, D_1, A_3) \cap \tau_2(P) = \langle \lambda_1(P), \{x_2\} \rangle,
\]

for some $x_2 \in \{A_2, B_2, C_2, D_2 \}$. WLOG assume $x_2 = A_2$. Then
A_2 = kP + X_1 + A_3, where X_1 \in St_1(p) - \{p\} and k = 0 or 1 (mod 2). WLOG assume X_1 = A_1. Then A_3 = kP + A_1 + A_2.

Similarly, by taking t(A_1, B_1, C_1, D_1, B_3) we can show that B_3 = k_1P + X_1 + X_2, for some X_1 \in St_1(p), i = 1, 2, k_1 = 0 or 1 (mod 2). \because P = A_3 + B_3 = (k + k_1)p + (A_1 + X_1) + (A_2 + X_2). Since \lambda_1(p) \cap \lambda_2(p) = \emptyset, k + k_1 = 1 (mod 2) and X_1 = A_1, i = 1, 2. WLOG assume A_3 = A_1 + A_2 and B_3 = P + A_1 + A_2. Since \lambda_1(p) \cap \lambda_2(p) = \{p\}, it is now easy to check that t(C_1, D_1, C_2, D_2) \cap <P, A_3, A_1> = \emptyset.

This is a contradiction. Hence the proposition is proved.

Let St_1(p) = \{p, A_1, B_1, C_1, D_1\}, i = 1, 2 and St_3(p) = \{p, A_3, B_3\}. We intend to show that St_3(p) \not\subseteq <St_1(p), St_2(p)>. By the proposition 5.1, <St_1(p), St_2(p)> is a 5-space which we denote by \Sigma_5.

By the theorem 2.9 and the proposition 5.1, \lambda_1(p) \cap \lambda_2(p) is a line through P. WLOG we assume \lambda_1(p) \cap \lambda_2(p) = \{p, A_1 + B_1 = A_2 + B_2, C_1 + D_1 = C_2 + D_2\}. Note that (\Sigma_3(p) - \Sigma(p)) \cap \Sigma_5 = \{kp + X_1 + X_2 | k = 0, 1 (mod 2), X_1 \in St_1(p) - \{p\}, i = 1, 2\}. We note that neither A_3 nor B_3 belongs to the set \{kp + X_1 + X_2 | k = 0, 1 (mod 2), X_1 \in \{A_i, B_i\} i = 1, 2\} or \{C_i, D_i\} i = 1, 2. For otherwise, we can assume (WLOG) that A_3 = A_1 + A_2 \because A_3 + B_3 = P, and for i = 1, 2, A_i and B_i as well as C_i and D_i can be interchanged]. But A_3 = A_1 + A_2 \implies B_3 = P + A_1 + A_2 \implies t(C_1, D_1) \cap <P, A_1, A_2> = \emptyset, which is a contradiction. If St_3(p) \subseteq \Sigma_5, we can, therefore, assume
(WLOG) that $A_3 \in \{A_1 + C_2, A_1 + D_2, C_1 + A_2, C_1 + B_2\}$ and $B_3 \in \{B_1 + D_2, B_1 + C_2, D_1 + B_2, D_1 + A_2\}$. Again, $A_i$ and $B_i$ as well as $C_i$ and $D_i$ can be interchanged for $i = 1, 2$; also the suffixes of $St_1(P)$ and $St_2(P)$ can be readjusted. WLOG we can assume that $A_3 = A_1 + C_2$, $B_3 = B_1 + D_2$, when $St_3(P) \subset \Sigma_5$ i.e. when $<St_1(P), St_2(P), St_3(P)>$ is a 5-space.

Remark 1. If $St_1(P) = \{P, A_1, B_1, C_i, D_i\}$, $i = 1, 2$, where $A_1 + B_1 = A_2 + B_2$ and $St_3(P) = \{P, A_3, B_3\}$, then in order to show that $St_3(P) \not\subset <St_1(P), St_2(P)> = \Sigma_5$, it is enough to show that $A_3 \neq A_1 + C_2$.

§2. A few polarising sets and some properties connected with them.

If $St_1(P)$ and $St_2(P)$ are both 5-stigms and $St_3(P)$ is a line contained in $<St_1(P), St_2(P)>$, then we find some polarising sets and study some of their properties.

Proposition 5.2. Let $St_1(P) = \{P, A_1, B_1, C_i, D_i\}$, $i = 1, 2$, $St_3(P) = \{P, A_3, B_3\}$, where $A_1 + B_1 = A_2 + B_2$ and $A_3 = A_1 + C_2$. Let $X_1 \in \{C_1, D_1\}$, $X_2 \in \{A_2, B_2\}$, $X_3 \in \{A_3, B_3\}$ and $\Sigma_5 = <A_1, B_1, C_i, D_i, A_2, C_2>$. Then the following results hold.

(i) $X_1 + X_3$, $X_2 + X_3$ and $X_1 + X_2 + X_3$ are all attenuation points with respect to $\Sigma_5$. 

(ii) \( \{X_1, X_2, X_3\} \) is a polarising set with respect to \( \Sigma_5 \).

(iii) None of \( \rho^i_3(X_1, X_2, X_3) \), \( i = 1, 2, 3 \), contain more than three points of \( \Sigma - \Sigma_5 \).

(iv) If for any \( i, i \in \{1, 2, 3\} \), \( \rho^i_3(X_1, X_2, X_3) \) contains three points \( S_1, S_2, S_3 \) of \( \Sigma - \Sigma_5 \), then \( S_i + S_j \in \langle X_1, X_2 \rangle \), \( i \neq j, i, j = 1, 2, 3 \).

(v) If \( S_k \in \Sigma - \Sigma_5 \), \( k = 1, 2, 3, 4 \), such that \( S_1, S_2 \in \rho^i_3(X_1, X_2, X_3) \) and \( S_3, S_4 \in \rho^j_3(X_1, X_2, X_3) \), \( i \neq j, i, j \in \{1, 2, 3\} \), then \( S_1 + S_2 \neq S_3 + S_4 \).

Proof: (i) By virtue of the theorem 2.3, the proof follows immediately from the following.

\[
\begin{align*}
\langle A_2, C_2, A_1 \rangle \cap t(B_2, D_2, X_1) &= \{A_2 + A_3\}, \\
\langle B_2, D_2, B_1 \rangle \cap t(B_2, C_2, X_1) &= \{A_2 + B_3\}, \\
\langle B_2, D_2, B_1 \rangle \cap t(A_2, C_2, X_1) &= \{B_2 + B_3\}, \\
\langle B_2, C_2, A_1 \rangle \cap t(A_2, D_2, X_1) &= \{B_2 + A_3\}, \\
\langle A_1, C_1, C_2 \rangle \cap t(B_1, D_1) &= \{C_1 + A_3\}, \\
\langle B_1, C_1, D_2 \rangle \cap t(A_1, D_1) &= \{C_1 + B_3\}, \\
\langle A_1, D_1, C_2 \rangle \cap t(B_1, C_1) &= \{D_1 + A_3\}, \\
\langle B_1, D_1, D_2 \rangle \cap t(A_1, C_1) &= \{D_1 + B_3\}.
\end{align*}
\]

(ii) First of all it is easy to see that for every choice of the triplet \( \{X_1, X_2, X_3\} \), \( t(X_1, X_2, X_3) \) does not meet either the line \( \{A_1, C_2, A_3\} \) or the line \( \{B_1, D_2, B_3\} \); so, \( t(X_1, X_2, X_3) \not\subset \Sigma_5 \). Let us now consider the following table.
\[ \Theta = t(x_1, x_2, x_3) \]

<table>
<thead>
<tr>
<th>[ t(C_1, A_2, A_3) ]</th>
<th>[ \Sigma_2 ]</th>
<th>[ \Theta \cap \Sigma_2 = \gamma ]</th>
<th>[ \text{dim} &lt; x_1, x_2, x_3, \gamma &gt; ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ &lt;p, A_1, B_2 &gt; ]</td>
<td>[ P + A_1 ]</td>
<td>[ 3 ]</td>
<td></td>
</tr>
<tr>
<td>[ &lt;p, B_1, B_2 &gt; ]</td>
<td>[ P + B_1 ]</td>
<td>[ 3 ]</td>
<td></td>
</tr>
<tr>
<td>[ &lt;p, A_1, A_2 &gt; ]</td>
<td>[ P + A_1 ]</td>
<td>[ 3 ]</td>
<td></td>
</tr>
<tr>
<td>[ &lt;p, B_1, A_2 &gt; ]</td>
<td>[ P + B_1 ]</td>
<td>[ 3 ]</td>
<td></td>
</tr>
<tr>
<td>[ &lt;p, A_1, B_2 &gt; ]</td>
<td>[ P + A_1 ]</td>
<td>[ 3 ]</td>
<td></td>
</tr>
<tr>
<td>[ &lt;p, B_1, B_2 &gt; ]</td>
<td>[ P + B_1 ]</td>
<td>[ 3 ]</td>
<td></td>
</tr>
<tr>
<td>[ &lt;p, A_1, A_2 &gt; ]</td>
<td>[ P + A_1 ]</td>
<td>[ 3 ]</td>
<td></td>
</tr>
<tr>
<td>[ &lt;p, B_1, A_2 &gt; ]</td>
<td>[ P + B_1 ]</td>
<td>[ 3 ]</td>
<td></td>
</tr>
</tbody>
</table>

Thus, from the above table and the theorem 2.5, we conclude that \( \{x_1, x_2, x_3\} \) is a polarising set with respect to \( \Sigma_5 \), for all \( x_1, x_2 \) and \( x_3 \) such that 

\[ x_1 \in \{C_1, D_1\}, \quad x_2 \in \{A_2, B_2\} \quad \text{and} \quad x_3 \in \{A_3, B_3\}. \]

(iii) If it is not true, WLOG we assume that 

\[ \rho_3(C_1, A_2, A_3) \] contains at least four points \( S_1, S_2, S_3 \) and \( S_4 \) of \( \Theta - \Sigma_5 \). Then \( S_i + S_j \in <C_1, A_2, A_3>, \) \( i \neq j, \ i, j, = 1, 2, 3, 4 \). Considering the distribution of \( S_1, S_2, S_3, S_4 \) among \( \rho_3(D_1, B_2, B_3) \), \( i = 1, 2, 3 \), we can find \( k, t \in \{1, 2, 3, 4\}, k \neq t \), and \( j \in \{1, 2, 3\} \) such that \( S_k, S_t \in \rho_j(D_1, B_2, B_3) \). So, \( S_k + S_t \in <D_1, B_2, B_3> \). Hence \( S_k + S_t \in <C_1, A_2, A_3> \cap <D_1, B_2, B_3> = \{C_1 + A_2 + A_3\} \), which violates the proposition 5.2 (i)

(iv) If it is not true, WLOG we assume that 

\[ \rho_3(C_1, A_2, A_3) \] contains three points \( S_1, S_2, S_3 \) of \( \Theta - \Sigma_5 \) such that \( S_1 + S_2 \notin <C_1, A_2> \). But \( S_1 + S_2 \in \)
\( <C_1, A_2, A_3>, \) so that by the proposition 5.2(i), it follows that \( S_1 + S_2 = A_3. \) Then it is clear that \( S_1 + S_3 \in \{C_1, A_2, C_1 + A_2\}. \) Therefore, \( S_2 + S_3 \in \{C_1 + A_3, A_2 + A_3, C_1 + A_2 + A_3\}, \) which violates the proposition 5.2(i).

(v) If it is not true, WLOG we can assume that \( S_1, S_2 \in \rho_{13}(C_1, A_2, A_3), S_3, S_4 \in \rho_{3}(C_1, A_2, A_3) \) and \( S_1 + S_2 = S_3 + S_4. \) Then \( S_1 + S_2 = S_3 + S_4 = X, \) where \( X \in \{C_1, A_2, C_1 + A_2, A_3\}. \) Now consider the distribution of \( S_1, S_2, S_3, \) and \( S_4 \) among \( \rho_{i1}(D_1, B_2, B_3), i = 1, 2, 3. \) Since \( <C_1, A_2, A_3> \cap <D_1, B_2, B_3> = \{C_1 + A_2 + A_3\} \) and \( S_1 + S_2 = S_3 + S_4 = X \in <C_1, A_2, A_3>, \) we get by virtue of the proposition 5.2(i) that none of the pairs \( \{S_1, S_2\} \) and \( \{S_3, S_4\} \) can be in the same \( \rho_{i1}(D_1, B_2, B_3), i \in \{1, 2, 3\}. \) Rearranging the suffixes we can assume (WLOG) that \( S_1, S_3 \in \rho_{i1}(D_1, B_2, B_3), \) for some \( i \in \{1, 2, 3\}. \) Thus \( S_1 + S_3 = S_2 + S_4 = Y, \) where \( Y \in \{D_1, B_2, D_1 + B_2, B_3\}. \) First we show that \( S_1 + S_2 \neq A_3. \) If possible, let \( S_1 + S_2 = A_3. \) By the proposition 5.2(i), we deduce that \( S_1 + S_3 = B_3. \) Then \( <A_3, S_1, S_3> \) is a Fano block contained in \( \mathcal{B}, \) which is a contradiction. Similarly, we show that \( S_1 + S_3 \neq B_3. \) We now consider the following table.
From the above table we arrive at a contradiction. This completes the proof of this proposition.

§3. Distribution of the points of $\Phi - \langle St_1(P), St_2(P) \rangle$ with respect to some polarising sets.

If $St_1(P) = \{P, A_1, B_1, C_1, D_1\}, i = 1, 2$, $St_3(P) = \{P, A_3, B_3\}$, where $A_1 + B_1 = A_2 + B_2$ and $A_3 = A_1 + C_2$, then we discuss how the points of $\Phi - \langle St_1(P), St_2(P) \rangle$ are distributed among $\bigcup_i^3(X_1, X_2, X_3), i = 1, 2, 3, X_1 \in \{C_1, D_1\}, X_2 \in \{A_2, B_2\}$ and $X_3 \in \{A_3, B_3\}$. In this connection we have the following propositions.

**Proposition 5.3.** Let $St_1(P) = \{P, A_1, B_1, C_1, D_1\}, i = 1, 2$ and $St_3(P) = \{P, A_3, B_3\}$, where $A_1 + B_1 = A_2 + B_2$ and $A_3 = A_1 + C_2$. Let $X_1 \in \{C_1, D_1\}, X_2 \in \{A_2, B_2\}$ and $X_3 \in \{A_3, B_3\}$ and let $S_i \subseteq \Phi - \sum_5, i = 1, 2, 3, 4, 5$ and

<table>
<thead>
<tr>
<th>$S_1 + S_2$</th>
<th>$S_1 + S_3$</th>
<th>$\sum_2$</th>
<th>$\Theta$</th>
<th>$\Theta \cap \sum_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1$</td>
<td>$D_1$</td>
<td>$\langle C_1, S_1, S_3 \rangle$</td>
<td>$t(C_2)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$C_1$</td>
<td>$B_2$</td>
<td>$\langle C_1, S_1, S_3 \rangle$</td>
<td>$t(D_1, A_2)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$C_1$</td>
<td>$D_1 + B_2$</td>
<td>$\langle C_1, S_1, S_3 \rangle$</td>
<td>$t(P, D_1)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$D_1$</td>
<td>$\langle A_2, S_1, S_3 \rangle$</td>
<td>$t(C_1, D_1)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$B_2$</td>
<td>$\langle A_2, S_1, S_3 \rangle$</td>
<td>$t(A_1)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$D_1 + B_2$</td>
<td>$\langle A_2, S_1, S_3 \rangle$</td>
<td>$t(P, D_1)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$C_1 + A_2$</td>
<td>$D_1$</td>
<td>$\langle C_1 + A_2, S_1, S_3 \rangle$</td>
<td>$t(P, C_1)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$C_1 + A_2$</td>
<td>$B_2$</td>
<td>$\langle C_1 + A_2, S_1, S_3 \rangle$</td>
<td>$t(P, C_1)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$C_1 + A_2$</td>
<td>$D_1 + B_2$</td>
<td>$\langle C_1 + A_2, S_1, S_3 \rangle$</td>
<td>$t(C_1, D_1)$</td>
<td>$\emptyset$</td>
</tr>
</tbody>
</table>
\[ \sum_5 = \langle A_1, B_1, C_1, D_1, A_2, C_2 \rangle. \] Then the following results hold.

(i) If \( S_i \in \mathcal{P}_3^k(x_1, x_2, x_3), i = 1, 2, 3, k \in \{1, 2, 3\}, \) then both \( S_4 \) and \( S_5 \) cannot be in the same \( \mathcal{P}_3^k(x_1, x_2, x_3), \) for any \( k \in \{1, 2, 3\}. \)

(ii) If \( S_i \in \mathcal{P}_3^j(x_1, x_2, x_3), i = 1, 2, 3, \) then \(| \sum_5 - \sum_5 | = 5.\)

Proof: (i) Suppose (i) is not true. WLOG we can assume that \( S_i \in \mathcal{P}_3^1(C_1, A_2, A_3), i = 1, 2, 3, \) and both \( S_4 \) and \( S_5 \) are in \( \mathcal{P}_3^j(C_1, A_2, A_3), \) for some \( j, j \in \{1, 2, 3\}. \)

By the proposition 5.2(iii), \( j \neq 1. \) WLOG we assume \( j = 2. \)

Now we consider the distribution of \( S_i \)'s among \( \mathcal{P}_3^1(D_1, B_2, B_3), i = 1, 2, 3. \) Since \( \langle C_1, A_2, A_3 \rangle \cap \langle D_1, B_2, B_3 \rangle = \{C_1 + A_2 + A_3 \} \) and \( C_1 + A_2 + A_3 \) is an attenuation point with respect to \( \sum_5, \) it follows that none of the pairs \( \{S_1, S_2, S_3\}, \{S_1, S_3\}, \{S_2, S_3\} \) and \( \{S_4, S_5\} \) is in the same \( \mathcal{P}_3^i(D_1, B_2, B_3), i = 1, 2, 3. \) WLOG we assume that \( S_i \in \mathcal{P}_3^1(D_1, B_2, B_3), i = 1, 2, 3, S_4 \in \mathcal{P}_3^1(D_1, B_2, B_3) \) and \( S_5 \in \mathcal{P}_3^2(D_1, B_2, B_3). \) Let \( S_1 + S_4 = X \) and \( S_2 + S_5 = Y. \) Then \( X, Y \in \langle D_1, B_2, B_3 \rangle \) and by the proposition 5.2(v), \( X \neq Y. \) Again \( S_1 + S_2 \) and \( S_4 + S_5 \) are in \( \langle C_1, A_2, A_3 \rangle. \) So, \( X + Y = S_1 + S_2 + S_4 + S_5 \in \langle C_1, A_2, A_3 \rangle \cap \langle D_1, B_2, B_3 \rangle = \{D_1 + B_2 + B_3 \}. \) But \( X, Y \in \{D, B_2, D_1 + B_2, B_3 \}. \) So either \( X \) or \( Y \) is \( B_3. \)

WLOG we assume that \( X = B_3 \) i.e. \( S_1 + S_4 = B_3. \) By
proposition 5.2(iv), \( S_1 + S_2 \in \langle C_1, A_2 \rangle \). \( \therefore S_2 + S_4 = Z + B_3 \), where \( Z \in \langle C_1, A_2 \rangle \), which violates the proposition 5.2(i).

(ii) Suppose (ii) is false. Since by theorem 2.7, \( |\mathcal{S} - \Sigma_5| \geq 5 \), there exists a point \( S_6 \in \mathcal{S} - \Sigma_5 \) such that \( S_6 \neq S_i \), \( i = 1, 2, 3, 4, 5 \). Now \( S_1, S_2, S_3 \in \mathcal{P}_3^{j}(C_1, A_2, A_3), j \in \{1, 2, 3\} \). WLOG we assume \( j = 1 \).

By proposition 5.2(iii), \( S_4, S_5, S_6 \notin \mathcal{P}_3^1(C_1, A_2, A_3) \).

Hence at least two of the points \( S_4, S_5 \) and \( S_6 \) must be in \( \mathcal{P}_3^j(C_1, A_2, A_3) \), for some \( j \in \{2, 3\} \). But this violates proposition 5.3(i). This completes the proof of the proposition.

**Proposition 5.4.** Let \( St_i(P) = \{P, A_i, B_i, C_i, D_i\} \), \( i = 1, 2 \), and \( St_3(P) = \{P, A_3, B_3\} \), where \( A_1 + B_1 = A_2 + B_2 \) and \( A_3 = A_1 + C_2 \). Let \( \Sigma_5 = \langle A_1, B_1, C_1, D_1, A_2, C_2 \rangle \) and \( S_i \in \mathcal{S} - \Sigma_5 \), \( i = 1, 2, 3, 4, 5 \), \( S_i \)'s being all distinct. Let \( \Gamma_1 = \langle A_1, B_1, C_1, D_1, A_2, S_1 \rangle \) and \( \Gamma_2 = \langle A_2, B_2, C_2, D_2, C_1, S_1 \rangle \). If \( \mathcal{S} - \Sigma_5 \subset \Gamma_i \), \( i = 1, 2 \), then there exist exactly two distinct points \( T_1 \) and \( T_2 \) such that \( T_1 \in \mathcal{S} - \Gamma_1 \cup \{C_2, D_2, A_3, B_3\} \) and \( T_2 \in \mathcal{S} - \Gamma_2 \cup \{A_1, B_1, A_3, B_3\} \). Furthermore, \( T_1 \in \{C_1 + C_2, C_1 + D_2\} \) and \( T_2 \in \{A_1 + A_2, A_1 + B_2\} \), and if \( Q_1 \in \Sigma_5 - \langle A_1, B_1, C_1, D_1, A_2 \rangle \) and \( Q_2 \in \Sigma_5 - \langle A_2, B_2, C_2, D_2, C_1 \rangle \), then \( Q_1 = T_1 \) and \( Q_2 = T_2 \).
Proof: Clearly, $\Gamma_i$ is a 5-space, $i = 1, 2$. By the theorem 2.7, $|\mathcal{S} - \Gamma_i| \geq 5$, $i = 1, 2$. Since $\mathcal{S} - \Sigma_5 \subset \Gamma_i$, $i = 1, 2$, there are only four known points of $\mathcal{S}$ lying outside each of $\Gamma_1$ and $\Gamma_2$; more precisely, the only known four points lying outside $\Gamma_1$ are $C_2$, $D_2$, $A_3$ and $B_3$ and those outside $\Gamma_2$ are $A_1$, $B_1$, $A_3$ and $B_3$. So, there exist points $T_1$ and $T_2$ such that

$$T_1 \in \mathcal{S} - \Gamma_1 \cup \{C_2, D_2, A_3, B_3\}$$

and

$$T_2 \in \mathcal{S} - \Gamma_2 \cup \{A_1, B_1, A_3, B_3\}.$$

First, we look for $T_1$. Since $\mathcal{S} - \Sigma_5 \subset \Gamma_1$, it follows that $T_1 \in \mathcal{S} \cap \Sigma_5 - \langle A_1, B_1, C_1, D_1, A_2 \rangle$, i.e., $T_1 \in \mathcal{S} \cap \langle A_1, B_1, C_1, D_1, A_2 \rangle - \langle A_1, B_1, C_1, D_1, A_2 \rangle$, i.e. $T_1$ is a point of $\mathcal{S}$ and $T_1 = C_2 + X$, where $X \in \langle A_1, B_1, C_1, D_1, A_2 \rangle$. We assert that $X \in \{C_1, D_1\}$.

First of all, by the proposition 5.2(i) and the fact that $T_1 \in \mathcal{S}$, we can easily conclude that $X \not\in \{A_1 + C_1, B_1 + D_1, A_1 + D_1, B_1 + C_1, A_1 + C_1 + A_2, B_1 + D_1 + A_2, A_1 + D_1 + A_2, B_1 + C_1 + A_2, A_1 + A_2 + C_2, P + A_1 + A_2 + C_2, B_1 + A_2 + C_2, P + B_1 + A_2 + C_2 \}$. Since $X \not\in A_3, B_3, D_2$, we get that $X \not\in \{A_1, P + A_1, C_1 + D_1\}$. Since $T_1 \in \mathcal{S}$ and $T_1 \not\in P$, $X \not\in \tau_2(P) - \pi(P)$. Also, note that $\langle A_1, B_1, C_1, D_1, A_2 \rangle \cap \pi(P) = \lambda_1(P)$. Now we have the following table.
From the above table and previous remarks, we conclude that \( X \in \{C_1, D_1, C_2\} \), i.e. \( T_1 \in \{C_1 + C_2, C_1 + D_2\} \). Again, both \( C_1 + C_2 \) and \( C_1 + D_2 \) are not in \( \emptyset \); for, \( C_1 + C_2, C_1 + D_2 \in \emptyset \Rightarrow t(A_1, B_1) \cap <C_1, D_1, C_2> = \emptyset \). Thus there exist exactly one point \( T_1 \) that belongs to \( \emptyset \) - \( \Gamma_1 \cup \{C_2, D_2, A_3, B_3\} \). Similarly, we can show that there exists exactly one point \( T_2 \) such that \( T_2 \in \emptyset \) - \( \Gamma_2 \cup \{A_1, B_1, A_3, B_3\} \) and furthermore, \( T_2 \in \{A_1 + A_2, A_1 + B_2\} \).

Now, from the way we have found out \( T_i, i = 1, 2 \), it follows clearly that if \( Q_1 \in \Sigma_5 - <A_1, B_1, C_1, D_1, A_2> \) and \( Q_2 \in \Sigma_5 - <A_2, B_2, C_2, D_2, C_1> \), then \( Q_1 = T_1 \) and \( Q_2 = T_2 \).

**Proposition 5.5.** Let \( St_i(p) = \{p, A_i, B_i, C_i, D_i\} \), i = 1, 2 and \( St_3(p) = \{p, A_3, B_3\} \), where \( A_1 + B_1 = A_2 + B_2 \) and
\( A_3 = A_1 + C_2 \). Let \( X_1, Y_1 \in \{C_1, D_1^3, X_2, Y_2 \in \{A_2, B_2\}, X_3, Y_3 \in \{A_3, B_3\}, S_i \in S - \Sigma_5, i = 1, 2, 3, 4, 5 \) and 
\[ \Sigma_5 = \langle A_1, B_1, C_1, D_1, A_2, C_2 \rangle \). If \( \langle Y_1, Y_2, Y_3 \rangle \cap \langle X_1, X_2, X_3 \rangle = \{X_1 + X_2 + X_3\} \) and \( S_i \in \rho^j(X_1, X_2, X_3), i = 1, 2, 3, j \in \{1, 2, 3\}, \) then both \( S_4 \) and \( S_5 \) cannot be in the same \( \rho^K(Y_1, Y_2, Y_3) \), for any \( k \in \{1, 2, 3\} \).

**Proof:** (i) Suppose it is false. WLOG we assume that \( S_1 \in \rho^i(C_1, A_2, A_3), i = 1, 2, 3 \) and \( \{Y_1, Y_2, Y_3\} = \{D_1, B_2, B_3\} \) and both \( S_4 \) and \( S_5 \) are in the same \( \rho^K(D_1, B_2, B_3) \), for some \( k \in \{1, 2, 3\} \). WLOG we assume that \( S_4 + S_2 = C_1, S_4 + S_3 = A_2 \) and \( S_2 + S_3 = C_1 + A_2 \) and (by proposition 5.3(i)) \( S_4 \in \rho^2(C_1, A_2, A_3), S_5 \in \rho^3(C_1, A_2, A_3) \). Since
\[ \langle C_1, A_2, A_3 \rangle \cap \langle B_1, B_2, B_3 \rangle = \{C_1 + A_2 + A_3\} \) so that by proposition 5.2(i), none of the pairs \( \{S_i, S_j\}, i \neq j, i, j = 1, 2, 3 \), is in \( \rho^i(D_1, B_2, B_3), \forall i \in \{1, 2, 3\} \), we can assume (WLOG) that \( S_i \in \rho^3(D_1, B_2, B_3), i = 1, 2, 3 \).

We now consider the following three cases:

(a) \( S_4, S_5 \in \rho^1(D_1, B_2, B_3) \)

(b) \( S_4, S_5 \in \rho^2(D_1, B_2, B_3) \)

(c) \( S_4, S_5 \in \rho^3(D_1, B_2, B_3) \)

**Case (a).** Let \( S_4, S_5 \in \rho^1(D_1, B_2, B_3) \). Also we have \( S_1 \in \rho^3(D_1, B_2, B_3) \). Therefore both \( S_1 + S_4 \) and \( S_4 + S_5 \) are in \( \{D_1, B_2, D_1 + B_2\} \) [by proposition 5.2(iv)]. Now consider the following table.
The above table leads to a contradiction. Thus case (a) is not possible.

Case (b). Let \( S_4, S_5 \in \binom{2}{3}(D_1, B_2, B_3) \). Also \( S_2 \in \binom{2}{3}(D_1, B_2, B_3) \). So \( S_i + S_j \in \{D_1, B_2, D_1 + B_2\} \), \( i \neq j \), \( i, j = 2, 4, 5 \) [by the proposition 5.2(iv)].

Consider the following table.

<table>
<thead>
<tr>
<th>( S_4 + S_5 )</th>
<th>( S_2 + S_4 )</th>
<th>( S_2 + S_5 )</th>
<th>( \Sigma_2 )</th>
<th>( \emptyset )</th>
<th>( \Sigma_2 \cap \emptyset )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D_1 + B_2 )</td>
<td>( D_1 )</td>
<td>( B_2 )</td>
<td>( \langle S_1, S_2, S_4 \rangle )</td>
<td>( t(S_3, P, C_2) )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( D_1 )</td>
<td>( B_2 )</td>
<td>( D_1 + B_2 )</td>
<td>( \langle S_1, S_3, S_4 \rangle )</td>
<td>( t(S_3, P, C_2) )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( B_2 )</td>
<td>( D_1 )</td>
<td>( D_1 + B_2 )</td>
<td>( \langle S_1, S_2, S_4 \rangle )</td>
<td>( t(S_3, P, C_2) )</td>
<td>( \emptyset )</td>
</tr>
</tbody>
</table>

From the above table we conclude (WLOG) that \( S_4 + S_5 = D_1 + B_2 \). We deduce that

\[
\begin{align*}
S_1 + S_2 &= C_1, \quad S_1 + S_3 = A_2, \quad S_1 + S_4 = C_1 + B_2, \quad S_1 + S_5 = P + A_2 \\
S_2 + S_3 &= C_1 + A_2, \quad S_2 + S_4 = B_2, \quad S_2 + S_5 = D_1 + B_2 \\
S_3 + S_4 &= P + D_1, \quad S_3 + S_5 = P, \quad S_4 + S_5 = D_1.
\end{align*}
\]

Let \( \Gamma_1 = \langle A_1, B_1, C_1, D_1, A_2, S_1 \rangle \) and \( \Gamma_2 = \langle A_2, B_2, C_2, D_2, C_1, S_1 \rangle \). From relations (1) it follows that \( S_i \in \Gamma_j, \forall i, j = 1, 2, 3, 4, 5 \) and \( j = 1, 2 \). So, by proposition 5.3(ii), we have \( \emptyset - \Sigma_5 \subset \Gamma_i, i = 1, 2 \). So, by the proposition 5.4, there exist exactly two points \( T_i, i = 1, 2 \) such that \( T_1 \in \emptyset - \Gamma_1 \cup \{C_2, D_2, A_3, B_3\} \) and \( T_2 \in \emptyset - \Gamma_2 \cup \{A_1, B_1, A_3, B_3\} \) and furthermore, \( T_1 \in \{C_1 + C_2, C_1 + D_2\} \) and \( T_2 \in \{C_1 + C_2, C_1 + D_2\} \).\]
\[ \{A_1 + A_2, A_1 + B_2\} \]. We now assert that \( \mathcal{S} = \left( \bigcup_{i=1}^{3} St_i(P) \right) \cup \{S_i \mid i = 1, 2, 3, 4, 5\} \cup \{T_1, T_2\}. \] If possible, let \( Q \in \mathcal{S} - \bigcup_{i=1}^{3} St_i(P) \cup \{S_i \mid i = 1, 2, 3, 4, 5\} \cup \{T_1, T_2\}. \) By proposition 5.3(ii), \( Q \in \mathcal{S} \cap S_5. \) But \( Q \in \langle A_1, B_1, C_1, D_1, A_2, C_2 \rangle \Rightarrow \langle A_1, B_1, C_1, D_1, A_2 \rangle \Rightarrow Q = T_1 \) [by proposition 5.3(ii)], which contradicts our choice of \( Q. \) So, \( Q \in \langle A_1, B_1, C_1, D_1, A_2 \rangle. \) Since \( Q \in \mathcal{S}, \) we have the following table

<table>
<thead>
<tr>
<th>( Q )</th>
<th>( \Sigma_2 )</th>
<th>( \mathcal{S} )</th>
<th>( \Sigma_2 \cap \mathcal{S} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p + A_1 )</td>
<td>( \langle p, A_1, A_3 \rangle )</td>
<td>( t(B_2, D_2) )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( p + B_1 )</td>
<td>( \langle p, B_1, B_3 \rangle )</td>
<td>( t(B_2, C_2) )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( p + C_1 )</td>
<td>( \langle p, C_1, D_1 \rangle )</td>
<td>( t(S_4, A_2, C_2) )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( p + D_1 )</td>
<td>( \langle S_4, S_3, S_5 \rangle )</td>
<td>( t(S_2, A_2, C_2) )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( p + A_2 )</td>
<td>( \langle S_3, S_1, S_5 \rangle )</td>
<td>( t(S_2, A_1, D_1) )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( p + B_2 )</td>
<td>( \langle P, A_2, B_2 \rangle )</td>
<td>( t(S_1, A_1, D_1) )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( C_1 + A_2 )</td>
<td>( \langle C_1, A_2, B_2 \rangle )</td>
<td>( t(S_1, A_1, D_1) )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( C_1 + B_2 )</td>
<td>( \langle C_1, D_1, B_2 \rangle )</td>
<td>( t(S_2, A_2, C_2) )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( D_1 + A_2 )</td>
<td>( \langle C_1, D_1, A_2 \rangle )</td>
<td>( t(S_3, B_2, C_2) )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( D_1 + B_2 )</td>
<td>( \langle B_2, C_1, D_1 \rangle )</td>
<td>( t(S_4, A_2, C_2) )</td>
<td>( \emptyset )</td>
</tr>
</tbody>
</table>

From the above table we deduce that no such point \( Q \) of \( \mathcal{S} \) can exist. Thus \( \mathcal{S} = \left( \bigcup_{i=1}^{3} St_i(P) \right) \cup \{S_i \mid i = 1, 2, 3, 4, 5\} \cup \{T_1, T_2\}. \) Let us suppose \( T_1 = C_1 + C_2 \) and \( T_2 = A_1 + A_2. \) It is easy to check that \( C_1 + C_2 \) cannot be on any other line of \( \mathcal{S} \) except the lines \( \{C_1, C_2, C_1 + C_2\} \) and \( \{D_1, D_2, C_1 + C_2\}. \) Let us now consider the distribution of the points of \( \mathcal{S} \) among \( T_i(C_1 + C_2), i = 1, 2, 3. \) Every
point of $\emptyset$ other than $C_1 + C_2$ must belong to $\tau_i(C_1 + C_2) - \Pi(C_1 + C_2)$, for some $i \in \{1, 2, 3\}$. Also we note that two points of $\emptyset$ are in the same $\tau_i(C_1 + C_2)$, for some $i \in \{1, 2, 3\}$, iff their sum is in $\Pi(C_1 + C_2)$. WLOG we assume $C_1, C_2 \in \tau_1(C_1 + C_2)$.

We consider the following cases:

1. $D_1, D_2 \in \tau_2(C_1 + C_2)$,
2. $D_1, D_2 \in \tau_1(C_1 + C_2)$.

**Case 1.** Let $D_1, D_2 \in \tau_2(C_1 + C_2)$. Since $S_1 + S_2 = C_1$ and $C_1 \in \tau_1(C_1 + C_2) - \Pi(C_1 + C_2)$, one of the points $S_1$ and $S_2$ is in $\tau_2(C_1 + C_2)$, while the other one is in $\tau_3(C_1 + C_2)$. We consider the following cases:

1.1. $S_1 \in \tau_3(C_1 + C_2)$,
1.2. $S_4 \in \tau_3(C_1 + C_2)$.

**Case 1.1.** Let $S_1 \in \tau_3(C_1 + C_2)$. So, $S_2 \in \tau_2(C_1 + C_2)$. Since $S_4 + S_5 = D_1$ and $D_1 \in \tau_2(C_1 + C_2)$, it follows that one of the points $S_4$ and $S_5$ is in $\tau_1(C_1 + C_2)$, while the other is in $\tau_3(C_1 + C_2)$. We consider the following cases:

1.1.1. $S_4 \in \tau_3(C_1 + C_2)$,
1.1.2. $S_5 \in \tau_1(C_1 + C_2)$.

**Case 1.1.1.** Let $S_4 \in \tau_3(C_1 + C_2)$. So, $S_5 \in \tau_1(C_1 + C_2)$. Since $S_1 + S_4 = A_2$ and $S_3 + S_5 = P$, $S_3 \not\in \tau_1(C_1 + C_2)$, $i = 1, 2$. \[ S_3 \in \tau_2(C_1 + C_2). \] Thus $S_1 + S_3 = A_2 \in \tau_1(C_1 + C_2), S_2 + S_4 = B_2 \in \tau_1(C_1 + C_2)$. Since $A_1 + C_2 = A_3$ and $B_1 + D_2 = B_3$ and $A_2 + B_2 = A_1 + B_1 \in \Pi(C_1 + C_2)$, it follows that $A_1, B_1 \in \tau_3(C_1 + C_2)$. So, $\emptyset \cap \tau_3(C_1 + C_2)$
\[ \{ S_1, S_4, P, A_1, B_1, C_1 + C_2 \} \], which does not contain any odd stigm \( S_3(C_1 + C_2) \). This is a contradiction.

**Case 1.1.2.** Let \( S_4 \in \tau_1(C_1 + C_2) \). So, \( S_5 \in \tau_3(C_1 + C_2) \).

Since \( S_1 + S_3 = A_2 \), \( S_3 \notin \tau_1(C_1 + C_2) \). Suppose \( S_3 \in \tau_2(C_1 + C_2) \). Using relations (1) and the fact that \( A_1 + C_2 = A_3 \) and \( B_1 + D_2 = B_3 \), we deduce that

\[ \emptyset \cap \tau_3(C_1 + C_2) = \{ S_1, S_5, A_1, B_2, C_1 + C_2 \} \], which does not contain an odd stigm \( S_3(C_1 + C_2) \). So, \( S_3 \notin \tau_2(C_1 + C_2) \). \( \therefore \) \( S_3 \in \tau_1(C_1 + C_2) \). Then it follows

\[ \emptyset \cap \tau_3(C_1 + C_2) = \{ S_1, S_5, B_1, B_2, C_1 + C_2 \} \], which does not contain an odd stigm \( S_3(C_1 + C_2) \). This is a contradiction.

Using this type of argument we can show contradiction in all the remaining cases. This shows that case (b) cannot happen.

**Case (c).** Let \( S_4, S_5 \in \rho_3(D_1, B_2, B_3) \). Also \( S_3 \in \rho_3(D_1, B_2, B_3) \). We have the following table.

<table>
<thead>
<tr>
<th>( S_4 + S_5 )</th>
<th>( S_3 + S_4 )</th>
<th>( S_3 + S_5 )</th>
<th>( \Sigma_2 )</th>
<th>( \emptyset )</th>
<th>( \emptyset \cap \Sigma_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D_1 + B_2 )</td>
<td>( B_2 )</td>
<td>( D_1 )</td>
<td>( &lt;S_3, S_1, S_4&gt;)</td>
<td>( t(S_5, A_1, C_1) )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( D_1 )</td>
<td>( B_2 )</td>
<td>( D_1 + B_2 )</td>
<td>( &lt;S_3, S_1, S_4&gt;)</td>
<td>( t(S_2, A_1, D_1) )</td>
<td>( \emptyset )</td>
</tr>
</tbody>
</table>

From the above table, we have (wlog) \( S_4 + S_5 = B_2, S_3 + S_4 = D_1, S_3 + S_5 = D_1 + B_2 \). So we get

\[
\begin{align*}
S_1 + S_2 &= C_1, S_1 + S_3 = A_2, S_1 + S_4 = D_1 + A_2, S_1 + S_5 = P + C_1 \\
S_2 + S_3 &= C_1 + A_2, S_2 + S_4 = P + B_2, S_2 + S_5 = P, \\
S_3 + S_4 &= D_1, S_3 + S_5 = D_1 + B_2, S_4 + S_5 = B_2.
\end{align*}
\]

(2)

Now we proceed in the way exactly similar to what we did in case (b) and finally arrive at a contradiction. Hence
case (c) cannot happen. This completes the proof of the proposition.

**Proposition 5.6.** Let $S^j_t(p) = \{P, A_1, B_1, C_1, D_1\}$, $i = 1, 2$, $S^3_t(p) = \{P, A_3, B_3\}$, where $A_1 + B_1 = A_2 + B_2$ and $A_3 = A_1 + C_2$. Let $X_1 \in \{C_1, D_1\}$, $X_2 \in \{A_2, B_2, B_3\}$, $X_3 \in \{A_3, B_3\}$, $S_i \in \mathbb{R} - \Sigma_5$, $i = 1, 2, 3, 4, 5$ and $\Sigma_5 = \langle A_1, B_1, C_1, D_1, A_2, C_2 \rangle$. Then none of $\rho_3^j(X_1, X_2, X_3)$, $i = 1, 2, 3$, contains more than two points of $S_1, S_2, S_3, S_4$ and $S_5$.

**Proof:** Suppose the proposition is false. WLOG we assume that $S_i \in \rho_3^j(C_1, A_2, A_3)$, $i = 1, 2, 3$. So, we have $(\text{WLOG}) S_1 + S_2 = C_1$, $S_1 + S_3 = A_3$, $S_2 + S_3 = C_1 + A_2$.

Considering the distribution of $S_1, S_2, S_3$ among $\rho_3^j(D_1, B_2, B_3)$, we assume $(\text{WLOG})$ that $S_i \in \rho_3^j(D_1, B_2, B_3)$, $i = 1, 2, 3$. Suppose $S_4 \in \rho_3^j(D_1, B_2, B_3)$. By the proposition 5.2(i), we conclude that $S_1 + S_4 \neq B_3$ [\because S_1 + S_2 = C_1]. We consider the following table.

<table>
<thead>
<tr>
<th>$S_1 + S_4$</th>
<th>$\Sigma_2$</th>
<th>$\Theta$</th>
<th>$\Sigma_2 \cap \Theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_1$</td>
<td>$\langle S_1, S_2, S_4 \rangle$</td>
<td>$t(S_3, B_2, D_2)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$B_2$</td>
<td>$\langle S_1, S_3, S_4 \rangle$</td>
<td>$t(S_2, A_1, D_1)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$D_1 + B_2$</td>
<td>$\langle S_1, S_2, S_4 \rangle$</td>
<td>$t(P, B_2)$</td>
<td>$\emptyset$</td>
</tr>
</tbody>
</table>

From the above table, we get a contradiction. Hence $S_4 \notin \rho_3^j(D_1, B_2, B_3)$. Similarly, $S_5 \notin \rho_3^j(D_1, B_2, B_3)$. WLOG we assume $S_4 \in \rho_3^j(D_1, B_2, B_3)$ and $S_5 \in \rho_3^j(D_1, B_2, B_3)$. Also we have $S_i \in \rho_3^j(D_1, B_2, B_3)$, $i = 2, 3$. By the proposition 5.2(iv), $S_2 + S_4 \neq S_3 + S_5$. We consider the
the following table:

<table>
<thead>
<tr>
<th>$S_2 + S_4$</th>
<th>$S_3 + S_5$</th>
<th>$\Sigma_2$</th>
<th>$\Theta$</th>
<th>$\Theta \cap \Sigma_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_1 + B_2$</td>
<td>$D_1$</td>
<td>$&lt;S_2, S_4, S_3&gt;$</td>
<td>$t(A_2, B_2)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$D_1 + B_2$</td>
<td>$B_2$</td>
<td>$&lt;S_2, S_3, S_4&gt;$</td>
<td>$t(C_1, D_1)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$D_2$</td>
<td>$D_1 + B_2$</td>
<td>$&lt;S_2, S_3, S_5&gt;$</td>
<td>$t(C_1, D_1)$</td>
<td>$\emptyset$</td>
</tr>
</tbody>
</table>

It follows from the above table that we are left with the following two cases.

1. $S_2 + S_4 = D_1$, $S_3 + S_5 = B_2$;
2. $S_2 + S_4 = B_2$, $S_3 + S_5 = D_1$.

**Case 1.** Let $S_2 + S_4 = D_1$, $S_3 + S_5 = B_2$. Also, we have

\[
\begin{align*}
S_1 + S_2 &= C_1, 
S_1 + S_3 &= A_2, 
S_2 + S_3 &= C_1 + A_2.
\end{align*}
\]

So we get

\[
\begin{align*}
S_1 + S_2 &= C_1, 
S_1 + S_3 &= A_2, 
S_1 + S_4 &= C_1 + D_1, 
S_1 + S_5 &= A_2 + B_2, \\
S_2 + S_3 &= C_1 + A_2, 
S_2 + S_4 &= D_1, 
S_2 + S_5 &= P + D_1, \\
S_3 + S_4 &= P + B_2, 
S_3 + S_5 &= B_2, 
S_4 + S_5 &= P.
\end{align*}
\]

Let $\Gamma_1 = \langle A_1, B_1, C_1, D_1, A_2, S_1 \rangle$ and $\Gamma_2 = \langle A_2, B_2, C_2, D_2, C_1, S_1 \rangle$. The relations (1) show that both $\Gamma_1$ and $\Gamma_2$ are $5$-spaces. By the proposition 5.3(ii), $A_3 = \Sigma_5 \subset \Gamma_i$, $i = 1, 2$. Now if $Q \in A_3 = \Sigma_5 \cap \Gamma_i$, $i = 1, 2$, then we have the following table.
<table>
<thead>
<tr>
<th>Q</th>
<th>$\Sigma_2$</th>
<th>$Q$</th>
<th>$\Sigma_2 \cap Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p + A_1$</td>
<td>$\langle p, A_1, A_2 \rangle$</td>
<td>$t(B_2, D_2)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$p + B_1$</td>
<td>$\langle p, B_1, B_3 \rangle$</td>
<td>$t(B_2, C_2)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$p + C_1$</td>
<td>$\langle p, C_1, D_1 \rangle$</td>
<td>$t(S_2, B_2, D_2)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$p + D_1$</td>
<td>$\langle S_1, S_2, S_5 \rangle$</td>
<td>$t(S_4, C_2, D_2)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$p + A_2$</td>
<td>$\langle p, A_2, B_2 \rangle$</td>
<td>$t(S_3, A_1, D_1)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$p + B_2$</td>
<td>$\langle S_1, S_3, S_4 \rangle$</td>
<td>$t(S_5, A_1, D_1)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$c_1 + B_2$</td>
<td>$\langle C_1, A_2, B_2 \rangle$</td>
<td>$t(S_3, A_1, D_1)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$d_1 + A_2$</td>
<td>$\langle A_2, C_1, D_1 \rangle$</td>
<td>$t(S_3, B_2, D_2)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$d_1 + B_2$</td>
<td>$\langle D_1, B_2, C_1 + A_2 \rangle$</td>
<td>$t(S_2, C_2, D_2)$</td>
<td>$\emptyset$</td>
</tr>
</tbody>
</table>

Also $Q = C_1 + A_2 \in \bigcap \langle S_1, S_2, S_3 \rangle = \{S_3 + C_1\}$ and $t(B_2, C_2) \cap \langle S_1, S_2, S_4 \rangle = \{S_2 + C_1 + D_1\}$

Now the rest of the arguments are exactly similar to those in case (b) in the proof of proposition 5.5 and we arrive at a contradiction. Thus case 1 cannot happen.

**Case 2.** Let $S_2 + S_4 = B_2$, $S_3 + S_5 = D_1$. Then we get

$S_1 + S_2 = C_1$, $S_1 + S_3 = A_2$, $S_1 + S_4 = C_1 + B_2$, $S_1 + S_5 = D_1 + A_2$, $S_2 + S_3 = C_1 + A_2$, $S_2 + S_4 = B_2$, $S_2 + S_5 = P + B_2$,

$S_3 + S_4 = P + D_1$, $S_3 + S_5 = D_1$, $S_4 + S_5 = P$. Again we use arguments exactly similar to those used for case (b) in the proof of the proposition 5.5 and finally arrive at a contradiction. Thus case 2 cannot happen. This completes the proof of the proposition.
§4. Non-existence of a (5, 5, 3)-tangential stigm system where the 3-stigm is contained in the space generated by the two 5-stigms.

If \( St_1(p) \) and \( St_2(p) \) are 5-stigms and \( St_3(p) \) is a 3-stigm, then we intend to show that \( St_3(p) \notin \langle St_1(p), St_2(p) \rangle \).

**Proposition 5.7.** Let \( St_i(p) = \{P, A_i, B_i, C_i, D_i\} \), \( i = 1, 2, St_3(p) = \{P, A_3, B_3\} \) and \( A_1 + B_1 = A_2 + B_2 \) and \( A_3 = A_1 + C_2 \). Let \( X_1 \in \{A_1, B_1\} \), \( X_2 \in \{A_2, B_2\} \) and \( X_3 \in \{A_3, B_3\} \).

Let \( \Sigma_5 = \langle A_1, B_1, C_1, D_1, A_2, C_2 \rangle \) and let \( S_i \subseteq \Sigma_5 \), \( i = 1, 2, 3, 4, 5 \). If \( S_u, S_t \in \rho_3^i(X_1, X_2, X_3) \), \( i \in \{1, 2, 3\} \) and \( u, t \in \{1, 2, 3, 4, 5\} \), \( u \neq t \), then

(i) \( S_u + S_t \neq X_3 \),
(ii) \( S_u + S_t \neq X_1 \), and
(iii) \( S_u + S_t \neq X_2 \).

**Proof:** (i) Suppose (i) is false. WLOG we assume that \( S_1, S_2 \in \rho_3^1(C_1, A_2, A_3) \) and \( S_1 + S_2 = A_3 \). By proposition 5.6, we assume (WLOG) that \( S_3, S_4 \in \rho_3^2(C_1, A_2, A_3) \).

Since \( \langle C_1, A_2, A_3 \rangle \cap \langle D_1, B_2, B_3 \rangle = \{C_1 + A_2 + A_3\} \) and \( C_1 + A_2 + A_3 \) is an attenuation point (by proposition 5.2(i)), it follows that none of the pairs \( \{S_1, S_2\} \) and \( \{S_3, S_4\} \) is in \( \rho_3^i(D_1, B_2, B_3) \), \( i \in \{1, 2, 3\} \). So, readjusting the suffixes of the same we can assume (WLOG) that \( S_1, S_3 \in \rho_3^i(D_1, B_2, B_3) \), for some \( i \in \{1, 2, 3\} \). Now \( S_1 + S_2 = A_3, S_3 + S_4 \in \langle C_1, A_2, A_3 \rangle \) and \( S_1 + S_3 \in \rho_3^1(D_1, B_2, B_3) \). Now
\(< D_1, B_2, B_3 >\). But \(S_1 + S_2 = A_3\) \(\implies S_1 + S_3 = B_3\) [by proposition 5.2(i)] \(\implies S_3 + S_4 = A_3\) [by proposition 5.2(i)]. \(< A_3, S_1, S_3 >\) is a Fano-block contained in \(\mathcal{B}\), which is a contradiction.

(ii) Suppose (ii) is false. Without loss of generality, assume that 
\(S_1, S_2 \in \mathcal{P}_3(C_1, A_2, A_3)\) and \(S_1 + S_2 = C_1\). Without loss of generality, assume \(S_3, S_4 \in \mathcal{P}_3(C_1, A_2, A_3)\) [by proposition 5.6]. As in the proof of (i), we can assume (without loss of generality) that \(S_1, S_3 \in \mathcal{P}_3(D_1, B_2, B_3)\). Without loss of generality, we assume that \(S_2 \in \mathcal{P}_3(D_1, B_2, B_3)\), \(i = 1, 2\). Clearly \(S_4 \not\in \mathcal{P}_3(D_1, B_2, B_3)\). Suppose \(S_4 \in \mathcal{P}_3(D_1, B_2, B_3)\). Then by proposition 5.7(i), 
\(S_1 + S_3, S_2 + S_4 \not\in \mathcal{P}_3(D_1, B_2)\) and \(S_1 + S_2, S_3 + S_4 \not\in \mathcal{P}_3(C_1, A_2);\) also \(S_1 + S_2 \not\in S_3 + S_4\) [by proposition 5.2(iv)]; so, \(S_1 + S_2 + S_3 + S_4 \not\in \mathcal{P}_3(C_1, A_2) \cap \mathcal{P}_3(D_1, B_2) = \emptyset\), which is a contradiction. Therefore, \(S_4 \not\in \mathcal{P}_3(D_1, B_2, B_3)\).

Case 1. Let \(S_3 + S_4 = A_2\). Also we have \(S_1 + S_2 = C_1\) and \(S_1 + S_3 \not\in \mathcal{P}_3(D_1, B_2)\). But \(S_1 + S_3 = D_1 \implies S_1, S_3, S_4 \in \mathcal{P}_3(D_1, A_2, A_3)\) for some \(i \in \{1, 2, 3\}\), which violates proposition 5.6; \(S_1 + S_3 = B_2 \implies S_1, S_2, S_3 \in \mathcal{P}_3(D_1, B_2, A_3)\), for some \(i \in \{1, 2, 3\}\), which violates proposition 5.6. \(\therefore S_1 + S_3 = D_1 + B_2\).

Clearly, \(S_5 \in \mathcal{P}_3(D_1, B_2, B_3)\), \(i = 2\) or 3. Suppose \(S_5 \in \mathcal{P}_3(D_1, B_2, B_3)\). Also \(S_2 \in \mathcal{P}_3(D_1, B_2, B_3)\). So, \(S_2 + S_5 = D_1\) or \(B_2\) [by proposition 5.2(iv)]. But \(S_2 + S_5 =
\( B_2 \implies S_1, S_2, S_5 \in \rho_3^1(C_1, B_2, A_3), \) which violates proposition 5.6. So, \( S_2 + S_5 = D_1. \) But \( S_2 + S_5 = D_1 \implies t(S_3, C_2) \cap <S_1, S_2, S_5> = \emptyset. \) Thus we arrive at a contradiction. So, \( S_5 \notin \rho_3^2(D_1, B_2, B_3). \) Hence \( S_5 \in \rho_3^3(D_1, B_2, B_3); \) also \( S_4 \in \rho_3^3(D_1, B_2, B_3). \) So, \( S_4 + S_5 \in <D_1, B_2>. \) Since \( S_1 + S_3 = D_1 + B_2, \) \( S_4 + S_5 \neq D_1 + B_2 \) [by proposition 5.2(iv)]. But \( S_4 + S_5 = D_1 \implies S_3, S_4, S_5 \in \rho_3^1(D_1, A_2, A_3), \) for some \( i \in \{1, 2, 3\}, \) which violates proposition 5.6. So, \( S_4 + S_5 = B_2; \) also \( S_3 + S_4 = A_2; \) but then \( t(S_1, A_1) \cap <S_4, S_3, S_5> = \emptyset, \) which is a contradiction. Thus case 1 cannot happen.

Case 2. Let \( S_3 + S_4 = C_1 + A_2. \) Now \( S_1 + S_3 \notin B_2, \) for, otherwise, \( S_1, S_2, S_3 \in \rho_3^1(C_1, B_2, A_3), \) for some \( i \in \{1, 2, 3\}. \) But \( S_1 + S_3 = D_1 \implies t(S_4, C_2) \cap <S_1, S_2, S_3> = \emptyset, \) which is a contradiction. So, \( S_1 + S_3 = D_1 + B_2. \) But then \( t(A_2, B_2) \cap <S_3, S_1, S_4> = \emptyset, \) which is a contradiction. Therefore, case 2 cannot happen. Thus we arrive at a contradiction due to the assumption that \( S_1 + S_2 = C_1. \)

(iii) Suppose (iii) is false. WLOG we assume that \( S_1, S_2 \in \rho_3^1(C_1, A_2, A_3) \) and \( S_1 + S_2 = A_2. \) WLOG we can assume that \( S_3, S_4 \in \rho_3^1(C_1, A_2, A_3) \) and \( S_1, S_3 \in \rho_3^1(D_1, B_2, B_3). \) By propositions 5.2(iv) and 5.7(i) and 5.7(ii), \( S_3 + S_4 = C_1 + A_2 \) and \( S_1 + S_3 \in \{B_2, D_1 + B_2\}. \) But \( S_1 + S_3 = B_2 \implies t(A_1, S_4) \cap <S_1, S_2, S_3> = \emptyset, \)
which is a contradiction. Also \( S_1 + S_3 = D_1 + B_2 \Rightarrow S_2 + S_3 = P + C_1 \Rightarrow S_3 + P = S_2 + C_1 \Rightarrow t(C_1, D_1) \cap \langle S_3, S_1, S_4 \rangle = \emptyset \), which is a contradiction.

This completes the proof of the proposition.

**Proposition 5.8.** If \( St_1(P) = \{ P, A_i, B_i, C_i, D_i \} \), \( i = 1, 2 \) and \( St_3(P) = \{ P, A_3, B_3 \} \), then \( St_3(P) \not\subset \langle St_1(P), St_2(P) \rangle \).

**Proof.** By theorem 2.9 and proposition 5.1, we assume (WLOG) that \( \lambda_1(P) \cap \lambda_2(P) = \{ P, A_1 + B_1 = A_2 + B_2, C_1 + D_1 = C_2 + D_2 \} \). Because of remark 1, made in the beginning of the section 1, it is enough to show that \( A_3 \neq A_1 + C_2 \). Let \( S_1 \in \mathcal{S} - \Sigma_5 \), where \( \Sigma_5 = \langle St_1(P), St_2(P) \rangle = \langle A_1, B_1, C_1, D_1, A_2, C_2 \rangle \) [by theorem 2.7]. Suppose \( A_3 = A_1 + C_2 \). Then by proposition 5.2(i), \( \{ C_1, A_2, A_3 \} \) is a polarising set with respect to \( \Sigma_5 \). By proposition 5.6, we can assume (WLOG) that \( S_1, S_2 \in \mathcal{P}_1(C_1, A_2, A_3) \) and \( S_3, S_4 \in \mathcal{P}_3(C_1, A_2, A_3) \).

\( S_1 + S_2, S_3 + S_4 \in \langle C_1, A_2, A_3 \rangle \). By the proposition 5.7, \( S_1 + S_2 = C_1 + A_2 = S_3 + S_4 \), which violates the proposition 5.2(iv). This completes the proof.

§5. A 5-space and an odd stigm of \( \mathcal{P} \).

Let \( St_1(P) = \{ P, A_i, B_i, C_i, D_i \}, i = 1, 2 \) and \( St_3(P) = \{ P, A_3, B_3 \} \). Then we know from proposition 5.8 that \( St_3(P) \not\subset \langle St_1(P), St_2(P) \rangle \). Also, by proposition 5.1, we can assume (WLOG) that \( \lambda_1(P) \cap \lambda_2(P) = \)
\{p, A_1 + B_1 = A_2 + B_2, C_1 + D_1 = C_2 + D_2\}. Now it is easy to check that \(\langle A_1 + A_2, A_1 + B_2, A_1 + C_2, A_1 + D_2, C_1 + A_2 \rangle\) is a 4-space and from now on we shall denote this 4-space by \(\Sigma_4\) and the 5-space \(\langle St_1(p), St_2(p) \rangle\) by \(\Sigma_5\). Let \(\tau_i', i = 1, 2, 3\) be the three 5-spaces on \(\Sigma_4\), one of which must be \(\Sigma_5\). WLOG we assume that \(\tau_1' = \Sigma_5\). By proposition 5.8, \(St_3(p) \not\subset \tau_1'\). So, \(St_3(p) \subset \tau_i', i = 2 \text{ or } 3\). WLOG we assume that \(St_3(p) \subset \tau_3^{'\prime}\). Note that \(St_3(p) \cap \Sigma_4 = \{p\}\). Since every 5-space contains an odd stigm of \(B\) [by propos. 1.10], \(\tau_3^{'\prime}\) must contain an odd stigm of \(B\). The rest of this chapter is devoted to the investigation into the existence of an odd stigm, always denoted by \(St_2'^\prime\), in the 5-space \(\tau_2^{'\prime}\). Before we start investigation into the nature of the odd stigm \(St_2'^\prime\) in \(\tau_2^{'\prime}\), we prove the following results.

**Proposition 5.9.** \(\Sigma_4\) does not contain any odd stigm of \(B\).

**Proof.** Suppose \(St_2'^\prime\) is an odd stigm of \(B\), contained in \(\Sigma_4\). Clearly \(\Sigma_4 \subset \tau_3(p) \cap \Sigma_5\). \(\therefore St_2'^\prime \subset \Sigma_4\) \(\Rightarrow St_2'^\prime \subset \Sigma_5 \cap \tau_3(p) \Rightarrow St_2'^\prime \cap \pi(p) \neq \emptyset \) [\(\because St_2'^\prime\) is a 1-block in \(\tau_3(p)\)] \(\Rightarrow p \in St_2'^\prime\) and \(St_2'^\prime \subset \Sigma_5 \cap \tau_3(p)\), which is not possible [by theorem 4.1 and proposition 5.8]. This completes the proof.

**Proposition 5.10.** \(\tau_2^{'\prime}\) does not contain a line of \(B\) of the form \(\{p, S_1, S_2^{'\prime}\}\), where \(S_1 \in \emptyset \cap \tau_2^{'\prime} \setminus \Sigma_4\).

**Proof.** Suppose \(S_i \in \emptyset \cap \tau_i^{'\prime} \setminus \Sigma_4, i = 1, 2\), where \(S_1 + S_2 = p\). Let \(\Sigma_2 = \langle p, S_1, A_3 \rangle = \cdots\)
\[\{P, A_3, B_3, S_1, S_2, S_1 + A_3, S_1 + B_3\}.\] Now \(S_1 + A_3 \in \tau_1' = \Sigma_4 = \Sigma_5 - \Sigma_4.\) Since \(\Sigma_5 \cap \tau_3(p) = \Sigma_4,\)
\(S_1 + A_3 \in \Sigma_5 \cap \tau_i(p), \ i = 1 \text{ or } 2.\) Now \(\Sigma_5 =
< A_1, B_1, C_1, D_1, A_2, C_2 >.\) Suppose \(S_1 + A_3 \in < St_1(p) >.\)
Then \(S_1 + A_3 = kP + X_1,\) where \(X_1 \in St_1(P) - \{P\} \) and \(k = 0\)
or \(1 \mod 2).\) Then it is always possible to choose two
distinct points \(Y_1, Z_1\) of \(St_1(p),\) different from \(P\) and
\(X_1,\) such that \(t(Y_1, Z_1) \cap \Sigma_2 = \emptyset.\) But this is not
possible. So, \(S_1 + A_3 \notin < St_1(p) >.\) Similarly, \(S_1 + A_3 \notin
< St_2(p) >.\) Again, suppose that \(S_1 + A_3 \in \Sigma_5 \cap \tau_2(p) -
< St_2(p) >,\) then \(S_1 + A_3 = X_2 + Y,\) where \(X_2 \in St_2(P) - \{P\}\)
and \(Y \in \lambda_4(P) - \lambda_1(P) \cap \lambda_2(P).\) Since we can interchange
\(A_1\) and \(B_1\) and well as \(C_1\) and \(D_1,\) we assume \((\text{WLOG})\) that
\(Y = A_1 + C_1.\) : : \(S_1 + A_3 = X_2 + A_1 + C_1.\) Then \(t(X_2, A_1, D_1) \cap \Sigma_2 =
\emptyset,\) which is a contradiction. Similarly, \(S_1 + A_3 \notin \Sigma_5 \cap \tau_2(p) -
< St_1(p) >.\) Hence \(S_1 + A_3 \notin \Sigma_5 \cap \tau_i(p),\)
\(i = 1, 2,\) which is a contradiction. This completes the
proof.

**Lemma 5.1.** Let \(S_i \in \emptyset \cap \tau_i' \subset \Sigma_4, \ i = 1, 2, 3, 4,\) where
\(S_1 + S_2 = A_1 + B_1\) and \(S_3 + S_4 = C_1 + D_1.\) Then \(S_1 = X_1 + A_3,\)
\(S_2 = Y_1 + B_3, S_3 = U_j + A_3\) and \(S_4 = V_j + B_3,\) where \(\{X_1, Y_1\} =
\{C_1, D_1\}\) and \(\{U_j, V_j\} = \{A_j, B_j, i, j \in \{1, 2\}.\)

**Proof.** First of all, we prove the result for \(S_1\) and \(S_2.\)
Clearly, \(S_1 = A_1 + A_3 + X\) and \(S_2 = A_1 + A_3 + Y,\) where either
both \(X\) and \(Y\) are in \(\Sigma_4\) or one of \(X\) and \(Y\) is zero, while
the remaining one is in \(\Sigma_4.\) : : \(X + Y = A_1 + B_1.\) But
\[ X = 0 \implies S_1 = A_1 + A_3 \implies S_2 = B_1 + A_3 \quad \therefore \quad S_1 + S_2 = A_1 + B_1 \] \implies t(C_1, D_1) \cap <A_3, S_1, S_2> = \emptyset, \text{ which is a contradiction.} \quad \therefore \quad X \neq 0. \quad \therefore \quad \text{Similarly, } Y \neq 0. \quad \therefore \quad \text{both} \ X, Y \in \sum_4 = <A_1 + A_2, A_1 + B_2, A_1 + C_2, A_1 + D_2, C_1 + A_2>.

\therefore \quad X = \lambda_1 (A_1 + A_2) + \lambda_2 (A_1 + B_2) + \lambda_3 (A_1 + C_2) + \lambda_4 (A_1 + D_2) + \lambda_5 (C_1 + A_2) \text{ and } Y = \lambda_1' (A_1 + A_2) + \lambda_2' (A_1 + B_2) + \lambda_3' (A_1 + C_2) + \lambda_4' (A_1 + D_2) + \lambda_5' (C_1 + A_2) \text{ where } \lambda_i, \lambda_i' \in \{0, 1\}, \ i = 1, 2, 3, 4, 5. \text{ But } X + Y = S_1 + S_2 = A_1 + B_1 \implies \lambda_i + \lambda_i' = 1, \ i = 1, 2 \text{ and } \lambda_j + \lambda_j' = 0, \ j = 3, 4, 5. \text{ Considering all the admissible values for } \lambda_i \text{ and } \lambda_i', \ i = 1, 2, 3, 4, 5, \text{ it is enough for our purpose to consider the following essentially distinct possibilities for } X (\text{or } Y).

Let \( X = \)

1. \( A_1 + A_2 \)
2. \( A_1 + A_2 + A_1 + C_2 \)
3. \( A_1 + A_2 + C_1 + A_2 \)
4. \( A_1 + A_2 + A_1 + C_2 + A_1 + D_2 \)
5. \( A_1 + A_2 + A_1 + C_2 + C_1 + A_2 \)
6. \( A_1 + A_2 + A_1 + C_2 + A_1 + D_2 + C_1 + A_2 \)
7. \( A_1 + A_2 + A_1 + B_2 + A_1 + C_2 \)
8. \( A_1 + A_2 + A_1 + B_2 + C_1 + A_2 \)
9. \( A_1 + A_2 + A_1 + B_2 + A_1 + C_2 + A_1 + D_2 \)
10. \( A_1 + A_2 + A_1 + B_2 + A_1 + C_2 + C_1 + A_2 \)
11. \( A_1 + A_2 + A_1 + B_2 + A_1 + C_2 + A_1 + D_2 + C_1 + A_2 \)

We have \( Y = X + A_1 + B_1 \) and \( S_1 = A_1 + A_3 + X \) and \( S_2 = A_1 + \)
Thus we calculate $S_1$ and $S_2$ corresponding to the eleven values of $X$ and form the following table.

<table>
<thead>
<tr>
<th></th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$\Sigma_2$</th>
<th>$\Theta$</th>
<th>$\Sigma_2 \cap \Theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$A_1 + A_2$</td>
<td>$B_2 + A_3$</td>
<td>$\langle A_3, S_1, S_2 \rangle$</td>
<td>$t(C_2, D_2)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>2.</td>
<td>$A_1 + A_2 + C_2 + A_3$</td>
<td>$A_1 + B_2 + C_2 + A_3$</td>
<td>$\langle A_3, S_1, S_2 \rangle$</td>
<td>$t(C_2, D_2, A_1)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>3.</td>
<td>$B_2 + S_3$</td>
<td>$D_1 + B_3$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>4.</td>
<td>$A_1 + C_1 + C_2 + A_3$</td>
<td>$B_1 + C_1 + C_2 + A_3$</td>
<td>$\langle A_3, S_1, S_2 \rangle$</td>
<td>$t(C_1, C_2, D_2)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>5.</td>
<td>$D_1 + A_3$</td>
<td>$C_1 + B_3$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>6.</td>
<td>$B_1 + D_1 + A_2 + A_3$</td>
<td>$A_1 + D_1 + A_2 + A_3$</td>
<td>$\langle A_3, S_1, S_2 \rangle$</td>
<td>$t(C_1, D_1, A_2)$</td>
<td>$\emptyset$</td>
</tr>
</tbody>
</table>

Thus from the above table we conclude that the lemma is true for $S_1$ and $S_2$. Similarly, we do it for $S_3$ and $S_4$.

This completes the proof.

**Lemma 5.2.** Let $S_i \in \emptyset \cap \Sigma_2 = \Sigma_4$, $i = 1, 2, 3, 4$, where $S_1 + S_2 = A_1 + C_1$ and $S_3 + S_4 = B_1 + D_1$. Then $S_1 = X_1 + A_3$, $S_2 = Y_1 + B_3$, $S_3 = U_1 + A_3$ and $S_4 = V_1 + B_3$, where $\{X_1, Y_1\} = \{B_1, D_1\}$ and $\{U_1, V_1\} = \{A_1, C_1\}$.

**Proof.** First, we prove the lemma for $S_1$ and $S_2$. Clearly, $S_1 = A_1 + A_3 + X$ and $S_2 = A_1 + A_3 + Y$, where either both $X$ and $Y$ are in $\Sigma_4$ or one of $X$ and $Y$ is zero, while the remaining one is in $\Sigma_4$. \[ \therefore X + Y = A_1 + C_1. \] But $X = 0 \implies Y = A_1 + C_1$. Thus $Y = A_1 + C_1$. Hence, $S_1 = A_1 + A_3 + X$ and $S_2 = A_1 + A_3 + Y$. Then $S_1 + S_2 = A_1 + A_3 + X + Y = A_1 + A_3 + (A_1 + A_3) = 2A_1 + 2A_3$. But $S_1 + S_2 = A_1 + C_1$, which implies $2A_1 + 2A_3 = A_1 + C_1$. Thus $A_3 = C_1$. Therefore, $S_1 = A_1 + A_3 + X = A_1 + A_3 + (A_1 + A_3) = 2A_1 + 2A_3$. This completes the proof.
\[ A_1 + C_1 \implies S_1 = A_1 + A_3 \quad \text{and} \quad S_2 = C_1 + A_3 \implies t(B_1, D_1) \cap \langle A_3, S_1, S_2 \rangle = \emptyset, \text{which is a contradiction.} \]

\[ \therefore X \neq 0. \quad \text{Similarly,} \quad Y \neq 0. \quad \therefore \text{Both X and Y are points in} \]
\[ \Sigma_4 = \langle A_1 + A_2, A_1 + B_2, A_1 + C_2, A_1 + D_2, C_1 + A_2 \rangle. \quad \therefore X = \lambda_1(A_1 + A_2) + \lambda_2(A_1 + B_2) + \lambda_3(A_1 + C_2) + \lambda_4(A_1 + D_2) + \lambda_5(C_1 + A_2); \quad Y = \lambda_1'(A_1 + A_2) + \lambda_2'(A_1 + B_2) + \lambda_3'(A_1 + C_2) + \lambda_4'(A_1 + D_2) + \lambda_5'(C_1 + A_2) \quad \text{where} \quad \lambda_i, \lambda_i' \in \{0, 1\}, i = 1, 2, 3, 4, 5. \]
\[ A_1 + C_1 = X + Y \implies \lambda_1 + \lambda_1' = 1, \lambda_5 + \lambda_5' = 1 \quad \text{and} \quad \lambda_j + \lambda_j' = 0, j = 2, 3, 4. \quad \text{Considering all the admissible values of} \ \lambda_i \ \text{and} \ \lambda_i', \ \text{it is enough for our purpose to consider the following essentially distinct possibilities for} \ X \ (\text{or} \ Y). \]

Let \( X = \)

1. \( A_1 + A_2 \)
2. \( A_1 + A_2 + A_1 + B_2 \)
3. \( A_1 + A_2 + A_1 + C_2 \)
4. \( A_1 + A_2 + A_1 + B_2 + A_1 + C_2 \)
5. \( A_1 + A_2 + A_1 + C_2 + A_1 + D_2 \)
6. \( A_1 + A_2 + A_1 + B_2 + A_1 + C_2 + A_1 + D_2 \)
7. \( A_1 + A_2 + C_1 + A_2 + A_1 + B_2 \)
8. \( A_1 + A_2 + C_1 + A_2 + A_1 + C_2 \)
9. \( A_1 + A_2 + C_1 + A_2 + A_1 + B_2 + A_1 + C_2 \)
10. \( A_1 + A_2 + C_1 + A_2 + A_1 + C_2 + A_1 + D_2 \)
11. \( A_1 + A_2 + C_1 + A_2 + A_1 + B_2 + A_1 + C_2 + A_1 + D_2 \).

We have \( Y = A_1 + C_1 + X, \ S_1 = A_1 + A_3 + X \) \text{and} \( S_2 = A_1 + A_3 + Y. \)

Corresponding to each of the eleven values of \( X, \) we
calculate $S_1$ and $S_2$ and make the following table.

<table>
<thead>
<tr>
<th></th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$\Sigma_2$</th>
<th>$\emptyset$</th>
<th>$\emptyset \cap \Sigma_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$A_2 + A_3$</td>
<td>$A_1 + C_1 + A_2 + A_3$</td>
<td>$\langle A_3, S_1, S_2 \rangle$</td>
<td>$t(B_1, D_1, B_2)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>2</td>
<td>$B_1 + A_3$</td>
<td>$D_1 + B_3$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$A_1 + A_2 + C_2 + A_3$</td>
<td>$C_1 + A_2 + C_2 + A_3$</td>
<td>$\langle A_3, S_1, S_2 \rangle$</td>
<td>$t(B_1, D_1, A_2, C_2)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>4</td>
<td>$A_1 + B_1 + C_2 + A_3$</td>
<td>$B_1 + C_1 + C_2 + A_3$</td>
<td>$\langle A_3, S_1, S_2 \rangle$</td>
<td>$t(B_1, D_1, C_2)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>5</td>
<td>$C_1 + D_1 + A_2 + A_3$</td>
<td>$A_1 + D_1 + A_2 + A_3$</td>
<td>$\langle A_3, S_1, S_2 \rangle$</td>
<td>$t(B_1, D_1, A_2)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>6</td>
<td>$A_1 + B_3$</td>
<td>$C_1 + B_3$</td>
<td>$\langle A_3, S_1, S_2 \rangle$</td>
<td>$t(B_1, D_1)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>7</td>
<td>$A_1 + C_1 + B_2 + A_3$</td>
<td>$B_2 + A_3$</td>
<td>$\langle A_3, S_1, S_2 \rangle$</td>
<td>$t(B_1, D_1, A_2)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>8</td>
<td>$A_1 + C_1 + C_2 + A_3$</td>
<td>$C_2 + A_3$</td>
<td>$\langle A_3, S_1, S_2 \rangle$</td>
<td>$t(B_1, D_1, D_2)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>9</td>
<td>$C_1 + B_2 + C_2 + A_3$</td>
<td>$A_1 + B_2 + C_2 + A_3$</td>
<td>$\langle A_3, S_1, S_2 \rangle$</td>
<td>$t(B_1, D_1, B_2, C_2)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>10</td>
<td>$D_1 + A_3$</td>
<td>$B_1 + B_3$</td>
<td></td>
<td>$t(B_1, D_1)$</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>$A_1 + C_1 + A_2 + B_3$</td>
<td>$A_2 + B_3$</td>
<td>$\langle B_3, S_1, S_2 \rangle$</td>
<td>$t(B_1, D_1, B_2)$</td>
<td>$\emptyset$</td>
</tr>
</tbody>
</table>

From the above table, we conclude that the lemma is true for $S_1$ and $S_2$. Similarly, we do it for $S_3$ and $S_4$. This completes the proof of the lemma.

Proposition 5.11. $\tau_2'$ does not contain any 5-stigm of the form $\{P, S_1, S_2, S_3, S_4\}$, where $S_i \in \emptyset \cap \tau_2' - \Sigma_4$.

Proof. Suppose $\tau_2'$ contains an odd stigm $St_2'$, where $St_2' = \{P, S_1, S_2, S_3, S_4\}$, $S_i \in \emptyset \cap \tau_2' - \Sigma_4$. Following the proof of proposition 5.1, we can show that $\langle St_1(P) \rangle \cap \langle St_2' \rangle$ contains a line of $\lambda_i(P)$ through $P$, for $i = 1, 2$. Now it is enough to consider the following two cases.
Case 1. Let \( \langle \text{St}_1(P) \rangle \cap \langle \text{St}_2 \rangle \) contains the line \( \{P, A^1 + B^1, C^1 + D^1 \}, i = 1, 2 \). But we have \( A^1 + B^1 = A^2 + B^2, C^1 + D^1 = C^2 + D^2 \). WLOG we assume \( S_1 + S_2 = A^1 + B^1 \), \( S_3 + S_4 = C^1 + D^1 \). By lemma 5.1, we assume (WLOG) that \( S_1 = C^1 + A^3 \) and \( S_2 = D^1 + B^3 \). But \( C^1 + A^3, D^1 + B^3 \in \Sigma \) 

\[ t(B_1, D_1) \cap \langle C_1, A_1, A_3 \rangle = \{A^1 + A^3, B^3 \} \]

\[ t(A_1, D_1) \cap \langle C_1, B_1, A_3 \rangle = \{B_1 + A^3, B^3, C_1 \} \]

\[ t(B_1, C_1) \cap \langle D_1, C_1, C_3 \rangle = \{B_1 + B^3, C_1 + D^3 \} \]

\[ X_1 + X_3 \notin \Sigma, \forall X_1 \in \{A_1, B^3\}, \forall X_3 \in \{A^3, B^3\}. \] So by lemma 5.1, we assume (WLOG) that \( S_3 = A^2 + A^3 \) and \( S_4 = B^2 + B^3 \). Then \( t(B_2, D_1) \cap \langle A^3, A_2, C_1 \rangle = \emptyset \), which is a contradiction. \( \therefore \) case 1 cannot happen.

Case 2. Let \( \langle \text{St}_1(P) \rangle \cap \langle \text{St}_2 \rangle \supseteq \{P, A^1 + C_1, B^1 + D^1 \} \). WLOG we assume that \( S_1 + S_2 = A^1 + C^1, S_3 + S_4 = B^1 + D^1 \). By lemma 5.1, we assume (WLOG) that \( S_1 = B^1 + A^3 \) and \( S_2 = D^1 + B^3 \); also, \( S_3 = X_1 + A^3 \), and \( S_4 = Y_1 + B^3 \), where \( X_1, Y_1 \in \{A^1, C_1^3\} \). But \( B^1 + A^3 \in \Sigma \) 

\[ t(C_1, D_1) \cap \langle A^3, A_1, B_1 \rangle = \{A^1 + A^3 \} \] and \( t(A_1, D_1) \cap \langle A^3, B_1, C_1 \rangle = \{C^1 + A^3 \} \); also \( D^1 + B^3 \in \Sigma \) 

\[ t(A_1, B_1) \cap \langle B^3, C_1, D_1 \rangle = \{C^1 + B^3 \} \] and \( t(B_1, C_1) \cap \langle B^3, A_1, D_1 \rangle = \{A^1 + B^3 \} \). So, \( X_1 + A^3, Y_1 + B^3 \notin \Sigma \), \( X_1, Y_1 \in \{A^1, C_1^3\} \), which is a contradiction. This completes the proof.

Proposition 5.12. Let \( X_i \in \{A^i, B^i\}, Y_i \in \{C^i, D^i\}, i = 1, 2 \). (i) If \( X_1 + X_2 \in \Sigma \), then \( \{C_1, D_1, C_2^3 \} \) form a polarising set with respect to \( \Sigma_5 \).
(ii) If \( Y_1 + Y_2 \in \mathcal{S} \), then \( \{A_1, B_1, A_2\} \) is a polarising set with respect to \( \Sigma_5 \).

(iii) If \( A_1 + C_2 \in \mathcal{S} \), then \( \{B_1, Y_1, X_2^3\} \) and \( \{Y_1, X_2, D_2\} \) are polarising sets with respect to \( \Sigma_5 \).

(iv) If \( C_1 + A_2 \in \mathcal{S} \), then \( \{D_1, X_1, Y_2^3\} \) is a polarising set with respect to \( \Sigma_5 \).

(v) If \( P + A_1 + A_2 \in \mathcal{S} \), then \( \{B_1, C_1, C_2^3\} \) is a polarising set with respect to \( \Sigma_5 \).

**Proof.** (i) Let \( X_1 + X_2 \in \mathcal{S} \). WLOG we assume that \( A_1 + A_2 \in \mathcal{S} \). But \( A_1 + A_2 \in \mathcal{S} \implies t(C_1, D_1, C_2) \cap \langle A_1, B_1, A_2 \rangle = \{A_1 + B_2^3\} \). Also \( t(C_1, D_1, C_2) \not\subseteq \Sigma_5 \), since \( t(C_1, D_1, C_2) \) does not meet the line \( \{A_1, A_2, A_1 + A_2^3\} \). By proposition 2.5, it follows that \( \{C_1, D_1, C_2\} \) is a polarising set with respect to \( \Sigma_5 \).

(ii) Let \( Y_1 + Y_2 \in \mathcal{S} \). WLOG we assume \( C_1 + C_2 \in \mathcal{S} \). But \( C_1 + C_2 \in \mathcal{S} \implies t(A_1, B_1, A_2) \cap \langle C_1, C_2, D_2 \rangle = \{C_1 + D_2^3\} \). Also, \( t(A_1, B_1, A_2) \not\subseteq \Sigma_5 \), since \( t(A_1, B_1, A_2) \) does not meet the line \( \{C_1, C_2, C_1 + C_2^3\} \). By proposition 2.5, we get that \( \{A_1, B_1, A_2^3\} \) is a polarising set with respect to \( \Sigma_5 \).

(iii) \( A_1 + C_2 \in \mathcal{S} \implies t(B_1, Y_1, X_2) \cap \langle A_1, C_2, D_2 \rangle = \{A_1 + D_2^3\} \). Also \( t(B_1, Y_1, X_2) \not\subseteq \Sigma_5 \), since \( t(B_1, Y_1, X_2) \) does not meet the line \( \{A_1, C_2, A_1 + C_2^3\} \). So, by proposition 2.5, we can deduce that \( \{B_1, Y_1, X_2^3\} \) is a polarising set with respect to \( \Sigma_5 \). Again, since \( t(Y_1, X_2, D_2) \cap \langle C_2, A_1, B_1 \rangle = \{B_1 + C_2^3\} \) and \( t(Y_1, X_2, D_2) \not\subseteq \Sigma_5 \), then \( \{Y_1, X_2, D_2\} \) is a polarising set with respect to \( \Sigma_5 \).
\( \emptyset \), we can show that \( \{Y_1, X_2, D_2\} \) is a polarising set with respect to \( \Sigma_5 \).

(iv) \( C_1 + A_2 \in \emptyset \implies t(D_1, X_1, Y_2) \cap \langle C_1, A_2, B_2 \rangle = \{C_1 + B_2\} \); also \( t(D_1, X_1, Y_2) \) does not meet the line \( \{C_1, A_2, C_1 + A_2\} \). So, by proposition 2.5, we deduce that \( \{D_1, X_1, Y_2\} \) is a polarising set with respect to \( \Sigma_5 \).

(v) \( P + A_1 + A_2 \in \emptyset \implies t(B_1, C_1, C_2) \cap \langle P, A_1, A_2 \rangle = \{P + A_2\} \); also \( t(B_1, C_1, C_2) \not\subset \Sigma_5 \), for otherwise, \( t(B_1, C_1, C_2) \cap \langle P, A_1, A_3 \rangle = \emptyset \). Hence \( \{B_1, C_1, C_2\} \) is a polarising set with respect to \( \Sigma_5 \).

**Proposition 5.13.**

(i) \( \emptyset \cap \{x_1 + x_2 \mid x_1 \in \{A_1, B_1\}, i = 1, 2 \} \leq 1 \)

(ii) \( \emptyset \cap \{y_1 + y_2 \mid y_1 \in \{C_1, D_1\}, i = 1, 2 \} \leq 1 \)

(iii) \( \emptyset \cap \{x_1 + y_2 \mid x_1 \in \{A_1, B_1\}, y_2 \in \{C_2, D_2\} \} \leq 1 \)

(iv) \( \emptyset \cap \{x_2 + y_1 \mid x_2 \in \{A_2, B_2\}, y_1 \in \{C_1, D_1\} \} \leq 1 \).

**Proof.** (i) WLOG we assume that \( A_1 + A_2 \in \emptyset \). Then
\[ t(C_1, D_1) \cap \langle A_1, A_2, B_2 \rangle = \{A_1 + B_2\} \], whence the proof follows immediately by proposition 2.4.

(ii) Proof is similar to that of (i).

(iii) WLOG we assume that \( A_1 + C_2 \in \emptyset \). Now
\[ t(P, C_1) \cap \langle A_1, C_2, D_2 \rangle = \{A_1 + D_2\} \implies P + A_1 + D_2 = B_1 + C_2 \in t(P, C_1) \]. Again \( B_1 + D_2 \not\in \emptyset \), otherwise \( \{P, A_1 + C_2, B_1 + D_2\} \) would be a line of \( \emptyset \), contradicting proposition 5.8. Now the proof follows easily.

(iv) Proof is similar to that of (iii).
Next, we want to show that \( P + X_1 + X_2 \notin \mathcal{B} \), where
\[ X_i \in \{ A_i, B_i \}, \quad i = 1, 2, \] or
\[ X_i \in \{ C_i, D_i \}, \quad i = 1, 2. \]
For this we need the following lemmas.

**Lemma 5.3.** Let \( P + X_1 + X_2 \in \mathcal{B} \), where
\[ X_i \in \{ A_i, B_i \}, \quad i = 1, 2 \] or
\[ X_i \in \{ C_i, D_i \}, \quad i = 1, 2. \]
Then in \( \tau_1 \) there does not exist any line of \( \mathcal{B} \) through \( P + X_1 + X_2 \).

**Proof.** Suppose the lemma is false. WLOG we assume \( P + A_1 + A_2 \in \mathcal{B} \) and in \( \tau_1 \) there exists a line \( m \) of \( \mathcal{B} \) through \( P + A_1 + A_2 \). Let \( m = \{ P + A_1 + A_2, X, Y \} \), where \( X, Y \in \tau_1 = \Sigma_5 \). By proposition 5.9, \( X, Y \notin \Sigma_4 \). We have the following table.

<table>
<thead>
<tr>
<th>( X )</th>
<th>( Y )</th>
<th>( \Sigma_2 )</th>
<th>( \Theta )</th>
<th>( \Theta \cap \Sigma_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_1 )</td>
<td>( P + A_2 )</td>
<td>( \langle P, A_1, A_2 \rangle )</td>
<td>( t(B_1, C_1) )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( B_1 )</td>
<td>( P + B_2 )</td>
<td>( \langle P, B_1, B_2 \rangle )</td>
<td>( t(A_1, C_1) )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( C_1 )</td>
<td>( B_1 + D_1 + A_2 )</td>
<td>( \langle B_1, A_2, B_2 \rangle )</td>
<td>( t(P, D_1) )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( P + C_1 )</td>
<td>( A_1 + C_1 + A_2 )</td>
<td>( \langle A_1, C_1, A_2 \rangle )</td>
<td>( t(P, D_1) )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( A_1 + C_1 + C_2 )</td>
<td>( C_1 + B_2 + D_2 )</td>
<td>( \langle A_1, C_1, C_2 \rangle )</td>
<td>( t(P, B_1, A_2) )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( B_1 + C_1 + C_2 )</td>
<td>( D_1 + A_2 + C_2 )</td>
<td>( \langle B_1, C_1, C_2 \rangle )</td>
<td>( t(P, A_1, B_2) )</td>
<td>( \emptyset )</td>
</tr>
</tbody>
</table>

Since we can interchange the suffixes 1 and 2 of \( St_1(P) \) as well as the letters \( C \) and \( D \), the above table leads to a contradiction. This completes the proof.

**Lemma 5.4.** Let \( P + X_1 + X_2 \in \mathcal{B} \), where
\[ X_i \in \{ A_i, B_i \}, \quad i = 1, 2, \] or
\[ X_i \in \{ C_i, D_i \}, \quad i = 1, 2. \]
Then there does not exist any line of \( \mathcal{B} \) of the form \( \{ P + X_1 + X_2, S_j, S_j \} \), where \( S_j \in \mathcal{B} \cap \tau_1 - \Sigma_4 \), \( j = 1, 2, \) or 3.
Proof. Suppose the lemma is false. WLOG we assume

\[ P + A_1 + A_2 \in \mathcal{S} \] and \[ \{P + A_1 + A_2, S_1, S_2\} \] is a line of \( \mathcal{S} \),

where \( S_1, S_2 \in \mathcal{S} \cap \mathcal{T}_i - \mathcal{T}_i, i = 2 \text{ or } 3 \). By

proposition 5.12 \( \{B_1, C_1, C_2\} \) form a polarising set with

respect to \( \mathcal{T}_5 \). Since \( A_3 + B_3 = P \) and \( S_1 + S_2 = P + A_1 + A_2 \), none of the pairs \( \{A_3, B_3\} \) and \( \{S_1, S_2\} \) is in the

same \( \mathcal{T}_i(B_1, C_1, C_2), i \in \{1, 2, 3\} \). So, rearranging

the suffixes of \( S \)'s and interchanging the letters \( A_3 \) and

\( B_3 \), we assume (WLOG) that \( S_1 + A_3 \in \mathcal{T}_i(B_1, C_1, C_2) \), for

some \( i \in \{1, 2, 3\} \). \( \therefore \) \( S_1 + A_3 \in \mathcal{T}_i(B_1, C_1, C_2) \).

If \( S_1, S_2 \in \mathcal{T}_2 \), then \( S_1 + A_3 \in \mathcal{T}_i(B_1, C_1, C_2, B_1 + C_1 + C_2) \)

and if \( S_1, S_2 \in \mathcal{T}_3 \), then \( S_1 + A_3 \in \mathcal{T}_i(B_1 + C_1, B_1 + C_2, C_1 + C_2) \).

We consider the following table.

<table>
<thead>
<tr>
<th>( S_1 + A_3 )</th>
<th>( \mathcal{T}_2 )</th>
<th>( \theta )</th>
<th>( \theta \cap \mathcal{T}_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B_1 )</td>
<td>( &lt;B_1, B_2, A_3&gt; )</td>
<td>( t(B_3, A_2, D_1) )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( C_1 )</td>
<td>( &lt;A_1, C_1, A_2&gt; )</td>
<td>( t(P, D_1, B_3) )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( C_2 )</td>
<td>( &lt;A_2, C_2, A_1&gt; )</td>
<td>( t(P, D_2, B_3) )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( B_1 + C_1 + C_2 )</td>
<td>( &lt;C_1, C_2, B_2&gt; )</td>
<td>( t(B_1, A_2, D_2, B_3) )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( B_1 + C_1 )</td>
<td>( &lt;C_1, A_3, B_3&gt; )</td>
<td>( t(B_1, D_1, B_2) )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( B_1 + C_2 )</td>
<td>( &lt;C_2, A_3, B_3&gt; )</td>
<td>( t(B_1, B_2, D_2) )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( C_1 + C_2 )</td>
<td>( &lt;C_1 + C_2, A_3, B_3&gt; )</td>
<td>( t(A_1, B_1, A_2, D_2) )</td>
<td>( \emptyset )</td>
</tr>
</tbody>
</table>

From the above table, we arrive at a contradiction. This

completes the proof.

Proposition 5.14. If \( X_i \in \{A_i, B_i\}, i = 1, 2 \) or \( X_i \in \{C_i, D_i\}, i = 1, 2 \), then \( P + X_1 + X_2 \notin \mathcal{S} \).
Proof. Suppose the proposition is not true. Without loss of generality (WLOG) we assume \( P + A_1 + A_2 \in \mathcal{S} \). By theorem 4.1, at least one of \( \text{St}_i(P + A_1 + A_2), i = 1, 2, 3 \), must be a line of \( \mathcal{S} \); but this is not possible by lemmas 5.3 and 5.4. This completes the proof.

Remark. If \( X_i \in \{A_i, B_i\}, Y_i \in \{C_i, D_i\}, i = 1, 2 \), then a point of \( \mathcal{S} \cap \Sigma_4 \) must be one of the following forms:

\[ X_1 + X_2, X_1 + Y_2, Y_1 + X_2, Y_1 + Y_2. \]

So, by propositions 5.13 and 5.14, it follows that

\[ |\mathcal{S} \cap \Sigma_4| \leq 5. \]

\( \S 6. \) **Non-existence of a line of \( \mathcal{S} \) in \( \Sigma_2' \).**

By proposition 5.9, a line of \( \mathcal{S} \) in \( \Sigma_2' \) must be of the form \( \{Z, S_1, S_2\} \), where \( Z \in \mathcal{S} \cap \Sigma_4 \) and \( S_1, S_2 \in \mathcal{S} \cap \Sigma_2' - \Sigma_4 \). By propositions 5.10 and 5.12, \( Z \in \Sigma_4 - \{P_i\} \). Let \( X_i \in \{A_i, B_i\}, i = 1, 2 \) and \( Y_i \in \{C_i, D_i\}, i = 1, 2 \). By proposition 5.13, \( Z \) is either \( X_1 + X_2, Y_1 + Y_2, X_1 + Y_2 \) or \( X_2 + Y_1 \), for some \( X_i \in \{A_i, B_i\}, i = 1, 2 \), and \( Y_i \in \{C_i, D_i\}, i = 1, 2 \). Since we can interchange \( A_i \) and \( B_i \) as well as \( C_i \) and \( D_i \), \( i = 1, 2 \), and also we can rearrange the suffixes 1 and 2 of \( \text{St}_i(P) \), we assume (WLOG) that \( Z \in \{A_1 + A_2, C_1 + C_2, A_1 + C_2\} \). Since \( A_1 + B_1 = A_2 + B_2 \) and \( C_1 + D_1 = C_2 + D_2 \), we can carry out simultaneous replacements of the pair \( \{A_1, B_1\} \) by \( \{C_1, D_1\} \) and the pair \( \{A_2, B_2\} \) by \( \{C_2, D_2\} \). So, we assume (WLOG) that \( Z \in \{A_1 + A_2, A_1 + C_2\} \).
Remark 2. In order to show that $\mathcal{T}_2'$ contains no line of $\mathfrak{B}$, it is enough to show that $\mathcal{T}_2'$ contains no line of $\mathfrak{B}$ of the form \{X, S_1, S_2\}, where $X \in \{A_1 + A_2, A_1 + C_2\}$ and $S_1, S_2 \in \mathfrak{B} \cap \mathcal{T}_2' - \Sigma_4$.

Lemma 5.5. $\mathcal{T}_2'$ does not contain any line of $\mathfrak{B}$ of the form \{A_1 + A_2, S_1, S_2\}, where $S_1 \in \mathfrak{B} \cap \mathcal{T}_2' - \Sigma_4$.

Proof. Suppose \{A_1 + A_2, S_1, S_2\} be a line of $\mathfrak{B}$ in $\mathcal{T}_2'$, where $S_1, S_2 \in \mathcal{T}_2' - \Sigma_4$. But $A_1 + A_2 \in \mathfrak{B} \Rightarrow \{C_1, D_1, C_2\}$ form a polarising set with respect to $\Sigma_5$. Since $S_1 + S_2 = A_1 + A_2$ and $A_3 + B_3 = P$, we assume (WLOG) $S_1, A_3 \in \mathfrak{t}_i(C_1, D_1, C_2)$, for some $i \in \{1, 2, 3\}$. Then $S_1 + A_3 \in \{C_1, D_1, C_2, D_2\}$. We now consider the following table.

<table>
<thead>
<tr>
<th>$S_1 + A_3$</th>
<th>$\Sigma_2$</th>
<th>$\mathfrak{B}$</th>
<th>$\mathfrak{B} \cap \Sigma_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>C_1</td>
<td>$&lt;A_1, C_1, A_2&gt;$</td>
<td>t(P, D_1, A_3)</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>D_1</td>
<td>$&lt;A_1, D_1, A_2&gt;$</td>
<td>t(P, C_1, A_3)</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>C_2</td>
<td>$&lt;A_1, C_2, A_2&gt;$</td>
<td>t(P, D_2, A_3)</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>D_2</td>
<td>$&lt;A_1, D_2, A_2&gt;$</td>
<td>t(P, C_2, A_3)</td>
<td>$\emptyset$</td>
</tr>
</tbody>
</table>

From the table, we arrive at a contradiction. This completes the proof.

Lemma 5.6. $\mathcal{T}_2'$ does not contain any line of $\mathfrak{B}$ of the form \{A_1 + C_2, S_1, S_2\}, where $S_1, S_2 \in \mathcal{T}_2' - \Sigma_4$.

Proof. Suppose \{A_1 + C_2, S_1, S_2\} be a line of $\mathfrak{B}$ in $\mathcal{T}_2'$, where $S_1, S_2 \in \mathcal{T}_2' - \Sigma_4$. Now $A_1 + C_2 \in \mathfrak{B} \Rightarrow \{B_1, X, X_2\}$ is a polarising set with respect to $\Sigma_5$, $V$ $X_1 \in \{C_1, D_1\}$ and $V$ $X_2 \in \{A_2, B_2\}$ (by proposition
5.12). Since $S_1 + S_2 = A_1 + C_2$ and $A_3 + B_3 = P$, we assume (WLOG) that $S_1, A_3 \in \mathcal{P}_3(B_1, C_1, A_2)$, for some $i \in \{1, 2, 3\}$. \therefore S_1 + A_3 \in \{B_1, C_1, A_2, B_1 + C_1 + A_2\}$. Now we consider the following table.

<table>
<thead>
<tr>
<th>$S_1 + A_3$</th>
<th>$\Sigma_2$</th>
<th>$\Theta$</th>
<th>$\Theta \cap \Sigma_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1$</td>
<td>$&lt;C_1, A_3, B_3&gt;$</td>
<td>$t(B_1, D_1, D_2)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$&lt;A_2, A_3, B_3&gt;$</td>
<td>$t(B_1, B_2, D_2)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$B_1 + C_1 + A_2$</td>
<td>$&lt;C_1, A_3, B_3&gt;$</td>
<td>$t(B_1, D_1, A_2, D_2)$</td>
<td>$\emptyset$</td>
</tr>
</tbody>
</table>

From the above table and the fact that $S_1 + A_3 \in \{B_1, C_1, A_2, B_1 + C_1 + A_2\}$, we conclude that $S_1 + A_3 = B_1$.

\therefore \forall x_1 \in \{C_1, D_1, B_2\}, \forall x_2 \in \{A_2, B_2, C_1\}$, we assume (WLOG) that $S_1, A_3 \in \mathcal{P}_3(B_1, X_1, X_2)$ (after rearranging the superscripts of $\mathcal{P}_3$'s) (WLOG) we assume that $S_2 \in \mathcal{P}_3(B_1, X_1, X_2)$, $\forall x_1 \in \{C_1, D_1, B_2\}$, $\forall x_2 \in \{A_2, B_2, C_1\}$.

[$\therefore S_1 + S_2 = A_1 + C_2 \notin <B_1, X_1, X_2>$.] If $B_3 \in \mathcal{P}_3(B_1, X_1, X_2)$ for some $x_1 \in \{C_1, D_1\}$ and $x_2 \in \{A_2, B_2\}$, then proceeding similarly as before we get $S_2 + B_3 = B_1$; also we have $S_1 + A_3 = B_1$ and thus

$t(C_1, D_1) \cap <B_1, A_3, B_3> = \emptyset$, which is a contradiction.

Thus $B_3 \in \mathcal{P}_3(B_1, X_1, X_2)$, $\forall x_1 \in \{C_1, D_1, B_2\}$, $\forall x_2 \in \{A_2, B_2\}$. By theorem 2.7, we have $|S_5 - \Sigma_5| \geq 5$. So, there exists a point $S \in S_5 - \Sigma_5 \cup \{S_1, S_2, A_3, B_3\}$.

There exists an $i$, $i \in \{1, 2, 3\}$, such that $S \in \mathcal{P}_3(B_1, C_1, A_2)$. Suppose that $S \in \mathcal{P}_3(B_1, C_1, A_2)$ also $B_3 \in \mathcal{P}_3(B_1, C_1, A_2)$. \therefore S + B_3 \in <B_1, C_1, A_3>$. But
\[ S + B_3 = B_1 \implies t(C_1, D_1) \cap \langle S_1, S, B_1 \rangle = \emptyset \]

\[ S_1 + A_3 = B_1 \], which is a contradiction. Now we consider the following table.

<table>
<thead>
<tr>
<th>( S + B_3 )</th>
<th>( S + S_1 )</th>
<th>( S + S_2 )</th>
<th>Observation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_1 )</td>
<td>( A_1 + D_1 )</td>
<td>( D_1 + C_2 )</td>
<td>( S \not\in \rho_3^1(B_1, D_1, A_2), i=1,2,3 ).</td>
</tr>
<tr>
<td>( A_2 )</td>
<td>( P + A_1 + B_2 )</td>
<td>( A_2 + D_2 )</td>
<td>( S \not\in \rho_3^2(B_1, C_1, B_2), i=1,2,3 ).</td>
</tr>
<tr>
<td>( B_1 + C_1 + A_2 )</td>
<td>( D_1 + B_2 )</td>
<td>( B_1 + C_1 + A_2 + D_2 )</td>
<td>( S \not\in \rho_3^3(B_1, D_1, A_2), i=1,2,3 ).</td>
</tr>
<tr>
<td>( B_1 + C_1 )</td>
<td>( P + C_1 )</td>
<td>( B_1 + D_1 + C_2 )</td>
<td>( S \not\in \rho_3^4(B_1, D_1, A_2), i=1,2,3 ).</td>
</tr>
<tr>
<td>( B_1 + A_2 )</td>
<td>( P + A_2 )</td>
<td>( A_1 + B_2 + D_2 )</td>
<td>( S \not\in \rho_3^5(B_1, C_1, B_2), i=1,2,3 ).</td>
</tr>
<tr>
<td>( C_1 + A_2 )</td>
<td>( A_1 + D_1 + A_2 )</td>
<td>( D_1 + A_2 + C_2 )</td>
<td>( S \not\in \rho_3^6(B_1, D_1, A_2), i=1,2,3 ).</td>
</tr>
</tbody>
</table>

From the above table and the fact that \( S + B_3 \neq B_1 \), we arrive at a contradiction. Hence \( S \not\in \rho_3^2(B_1, C_1, A_2) \).

Similarly, we can show that \( S \not\in \rho_3^3(B_1, X_1, X_2) \), \( \forall X_1 \in \{ C_1, D_1 \} \) and \( X_2 \in \{ A_2, B_3 \} \).

Again, suppose \( S \in \rho_3^2(B_1, C_1, A_2) \). Also, \( S_2 \in \rho_3^2(B_1, C_1, A_2) \). \( \forall \) \( S + S_2 \in <B_1, C_1, A_2> \). But \( S + S_2 = B_1 \implies t(D_2) \cap <S_1, S_2, A_3> = \emptyset \)

\[ S_1 + A_3 = B_1 \], which is a contradiction. \( \therefore S + S_2 \neq B_1 \).

We consider the following table.
<table>
<thead>
<tr>
<th>$S + S_2$</th>
<th>$S + S_1$</th>
<th>Observation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1$</td>
<td>$A_1 + C_1 + C_2$</td>
<td>$S \not\in R^1_3(B_1, D_1, A_2), i = 1,2,3$</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$A_1 + A_2 + C_2$</td>
<td>$S \not\in R^1_3(B_1, C_1, B_2), i = 1,2,3$</td>
</tr>
<tr>
<td>$B_1 + C_1$</td>
<td>$P + D_1 + C_2$</td>
<td>$S \not\in R^1_3(B_1, D_1, A_2), i = 1,2,3$</td>
</tr>
<tr>
<td>$B_1 + A_2$</td>
<td>$B_2 + C_2$</td>
<td>$S \not\in R^1_3(B_1, C_1, B_2), i = 1,2,3$</td>
</tr>
<tr>
<td>$C_1 + A_2$</td>
<td>$A_1 + C_1 + A_2 + C_2$</td>
<td>$S \not\in R^1_3(B_1, D_1, A_2), i = 1,2,3$</td>
</tr>
<tr>
<td>$B_1 + C_1 + A_2$</td>
<td>$D_1 + B_2 + D_2$</td>
<td>$S \not\in R^1_3(B_1, D_1, A_2), i = 1,2,3$</td>
</tr>
</tbody>
</table>

Since $S + S_2 \not\in B_1$, we now conclude from the above table that $S \not\in R^2_3(B_1, C_1, A_2)$. Similarly we can show that $S \not\in R^2_3(B_1, x_1, x_2), \forall x_1 \in \{C_1, D_1\}, \forall x_2 \in \{A_2, B_2\}$. Hence we finally get that $S \in R^1_3(B_1, x_1, x_2), \forall x_1 \in \{C_1, D_1\} \land \forall x_2 \in \{A_2, B_2\}$. $S_1 \in R^1_3(B_1, x_1, x_2), \forall x_1 \in \{C_1, D_1\}, \forall x_2 \in \{A_2, B_2\}$. \therefore $S + S_1 \in <B_1, C_1, A_2> \cap <B_1, D_1, B_2> = \{B_1\}$. \therefore $S + S_1 = B_1 = S_1 + A_3$, i.e., $S = A_3$, which contradicts the choice of $S$.

This completes the proof of the lemma.

**Proposition 5.15.** $\tau^1_2$ does not contain any line of $\mathcal{E}$.  
**Proof.** By virtue of remark 2, made in the beginning of this section, and the lemmas 5.5 and 5.6, proof follows immediately.

§7. **Non-existence of a 5-stigm $St^1_2$ of $\mathcal{E}$ in $\tau^1_2$**

with $|St^1_2 \cap \Sigma^1_4| = 1$.

We now want to show that in $\tau^1_2$ there does not exist any 5-stigm of $\mathcal{E}$ of the form $\{Z, S_1, S_2, S_3, S_4\}$, where
Z ∈ Σ_4 and S_i's ∈ B ∩ τ_2 ∩ Σ_4. As in section 6, here also we can make the following remark by virtue of proposition 5.11.

**Remark 3.** In order to show that there does not exist any 5-stigm St^' of B in τ_2 such that |St^' ∩ Σ_4| = 1, it is enough to show that t_2 does not contain any 5-stigm of the form \{Z, S_1, S_2, S_3, S_4\}, where Z ∈ \{A_1 + A_2, A_1 + C_2\} and S_i ∈ B ∩ τ_2 ∩ Σ_4.

Let us now prove the following lemmas.

**Lemma 5.7.** τ_2 does not contain any 5-stigm of B of the form \{A_1 + A_2, S_1, S_2, S_3, S_4\}, where S_i ∈ τ_2 ∩ Σ_4.

**Proof.** Suppose \{A_1 + A_2, S_1, S_2, S_3, S_4\} is a 5-stigm of in τ_2, where S_i ∈ τ_2 ∩ Σ_4, i = 1, 2, 3, 4. Then \(<S_1, S_2, S_3, S_4> \cap \lambda_i(P)\) contains a point Q_i, i = 1, 2. Assert that Q_i \⊊ \{P, A_1 + B_1, C_1 + D_1\}, i = 1, 2. Suppose Q_1 \⊊ \{P, A_1 + B_1, C_1 + D_1\}. Rearranging the suffixes of S's, we assume (WLOG) that S_1 + S_2 = Q_1. Now S_1 + S_2 ≠ P, by proposition 5.10. \(\therefore S_1 + S_2 = Q_1 \in \{A_1 + B_1, C_1 + D_1\}\). So, S_3 + S_4 ∈ \{B_1 + A_2, P + B_1 + A_2\}, which is not possible, since t(P, C_2) ∩ \(<A_1, B_1, A_2> = \{B_1 + A_2\}\) implies both B_1 + A_2 and P + B_1 + A_2 are attenuation points with respect to Σ_5. Thus Q_1 \⊊ \{P, A_1 + B_1, C_1 + D_1\}. Similarly, Q_2 \⊊ \{P, A_1 + B_1, C_1 + D_1\}. So, Q_1 ∈ \{A_1 + C_1, A_1 + D_1, B_1 + C_1, B_1 + D_1\}, i = 1, 2. Let Q_1 = A_1 + C_1, Q_2 = A_2 + C_2. Rearranging the suffixes of S's, we can assume (WLOG) that S_1 + S_2 = A_1 + C_1 and S_1 + S_3 =
$A_2 + C_2 \quad \therefore S_3 + S_4 = C_1 + A_2$ and $S_2 + S_4 = A_1 + C_2 \quad \because$

$S_1 + S_2 + S_3 + S_4 = A_1 + A_2$. Now $A_1 + A_2 \in \mathcal{B} \Rightarrow$

$\{C_1, D_1, C_2\}$ form a polarising set with respect to $\Sigma_5$

[by proposition 5.12]. Now $S_1 + S_2 = A_1 + C_1, S_1 + S_3 = A_2 + C_2, S_2 + S_3 = A_1 + C_1 + A_2 + C_2$. WLOG we can, therefore, assume that $S_i \in \rho^{i,3}(C_1, D_1, C_2), i = 1, 2, 3$. Since

$A_3 + B_3 = P \not\in \langle C_1, D_1, C_2 \rangle,$ there exist $i, j \in \{1, 2, 3\}, i \neq j$, such that $A_3 \in \rho^{i,3}(C_1, D_1, C_2)$ and $B_3 \in \rho^{j,3}(C_1, D_1, C_2)$.

So, $S_i + A_3, S_j + B_3 \in \{C_1, D_1, C_2, D_2\}$, i.e., $S_i + S_j + A_3, i \neq j$. But this is not possible, since $P + S_i + S_j \in 3\Sigma_3$, $i \neq j$. But this is not possible, since $P + S_i + S_j \in$

$\{B_i + D_1, B_2 + D_2, B_i + D_1 + A_2 + C_2\}$, $i, j \in \{1, 2, 3\}, i \neq j$.

Hence $Q_i \not\in A_1 + C_1, i = 1, 2$. Similarly, we can show that

$Q_i \not\in \{A_1 + C_1, A_i + D_1, B_i + C_1, B_i + D_i\}$. Thus we arrive at a contradiction. This completes the proof.

Lemma 5.8. $\mathcal{C}_2$ does not contain any 5-stigm of $\mathcal{B}$ of the form $\{A_1 + C_2, S_1, S_2, S_3, S_4\}$, where $S_i \in \mathcal{C}_2 - \Sigma_4, i = 1, 2, 3, 4$.

Proof. Suppose $\{A_1 + C_2, S_1, S_2, S_3, S_4\}$ is a 5-stigm of $\mathcal{B}$, where $S_i \in \mathcal{C}_2 - \Sigma_4, i = 1, 2, 3, 4$. Then

$\langle S_1, S_2, S_3, S_4 \rangle \cap \lambda_i(P)$ contains a point $Q_i, i = 1, 2$.

Now $A_1 + C_2 \in \mathcal{B} \Rightarrow t(A_2, D_2) \cap \langle A_1, B_1, C_2 \rangle = \{B_1 + C_2\}$ and $t(B_1, D_1) \cap \langle A_1, C_2, D_2 \rangle = \{A_1 + D_2\} \Rightarrow$

$\{B_1 + C_2\}$ and $\{A_1 + D_2\}$ are attenuation points with respect to $\Sigma_5$. So, as in the proof of lemma 5.7 we get $Q_i \not\in$

$\{P, A_1 + B_1, C_1 + D_1\}, i = 1, 2$. \therefore Q_i \in
\{A_1 + C_1, A_1 + D_1, B_1 + C_1, B_1 + D_1\}. Again A_1 + C_2 \in S \implies \{B_1, X_1, X_2^1\} is a polarising set for every \( X_1 \in \{C_1, D_1^2\} \) and for every \( X_2 \in \{A_2, B_2^2\} \) [by proposition 5.12]. If \( Q_1 = A_1 + C_1 \) and \( Q_2 = A_2 + C_2 \), we choose the polarising set \( \{B_1, C_1, B_2^2\} \) and carry out similar arguments as we did in the proof of lemma 5.7 and finally arrive at a contradiction. Choosing a suitable polarising set for each value of the pair \( \{Q_1, Q_2\} \), we can show similarly that \( Q_i \not\in \{A_1 + C_1, A_1 + D_1, B_1 + C_1, B_1 + D_1^2\}, i = 1, 2 \). But this is a contradiction. This completes the proof.

**Proposition 5.16.** \( \mathcal{T}_2 \) does not contain any 5-stigm \( \text{St}_2 \) of \( \mathcal{S} \), where \( |\text{St}_2 \cap \Sigma_4| = 1 \).

**Proof.** By virtue of remark 3, made in the beginning of this section, and the lemmas 5.7 and 5.8, the proof follows immediately.

\( \mathcal{S}_5 \). Non-existence of a 5-stigm \( \text{St}_2 \) of \( \mathcal{S} \) in \( \mathcal{T}_2 \) with \( |\text{St}_2 \cap \Sigma_4| = 3 \).

If \( \text{St}_2 \) be a 5-stigm of \( \mathcal{S} \) in \( \mathcal{T}_2 \), then by proposition 5.16, \( |\text{St}_2 \cap \Sigma_4| = 3 \). First, we prove the following proposition.

**Proposition 5.17.** Let \( \text{St}_2 \) be a 5-stigm of \( \mathcal{S} \) in \( \mathcal{T}_2 \), such that \( |\text{St}_2 \cap \Sigma_4| = 3 \). Then \( P \not\in \text{St}_2 \).

**Proof.** Suppose \( P \in \text{St}_2 \). Let \( Z \in \text{St}_2 \cap \Sigma_4 \setminus \{P\} \). Then following the argument made in the beginning of section 6, we assume (w.l.o.g) that \( Z \in \{A_1 + A_2, A_1 + C_2^2\} \).
**Case 1.** Let $Z = A_1 + A_2$. Since $|St_2^1 \cap \Sigma_4| = 3$, there exists a point $X \in St_2^1 = \{P, A_1 + A_2, X, S_1, S_2\}$. Again using arguments made in the beginning of section 6 and the propositions 5.13 and 5.14, we assume (WLOG) that $X \in \{A_1 + C_2, C_1 + C_2\}$. Let $St_2^1 = \{P, A_1 + A_2, X, S_1, S_2\}$, where $X \in \{A_1 + C_2, C_1 + C_2\}$ and $S_1, S_2 \in \Sigma_2$. Now $X = A_1 + C_2 \implies S_1 + S_2 = B_2 + D_2$. But $A_1 + A_2, A_1 + C_2 \in \Theta \implies t(P, C_1) \cap \langle A_1, A_2, B_2 \rangle = \{A_1 + B_2\}$ and $t(P, C_1) \cap \langle A_1, C_2, D_2 \rangle = \{A_1 + D_2\} \implies B_2 + D_2 \in t(P, C_1) \implies B_2 + D_2$ is an attenuation point with respect to $\Sigma_2$, which is a contradiction. $\because S_1 + S_2 = B_2 + D_2$. 

$\therefore X \neq A_1 + C_2$. So, $X = C_1 + C_2$. Thus $S_1 + S_2 = P + A_1 + A_2 + C_1 + C_2 = B_1 + D_1 + A_2 + C_2$. But $A_1 + A_2 \in \Theta \implies t(P, D_1, C_2) \cap \langle A_1, A_2, B_2 \rangle = \{A_1 + B_2\} \implies A_1 + B_2 + D_1 + C_2 = B_1 + D_1 + A_2 + C_2 \in t(P, D_1, C_2) \implies B_1 + D_1 + A_2 + C_2$ is an attenuation point with respect to $\Sigma_5$, which is a contradiction $\because S_1 + S_2 = B_1 + D_1 + A_2 + C_2$. So, case 1 cannot happen.

**Case 2.** $Z = A_1 + C_2$. Since $|St_2^1 \cap \Sigma_4| = 3$, there exists $X \in St_2^1 \cap \Sigma_4 = \{P, A_1 + C_2\}$. Here also we assume (WLOG) that $X \in \{A_1 + A_2, C_1 + A_2\}$ [by propositions 5.13 and 5.14]. By case 1, $X \neq A_1 + A_2$. $\therefore X = C_1 + A_2$. Let $St_2^1 = \{P, A_1 + C_2, C_1 + A_2, S_1, S_2\}$, where $S_1, S_2 \in \Sigma_2$. Then $S_1 + S_2 = B_1 + D_1 + A_2 + C_2$. But $A_1 + C_2, C_1 + A_2 \in \Theta \implies t(P, C_2) \cap \langle A_2, C_1, D_1 \rangle = \{A_2 + D_1\}$ and $t(P, C_2) \cap \langle C_2, A_1, B_1 \rangle = \{B_1 + C_2\} \implies B_1 + D_1 + A_2 + C_2$
t(P, C_2) \implies B_1 + D_1 + A_2 + C_2 is an attenuation point with respect to \( \Sigma_5 \), which is a contradiction \( \therefore S_1 + S_2 = B_1 + D_1 + A_2 + C_2 \). Hence case 2 cannot happen. This completes the proof.

Before we state and prove the main proposition of this section, let us prove the following lemma.

**Lemma 5.9.** Let \( \Delta_1 = \{X_1 + X_2 \mid X_i \in \{A_{1i}, B_{1i}\}, i = 1, 2\} \) and \( \Delta_2 = \{Y_1 + Y_2 \mid Y_i \in \{C_i, D_i\}, i = 1, 2\} \) and let \( \text{St}^i \) be a 5-stigm of \( \emptyset \) in \( \mathcal{T}_2 \) such that \( |\text{St}^i \cap \Sigma_4| = 3 \). Then \( \text{St}^i \cap \Delta_i = \emptyset, i = 1, 2 \).

**Proof.** Suppose the lemma is not true. WLOG we assume that \( \text{St}^i \cap \Delta_i \neq \emptyset \) and \( A_1 + A_2 \in \text{St}^i \). By proposition 5.17, \( P \notin \text{St}^i \). We now consider the following two cases:

1. \( \text{St}^i \cap \Delta_2 \neq \emptyset \),
2. \( \text{St}^i \cap \Delta_2 = \emptyset \).

**Case 1.** Let \( \text{St}^i \cap \Delta_2 \neq \emptyset \). WLOG we assume \( C_1 + C_2 \in \text{St}^i \).

Let \( \text{St}^i_2 = \{A_1 + A_2, C_1 + C_2, X, S_1, S_2\} \), where \( X \in \text{St}^i_2 \cap \Sigma_4 = \{P, A_1 + A_2, C_1 + C_2\} \) and \( S_1, S_2 \in \mathcal{T}_2 - \Sigma_4 \). By propositions 5.13 and 5.14, we can assume (WLOG) that \( X = A_1 + C_2 \).

So \( \text{St}^i_2 = \{A_1 + A_2, C_1 + C_2, A_1 + C_2, S_1, S_2\} \). \( \therefore S_1 + S_2 = C_1 + A_2 \). Since \( A_1 + A_2, C_1 + C_2, A_1 + C_2 \in \emptyset \), \( \{C_1, D_1, C_2\}, \{A_1, A_2, B_2\} \) and \( \{B_1, X_1, X_2\} \) are polarising sets with respect to \( \Sigma_5 \), where \( X_1 \in \{C_1, D_1\}, X_2 \in \{A_2, B_2\} \). Now we have \( S_1 + S_2 = C_1 + A_2, A_3 + B_3 \); so, using usual arguments, we can assume (WLOG) that \( S_1, A_3 \in \mathcal{P}_3(C_1, D_1, C_2) \). So, \( S_1 + A_3 \in \{C_1, D_1, C_2, D_2, B_2\} \). Let \( \Delta \) be a polarising set
with respect to $\Sigma_5$. Let us now consider the following table.

<table>
<thead>
<tr>
<th>$S_1 + S_2$</th>
<th>$S_1 + A_3$</th>
<th>$S_2 + A_3$</th>
<th>$S_1 + B_3$</th>
<th>$S_2 + B_3$</th>
<th>$\Delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1 + A_2$</td>
<td>$C_1$</td>
<td>$A_2$</td>
<td>$P + C_1$</td>
<td>$P + A_2$</td>
<td>${B_1, D_1, B_2}$</td>
</tr>
<tr>
<td>$C_1 + A_2$</td>
<td>$D_1$</td>
<td>$P + B_2$</td>
<td>$P + D_1$</td>
<td>$B_2$</td>
<td>${B_1, D_1, A_2}$</td>
</tr>
<tr>
<td>$C_1 + A_2$</td>
<td>$C_2$</td>
<td>$C_1 + A_2 + C_2$</td>
<td>$P + C_2$</td>
<td>$C_1 + B_2 + D_2$</td>
<td>${B_1, D_1, B_2}$</td>
</tr>
<tr>
<td>$C_1 + A_2$</td>
<td>$D_2$</td>
<td>$C_1 + A_2 + D_2$</td>
<td>$P + D_2$</td>
<td>$C_1 + B_2 + C_2$</td>
<td>${B_1, D_1, B_2}$</td>
</tr>
</tbody>
</table>

We consider the distribution of $S_1, S_2, A_3$ among $\mathcal{P}_3(\Delta)$, $i = 1, 2, 3$. By the above table, we can assume (WLOG) that $S_i \in \mathcal{P}_3(\Delta)$, $i = 1, 2$ and $A_3 \in \mathcal{P}_3(\Delta)$. Now using the above table and the fact that $A_3 + B_3 = P$, it follows clearly that $B_3 \notin \mathcal{P}_3(\Delta)$, $i = 1, 2, 3$. But this is a contradiction. ... case 1 cannot happen.

Case 2. Let $S_{12} \cap \triangle_2 = \emptyset$. Let $Z \in S_{12} - \{A_1 + A_2\}$.

Then by proposition 5.13, we can assume (WLOG) that $Z = A_1 + C_2$. Let $S_{12}' = \{A_1 + A_2, A_1 + C_2, X, S_1, S_2\}$, where $X \in \sum - \{P, A_1 + A_2, A_1 + C_2\}$ and $S_1, S_2 \in \tau - \sum$. Since case 1 does not happen, we assume (WLOG) that $X = C_1 + A_2$.

Then $S_{12}' = \{A_1 + A_2, A_1 + C_2, C_1 + A_2, S_1, S_2\}$, so that $S_1 + S_2 = C_1 + C_2$. Now $C_1 + A_2 \in \mathcal{B} \implies \{D_1, A_1, C_2\}$ is a polarising set with respect to $\sum_5$; also $A_1 + C_2 \notin \mathcal{B} \implies \{B_1, Y_1, X_2\}$ is a polarising set with respect to $\sum_5$, $Y_1 \in \{C_1, D_1\}$, $X_2 \in \{A_2, B_2\}$. Since $S_1 + S_2 = C_1 + C_2 \notin \langle D_1, A_1, C_2 \rangle$ and $A_3 + B_3 = P \notin \langle D_1, A_1, C_2 \rangle$, we can assume (WLOG) that $S_1, A_3 \in \mathcal{P}_3(D_1, A_1, C_2)$, for some
i \in \{1, 2, 3\}. \therefore S_1 + A_3 \in \{D_1, A_1, C_2, A_1 + D_1 + C_2\}.

Let \(\Lambda\) be a polarising set with respect to \(\Sigma_5\). We now consider the following table.

<table>
<thead>
<tr>
<th>(S_1 + S_2)</th>
<th>(S_1 + A_3)</th>
<th>(S_2 + A_3)</th>
<th>(S_1 + B_3)</th>
<th>(S_2 + B_3)</th>
<th>(\Delta)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C_1 + C_2)</td>
<td>(D_1)</td>
<td>(D_2)</td>
<td>(P + D_1)</td>
<td>(P + D_2)</td>
<td>({B_1, C_1, A_2})</td>
</tr>
<tr>
<td>(C_1 + C_2)</td>
<td>(A_1)</td>
<td>(A_1 + C_1 + C_2)</td>
<td>(P + A_1)</td>
<td>(B_1 + D_1 + C_2)</td>
<td>({B_1, C_1, A_2})</td>
</tr>
<tr>
<td>(C_1 + C_2)</td>
<td>(C_2)</td>
<td>(C_1)</td>
<td>(P + C_2)</td>
<td>(P + C_1)</td>
<td>({B_1, D_1, A_2})</td>
</tr>
<tr>
<td>(C_1 + C_2)</td>
<td>(A_1 + D_1 + C_2)</td>
<td>(P + B_1)</td>
<td>(B_1 + D_1 + C_2)</td>
<td>(B_1)</td>
<td>({C_1, A_2, D_2})</td>
</tr>
</tbody>
</table>

As in case 1, we now use the above table to make similar arguments and arrive at a contradiction. \(\therefore\) case 2 cannot happen. This completes the proof.

**Proposition 5.18.** \(\mathcal{T}_2\) does not contain any 5-stigm \(\text{St}_2^i\) of \(\mathcal{S}\) such that \(|\text{St}_2^i \cap \Sigma_4| = 3\).

**Proof.** By virtue of propositions 5.13, 5.14, 5.17, and lemma 5.9 and the remark made at the end of the proof of proposition 5.14, the proof follows immediately.

§9. **Non-existence of a 7-stigm of \(\mathcal{S}\) in \(\mathcal{T}_2\).**

In order to show that \(\mathcal{T}_2\) does not contain a 7-stigm of \(\mathcal{S}\), we need the following lemmas.

**Lemma 5.10.** Let \(\text{St}_2^i\) be a 7-stigm of \(\mathcal{S}\) in \(\mathcal{T}_2\) and let

\[ \Delta_1 = \{x_i + x_{i'} \mid x_i \in \{A_i, B_i\}, i = 1, 2\} \]

and

\[ \Delta_2 = \{y_i + y_{i'} \mid y_i = \{C_i, D_i\}, i = 1, 2\}. \]

Then \(\text{St}_2^i \cap \Delta_i = \emptyset\), \(i = 1, 2\).

**Proof.** Suppose \(\text{St}_2^i \cap \Delta_i \neq \emptyset\), for some \(i \in \{1, 2\}\).

WLOG we assume \(\text{St}_2^i \cap \Delta_1 \neq \emptyset\) and \(A_1 + A_2 \in \text{St}_2^i\). Now
through 1 + 2 there exist two lines of ☐, viz, 
\{1 + 2, 1, 2\} and \{1 + 2, 1\}, none of these two 
lines are contained in \(T_2\), where \(T_2 = <St_2>\). This 
contradicts theorem 2.8. Hence the proof is completed.

**Lemma 5.11.** Let St be a 7-stigm of ☐ in \(T_2\). Then 
P \(\not\in\) St.

**Proof.** Suppose St is a 7-stigm of ☐ such that P \(\in\) St. 
We assert that \(|St_2 \cap \Sigma_4| = 3\). First, we observe that 
\(\Sigma_4 \cap \emptyset \neq \emptyset\) [by theorem 3.1 and propositions 5.10 
and 5.11]. Now if \(|St_2 \cap \Sigma_4| = 1\), then for any point X \(\in\) 
\(\Sigma_4 \cap \emptyset \neq \emptyset\), X is the sum of either two or four points 
of ☐ \(\cap \) \(\emptyset \), which leads to a contradiction [by 
propositions 5.14 and 5.15]. By lemma 5.10 and proposition 
5.13, we get \(|St_2 \cap \Sigma_4| = 3\). Again by proposition 5.13 
and lemma 5.10, we assume (wlog) that St = 
\{P, 1 + 2, 1 + 2\}. Let St = 
\{P, 1 + 2, 1 + 2, 1 + 2, 1 + 2, 1 + 2, 1 + 2\}, where S \(\in\) \(\Sigma_2\) \(\subset\) \(\Sigma_4\), 
i = 1, 2, 3, 4. \(\therefore S_1 + S_2 + S_3 + S_4 = B + D + 1 + 2 + C_2\). Now 
\(1 + 2, 1 + 2 \in \emptyset \implies t(B_1, D_1) \cap <1, 2, 2, 2> = \{1 + 2, 2\} \text{ and } t(B_1, D_1) \cap <1, 2, 2, 2> = \{1 + 2, 2\} \implies \{1 + 2, 2\} \text{ and } \{1 + 2, 2\} \text{ are two attenuation points with}
respect to \(\Sigma_5\). Further, \(<B_1, D_1, 1 + 2, 1 + 2, 1 + 2> \text{ is}
a 3-space. Also t(B_1, D_1) \not\subset \Sigma_5\), since t(B_1, D_1) does not 
meet the line \{1 + 2, 2, 2\}. Hence by theorem 2.5,
\{1 + 2, 2\} is a polarising set with respect to \(\Sigma_5\). Since
A_1 + C_2, C_1 + A_2 \in \mathcal{B}, \ t(B_2, D_2) \cap <A_1, B_1, C_2> = \{B_1 + C_2\} \quad \text{and} \quad t(B_2, D_2) \cap <C_1, D_1, A_2> = \{D_1 + A_2\}.

Now it is easy to deduce that \(\{B_2, D_2\}\) is a polarising set with respect to \(\sum_5\).

Considering the distribution of \(S_1, S_2, S_3, S_4\) among \(\rho_1^i(B_1, D_1), i = 1, 2, 3\) and rearranging the suffixes of \(S\)'s, we get \((\text{WLOG})\) that \(S_1, S_2 \in \rho_2^i(B_1, D_1)\), for some \(i \in \{1, 2, 3\}\). \(\therefore S_1 + S_2 = B_1 + D_1\). So,
\[
S_3 + S_4 = A_2 + C_2 \quad \text{(\because S_1 + S_2 + S_3 + S_4 = B_1 + D_1 + A_2 + C_2).}
\]

So, none of the pairs \(\{S_1, S_2\}\) and \(\{S_3, S_4\}\) is in the same \(\rho_2^i(B_2, D_2), i \in \{1, 2, 3\}\). By adjusting the suffixes of \(S\)'s, we get \((\text{WLOG})\) that \(S_1, S_3 \in \rho_2^i(B_2, D_2)\), for some \(i \in \{1, 2, 3\}\). \(\therefore S_1 + S_3 = B_2 + D_2\). Also we have
\[
S_3 + S_4 = A_2 + C_2. \quad \text{So, S}_1 + S_4 = P, \text{ which is a contradiction.}
\]
This completes the proof.

**Proposition 5.19.** \(\tau_2^1\) does not contain any 7-stigm of \(\mathcal{B}\).

**Proof.** By virtue of lemmas 5.10 and 5.11 and the proposition 5.13, the proof follows immediately.

Let us now state and prove the main result of this chapter.

**Theorem 5.1.** If \(\mathcal{B}\) be a 6-dimensional tangential 2-block and if \(P \in \mathcal{B}\), then \(P\) does not have any \((5, 5, 3)\)-tangential stigm system.

**Proof.** Suppose the theorem is false. We assume \(\text{St}_i(P) = \{P, A_i, B_i, C_i, D_i\}, i = 1, 2\) and \(\text{St}_3(P) = \{P, A_3, B_3\}\).
By proposition 5.1 we assume (WLOG) $A_1 + B_1 = A_2 + B_2$. Let $\sum_5 = \langle A_1, B_1, C_1, D_1, A_2, C_2 \rangle$ and $\sum_4 = \langle A_1 + A_2, A_1 + B_2, A_1 + C_2, A_1 + D_2, C_1 + A_2 \rangle$. Let $\tau_i^1, i = 1, 2, 3$, be the three 5-spaces on $\sum_4$, one of which must be $\sum_5$. WLOG we assume $\tau_1^1 = \sum_5$. By proposition 5.8, $St_3(p) \not\subset \sum_5 = \tau_1^1$. WLOG we assume $St_3(p) \subset \tau_3^1$.

Now $\tau_2^1$ must contain an odd stigm of $\mathcal{B}$ [by proposition 1.10.]. By propositions 5.15, 5.16, 5.18 and 5.19, $\tau_2^1$ does not contain any odd stigm of $\mathcal{B}$. Thus we arrive at a contradiction. This completes the proof.
CHAPTER VI

Non-existence of (5, 3, 3)-tangential stigm system for a 6-dimensional tangential 2-block.

Let $\mathcal{S}$ be a 6-dimensional tangential 2-block in $\text{PG}(6, 2)$. Let $P \in \mathcal{S}$. In this chapter we want to show that $P$ cannot have a (5, 3, 3)-tangential stigm system.

§1. Two 5-spaces and their odd stigms.

First of all, we prove the following proposition.

Proposition 6.1. Let $\text{St}^1(P) = \{P, A_1, B_1, C_1, D_1, A_2, A_3\}$, $\text{St}^i(P) = \{P, A_1, B_2^i, i = 2, 3\}$. Then

(i) $<A_1, B_1, C_1, D_1, A_2, A_3>$ is a 5-space.

(ii) $<A_1 + A_2, B_1 + A_2, C_1 + A_2, D_1 + A_2, A_2 + A_3>$ is a 4-space.

Proof. (i) Clearly, $4 \leq \dim <A_1, B_1, C_1, D_1, A_2, A_3> \leq 5$. Suppose the dimension is 4. Then $A_3 \in <A_1, B_1, C_1, D_1, A_2>$; so, $A_3 = kP + X_1 + A_2$, where $k = 0 \text{ or } 1 \text{ (mod 2)}$ and $X_1 \in \text{St}^1(P) - \{P\}$. WLOG we assume that $X_1 = A_1$. $\therefore A_3 = kP + A_1 + A_2$. But $A_3 + B_3 = P$. $\therefore A_1 + A_2$ and $P + A_1 + A_2$ are both points of $\mathcal{S}$. So, $t(C_1, D_1) \cap <P, A_1, A_2> = \emptyset$, which is a contradiction.

(ii) The proof of part (ii) follows immediately from part (i). This completes the proof.
The 5-space and the 4-space, mentioned in proposition 6.1, will always be denoted in this chapter by $\Sigma_5$ and $\Sigma_4$ respectively. We observe that $\Sigma_4 =$ 

$\langle \lambda_1(p), \{A_1 + A_2^3\} \rangle \cup \langle \lambda_1(p), \{A_1 + A_3^2\} \rangle \cup$

$\langle \lambda_1(p), \{A_2 + A_3^3\} \rangle$. So, the points of $\Sigma_4$ are of the following types:

$P, X_1 + X_j, X_1 + Y_1, X_1 + Y_1 + X_2 + X_3, \text{ where } i \neq j,$

$i, j \in \{1, 2, 3\}, X_k, Y_k \in St_k(p) = \{p\}_k, k = 1, 2, 3.$

Further, we note that

$\Sigma_4 \cap \tau_1(p) - \eta(p) =$

$\{k(X_1 + Y_1) + X_2 + X_3 \mid k = 0, 1; X_i, Y_i \in St_i(p) = \{p\}_i, i = 1, 2, 3\}.$

$\Sigma_4 \cap \tau_2(p) - \eta(p) =$

$\{X_1 + X_3 \mid X_i \in St_i(p) = \{p\}_i, i = 1, 3\}.$

$\Sigma_4 \cap \tau_3(p) - \eta(p) =$

$\{X_1 + X_2 \mid X_i \in St_i(p) = \{p\}_i, i = 1, 2\}.$

Let us now introduce the following notations. Let $\tau_i^i, i = 1, 2, 3,$ be the three 5-spaces on $\Sigma_4$, one of which must be $\Sigma_5$. WLOG we assume that $\tau_1^i = \Sigma_5$. Now each of $\tau_i^i, i = 2, 3,$ must contain an odd stigm of $B$. Let $St_i^i$ be an odd stigm of $B$ in $\tau_i^i, i = 2, 3$. Now we shall investigate into the existence of the pair of odd stigms $St_i^i$ and $St_j^i$. Before we do that we prove the following proposition regarding some attenuation points and polarising sets with respect to $\Sigma_5$. 
Proposition 6.2. Let \( X_i, Y_i, Z_i, w_i \in St_i(p) - \{ p_i \}, \)
i \( i \in \{ 1, 2, 3 \} \). Then the following results hold.

(i) If \( X_i + X_j \in \emptyset \), \( i \neq j \), and \( i, j = 1, 2, 3 \),
then (a) \( Y_i + X_j \) is an attenuation point with respect to
\( \Sigma_5 \), \( \forall \ Y_i \in St_i(p) - \{ p, X_i \} \);
(b) \( Y_1 + Y_2 + Y_3 \) and \( Y_1 + Z_1 + Y_2 + Y_3 \) are both
attenuation points with respect to \( \Sigma_5 \), \( \forall \ Y_i, Z_i, Y_1 \neq 
Z_1 \), \( i = 1, 2, 3 \).

(ii) If any two of the points \( X_1 + X_2 \), \( Y_1 + X_3 \) and
\( Z_2 + Z_3 \) are points of \( \emptyset \), then the remaining one is an
attenuation point with respect to \( \Sigma_5 \).

(iii) If \( X_1 + Y_1 + X_2 + X_3 \in \emptyset \), then \( Z_1 + W_1 + Y_2 + 
Y_3 \) and \( Y_2 + Y_3 \) are both attenuation points with respect to
\( \Sigma_5 \), for all \( Y_2 \) and \( Y_3 \) and for all those \( Z_1, W_1 \), such that
\( Z_1 + W_1 + Y_2 + Y_3 \neq X_1 + Y_1 + X_2 + X_3 \).

(iv) \( | \emptyset \cap \Sigma_4 \cap T_i(p) - \{ p_i \} | \leq 1 \) and
\( | \emptyset \cap \Sigma_4 \cap T_j(p) - \{ p_i \} | \leq 2 \), \( j = 2, 3 \).

(v) If \( X_1 + X_j \in \emptyset \), \( j \in \{ 2, 3 \} \), then \( \{ X_1, Z_1, X_2 \} \)
is a polarising set with respect to \( \Sigma_5 \), where \( X_1 \neq Y_1 \neq 
Z_1 \neq X_1 \) and \( k \) equals 2 or 3 according as \( j = 3 \) or 2.

(vi) If \( X_2 + X_3 \in \emptyset \), then \( \{ X_1, Y_1, Z_1 \} \)
is a polarising set with respect to \( \Sigma_5 \), for all \( X_1, Y_1 \) and
\( Z_1 \).
(vii) If \(X_1 + X_2\) and \(Y_1 + X_3\) belong to \(\mathcal{B}\), then 
\[\{Z_1, W_1\}^2\] is a polarising set with respect to \(\Sigma_5\), where 
\(Z_1 \not= W_1\) and \(Z_1, W_1, X_1, Y_1^2 = \emptyset\).

**Proof:** (i) Suppose \(A_1 + A_2 \in \mathcal{B}\). To prove part (a) of (i) in this case, we need to show that \(Y_1 + A_2\) and \(B_2 + A_1\) are both attenuation points with respect to \(\Sigma_5\) for all \(Y_1 \in \text{St}_1(p) - \{p, A_1^2\}\). Since \(t(Z_1, W_1) \cap \langle p, A_1, Y_1 \rangle = \{Y_1 + A_2^2\}\), where \(Y_1, Z_1, W_1\) are three distinct points of \(\text{St}_1(p) - \{p, A_1^2\}\), it follows from theorem 2.4 that \(Y_1 + A_2\) is an attenuation point with respect to \(\Sigma_5\). Similarly, we can show that if \(X_1 + X_3 \in \mathcal{B}\), \(j = 2, 3, X_i \in \text{St}_3(p) - \{p\}\), then part (a) of (i) is true.

Let \(A_2 + A_3 \in \mathcal{B}\). To prove part (a) of (i) in this case we need to show that \(A_2 + B_3\) is an attenuation point with respect to \(\Sigma_5\). Since \(t(A_1) \cap \langle p, A_2, A_3 \rangle = \{A_2 + B_3^3\}\), \(A_2 + B_3\) is an attenuation point with respect to \(\Sigma_5\) [by theorem 2.4]. Similarly, if \(A_2 + B_3 \in \mathcal{B}\), then \(A_2 + A_3\) is an attenuation point.

To prove part (b) of (i), we first assume that \(A_1 + A_2 \in \mathcal{B}\). Let \(Y_1, Z_1\) be any two points of \(\{B_1, C_1, D_1\}\), then \(t(Y_1, Z_1, X_3) \cap \langle p, A_1, A_2 \rangle = \{A_1 + B_2^3\}\) and \(X_3 \in \{A_3, B_3\}\). Since \(A_2 + A_3 = B_2 + B_3\), we
can now easily deduce that \( Y_1 + Y_2 + Y_3 \) and \( Y_1 + Z_1 + Y_2 + Y_3 \) are attenuation points with respect to \( \sum_5 \), \( Y_1, Z_1 \in St_1(P) - \{P\} \). Similarly, if \( X_1 + X_j \in \emptyset \), \( j = 2, 3, X_1 \in St_1(P) - \{P\} \), then we can show that part (b) of (i) is true. Now, we suppose that \( A_2 + A_3 \in \emptyset \). Then \( t(X_1) \cap < P, A_2, A_3 > = \{A_2 + B_3\} \) and \( t(X_1, Y_1, Z_1) \cap < P, A_2, A_3 > = \{A_2 + B_3\} \), where \( X_1, Y_1, Z_1 \) belong to \( St_1(P) - \{P\} \), and are all different. Now it is easy to deduce that \( Y_1 + Y_2 + Y_3 \) and \( Y_1 + Z_1 + Y_2 + Y_3 \) are attenuation points with respect to \( \sum_5 \), \( Y_1, Z_1 \in St_1(P) - \{P\} \). Similarly, if \( A_2 + B_3 \in \emptyset \), then we can show that part (b) of (i) is true. This completes the proof of (i).

(ii) First, we suppose that \( A_1 + A_2 \) and \( Y_1 + A_3 \) are points of \( \emptyset \), where \( Y_1 \in St_1(P) - \{P\} \). Suppose \( Y_1 = A_1 \). Then \( t(C_1, D_1) \cap < P, A_1, A_2 > = \{A_1 + B_2\} \) and \( t(C_1, D_1, A_1 + A_3) \cap < P, A_1, A_2 > = \{A_1 + B_3\} \). So, \( t(C_1, D_1) \cap < A_1, A_2, A_3 > = \{A_2 + A_3\} \) and \( t(C_1, D_1, A_1 + A_3) \cap < A_1, A_2, B_3 > = \{A_2 + B_3\} \). \( \{A_2 + A_3\} \) and \( \{A_2 + B_3\} \) are both attenuation points with respect to \( \sum_5 \). Suppose \( Y_1 = B_1 \). Now \( t(C_1, D_1) \cap < P, A_1, A_2 > = \{A_1 + B_2\} \), \( t(C_1, D_1) \cap < P, B_1, A_3 > = \{B_1 + B_3\} \). \( A_1 + B_1 + B_2 + B_3 \in t(C_1, D_1) \) i.e. \( A_2 + B_3 \in t(C_1, D_1) \); also, \( t(C_1, D_1, B_1 + A_3) \cap < A_1, B_1, A_2 > = \{B_1 + A_2\} \) i.e.
$A_2 + A_3 \in t(C_1, D_1, B_1 + A_3)$. So, we can conclude that
$A_2 + A_3$ and $A_2 + B_3$ are both attenuation points with respect to $\Sigma_5$. Similarly, if $x_i \in \text{St}_i(P) - \{p_i\}$, and $x_1 + x_2$ and $y_1 + x_3$ are points of $\mathcal{S}$, then both $A_2 + A_3$ and $A_2 + B_3$ are attenuation points with respect to $\Sigma_5$.

Next, we suppose that $A_1 + A_2$ and $A_2 + A_3$ are points of $\mathcal{S}$. We now want to show that $x_1 + x_3$ is an attenuation point with respect to $\Sigma_5, \forall x_i \in \text{St}_i(P) - \{p_i\}, i = 1, 3$. Since $t(C_1, D_1) \cap < A_1, A_2, A_3 > = \{A_1 + A_3\}$,$A_1 + A_3$ and $B_1 + B_3$ are attenuation points with respect to $\mathcal{S}$. Suppose $A_1 + B_3$ is not an attenuation point. We observe that $x_1 + x_3$ is not a point of $\mathcal{S}, \forall x_i \in \text{St}_i(P) - \{p_i\}$, for, otherwise $A_2 + A_3$ would become an attenuation point with respect to $\Sigma_5$. So, we have $A_1 + B_3 = S_1 + S_2$, for some $S_i \in \mathcal{S} - \Sigma_5, i = 1, 2$. Therefore
\[ t(C_1, D_1, S_1, S_2) \cap < P, A_1, A_2 > = \{A_1 + B_3\} \text{ and hence} \]
\[ t(C_1, D_1, S_1, S_2) \cap < A_1, A_2, A_3 > = \{A_2 + A_3\}, \text{ which is a contradiction. [} \cdot \cdot A_2 + A_3 \in \mathcal{S} \text{ and } A_2 + A_3 \notin \]
\[ < C_1, D_1, S_1, S_2 > \} \]. So, $A_1 + B_3$ is an attenuation point with respect to $\Sigma_5$. Suppose $B_1 + A_3$ is not an attenuation point. Since $B_1 + A_3 \notin \mathcal{S}$, we get $B_1 + A_3 = S_1 + S_2$, for some $S_i \in \mathcal{S} - \Sigma_5, i = 1, 2$. So,
\[ t(C_1, D_1, S_1, S_2) \cap < A_1, B_1, A_2 > = \{B_1 + A_2\} \text{ and hence} \]
\[ t(C_1, D_1, S_1, S_2) \cap < B_1, B_2, B_3 > = \{A_2 + A_3\}, \text{ which is a contradiction. [} \cdot \cdot A_2 + A_3 \in \mathcal{S} \text{ and } A_2 + A_3 \notin \]
\[ \langle C_1, D_1, S_1, S_2 \rangle \] \Rightarrow B_1 + A_3 \text{ is an attenuation point with respect to } \Sigma_5. \text{ Replacing } B_1 \text{ by } C_1 \text{ and } D_1, we can show in a similar way that } C_1 + X_3 \text{ and } D_1 + X_3 \text{ are attenuation points with respect to } \Sigma_5, \text{ for all } X_3 \in \{A_3, B_3\}. \text{ Similarly, we can show that if } X_1 + X_j \text{ and } X_2 + X_3 \text{ belong to } \Sigma, \text{ then } Y_1 + X_k \text{ is an attenuation point with respect to } \Sigma_5, \text{ where } \{j, k\} = \{2, 3\}, X_1, Y_1 \in \text{St}_1(p) - \{p\}, i = 1, 2, 3. \text{ This completes the proof of (ii).}

(iii) Suppose } A_1 + B_1 + A_2 + A_3 \in \Sigma. \text{ Then } t(C_1, D_1, X_1) \cap \langle P, A_2, A_3 \rangle = \{A_2 + A_3\} \text{ and } t(A_1, B_1, Y_1) \cap \langle P, A_2, A_3 \rangle = \{A_2 + B_3\}, \forall X_1 \in \{A_1, B_1\}, \forall Y_1 \in \{C_1, D_1\}. \text{ So, we easily deduce that } Y_2 + Y_3 \text{ and } Z_1 + W_1 + Y_2 + Y_3 \text{ are attenuation points with respect to } \Sigma_5, \text{ for all } Y_i \in \{A_i, B_i\}, i = 2, 3, \text{ and for all } Z_1 + W_1 \in \lambda_1(p) - \{p\} \text{ such that } Z_1 + W_1 + Y_2 + Y_3 \neq A_1 + B_1 + A_2 + A_3. \text{ Similarly, we can deal with other cases and prove (iii).}

(iv) By virtue of (i), (ii) and (iii) of this proposition, the proof follows immediately.

(v) First, we suppose that } A_1 + A_2 \in \Sigma. \text{ Let } X_1, Y_1 \in \{B_1, C_1, D_1\}, X_1 \neq Y_1. \text{ Then } t(X_1, Y_1, X_3) \cap \langle P, A_1, A_2 \rangle = \{A_1 + B_2\} \text{ and by proposition 6.1 } \langle X_1, Y_1, X_3, A_1 + B_2 \rangle \text{ is a 3-space, } \forall X_3 \in \{A_3, B_3\}. \text{ Further } t(X_1, Y_1, X_3) \not\subset \Sigma_5, \forall X_3 \in \{A_3, B_3\}, \text{ for the tangent does not meet the line } \{P, A_2, B_2\} \text{ of } \Sigma_5. \text{ Hence by proposition 2.5, we get that}
\{X_1, Y_1, X_3\} is a polarising set with respect to \(\Sigma_5\),
\forall X_1, Y_1 \in \{B_1, C_1, D_1, \mathcal{B}_1\}, X_1 \neq Y_1 \text{ and } \forall X_3 \in \{A_3, B_3\}.
Similarly, we can deal with the other cases and prove (iii).

(vi) Suppose \(A_2 + A_3 \in \mathcal{B}\). Then
t\left( X_1, Y_1, Z_1 \right) \cap \langle p, A_2, A_3 \rangle = \{A_2 + B_3\} \text{ and also}
\langle X_1, Y_1, Z_1, A_2 + B_3 \rangle \text{ is a 3-space [by proposition 6.1].}
Further t\left( X_1, Y_1, Z_1 \right) \not\subseteq \Sigma_5, \text{ since the tangent does not meet the line } \{p, A_2, B_3\} \text{ of } \Sigma_5. \text{ So, by proposition 2.5 we get that } \{X_1, Y_1, Z_1\} \text{ is a polarising set with respect to } \Sigma_5. \text{ We can deal similarly with other cases and prove (vi).}

(vii) Suppose \(A_1 + A_2, Y_1 + A_3 \in \mathcal{B}\). Let
\{Z_1, W_1\} \cap \{A_1, Y_1\} = \emptyset. \text{ Then } t\left( Z_1, W_1 \right) \cap \langle p, A_1, A_2 \rangle = \{A_1 + B_2\} \text{ and } t\left( Z_1, W_1 \right) \cap \langle p, Y_1, A_3 \rangle = \{Y_1 + B_3\} \text{ and }
\langle Z_1, W_1, A_1 + B_2, Y_1 + B_3 \rangle \text{ is a 3-space. Further}
t\left( Z_1, W_1 \right) \not\subseteq \Sigma_5, \text{ for, the tangent does not meet the line } \{p, A_2, B_3\} \text{ of } \Sigma_5. \text{ So, by proposition 2.5, it follows that } \{Z_1, W_1\} \text{ is a polarising set with respect to } \Sigma_5. \text{ Other cases can be similarly dealt with and the proof is completed.}
§2. Existence of a point of $\mathcal{B}$ and non-existence of an odd stigm of $\mathcal{B}$ in $\Sigma_4$.

Here we prove the following propositions.

**Proposition 6.3(i).** $\mathcal{B} \cap \Sigma_4 \cap \tau_j(P) = \{P\} \neq \emptyset$, for some $j \in \{2, 3\}$.

(ii). $x_1 + y_1 + x_2 + x_3$ is an attenuation point, $x_1 \neq y_1$, $\forall \ x_i, y_i \in \text{St}_1(P) - \{P\}^j$, $i = 1, 2, 3$.

**Proof.** By virtue of theorems 3.1, 4.1 and 5.1, there exist at least two lines of $\mathcal{B}$ through each point of $\mathcal{B}$.

So, there are at least two lines of $\mathcal{B}$ through each point of $\text{St}_1(P) - \{P\}^j$. At most two points of $\{P + x_1 \mid x_1 \in \text{St}_1(P) - \{P\}^j\}$ can be points of $\mathcal{B}$. For, if $P + C_1, P + D_1 \in \mathcal{B}$, then $t(x_1) \cap \langle P, C_1, D_1 \rangle = \{C_1 + D_1, y \mid x_1 \in \{A_1, B_1\}\};$ so, it follows that $P + A_1, P + B_1 \notin \mathcal{B}$. WLOG we assume that no line of $\mathcal{B}$ through $A_1$ contains $P$. First, we assert that at least one line of $\mathcal{B}$ through $A_1$ must be contained in $\Sigma_5$. If possible, let $\{A_1, S_1, S_2\}$ and $\{A_1, S_3, S_4\}$ be two lines of $\mathcal{B}$ where $S_i \in \mathcal{B} - \Sigma_5, i = 1, 2, 3, 4$. Since $\langle A_1, S_1, S_3 \rangle \cap \Sigma_4 \neq \emptyset$, there exist $j \in \{3, 4\}$, such that $S_1 + S_j \in \Sigma_4$. WLOG we assume $S_1 + S_3 \in \Sigma_4$. Then we have the following table:
Now we can interchange $B_1$, $C_1$ and $D_1$ as well as $A_i$ and $B_i$, $i = 2, 3$; also we can rearrange the suffixes of $S_{i+2}(P)$ and $S_{i+3}(P)$. Further, any point of the form $X_1 + Y_1 + X_2 + X_3$ can be expressed in the form $A_1 + Y_1 + X_2 + X_3$, where $X_i, Y_i \in S_{i+1}(P) - \{p\}, i = 1, 2, 3, X_1 \neq Y_1$. So, by the above table, we arrive at a contradiction. Hence at least one line of $\mathcal{B}$ through $A_1$ must be in $\Sigma_5$ and such a line must meet $\Sigma_4$. Since $X_1 + Y_1, X_1 + X_2 + X_3 \in T(p) - \{p\}$, for all $X_i, Y_i \in S_{i+1}(P) - \{p\}, X_i \neq Y_i$, $i = 1, 2, 3$ and $P + A_1 \neq \emptyset$, it follows that there exists a point $X_1 + X_j, j \in \{2, 3\}, X_i \in S_{i+1}(P) - \{p\}, i = 1, 2, 3$, such that $\langle A_1, X_1 + X_j \rangle$ is a line of $\mathcal{B}$. But $X_1 + X_j \in T_i(p)$, where $i = 3$ or 2 according as $j = 2$ or 3. So, $\mathcal{B} \cap \Sigma_4 \cap T_j(p) - \{p\} \neq \emptyset, j = \{2, 3\}.$

(ii) By proposition 6.2 and part (i) of this proposition, proof follows immediately.
Proposition 6.4. $\sum_4$ does not contain any odd stigm of $\mathfrak{B}$.

Proof. If possible, let $\sum_4$ contain an odd stigm $St$ of $\mathfrak{B}$. Suppose $St$ is a 3-stigm. By propositions 6.2 and 6.3 (i) we can easily conclude that $St = \{p, X, P + x\}$, where $X \subseteq \{X_i + X_j \mid i \neq j, i, j = 1, 2, 3\}$ and $X_k \in St_k(p) - \{p\}, k \in \{1, 2, 3\}$. Suppose $X = A_1 + A_2$.

Then $A_1 + A_2, P + A_1 + A_2 \in \mathfrak{B}$. So,

$$\text{t}(C_1, D_1) \cap \langle p, A_1, A_2 \rangle = \emptyset,$$

which is a contradiction. Hence $St$ is not a 3-stigm. Thus we arrive at a contradiction. Hence $St$ must be a 5-stigm. By virtue of proposition 6.2, we can assume $\{A_1, C_1, B_1, D_1, B_3\}$.

By proposition 6.2, $\{A_1, C_1, B_1, D_1, B_3\}$ are polarising sets with respect to $\sum_5$. Since

$$\text{t}(B_1 + B_3, D_1 + B_2) \cap \langle A_1 + A_3, A_1, P \rangle = \{p + A_1\},$$

$$\text{t}(B_1 + B_3, D_1 + B_2) \cap \langle C_1 + A_2, A_2, P \rangle = \{p + C_1 \}$$

and

$$\text{t}(B_1 + B_3, D_1 + B_2) \subset \sum_5 [\therefore \text{t}(B_1 + B_3, D_1 + B_2) \cap \{p, A_2, B_2\} = \emptyset],$$

it follows easily that $\{B_1 + B_3, D_1 + B_2\}$ is a polarising set with respect to $\sum_5$. Since
\[ |S - \Sigma_5| \geq 5 \text{ (by theorem 2.7), there exist } S_i \in S - \Sigma_5, \]
i = 1, 2, 3, 4, 5, where the \( S_i \)'s are all distinct. Now we consider the distribution of \( S_1, S_2, S_3, S_4 \) and \( S_5 \) among \( \rho^i_2(A_1, C_1) \) and \( \rho^i_2(B_1, D_1) \). We consider the following two cases:

1. \( \exists i \in \{1, 2, 3\} \text{ such that } \rho^i_2(A_1, C_1) \text{ contains at least three points of } \{S_j \mid j = 1, 2, 3, 4, 5\} \).

2. None of \( \rho^i_2(A_1, C_1), i = 1, 2, 3, \) contains more than two points of \( \{S_j \mid j = 1, 2, 3, 4, 5\} \). Since \( \langle A_1, C_1 \rangle \cap \langle B_1, D_1 \rangle = \emptyset \), we can easily show that in both the cases there exist four points of \( \{S_i \mid i = 1, 2, 3, 4, 5\} \), say \( S_1, S_2, S_3, S_4 \), such that \( S_i + S_j \in \langle A_1, B_1, C_1, D_1 \rangle, i \neq j, i, j = 1, 2, 3, 4 \). Since \( \langle A_1, B_1, C_1, D_1 \rangle \cap \langle B_1 + B_3, D_1 + B_2 \rangle = \emptyset \), we arrive at a contradiction as soon as we consider the distribution of \( S_1, S_2, S_3 \) and \( S_4 \) among \( \rho^i_2(B_1 + B_3, D_1 + B_2), i = 1, 2, 3 \). This completes the proof.

§3. Non-existence of a pair of lines of \( \mathcal{S} \), one being in \( \mathcal{T}_2 \) and the other in \( \mathcal{T}_3 \).

First of all we prove the following proposition.

**Proposition 6.5.** \( \mathcal{T}_i \) does not contain any line of \( \mathcal{S} \) through \( X_2 + X_3 \), where \( X_i \in \text{St} \_i(P) - \{P\}, \) for \( i = 1, 2, 3 \).

**Proof.** Suppose the lemma is false. WLOG we assume that \( \mathcal{T}_i \) contains a line of \( \mathcal{S} \) through \( A_2 + A_3 \). By proposition
6.4, this line must be \(\{A_2 + A_3, S_1, S_2\}\), for some \(S_1 \in \mathcal{B} \cap \tau'_2 - \Sigma_4\), \(i = 1, 2\) and there exist at least two points of \(S_3\) and \(S_4\) in \(\mathcal{B} \cap \tau'_3 - \Sigma_4\) [\(\therefore \tau'_3\) contains an odd stigm of \(\mathcal{B}\)]. By proposition 6.3, we assume (WLOG) that \(A_1 + A_2 \in \mathcal{B}\). Now, \(A_2 + A_3 \in \mathcal{B} \implies \{X_1, Y_1, Z_1\}\) is a polarising set, \(\forall X_1, Y_1, Z_1 \in \text{St}_i(p) - \{p\}\), \(X_1, Y_1, Z_1\) are all different. [by proposition 6.2].

Since \(S_3 + S_4 \in \Sigma_4\), we can always choose three distinct points \(X_1, Y_1, Z_1\) from \(\text{St}_i(p) - \{p\}\) such that \(S_3 + S_4 \not\in \angle X_1, Y_1, Z_1\) \(\bowtie \Sigma_4\). Also \(S_1 + S_2 = A_2 + A_3 \not\in \angle X_1, Y_1, Z_1\). Considering the distribution of \(S_i\)'s, we assume (WLOG) that \(S_1, S_3 \in \mathcal{P}_i(X_1, Y_1, Z_1)\), for some \(i \in \{1, 2, 3\}\). So, \(S_1 + S_3 \in \{X_1, Y_1, Z_1, X_1 + Y_1 + Z_1\}\), which shows that \(S_2 + S_3 = W_1 + W_2 + W_3\), for some \(W_i \in \text{St}_i(p) - \{p\}, i = 1, 2, 3\). But \(A_1 + A_2 \in \mathcal{B} \implies W_1 + W_2 + W_3\) is an attenuation point with respect to \(\Sigma_5\) [by proposition 6.2]. Thus we arrive at a contradiction.

This completes the proof.

Now we proceed to show that for \(i = 2, 3\), \(\tau'_i\) does not contain any line of \(\mathcal{B}\) through \(X_1 + X_j, j = 2, 3\) and \(X_k \in \text{St}_k(p) - \{p\}, k = 1, 2, 3\). For this purpose, let us prove the following lemma.

**Lemma 6.1.** Let \(\{A_1 + A_2, S_1, S_2\}\) be a line of \(\mathcal{B}\) in \(\tau'_2\), where \(S_1, S_2 \in \tau'_2 - \Sigma_4\). Let \(X_1, Y_1 \in \mathcal{B}_1, C_1, D_1\), \(X_1 \neq Y_1\), and \(S_3, S_4 \in \mathcal{B} \cap \tau'_3 - \Sigma_4\). Then
(i) there exist \( i, j, i \in \{1, 2\}, j \in \{3, 4\} \) such that \( S_i + S_j \in \{X_1, Y_1^3\} \), whenever \( S_3 + S_4 \neq X_1 + Y_1 \);

(ii) \( S_3 + S_4 \in (\lambda_1(p) - \{p_1^3\}) \cup \{Z_1 + B_2 \mid Z_1 \in \{B_1, C_1, D_1^3\}\}; \)

(iii) \( S + S_j \in (\lambda_1(p) - \{p_1^3\}) \cup \{Z_1 + B_2 \mid Z_1 \in \{B_1, C_1, D_1^3\}\}, \)

\( j = 1, 2 \), where \( S \in \mathcal{B} \cap \mathcal{T}_2 - \Sigma_4 \).

**Proof.** (i) Suppose \( S_3 + S_4 \neq X_1 + Y_1 \), for some \( X_1, Y_1 \in \{B_1, C_1, D_1^3\} \). Then we can always choose \( X_3 \in \{A_3, B_3^3\} \), such that \( S_3 + S_4 \notin (X_1, Y_1, X_3) \), i.e. \( S_3 + S_4 \notin (X_1, Y_1, X_3) \cap \Sigma_4 \). By proposition 6.2, \( (X_1, Y_1, X_3) \) is a polarising set with respect to \( \Sigma_5 \). Then \( S_1 + S_2 = A_1 + A_2 \notin (X_1, Y_1, X_3) \) and \( S_3 + S_4 \notin (X_1, Y_1, X_3) \).

So, considering the distribution of \( S_1, S_2, S_3 \) and \( S_4 \) among \( \rho_3^k(X_1, Y_1, X_3) \), \( k = 1, 2, 3 \), we can find \( i, j, i \in \{1, 2\}, j \in \{3, 4\} \) such that \( S_i, S_j \in \rho_3^k(X_1, Y_1, X_3) \), for some \( k \in \{1, 2, 3\} \). \( \therefore S_i + S_j \in (X_1, Y_1, X_3) \) i.e. \( S_i + S_j \in \{X_1, Y_1, X_3, X_1 + Y_1 + X_3^3\} \), since \( S_i + S_j \notin \Sigma_4 \). WLOG we assume \( i = 1, j = 3 \). \( \therefore S_1 + S_3 \in \{X_1, Y_1, X_3, X_1 + Y_1 + X_3^3\} \). But \( S_1 + S_3 \in \{X_3, X_1 + Y_1 + X_3^3\} \implies S_2 + S_3 \in \{A_1 + A_2 + X_3, A_1 + X_1 + Y_1 + A_2 + X_3^3\} \), which contradicts proposition 6.2 [\( \therefore A_1 + A_2 \notin \mathcal{B} \)]. \( \therefore S_1 + S_3 \in \{X_1, Y_1\} \). This completes the proof of the first part.

(ii) Suppose \( S_3 + S_4 = P \). By part (i), we assume (WLOG) \( S_1 + S_3 \in \{B_1, C_1^3\} \). Suppose \( S_1 + S_3 = B_1 \).
Then $S_2 + S_3 = A_1 + B_1 + A_2$ and $S_2 + S_4 = A_1 + B_1 + B_2$. Now
$t(C_1, D_1) \cap <P, A_1, A_2> = \{A_1 + B_2\}$ and
$t(C_1, D_1) \cap <S_2, S_3, S_4> = \{P + S_2\}$; so, $P + S_2 + A_1 + B_2
= A_1 + A_2 + S_2 = S_1 \in t(C_1, D_1)$, which is a contradiction.
So, $S_1 + S_3 \neq B_1$. Similarly, $S_1 + S_3 \neq C_1$. Thus we get a
contradiction. So, $S_3 + S_4 \neq P$. Since $A_1 + A_2 \in \mathbb{S}$, $S_3 + S_4
\neq X_1 + A_2$, $X_1 \in \{B_1, C_1, D_1\}$, and $S_3 + S_4 \neq A_1 + B_2$ [by
proposition 6.2]. Suppose $S_3 + S_4 = A_1 + A_2$. Again by part
(i), we assume (WLOG) that $S_1 + S_3 \in \{B_1, C_1\}$. Suppose
$S_1 + S_3 = B_1$. Then $t(C_1, D_1) \cap <A_1 + A_2, S_1, S_3> = \emptyset$,
which is a contradiction. $\therefore S_1 + S_3 \neq B_1$. Similarly,
$S_1 + S_3 \neq C_1$. Thus we arrive at a contradiction. Hence
$S_3 + S_4 \neq A_1 + A_2$. Suppose $S_3 + S_4 = X_1 + X_3$, for some
$X_i \in St_i(P) - \{P\}, i = 1, 3$. By part (i), we assume
(WLOG) that $S_1 + S_3 \in \{B_1, C_1\}$. Suppose $S_1 + S_3 = B_1$. If
$X_1 \neq B_1$, then $S_2 + S_4 = X_1 + B_1 + A_1 + A_2 + X_3$, which leads
to a contradiction [by proposition 6.2]. Suppose $X_1 = B_1$.
Then $S_2 + S_4 = A_1 + A_2 + X_3$, which contradicts
proposition 6.2. Thus $S_1 + S_3 \neq B_1$. Similarly, $S_1 + S_3 \neq
C_1$. Thus we arrive at a contradiction. Hence $S_3 + S_4 \neq
X_1 + X_3$, $\forall X_i \in St_i(P) - \{P\}, i = 1, 3$. By proposition
6.2, $S_3 + S_4 \neq X_1 + Y_1 + X_2 + X_3$, $\forall X_i, Y_i \in St_i(P) - \{P\}$,
i = 1, 2, 3. Hence $S_3 + S_4 \in $(\lambda_1(P) - \{P\}) \cup \{Z_1 + B_2 | Z_1 \in \{B_1, C_1, D_1\}\} .
(iii) Suppose $S + S_j = P$, for some $j \in \{1, 2, 3\}$. WLOG we assume $j = 1$. $S + S_1 = P \implies S + S_2 = P + A_1 + A_2 = A_1 + B_2$, which contradicts proposition 6.2 [\because A_1 + A_2 \in \mathcal{B}] .

By proposition 6.2, it follows that $S + S_j \notin \{X_1 + A_2, A_1 + X_2\}$, $\forall X_i \in \text{St}_i(P) - \{P_i\}$, $i = 1, 2$, and $j = 1, 2$. Suppose $S + S_j = X_1 + X_2$, for some $X_i \in \text{St}_i(P) - \{P_i\}$ and for some $j \in \{1, 2\}$. WLOG we assume $j = 1$ and

$X_3 = A_3$. $\therefore S + S_1 = X_1 + A_3$. $\therefore S + S_2 = A_1 + X_1 + A_2 + A_3$.

By proposition 6.3, we conclude that $X_1 = A_1$. $\therefore S + S_2 = A_2 + A_3$ and $S + S_1 = A_1 + A_3$. Since $t(S, C_1, D_1) \cap \langle p, A_1, A_2 \rangle = \{A_1 + B_2\}$, it follows that $t(S, C_1, D_1) \cap \langle A_1, A_2, A_3 \rangle = \emptyset$, which is a contradiction. Hence $S + S_j \notin (\lambda_1(P) - \{P_i\} \cup \{Z_1 + B_2 \mid Z_1 \in \{B_1, C_1, \ldots\}\}$. This completes the proof.

Now we have the following proposition.

**Proposition 6.6.** Let $X_i \in \text{St}_i(P) - \{P_i\}$, $i = 1, 2, 3$. Then neither $\mathcal{T}_2$ nor $\mathcal{T}_3$ contains any line of $\mathcal{B}$ through $X_1 + X_j$, $j = 2, 3$.

**Proof:** Suppose the proposition is false. WLOG we assume $\{A_1 + A_2, S_1, S_2\}$ is a line of $\mathcal{B}$ in $\mathcal{T}_2$. By proposition 6.4, $S_1, S_2 \in \mathcal{T}_2 - \Sigma_4$. Since $\mathcal{T}_3$ must contain an odd stigm of $\mathcal{B}$ and $\Sigma_4$ does not contain any odd stigm of $\mathcal{B}$ [by proposition 6.4], there exist at least two points say $S_3$ and $S_4$, in $\mathcal{B} \cap \mathcal{T}_3 - \Sigma_4$. Again by theorem 2.7, $|\mathcal{B} - \Sigma_5| = |\mathcal{B} - \mathcal{T}_i| \geq 5$; so, there exists a point $S$ such that
$S \neq S_i$, $i = 1, 2, 3, 4$ and $S \in \sum - \sum_5$. Then $S \in \tau - \sum_4$, $i = 2$ or 3.

**Case 1.** Let $S \in \tau_2 - \sum_4$. By lemma 6.1, we assume (WLOG) that $S + S_1 = B_1 + B_2$. So, we have $S_1 + S_2 = A_1 + A_2$, $S + S_1 = B_1 + B_2$ and $S + S_2 = C_1 + D_1$. By lemma 6.1, $S_3 + S_4 \in (\Lambda(p) - \{p_3^2\}) \cup \{Z_1 + B_2 | Z_1 \in \{B_1, C_1, D_1\}\}$. Suppose $S_3 + S_4 = B_1 + B_2$. By lemma 6.1, we assume (WLOG) that $S_j + S_3 \in \{C_1, D_1\}$, where $j = 1$ or $2$. If $j = 1$, then $\tau(B_2, A_1, X_1) \cap <B_1 + B_2, S_1, S_3> = \emptyset$, where $X_1 = D_1$ or $C_1$ according as $S_1 + S_3 = C_1$ or $D_1$. This is a contradiction. If $j = 2$, then $S_2 + S_3 \in \{C_1, D_1\}$; so, $S_1 + S_3 \in \{A_1 + C_1 + A_2, A_1 + D_1 + A_2\}$.

$t(A_1, B_1, X_1) \cap <B_1 + B_2, S_1, S_3> = \emptyset$, where $X_1 = C_1$ or $D_1$ according as $S_2 + S_3 = C_1$ or $D_1$. Thus $S_3 + S_4 \neq B_1 + B_2$.

Now, suppose $S_3 + S_4 = C_1 + B_2$. By lemma 6.1, we assume (WLOG) that $S_j + S_3 \in \{B_1, D_1\}$, for some $j \in \{1, 2\}$. Suppose $j = 1$. So, $S_1 + S_3 \in \{B_1, D_1\}$. Suppose $S_1 + S_3 = B_1$. Then $S_1 + S_4 = B_1 + C_1 + B_2$; so,

$t(S_4, P, A_1) \cap <B_2, C_1, D_1> = \emptyset$, which is a contradiction. Suppose $S_1 + S_3 = D_1$; so, $S_1 + S_4 = B_1 + D_1 + B_2$ and hence $t(S_4, P, B_1) \cap <B_2, C_1, D_1> = \emptyset$, which is a contradiction. \[\therefore j \neq 1.\] Hence $j = 2$ i.e. $S_2 + S_3 \in \{B_1, D_1\}$. Suppose $S_2 + S_3 = B_1$. Then $S_2 + S_4 = B_1 + C_1 + B_2$, so that $t(S_4, P, A_1) \cap <B_2, C_1, D_1> = \emptyset$. \[\therefore S_2 + S_3 \neq B_1.\] If $S_2 + S_3 = D_1$, then $S_2 + S_4 = B_1 + D_1 + B_2$, so that
\( t(S_4, P, B_1) \cap \langle B_2, C_1, D_1 \rangle = \emptyset. \therefore S_2 + S_3 \not\in D_1. \) Thus we arrive at a contradiction. Hence \( S_3 + S_4 \not\in C_1 + B_2. \)

Similarly \( S_3 + S_4 \not\in D_1 + B_2. \) Thus we get that \( S_3 + S_4 \in \lambda_1(P) - \{P\}. \) Suppose \( S_3 + S_4 \in C_1 + D_1. \) Also we have \( S + S_2 = C_1 + D_1. \) By lemma 6.2, we can assume (w.l.o.g.) that \( S_3 + S_2 \in \{B_1, C_1, D_1\} \) for some \( j \in \{1, 2\}. \) If \( j = 2, \) then \( t(A_1, X_1) \cap \langle C_1 + D_1, S_2, S_3 \rangle = \emptyset, \) where \( X_1 = C_1 \) or \( B_1 \) according as \( S_3 + S_2 = B_1 \) or \( C_1. \) If \( j = 1, \) then \( S_1 + S_3 \in \{B_1, C_1\}; \) so, \( S_2 + S_3 \in \{A_1 + B_1 + A_2, A_1 + C_1 + A_2\}. \) Then \( t(B_1, X_1) \cap \langle C_1 + D_1, S_2, S_3 \rangle = \emptyset, \) where \( X_1 = A_1 \) or \( D_1 \) according as \( S_2 + S_3 = A_1 + B_1 + A_2 \) or \( A_1 + C_1 + A_2, \) which is a contradiction. \( \therefore S_3 + S_4 \in \lambda_1(P) - \{P, C_1 + D_1\}. \) WLOG we assume \( S_3 + S_4 \in \{A_1 + B_1, A_1 + C_1, B_1 + C_1\}. \) By lemma 6.1, we assume (w.l.o.g.) \( S_j + S_3 \in \{B_1, D_1\}, \) for some \( j \in \{1, 2\}. \) Let \( \triangle \) be a polarising set. Let us consider the following table.
For each value of \( S_j + S_3 \), \( j \in \{1, 2, 3, 4\} \), we calculate \( S_u + S_v \) and \( S + S_u \), \( u \neq v \) and \( u, v = 1, 2, 3, 4 \). Then from the above table, we arrive at a contradiction, when we consider the distribution of \( S \) and \( S_j \)'s among \( P^i(\Delta) \), \( i = 1, 2, 3 \).

\[ \therefore \text{we get } S_2 + S_3 = B_1. \quad \therefore S_2 + S_4 = C_1 \quad \text{and } S_1 + S_3 = A_1 + B_1 + A_2. \quad \text{Now } t(P, D_1, A_2) \cap \langle S_2, S_3, S_4 \rangle = \{S_3 + C_1\}. \quad \text{So,} \]

\[ S_3 + C_1 + P + D_1 + A_2 = S_3 + A_1 + B_1 + A_2 = S_1 \in t(P, D_1, A_2), \]

which is a contradiction. Hence case 1 cannot happen.

**Case 2.** Let \( S \in C_3^1 - \Sigma_4 \). By lemma 6.1, \( S_3 + S_4 \), \( S + S_3 \) and \( S + S_4 \) are all in

\[ (\lambda_1(P) - \{P\}) \cup \{z_1 + B_2 \mid z_1 \in \{B_1, C_1, D_1\} \}. \quad \text{Suppose} \]

\[ S_3 + S_4 \in \{z_1 + B_2 \mid z_1 \in \{B_1, C_1, D_1\} \}. \quad \text{WLOG we assume} \]
\[ S + S_3 = B_1 + B_2. \] Then we assume (WLOG) \[ S + S_3 = C_1 + B_2. \] So, \( t(S_3, A_1, D_1) \cap < B_2, B_1, C_1 > = \emptyset, \) which is a contradiction. Similarly, \( S + S_j \neq Z_1 + B_2, \forall Z_1 \in \{ B_1, C_1, D_1 \} \) and \( \forall j \in \{ 3, 4 \} \). Thus \( S + S_3, S + S_3 \) and \( S + S_4 \) are all in \( \lambda_1(p) - \{ p \} \). Since we can interchange \( B_1, C_1 \) and \( D_1 \), it is enough to consider the following cases:

(a) \( S_3 + S_4 = A_1 + B_1, S_3 + S = A_1 + C_1 \).

(b) \( S_3 + S_4 = B_1 + C_1, S_3 + S = B_1 + D_1 \).

**Case (a).** Let \( S_3 + S_4 = A_1 + B_1 \) and \( S_3 + S = A_1 + C_1 \). So, \( S_4 + S = B_1 + C_1. \) By lemma 6.1, we assume (WLOG) that \( S_1 + S_j \in \{ B_1, D_1 \} \), for some \( j \in \{ 3, 4 \} \). Suppose \( j = 3 \). \( S_3 + S_3 = B_1 \implies S + S = P + D_1 \implies S_2 + S = A_1 + D_1 + B_2. \) But \( t(B_1, D_1) \cap < P, A_1, A_2 > = \{ A_1 + B_2 \} \), whence it follows that \( A_1 + D_1 + B_2 \) is an attenuation point with respect to \( \Sigma_5 \). So we get a contradiction. \( \therefore S_1 + S_3 \neq B_1. \) Now, \( S_1 + S_3 = D_1 \implies S_1 + S = P + B_1 \implies S_2 + S = A_1 + B_1 + B_2. \) But \( t(B_1, D_1) \cap < P, A_1, A_2 > = \{ A_1 + B_2 \} \), so that \( A_1 + B_1 + B_2 \) is an attenuation point with respect to \( \Sigma_5 \). \( \therefore S_1 + S_3 \neq D_1. \) Thus we get a contradiction. \( \therefore j \neq 3. \) Suppose \( j = 4 \). Now \( S_1 + S_4 = B_1 \implies S_1 + S_3 = A_1 \implies t(C_1, D_1) \cap < S_1, S_3, S_4 > = \emptyset [\because B_1 + A_2 \in t(C_1, D_1)]; \) and \( S_1 + S_4 = D_1 \implies S_1 + S_4 = P + C_1 \implies S_2 + S_4 = A_1 + C_1 + B_2, \) but \( A_1 + C_1 + B_2 \) is an attenuation point with respect to \( \Sigma_5 \). \( \therefore t(C_1, D_1) \cap < P, A_1, A_2 > = \{ A_1 + B_2 \}. \) \( \therefore \neq 4. \)
Thus we arrive at a contradiction. So, case (a) cannot happen.

Case (b). Let $S_3 + S_4 = B_1 + C_1$, $S_3 + S = B_1 + D_1$. So, $S_4 + S = C_1 + D_1$. By lemma 6.1, we assume (WLOG) that $S_1 + S_j \in \{C_1, D_1\}$, for some $j \in \{3, 4\}$. Suppose $j = 3$. $S_1 + S_3 \in \{C_1, D_1\}$. $S_1 + S_3 = C_1 \implies S_1 + S = P + A_1$ and $S_1 + S_4 = B_1 \implies t(P, D_1) \cap <S_1, B_1, C_1> = \emptyset$, which is a contradiction. $S_1 + S_3 \not\in C_1$. Again, $S_1 + S_3 = D_1 \implies S_1 + S = B_1$ and $S_1 + S_4 = P + A_1 \implies t(P, C_1) \cap <S_1, B_1, D_1> = \emptyset$, which is a contradiction. $S_1 + S_3 \not\in \{C_1, D_1\}$, which is a contradiction.

So, case (b) is not possible. So, case 2 cannot happen.

Thus the proof is completed.

We recall that $St_i$ is an odd stigm of $\varnothing$ in $\tau'_1$, $i = 2, 3$. Let us now prove the following proposition.

Proposition 6.7. $St_2$ and $St_3$ are not both lines of $\varnothing$.

Proof. Suppose $St_2$ and $St_3$ are both lines of $\varnothing$. Then by propositions 6.3, 6.4, 6.5 and 6.7, we conclude that $St_i \cap \Sigma_4 = \{P_i\}$, $i = 2, 3$. Let $St_i = \{P, S_i, S_4\}$ and $St_3 = \{P, S_3, S_4\}$. By virtue of proposition 6.3 we assume (WLOG) that $A_1 + A_2 \in \varnothing$. By proposition 6.2, $\{B_1, C_1, A_2\}$
is a polarising set with respect to $\Sigma_j$. Considering the distribution of $S_i$'s among $\rho_j^3(B_1, C_1, A_3)$, $j = 1, 2, 3$, we can assume (WLOG) that $S_1, S_3 \in \rho_j^3(B_1, C_1, A_3)$, for some $j \in \{1, 2, 3\}$. \therefore S_1 + S_3 \in \rho_j^3(B_1, C_1, A_3, B_1 + C_1 + A_3).$

Now we consider the following table.

<table>
<thead>
<tr>
<th>$S_1 + S_3$</th>
<th>$\Sigma_2$</th>
<th>$\Theta$</th>
<th>$\Theta \cap \Sigma_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_1$</td>
<td>$&lt;p, S_1, S_3&gt;$</td>
<td>$t(C_1, D_1)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$C_1$</td>
<td>$&lt;p, S_1, S_3&gt;$</td>
<td>$t(A_1, B_1)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$A_3$</td>
<td>$&lt;p, S_1, S_3&gt;$</td>
<td>$t(A_1)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$B_1 + C_1 + A_3$</td>
<td>$&lt;p, S_1, S_3&gt;$</td>
<td>$t(A_1, D_1)$</td>
<td>$\emptyset$</td>
</tr>
</tbody>
</table>

From the above table, we arrive at a contradiction. This completes the proof.

§4. Non-existence of odd stigm $\text{St}_i$ in $\tau_j$ with $|\text{St}_i| = 5$ and $|\text{St}_i \cap \Sigma_j| = 1$, $i = 2, 3$.

We now proceed to show that neither $\tau_2$ nor $\tau_3$ can contain a 5-stigm of the form $\{Q, S_1, S_2, S_3, S_4\}$, where $Q \in \mathcal{B} \cap \Sigma_4$ and $S_i \in \mathcal{B} \cap \tau_i - \Sigma_4$, $i = 1, 2, 3, 4$ and $j \in \{2, 3\}$. For this purpose we need the following results.

**Lemma 6.2.** Let $\text{St}_i = \{p, S_1, S_2, S_3, S_4\}$, where $S_j \in \mathcal{B} \cap \tau_i - \Sigma_j$, $j = 1, 2, 3, 4$, $i \in \{2, 3\}$. Then $<S_1, S_2, S_3, S_4> \cap \lambda_1(p) = \lambda_1(p)$.

**Proof.** By virtue of proposition 6.4 we assume (WLOG) that $A_1 + A_2 \in \mathcal{B}$. So, $\{X_1, Y_1, X_3\}$ is a polarising set with respect to $\Sigma_5$, where $X_1, Y_1 \in \{B_1, C_1, D_1\}$, $X_1 \neq Y_1$, $X_3 \in \{A_3, B_3\}$. First of all, we assert that
\(<S_1, S_2, S_3, S_4> \cap \lambda_1(P)\) contains a line of \(\lambda_1(P)\) through \(P\). Consider the distribution of \(S_1, S_2, S_3\) and \(S_4\) among \(\rho_i^3(B_1, C_1, A_3), i = 1, 2, 3\), and rearranging the suffixes of \(S\)'s, we can assume (WLOG) that \(S_1, S_2 \in \rho_i^3(B_1, C_1, A_3), \text{ for some } i \in \{1, 2, 3\}\). \(\because S_1 + S_2 \in \{B_1 + C_1, B_1 + A_3, C_1 + A_3\}\). If \(S_1 + S_2 = B_1 + C_1\), then our assertion is justified. So, we assume \(S_1 + S_2 = B_1 + A_3\).

So, \(S_3 + S_4 = B_1 + B_3\). Since \(S_1 + S_2\) and \(S_3 + S_4 \notin \langle C_1, D_1, A_3 \rangle\) and we can interchange \(S_1\) and \(S_2\) as well as \(S_3\) and \(S_4\), we assume (WLOG) that \(S_1, S_3 \in \rho_i^3(C_1, D_1, A_3), \text{ for some } i \in \{1, 2, 3\}\). \(\because S_1 + S_3 \in \{C_1 + D_1, C_1 + A_3, D_1 + A_3\}\). If \(S_1 + S_3 = C_1 + D_1\), then our assertion is justified. If \(S_1 + S_3 \notin \{C_1 + A_3, D_1 + A_3\}\), then \(S_2 + S_3 \in \{B_1 + C_1, B_1 + D_1\}\) \(\because S_1 + S_2 = B_1 + B_3\); so, our assertion is justified. Similarly, if \(S_1 + S_2 = C_1 + A_3\), then we can show that our assertion is justified. Thus \(<S_1, S_2, S_3, S_4> \cap \lambda_1(P)\) contains a line of \(\lambda_1(P)\) through \(P\). WLOG we can assume that this line be \(\{P, A_1 + B_1, C_1 + D_1\}\) \(\because \text{ any line of } \lambda_1(P)\) through \(P\) must contain a point of the form \(A_1 + X_1, X_1 \in \{B_1, C_1, D_1\}\), and we can interchange \(B_1, C_1\) and \(D_1\). Rearranging the suffixes of \(S_i\)'s, we assume (WLOG) that \(S_1 + S_2 = A_1 + B_1\) and \(S_3 + S_4 = C_1 + D_1\). Now considering the distribution of \(S_1, S_2, S_3\) and \(S_4\) among \(\rho_i^3(B_1, C_1, A_3), i = 1, 2, 3\), we can assume (WLOG) that \(S_1, S_3 \in \rho_i^3(B_1, C_1, A_3), i = 1, 2, 3\), and we can interchange \(B_1, C_1\) and \(D_1\).
Lemma 6.3. Let \( \mathcal{S}_t = \{\mathcal{Q}, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4\} \), where \( \mathcal{Q} \in \mathcal{P} \cap \Sigma_4 \), \( \mathcal{S}_j \in \mathcal{P} \cap \Sigma_4 \), \( j = 1, 2, 3, 4, i = 2, 3 \). Then \( \mathcal{Q} \neq \mathcal{P} \).

**Proof.** Suppose \( \mathcal{Q} = \mathcal{P} \). Then by lemma 6.3 and by arguments used in the proof of theorem 2.9, we can assume (WLOG) that \( \mathcal{S}_1 + \mathcal{S}_2 = \mathcal{A}_1 + \mathcal{B}_1 \), \( \mathcal{S}_3 + \mathcal{S}_4 = \mathcal{C}_1 + \mathcal{D}_1 \), \( \mathcal{S}_1 + \mathcal{S}_3 = \mathcal{A}_1 + \mathcal{C}_1 \), \( \mathcal{S}_2 + \mathcal{S}_4 = \mathcal{B}_1 + \mathcal{D}_1 \), \( \mathcal{S}_1 + \mathcal{S}_4 = \mathcal{A}_1 + \mathcal{D}_1 \), and \( \mathcal{S}_2 + \mathcal{S}_3 = \mathcal{B}_1 + \mathcal{C}_1 \). \( \mathcal{T}(\mathcal{P}, \mathcal{S}_4) \cap \langle \mathcal{A}_1, \mathcal{S}_1, \mathcal{S}_2 \rangle = \emptyset \), which is a contradiction. This completes the proof.

Lemma 6.4. Let \( \mathcal{S}_t = \{\mathcal{Q}, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4\} \), where \( \mathcal{Q} \in \mathcal{P} \cap \Sigma_4 \), \( \mathcal{S}_j \in \mathcal{P} \cap \Sigma_4 \), \( j = 1, 2, 3, 4, i \in \{2, 3\} \). Then \( \mathcal{Q} \neq \mathcal{X}_1 + \mathcal{X}_j \), \( \mathcal{X}_i \in \mathcal{P}_t(\mathcal{P}) - \{\mathcal{P}\}, i = 1, 2, 3, j = 2, 3 \).

**Proof.** Suppose the lemma is false. Then we assume (WLOG) that \( \mathcal{Q} = \mathcal{A}_1 + \mathcal{A}_2 \). Now \( \langle \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4 \rangle \cap \lambda_1(\mathcal{P}) \) must contain a point \( \mathcal{Z} \) of \( \lambda_1(\mathcal{P}) \). Assert \( \mathcal{Z} \neq \mathcal{P}, \mathcal{A}_1 + \mathcal{X}_1 \), \( \mathcal{X}_1 \in \{\mathcal{B}_1, \mathcal{C}_1, \mathcal{D}_1\} \). If \( \mathcal{Z} = \mathcal{P} \), then (WLOG) we can assume that \( \mathcal{S}_1 + \mathcal{S}_2 = \mathcal{P} \); so, \( \mathcal{S}_3 + \mathcal{S}_4 = \mathcal{P} + \mathcal{A}_1 + \mathcal{A}_2 = \mathcal{A}_1 + \mathcal{B}_2 \), which contradicts proposition 6.2 [\( \vdash \mathcal{A}_1 + \mathcal{A}_2 \in \mathcal{P} \)]. Suppose
$Z = A_1 + B_1$; then we assume (WLOG) that $S_1 + S_2 = A_1 + B_1$, so that $S_3 + S_4 = B_1 + A_2$, which again contradicts proposition 6.2. Since we can interchange $B_1, C_1$ and $D_1$, we get that $S_1 + S_2 \neq A_1 + X_1$, $X_1 \in \{B_1, C_1, D_1\}$ and we assume (WLOG) that $Z = B_1 + C_1$. We assume (WLOG) that $S_1 + S_2 = B_1 + C_1$. So, $S_3 + S_4 = D_1 + B_2$. Considering the distribution of $S_1, S_2, S_3$ and $S$ among $P_i(B_1, D_1, A_3)$, $i = 1, 2, 3$, we assume (WLOG) that $S_1, S_2 \in P_i(B_1, D_1, A_3)$, for some $i \in \{1, 2, 3\}$. Thus, $S_1 + S_3 \in \{B_1 + D_1, B_1 + A_3, D_1 + A_3\}$. But $S_1 + S_3 = B_1 + A_3 \implies S_1 + S_4 = B_1 + D_1 + B_2 + A_3$ and $S_1 + S_3 = D_1 + A_3 \implies S_1 + S_4 = B_2 + A_3 \implies S_2 + S_4 = B_1 + C_1 + B_2 + A_3$; so, in both cases we get a contradiction [by proposition 6.3]. Again, $S_1 + S_3 = B_1 + D_1 \implies S_1 + S_4 = B_1 + B_2 \implies t(A_1, C_1, S_4) \cap \langle B_2, B_1, D_1 \rangle = \emptyset$ [$\vdots$ $S_3 + S_4 = D_1 + B_2$]. Thus we arrive at a contradiction. This completes the proof.

Lemma 6.5. Let $S_{i} = \{Q, S_1, S_2, S_3, S_4\}$, where $Q \in \emptyset \cap \Sigma_{4}, S_1 \in \emptyset \cap \Sigma_{4}^{i} - \Sigma_{4}, j = 1, 2, 3, 4$ and $i \in \{2, 3\}$. Then $Q \neq X_2 + X_3$, where $X_i \in \{A_i, B_i\}^i, i = 2, 3$.

Proof. Suppose the lemma is false. WLOG we assume that $Q = A_2 + A_3$. Now $\langle S_1, S_2, S_3, S_4 \rangle \cap \lambda_1(P)$ contains at least a point $Z$ of $\lambda_1(P)$. WLOG we assume that $S_1 + S_2 = Z$. Now $S_1 + S_2 = Z = P \implies S_3 + S_4 = A_2 + B_3$, which contradicts proposition 6.2. So, $S_3 + S_4 = Z + A_2 + A_3$, where $Z \in \lambda_1(P) - \{P_3\}$, which again contradicts
proposition 6.2 or proposition 6.3. This completes the proof.

Now we have the following proposition.

**Proposition 6.8.** If \( S^I \) be a 5-stigm in \( \mathcal{T}^I \), then 
\[
|S^I \cap \Sigma_4| = 3, \ i = 2, 3.
\]

**Proof.** By virtue of proposition 6.3 and lemmas 6.2, 6.3 and 6.4, we conclude that \( |S^I \cap \Sigma_4| \neq 1 \). By 6.4, it follows clearly that \( |S^I \cap \Sigma_4| = 3 \).

\( \phi \) 5. **Non-existence of a pair of odd stigms**

\( S^I \) of \( \mathcal{P} \) in \( \mathcal{T}^I \), \( i = 1, 2 \) with one as a 5-stigm and the other as a 3-stigm.

First of all we prove the following lemmas.

**Lemma 6.6.** Let \( S^I \) be a 5-stigm in \( \mathcal{T}^I \), such that \( P \in S^I \), \( i \in \{2, 3\} \). Then \( S^I \cap \Sigma_4 \subset \mathcal{T}^I(P) \), \( j = 2 \) or 3.

**Proof.** Suppose \( S^I \) be a 5-stigm in \( \mathcal{T}^I \), such that \( P \in S^I \). By proposition 6.8, \( |S^I \cap \Sigma_4| = 3 \). So, by proposition 6.2, it follows easily that \( S^I \cap \Sigma_4 \cap \mathcal{T}^I(P) - \{P\} \neq \emptyset \), for some \( j \in \{2, 3\} \). WLOG we assume \( j = 3 \). We assume (WLOG) that \( A_1 + A_2 \in S^I \). We now want to show that \( S^I \cap \Sigma_4 \subset \mathcal{T}_3(P) \). Let \( S^I \cap \Sigma_4 \subset \mathcal{T}_3(P) \). Now if \( Z = X_1 + X_3, X_i \in \mathcal{T}_i(P) - \{P\}, i = 1, 3 \), then \( S_1 + S_2 = P + A_1 + A_2 + Z = P + A_1 + A_2 + X_1 + X_3 \), which leads to a contradiction [by proposition 6.2]. If \( Z = X_2 + X_3, X_i \in \mathcal{T}_i(P) - \{P\}, i = 2, 3 \), then \( S_1 + S_2 = P + A_1 + A_2 + X_2 + X_3 = A_1 + Y_3 \), for
some $Y_3 \in \{A_3, B_3\}$, which again violates proposition 6.2. So, by proposition 6.3, we conclude that $Z \in \sum_4 \cap \tau_3(P)$. Hence $S;i \subset \tau_3(P)$. This completes the proof.

**Lemma 6.7.** Let $\{i, j\} = \{2, 3\}$. If $S;i$ be a 5-stigm and $S;j$ a 3-stigm, then $P \notin S;i$.

**Proof.** Suppose $S;i$ be a 5-stigm such that $P \in St_i$ and $S;j$ a 3-stigm. By proposition 6.8, $|St_i \cap \sum_4| = 3$ and by propositions 6.3, 6.4, 6.5 and 6.7, we get $St_i \cap \sum_4 = \{p\}$. By lemma 6.6, $St_i \cap \sum_4 \subset \tau_j(P)$, $j = 2$ or 3. WLOG we assume $St_i \cap \sum_4 \subset \tau_3(P)$. So, by proposition 6.2, we can assume (WLOG) that $St_i = \{P, A_1 + A_2, B_1 + B_2, S_1, S_2, S_3\}$, where $S_1, S_2 \in \tau_2 - \sum_4$. Let $St_3 = \{P, S_3, S_4\}$, where $S_3, S_4 \in \tau_3 - \sum_4$. \[ \]

Let $S_1 + S_2 = A_1 + B_1$ and $S_3 + S_4 = P$. Now $A_1 + A_2, B_1 + B_2 \in \sum_4 \implies \{X_1, Y_1, X_3\}$ is a polarising set with respect to $\sum_5$, where $X_1 \neq Y_1, X_1, Y_1 \in \{B_1, C_1, D_3\}$ or $X_1, Y_1 \in \{A_1, C_1, D_3\}$ and $X_3 \in \{A_3, B_3\}$. Considering the distribution of $S_1, S_2, S_3$ and $S_4$ among $\rho_3(B_1, C_1, A_3)$, $i = 1, 2, 3$, we assume (WLOG) that $S_1, S_3 \in \rho_3(B_1, C_1, A_3)$, for some $i \in \{1, 2, 3\}$. \[ \]

Let us consider the following table.
From the table, it follows that $S_1 + S_3 = C_1$. \therefore $S_1 + S_4 = P + C_1$, $S_2 + S_3 = P + D_1$ and $S_2 + S_4 = D_1$. We assert that neither $P + C_1$ nor $P + D_1$ is a point of $\mathcal{B}$. Suppose $P + C_1 \in \mathcal{B}$. Then $t(B_1, B_2) \cap <P, C_1, D_1> = \{P + C_1\}$, whence it follows that $P + D_1$ is an attenuation point with respect to $\Sigma_5$. But $S_2 + S_3 = P + D_1$. This is a contradiction. If $P + D_1 \in \mathcal{B}$, then $t(B_1, B_2) \cap <P, C_1, D_1> = \{P + D_1\}$; since $S_1 + S_4 = P + C_1$, we again arrive at a contradiction. Since $St_1$ is a line of $\mathcal{B}$ in $\mathcal{T}_3$, there does not exist any line of $\mathcal{B}$ through $S_1$ in $\mathcal{T}_2$, $i \in \{1, 2\}$. Also by theorems 3.1, 4.1 and 5.1, there exist two lines of $\mathcal{B}$ through each point of $\mathcal{B}$. So, there exist at least two lines of $\mathcal{B}$ through $S_1$, one of which is $\{C_1, S_1, S_3\}$. Since $S_1 + S_4 = P + C_1 \notin \mathcal{B}$, we conclude that there exists a point $S$ of $\mathcal{B}$ such that $S_1 + S \in \Sigma_5 \cap \mathcal{B} - \Sigma_4$. It follows that $S \in \mathcal{B} \cap \mathcal{T}_3 - \Sigma_4$. We assert that $S + S_j \notin X_1 + X_3$, $X_i \in St_i(P) - \{P\}$, $i = 1, 3$, $j = 3, 4$. Suppose $S + S_3 = A_1 + A_3$. Then $S + S_4 = A_1 + B_3$; so, $t(S, C_1, D_1) \cap <A_1, A_3, B_3> = \emptyset$, which is a contradiction. Similarly, we can show that
$S + S_j \notin X_1 + X_3, \forall X_i \in St_1(p) - \{p_i\}, i = 1, 3,$ and $j = 3, 4$. Now by proposition 6.2, it is easy to check that $S + S_j \in \lambda_1(p) - \{p_j\}, j = 3, 4$. Also, we assert $S + S_j \notin \{A_1 + B_1, C_1 + D_1\}$. Suppose $S + S_3 = A_1 + B_1$; then $t(A_1, D_1) \cap \langle A_1 + B_1, S_1, S_3 \rangle = \emptyset$, which is a contradiction. Similarly, $S + S_4 \notin A_1 + B_1$. Now $S + S_3 = C_1 + D_1 \Longrightarrow S + S_4 = A_1 + B_1$, which is not possible. Similarly, $S + S_4 \notin C_1 + D_1$. Thus our assertion is justified. Since we can interchange $C_1$ and $D_1$, we assume (WLOG) that $S + S_3 = A_1 + D_1$ and $S + S_4 = B_1 + C_1$. But $S_1 + S_3 = C_1$; so, $S + S_1 = P + B_1$. Thus we arrive at a contradiction as soon as we consider the distribution of $S_1, S_3, S_4$ and $S$ among $\mathcal{P}_3(B_1, D_1, A_3), i = 1, 2, 3$. This completes the proof of the lemma.

**Proposition 6.9.** Let $\{i, j\} = \{2, 3\}$. If $St_i^j$ be a 5-stigm, then $St_i^j$ cannot be a 3-stigm.

**Proof.** Suppose $St_2^i$ be a 5-stigm and $St_3^i$ a 3-stigm. Then $St_3^i \cap \Sigma_4 = \{p_i\}$ and by proposition 6.8 and lemma 6.7, $|St_2^i \cap \Sigma_4| = 3$ and $P \notin St_2^i$. By virtue of proposition 6.2, it is easy to check that $St_2^i \cap \tau_1(p) = \emptyset$; so, we assume (WLOG) that $St_2^i = \{A_1 + A_2, B_1 + B_2, X_1 + A_3, S_1, S_2\}$, for some $X_1 \in St_1(p) - \{p_1\}$, and $S_1, S_2 \in \tau_2 - \Sigma_4$.

Since $S_1 + S_2 = A_1 + B_1 + X_1 + B_3, X_1 \in \{A_1, B_1\}$; for, $S_1 + S_2 = D_1 + A_3$ or $C_1 + A_3$ according as $X_1 = C_1$ or $D_1$, which contradicts proposition 6.2 [$\because X_1 + A_3 \notin \emptyset$]. WLOG we
assume that $X_1 = A_1$. \[\therefore St'_2 = \{A_1 + A_2, B_1 + B_2, A_1 + A_3, S_1, S_2\text{ and } S_1 + S_2 = B_1 + B_3.\]
Let $St'_3 = \{P, S_3, S_4\}$, $S_3, S_4 \in \tau_3 - \{A_2\}$. Now $A_1 + A_2$, $A_1 + A_3 \in \mathcal{B} \implies \{X_1, Y_1\}$ is a polarising set with respect to $\sum$, $X_1, Y_1 \in \{B_1, C_1, D_1\}$, $X_1 \neq Y_1$. Considering the distribution of $S_1, S_2, S_3$ and $S_4$ among $\rho_i(C_1, D_1)$, $i = 1, 2, 3$, we assume (WLOG) that $S_1, S_3 \in \rho_i(C_1, D_1)$, for some $i \in \{1, 2, 3\}$, \[\therefore S_1 + S_3 \in \{C_1, D_1\}.\]
Suppose $S_1 + S_3 = C_1$. Then $S_1 + S_4 = P + C_1$, $S_2 + S_3 = B_1 + C_1 + B_3$,
$S_2 + S_4 = B_1 + C_1 + A_3$. Now we get a contradiction by considering the distribution of $S_1, S_2, S_3$ and $S_4$ among $\rho_i(B_1, D_1)$, $i = 1, 2, 3$. \[\therefore S_1 + S_3 \not\in C_1.\] Similarly, $S_1 + S_3 \not\in D_1$. Thus we get a contradiction. This completes the proof.

§6. Non-existence of a pair of 5-stigms, one being in $\tau'_2$ and the other in $\tau'_3$.

We now want to show that both $St'_2$ and $St'_3$ are not 5-stigms. In this connection we prove the following results.

Lemma 6.8. Let $\{i, j\} = \{2, 3\}$ and let $St'_2$ and $St'_3$ be both 5-stigms. If $P \not\in St'_i$, then $P \in St'_j$.

Proof. Suppose $St'_2$ and $St'_3$ are both 5-stigms and $P \not\in St'_2$. Then by proposition 6.8, $|St'_k \cap \sum_k| = 3$, $k = 2, 3$. But $P \not\in St'_2$; so, as in the proof of proposition 6.9, we can assume (WLOG) that $St'_2 = \}$
\[ \{A_1 + A_2, B_1 + B_2, A_1 + A_3, S_1, S_2, S_1, S_2 \in \tau_2 - \sum_4. \]

\[ \therefore S_1 + S_2 = B_1 + B_3; \text{ so, by proposition 6.6, } B_1 + B_3 \notin \sum. \]

Also \( X_1 + B_3 \notin \sum \), \( X_1 \notin \{ A_1, C_1, D_1 \} \), for otherwise \( B_1 + B_3 \) would become an attenuation point with respect to \( \sum_5 \). So, by proposition 6.2, we conclude that the only points of \( \sum \) in \( \sum_4 \) are \( P, A_1 + A_2, B_1 + B_2 \) and \( A_1 + A_3 \). Now, suppose \( P \notin \sum_3 \). Then clearly, \( \sum_3 = \{A_1 + A_2, B_1 + B_2, A_1 + A_3, S_3, S_4 \} \), \( S_3, S_4 \in \tau_3 - \sum_4 \). \[ \therefore S_3 + S_4 = B_1 + B_3. \]

Considering the distribution of \( S_1, S_2, S_3 \) and \( S_4 \) among \( \rho^i(\mathcal{C}_1, D_1) \), \( i = 1, 2, 3 \), we assume (WLOG) that \( S_1, S_3 \in \rho^i(\mathcal{C}_1, D_1) \), for some \( i \in \{1, 2, 3\} \). \[ \therefore S_1 + S_3 \in \{C_1, D_1\}. \]

Then \( t(A_1, X_1, B_3) \cap <B_1 + B_3, S_1, S_3> = \emptyset \), where \( X_1 = D_1 \) or \( C_1 \) according as \( S_1 + S_3 = C_1 \) or \( D_1 \).

Thus we arrive at a contradiction. So, \( P \in \sum_3 \). This completes the proof.

**Lemma 6.9.** If \( \sum_1, i = 2, 3 \), are both 5-stigms, then \( P \in \sum_1, i = 2, 3 \).

**Proof.** Suppose \( \sum_2 \) and \( \sum_3 \) are both 5-stigms and \( P \notin \sum_2 \). By lemma 6.8 above, we get \( P \in \sum_3 \). As in the proof of lemma 6.8, we can write (WLOG) that \( \sum_2 = \{A_1 + A_2, B_1 + B_2, A_1 + A_3, S_1, S_2, S_1, S_2 \in \tau_2 - \sum_4 \}

\[ \text{and } \sum_3 = \{P, A_1 + A_2, B_1 + B_2, S_3, S_4 \} \], \( S_3, S_4 \in \tau_3 - \sum_4 \).

So, \( S_1 + S_2 = B_1 + B_3 \) and \( S_3 + S_4 = A_1 + B_1 \). Considering the distribution of \( S_1, S_2, S_3 \) and \( S_4 \) among \( \rho^i(\mathcal{C}_1, D_1) \), \( i = \)}
1, 2, 3, we assume (w.l.o.g.) that $S_1$, $S_3 \in \mathcal{P}(C_1, D_1)$, for some $i \in \{1, 2\}$: \( S_1 + S_3 \notin \mathcal{P}(C_1, D_1) \). Now for each value of $S_1 + S_3$, we calculate $S_i + S_j$, $i \neq j$, and $i, j = 1, 2, 3, 4$ and arrive at a contradiction by considering the distribution of $S_i$'s among $\mathcal{P}(B_1, X_1)$, $i = 1, 2, 3$, where $X_1 = D_1$ or $C_1$ according as $S_1 + S_3 = C_1$ or $D_1$. This completes the proof.

**Proposition 6.10.** $S_1^i$ and $S_3^i$ are not both 5-stigms.

**Proof.** Suppose $S_1^i$ and $S_3^i$ are both 5-stigms. By lemma 6.9, $p \in S_1^i$, $i = 2, 3$. By virtue of lemma 6.6, we can assume (w.l.o.g.) that $S_1^i = \{P, A_1 + A_2, B_1 + B_2, S_1, S_2\}$, $S_1, S_2 \in \mathcal{T}_2 \setminus \Sigma_4 \therefore S_1 + S_2 = A_1 + B_1$. We assert that $\Sigma_4 \cap S_3^i \subset \mathcal{T}_2(P)$. If it is not true, then by lemma 6.6, $\Sigma_4 \cap S_3^i \subset \mathcal{T}_3(P)$. So, $S_3^i = \{P, A_1 + A_2, B_1 + B_2, S_1, S_2\}$, $S_1, S_2 \in \mathcal{T}_3 \setminus \Sigma_4$ [by proposition 6.2]. \( \therefore S_3 + S_4 = A_1 + B_1 \). Now considering the distribution of $S_1$, $S_2$, $S_3$ and $S_4$ among $\mathcal{P}(C_1, D_1, 3)$, $i = 1, 2, 3$, we assume (w.l.o.g.) that $S_1, S_3 \in \mathcal{P}(C_1, D_1, 3)$. Also, $S_1 + S_3 = S_3 + S_4 = A_1 + B_1$. We now consider the following table.

<table>
<thead>
<tr>
<th>$S_1 + S_3$</th>
<th>$\sum_2$</th>
<th>$\Theta$</th>
<th>$\Theta \cap \sum_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1$</td>
<td>$\langle A_1 + B_1, S_1, S_3 \rangle$</td>
<td>$t(A_1, D_1)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$D_1$</td>
<td>$\langle A_1 + B_1, S_1, S_3 \rangle$</td>
<td>$t(A_1, C_1)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$A_3$</td>
<td>$\langle A_1 + B_1, S_1, S_3 \rangle$</td>
<td>$t(C_1, D_1)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$C_1 + D_1 + A_3$</td>
<td>$\langle A_1 + B_1, S_1, S_3 \rangle$</td>
<td>$t(C_1, D_1)$</td>
<td>$\emptyset$</td>
</tr>
</tbody>
</table>
From the above table, we arrive at a contradiction. Hence \( \sum_4 \cap \text{St}^!_3 \subset \tau_2(p) \). Let us now consider the following three cases.

1. \( \text{St}^!_3 = \{ p, A_1 + A_3, B_1 + B_3, S_3, S_4 \} \)
2. \( \text{St}^!_3 = \{ p, C_1 + A_3, D_1 + B_3, S_3, S_4 \} \)
3. \( \text{St}^!_3 = \{ p, A_1 + A_3, C_1 + B_3, S_3, S_4 \} \)

where \( S_3, S_4 \in \tau^!_3 - \sum_4 \).

**Case 1.** Suppose \( \text{St}^!_3 = \{ p, A_1 + A_3, B_1 + B_3, S_3, S_4 \} \). Then

\[
S_3 + S_4 = A_1 + B_1 = S_1 + S_2.
\]

Now \( \{ C_1, D_1 \} \) is a polarising set with respect to \( \sum_5 \); so considering the distribution of \( S_1, S_2, S_3, S_4 \) among \( \rho^i_2(C_1, D_1) \), \( i = 1, 2, 3 \), we assume (WLOG) that \( S_1, S_3 \in \rho^i_2(C_1, D_1) \), for some \( i \in \{1, 2, 3\} \). So, \( S_1 + S_3 \in \{ C_1, D_1 \} \). Now using the above table, we arrive at a contradiction. Hence case 1 cannot happen.

**Case 2.** Suppose \( \text{St}^!_3 = \{ p, C_1 + A_3, D_1 + B_3, S_3, S_4 \} \). Then

\[
\{ p, A_1 + A_2, B_1 + B_2, C_1 + A_3, D_1 + B_3 \}
\]
form a 5-stigm of \( \Theta \) in \( \sum_4 \), which violates proposition 6.4. Hence case 2 cannot happen.

**Case 3.** Suppose \( \text{St}^!_3 = \{ p, A_1 + A_3, C_1 + B_3, S_3, S_4 \} \). Then

\[
S_3 + S_4 = A_1 + C_1.
\]

Also we have \( S_1 + S_2 = A_1 + B_1 \). Considering the distribution of \( S_1, S_2, S_3 \) and \( S_4 \) among \( \rho^i_2(C_1, D_1) \), \( i = 1, 2, 3 \), we assume (WLOG) that \( S_1, S_3 \in \rho^i_2(C_1, D_1) \), for some \( i \in \{1, 2, 3\} \). Then \( S_1 + S_3 \in \{ C_1, D_1 \} \). Suppose \( S_1 + S_3 = D_1 \). Then \( S_2 + S_3 = p + C_1 \), \( S_2 + S_4 = p + A_1 \),
$S_1 + S_4 = P + B_1$. Now considering the distribution of $S_1$, $S_2$, $S_3$ and $S_4$ among $\rho_i(B_1, C_1)$, $i = 1, 2, 3$, we readily arrive at a contradiction. So, $S_1 + S_3 \neq D_1$.

$\therefore S_1 + S_3 = C_1$.

By theorem 2.7, we have $|\mathcal{B} - \Sigma_5| \geq 5$. So, there exists a point $S$ such that $S \in \mathcal{B} - \Sigma_5$ and $S \notin S_i$, $i = 1, 2, 3, 4$. WLOG we assume that $S \in \mathcal{C}_2 - \Sigma_4$. Also $S_1, S_2 \in \mathcal{C}_2 - \Sigma_4$. By virtue of propositions 6.2, 6.3 and 6.9, we conclude that $S + S_i \in \lambda_1(p) - \{p_i^2\}, i = 1, 2$. Clearly $S + S_i \neq A_1 + B_1$. Assert that $S + S_i \neq A_1 + C_1, i = 1, 2$. Suppose $S + S_1 = A_1 + C_1$. Also we have $S_1 + S_3 = C_1$. $\therefore S + S_3 = A_1$ and $S_1 + S_4 = A_1$. $\therefore t(B_1, D_1) \cap \langle A_1, S_1, S_3 \rangle = \emptyset$, which is a contradiction.

$\therefore S_1 + S \neq A_1 + C_1$. Suppose $S_2 + S = A_1 + C_1$. Since $S_1 + S_3 = C_1$, $S_2 + S_3 = P + D_1$, $S + S_3 = B_1$. But $S_2 + S = A_1 + C_1 = S_3 + S_4$; so, $S + S_3 = B_1 = S_2 + S_4$. $\therefore t(C_1, D_1) \cap \langle B_1, S_3, S_4 \rangle = \emptyset$, which is a contradiction.

$\therefore S_2 + S \neq A_1 + C_1$. Thus we get that $S + S_i \in \lambda_1(p) - \{p, A_1 + B_1, A_1 + C_1\}$. Also, $S + S_1 \neq C_1 + D_1$, for otherwise $S + S_2 = P$, which contradicts proposition 6.9. Similarly, $S + S_i \neq C_1 + D_1, i = 1, 2$. Since $S + S_i \neq A_1 + C_1, i = 1, 2$, it follows that $S + S_i \neq B_1 + C_1, i = 1, 2$. Hence we can assume (WLOG) that $S + S_1 = A_1 + D_1$ and $S + S_2 = B_1 + D_1$. Since $S_1 + S_3 = C_1$, it follows that $S + S_3 = P + B_1$ and $S_2 + S_3 = P + D_1$. $\therefore$
t(\(P, A_1\)) \(\cap <S_3, P + B_1, P + D_1> = \emptyset\), which is a contradiction. Hence case 3 is also not possible. Now, following the type of arguments used in case 3, we can disprove any other case that is not explicitly covered by the above three cases. This completes the proof.

§7 Non-existence of \((5, 3, 3)\)-tangential stigm system.

Let us prove the following proposition.

Proposition 6.11. None of \(St_2\) and \(St_3\) is a 7-stigm.

Proof. Suppose one of \(St_2\) and \(St_3\) be a 7-stigm. WLOG we assume \(St_2\) to be a 7-stigm. Let \(Q \in St_2 \cap \Sigma_4\). If \(Q = P\), then there exist two lines of \(\mathcal{S}\) through \(P\), which are contained in \(\mathcal{T}_1\); so, we get a contradiction [by theorem 2.8]. Suppose \(Q \neq P\). By theorem 3.1, 4.1 and 5.1, we get that there are two lines of \(\mathcal{S}\) through \(Q\). But by propositions 6.5 and 6.6, we conclude that these two lines of \(\mathcal{S}\) are in \(\mathcal{T}_1\). So, by theorem 2.8, we again arrive at a contradiction. This completes the proof.

Let us now state and prove the main result of this chapter.

Theorem 6.1. Let \(\mathcal{B}\) be a 6-dimensional tangential 2-block in \(PG(6, 2)\) and let \(P \in \mathcal{B}\). Then \(P\) cannot have a \((5, 3, 3)\)-tangential stigm system.

Proof. Suppose \(P\) has a \((5, 3, 3)\)-tangential stigm system. Let \(St_1(P) = \{P, A_1, B_1, C_1, D_1\}\) and \(St_2(P) = \{P, A_2, B_2, C_2, D_2\}\). Next, consider the set \(t(P, A_1) \cap \Sigma_4\). If this set is empty, then \(P\) cannot have a \((5, 3, 3)\)-tangential stigm system.
\{p, A_1, B_1, i = 2, 3. Then the space generated by
St_i(p), i = 1, 2, 3, is a 5-space denoted by \( \Sigma_5 \)
[by proposition 6.1]. Also

\[<A_1 + A_2, B_1 + A_2, C_1 + A_2, D_1 + A_2, A_2 + A_3> \]
is a 4-space denoted by \( \Sigma_4 \). Let \( \mathcal{T}_1, i = 1, 2, 3 \), be the three 5-spaces
on \( \Sigma_4 \). WLOG we assume \( \Sigma_5 = \mathcal{T}_1 \). Now each of \( \mathcal{T}_2 \) and
\( \mathcal{T}_3 \) must contain an odd stigm of \( \mathcal{B} \). Let \( St_i \) be an odd
stigm of \( \mathcal{B} \) in \( \mathcal{T}_i, i = 2, 3 \). Then by virtue of proposi-
tions 6.4, 6.7, 6.9, 6.10 and 6.11 we get a contradiction.
This completes the proof.
CHAPTER VII

Non-existence of 6-dimensional tangential 2-block in PG(6, 2)

In this chapter we prove the main result, viz, the non-existence of 6-dimensional tangential 2-block in PG(6, 2).

§1. A 5-space and some attenuation points.

Let $\mathcal{B}$ be a 6-dimensional tangential 2-block in PG(6, 2). By virtue of theorems 3.1, 4.1, 5.1 and 6.1, we conclude that $\text{St}_i(P)$, $i = 1, 2, 3$, are all lines of $\mathcal{B}$, $\forall P \in \mathcal{B}$. So, for each point $P$ of $\mathcal{B}$, the three stigms $\text{St}_i(P)$, $i = 1, 2, 3$ generate a 3-space, which we denote by $C_3(P)$ [Refer to section 4, Chapter II]. Let $\text{St}_i(P) = \{P, A_i, B_i\}$, $i = 1, 2, 3$. Then $\nu_2(P)$ denotes the plane $\langle P, A_1 + A_2, A_1 + A_3 \rangle$, which meets each of $\text{St}_i(P)$, $i = 1, 2, 3$, only at the point $P$. Let $\alpha(P)$ denote the line $\{P, A_1 + A_2 + A_3, A_1 + A_2 + B_3\}$. We are now interested in constructing a 5-space of special kind and for this purpose, we need the following lemmas.
Lemma 7.1. There exists a point $P \in \mathcal{S}$ such that $\nu_2(P) \cap \mathcal{S} = \{P\}$.

**Proof.** Let $T \in \mathcal{S}$ and $St_1(T) = \{T, A_1, B_1\}$, $i = 1, 2, 3$. Then $\nu_2(T) =$

$\{T, A_1 + A_2, A_1 + B_2, A_1 + A_3, A_1 + B_3, A_2 + A_3, A_2 + B_3\}$. If $\nu_2(T) \cap \mathcal{S} = \{T\}$, then we are done. So, we assume that $\nu_2(T) \cap \mathcal{S} \neq \emptyset$. Since we can interchange the suffixes 1, 2, and 3 of $St(T)$, as well as the letters $A$ and $B$, we assume (WLOG) that $A_1 + A_2 \in \nu_2(T)$. Then $A_1 + B_2 \notin \mathcal{S}$, for, otherwise, $<P, A_1, A_2>$ would be a Fano block contained in $\mathcal{S}$, leading to a contradiction. $A_1 + X_3 \notin \mathcal{S}, \forall X_3 \in \{A_3, B_3\}$, for otherwise, $<P, A_1, A_2>$ and $<P, A_1, X_3>$ would be two distinct planes, each having more than five points of $\mathcal{S}$; but by theorem 2.10 and corollary 2.3 of theorem 2.1, we arrive at a contradiction. Similarly, we can show that $A_2 + A_3$ and $A_2 + B_3$ are not points of $\mathcal{S}$. Since every line through $A_3$ in $C_3(T)$ must meet $\nu_2(T)$ in a point, $C_3(T)$ contains exactly one line of $\mathcal{S}$ through $A_3$, viz, $\{A_3, B_3, P\}$. So, we can suppose (WLOG) that $St_1(A_3) = \{A_3, B_3, P\}$, $St_j(A_3) = \{A_3, Q_j, R_j\}, Q_j \notin C_3(T), j = 2, 3$. Now $\nu_2(A_3) =$

$<A_3, Q_2 + Q_3, Q_2 + B_3>$. We assert that $\nu_2(A_3) \cap \mathcal{S} = \{A_3\}$. Suppose $Q_2 + Q_3 \in \mathcal{S}$. Also, we have $A_1 + A_2 \in \mathcal{S}$. So, $<P, A_1, A_2>$ and $<A_3, Q_2, Q_3>$ are two distinct planes, each having more than five points of $\mathcal{S}$. By
Theorem 2.10 and corollary 2.3 of theorem 2.1, we arrive at a contradiction. So, $Q_2 + Q_3 \notin \mathcal{B}$. Again, $Q_2 + B_3 \notin \mathcal{B}$, for otherwise $\langle P, A_1, A_2 \rangle$ and $\langle A_3, Q_2, B_3 \rangle$ would be two distinct planes, each having more than five points of $\mathcal{B}$. Also $P + Q_2 \notin \mathcal{B}$, for otherwise, $\langle P, A_1, A_2 \rangle$ and $\langle A_3, P, Q_2 \rangle$ would be two distinct planes, each having more than five points of $\mathcal{B}$. Similarly, we can show that no point of $\mathcal{V}_2(A_3) - \{A_3\}$ belongs to $\mathcal{B}$, i.e. $\mathcal{V}_2(A_3) \cap \mathcal{B} = \{A_3\}$. This completes the proof.

Lemma 7.2. Let $P \in \mathcal{B}$ and $\text{St}_1(P) = \{P, A_i, B_i\}, i = 1, 2, 3$. Let $\langle A_3, Q_i, R_i \rangle, i = 1, 2$, be two lines of $\mathcal{B}$ such that $Q_i, R_i \not\in C_3(P), i = 1, 2$. Then $\langle C_3(P), \{Q_1, Q_2\} \rangle$ is a 5-space.

Proof. Clearly, $4 \leq \dim \langle C_3(P), \{Q_1, Q_2\} \rangle \leq 5$. Suppose $\dim \langle C_3(P), \{Q_1, Q_2\} \rangle = 4$. Since $Q_i \not\in C_3(P)$, it follows that $Q_1 + Q_2 \in C_3(P)$. We now consider the following table.

<table>
<thead>
<tr>
<th>$Q_1 + Q_2$</th>
<th>$\Sigma_2$</th>
<th>$\Theta$</th>
<th>$\Theta \cap \Sigma_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P$</td>
<td>$\langle A_3, Q_1, Q_2 \rangle$</td>
<td>$t(A_1)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$A_1 + A_3$</td>
<td>$\langle A_3, Q_1, Q_2 \rangle$</td>
<td>$t(B_3)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$A_1 + A_2 + A_3$</td>
<td>$\langle A_3, Q_1, Q_2 \rangle$</td>
<td>$t(B_2, B_3)$</td>
<td>$\emptyset$</td>
</tr>
</tbody>
</table>

From the above table, it follows that $Q_1 + Q_2 \notin \{P, A_1 + A_3, A_1 + A_2 + A_3\}$. Similarly, for other values of $Q_1 + Q_2$, we show contradiction by showing that the intersection of a suitable tangent $\Theta$ with the plane $\langle A_3, Q_1, Q_2 \rangle$
is empty. Thus we arrive at a contradiction. Hence \( \dim \langle C_3(P), \{Q_1, Q_2\} \rangle \) is a 5-space. This completes the proof.

**Construction of a 5-space.** Let \( P \) be a point of \( \mathcal{B} \) and \( \text{St}_1(P) = \{P, A_1, B_1\}, i = 1, 2, 3 \), such that \( \nu_2(P) \cap \mathcal{B} = \{P\} \) [by Lemma 7.1]. Since \( \nu_2(P) \cap \mathcal{B} = \{P\} \), there exist two lines of \( \mathcal{B} \) through \( A_3 \) such that each of these two lines meets \( C_3(P) \) only at \( P \). Let these two lines of \( \mathcal{B} \) be \( \{Q_1, R_1\} \), \( i = 1, 2 \), where \( Q_i, R_i \notin C_3(P) \). Then by lemma 7.2, we conclude that \( \langle C_3(P), \{Q_1, Q_2\} \rangle \) is a 5-space. We denote this 5-space by \( \Sigma_5 \). Let us prove the following proposition.

**Proposition 7.1.** In the 5-space \( \Sigma_5 \) there exist points \( X \in \{Q_1, R_1\} \) and \( Y \in \{Q_2, R_2\} \) such that \( P + X, P + Y \) and \( X + Y \) are not members of \( \mathcal{B} \).

**Proof.** Let \( i \in \{1, 2\} \). Then both \( P + Q_i \) and \( P + R_i \) are not points of \( \mathcal{B} \); for, otherwise, \( \langle P, Q_i, R_i \rangle \) would be a Fano block contained in \( \mathcal{B} \).

**Case 1.** Let each of the pairs \( \{P + Q_i, P + R_i\} \), \( i = 1, 2 \), contains a point of \( \mathcal{B} \). WLOG we assume that \( P + Q_i \in \mathcal{B}, i = 1, 2 \). Then \( P + R_i \notin \mathcal{B}, i = 1, 2 \). Now we assert that \( R_1 + R_2 \notin \mathcal{B} \). Suppose \( R_1 + R_2 \in \mathcal{B} \). Let \( C_3(A_3) \) be the 3-space generated by the three lines \( \{A_3, B_3, P_3\} \) and \( \{A_3, Q_i, R_i\}, i = 1, 2 \). Let \( T \in \mathcal{B} - C_3(A_3) \). Then \( R_1 + R_2 \in \mathcal{B} \Rightarrow t(T) \cap \langle P, Q_1, Q_2 \rangle = \{P + Q_1 + Q_2\} \). But \( P + Q_1 + Q_2 \leq \mathcal{B} \).
€ Α(Α₃), where Α(Α₃) denotes the axis of the cone generated by \{Α₃, Λ, Α\} and \{Α₃, Α₁, Α₂\}, i = 1, 2. So, we arrive at a contradiction (by theorem 2.6 (iii)). \(\therefore\)

\(R_1 + R_2 \not\in \mathcal{B}\). So, we can choose \(X = R_1\) and \(Y = R_2\) such that \(P + X, P + Y\) and \(X + Y\) are not members of \(\mathcal{B}\).

**Case 2.** Suppose that only one of the pairs \(\{P + Q_1, P + R_1\}, i = 1, 2\), contains a point of \(\mathcal{B}\). WLOG we assume that \(P + Q_1 \in \mathcal{B}\). Then \(P + Q_2, P + R_1\) and \(P + R_2\) are not members of \(\mathcal{B}\). Now, both \(Q_1 + Q_2\) and \(Q_1 + R_2\) cannot be points of \(\mathcal{B}\); for, otherwise, \(\langle Α₃, Α₁, Α₂ \rangle\) would be a Fano block contained in \(\mathcal{B}\). Suppose \(Q_1 + Q_2 \not\in \mathcal{B}\). But \(Q_1 + Q_2 = R_1 + R_2\). We choose \(X = R_1, Y = R_2\), so that \(P + X, P + Y\) and \(X + Y\) are not members of \(\mathcal{B}\). Suppose \(Q_1 + R_2 \not\in \mathcal{B}\). But \(Q_1 + R_2 = R_1 + Q_2\). We choose \(X = R_1, Y = Q_2\), so that \(P + X, P + Y, X + Y\) are not members of \(\mathcal{B}\).

**Case 3.** Suppose none of the pairs \(\{P + Q_1, P + R_1\}, i = 1, 2\), contains a point of \(\mathcal{B}\). Since both \(R_1 + R_2\) and \(R_1 + Q_2\) are not points of \(\mathcal{B}\), we can easily choose \(X \in \{Q_1, R_1\}\) and \(Y \in \{Q_2, R_2\}\) such that \(P + X, P + Y\) and \(X + Y\) are not points of \(\mathcal{B}\). This completes the proof.

By virtue of proposition 7.2, we can assume (WLOG) that in \(Σ_5\), the points \(P + R_1, P + R_2\) and \(R_1 + R_2\) are not members of \(\mathcal{B}\). We note that

\(\langle P, A_1 + A_2, A_1 + A_3, A_1 + Q_1, A_1 + Q_2 \rangle\) is a 4-space. We denote this 4-space by \(Σ_4\). We observe that \(P + R_1, P + R_2\)
and \( R_1 + R_2 \) are all in \( \Sigma_4 \); but none of the points \( P + Q_1 \), \( P + Q_2 \) and \( Q_1 + R_2 \) is in \( \Sigma_4 \). Also \( \Sigma_4 = \langle \nu_2(P), \langle A_1, Q_1 \rangle \cup \langle \nu_2(P), \langle A_1, Q_2 \rangle \cup \langle \nu_2(P), \langle Q_1, Q_2 \rangle \rangle \). So, if \( X_i \in \{ A_i, B_i \}, i = 1, 2, 3 \), then points of \( \Sigma_4 \) are of the following types:

\[ P; \ X_i + X_j, \ i \neq j, \ i, j = 1, 2, 3; \ X_i + Q_j, \ i, j = 1, 2; \]

\[ R_1; \ P + R_1, \ i = 1, 2; \ R_1 + R_2; \ P + R_1 + R_2; \ X_1 + X_2 + X_3 + Q_i, \ i = 1, 2; \]

\[ X_1 + X_2 + Q_1 + Q_2; \ X_1 + X_3 + Q_1 + Q_2, \ i = 1, 2. \]

Let us prove the following proposition regarding some attenuation points with respect to \( \Sigma_5 \).

**Proposition 7.2.** Let \( X_i \) be any point belonging to \( \{ A_i, B_i \}, \ i = 1, 2, 3 \). Then the following points of \( \Sigma_5 \) are attenuation points with respect to \( \Sigma_5 \):

\[ X_1 + X_2 + X_3; \ P + Q_1 + Q_2; \ B_3 + Q_1 + Q_2; \]

\[ X_1 + X_2 + X_3 + Q_i, \ i = 1, 2; \ X_1 + X_i + Q_1 + Q_2, \ i = 2, 3; \]

\[ X_2 + X_3 + Q_1 + Q_2; \ X_1 + X_2 + Q_1, \ i = 1, 2; \]

\[ X_i + Q_1 + Q_2, \ i = 1, 2; \ X_1 + X_2 + X_3 + Q_1 + Q_2. \]

**Proof.** Let \( C_3(P) \) be the 3-space generated by \( \{ P, A_i, B_i \}, \ i = 1, 2, 3 \) and \( C_3(A_3) \) the 3-space generated by \( \{ A_3, B_3, P \_i \} \) and \( \{ A_3, Q_i, R_i \}, \ i = 1, 2 \). Let \( C(P) = \{ P, A_1 + A_2 + A_3, A_1 + A_2 + B_3 \} \) and \( C(A_3) = \{ A_3, P + Q_1 + Q_2, B_3 + Q_1 + Q_2 \} \). So, by theorem 2.6 (vi), we conclude that \( X_1 + X_2 + X_3, \ P + Q_1 + Q_2 \) and \( B_3 + Q_1 + Q_2 \) are all attenuation points with respect to \( \Sigma_5 \), for all \( X_i \in \{ A_i, B_i \}, \ i = 1, 2, 3 \).
Let $\Sigma_3$ be a 3-space, $T$ a point of $\mathcal{B} \cap \Sigma_5$. Let $\mathcal{A}$ be a subset of $\mathcal{B}$ of dimension $\leq 4$ such that $\mathcal{A} \subseteq \mathcal{B} - \Sigma_3$. Let $x_i$ be any point of $\{A_i, B_i\}$, $i = 1, 2, 3$. Now we have the following table [the suffix $i$ that appears in the table will have the integral value either 1 or 2].

<table>
<thead>
<tr>
<th>$T$</th>
<th>$\Sigma_3$</th>
<th>$\mathcal{A}$</th>
<th>$\mathcal{A} \cap \Sigma_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1 + x_2 + x_3 + q_i$</td>
<td>$c_3(p)$</td>
<td>${t, q_{i}^3}$</td>
<td>${x_1 + x_2 + x_3^3}$</td>
</tr>
<tr>
<td>$x_1 + x_2 + q_1 + q_2$</td>
<td>$c_3(p)$</td>
<td>${t, r_1, q_2^3}$</td>
<td>${x_1 + x_2 + A_3^3}$</td>
</tr>
<tr>
<td>$x_1 + A_3 + q_1 + q_2$</td>
<td>$c_3(A_3)$</td>
<td>${t, p + x_1^3}$</td>
<td>${B_3 + q_1 + q_2^3}$</td>
</tr>
<tr>
<td>$x_1 + B_3 + q_1 + q_2$</td>
<td>$c_3(A_3)$</td>
<td>${t, x_1^3}$</td>
<td>${B_3 + q_1 + q_2^3}$</td>
</tr>
<tr>
<td>$x_2 + A_3 + q_1 + q_2$</td>
<td>$c_3(A_3)$</td>
<td>${t, p + x_2^3}$</td>
<td>${B_3 + q_1 + q_2^3}$</td>
</tr>
<tr>
<td>$x_2 + B_3 + q_1 + q_2$</td>
<td>$c_3(A_3)$</td>
<td>${t, x_2^3}$</td>
<td>${B_3 + q_1 + q_2^3}$</td>
</tr>
<tr>
<td>$x_1 + x_2 + q_i$</td>
<td>$c_3(p)$</td>
<td>${t, r_i^3}$</td>
<td>${x_1 + x_2 + A_3^3}$</td>
</tr>
<tr>
<td>$x_1 + q_1 + q_2$</td>
<td>$c_3(A_3)$</td>
<td>${t, p + x_1^3}$</td>
<td>${p + q_1 + q_2^3}$</td>
</tr>
<tr>
<td>$x_1 + x_2 + x_3 + q_1 + q_2$</td>
<td>$c_3(p)$</td>
<td>${t, q_1, q_2^3}$</td>
<td>${x_1 + x_2 + x_3^3}$</td>
</tr>
</tbody>
</table>

If any one of the points in question is a point of $\mathcal{B}$, we arrive at a contradiction by virtue of the above table and theorem 2.6(iv).

Instead of assuming that $T \in \mathcal{B}$, we now assume that $T$ can be expressed as the sum of two points of $\mathcal{B} - \Sigma_5$, i.e., we assume $T = S_1 + S_2$, for some $S_i \in \mathcal{B} - \Sigma_5$, $i = 1, 2$. In the column for $\mathcal{A}$, as it appears in the above table, we replace $T$ by the two points $S_1$ and
It can be easily checked that this replacement can be accomplished without affecting the remaining columns of the above table. So, if any of the points in question is the sum of two points of \( \mathfrak{B} - \Sigma_5 \), then the new table, thus obtained from the above table after the above-mentioned replacement, lead to a contradiction by virtue of theorem 2.6 (iv). Hence all the points, mentioned in the proposition 7.2, are attenuation points with respect to \( \Sigma_5 \). This completes the proof.

§2. Non-existence of any odd stigm of \( \mathfrak{B} \) in \( \Sigma_4 \).

We recall that \( \Sigma_4 = \langle p, A_1 + A_2, A_1 + A_3, A_1 + Q_1, A_1 + Q_2 \rangle \), where \( p + R_1, p + R_2 \) and \( R_1 + R_2 \) are not points of \( \mathfrak{B} \). We observe that \( p + R_1, p + R_2 \) and \( R_1 + R_2 \) are, however, points of \( \Sigma_4 \). We now want to show that in \( \Sigma_4 \), there does not exist any odd stigm of \( \mathfrak{B} \). For this purpose we need the following lemma.

Lemma 7.3. Let \( \Delta = \{X_i + Q_j \mid X_i \in \{A_i, B_i^2\}, i, j = 1, 2 \} \). Let \( \{X_i, Y_i^3 = \{A_i, B_i^2\}, i = 1, 2 \).

(i) If \( X_i + Q_j \in \mathfrak{B} \), for some \( j \in \{1, 2\} \), then \( Y_i + Q_j \) is an attenuation point with respect to \( \Sigma_5 \), for \( i = 1, 2 \) and \( k = 1, 2 \).

(ii) \(|\Delta \cap \mathfrak{B}| \leq 4\).
Proof. (i) Let $A_1 + Q_1 \in \mathcal{B}$. Then $t(B_3) \cap < Q_1, A_1, B_1 > = \{B_1 + Q_1\}$ implies $B_1 + Q_1$ is an attenuation point with respect to $\Sigma_5$ [by theorem 2.4]. Also $t(A_1 + Q_1, R_2) \cap < B_1, Q_1, Q_2 > = \{B_1 + Q_2\}$. So, $B_1 + Q_2$ is an attenuation point with respect to $\Sigma_5$ [by theorem 2.4] Similarly, we can show that if $X_i + Q_j \in \mathcal{B}$, for some $j \in \{1, 2\}$, then $Y_i + Q_k$ is an attenuation point with respect to $\Sigma_5$, for $i = 1, 2$, $k = 1, 2$.

(ii) We observe that $|\Delta| = 8$. Now by part (i) of this proposition, we can easily show that $|\Delta \cap \mathcal{B}| \leq 4$.

Proposition 7.3. $\Sigma_4$ does not contain any odd stigm of $\mathcal{B}$. 

Proof. By virtue of proposition 7.2 and the fact that $P + R_1$, $P + R_2$ and $R_1 + R_2 \notin \mathcal{B}$, we can easily conclude that $\mathcal{B} \cap \Sigma_4 \subset \{P, R_1, R_2\} \cup \Delta$, where $\Delta = \{X_i + Q_j \mid X_i \in \{A_i, B_i\}, i, j = 1, 2\}$. Let $St$ be an odd stigm of $\mathcal{B}$. First we show that $St$ is not a 3-stigm. If possible, let $St$ be a 3-stigm. We assert that $P \notin St$. Suppose $P \in St$. Let $St = \{P, X, P + X\}, X \in \Sigma_4 \cap \mathcal{B}$. Then it follows that $X \in \Delta$. WLOG we assume that $X = A_1 + Q_1$. So, $P + X = B_1 + Q_1 \in \mathcal{B}$, which contradicts lemma 7.3. $\therefore P \notin St$. We assert that $R_i \notin St$, $i = 1, 2$.

Suppose $R_1 \in St$. Let $St = \{R_1, X, R_1 + X\}$, for some $X \in \mathcal{B} \cap \Sigma_4$. It is clear that $X \in \Delta$. Assert that $X \notin X_i + Q_1$, $\forall X_i \in \{A_i, B_i\}$, $i = 1, 2$. Suppose $X =$
\(X_i + Q_i\), for some \(X_i \in \{A_i, B_i\}\), and some \(i \in \{1, 2\}\).

WLOG we assume that \(X = A_1 + Q_1\). Then \(S_t = \{R_1, A_1 + Q_1, A_1 + A_3\} \implies A_1 + A_3 \in \mathcal{U}_2(P) \cap \mathcal{G}\), which contradicts our choice of \(P\) in the construction of \(\Sigma_5\).

Again, if \(X = X_1 + Q_2\), for some \(X_1 \in \{A_1, B_1\}\), and some \(i \in \{1, 2\}\), then \(X_1 + R_1 + Q_2 \in \mathcal{G}\), which contradicts the proposition 7.2. Thus \(R_1 \notin S_t\). Similarly, \(R_2 \notin S_t\).

\(S_t \subseteq \triangle\). WLOG we assume that \(A_1 + Q_1 \in S_t\). Let \(S_t = \{A_1 + Q_1, X, A_1 + Q_1 + X_3, X \in \Sigma_4 \cap \mathcal{G}\}\). Since \(Q_1 + Q_2 \notin \mathcal{G}\), and \(A_1 + Q_1 \in \mathcal{G} \implies B_1 + Q_1 \notin \mathcal{G}\), \(i = 1, 2\) (by lemma 7.3), it follows that \(X \in \{A_2 + Q_1, B_2 + Q_1, A_2 + Q_2, B_2 + Q_2\}\).

But \(A_1 + Q_1 + X \notin \mathcal{G}\). So, we arrive at a contradiction (by lemma 7.1 and proposition 7.2). Thus \(S_t\) is not a 3-stigm of \(\mathcal{G}\).

Thus we get that \(S_t\) is a 5-stigm of \(\mathcal{G}\). We assert that \(P \notin S_t\). If possible, let \(P \in S_t\). Then both \(R_1\) and \(R_2\) do not belong to \(S_t\); for, otherwise, we can assume (WLOG) that \(S_t = \{P, R_1, R_2, A_1 + Q_1, Y_2\}\), where \(Y = P + R_1 + R_2 + A_1 + Q_1 = B_1 + Q_2\); but \(A_1 + Q_1 \in \mathcal{G} \implies B_1 + Q_2 \notin \mathcal{G}\) (by lemma 7.3); so, we arrive at a contradiction. We now assert that \(R_i \notin S_t\), \(i = 1, 2\). Suppose \(R_1 \in S_t\). WLOG we assume that \(S_t = \{P, R_1, A_1 + Q_1, X, Y_2\}\), where \(X, Y \in \triangle \cap \mathcal{G}\). By lemma 7.3, we assume (WLOG) that \(X \in \{A_1 + Q_2, A_2 + Q_1, A_2 + Q_2\}\). But \(X = A_1 + Q_2 \Rightarrow Y = P + R_2 \notin \mathcal{G}\), which contradicts our choice of \(R_1\) and \(R_2\) in
the definition of \( \Sigma_4 \). Again, \( X = A_2 + Q_1 \Rightarrow Y = A_1 + B_2 + R_1 \in \mathcal{B} \), which contradicts proposition 7.2.

Finally, \( X = A_2 + Q_2 \Rightarrow A_1 + A_2 + B_3 + Q_2 \in \mathcal{B} \), which contradicts proposition 7.2. Hence \( R_1 \notin \mathcal{S}_T \). Similarly, \( R_2 \notin \mathcal{S}_T \). So, \( \mathcal{S}_T = \{P \} \subseteq \triangle \). WLOG we assume that \( \mathcal{S}_T = \{P, A_1 + Q_1, X, Y, Z_3\} \), \( X, Y, Z \in \mathcal{B} \cap \triangle \). By lemma 7.3, we assume (WLOG) that \( \{X, Y, Z_3\} = \{A_1 + Q_2, A_2 + Q_1, A_2 + Q_2\} \). So, \( \mathcal{S}_T = \{P, A_1 + Q_1, A_1 + Q_2, A_2 + Q_1, A_2 + Q_2\} \). Therefore, \( P = A_1 + Q_1 + A_1 + Q_2 + A_2 + Q_1 + A_2 + Q_2 \), which is not possible. So, \( P \notin \mathcal{S}_T \). By lemma 7.1, 7.3 and propositions 7.2 and the fact that \( P + R_1, P + R_2 \notin \mathcal{B} \), we can show by using similar arguments that \( R_1, R_2 \notin \mathcal{S}_T \). : \( \mathcal{S}_T \) is a 5-stigm such that \( \mathcal{S}_T \subseteq \triangle \cap \mathcal{B} \). But by lemma 7.3, \( |\triangle \cap \mathcal{B}| \leq 4 \). Hence we arrive at a contradiction. So, \( \Sigma_4 \) does not contain any odd stigm of \( \mathcal{B} \). This completes the proof.

\( \xi 3. \) Points of \( \mathcal{B} \) in \( \Sigma_4 \).

Here we intend to find exactly what the points of \( \mathcal{B} \cap \Sigma_4 \) are. We now introduce the following notations.

Let \( \mathcal{T}_i \), \( i = 1, 2, 3 \) be the three 5-spaces on \( \Sigma_4 \). One of these \( \mathcal{T}_i \)'s must be \( \Sigma_5 \). WLOG we assume \( \mathcal{T}_1 = \Sigma_5 \).

Let us now prove the following lemma.

**Lemma 7.4.** Let \( \triangle = \{X_i + Q_j \mid X_i \in \{A_1, B_1\}, i, j = 1, 2\} \) and let \( T \in \mathcal{B} \cap \triangle \). Then there exist two lines of \( \mathcal{B} \)
through $T$ such that both these lines are in $\Sigma_5$.

Proof. WLOG we assume $T = A_1 + Q_1 \in \mathfrak{B}$. We want to show that there are two lines of $\mathfrak{B}$ through $A_1 + Q_1$, which are in $\Sigma_5$. By theorems 3.1, 4.1, 5.1 and 6.1, $\text{St}_i(A_1 + Q_1)$, $i = 1, 2, 3$, are all lines of $\mathfrak{B}$, one of which can be taken (WLOG) as $\{A_1, Q_1, A_1 + Q_1\}$. Suppose the remaining two lines of $\mathfrak{B}$ be $\{A_1 + Q_1, S_1, S_2\}$ and $\{A_1 + Q_1, S_3, S_4\}$ and none of them is in $\Sigma_5$. Then $S_i \in \mathfrak{B} - \Sigma_5 = \mathfrak{B} - \tau_i, i = 1, 2, 3, 4$. WLOG we assume that the line $\{A_1 + Q_1, S_1, S_2\}$ is in $\tau_1$. We now consider the following two cases.

1. $\{A_1 + Q_1, S_3, S_4\} \subset \tau_2$
2. $\{A_1 + Q_1, S_3, S_4\} \subset \tau_3$

Case 1. Let $\{A_1 + Q_1, S_3, S_4\} \subset \tau_2$. Since $A_1 + Q_1 \in \mathfrak{B}$, $\langle B_2, B_3, Q_2 \rangle \cap \langle A_1, Q_1, A_3 \rangle = \{A_1 + R_1\}$. Now $\langle B_2, B_3, Q_2, A_1 + R_1 \rangle$ is a 3-space; also, $\langle B_2, B_3, Q_2 \rangle \not\subset \Sigma_5$, since it does not meet the line $\langle P, A_1, B_3 \rangle$ of $\Sigma_5$.

So, $\{B_2, B_3, Q_2\}$ is a polarising set with respect to $\Sigma_5$ (by theorem 2.5). We have $S_i \in \tau_2 - \Sigma_4, i = 1, 2, 3, 4$. Consider the distribution of $S_i$'s among $\rho_3(B_2, B_3, Q_2)$, $i = 1, 2, 3$; we get after rearranging the suffixes of $S_i$'s, that $S_1, S_3 \in \rho_3(B_2, B_3, Q_2)$, for some $i \in \{1, 2, 3\}$. Thus $S_1 + S_3 \in \{B_2 + B_3, B_2 + Q_2, B_3 + Q_2\}$ and $S_2 + S_3 \in \{A_1 + Q_1 + B_2 + B_3, A_1 + Q_1 + B_2 + Q_2, A_1 + Q_1 + B_3 + Q_2\}$.
= A_1 + Q_1]. So, we arrive at a contradiction (by proposition 7.2). So, case 1 cannot happen.

**Case 2.** Let \( \{A_1 + Q_1, S_3, S_4\} \subseteq \mathcal{T}_3' \). Also we have \( \{A_1 + Q_1, S_1, S_2\} \subseteq \mathcal{T}_2' \). Considering the distribution of \( S_i \)'s among \( \rho_i^3(B_2, B_3, Q_2), i = 1, 2, 3 \), we assume (WLOG) that \( S_1, S_3 \in \rho_i^3(B_2, B_3, Q_2) \), for some \( i \in \{1, 2, 3\} \). \( \therefore S_1 + S_3 \in \{B_2, B_3, Q_2, B_2 + B_3 + Q_2\} \) and hence \( S_2 + S_3 \in \{A_1 + Q_1 + B_2, A_1 + Q_1 + B_3, A_1 + Q_1 + Q_2, A_1 + Q_1 + B_2 + B_3 + Q_2\} \).

By proposition 7.2, we conclude that \( S_2 + S_3 = A_1 + Q_1 + B_3 \) i.e. \( S_1 + S_3 = B_3 \). So, \( t(A_1, R_1) \cap <A_1 + Q_1, S_1, S_3> = \emptyset \), which is a contradiction. \( \therefore \) case 2 cannot happen. Hence at least two of the lines \( S_i(A_1 + Q_1), i = 1, 2, 3 \), must be in \( \mathcal{T}_1' \), i.e., in \( \Sigma_5' \). This completes the proof.

**Lemma 7.5.** Let \( \Delta = \{x_i + Q_j \mid x_i \in \{A_1, B_1\}, i, j = 1, 2\} \) and let \( T \in \mathcal{B} \cap \Delta \). Then every line of \( \mathcal{B} \) through \( T \) lies in \( \Sigma_5' \).

**Proof.** WLOG we assume \( T = A_1 + Q_1 \in \mathcal{B} \). Suppose \( \{A_1 + Q_1, S_1, S_2\} \) to be a line of \( \mathcal{B} \), which does not lie in \( \mathcal{T}_1' = \Sigma_5' \). WLOG we assume that \( \{A_1 + Q_1, S_1, S_2\} \subseteq \mathcal{T}_2' \), i.e., \( S_1, S_2 \in \mathcal{T}_2' \subseteq \Sigma_4 \). By lemma 7.4, there exist two lines of \( \mathcal{B} \) through \( A_1 + Q_1 \), which are in \( \Sigma_5' \), i.e., in \( \mathcal{T}_1' \). One of these lines may be taken as \( \{A_1 + Q_1, A_1, Q_1\} \).

Let \( \{A_1 + Q_1, X, A_1 + Q_1 + X_2\} \) be the other line of \( \mathcal{B} \) in \( \mathcal{T}_1' \). Then \( X \notin \Sigma_4 \) (by proposition 7.3). \( \therefore X \in \mathcal{T}_1' \subseteq \Sigma_4 \).
By proposition 7.2, we conclude that \( X \) is one of the following points.

\[ X_i^+ R_j \quad X_i \in \{ A_i, B_i^3 \}, \quad i = 1, 2; \text{ and } R_1 + Q_2. \]

Now, \( A_1 + Q_1 \in \mathfrak{T} \implies A_1 + R_1 \notin \mathfrak{T} \quad [\because t(B_3) \cap \langle A_1, Q_1, R_1 \rangle = \{ A_1 + R_1 \}]. \) Also, \( P + R_1, P + R_2 \) and \( R_1 + R_2 \notin \mathfrak{T}. \) So, by proposition 7.2, we get only the following possibilities for \( X. \)

1. \( X = B_4, \)
2. \( X = B_3, \) and
3. \( X = R_1 + Q_2. \)

We have \( \{ A_1 + Q_1, S_1, S_2 \} \subseteq \mathcal{T}_2. \) Since \( \mathcal{T}_3 \) must contain an odd stigm of \( \mathfrak{T} \) (by proposition 1.10) and \( \mathcal{T}_4 \) does not contain any odd stigm of \( \mathfrak{T}, \) there exist at least two points \( S_3 \) and \( S_4 \) such that \( S_3, S_4 \in \mathfrak{T} \cap \mathcal{T}_3 - \mathcal{T}_4. \)

Since \( A_1 + Q_1 \in \mathfrak{T}, \ t(X_2, B_3, Y) \cap \langle A_1, Q_1, R_1 \rangle = \{ A_1 + R_1 \}, \) and \( t(X_2, B_3, Y) \cap \langle P, A_1, B_1 \rangle = \emptyset, \ \forall \ Y \in \{ A_2, B_2^3 \} \) and \( \forall \ Y \in \{ Q_2, R_2 \}. \) So, by theorem 2.5, we can easily deduce that \( \{ X_2, B_3, Y \} \) is a polarising set with respect to \( \mathcal{T}_5, \) for all \( X_2 \in \{ A_2, B_2^3 \} \) and all \( Y \in \{ Q_2, R_2 \}. \) Since \( S_3 + S_4 \in \mathcal{T}_4 \) and \( B_3 \notin \mathcal{T}_4, \) we can choose \( X_2 \) and \( Y_2, X_2 \in \{ A_2, B_2^3 \}, Y \in \{ Q_2, R_2 \}, \) such that \( S_3 + S_4 \notin \langle X_2, B_3, Y \rangle \cap \mathcal{T}_4. \) Also, \( S_1 + S_2 = A_1 + Q_1 \notin \langle X_2, B_3, Y \rangle. \) So, considering the distribution of \( S_1, S_2, S_3 \) and \( S_4 \) among \( \mathcal{E}_3^i(X_2, B_3, Y), \ i = 1, 2, 3, \) we
assume (WLOG) that $S_1, S_3 \in \{g_i(x_2, B_3, Y) \mid i \in \{1, 2, 3\}$, if $Y = Q_2$, then $S_1 + S_3 \in \{x_2, B_3, x_2 + B_3 + q_2, A_1 + q_1 + B_3, A_1 + q_1 + q_2, A_1 + q_1 + B_3 + q_2\}$. By proposition 7.2, we conclude that $S_1 + S_3 = A_1 + q_1 + B_3$, i.e., $S_1 + S_3 = B_3$.

Again, if $Y = R_2$, then $S_1 + S_3 \in \{x_2, B_3, x_2 + R_2, B_3 + R_2, A_1 + q_1 + B_3 + R_2, A_1 + q_1 + p + q_2, A_1 + q_1 + B_3 + R_2\}$. By proposition 7.2, we conclude that $S_1 + S_3 = A_1 + q_1 + B_3$.

So, in both the cases we get $S_1 + S_3 = B_3$ and $S_2 + S_3 = A_1 + q_1 + B_3$. We have already seen

$\{A_1 + q_1, x, A_1 + q_1 + x\}$ is a line of $\mathbb{B}$, where $X \in \{B_1, B_3, R_1 + q_2\}$.

**Case 1.** Let $X = B_1$. Then $\{A_1 + q_1, B_1, P + q_1, B_3\}$ is a line of $\mathbb{B}$. So, $t(S_2, S_3) \cap \langle P, A_1, q_1 \rangle = \{B_1 + q_1, B_3\}$. $	herefore$

$B_1 + q_1 + S_2 + S_3 = B_1 + q_1 + B_1 + R_1 = A_3 \in t(S_2, S_3)$, which is not possible. $	herefore$ case 1 cannot happen.

**Case 2.** Let $X = B_3$. So, $\{A_1 + q_1, B_3, B_1 + R_1\}$ is a line of $\mathbb{B}$. But $A_1 + q_1 \in \mathbb{B} \implies \pi(P) \cap \langle A_1, q_1, R_1 \rangle = \{A_1 + R_1\}$ [*: $A_1 + A_3 \not\in \pi(P)$] $\implies P + A_1 + R_1 = B_1 + R_1 \in \pi(P) \implies B_1 + R_1 \not\in \mathbb{B}$. Thus we get a contradiction. $	herefore$ case 2 cannot happen.

**Case 3.** Let $X = R_1 + q_2$. So, $\{A_1 + q_1, R_1 + q_2, A_1 + R_2\}$ is a line of $\mathbb{B}$. But $R_1 + q_2, A_1 + R_2 \in \mathbb{B} \implies t(S_2, S_3, P) \cap \langle A_1, q_1, R_2 \rangle = \{A_1 + q_1 + R_2\}$. But
\( \langle S_2, S_3, P \rangle \cap \sum_5 = \{P, B_1 + R_1, A_1 + R_1^3\}. \)

\((A_1 + R_1) + (A_1 + Q_1 + R_2) = Q_2 \in t(S_2, S_3, P). \)

\( \langle S_2, S_3, P \rangle \cap \sum_5, \) i.e., \( Q_2 \in \{P, B_1 + R_1, A_1 + R_1^3\}, \)

which is a contradiction. \( \therefore \) case 3 cannot happen. Thus

we arrive at a contradiction because of the assumption

that there exists a line of \( \mathcal{B} \) through \( A_1 + Q_1 \), which does

not lie in \( \sum_5 \). This completes the proof.

**Proposition 7.4.** Let \( \triangle = \{X_i + Q_j \mid X_i \in \{A, B_i^3\}, i, j = 1, 2, 3\}. \)

Then

(i) \( \triangle \cap \mathcal{B} = \emptyset \)

(ii) \( \triangle \cap \sum_4 = \{P, R_1, R_2^3\}. \)

**Proof.** (i) Suppose \( \triangle \cap \mathcal{B} \neq \emptyset. \) WLOG we assume \( A_1 + Q_1 \in \mathcal{B}. \)

Then by theorems 3.1, 4.1, 5.1 and 6.1, \( \text{St}_i(A_1 + Q_1), i = 1, 2, 3, \) are lines of \( \mathcal{B}. \) By proposition 7.4,

\( \text{St}_i(A_1 + Q_1) \subset \sum_5, \forall i \in \{1, 2, 3\}. \) WLOG we assume

\( \text{St}_1(A_1 + Q_1) = \{A_1 + Q_1, A_i^3 \}. \) Let \( \text{St}_2(A_1 + Q_1) = \{A_1 + Q_1, X, X + A_1 + Q_1^3 \} \) and \( \text{St}_3(A_1 + Q_1) = \{A_1 + Q_1, Y, Y + A_1 + Q_1^3 \}, \)

where \( X \neq Y \) and \( X, Y \in \sum_5 - \sum_4 \) [by proposition 7.3]. In the proof of lemma 7.5, we have

already seen that \( X \) as well as \( Y \) must be one of the three

points \( B_1, B_3 \) and \( Q_1 + R_2 \). But neither \( X \) nor \( Y \) is \( B_3 \), for

otherwise, \( \{A_1 + Q_1, B_3, B_1 + R_1^3 \} \) would be a line of \( \mathcal{B} \) and

would lead to a contradiction [see case 2 in the proof of

lemma 7.5]. Hence it follows that \( \{A_1 + Q_1, B_1, P + Q_1^3 \} \) and

\( \{A_1 + Q_1, Q_1 + R_2, A_1 + R_2^3 \} \) are both lines of \( \mathcal{B}. \) So, \( P + Q_1, \)
$ R_1 + Q_2 \in \mathcal{S} \implies \langle A_1, Q_1, B_1 \rangle \text{ and } \langle A_3, Q_1, Q_2 \rangle \text{ are two distinct planes having more than six points of } \mathcal{S}. \text{ But this is a contradiction [by theorem 2.10 and corollary 2.3 of theorem 2.1]. Hence } \triangle \cap \mathcal{S} = \emptyset.$

(ii) We know that $P + R_1, P + R_2$ and $R_1 + R_2$ are not points of $\mathcal{S}$. So, by proposition 7.2, we conclude that $\mathcal{S} \cap \Sigma_4 \subseteq \{P, R_1, R_2, \triangle\}$. But by part (i) of this proposition, $\mathcal{S} \cap \triangle = \emptyset \implies \mathcal{S} \cap \Sigma_4 = \{P, R_1, R_2, \triangle\}$. This completes the proof.


First of all we prove the following proposition.

Proposition 7.5. There exists a point $T \in \{A_1, B_1, \triangle\}$ such that $T$ has two lines of $\mathcal{S}$, both of which meet $\Sigma_5$ only at the point $T$.

Proof. By theorems 3.1, 4.1, 5.1 and 6.1, $S_1(X), i = 1, 2, 3$ are all lines of $\mathcal{S}, \forall X \in \{A_1, B_1, \triangle\}$. Let $m_1(X)$ and $m_2(X)$ denote two distinct lines of $\mathcal{S}$ through $X$, each being different from the line $\{P, X, P + X, B_1, A_1\}$. If $m_1(A_1) \cap \Sigma_5 = \{A_1, \}, i = 1, 2$, then we are done. So, we assume that $m_1(A_1) \subset \Sigma_5$, for some $i \in \{1, 2\}$. WLOG we assume that $m_1(A_1) \subset \Sigma_5$. Then $m_1(A_1) \cap \Sigma_4 = \emptyset$. By proposition 7.4, $\mathcal{S} \cap \Sigma_4 = \{P, R_1, R_2, \triangle\}$. By definition of $m_1(A_1), p \notin m_1(A_1).$ WLOG we assume that $m_1(A_1) = \{A_1, R_1, A_1 + R_1, \triangle\} \implies A_1 + R_1 \in \mathcal{S}.$
Now, we assert that $m_i(B_1) \cap \Sigma_5 = \{B_1^3\}$. Suppose $m_i(B_1)$
$\subset \Sigma_5$, for some $i \in \{1, 2\}$. Then we conclude that $m_i(B_1)$
is either $\{B_1, R_1, B_1 + R_1\}$ or $\{B_1, R_2, B_1 + R_2\}$. In other
words, either $B_1 + R_1 \in \mathcal{B}$ or $B_1 + R_2 \in \mathcal{B}$. We have $A_1 + R_1$
$\in \mathcal{B}$. So, $B_1 + R_1 \in \mathcal{B} \implies t(Q_1) \cap \langle A_1, B_1, R_1 \rangle = \emptyset$
and $B_1 + R_2 \in \mathcal{B} \implies t(Q_1, B_1 + R_2) \cap \langle A_1, R_1, R_2 \rangle = \emptyset$.
Thus we get a contradiction. Hence $m_i(B_1) \cap \Sigma_5 = \{B_1^3\}$.
This completes the proof.

Let us now state and prove our main theorem.

**Theorem 7.1.** There does not exist any 6-dimensional
tangential 2-block in $PG(6, 2)$.

**Proof.** Let $\mathcal{B}$ be a 6-dimensional tangential 2-block. By
theorems 3.1, 4.1, 5.1 and 6.1, $St_i(p)$, $i = 1, 2, 3$, are all
lines of $\mathcal{B}$, $\forall p \in \mathcal{B}$. We now choose $p \in \mathcal{B}$ such that
$V_2(p) \cap \mathcal{B} = \{p\}$, where $St_i(p) = \{p, A_i, B_i^3\}$, $i = 1, 2, 3$
[See Lemma 7.1]. Now using the notations of section 1
of this chapter, we have the 5-space $\Sigma_5$ and the 4-space
$\Sigma_4$. Also $\tau_i^1$, $i = 1, 2, 3$, denote the three 5-spaces on
$\Sigma_4$, where $\tau_i^1 = \Sigma_5$. Now by proposition 7.5, there exist
two lines of $\mathcal{B}$ through a point $T$, $T \in \{A_1, B_1^3\}$, such that
both these lines meet $\Sigma_5$ only at the point $T$. WLOG we
assume that $T = A_1$ and let the two lines of $\mathcal{B}$ be
$\{A_1, S_2, S_3\}$ and $\{A_1, S_3', S_3\}$, where $S_i, S_i' \in \tau_i - \Sigma_4$,
i = 2, 3. So, $S_2 + S_2' = S_3 + S_3' \in \Sigma_4$. Let $X_i \in \{A_i, B_i^3\}$,
i = 1, 2, 3. By proposition 7.2, we conclude that $S_3 + S_3'$
must be one of the following points.

\[ P; X_i + X_j, i \neq j, i, j = 1, 2, 3; X_i + Q_j, i, j = 1, 2; \]

\[ R_i, i = 1, 2 \text{ and } P + R_i, i = 1, 2. \]

Let us now consider the following table.

<table>
<thead>
<tr>
<th>( S_3 + S_3' )</th>
<th>( S_2 + S_2' )</th>
<th>( \Sigma_2 )</th>
<th>( \Theta )</th>
<th>( \Theta \cap \Sigma_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P )</td>
<td>( B_1 )</td>
<td>( \langle A_1, S_2, S_2' \rangle )</td>
<td>( t(A_3) )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( A_1 + A_2 )</td>
<td>( A_2 )</td>
<td>( \langle A_1, S_2, S_2' \rangle )</td>
<td>( t(B_1) )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( A_2 + A_3 )</td>
<td>( A_1 + A_2 + A_3 )</td>
<td>( \langle A_1, S_2, S_2' \rangle )</td>
<td>( t(B_1, B_2) )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( A_1 + Q_1 )</td>
<td>( Q_1 )</td>
<td>( \langle A_1, S_2, S_2' \rangle )</td>
<td>( t(B_3, R_1) )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( B_1 + Q_1 )</td>
<td>( P + Q_1 )</td>
<td>( \langle A_1, S_2, S_2' \rangle )</td>
<td>( t(Q_1) )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( A_2 + Q_1 )</td>
<td>( A_1 + A_2 + Q_1 )</td>
<td>( \langle A_1, S_2, S_2' \rangle )</td>
<td>( t(B_1, Q_1) )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( R_1 )</td>
<td>( A_1 + R_1 )</td>
<td>( \langle A_1, S_2, S_2' \rangle )</td>
<td>( t(B_3, Q_1) )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( P + R_1 )</td>
<td>( B_1 + R_1 )</td>
<td>( \langle A_1, S_2, S_2' \rangle )</td>
<td>( t(R_1) )</td>
<td>( \emptyset )</td>
</tr>
</tbody>
</table>

From the above table, it follows that \( S_3 + S_3' \notin \{ P, A_1 + A_2, A_2 + A_3, A_1 + Q_1, B_1 + Q_1, A_2 + Q_1, R_1, P + R_1 \}. \)

Similarly, we can deal with the remaining possibilities for \( S_3 + S_3' \) and finally we can conclude that \( S_3 + S_3' \notin \Sigma_4'. \)

So, we arrive at a contradiction. Hence there cannot be any 6-dimensional tangential 2-block. This completes the proof.
BIBLIOGRAPHY


