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ANALYSIS OF A LINEAR VISCOELASTIC LAYER PAVEMENT SYSTEM

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

Ping-Kuen Chen, B.S., M.Sc.

The Ohio State University
1970

Approved by

[Signature]

Adviser
Department of Civil Engineering
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VITA

November 18, 1941    Born - Shanghai, China

1963               B.S. - National Taiwan University, Taipie, Taiwan

1963-1964          Maintenance Engineer - Chinese Air Force

1964-1966         Research Assistant, Engineering Experiment Station
                  Ohio State University, Columbus, Ohio

1966               M.Sc., The Ohio State University, Columbus, Ohio

1966-1970         Research Associate, Engineering Experiment Station,
                  Ohio State University, Columbus, Ohio

FIELDS OF STUDY

Major Field: Civil Engineering

Studies in Materials: Professors Kamran Majidzadeh, Charles A. Pagen


Studies in Mathematics: Professors S. Drobot, Jack P. Tull

Studies in Engineering Mechanics: Professor Peter E. Korda

Studies in Statistics: Professor Thomas A. Willke
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NOTATIONS

\begin{align*}
a & \quad \text{radius of load area} \\
\nabla & \quad \text{Laplace operator} \\
\phi & \quad \text{Stress function} \\
E & \quad \text{Elastic modulus} \\
\mu & \quad \text{Poisson's ratio} \\
u, w & \quad \text{Components of displacements} \\
c & \quad \text{Strain (with appropriate subscripts)} \\
\sigma & \quad \text{Stress (with appropriate subscripts)} \\
\Sigma & \quad \text{summation variable} \\
K & \quad \text{bulk modulus} \\
G & \quad \text{shear modulus} \\
J_0(m\rho) & \quad \text{Bessel function of zero degree} \\
S_{ij} & \quad \text{deviatoric stress} \\
e_{ij} & \quad \text{deviatoric strain} \\
S_{ij} & \quad \text{Kronecker's delta} \\
Q, R, Q', R' & \quad \text{linear operator}
\end{align*}
\( p \)  
\text{transform of t coordinate}

\( E_r(t) \)  
\text{relaxation modulus}

\( J(t) \)  
\text{creep compliance}

\( q \)  
\text{intensity of load}

\( \gamma \)  
\text{multiple convolution integral}
CHAPTER I
INTRODUCTION

The rational method of analysis and design of pavements, as with any other structure, is concerned with the determination of the critical stresses and displacements in the pavement system with proper consideration to boundary conditions, loading history and realistic material properties. The analysis of stresses and displacements in a pavement system has been investigated by many researchers either theoretically or by actual field measurement of load-deflection and stress-strain distribution characteristics.

In the theoretical analysis of stresses and displacements, pavement systems are treated as a multi-layered system of semi-infinite extent, subjected to quasi-static and moving loads. These theoretical analyses have been primarily concerned with an elastic stress distribution. However, the observed behavior of many engineering materials used in highway construction such as soils, portland cement concrete and bituminous mixtures deviates from the idealized behavior assumed by the theory of elasticity. These materials exhibit various degrees of time dependency, non-homogeneity, and anistropy as
well as temperature dependency.

The effect of the time and temperature dependency of paving materials on the stress-displacement relations becomes critical when the pavement systems are subjected to static loads of long duration and slow vehicular speeds. At short loading times, corresponding to fast vehicular speed on the roadway, the behavior of time-dependent paving materials can be analyzed utilizing the theory of elasticity.

The stress-deformation characteristics of time and temperature dependent paving materials can be treated by the theory of linear viscoelasticity. Over recent years numerous investigators from different branches of science have studied linear viscoelastic stress analysis and have solved many problems of practical importance. In order to develop a rational method for the analysis and design of pavements under a wide range of loading and environmental conditions, it seems necessary to examine the use of linear viscoelastic stress and displacement analysis. Up to now, most published works on linear viscoelastic layered systems have been confined to the investigation of the theoretical behavior of the systems. The purpose of this study is to conduct a comprehensive experimental investigation of the stresses and displacements of a linear viscoelastic system and to evaluate the reliability of theoretical formulations for predicting the primary response behavior of flexible pavements.
CHAPTER II

ANALYSIS OF STRESSES AND DISPLACEMENTS IN LAYERED PAVEMENT SYSTEMS

Elastic Analysis

The theoretical analysis of stresses and displacements in a highway pavement system is a rather well-discussed problem. As a result of the inherent variability and complex properties of construction materials used in highway pavements, most analysis involves a simplified approach in which idealized boundary conditions and constitutive equations are used. The significance of the various theoretical approaches which have been advanced to date and their limitations for the prediction of pavement system responses are reviewed below.

Boussinesq Elastic Theory: In 1883, Boussinesq (1)* analyzed the stress distribution in a homogeneous, isotropic, linearly elastic compressible and incompressible solid of semi-infinite extent subjected to a point load applied normally to the surface. The resulting theory can only be used to describe the behavior of a single layered system under a point load. For certain other load

*Numbers in parentheses indicate corresponding references in bibliography.
configuration such as circular and rectangular loadings, the solution for the stresses distribution can be obtained by integration.

**Burmister Theory:** A highway pavement system obviously does not satisfy the assumptions of Boussinesq's layer of semi-infinite extent. In 1944, Burmister (2), (3) developed a theory for two and three layered systems. He assumed each layer to be isotropic and homogeneous and solved the problem by the classical method of elasticity using two sets of boundary conditions at each interface between layers. First, the two layers of the system are assumed to be continuously in contact with shearing resistance fully mobilized at the interface. Therefore, two layers act together as an elastic medium of composite nature with full continuity of stress and displacement across the interface between layers. Second, it is assumed that the layers of the system are continuously in contact across a frictionless interface, with only the continuity of normal stress and normal displacement satisfied. In reality, a pavement system does not fully satisfy either of these two boundary conditions. The assumption of full continuity however is much closer than the assumption of the frictionless interface. Therefore, the majority of methods for analyzing layered systems are based on the assumption of full active shearing resistance. In 1948, Fox (4) used the Burmister solution to calculate stresses at the interface and at points in the lower layer of a two-layer system.

The elastic three-layer system for the analysis of pavements is credited to Nijboer (5), Burmister (3), Acum and Fox (6), Jeuffrey and Bachelez...
From Nijboer's solutions the compressive strain on the surface of the top layer can be determined. The upper two layers are assumed to be one composite layer, and a neutral axis is determined based on the strength values of the layers. A "two-layer" system analysis is then used with this equivalent layer and the subgrade. Odemark also discussed the use of radius of curvature for determination of compressive strain. In Jeuffrey's and Bachelez's analyses, the top layer was assumed to be an "elastic plate," and the middle layer to be an "elastic layer," with both lying on a semi-infinite elastic mass. No friction was assumed at the upper interface, whereas a perfect continuity condition was considered for the lower interface. These analyses allow evaluation of vertical stresses on the subgrade, horizontal stresses at the base of the top layer and the surface deflections.

For the three-layer elastic analysis, Acun and Fox extended Burmister method to give stress equations for the two interfaces assuming with full friction at each interface. Computations for the three-layer system are dependent on the following parameters:

\[
\begin{align*}
A &= \frac{a}{h_2} \\
H &= \frac{h_1}{h_2} \\
K_1 &= \frac{E_1}{E_2} \\
K_2 &= \frac{E_2}{E_3}
\end{align*}
\]

where

- \( a \) = radius of the loaded area
- \( h_1 \) = thickness of the top layer
- \( h_2 \) = thickness of the intermediate layer or layer 2
\[ E_1 = \text{modulus of elasticity of the top layer} \]
\[ E_2 = \text{modulus of elasticity of the intermediate layer} \]
\[ E_3 = \text{modulus of elasticity of the lower layer} \]

With these parameters it is possible to evaluate the vertical stresses at (a) the interface of the surface and the base, and (b) the interface between the base and subgrade. It is also possible to evaluate the tensile stress and strain at the bottom of the asphaltic concrete. The deflection of the uppermost surface and any interface below the surface can be evaluated according to the theory; however, solutions are only available for points on the center axis of the loaded area.

The Shell Oil Co. (13) has also provided solutions based on the basic equations of Burmister, but has extended these solutions to encompass more conditions than were considered by Acum and Fox. In these analyses the upper two layers are assumed to be infinite in extent and finite in thickness, while the third layer is assumed to be infinite in both extent and depth. Further modifications have been made by Venstraeten (14) and Chargular and Shealer (15). Venstraeten has extended the solution to a four layer elastic system assuming uniformly distributed shear stress and continuity across each interface. Chargular also made calculations on a four layer elastic system. For the analysis, a parabolic distribution of load over a circular area was assumed. All the above refinements are a result of recent advancements in computer technology which make the solution of complicated numerical problems feasible.
Plate Theory: This is another approach to the solution of the layered system. In the plate theory it is assumed that the top layer behaves as a plate. Plate behavior is the behavior of an elastic body which has one dimension which is small compared to its other dimensions. These analyses assume that deflections are small in relation to plate thickness, and that bending is the only mode of deformation. Several authors have used the assumptions of plate behavior. In 1926, Westergaard (16) developed a solution for an elastic plate resting on an elastic subgrade designated as Winkler's foundation which could only undergo vertical displacements or provide vertical reactions. The subgrade characteristic is represented by "the modulus of subgrade reaction," which is a proportional constant between strain and deformation. From the Westergaard analysis the tensile stresses and deflections of a plate can be calculated. Hogg (17) in 1938 presented an analysis for the case of an elastic plate on an elastic semi-infinite subgrade. Odenmark (10) used this solution in 1949 to obtain curves for the tensile stresses in the upper layer, and Pickett and Auidaniel (18) modified the analysis to account for shear deformations. Since not it is possible to solve numerically the more exact elastic layered systems, there seems to be little reason to use the plate assumption especially when more than two layers are present in the pavement system.

Limitations

In the foregoing elastic layered analysis, it has been assumed that the pavement materials for each layer and the subgrade soil can be treated
as homogeneous isotropic, linear elastic solids. Although the assumption is not strictly true, model tests in addition to laboratory and field observations indicate that at low stress and strain levels, at fast rates of loading, and after a considerable number of load repetitions the characteristics of these materials can be approximated by those of linear elastic bodies.

With respect to the assumption of isotropy, Busching et al (19) has shown that paving slabs compacted in the laboratory have a higher stiffness in the compaction direction (vertical). The ratio of vertical to horizontal stiffness \( \frac{E_v}{E_h} \) measured under quasi-static conditions has been reported to about 1.3 to 1. Moriarty (20) also has measured higher stiffnesses in the vertical direction for cores from a rolled asphalt pavement. The evidence of anisotropy in soils and unbound materials is limited because of the small number of tests carried out in which appropriate measurements have been taken. Up to now, the theoretical solution of the anisotropic analysis is based on cross-anisotropy which could represent most soils and asphaltic materials (21), (22).

It has been shown that materials deposited on a horizontal plane and compacted by rolling or tamping methods exhibit this type of anisotropy. The anisotropic analysis requires five independent elastic constants. The constants defining the behavior are:

- \( E_v \) = Young's modulus in the vertical direction
- \( E_h \) = Young's modulus in the horizontal direction
- \( \mu_{vh} \) = Poisson's ratio for influence of vertical strain on a horizontal strain
\[ \mu_h = \text{Poisson's ratio for influence of horizontal strain in one direction on horizontal strain in a complementary direction, and} \]
\[ G = \text{shear modulus relating shear stress and shear strain in a vertical plane.} \]

Furthermore, the results of the elastic stress analysis also depend on the magnitude of Poisson's ratio, \( \mu \), for the various pavement layers. In most analysis, it is assumed that \( \mu = 0.5 \). The effect of such an assumption on the vertical stress and displacements is rather small as compared to the effect of Young's modulus; however, it has a large influence on the radial stress (in the single layer case). From K.R. Peatties' work (23), the importance of the choice of a value of Poisson's ratio is illustrated by the following effects resulting from a change in the value \( \mu \), from 0.5 to 0.35 in all three layers.

1) the factor for the vertical stress at the bottom of the second layer is increased by 2% to 10%. (With an average increase of about 9%).

2) the factor for the horizontal stress at the base of the top layer is reduced by 15% to 22% with an average decrease of about 19%.

3) the factor for the horizontal strain, \( \varepsilon_1 \), at the base of the first layer is decreased by 4% to 17% with an average of 8%.

4) the factor for the surface deflection is increased by 6% to 9% with an average increase of 7%.

Theoretically, the upper limit of \( \mu \) is 0.5 corresponding to an incompressible material. This value appears appropriate for a saturated clay loaded under undrained conditions, while sands are usually assigned a value in the range of 0.3 to 0.4 and asphaltic mixtures attain values between 0.4 to 0.5.

It should be noted that most of the values reported in the literature have been
determined on the assumption of material isotropy.

Verifications

The preceding discussions have indicated many aspects of real material behavior which may require recognition if confidence is to be placed in analytical predictions. Alternatively, it may be that the differences between idealized and real material behavior lead only to insignificant errors in the prediction of the most important stresses, strains or deflections. Ultimately, the decision about the applicability of elastic theory to pavement design must be based on the comparison of predicted and observed behaviors. At present, information is readily available on (a) stresses in single and multi-layered systems, and (b) surface and vertical deflections and strains.

The measurement of stresses would seem to be one of the most straightforward ways for evaluating the usefulness of elastic theory. Stress measurements have been reported in both uniformly prepared sand mass (24) and in fine grained soils (25, 26, 27). Because of the bulk of the results, only those which appear to be of the greatest significance will be considered here. The stresses discussed are those resulting only as a consequence of the load application and take no account of any initial stresses in the material due to self-weight or pre-stress. There is very little information available on the magnitude of these latter stresses although it is known that they can have a significant influence on material behavior, particularly in sands. Within the range of loads applied in the tests, the load-stress relationship can be assumed linear, and hence the results may be expressed as percentages of the applied surface load.
These experiments have been carried out in sands differing in gradation, density and method of compaction. A variety of loading plates were used and with stresses measured using several different pressure cells. There is substantial similarities between the measured and predicted results. In particular, the deviations of the observed vertical stresses from the predicted values are not much greater than could be expected from experimental error. However, the radial stress distributions bear little resemblance to the predicted values. The overestimate near the surface probably results from the experimental difficulties involved in measuring a rapidly varying stress and the lack of resistance to sideways shoving at this location. However, there are no reasons to suspect the trends shown at greater depths where the observed stresses are consistently underestimated by the theory.

The observed discrepancy has been attributed by Morgan and Gerrard (28) to anisotropic material behavior. Independent experimental work indicates that similarly prepared sand samples were anisotropic with horizontal stiffness being less than the vertical (29). The best agreement was obtained for $E_h/E_v = 0.6$.

The results of the three series of vertical stress measurements carried out in cohesive soils (25, 26, 27) show reasonable agreement with the Boussinesq predictions. Brown and Pell (25), however, again found lack of agreement for the radial stresses and similar to that found for sand. On the contrary, the Waterway Experiment Station results (27) show reasonable agreement between observed radial stresses and predicted values. In both
cases the fine grained soils considered were of low plasticity, and were compacted at about optimum moisture content. Loading conditions were different in the two cases. Brown and Pell used a dynamic loading system, and the Waterway Experiment Station tests were carried out using a quasi-static load. Stress measurements away from the centerline lead to conclusions similar to the above. In view of the expected deviations of soils from the assumptions of isotropy and homogeneity involved in the Boussinesq theory, the agreement obtained for the vertical stress is surprising. Also it indicates that the vertical stress distribution is not greatly different for an anisotropic material. Furthermore, Huang (30) in his investigation of stress distribution in material with stress-dependent properties of the form \( E = E_0 (1 + \beta \sigma) \) found that the greatest deviation from the Boussinesq case was about 30%.

Measurements of stresses in multi-layer systems are of more interest because of the resemblance of these systems to pavement systems. Vesic (31) assembled the results of tests carried out by various workers and compared them with the Boussinesq single layer and the Burmister two-layer systems. The results can be grouped into three types (a) values exceeding the Boussinesq stresses; (b) values close to the Boussinesq solution which would be expected to correspond instead to the layered stresses for \( E_1/E_2 = 10 \); (c) a very small group of values close to layered stresses for \( E_1/E_2 = 100 \). Vesic suggested that the observed differences can be attributed to anisotropy of base course and surface course materials and/or to differences in moduli found in com-
pression and tension of the surface and base course. For pavements lacking tensile strength at the upper layer as in the case of soil bound-macadam and silty sand base, even though $E_1/E_2 = 10$, the stress distribution pattern is close to that computed by the Boussinesq theory. This indicates that the load spreading ability of such a layer is no better than that of a homogenous soil. However, for a layered pavement with soil-cement base ($E_1/E_2 = 100$) and a high tensile strength, the stress distribution is close to that predicted by Burmister's layered theory. The failure of two-layer theory in many cases can also be attributed to lack of tensile strength in the upper layers. The recent work on the influence of confining pressure on the elastic modulus of granular materials suggests that degree of confinement could also be an important factor. Unfortunately, for many of the tests reported, little or no information is given on the method used to find $E$. A common procedure for determining $E$ is to carry out a triaxial test, using a value of confining pressure that is arbitrary selected and considerably greater than the corresponding radial stress in the layered system. This procedure may result in an error in estimating the modulus of certain granular materials, and in the modular ratio needed for layered stress analysis.

Brown and Pell (32) measured, among other values, the radial stresses in a two layer system composed of a fine crushed rock base overlying a stiff clay subgrade. For such a system the modular ratio was calculated to be close to unity from separate in-situ modulus determinations. Therefore, comparisons were made with the two layer theory for values of $E_1/E_2$ in the range
of 0.5 to 1.3. The comparison was somewhat inconclusive because of the small difference from the Boussinesq values, but they concluded that the prediction of vertical stresses from the two layer theory was satisfactory, while the radial stresses, except at the interface, were underestimated.

Thus, considering the results of measurements in both single and multi-layer systems, the following conclusions would seem reasonable:

(1) Vertical stress distributions are given with reasonable accuracy by either the Boussinesq single layer or the Burmister multi-layer theories. For two layer systems, the modular ratio used for unbound bases is probably only two or three to one so that the difference between the stresses predicted by the theories is small. Only when the base has significant unconfined stiffness is a higher modular ratio and, hence a higher stress reduction in the subgrade achieved. Variations from the assumed conditions of isotropy and homogeneity are unlikely to influence the vertical stress significantly.

(2) Radial stresses (except close to the surface) in single layer systems are significantly underestimated by both the single and multi-layer theories. In one single layer case for a sand mass, better agreement has been obtained by taking into account the known anisotropy of the material.

In assessing the usefulness of the elastic theory for predicting strains and deflections, it is sufficient to consider the vertical direction only. There is justification for this in that vertical deflections give a good indication of the overall structural strength of a pavement and furthermore may be readily measured in the field by the Benkleman beam and other devices. The most
complete sets of data available are for single layer systems, sands having been tested by the Waterway Experiment Station (24) and clays by the Waterway Experiment Station (27), and Brown and Pell (25). In some cases, difficulties were found in measuring small strains and deflections but the trends appear consistent and reliable. In comparing observed and predicted values, an immediate difficulty arises because of the need to specify an absolute value for $E$, Young's modulus. This difficulty does not arise in stress predictions because they are independent of $E$ in the single layer system and depend only on the modular ratio in the multi-layer system.

The most consistent data for cohesive soils appear to be those given by Brown and Pell (27). The vertical strains were measured with a 'spool' type gauge having a length of 2-1/2 inches and a diameter of 2 inches. These gauges gave reliable results except at low strain levels. The experimental results show that the strain distribution differs remarkably from the predicted. In an attempt to take into account the non-homogeneity of the soil, Brown and Pell calculated strains using values of 'local modulus'. These local moduli were calculated for each point from the stresses and strains measured there, and were used in the Boussinesq theory to calculate the strains assuming that the whole mass has these properties. The real significance of these local moduli is open to discussion, but it appears that their use gives better agreement with measured strains than the Boussinesq theory. Of course, the approach is of limited general use because $E$ and $\mu$ were determined from measurements inside the mass and not from the simple model
tests.

Further evidence can be produced showing that deflections are not satisfactorily predicted using the assumption of isotropic behavior. Dehlen (33) carried out deflection and curvature measurements on a number of flexible pavements and compared the results with both the single and multi-layer theories. He found that the deflection decreased more rapidly with depth than was predicted. Dehlen found that both at the pavement surface and at subgrade level the computed radii of curvature were considerably greater than those measured. That is, the deflections were more concentrated about the load than expected from theory. The same behavior may be seen from the Waterway Experiment Station results (24) i.e., the vertical deflections dropping off very rapidly with distance from the load centerline.

As a result of these tests, it was concluded that for vertical strain or deflection prediction, the Boussinesq theory is quite inadequate. This conclusion is based on the differing distribution with depth of the measured and observed values, and does not take into account the problems associated with determining a value of elastic modulus in order to find an absolute surface deflection. Attempts to take into account the non-homogeneity of the soil resulting from stress application cannot yet be considered successful.

Viscoelastic Analysis

In the elastic analysis of layered pavement systems discussed previously, it is assumed that pavement materials are characterized by isotropic, homogeneous, linear elastic solids. However, for more exact analysis of
pavement response, it is necessary to consider more realistic constitutive equations for the materials, i.e., to include time and temperature dependency of the materials. The viscoelastic stress analysis, in which the time effects are introduced by the stress-strain relations, was first used by Alfrey (34) for incompressible bodies in 1944.

Tsien (35) generalized Alfrey's principle in 1950 to include bodies with the same time characteristics in shear and dilatation, and then Lee (36) formulated in 1955 the "correspondence principle" so that it included any linear viscoelastic body. The essence of the corresponding principle is that if the equations of viscoelasticity (equilibrium, stress-strain, strain-displacement and the boundary conditions) are transformed from the time domain to the Laplace domain through the application of the Laplace transform, the partial differential equations with respect to the time variable will be transformed into algebraic equations in the variable \( P \) (Laplace parameter), which are in the same form as those of an associated elastic problem. Therefore, all viscoelastic problems can be solved through the Laplace inversion of the associated elastic solution.

An alternative approach to the problem was suggested by Lee and Rogers (37) in 1963 based upon the use of creep or relaxation functions in the form of hereditary integrals. The method results in integral equations which may be solved numerically.

The Laplace transform method has been used very often in early published works on viscoelastic stress and displacement analysis. However, the
viscoelastic analysis using this method has been limited to materials characterized by simple discrete models where the inverse Laplace transform is easily obtained. Lee (38) solved the problem for a fixed and moving point load on a viscoelastic halfspace which was assumed to behave as Voigt's model in shear and to behave elastically in hydrostatic tension or compression. Pister (39) presented the solution for a viscoelastic plate on a viscoelastic foundation under a uniform circular load where both the plate and the foundation are assumed to behave as incompressible Maxwell materials. Pister and Westmann (40) used a three-element model to characterize the behavior of a beam on a Winkler foundation, and analyzed this for a moving point load. Kraft (41) presented an analysis of the deflection of a two-layer half-space system in which the layers were each composed of three-element models, and the volumetric behavior was assumed to be elastic. Basic stress and deformation analyses of viscoelastic materials and all the pertinent mathematical representation and characterization methods have been fully discussed in the literature (42), (43), (44), (45).

To find the stresses and displacements in a viscoelastic system, such as a pavement layered structure, the correspondence principle formulated by Lee requires, as the first step, determination of the elastic solution for the problem. In the case of two and three layer pavement structures, the elastic solution was formulated by Burmister as discussed previously.

Following the classical theory of elasticity, a stress function $\phi$ which satisfies the governing differential equation:
\[ \nabla^4 \phi = 0 \] (1)

is assumed for each of the layers. For systems with an axially symmetrical stress distribution \( \nabla^4 \) is written as:

\[
\nabla^4 = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\lambda^2}{\partial z^2} \right) \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right)
\]

where \( r \) and \( z \) are the cylindrical coordinates for the vertical and radial directions respectively. After a suitable stress function is found, the stresses and displacements in the system can be determined by conventional means. Since equation (1) is a fourth order differential equation, the stresses and displacements thus determined will consist of four constants of integration which must be evaluated from the boundary and continuity conditions. In Figure 1, an \( n \)-layer elastic system is shown. The Young's modulus and the Poisson's ratio of the \( j \)th layer are \( E_j \) and \( \mu_j \) respectively.

Let \( \rho = r/H \)

\[
\lambda = z/H
\]

To satisfy equation 1, Burmister used the following stress function,

\[
\phi_j = \frac{H^3 J_0 (m \rho)}{m^2} \left[ A_j e^{-m (\lambda_j - \lambda)} - B_j e^{-m (\lambda_j - \lambda - \lambda_j - 1)} \right. \\
\left. + C_j m e^{-m (\lambda_j - \lambda)} - D_j m e^{-m (\lambda_j - \lambda - \lambda_j - 1)} \right] \quad (2)
\]

where \( J_0 = \) Bessel function of the first kind and zero order and

\[ A, B, C, D = \text{constants.} \]
Figure 1. An n-layer system.
In order to determine $A_j$, $B_j$, $C_j$, $D_j$, two boundary conditions must be satisfied.

At surface \( (\sigma_z)_1 = -m J (m \rho) \), \( (\tau_{rz})_1 = 0 \)

At Infinite Depth \( (\sigma_z)_n = 0 \), \( (W)_n = 0 \)

and At the Interfaces \( (\sigma_z)_j = (\sigma_z)_{j+1} \), \( (\tau_{rz})_j = (\tau_{rz})_{j+1} \), \( (W)_j = (W)_{j+1} \), \( (U)_j = (U)_{j+1} \)

Then from Equation 2, \((4n-2)\) linear equations were obtained and \((4n-2)\) constants \((A_j, B_j, C_j, D_j)\) for \(j = 1, n\) can be determined.

This is the solution due to \(-m J_0 (m \rho)\) loading conditions. In order to find the solution for a load \(q\) distributed over a circular area of radius \(a\), Henkel's transformation is used.

In the elastic solution, for isotropic linear systems, two time-independent material constants \(E\) and \(\mu\) enter into the analysis. These two constants are in turn related to the bulk modulus, \(K\), and shear modulus, \(G\), by

\[
E = \frac{9KG}{3K+G} \\
\mu = \frac{3K - 2G}{2(3K+G)}
\]

(3)

\(G\) and \(K\) are material constants relating stresses and strains in deviatoric and volumetric equations of states as;

\[
S_{ij} = 2G e_{ij} \\
\sigma_{ii} = 3K \epsilon_{ii}
\]

(4)
where \( S_{ij} \) = deviatoric stress \( = \sigma_{ij} - (\frac{1}{3}) \varepsilon_{ij} \sigma_{kk} \)

\( e_{ij} \) = deviatoric strain \( = \varepsilon_{ij} - (\frac{1}{3}) \varepsilon_{ij} \varepsilon_{kk} \)

\( \sigma_{ij} \) = stress tensors

\( \varepsilon_{ij} \) = strain tensors

\( \delta_{ij} \) = Kronecker's delta; i.e., \( \delta_{ij} = 1 \) when \( i = j \)

\( \delta = 0 \) when \( i \neq j \)

The above two equations (3) and (4) also hold for viscoelastic materials. However for a viscoelastic material, \( G \) and \( K \) are not constants but are the quotients of two linear differential operators.

\[
G = \frac{Q}{R} \\
K = \frac{Q'}{R'}
\]  

(5)

where \( Q, R, Q' \) and \( R' \) are linear operators of the form

\[
\sum_{k=0}^{n} a_k D_k, \quad \text{and}
\]

(6)

where \( D_k \) is the time derivative \( \frac{\partial}{\partial t} \). Then for viscoelastic materials Equation (4) can be written as

\[
RS_{ij} = 2Qe_{ij}
\]

(7)

\[
R'\sigma_{ii} = 3Q'\varepsilon_{ii}
\]

In order to eliminate the time variable \( t \), the Laplace transform is applied to equation (7). The transforms of stresses and strains are designated by a bar on the corresponding functions given by;
\[ R(p) \bar{S}_{ij} = 2Q(p) \bar{e}_{ij} \] (8)
\[ R(p) \bar{e}_{ii} = 3Q'(p) \bar{e}_{ii} \]

where \( R(p), Q(p), R'(p), Q'(p) \) are polynomials of the form
\[ \sum_{k=0}^{n} a_k p^k \] (9)

Then, the shear and bulk moduli in the Laplace domain are defined as:
\[ \bar{G} = \frac{Q(p)}{R(p)} \]
\[ \bar{K} = \frac{Q'(p)}{R'(p)} \] (10)

To obtain the viscoelastic solution, the material constants \( G, K, E, \mu \) and the load intensity \( q \) in the elastic equations for stresses and displacements are replaced by \( \bar{G}, \bar{K}, \bar{E}, \bar{\mu}, \) and \( \bar{q} \) respectively leading to the equations for stresses and displacements in Laplace domain. Then all the solutions are functions of the transformed variable \( p \) and their inversions will give the stresses and displacements as functions of time.

Using the technique mentioned above, Kraft (41) solved a two layer viscoelastic system by assuming \( \mu = 1/2 \) in each layer and that \( E \) is presented by a linear operator of second degree. The final inversion of the Laplace transform led to an eighth degree polynomial. However, it is obvious that when the pavement materials are characterized with higher order linear operators, the Laplace inversion of the transformed equation becomes a rather difficult task (Huang-36).

**Convolution Integral Method**

The convolution integral method was used by Ashton (47) to overcome the difficulty of the inversion of the Laplace transform. The viscoelastic
stress-strain relations are expressed by hereditary integrals such as:

\[ \sigma(t) = \int_{-\infty}^{t} E_r(t-\tau) \frac{\partial \varepsilon(\tau)}{\partial \tau} \, d\tau \]  

(11)

where \( E(t) \) is the relaxation modulus and \( \varepsilon(t) \) is the strain. For a system at rest for time \( t < 0 \), equation (11), can be written as

\[ \sigma(t) = E_r(0) \varepsilon(t) + \int_{0}^{t} \frac{\partial E_r(t-\tau)}{\partial \tau} \varepsilon(\tau) \, d\tau \]  

(12)

or

\[ \sigma(t) = I_1 \varepsilon(t) \]

where \( I_1 \) is a linear operator, analogous to Young's modulus for an elastic body. This operator can be rewritten as:

\[ E_{\text{equiv.}} = \left[ E_r(0) + \int_{0}^{t} (\cdot) \frac{\partial E_r(t-\tau)}{\partial \tau} \, d\tau \right] \]  

(13)

Such an operator is then substituted for the equivalent elastic constants in the elastic solution, yielding an integro-differential equation for the viscoelastic system as suggested by Radok (48).

In order to use the operator approach in a straightforward manner, the equivalent elastic solution is arranged by appropriate algebraic operations, into the following form

\[ \psi_t = \sum_{i=1}^{n} \theta_i \alpha_i \psi_i(t) \]  

\[ \sum_{i=1}^{m} \phi_i \beta_i \]  

\[ m \]

(14)

where \( \psi_t = \) the desired stress or displacement

\( \theta_i, \phi_i = \) constants with respect to time
\( a_i, \beta_i = \) products of elastic constants

\( f_i(t) = \) functions of time introduced through time variations in the boundary conditions

If each of the elastic constants in the \( a_i \) and \( \beta_i \) terms can be replaced by its viscoelastic operator equivalents, then equation (14) can be converted to the viscoelastic solution. Equation (14) may be rearranged to the following form

\[
\sum_{i=1}^{m} \phi_i \beta_i \psi(t) = \sum_{i=1}^{n} \theta_i a_i f_i(t)
\]  

(15)

Now to obtain the viscoelastic solution, the operator equivalent of the elastic constants are substituted in equation (15) leading to equation (16)

\[
\begin{align*}
\sum_{i=1}^{m} \phi_i & \left\{ \int_{0^+}^{t} \psi(t-\tau) \frac{\delta \beta_i(\tau)}{\delta \tau} d\tau + \psi(t) \beta_i(0) \right\} \\
= \sum_{i=1}^{n} \theta_i & \left\{ \int_{0^+}^{t} f_i(t-\tau) \frac{\delta a_i(\tau)}{\delta \tau} d\tau + f_i(t) a_i(0) \right\}
\end{align*}
\]

(16)

Equation 16 is Voltera's integral equation. The solution of this equation yields \( \psi(t) \), the desired stress or displacement for the viscoelastic body. It should be noted that the \( a(t) \) and \( \beta(t) \) terms in Equation 16 are multiple convolution integrals in the form of

\[
\begin{align*}
\gamma(\tau) &= \int_{0^+}^{T} \gamma_1(\tau-\lambda) \frac{\delta}{\delta \lambda} \int_{0^+}^{\lambda} \cdots \int_{0^+}^{\xi} \gamma_{K-1}(\xi-\eta) \frac{\delta V(\eta)}{\delta \eta} d\eta \\
+ \gamma_{K-1}(\xi) \gamma_K(0) d\xi + \cdots + \gamma_1(\tau) \gamma_2(0) - \cdots - \gamma_K(0)
\end{align*}
\]

(17)
The above approach has three main advantages. First of all, the Laplace transform is not used, and thus it is not necessary that all of the equations and boundary conditions have Laplace transforms. Secondly, the application of the above method, although possibly appearing complex because of its abstract form in the above presentation, is straightforward. Thirdly, due to the general approaches used to evaluate the multiple convolution integrals, and since the integral equation is solved numerically, the relaxation or creep functions which appear in the solution can be kept realistic and representative of real materials.

The above method will be used in this thesis to analyze the stress-deflection response of a three layer pavement system.
CHAPTER III
MATERIALS AND TESTING PROCEDURES

Materials

In order to evaluate the reliability of the theoretical formulation of the viscoelastic layered analysis, experiments were conducted on a viscoelastic slab resting on an elastic layered media (Figures 2, 3 and 4).

The pavement model used in this investigation consisted of a viscoelastic slab, an elastic base and an elastic subgrade. A commercially manufactured gum 4 feet in diameter and 12 inches in height was used to simulate the elastic subgrade. The characterization experiments on this gum indicated that the material is linear elastic with Young's modulus of 210 lb./in.².

The base course in this model was simulated with a sheet of rubber 1 inch thick and 4 feet in diameter. Tests on the rubber indicated that it is linear elastic with Young's modulus of 750 lbs./in.².

To satisfy the boundary condition used in the theoretical formulation, the base course was bonded to the elastic subgrade with Bondmaster rubber cement. The sand asphalt slab was then bonded to the rubber with a thin tack coat of 60/70 penetration asphalt.
Figure 2. Schematic Diagram of the Testing System
Figure 3. Photograph of the Testing System I
Figure 4. Photograph of the Testing System II
In this pavement model, the asphaltic slab consisted of a 44" diameter circular plate with a thickness of 2" or 1-1/2". The mixture characteristics used in the preparation of these slabs are presented in Table 1. The asphaltic slabs were compacted in the laboratory using a vibrating compactor and a circular mold assembly. First, the asphaltic mixture was distributed very uniformly in the circular mold by means of special dividers and rakes. Then the compaction energy was introduced through a compaction head into the asphaltic mixture.

It should be noted that each specimen was visually examined as well as X-rayed to detect possible flaws and non-uniformities before being accepted for testing.

Testing Procedures

In this study several types of instrumentation were used to control environment, to apply load and to measure the response of the layered system.

To obtain the consistent results necessary, temperature was maintained by the normal room control system to within ± 0.5°F of the desired room temperature, 78.5°F. During the tests, the temperature of the sand asphalt was measured by thermocouples at the surface as shown in Figure 3.

To measure the response of the layered system, strain gages and LVDT's were utilized. Radial and tangential strains on the surface were measured by strain gages. The gages employed were SRA-5 paperback gages, 0.5 inches long, with a standard resistance of 120 ohms. An exact layout of the strain gages and other instrumentation is shown in Figure 5. The strain gages
TABLE 1

Test Characteristics of Sand-Asphalt Mixtures

A. Asphalt Cement

<table>
<thead>
<tr>
<th>Test</th>
<th>60/70</th>
</tr>
</thead>
<tbody>
<tr>
<td>Specific Gravity</td>
<td>1.010</td>
</tr>
<tr>
<td>Softening Point, Ring and Ball</td>
<td>123°F</td>
</tr>
<tr>
<td>Ductility 77°F</td>
<td>150 + cm</td>
</tr>
<tr>
<td>Penetration, 100 gm., 5 sec., 77°F</td>
<td>63</td>
</tr>
<tr>
<td>200 gm., 60 sec., 39.4°F</td>
<td>23.5</td>
</tr>
<tr>
<td>Flash Point, Cleveland Open Cup</td>
<td>455°F</td>
</tr>
</tbody>
</table>

B. Aggregate

<table>
<thead>
<tr>
<th>Sieve Number</th>
<th>% Passing (samples)</th>
<th>ASTM Specification D1663-59T</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>100.0</td>
<td>85-100</td>
</tr>
<tr>
<td>30</td>
<td>88.0</td>
<td>70-95</td>
</tr>
<tr>
<td>50</td>
<td>64.2</td>
<td>45-75</td>
</tr>
<tr>
<td>100</td>
<td>30.1</td>
<td>20-40</td>
</tr>
<tr>
<td>200</td>
<td>15.7</td>
<td>9-20</td>
</tr>
</tbody>
</table>

C. Mixture

- % Asphalt Cement (by weight of aggregate): 6%
- Average Density: \(2.052 \pm 0.011 \text{ gm/cm}^3\)
- Average Air Voids: 17.0 ± 0.5%
were attached to the sand asphalt slab with an epoxy-150 cement. The output of the strain gages was measured by including the gage as the active arm in a four-arm wheatstone bridge. In order to provide temperature compensation for the active gages, similar strain gages were mounted on a dummy sand asphalt slab to form the opposite arm of the bridge. The remainder of the bridge was composed of the same strain gages mounted on an aluminum plate. No drift occurred in the gages at the working temperature. Strain gage output was amplified by Sanborn and Brush amplifiers and recorded by Sanborn and Brush recorders.

Deflections of the surface of the pavement slab were measured by linear variable differential transformers (LVDT). The LVDT's used were Datronic LVDT's, made readable to 0.001 of an inch by increasing the input voltage above that normally used. Linearity was checked by a micrometer while the voltage was amplified by a Sanborn amplifier and recorded on a Sanborn recorder. The arrangement of the LVDT and a view of the supporting frame for the LVDT is shown in Figures 2 and 3. Figure 5 designates the location of the three LVDT's.

In this investigation the input load functions were applied to the material by a MTS closed loop system. This equipment consists of a control console onto which the waveform, amplitude, and frequency of a repetitive load are set, and a load cell and a hydraulic actuator which is connected electrically to the control servo-amplifier and hydraulically to a high pressure pump. The actuator applies the load in the form and frequency set on the function generator of the console. When the maximum load set on the servo-amplifier
Figure 5. Configuration of the LVDT, Strain Gages Thermocouple on the System
at the console is sensed by the load cell on the actuator, the system unloads hydraulically according to the electrically set function. The test pavement was loaded through a 1" thick sieves of aluminum plates, and 1.5", 2", and 2.5" in radius. A 1" thick rubber pad was bonded to the aluminum plate in order to give uniform load distribution. A thin teflon sheet was placed under the rubber pad in order to reduce the friction between the pavement slab and the loading system. This load application system is illustrated by Figure 2.

Two modes of loading and three different magnitudes of load were applied to the pavement during the test. These modes of loading were:

1) A half-sine load function applied at the rate of ten cycles per second with nine cycles suppressed as shown in Figure 6. The duration of loading was 0.1 second and was repeated each second allowing 0.9 seconds of rebound between load applications. This type of test will be referred to as a type A test. Each type A test was run for 400 load repetitions.

2) A creep test with the load applied as instantaneously as mechanically possible (0.02 seconds) was maintained for 30 minutes. This type of static test will be referred to as a type B test.

In order to prevent possible damage to the system and to eliminate any impact loads, a surcharge load was maintained on the pavement during the repeated load tests. Tests were conducted at three levels of load intensity, 10, 15, and 20 pounds per square inch, for each loading plate. The tests at each level of load were run in the order of Type A, Type B; then a repeat of that sequence; then changed to the next higher load.
Figure 6. Dynamic Loading Function

Deflection
CHAPTER IV

EXPERIMENTAL RESULTS

Material Characterization

The theoretical analysis of linear or viscoelastic layered systems, requires, besides geometrical and boundary conditions, consideration to the material characteristics of each layer. In the linear elastic analysis, only the Young's modulus $E$ and Poisson's ratio $\mu$ of each layer are needed. The viscoelastic stress analysis however requires knowledge of time dependent material constants which are commonly obtained from creep, relaxation or dynamic experiments. The material characterization, however, does not necessarily require a complete analysis of all three responses, since according to the theory of linear viscoelasticity, the creep, relaxation, dynamic and other responses are interrelated by means of transformation techniques. However, when the validity of the linearity of the material response is examined, it requires that the interrelation among various factors be verified. Therefore in this investigation, in order to get the proper representation of the material response, and to verify the assumption of linearity, three types of experiments were conducted; namely, creep tests under different stress levels,
relaxation tests under different strain levels and dynamic tests under different
stress levels and at different frequency ranges.

These experiments also included tests on cylindrical samples sub-
jected to compression and tension states of stress and simply supported beam
samples subjected to concentrated point loads at the center. The details of
the tests and test results are discussed as follows:

Compressive Creep Tests on Cylinders

The constant-load creep test was used as one of the principal tests in
this study to evaluate the properties of sand asphalt mixtures. The length of
the cylindrical samples was 5-5/8 inches and the radius was 1-13/32 inches.
These tests were performed on specimens at an experimental temperature of
78.5°F using MTS equipment.

In most of the tests, three or more experiments were performed
under identical conditions to obtain reproducible data. The average of the
experimental observations are reported here.

The experiments were conducted under three axial stress levels of
5, 10 and 15 psi. These stress levels were relatively low when compared to
the ultimate unconfined compressive strength of the material and were selected
because they were within the linear viscoelastic stress range for the materials
at the conditions investigated. In Figure 7, the linearity of the system under
various stress levels is presented.

To obtain the analytical expressions from the creep data, a series
expansion of exponential functions (prony series) can be used (49). The
criterion for the selection of the coefficients of each series, and the conditions for their positiveness has been obtained by Schapery (50) and Moavenzadeh (51). Therefore, the creep data can be represented by

\[ J(t) = \sum_{i=1}^{n} A_i e^{-\alpha_i t} \]  \hspace{1cm} (18)

Where \( A_i \) and \( \alpha_i \) are material constants and can be determined by the procedure suggested by Brisbane (52) in Appendix I. The comparison of a typical experimental curve with the creep compliance calculated from equation (18) is shown in Figure 8.

The creep response is then transformed into the frequency domain to yield the magnitudes of the complex dynamic modulus, \( |E^*| \), and the relaxation modulus, \( E_r(t) \), under constant strain. These predicted results are later compared with the experimental relaxation modulus and the complex dynamic modulus.

**Compressive Relaxation Tests on Cylinders**

The constant strain relaxation tests were also used to evaluate the rheological properties of sand asphalt mixtures. The tests were performed on the samples at an experimental temperature of 78.5°F using a MTS machine. Similar to the creep experiments, three or more tests were performed under identical conditions and the data were then averaged. The axial strain levels used were 4.5, 6.75, 9.0 x 10^{-3} in./in. The linearity of the test results under these strain levels were checked similar to the creep tests (see Fig. 7).
The comparison of the experimental relaxation modulus with the relaxation modulus calculated from the transformation of the creep data is shown in Figure 9. The agreement is found to be excellent indicating the validity of the assumption of linear response.

Compressive Dynamic Tests on Cylinders

Dynamic tests were also performed on the sand asphalt specimens at an experimental temperature of 78.5°F. These specimens were loaded such that the actual force on the specimen being tested varied sinusoidally from zero to a maximum compressive stress, \( \sigma_0 \), according to the following equation:

\[
\sigma = \sigma_0 + \sigma_0 \sin \omega t
\]

Because all stresses were small compared with the strength of the material, it is assumed that the presence of the static stress did not influence the dynamic response, and that the principle of superposition is valid within the range of stresses used. The stress amplitude, \( \sigma_0 \), varied between 5, 10 and 15 psi and frequency values of 0.01, 0.1, 0.5, 1, 10, 20 cycles/sec. were used in the tests. Similar to the creep and relaxation experiments, the reproducibility of the tests data and their linearity under stresses were investigated. The comparison of the experimental dynamic modulus with the dynamic modulus determined from transformation of the creep tests is shown in Figure 10, indicating excellent agreement.

Creep Tests on Beams

These tests were performed on beam specimens with a width of three
inches, depth of two inches and length of eight inches at an experimental temperature of 78.5°F by using MTS equipment. The load levels were 5, 10 and 15 lbs. and were selected to be within the linear viscoelastic range for the materials. The averaged creep test data was fitted by an 11-element model as before (Equation 18) and the comparison of the theoretical response and the experimental curves is shown in Figure 11. The creep compliance response represented by Equation 18 was then transformed into the frequency domain to yield the magnitudes of the complex dynamic modulus and the relaxation modulus under constant deflection.

**Relaxation Tests on Beams**

Constant deflection relaxation tests were also performed on identical beams at an experimental temperature of 78.5°F. Three deflection levels of 2.5, 5 and 7.5 x 10^{-2} inches were used and the linearity of the test results under these deflection levels were verified. The average experimental relaxation modulus was then compared with the transformed relaxation modulus obtained from the creep data as shown in Figure 12.

**Dynamic Tests on Beams**

Similar to the relaxation experiments, dynamic tests were performed on beam specimens at a temperature of 78.5°F. These specimens were loaded such that the actual load on the specimen being tested varied sinusoidally from zero to a maximum as shown below

\[ P = P_0 + P_0 \sin \omega t \]

The load amplitude \( P_0 \) was selected to be small as compared with the strength
of the material so that the principle of superposition of linear viscoelastic behavior could be considered valid. Three load levels ($P_0$) of 5, 10 and 15 lbs. and six frequency values of 0.01, 0.1, 0.5, 1.0, 10 and 20 cycles/sec. were used. The average experimental dynamic modulus was then compared with the transformed dynamic modulus obtained from the creep response. The comparison of the data is shown in Figure 13.

**Tensile Creep Tests on Cylinders**

Tensile creep experiments were also conducted on cylindrical samples. To measure strains, two SRA-4 strain gages were mounted on opposite sides of each specimen. The test conditions were exactly the same as those of the compressive creep experiments. In Figure 14 the creep compliance for tension and compression and the bending creep compliance are presented. It is observed that the bending creep compliance falls in between that for tension and compression. It is also noted that, at short loading times, the creep compliances of these specimens are very close whereas, at long loading times the difference is substantial.

**Tests on Viscoelastic Layered Systems**

**Creep Test**

The experimental set up for the viscoelastic system was mentioned in Chapter IV and consists of two slabs of thickness 1.5 inches and 2 inches resting on an elastic foundation. Tests were carried out at a temperature of 78.5°F. For each slab, three LVDT readings were taken at the center and at r's of three inches and eight inches away from the center of the loaded plate.
Four radial strains and four tangential strains at $r=4''$, 6'', 8'' and 10'' away from the center were also measured. Three stress levels of 10, 15 and 20 psi were applied. Under each stress level, three different loading plates with a radii of 1.5'', 2'' and 2.5'' were used. The repeatability of the test results and the linearity under different stress levels were investigated. The averaged data were used in this investigation. The experimental results for all the measurements on the 1.5'' slab are shown in Figure 15 to Figure 25 and those for the 2.0'' slab are shown from Figure 26 to Figure 36.

**Dynamic Test**

The dynamic tests were conducted under conditions similar to those of the creep tests. The type of loading function used is shown in Figure 6. Three stress levels of 10, 15 and 20 psi were applied. The dynamic response has been analyzed. Due to the low stress levels used, it was found that the strain and deflection responses are almost constant during the 400 cycles. These responses are composed of two parts: a dynamic part due to a half sine load plus a static part due to a small static load. It appears that there is no accumulation of pavement deformation during the loading cycles. This can be attributed partly to the idealized elastic subgrade and to the low stress levels. Furthermore, it appears that the dynamic deflection or strain amplitude does not change during the 400 cycles. This indicates that there is no damage occurring in the pavement system.
Figure 7. Linearity of Compressive Sample Response.
Figure 8. Comparison of Experimental and Calculated Creep Compliance Curve vs. Time from the Model (Cylindrical Sample)

- Experimental
- Calculated from the Mechanical Model

Creep Compliance J (t) (in./in./psi)

<table>
<thead>
<tr>
<th>Time, secs.</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>5 x 10^-4</td>
</tr>
<tr>
<td>0.1</td>
<td>10 x 10^-4</td>
</tr>
<tr>
<td>1.0</td>
<td>20 x 10^-4</td>
</tr>
<tr>
<td>10.0</td>
<td>30 x 10^-4</td>
</tr>
<tr>
<td>100.0</td>
<td>40 x 10^-4</td>
</tr>
<tr>
<td>1,000.0</td>
<td>50 x 10^-4</td>
</tr>
<tr>
<td>10,000.0</td>
<td>60 x 10^-4</td>
</tr>
</tbody>
</table>

Constants:

- $E_0 = 5.556 \times 10^4$ psi
- $E_1 = 4.025 \times 10^4$ psi
- $E_2 = 3.20 \times 10^4$ psi
- $E_3 = 6.024 \times 10^4$ psi
- $E_4 = 0.126 \times 10^4$ psi
- $E_5 = 5.297 \times 10^4$ psi

- $\lambda_1 = 1.242 \times 10^4$ psi-sec.
- $\lambda_2 = 1.562 \times 10^4$ psi-sec.
- $\lambda_3 = 0.083 \times 10^3$ psi-sec.
- $\lambda_4 = 0.396 \times 10^3$ psi-sec.
- $\lambda_5 = 0.800 \times 10^3$ psi-sec.
Figure 9. Comparison of Experimental and Predicted Relaxation Response - vs. Time from Experimental Creep Data (Cylindrical Sample).
Figure 10. Comparison of Experimental and Predicted Complex Elastic Modulus vs. Frequency for Cylindrical Samples.
Figure 11. Comparison of Experimental and Calculated Creep Compliance Curve vs. Time from the Model
(Beam Specimens)

- Experimental
- Calculated from the Mechanical Model

Creep Compliance J(t) (in./ln. /psi)

<table>
<thead>
<tr>
<th>Compliance</th>
<th>Value</th>
<th>Time, secs.</th>
</tr>
</thead>
<tbody>
<tr>
<td>E₀</td>
<td>3.333 x 10⁴ psi</td>
<td>0.01</td>
</tr>
<tr>
<td>E₁</td>
<td>0.451 x 10⁴ psi</td>
<td>0.1</td>
</tr>
<tr>
<td>E₂</td>
<td>1.551 x 10⁴ psi</td>
<td>1.0</td>
</tr>
<tr>
<td>E₃</td>
<td>0.065 x 10⁴ psi</td>
<td>10.0</td>
</tr>
<tr>
<td>E₄</td>
<td>0.114 x 10⁴ psi</td>
<td>100.0</td>
</tr>
<tr>
<td>E₅</td>
<td>5.297 x 10⁴ psi</td>
<td>1,000.0</td>
</tr>
<tr>
<td>E₆</td>
<td>5.297 x 10⁴ psi</td>
<td>10,000.0</td>
</tr>
</tbody>
</table>

- E₀ = 3.333 x 10⁴ psi
- E₁ = 0.451 x 10⁴ psi
- E₂ = 1.551 x 10⁴ psi
- E₃ = 0.065 x 10⁴ psi
- E₄ = 0.114 x 10⁴ psi
- E₅ = 5.297 x 10⁴ psi

λ₁ = 0.772 x 10⁴ psi-sec.
λ₂ = 0.226 x 10³ psi-sec.
λ₃ = 0.536 x 10³ psi-sec.
λ₄ = 0.608 x 10² psi-sec.
λ₅ = 0.560 x 10¹ psi-sec.
Figure 12. Comparison of Experimental and Predicted Relaxation Response vs. Time for (Beam Specimens)
Figure 13. Comparison of Experimental and Predicted Complex Elastic Modulus for Beam Samples.
Figure 14. Comparison of Creep Compliance vs. Time for Tension, Compression and Bending Load
Figure 15. Comparison of Theoretical and Experimental Deflections vs. Time at the Center of Slab ($h_1 = 1.5''$)
Figure 16. Comparison of Theoretical and Experimental Deflections vs. Time at $r = 3^{"}$ of Slab ($h_1 = 1.5^{"}$)
Figure 17. Comparison of Theoretical and Experimental Deflections vs. Time at $r = 8''$ of Slab ($h_1 = 1.5''$)
Figure 18. Comparison of Theoretical and Experimental Radial Strains vs. Time at $r=4''$ of Slab ($h_1=1.5''$)
Figure 19. Comparison of Theoretical and Experimental Radial Strains vs. Time at \(r = 6''\) of Slab \((h_1 = 1.5'')\)
Figure 20. Comparison of Theoretical and Experimental Radial Strains vs. Time at \( r = 8'' \) of Slab (\( h_1 = 1.5'' \))
Figure 21. Comparison of Theoretical and Experimental Radial Strains vs. Time at $r=10''$ of Slab ($h_1 = 1.5''$)
Figure 22. Comparison of Theoretical and Experimental Tangential Strains vs. Time at \( r = 4'' \) of Slab (\( h_1 = 1.5'' \))
Figure 23. Comparison of Theoretical and Experimental Tangential Strains vs. Time at $r = 6''$ of Slab ($h_1 = 1.5''$)
Figure 24. Comparison of Theoretical and Experimental Tangential Strains vs. Time at $r = 8''$ of slab ($h_1 = 1.5''$)
Figure 25. Comparison of Theoretical and Experimental Tangential Strains vs. Time at \( r=10'' \) of Slab (\( h_1=1.5'' \))
Figure 26. Comparison of Theoretical and Experimental Deflections vs. Time at Center of Slab (h₁ = 2.0")
Figure 27. Comparison of Theoretical and Experimental Deflections vs. Time at $r = 3''$ of Slab ($h_1 = 2''$)
Figure 28. Comparison of Theoretical and Experimental Deflections vs. Time at r = 8" of Slab (h1 = 2")
Figure 29. Comparison of Theoretical and Experimental Radial Strains vs. Time at $r=4''$ of Slab ($h_1 = 2.0''$)
Figure 30. Comparison of Theoretical and Experimental Radial Strain vs. Time at $r = 6''$ of slab ($h = 2.0''$)
Figure 31. Comparison of Theoretical and Experimental Radial Strains vs. Time at $r = 8''$ of slab ($h_1 = 2.0''$)
Figure 32. Comparison of Theoretical and Experimental Radial Strains vs. Time at $r = 10''$ of slab ($h_1 = 2.0''$)
Figure 33. Comparison of Theoretical and Experimental Tangential Strains vs. Time at r = 4" of Slab (h₁=2.0"")
Figure 34. Comparison of Theoretical and Experimental Tangential Strains vs. Time at r=6" of Slab (h₁ = 2.0")
Figure 35. Comparison of Theoretical and Experimental Tangential Strains vs. Time at \( r = 8'' \) of Slab \( (h_1 = 2.0'') \)
Figure 36. Comparison of Theoretical and Experimental Tangential Strains vs. Time at $r = 10''$ of Slab ($h_1=2.0''$)
CHAPTER V

DISCUSSION OF TEST RESULTS

Material Characterization

In this study in order to obtain material properties needed for the theoretical analysis, a series of creep, dynamic and relaxation tests were conducted. The characteristic tests were carried out to establish: (i) the linear range of viscoelastic response of sand-asphalt mixtures; (ii) the validity and accuracy of interrelating various viscoelastic functions; and (iii) the selection of appropriate material constants needed for the analysis.

The experimental results presented in Chapter IV (Figure 7) indicate that the sand asphalt mixture used in this study and under the described test conditions can be treated as a linear viscoelastic material. Furthermore, the presented data (Figure 8, 9, 10) show clearly that viscoelastic response can be accurately transformed from one response to another as assumed by the theory of linear viscoelasticity.

However, as a matter of great interest, the comparison of creep response under tensile, bending and compressive loadings indicates that the material response differs under these modes of loading.
A decision should then be made with respect to the most appropriate and correct creep function to be used in the analysis. According to the data shown in Figure 14 at short loading time, corresponding to high frequency load application, there is no significant difference between creep functions in compression, bending and tension. However at long loading time, there is a substantial difference, i.e. the creep function for bending falls in between those for compression and tension.

The pressure bulb concept shows that the influence of the surface load intensity \( q \) may be effective through a depth of approximately 1.5" to 2.0" times the diameter plates. The three layer system used in the laboratory was made up of 1.5" or 2.0" top layer. Thus, three-inch, four-inch and five-inch diameter plates were used for loading the system and could extend the influence of the surface loading intensity \( q \) below this layer. In addition, the modulus of sand-asphalt mixtures is relatively higher than the modulus of the foundation. Under these conditions, it is more likely that the first layer is subjected to the flexure mode. Therefore, the bending creep compliance of the first layer was used to calculate the deflections and strains at various points of the viscoelastic three layer system.

**Theoretical Analysis of Linear Viscoelastic Layered Systems**

The theoretical values of deflections and strains were obtained from the three layer linear viscoelastic computer program shown in Appendix II. This program which was originally developed by Ashton has been modified to suit the IBM Computer System/360 available at The Ohio State University.
The input for the computer program is the creep compliance $J(t)$, Poisson's ratio $\mu = 0.5$, thickness $h$ of each layer, load plate radius $r$, and load intensity $q$. The output of this computer program generates only deflections and stresses at various points in the linear viscoelastic layered system; whereas the experimental data produces information about deflection and strains instead of stresses. Therefore, the radial strains $\varepsilon_r$ were calculated as:

$$\varepsilon_r = \frac{u}{r} \quad (19)$$

for axially symmetric load systems in cylindrical coordinates as shown in Figure 1.

This formula can be approximated for a small increment $\Delta r$ as follows:

$$\varepsilon_r \approx \frac{\Delta u}{\Delta r} \quad (20)$$

In this study $\Delta r$ was fixed by the length (0.5") of the strain gages (SR4A-5-513) used to measure the strains.

Let $U_1$ and $U_2$ be the radial displacements corresponding to the radii $R_1$ and $R_2$ (measured along the same radius) respectively.

Then $\Delta U = U_2 - U_1$

and $\Delta r = R_2 - R_1$ assuming $R_2 > R_1$

If we select $R_2 = r + \Delta r/2$

and $R_1 = r - \Delta r/2$

then we get the average distance $r = \frac{1}{2} (R_2 + R_1)$

Now, the radial strain $\varepsilon_r$ at a radius $r$ can be calculated as:
\[ \epsilon_r \approx \frac{\Delta u}{\Delta r} \]

as explained in Equation 20.

The tangential strain \( \epsilon_t \) (for axially symmetric load system in cylindrical coordinates as shown in Figure 1) is calculated as:

\[ \epsilon_t = \frac{u}{r} \]

where \( u \) represents the radial displacements at a radius \( r \).

Both the theoretical and experimental results were plotted on semi-logarithmic graphs as shown in Figures 15 through 36. Results for systems with 1.5" thick top layers are shown in Figures 15 to 25. The rest of the figures (26-36) show the results for systems with 2.0" thick top layers. Each set of the three curves in each figure corresponds to the three different load plates used in this investigation.

As already mentioned in Chapter I, the purpose of this investigation was as follows:

1) to investigate experimentally the strains and displacements in a linear viscoelastic system, and

2) to evaluate the reliability of the theoretical analysis in predicting the primary response behavior of flexible pavements.

These objectives are attained by taking measurements of pavement surface deflections and strains under different combinations of the variables involved as described in Chapter IV. The results of this investigation were analyzed as mentioned earlier in this chapter. A discussion of the results follows in subsequent paragraphs.
Analysis of the Deflection Data

Typical deflection versus time diagrams are shown in Figure 15-17 and 26-28. It was observed that the surface deflection under the center of the loaded area increased with increasing time for different load plates used (see Figures 15 and 26). A similar trend was observed for deflections measured at a distance $r = 3''$ away from the center of the load, (see Figures 16 and 27). The trend, however, reversed itself at a distance $r = 8''$ away from the center of the load (see Figures 17 and 28).

In all these cases, the experimental values were in good agreement with the theoretical values obtained using the computer program. In general, the theoretical values are larger than the experimental values, the variation being less than 15% in most of the cases investigated except in one case where the variation was much larger (Figure 17).

Analysis of the Surface Strain Data

Radial Strains: Radial strain versus time diagrams was plotted for all the cases described above, and typical results are shown in Figures 18 to 21 and 29 to 32. It was observed that at a radial distance $r = 4''$, the radial strain $\epsilon_r$ increased with increasing time $t$ (see Figures 18 and 29). In other cases, for $\epsilon_r$ measured at $r = 6''$, $8''$ and $10''$, the usual trend was such that $\epsilon_r$ increased with increasing $t$ until a peak value of $\epsilon_r$ was reached. After this peak, the value of $\epsilon_r$ decreased with increasing time $t$. Usually the peak value of $\epsilon_r$ occurred between $t = 1.0$ sec. and $t = 10.0$ sec.
In general it was observed that the experimental results followed the theoretical pattern although the values were not in good agreement. For the case of \( r = 4'' \), the difference between the experimental and theoretical values ranged around 30\% or more. But in cases where \( r = 6'' \), 8'' and 10'', the difference usually was within 15\%.

**Tangential Strains:** Tangential strains were analyzed in a similar way as the radial strains. Results have been plotted in Figures 22-25 and 33-37. As in the previous cases, the strains \( \epsilon_t \) also increase with increasing time \( t \) until a maximum value of \( \epsilon_t \) is reached; then it starts decreasing with increasing time \( t \). In most of the cases, the maximum value of \( \epsilon_t \) is observed within 1.0 second of the loading. Unlike radial strains, tangential strains do not change significantly after the maximum \( \epsilon_t \) is reached.

It was observed that the experimental results were generally in good agreement with the theoretical values. In this case also, as in the previous cases, the theoretical values were larger than the measured values. The maximum deviation observed was about 30\%.

**Discussion of the Test Results**

It should be noted that the test results reported above are subject to the various constraints of experimentation, and it is very difficult to match the assumptions made in the theoretical analysis of the system. This is true for every phase of the investigation whether it relates to the characterization of the materials (homogeneous, isotropic, etc.) or instrumentation of the models or the analysis of the systems using numerical methods. Thus,
certain discrepancies may be expected between the theoretical and experimental results and could possibly be explained as follows:

1) The theoretical analysis used is based on the assumption of complete bond at the interface of the adjoining layers. The asphalt tack coat used as a bond between the rubber and the sand-asphalt slab may not completely satisfy the theoretical assumptions, and the shear force may have developed only partially. So far no theoretical solutions are available which include this factor.

2) The theoretical results were based on Poisson's ratio of 0.5\(^\text{\textdegree}\) for each layer. In the laboratory setup, the assumption of \(\mu = 0.5\) for rubber and gum may be close to reality. However, Poisson's ratio for sand asphalt may be less than 0.5. Therefore, an investigation was made by using \(\mu = 0.4\) and \(\mu = 0.5\) for sand asphalt in an elastic system and using \(\mu = 0.5\) for rubber and gum layers. The two results did not indicate much difference in the calculated deflections and strains. However, the exact analysis should be based on time-dependent Poisson's ratio \(\mu (t)\) for sand asphalt.

3) The three-layer system used for theoretical analysis is based on the assumption of a semi-infinite third layer. In the experimental setup, the third layer was supported on a rigid base and was only 12\(^\text{\textdegree}\) thick. Since visco-elastic analysis at present is available for three layers only, it was not possible to investigate the effect of the fourth layer present in the system.

4) The stress distribution may not be uniform under the load plate, although a rubber pad was attached to the aluminum plate. The actual stress
distribution cannot be measured due to the limitations of the equipment. Also, the effect of this factor on the theoretical analysis of strains and deflections cannot be examined.

5) An initial check on the strain gauges was made by using them on a compression test sample along with an LVDT. Although this test indicated good reliability on strain gauges as compared to the LVDT, the bonding cement still may cause local stiffening on the sand asphalt slab. This means that a higher Young's modulus may occur at the spot measured, which may be responsible for the smaller measured strains as compared to the theoretical values.

6) The discrepancies may be partly due to the anisotropic behavior of the sand asphalt slab. Laboratory tests have been conducted on the sand asphalt mixtures in order to examine the anisotropy. It has been found that the ratio of $E_v / E_H$ was 1.25. Unfortunately, theoretical analysis for multi-layer systems of anisotropic materials is not available yet.

7) In most cases it was observed that the discrepancies between the measured and theoretical values of radial strains were much larger at $r = 4''$ than at $r = 6''$, $8''$, or $10''$. This may be due to the approximations used in the analysis involving the relationship $\epsilon_r = \frac{\Delta u}{\Delta r}$. Herein a constant value of 0.5'' was used for $\Delta r$, whereas the slope of U-$r$ curve ($\Delta u/\Delta r$) is likely to change more at $r = 4''$ than $r > 4''$. Therefore, the approximation used for calculating $\epsilon_r$ will produce better results at $r > 4''$ than at $r = 4''$. 
CHAPTER VI

CONCLUSIONS

The conclusions presented are specially applicable only to the particular type of systems considered in this investigation. Within the range of this study, the following conclusions appear to be justified.

1) In this study in order to characterize material properties accurately, it requires a complete series of creep tests, relaxation tests and dynamic tests to get a representative viscoelastic model for the material.

2) The sand asphalt mixture used here has different behavior patterns under creep tests of tension and compression. It has been found that the creep compliance of tension is almost double that of compression except for short time intervals. The bending creep compliance, however, falls in between them.

3) It seems that if the flexure mode occurs in the first layer, the predictions of strains and deflections on the viscoelastic layered system would be better based on the bending creep compliance instead of conventional compression creep compliance, unless the material has the same characteristics in tension and compression.
4) The comparison between theoretical and experimental results indicates that deflections and tangential strains can be predicted fairly accurately.

5) The comparison between the theoretical and experimental results of radial strains are not in good agreement. Experimental limitation of the model can account for the discrepancy.

6) Finally, from this study, it may be concluded that the linear visco-elastic theory can be used to predict the time-dependent behavior of a layered pavement system under creep loading.

This study was the preliminary laboratory investigation into the linear viscoelastic layered system under creep loading. It is suggested that the following research be conducted to continue this investigation.

a) Investigations should be conducted on the behavior of linear visco-elastic layered systems under various temperature ranges instead of under one single temperature.

b) Investigation of the layered system by using real highway materials. This means that a soil subgrade, granular subbase and asphalt concrete full scale setup should be tested.

c) Development of a computer program for more accurate analysis of linear viscoelastic multi-layered systems.
Appendix A

A creep compliance curve can be fitted by prony series as:

\[ J(t) = \sum_{i=1}^{n} A_i e^{-\alpha_i t} \]

Considerable difficulties arise in the above equation because it appears that a set of \(2n+1\) simultaneous transcendental equations must be solved to evaluate the constants \(A_i\) and \(\alpha_i\).

In this method, the constants \(\alpha_i\) are judiciously chosen and then the corresponding set of constants \(A_i\) are found from a set of simultaneous linear equations. The \(\alpha_i\) are chosen by means of the formula:

\[ \alpha_i = \frac{1}{2t_i} \]

when \(t_i\) represents the values of time in the data points. The data points were chosen one decade apart. Because of smoothly varying functions of logarithmic time of the general type represented by creep curves, the change of the function over a decade of time can often be adequately represented by a single exponential term.
By substituting $t_1$ and $J(t_1)$ a set of linear equations formed. Then $A_1, A_2, A_3$, $A_4, A_5$, and $A_n$ can be solved very easily. In this study only six terms have been used to represent the creep function.
Appendix B

(Computer Program)
MAIN PROGRAM FOR HALF-SPACE ANALYSIS USING EXACT INTEGRATION
C THIS IS THE MAIN PROGRAM FOR THE ANALYSIS OF A LINEAR VISCOELASTIC
C THREE-LAYER HALF-SPACE UNDER A UNIFORM CIRCULAR LOAD, FOR THE CASE
C THAT THE MULTIPLE CONVOLUTION INTEGRALS ARE EVALUATED EXACTLY.
C THE NECESSARY SUBROUTINES ARE CNSTNT, TIME, SOLVE, TERPO, AND
C INTEGR (EXACT). ALSO NECESSARY IS THE FUNCTION SUBPROGRAM BESSEL.
C THE INPUT IS IST, H, A, R, ZZ, ILAYER, IDEFL, NJJJ, DELTX, DELXX, AND THE
C VECTORS E1(I), E2(J), AND E3(J). IST IS A DUMMY WHICH, TOGETHER
C WITH IDEFL DETERMINES WHICH STRESS OR DISPLACEMENT IS DESIRED.
C IST IS 1 FOR NORMAL STRESS OR NORMAL DEFLECTION, IS 2 FOR SHEAR
C STRESS OR RADIAL DEFLECTION, AND IS 3 FOR RADIAL STRESS. H IS THE
C THICKNESS OF THE SECOND LAYER (THICKNESS OF THE FIRST LAYER IS
C ONE). A IS THE RADIUS OF THE LOAD. R IS THE OFFSET AT WHICH THE
C STRESS OR DISPLACEMENT IS DESIRED. ZZ IS THE DEPTH AT WHICH THE
C SOLUTION IS DESIRED. ILAYER IS THE LAYER OF INTEREST (1, 2, OR 3)
C IDEFL IS POSITIVE IF A DEFLECTION IS TO BE DONE, ZERO OTHERWISE.
C NJJJ IS AN INPUT TO THE SUBROUTINE SOLVE, AND IS EXPLAINED IN
C DETAIL THERE. DELTX AND DELXX ARE INPUTS TO THE SUBROUTINE TIME
C AND ARE EXPLAINED IN DETAIL THERE. N AND NNN ARE ALSO INPUT. N
C IS THE NUMBER OF TERMS IN THE DIRICHLET SERIES REPRESENTATIONS OF
C THE INPUT CREEP FUNCTIONS. NNN IS THE NUMBER OF POINTS IN TIME AT
C WHICH THE SOLUTION IS DESIRED. THE VECTORS E1(I), E2(J), AND E3(J)
C CONTAIN THE CONSTANTS FOR THE SERIES REPRESENTATIONS OF THE CREEP
C FUNCTIONS FOR THE FIRST, SECOND, AND THIRD LAYERS RESPECTIVELY.
C THE RESULT OF THE PROGRAM IS THE DESIRED STRESS OR DISPLACEMENT
C AT EACH OF THE NNN TIMES.

0001 DIMENSION E1(12), E2(12), E3(12), EM(13), G(7, 12, 18), GG(7, 12, 9),
     IE(8, 12), PH(18), PHJ(18), TH(9), SII(13, 201), SII(13, 201), S(13),
     BESS(91), BESS(91)
0002 COMMON X(20), DB(8, 20), T(201), DELTA(201), BETA(201), B(8, 20),
     SII(201), WI, DELTX, DELXX, NJ, NJJ
C THE LOOP THROUGH 1000 ALLOWS MULTIPLE SETS OF DATA TO BE RUN.
0003 DO 1000 I1=1, 100
0004 DO 1234 I=1, 8
0005 DO 1234 J=1, 12
0006 1234 E1(I, J)=0.0
0007 READ(5, 52) IST, H, A, R, ZZ
IF(IDFLE)55,55,53

IOWA=IST
GO TO 54

IF(IDFLE)53,55,53

IOWA=4+IST
CONTINUE

READ(5,20)NJJJ
READ(5,1)DELTX,DELXX

NJ AND NJJ ARE INPUT TO THE SUBROUTINE SOLVE. THEY HAVE NO SIGNIFICANCE IN THE PRESENT USE OF THAT SUBROUTINE AND ARE GIVEN ARBITRARY VALUES.

NJ=10
NJJ=8

READ(5,20)N,N,NNN

READ(5,1)(E1(I),I=1,N)
READ(5,1)(E2(I),I=1,N)
READ(5,1)(E3(I),I=1,N)
WRITE(6,2)(E1(I),I=1,N)
WRITE(6,2)(E2(I),I=1,N)
WRITE(6,2)(E3(I),I=1,N)

CALL TIME(NNN)

THE APPROPRIATE NNN VALUES OF TIME ARE CALCULATED AND STORED IN THE VECTOR T(I) USING SUBROUTINE TIME. ALSO, CALCULATED WITH THIS SUBROUTINE ARE THE INVERSES OF THE RELAXATION TIMES, WHICH ARE STORED IN THE VECTOR DELTA(I). THE VECTOR EM(I) SERVES AS INTERMEDIATE STORAGE OF THE VALUES OF...
THE DUMMY INTEGRATION VARIABLE M FOR WHICH THE INTEGRAL EQUATION IS SOLVED. THESE VALUES OF M ARE 0.0, 2.0, 4.0, 7.0, 1.0, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0, AND 9.0.

0032 EM(10)=6.0
0033 EM(11)=7.0
0034 EM(12)=8.0
0035 EM(13)=9.0

THE LOOP FROM HERE TO THREE ARRANGES EACH OF THE POSSIBLE COMBINATIONS OF THE FIRST FOUR CREEP FUNCTIONS FOR THE MULTIPLE CONVOLUTION INTEGRATIONS AND COMPUTES THE THREE-FOLD INTEGRAL OF THESE FOUR FUNCTIONS.

DO 3 I=1,9

DO 19 J=1,N
C THERE ARE NINE COMBINATIONS OF THESE RELAXATION FUNCTIONS.

0038 IF(I-2)12,11,15
0039 15 IF(I-4)-10,9,16
0040 16 IF(I-6)-8,7,17
0041 17 IF(I-8)-6,5,4

SOME OF THE M VALUES ARE STORED DURING THIS ARRANGEMENT.

0042 4 EM(I)=5.0
0043 E(1,J)=E1(J)
0044 E(2,J)=E1(J)
0045 E(3,J)=E3(J)
0046 E(4,J)=E3(J)
0047 GO TO 19

0048 5 EM(I)=4.0
0049 E(1,J)=E1(J)
0050 E(2,J)=E1(J)
0051 E(3,J)=E2(J)
0052 E(4,J)=E3(J)
0053 GO TO 19

0054 6 EM(I)=3.0
0055 E(1,J)=E1(J)
0056 E(2,J)=E1(J)
E(3,J)=E2(J)
E(4,J)=E2(J)

THE ITH INTEGRAL IS CALCULATED AS A SERIES OF N EXPONENTIAL TERMS
EACH MULTIPLIED BY A THIRD DEGREE POLYNOMIAL. THE CONSTANTS ARE
TRANSFERRED INTO G(i, j, i).

CALL INTEGR(E, N, 3, 0)
DO 21 L=1, N
DO 21 J=1, 4
21 G(J,L,I)=B(J,L)

CONTINUE
CONTINUE

102 FORMAT(24H INTEGRAL RESULT FOLLOWS/(E15.8))

IF ARE IN FIRST LAYER, HAVE ONLY 9 DIFFERENT MULTIPLE INTEGRALS
IN THE 'NUMERATOR'. IF IN THE SECOND OR THIRD LAYER, HAVE 18 SUCH
DIFFERENT INTEGRATIONS.

IF(ILAYER-2)22,23,23
IF IN THE FIRST LAYER, THEN THE 'NUMERATOR' AND 'DENOMINATOR' EACH
HAVE ONLY 9 SEPARATE INTEGRAL RESULTS.

22 IF(IDEFE)24,24,25
IF DOING A STRESS, THE NUMERATOR AND DENOMINATOR INTEGRAL RESULTS
ARE THE SAME. CONSEQUENTLY, THE RESULTS STORED IN G(i, j, i) ARE
ALSO TRANSFERRED INTO GG(i, j, i).

DO 38 I=1, 9
DO 38 L=1, N
DO 38 J=1, 4
38 GG(J,L,I)=G(J,L,I)

N9 = NUMBER OF INTEGRAL RESULTS IN THE 'NUMERATOR'. N7 TELLS HOW
MANY TERMS IN THE POLYNOMIALS MULTIPLYING THE EXPONENTIALS IN THE
'NUMERATOR' WHILE N8 CONTAINS HOW MANY FOR THE 'DENOMINATOR'.

N7=4
N8=4
N9=9

GO TO 50

WHEN DOING A DEFLECTION IN THE FIRST LAYER, THE 'NUMERATOR' INTEGRATIONS
CONTAIN ONE ADDITIONAL INTEGRATION INVOLVING E1(I). THUS
THE PRESENT CONTENTS OF G(i, j, i) ARE FIRST TRANSFERRED TO GG(i, j, i)

GO TO 50
WHICH IS THE DENOMINATOR ARRAY, THEN THE ADDITIONAL INTEGRATION IS CARRIED OUT BY PUTTING E1(J) IN E(8,)
AND USING THE OPTION OF SUBROUTINE INTEGR FOR EXECUTING ONE ADDITIONAL INTEGRATION GIVEN THE RESULTS OF PREVIOUS INTEGRATIONS OF SERIES. THE FINAL RESULT IS STORED BACK IN G( , , ).

```
0112  25 DO 26 J=1,N
0113  26 E(8,J)=E1(J)
0114  DO 111 I=1,9,
0115  DO 112 L=1,N
0116  DO 113 J=1,4
0117       111 GG(J,L,I)=G(J,L,I)
0118  DO 27 I=1,9
0119  DO 28 L=1,N
0120       DO 28 K=1,4
0121       28 E(K,L)=G(K,L,I)
0122       CALL INTEGR(E,N,4,1)
0123  DO 29 L=1,N
0124  DO 29 J=1,5
0125       29 G(J,L,I)=B(J,L)
0126       27 CONTINUE
0127       N7=5
0128       N8=4
0129       N9=9
0130  GO TO 50
```

WHEN IN THE SECOND OR THIRD LAYER, THE 'NUMERATOR' AND 'DENOMINATOR' CONTAIN ONE ADDITIONAL INTEGRATION. IN ADDITION, THE 'NUMERATOR' CONTAINS 9 ADDITIONAL INTEGRAL RESULTS. TO CALCULATE THESE, USE IS AGAIN MADE OF THE SPECIAL OPTION FOR EXECUTING A SINGLE ADDITIONAL INTEGRATION USING SUBROUTINE INTEGR. FIRST THE EIGHTH ROW OF E( , ) IS FILLED WITH E1(J) AND USING THE RESULTS STORED IN G( , , ) THE TENTH THROUGH EIGHTEENTH INTEGRAL RESULTS ARE FOUND USING SUBROUTINE INTEGR. THEN, THESE RESULTS ARE STORED IN G( , , ). NEXT THE EIGHTH ROW OF E( , ) IS REPLACED WITH E2(I) AND INTEGRAL RESULTS ONE TO NINE ARE CALCULATED. THESE ARE ALSO STORED IN G( , , ).
0132 32 DO 35 J=1,N
0133 35  E(8,J)=E1(J).
0134      IJ=I+9
0135 34 DO 36 J=1,N
0136 36 DO 36 K=1,4
0137 36 E(K,J)=G(K,J,I)
0138 1001 CALL INTEGR(E,N,4,1)
0139 37 DO 37 L=1,N
0140 37 J=1,5
0141 37 G(J,L,IJ)=G(J,L)
0142 31 DO 33 J=1,N
0143 33 E(8,J)=E2(J)
0144      IJ=I
0145 146 GO TO 1001
0147 30 CONTINUE
0148  N8=5
0149  N9=18
0150 IF(IDFLE)39,39,40
0151 39 DO 41 I=1,9
0152 41 DO 41 J=1,5
0153 41 DO 41 L=1,N
0154 41 G(J,L,I) = G(J,L,I)
0155  N7=5
0156  GO TO 50
0157 IF A DEFORMATION IS DESIRED, THE 'NUMERATOR' INTEGRAL RESULTS MUST
0158 BE INTEGRATED WITH EITHER E2( ) OR E3( ) YET. FIRST THE PRESENT
0159 FIRST NINE INTEGRAL RESULTS ARE TRANSFERRED INTO THE DENOMINATOR
0160 ARRAY GG( , , ). THEN THE INTEGRATION OF THE NUMERATOR RESULTS
0161 AND E2( ) OR E3( ) IS CARRIED OUT BY STORING E2( ) OR E3( ) IN
0162 THE EIGHTH ROW OF E( , ) AND USING SUBROUTINE INTEGR WITH THE
0163 SINGLE ADDITIONAL INTEGRATION OPTION. THE RESULTS ARE STORED BACK
0164 IN THE G( , , ) ARRAY.
0157  IF1ILAYER-2)42,42,43
0158   DO 44 J=1,N
0159   E(8,J)=E21(J)
0160   GO TO 45
0161   DO 46 J=1,N
0162   E(8,J)=E31(J)
0163   DO 112 I=1,9
0164   DO 112 I=1,9
0165   DO 112 J=1,5
0166   G(I,J,L,I)=G(I,J,L,I)
0167   DO 47 I=1,18
0168   DO 48 J=1,N
0169   DO 48 I=1,5
0170   E(L,J)=G(L,J,I)
0171   CALL INTEGR(E,N,5,1)
0172   DO 49 L=1,N
0173   DO 49 J=1,6
0174   G(J,L,I)=B(I,J,L)
0175   47 CONTINUE
0176   N7=6
0177   50 CONTINUE
C    ALL NECESSARY INTEGRALS ARE NOW STORED. THE NUMERATOR RESULTS
C    ARE STORED IN THE G ARRAY, DENOMINATOR RESULTS IN GG ARRAY
0178   NNX=NNN
C    THE LOOP TO STATEMENT 56 SOLVES THE INTEGRAL EQUATION FOR EACH
C    OF THE THIRTEEN VALUES OF M.
0179   DO 56 K=1,13
C    THE CONSTANTS IN THE INTEGRAL EQUATION ARE CALCULATED FOR THIS
C    VALUE OF M USING THE SUBROUTINE CNSTNT. THE RESULTS ARE STORED
C    IN THE VECTORS PH( ), PHJ( ), AND TH( ).
0180   EMM=FMK
C    CALL CNSTNT(EMM,H,ZZ,IOWA,PH,PHJ,TH,ILAYER)
C    THE TOTAL RIGHT HAND SIDE OF THE INTEGRAL EQUATION IS REDUCED TO
C    A SERIES OF EXPONENTIALS EACH MULTIPLIED BY A POLYNOMIAL CONTAIN-
0181   C    ING N7 TERMS. THE CONSTANTS IN THIS SERIES REPRESENTATION ARE ALL
C    STORED IN THE BB( , ) ARRAY.
THE KERNEL OF THE INTEGRAL OF THE LEFT-HAND SIDE OF THE INTEGRAL
EQUATION IS REDUCED TO A SERIES OF EXPONENTIALS EACH MULTIPLIED BY
A POLYNOMIAL CONTAINING N\(_R\) TERMS. THE CONSTANTS IN THIS SERIES
REPRESENTATION ARE ALL STORED IN THE B(I,J) ARRAY.

THE INTEGRAL EQUATION IS SOLVED FOR THIS VALUE OF M USING SUBROUTINE
SOLVE. THE RESULTS ARE STORED IN THE VECTOR SII( ).

THE RESULT IN SII( ) IS TRANSFERRED INTO THE KTH ROW OF SII( ).

IF (IST=3), THEN MUST SOLVE A SECOND INTEGRAL
EQUATION FOR EACH M. THIS IS DONE IN THE SAME WAY AS THE FIRST
ONE. THE CONSTANTS ARE ALREADY AVAILABLE, IN PHJ( ) AND TH( ).
THE FINAL RESULT IS STORED IN SII( ).

NEXT THE BESSEL MULTIPLIERS MUST BE CALCULATED. THESE VARY.
DEPENDING ON WHICH STRESS OR DEFLECTION IS BEING DONE.
THE BESSEL MULTIPLIERS ARE DIVIDED BY M FOR DEFLECTION ONLY. THE
VARIABLE DIVIDE IS UNITY UNLESS DOING A DEFLECTION.

DIVIDE = 1.
IF(IST = 2) 78, 79, 78
INDEX IS A DUMMY USED FOR SELECTING EITHER J0(MR) OR J1(MR).
79 INDEX = 1
INDEX IS A DUMMY USED TO STORE THE FIRST BESSEL TERM. SINCE J1(MR)
IS ZERO FOR R = 0, AND J0(MR) IS 1 FOR R = 0, TM1 IS SET ACCORDINGLY.
INDEX = 1
0210 GO TO 80
0211 78 INDEX = 0
0212 TM1 = 1.
80 IF(IDEFLE) 81, 81, 82
THE LIMIT OF J1(MA) AS M TENDS TO ZERO IS 0. SO THE FIRST TERM FOR
ALL STRESSES IS ZERO.
81 BESS(1) = 0.
0214 GO TO 83
0215 THE LIMIT OF J1(MA)/M AS M TENDS TO ZERO IS A/2. SO BESS(1) IS
A/2 FOR DEFLECTIONS.
82 BESS(1) = A/2.
0216 DDD TAKES ON THE VALUES OF M*. 91 VALUES OF THE BESSEL MULTIPLIERS
ARE COMPUTED, AT VALUES OF M*. 1 M APART.
83 DDD = 0.
0217 DO 86 I = 2, 91
0218 DDD = DDD + 1
0220 RM = R * DDD
0221 AM = A * DDD
0222 IF(RM = .0001) 84, 84, 85
0223 85 TM1 = BESS(1, INDEX, RM)
0224 84 TM2 = BESS(1, AM)
0225 IF(IDEFLE) 86, 86, 87
0226 87 DIVIDE = DDD
0227 86 BESS(1) = TM1 * TM2 / DIVIDE
0228 85 IF(IST = 3) 70, 71, 71
C IF DOING RADIAL STRESS, MUST COMPUTE A SECOND SET OF BESSEL_MUL-
TIPLERS, WHICH ARE STORED IN BESSS().

THE LIMIT OF J1(MR)JR1(MA)/MR AS M TENDS TO ZERO IS J1(MA)/2.

IF (RR = .0001) 250, 250, 799

71 BESSS(1) = 0.
0230 DDD = 0.
0231 RR = R
0232 DO 77 I = 2, 91
0233 DDD = DDD + .1
0234 RM = R * DDD
0235 AM = A * DDD

THE LIMIT OF J1(MR)JR1(MA)/MR AS R TENDS TO ZERO IS MJ1(MA)/2.

0236 IF (RR = .0001) 250, 250, 799

250 R = 1.
0238 TM1 = DDD / 2.
0239 GO TO 76
0240 799 TM1 = BESS1(1, RM)
0241 76 TM2 = BESS1(1, AM)
0242 77 BESSS(I) = TM1 * TM2 / R / DDD

TWO DIFFERENT INTEGRATIONS ON M ARE CARRIED OUT WHEN DOING THE
RADIAL STRESS. FIRST, AT EACH VALUE OF TIME, 13 VALUES ARE TRANS-
FERRED FROM SII() INTO THE VECTOR S(). THESE RESULTS ARE
USED WITH BESS() IN SUBROUTINE TERPO TO COMPUTE THIS INTEGRAL
RESULT. THIS IS STORED IN WII. THEN 13 VALUES (FOR THE SAME
TIME) ARE TRANSFERRED FROM SIII( ) INTO S() AND USED WITH
BESS( ) TO COMPUTE THE SECOND INTEGRAL RESULT. THIS IS ADDED
INTO WII, THE TOTAL RESULT MULTIPLIED BY A, AND THEN THIS ANSWER
IS PRINTED ALONG WITH THE CORRESPONDING TIME.

0243 DO 72 I = 1, NNN
0244 DO 73 J = 1, 13
0245 73 S(J) = SII(J, I)
0246 CALL TERPO(S, BESS)
0247 WII = WI
0248 WRITE(6, 102) WI
0249 DO 74 J = 1, 13
0250 74 S(J) = SIII(J, I)
0251 CALL TERPO(S, BESSS)
0252 WRITE(6, 102) WI
0253      WII=WII+WII
0254      WII=WII*A
0255      72 WRITE(6,93)T(I),WII
0256      GO TO 1000
0257      C CONTROL ENTERS HERE FOR ALL BUT RADIAL STRESS FOR THE FINAL INTEGRATION ON M. THIS IS DONE AT EACH OF THE NNN VALUES OF TIME.
0258      70 DO 91 I=1,NNN
0259      C THE 13 VALUES OF THE SOLUTION AT THE 13 VALUES OF M ARE TRANSFERRED INTO THE S() VECTOR FROM SI().
0260      91 S(J)=SI(I,J)
0261      DO 92 J=1,13
0262      92 S(J)=S(J-1)*S(J)
0263      CONTINUE
0264      IF(<S(J-1)*S(J))>784,784,783
0265      S(J)=0.
0266      783 CONTINUE
0267      C THE SOLUTION IS CALCULATED FOR THIS TIME USING SUBROUTINE TERPO AND THE CONTENTS OF S() BESS(). THIS RESULT IS MULTIPLIED BY A AND PRINTED ALONG WITH THE CORRESPONDING TIME.
0268      CALL TERPO(S,BESS)
0269      WI=WI*A
0270      91 WRITE(6,93)T(I),WI
0271      93 FORMAT(3H TIME = E15.8,12H SOLUTION = E15.8)
0272      1000 CONTINUE
0273      END
SUBROUTINE INTEGR (EXACT)
SUBROUTINE INTEGR(G,N,ITEST,ISIB)

C THIS SUBROUTINE PERFORMS THE EXACT INTEGRATIONS FOR THE CASE THAT
C THE CREEP OR RELAXATION FUNCTIONS CAN BE REPRESENTED BY DIRICHEL
C SERIES. THE INPUT IS THE ARRAY G(,), AND THE INTEGERS N, ITEST
C AND ISIB. THE ARRAY G(,) CONTAINS THE RELAXATION FUNCTIONS FOR
C THE MULTIPLE CONVOLUTION INTEGRATIONS IN THE FORM OF SERIES. EACH
C COLUMN OF G(,) CONTAINS THE CONSTANTS FOR ONE OF THE SERIES.
C N IS THE NUMBER OF TERMS IN THE SERIES REPRESENTATIONS. ITEST IS
C THE NUMBER OF MULTIPLE CONVOLUTION INTEGRATIONS INVOLVED. THE
C MAXIMUM NUMBER FOR THIS PROGRAM IS 6 (THAT IS, THE INTEGRATION OF
C 7 RELAXATION OR CREEP FUNCTIONS. ISIB IS A DUMMY WITH THE VALUE
C OF EITHER ZERO OR ONE. IF ISIB=0, THEN THE MULTIPLE CONVOLUTION
C INTEGRATIONS ARE TO BE PERFORMED FROM THE BEGINNING. IF ISIB=1,
C THEN THE RESULT OF ITEST-1 INTEGRATIONS IS STORED IN G(+,), AND
C THE ONE NEW RELAXATION OR CREEP FUNCTION SERIES IS STORED IN G(8,)
C AND IN THIS CASE ONLY ONE INTEGRATION IS PERFORMED. THE RESULT
C FROM THIS PROGRAM, STORED IN THE ARRAY B(+,), IS A FINITE SERIES
C OF EXPONENTIALS EACH MULTIPLIED BY A FINITE POLYNOMIAL. THE CON-
C STANTS OF THESE POLYNOMIALS ARE STORED IN THE COLUMNS OF B(,).
C THE DELTA TERMS (THE INVERSE OF THE RELAXATION TIMES) IS INPUT
C TO THIS PROGRAM THROUGH STORAGE IN THE VECTOR DELTA( ), WHICH IS
C CALCULATED IN THE SUBROUTINE TIME. THE NOTATION OF THIS PROGRAM
C IS DIFFERENT THAN THAT OF THE TEXT.

DIMENSION G(8,20),DEL(20),AK(20),AL(20),AM(20),AP(20),AR(20),
1D(20),C(20),B(20),C1(20),D2(20),D1(20),E1(20),C2(20),B2(20),
1E3(20),F3(20),C3(20),B3(20),D3(20),
2H4(20),C4(20),B4(20),D4(20),E4(20),H5(20),P5(20)
COMMON X(20),BB(8,20),T(201),DELTA(20),BETA(201),B(8,20),
1SI(201),WI,DELTX,DELXX,NJ,NJJ

0003 NN=ITEST+1
C THE DELTA( ) TERMS ARE TRANSFERRED TO THE VECTOR DEL( ).

DO 250 I=1,N
0005 250 DEL(I)=DELTA(I)
C ISIG, ISIG1, ISIG2, ISIG3, AND ISIG4 ARE DUMMY VARIABLES USED TO
C DETERMINE WHEN THE PROPER NUMBER OF INTEGRATIONS HAVE BEEN PER-
C FORMED.
ISIG = 1

ISIG1 = 1

ISIG2 = 1

ISIG3 = 1

ISIG4 = 1

IF(ISIG)200,200,201

200 ISIG = 0

C IF ISIG IS ZERO, THEN ITEST INTEGRATIONS ARE PERFORMED, AND ALL
C THE ISIGS ARE ZEROED UP TO THE LAST ONE.

IF(I TEST -1)202,202,203

203 ISIG1 = 0

IF(I TEST -2)202,202,204

204 ISIG2 = 0

IF(I TEST -3)202,202,205

205 ISIG3 = 0

IF(I TEST -4)202,202,206

206 ISIG4 = 0

C THE SERIES REPRESENTATIONS OF THE FIRST TWO RELAXATION OR CREEP
C FUNCTIONS ARE STORED IN THE VECTORS AK(I), AND AL(J).

DO 1 J = 1, N

1 AX(J) = G(1, J)

AL(J) = G(2, J)

GO TO 7

C IF ONLY ONE CERTAIN ADDITIONAL MULTIPLE CONVOLUTION INTEGRATION
C IS TO BE PERFORMED, THEN ONLY A SINGLE ISIG IS ZEROED.

IF(I TEST -2)207,208,209

207 ISIG = 0

GO TO 215

208 ISIG1 = 0

GO TO 215

209 ISIG2 = 0

GO TO 215

210 ISIG3 = 0

GO TO 215

211 ISIG4 = 0

GO TO 215

212 IF(I TEST -5)213,213,215
IF ONLY ONE CERTAIN ADDITIONAL CONVOLUTION INTEGRATION IS TO BE
PERFORMED, THEN THE RESULT OF THE LAST INTEGRATION MUST BE STORED
IN THE VECTORS AK( ), AM( ), AP( ), AR( ), AS( ), AND AT( ). SOME
OF THESE MAY NOT BE USED. THE NEW SERIES IS STORED IN THE VECTOR
AL( ) ALSO.

AM(J)=G(2,J)

DO 216 J=1,N

AK(J)=G(1,J)

AP(J)=G(3,J)

AR(J)=G(4,J)

AS(J)=G(5,J)

AT(J)=G(6,J)

AL(J)=G(3,J)

STATEMENT SEVEN BEGINS THE FIRST INTEGRATION, AND ALSO BEGINS THE
EVALUATION OF THE CONSTANTS RELATED TO A FIRST INTEGRATION FOR THE
LATER INTEGRATIONS (SEE TEXT).

DO 2 J=1,N

THE RESULT OF THE FIRST INTEGRATION WILL BE STORED IN THE VECTORS
C( ), AND D( ). THE VARIABLES ADUM1 AND ADUM2 ARE USED FOR INTER-
MEDIATE STORAGE.

D(J)=-DEL(J)*AL(J)*AK(J)

C(J)=AL(J)*AK(J)

ADUM1=0.

ADUM2=0.

DO 3 I=1,N

IF(I-J)\leq3,21

ADUM1=ADUM1-DEL(J)*AK(I)/(DEL(I)-DEL(J))

ADUM2=ADUM2-DEL(J)*AL(I)/(DEL(I)-DEL(J))

CONTINUE

2 C(J)=C(J)+AL(J)*ADUM1+AK(J)*ADUM2

IF ISIG IS EQUAL TO 1, THEN EITHER THIS IS THE SECOND OR MORE TIME
THROUGH THIS PATH OR ELSE A SINGLE INTEGRATION IS TO BE DONE,
WHERE THERE WERE PREVIOUSLY DONE INTEGRATIONS.

IF(ISIG\neq1)-16,9

THE FIRST TIME THROUGH, THE RESULTS OF THE FIRST INTEGRATION ARE
C STORED IN AK( ) AND AM( ), AND THE NEXT SERIES IS STORED IN AL( ).
C THE RESULTS ARE ALSO STORED IN THE BL( ) ARRAY.

C ISIG IS SET EQUAL TO 1 SO THAT THE BRANCH TO SIX WILL NOT BE TAKEN
C AGAIN. AND IF MORE INTEGRATIONS ARE TO BE DONE CONTROL RETURNS TO
C SEVEN. IF NO MORE ARE TO BE DONE, THE SUBROUTINE IS ENDED.

0058 6 DO 5 J=1,N
0059 5 B(I,J)=C(J)
0060 B(I,J)=D(I)
0061 AK(J)=C(J)
0062 AL(J)=G(3,J)
0063 AM(J)=D(J)

0064 DO 8 J=1,N
0065 8 BI(J)=AL(J)AM(J)
0066 DI(J)=-BI(J)*DEL(J)/2.
0067 ADUM1=0.
0068 ADUM2=0.
0069 ADUM3=0.
0070 DO 10 I=1,N
0071 10 IF(I-J)<22,10,22
0072 22 ADUM1=ADUM1+AL(I)DEL(J)/(DEL(J)-DEL(I))
0073 ADUM2=ADUM2-AM(I)DEL(J)/(DEL(I)-DEL(J))**2
0074 ADUM3=ADUM3+AL(I)DEL(I)/(DEL(J)-DEL(I))**2
0075 10 CONTINUE
0076 BI(J)=BI(J)+AM(J)*ADUM1
0077 CI(J)=AL(J)ADUM2+AM(J)*ADUM3

C CONTROL BRANCHES TO 11 OR 23 DEPENDING ON WHICH INTEGRATION HAS
C BEEN COMPLETED.
C THE REMAINDER OF THE PROGRAM FOLLOWS THE SAME TYPE OF LOGIC. THE
C INTEGRATIONS ARE SUCCESSFULLY CARRIED OUT, RETURNING ALWAYS TO
C STATEMENT SEVEN IF NOT A SUFFICIENT NUMBER HAVE BEEN EXECUTED.
WHEN THE APPROPRIATE NUMBER HAVE BEEN CALCULATED THEN THE CONTROL
IS SENT TO STATEMENT 151 AND THE PROGRAM IS ENDED.

0080 IF(ISIG1=1)11,23,23
0081 11 DO 12 J=1,N
0082 12 B1(J)=C(J)+C1(J)
0083 B2(J)=D(J)+B1(J)
0084 B3(J)=D1(J)
0085 AK(J)=C1(J)+C1(J)
0086 AL(J)=G4,J)
0087 AM(J)=D(J)+B1(J)
0088 AP(J)=D1(J)
0089 ISIG1=1
0090 IF(ITEST-2)7,151,7
0091 7 DO 13 J=1,N
0092 13 D2(J)=AL(J)*AP(J)
0093 E1(J)=-D2(J)*DEL(J)/3.
0094 ADUM1=0.
0095 ADUM2=0.
0096 ADUM3=0.
0097 ADUM4=0.
0098 14 I=1,N
0099 14 IF(I-J)24,14,24
0100 24 ADUM=DEL(I)-DEL(J)
0101 ADUM1=ADUM1-AP(I)*2.*DEL(J)/(ADUM**3).
0102 ADUM2=ADUM2+AL(I)*(-2.)*DEL(I)/(ADUM**3)
0103 ADUM3=ADUM3+AL(I)*2.*DEL(I)/(ADUM**2)
0104 ADUM4=ADUM4+AL(I)*DEL(J)/(-ADUM)
0105 CONTINUE
0106 C2(J)=AL(J)*ADUM1+AP(J)*ADUM2
0107 B2(J)=AP(J)*ADUM3
0108 D2(J)=D2(J)+AP(J)*ADUM2
0109 15 J=1,N
0110 15 IF(ISIG2=1)55,26,26
0111 55 DO 15 J=1,N
0112 15 AK(J)=C(J)+C1(J)+C2(J)
0113 AM(J)=D(J)+B1(J)+B2(J)
0114 AP(J)=D1(J)+D2(J)
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<td>3544 = 4 + (J) + C2 + (J) + C3 + (J) + C4 + (J)</td>
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0222 \[ B(4,J) = AT(J) \times ADUM_5 + E_1(J) + E_3(J) + E_4(J) \]

0223 \[ B(5,J) = AT(J) \times ADUM_6 + F_3(J) + F_4(J) \]

0224 \[ B(6,J) = H_5(J) + AT(J) \times ADUM_7 + H_4(J) \]

0225 \[ B(7,J) = P_5(J) \]

0226 151 CONTINUE

0227 RETURN

0228 END
SUBROUTINE CNSTNT
SUBROUTINE CNSTNT(XM,HH,ZZZ,IOWA,PH,PHJ,TH,ILAYER)

THIS SUBROUTINE CALCULATES THE CONSTANTS FOR THE THREE LAYER HALF-
SPACE, USING THE EQUATIONS PRESENTED IN THE TEXT. THE NOTATION
USED IS ESSENTIALLY THE SAME THROUGH-OUT AS THE TEXT. THE INPUT
IS XM=EM=M, THE DUMMY INTEGRATION VARIABLE, HH = H, THE THICKNESS
OF THE SECOND LAYER EXPRESSED AS MULTIPLES OF THE FIRST LAYER
THICKNESS, ZZZ=ZZ=Z OF TEXT, THE DEPTH OF INTEREST, IOWA= INTEGER
1 OR 2 OR 3 OR ... OR 6 DEPENDING ON WHICH PHI S ARE DESIRED (THAT
IS, WHICH STRESS OR DISPLACEMENT IS BEING CONSIDERED--IOWA WILL
BE 1 FOR NORMAL STRESS, 2 FOR SHEAR STRESS, 3 FOR RADIAL STRESS,
5 FOR VERTICAL DEFLECTION, OR 6 FOR RADIAL DEFLECTION), ILAYER=
THE LAYER OF INTEREST. ALSO READ IN ARE THE VECTORS PHI ), PHJ( )
AND TH( ). THESE ARE READ IN ONLY SO THE RESULTS, WHICH ARE
STORED IN THESE VECTORS WILL BE RETURNED TO THE MAIN PROGRAM (TO
SAVE COMMON STORAGE).

DIMENSION PHI(18),PHJ(18),TH(9)
COMMON X(20),RB(9,20), T(201) ,DELTA(20),BETA(201),B(8,20),
ISL(201),W1,DELTX,DELX2,NJ,NUJ

ALL THE OPERATIONS ARE EXECUTED IN DOUBLE PRECISION SINCE IT WAS
FOUND THAT THIS IS NECESSARY TO MAINTAIN REASONABLE ACCURACY AT
LARGE VALUES OF M.

DOUBLE PRECISION S,EM,H,ZZ,G(9),V(9),PHI(6,3,18),ALAM(6,4),
1Q(4,3,18),Z,Z1,Z2,Z3,Z4,Z5,Z6,A1,A2,A3,A4,A5,A6,A7,A8,B1,B2,B3,
2B4,B5,B6,B7,B8,B9,03,04,EZ,EZ1,EZ2,G1,G2,G3,G4,G5,G6,G7,G8,G9,G10,
3G11,G12,G13,G14,G15,G16,G17,G18,G19,G20,G21,G22,G23,G24,G25,G26,
3G27,G28,G29,G30,G31,G32,G33,G34,G35,G36,G37,G38,G39,G40,G41,G42,
4G43,G44,G45,G46,G47,G48,G49,G50,G51,G52,G53,G54,G55,G56,G57,G58,
5G59,G60,G61,G62,G63,G64,G65,G66,G67,G68,DEXP

THE NOTATION IN ALL THE FOLLOWING IS THE SAME AS THE TEXT, WITH
Z = ZZ AND M = EM, AND AN OCCASIONAL INTERMEDIATE VARIABLE DEFINED
TO SAVE EXECUTION TIME.

DO 4567 I=1,18
DO 4567 K=1,3
DO 4567 M=1,4
4567 0(M,K,I)=0.0
EM=XM
ALAM(2,1) = -ALAM(1,1)
ALAM(2,2) = ALAM(1,2)
ALAM(2,3) = ALAM(2,1) - ALAM(1,3)
ALAM(2,4) = -ALAM(1,2) + ALAM(1,4)
ALAM(3,1) = ALAM(2,1)
ALAM(3,2) = -ALAM(2,2)
ALAM(3,3) = 2*ALAM(3,1) - ALAM(1,3)
ALAM(3,4) = 2*ALAM(2,2) - ALAM(1,4)
ALAM(4,1) = ALAM(1,1)
ALAM(4,2) = ALAM(1,2)
ALAM(4,3) = -ALAM(2,3)
ALAM(4,4) = ALAM(2,4)
ALAM(5,1) = -1.5*EZ1
ALAM(5,2) = 1.5*EZ2
ALAM(5,3) = -1.5*EZ*EZ1
ALAM(5,4) = -1.5*ALAM(1,4)
ALAM(6,1) = 1.5*EZ1
ALAM(6,2) = 1.5*EZ2
ALAM(6,3) = 1.5*ALAM(2,3)
ALAM(6,4) = 1.5*ALAM(2,4)
DO 107 I = 10, 18
TH(I-9) = V(I-9)
THE UNDEFINED Q(I,I) S ARE ZEROED.
DO 107 J = 1, 4
Q(J,1,I) = 0.
THE PHI S ARE CALCULATED FOR ALL POSSIBILITIES.
DO 106 J = 1, 6
DO 106 I = 1, 18
DO 106 K = 1, 3
PHI(J,K,I) = 0.
PHI(J,K,I) = PHI(J,K,I) + Q(M,K,I)*ALAM(J,M)
THE PROPER PHI S ARE STORED IN PHI(J,J) FOR RETURN TO THE MAIN PROGRAM. SINCE THE RADIAL STRESS INVOLVES TWO SETS OF PHI S, ONE SET IS ALWAYS STORED IN THE PHI(J) VECTOR FOR RETURN TO THE MAIN PROGRAM. THE THETA(I) S ARE ALSO RETURNED TO THE MAIN PROGRAM.
C IN THE TH() VECTOR.

0218 DO_50 I=1,18

0219 PHI(I)=PHI(IOWA,ILAYER,I)

0220 50 PHI(J)=PHI(J,ILAYER,I)

0221 RETURN

0222 END
SUBROUTINE SOLVE
SUBROUTINE \textsc{solve}(N,M,MM,NNN,NJJJ)


0002 DIMENSION T1(20)
0003 COMMON X(20),BB(8,20), T(201),DELTA(20),BETA(201),B(8,20),
1SI(201),XI,DELTX,DELTXX,NJ,NJJ
2 THE FIRST POINT, \( t = 0.0 \), IS CALCULATED FIRST. ITRequires ONLY THE FIRST COLUMN OF THE ARRAYS B(,,) AND BB(,,).

0004 DO 123 I=1,201
0005 123 SI(I)=0.0
0006 BETA(I)=0.0
0007 SUMM=0.0
0008 DO 2 I=1,N
0009 2 SUMM=SUMM+BB(1,I)
0010 2 BETA(1)=BETA(1)+B(1,I)
$S_i(1) = \text{SUMM/BETA}(1)$

- THE VECTOR $T(1)$ IS USED TO STORE PRODUCTS OF TIMES. $T(1)$ IS T**0, $T(1(2)$ IS T**1, $T(1(3)$ IS T**2, ETC.

- $T(1(1)$ = 1.

- SINCE THE TIME SPACING IS LOGARITHMIC, SUCCESSIVE ANSWERS DEPEND LESS AND LESS ON THE FIRST ANSWERS. FOR THIS REASON, IT IS POSSIBLE TO NEGLECT SOME TERMS WHEN COMPUTING THE RESULTS. IN GENERAL $NJ$ TERMS OF THE SOLUTION VECTOR WILL BE USED TO CALCULATE THE NEXT TERM, AFTER THE FIRST $NJ$ TERMS HAVE BEEN CALCULATED. THIS ALLOWS SUCCESSIVE STEPS TO TAKE A CONSTANT AMOUNT OF EXECUTION TIME, RATHER THAN A CONTINUALLY INCREASING AMOUNT. FURTHERMORE, THE APPROXIMATION INVOLVED IS WELL WITHIN THE APPROXIMATION THAT IS MADE USING THE INTERVAL OF SOME OR MOST OF THE OTHER SOLUTION POINTS, DUE TO THE LOG SPACING. IN THE ANALYSES REPORTED IN THE TEXT, $NJ$ HAS ALWAYS BEEN TAKEN AS 31, WHICH SEEMS TO BE ADEQUATELY LARGE. $N5$, $N6$, AND $N4$ ARE INTEGERS USED TO PROPERLY SELECT THE POINTS OF THE SOLUTION VECTOR TO BE USED. THEY ARE TAKEN AS 1, 1, AND 4 UNTIL $NJ$ SOLUTION POINTS HAVE BEEN OBTAINED.

0013

- $N5 = 1$
- $N6 = 1$
- $N4 = 4$

0016

- THE LOOP UP TO 3 CALCULATES THE $NN$ SOLUTIONS (EXCEPT FOR $T=0$.)
- DO 3 $K=2,NNN$
- IF $K$ IS GREATER THAN $NJ$, THEN INCREMENT $N5$ AND $N4$ BY 1, AND PUT A NEGATIVE NUMBER IN $N6$.

0017

- IF($K-NJ$) > 7, 13
- $N6 = -5$
- $N5 = N5 + 1$
- $N4 = N4 + 1$

0021

- $T2=T(K)$
- $K1 = K-1$

0022

- THE LOOP UP TO 4 CALCULATES THE VALUES OF THE KERNEL FUNCTION WHICH IS A RESULT OF MULTIPLE CONVOLUTION INTEGRATIONS AND IS STORED IN THE ARRAY $B(.,1)$ NECESSARY FOR THE NEXT SOLUTION. THEY ARE AT THE TIMES $T2-T(L)$ WHERE $L$ GOES FROM ZERO TO $K$. IF $K$ IS
GREATER THAN NJJJ, THEN K-NJJJ POINTS ARE SKIPPED. THESE ARE THE TIMES T2-T(L) CORRESPONDING TO T(I). SMALL, EXCEPT INCLUDING ALWAYS ZERO TIME. THE VALUE OF L IS SELECTED THUS EQUAL TO LL EXCEPT AT THE FIRST POINT, WHEN IT IS SET EQUAL TO 1 (T=0) AND N6 IS MADE POSITIVE.

```plaintext
0023 DO 4 LL=N5, K
0024 L=LL
0025 IF(N5*N6-1) 6,8,8
0026 6 L=1
0027 N6=1
0028 8 BETA(L)=0.

THE LOOP TO 5 STORES THE PROPER PRODUCTS OF THE TIME IN THE VECTOR T1(I).

0029 DO 5 I=2, M
0030 5 T1(I)=T1(I-1)*(T2-T(L))

THE TERM MULTIPLYING EACH EXPONENTIAL TERM IS CALCULATED AND STORED IN SUM, THEN MULTIPLIED BY THE EXPONENTIAL TERM AND ADDED INTO BETA(L).

0031 DO 18 J=1, N
0032 SUM=0.
0033 DO 9 I=1, M
0034 9 SUM=SUM+B(I,J)*T1(I)
0035 PPP=DELTA(J)*(T2-T(L))
0036 IF(PPP=24)18,18,444
0037 444 PPP=24.0
0038 18 BETA(L)=BETA(L)+SUM*EXP(-PPP)
0039 4 CONTINUE

FROM HERE TO STATEMENT 21 CALCULATES THE RIGHT-HAND-SIDE RESULT FROM THE INPUT ARRAY BB(I,J) ANALOGOUS TO THE ABOVE CALCULATIONS FOR THE KERNEL FUNCTION, EXCEPT AT ONLY THE ONE TIME T2, AND STORES THE RESULT IN SUM.

0040 DO 23 I=2, MM
0041 23 T1(I)=T1(I-1)*T2
0042 SUMM=0.
0043 DO 21 J=1, N
0044 SUM=0.
```
DO 22 I=1,MM
22 SUM=SUM+BB(I,J)*T(I)

TTT=DELTA(J)*T2
IF(TTT-24.0) 21,21,222
222 TTT=24.0

SUMM=SUMM+SUM*EXP(-TTT)

THE NUMERATOR OF THE SOLUTION IS NOW CALCULATED AND STORED IN BUM.
THE TERMS IN THIS NUMERATOR VARY DEPENDING ON THE SIZE OF K.

BUM=SUMM-.5*SI(K-1)*(BETA(K-1)-BETA(K))

IF(K-2)10,10,11
11 IF(N4-K)15,15,14
15 DO 12 LL=N4,K
12 BUM=BUM-.5*(SI(L-2)+SI(L-1))*(BETA(L-2)-BETA(L-1))
14 BUM=BUM-.5*(SI(1)+SI(N4-2))*(BETA(1)-BETA(N4-2))

THE SOLUTION AT THIS TIME IS CALCULATED AND STORED IN SI(K).

SI(K)=BUM/(-.5*(BETA(K)+BETA(K-1)))

3 CONTINUE

THE SOLUTION AT ZERO TIME IS STORED IN X(1)

X(1)=SI(1)

THE SOLUTION CORRESPONDING TO DELTA(J)*T(J)=1. FOR EACH DELTA(J)
IS CALCULATED AND STORED IN X(J) FOR USE IN THE SUBROUTINE CVEFIT.
THIS IS TRUE BECAUSE THE INPUT NJ AND NJJ ARE SELECTED APPROPRIATELY.

K=NJ
DO 20 I=2,12
20 X(I)=SI(K)
RETURN
END
SUBROUTINE TIME
SUBROUTINE TIME(NNN)

C THIS_SUBROUTINE_CALCULATES_THE_TIMES_THAT_THE_SOLUTION, FOR_THE,
C CASE_THAT_THE_INTEGRATIONS_ARE_PERFORMED_EXACTLY, ARE_DESIR ED.
C IT ALSO CALCULATES THE INVERSE_OF_THE_RELAXATION_TIMES (THE_DELTA
C TERMS_OF_THE_TEXT) AND STORES THIS_RESULT_IN_THE_VECTOR_DELTA( ).
C THE_INPUT_CONSISTS_OF_NNN_NUMBER_OF_TIMES_DESIR ED. ALSO, DELTX
C AND DELXX_ARE_REQUIRED, WHICH_ARE_INCOMMON_STORAGE. DELXX
C SPECIFIES THE LOGARITHMIC_INCREMENT_OF_TIME (IT HAS BEEN TAKEN AS
C .0625_IN_THE_APPLICATIONS_IN_THIS_THESIS) AND DELTX SPECIFIES THE
C LOG_OF_THE_FIRST_FINE_TIME_MINUS DELTX (TAKEN AS -2.0625
C OR -2.5625_DEPENDING_ON_THE_SIZE_OF_SHORT_TIME_VARIATION_IN_THE
C RESPONSE_THAT_WAS_EXPECTED).

COMMON X(20), BB(8,20), T(201), DELTA(20), BETA(201), B(8,20),
1SI(201), WI, DELTX, DELXX, NJ, NJJ

N=12

THE_FIRST_TIME_IS_SET_EQUAL_TO_ZERO, AND THEN THE_OTHER_NNN-1
TIMES_ARE_CALCULATED_BY_RAISING_10. TO_THE_DELT POWER, WHERE DELT
IS_INCREMENTED_BY_DEL_AT_EACH_STEP.

DELT=DELTX
0005 DELT=DELXX
0006 T(11)=0.
0007 NNNN=NNN-1
0008 DC 7 K=1, NNNN
0009 DELT=DELT+DEL
7 T(K+1)=10. ** DELT

THE_FIRST_DELTA_IS_SET_EQUAL_TO_ZERO, THE_SECOND_EQUAL TO_10.-, AND
10 ADDITIONAL ONES_ARE_CALCULATED_BY_SUCCESSIVELY_DIVID ING BY THE
SQUARE_ROOT_OF_TEN.

DELT(1)=0.
0012 READ(5,15) DELTA(2)
0013 IF (10-DELT(2))14, 12, 14
0014 DO 6 J=3,N
0015 6 DELTA(J)=DELT(A(J-1)/(10. **.5)
0016 RETURN
0017 READ(5,16)(DELTA(J), J=3,8)
0018 WRITE(6,17)(DELTA(J), J=1,8)
0019   15   FORMAT(F10.5)
0020   16   FORMAT(9F10.5)
0021   17   FORMAT(4X,8F10.5)
0022   RETURN
0023   END
SUBROUTINE BESSEL
FUNCTION BESSEL(NN, S)

C THIS IS A FUNCTION SUB-PROGRAM TO CALCULATE BESSEL FUNCTIONS OF 
C THE ZEROETH AND FIRST ORDER, OF THE FIRST KIND. THE INPUT IS NN, 
C AND S. NN IS THE ORDER DESIRED (EITHER ZERO OR ONE) AND S IS THE 
C ARGUMENT OF THE BESSEL FUNCTION. IF THE ARGUMENT IS LESS THAN OR 
C EQUAL TO 12, THE FUNCTION IS EVALUATED USING THE INFINITE SERIES 
C REPRESENTATION. IF THE ARGUMENT IS GREATER THAN 12., THEN THE 
C ASYMPTOTIC EXPANSION FORMULAS ARE USED. THE OUTPUT IS THE SINGLE 
C NUMBER STORED IN BESSEL.

COMMON X(20), BB(8, 20), T(201), DELTA(20), BETA(201), R(8, 20), 
ISI(201), W, DELTX, DELXX, NJ, NJ

0003 N = NN
C THE SIZE OF THE ARGUMENT DETERMINES WHETHER THE ASYMPTOTIC EXPAN- 
0004 KK = N
C SIONS CAN BE USED.

0005 IF (S - 12.16, 16, 17
C THE FORM OF THE ASYMPTOTIC EXPANSION DEPENDS ON WHICH FUNCTION IS 
C TO BE EVALUATED.

0006 17 IF (N) 18, 19, 18
0007 19 PHI = S - 3.14159 * .25
0008 GO TO 20
0009 18 PHI = S - 3.14159 * .75
0010 20 RES = ((2./3.14159/S)**.5)*COS(PHI)
0011 GO TO 15
C THE PROGRAM FROM HERE TO THE END IS THE SAME AS GIVEN IN THE 
C REFERENCE CITED IN THE TEXT.

0012 16 IF (N) 2, 1, 2
0013 1 BESSEL = 1.
0014 GO TO 6
0015 2 FACT = N
0016 3 N = N - 1
0017 4 IF (N - 1) 5, 5, 4
0018 5 XN = N
0019 4 FACT = FACT * XN
0020 GO TO 3
0021 5 XFACT = FACT
END
RETURN
15 BESSSEL=BE
GO TO 7
K=K+1
14 BESSSEL=BE
IF (ABSD(ABSS-BESSSEL)- 0001)15,14
BE=BE*SUM
SUM=(1-K)/(K-1)*SUM*SUM
SUM=(1-K)/(K-1)*SUM*SUM
SUM=(1-K)/(K-1)*SUM*SUM
13 FACT2=FACT2*X
GO TO 11
FACT2=FACT2*X
12 XN=X+2
IF (K2-1)=13,13,12
11 XN=X-1
5 FACT2=FACT
GO TO 8
FACT2=FACT
10 XN=X-1
IF (K1-1)=10,10,9
9 XN=X-1
8 K1=K-1
7 FACT1=1
K2=K+X
6 K1=K
5 EXP=EXP/2
4 EXP=2*XK
3 EXP=EXP/2
2 EXP=EXP/2
1 EXP=EXP/2
BESSSEL=(S/2)K**K)/FACT
SUBROUTINE TERPO
SUBROUTINE TERPO(S, BESS)


DIMENSION S(13), BESS(91), FUN(91)

COMMON X(20), BB(8,20), T(201), DELTA(20), BETA(201), B(8,20),

ISI(201), WI, DELTX, DELXX, NJ, NJJ

THE VECTOR FUN() IS USED TO STORE THE ORIGINAL POINTS AND THE INTERPOLATED VALUES OF THE FUNCTION DESCRIBED BY THE CONTENTS OF S FIRST THE INPUT VALUES ARE STORED IN THE APPROPRIATE LOCATIONS

F(U(1))=S(1)
F(U(3))=S(2)
F(U(5))=S(3)
F(U(8))=S(4)
F(U(11))=S(5)

K=11
DO 1 I=6,13
1 K=K+10
F(U(K))=S(I)

THE INTERPOLATION IS PERFORMED BY FITTING A PARABOLA TO THREE CONSECUTIVE POINTS, AND THEN EVALUATING THIS PARABOLA AT THE INTERMEDIATE POINTS. THE EQUATION OF THE PARABOLA IS AX**X+BX+C. THE CENTER VALUE IS USED AS OM IN ALL CASES. NY IS A DUMMY USED TO DIRECT THE FLOW TO TAKE CARE OF THE THREE DIFFERENT SPACINGS OF THE THREE POINTS.

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IF(K=13)8,7,7
AT THIS POINT THE INTERPOLATED VALUES HAVE ALL BEEN STORED IN FUN.
AND THE INTEGRATION OF THE PRODUCTS FUN(I)*BEES(I) IS NOW CARRIED
OUT.

7 WI=0.
DO 70 J=2,88,2
70 WI=WI+4.*BEES(J)*FUN(J)+2.*BEES(J+1)*FUN(J+1)
WI=WI+BEES(1)*FUN(1)+4.*BEES(90)*FUN(90)
WI=WI+BEES(91)*FUN(91)
WI=WI*.1/3.
RETURN
END
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