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UNDER A GENERALIZED VORTEX

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By
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* * * * *

The Ohio State University
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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACKNOWLEDGMENTS</td>
<td>11</td>
</tr>
<tr>
<td>VITA</td>
<td>iii</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>v</td>
</tr>
<tr>
<td>LIST OF ILLUSTRATIONS</td>
<td>vi</td>
</tr>
<tr>
<td>NOMENCLATURE</td>
<td>ix</td>
</tr>
<tr>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>Chapter</td>
<td></td>
</tr>
<tr>
<td>I. EQUATIONS OF MOTION</td>
<td>10</td>
</tr>
<tr>
<td>II. REVIEW OF POTENTIAL VORTEX</td>
<td>16</td>
</tr>
<tr>
<td>III. NUMERICAL PROCEDURES</td>
<td>25</td>
</tr>
<tr>
<td>IV. NUMERICAL RESULTS</td>
<td>43</td>
</tr>
<tr>
<td>V. INNER SIMILARITY SOLUTION</td>
<td>99</td>
</tr>
<tr>
<td>VI. OUTER SIMILARITY SOLUTION</td>
<td>114</td>
</tr>
<tr>
<td>CONCLUSIONS</td>
<td>133</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>139</td>
</tr>
<tr>
<td>Table</td>
<td>Description</td>
</tr>
<tr>
<td>-------</td>
<td>-----------------------------------------------------------------------------</td>
</tr>
<tr>
<td>1</td>
<td>Comparison of Wall Shear by Finite Difference and Momentum Integral, n = - 1.0</td>
</tr>
<tr>
<td>2</td>
<td>Comparison of Wall Shear by Finite Difference and Momentum Integral, n = 0</td>
</tr>
<tr>
<td>3</td>
<td>Comparison of Wall Shear by Finite Difference and Momentum Integral, n = + 1.0</td>
</tr>
<tr>
<td>4</td>
<td>Comparison of Wall Shear by Finite Difference and Momentum Integral, n = 0.5</td>
</tr>
<tr>
<td>5</td>
<td>Eigenvalues, Inner Tangential Expansion</td>
</tr>
<tr>
<td>6</td>
<td>Eigenvalues, Inner Radial Expansion</td>
</tr>
<tr>
<td>7</td>
<td>The Terminal Structure of the Outer Flow, n = 0.5</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------------</td>
</tr>
<tr>
<td>1</td>
<td>Numerical Solution for the Potential Vortex, Radial Velocity</td>
</tr>
<tr>
<td>2</td>
<td>Numerical Solution for the Potential Vortex, Tangential Velocity</td>
</tr>
<tr>
<td>3</td>
<td>Potential Vortex Inner Solution, Radial Velocity</td>
</tr>
<tr>
<td>4</td>
<td>Potential Vortex Inner Solution, Tangential Velocity</td>
</tr>
<tr>
<td>5</td>
<td>Potential Vortex Outer Solution, Radial Velocity</td>
</tr>
<tr>
<td>6</td>
<td>Potential Vortex Outer Solution, Tangential Velocity</td>
</tr>
<tr>
<td>7</td>
<td>Schematic of Boundary Conditions</td>
</tr>
<tr>
<td>8</td>
<td>Time Development of the Radial Velocity at x = 1.0 for n = -1.0</td>
</tr>
<tr>
<td>9</td>
<td>Time Development of the Tangential Velocity at x = 1.0 for n = -1.0</td>
</tr>
<tr>
<td>10</td>
<td>Time Development of the Radial Velocity at x = 2.0 for n = -1.0</td>
</tr>
<tr>
<td>11</td>
<td>Time Development of the Tangential Velocity at x = 2.0 for n = -1.0</td>
</tr>
<tr>
<td>12</td>
<td>Radial Development of the Radial Velocity, n = -1.0</td>
</tr>
<tr>
<td>13</td>
<td>Radial Development of the Tangential Velocity, n = -1.0</td>
</tr>
<tr>
<td>14</td>
<td>Radial Development of the Vertical Velocity, n = -1.0</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------------</td>
</tr>
<tr>
<td>15</td>
<td>Time Development of the Radial Velocity at ( x = 1.0 ) for ( n = 0 )</td>
</tr>
<tr>
<td>16</td>
<td>Time Development of the Tangential Velocity at ( x = 1.0 ) for ( n = 0 )</td>
</tr>
<tr>
<td>17</td>
<td>Time Development of the Radial Velocity at ( x = 2.0 ) for ( n = 0 )</td>
</tr>
<tr>
<td>18</td>
<td>Time Development of the Tangential Velocity at ( x = 2.0 ) for ( n = 0 )</td>
</tr>
<tr>
<td>19</td>
<td>Radial Development of the Radial Velocity, ( n = 0 )</td>
</tr>
<tr>
<td>20</td>
<td>Radial Development of the Tangential Velocity, ( n = 0 )</td>
</tr>
<tr>
<td>21</td>
<td>Radial Development of the Vertical Velocity, ( n = 0 )</td>
</tr>
<tr>
<td>22</td>
<td>Time Development of the Radial Velocity at ( x = 1.0 ) for ( n = + 1.0 )</td>
</tr>
<tr>
<td>23</td>
<td>Time Development of the Tangential Velocity at ( x = 1.0 ) for ( n = + 1.0 )</td>
</tr>
<tr>
<td>24</td>
<td>Time Development of the Radial Velocity at ( x = 2.0 ) for ( n = + 1.0 )</td>
</tr>
<tr>
<td>25</td>
<td>Time Development of the Tangential Velocity at ( x = 2.0 ) for ( n = + 1.0 )</td>
</tr>
<tr>
<td>26</td>
<td>Radial Development of the Radial Velocity, ( n = + 1.0 )</td>
</tr>
<tr>
<td>27</td>
<td>Radial Development of the Tangential Velocity, ( n = + 1.0 )</td>
</tr>
<tr>
<td>28</td>
<td>Radial Development of the Vertical Velocity, ( n = + 1.0 )</td>
</tr>
<tr>
<td>29</td>
<td>Potential Vortex Inner Solution, Unsteady Results, Radial Velocity</td>
</tr>
<tr>
<td>30</td>
<td>Potential Vortex Inner Solution, Unsteady Results, Tangential Velocity</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>31</td>
<td>Time Development of the Radial Velocity at $x = 1.0$ for $n = 0.5$</td>
</tr>
<tr>
<td>32</td>
<td>Time Development of the Tangential Velocity at $x = 1.0$ for $n = 0.5$</td>
</tr>
<tr>
<td>33</td>
<td>Time Development of the Radial Velocity at $x = 2.0$ for $n = 0.5$</td>
</tr>
<tr>
<td>34</td>
<td>Time Development of the Tangential Velocity at $x = 2.0$ for $n = 0.5$</td>
</tr>
<tr>
<td>35</td>
<td>Radial Development of the Radial Velocity, $n = 0.5$</td>
</tr>
<tr>
<td>36</td>
<td>Radial Development of the Tangential Velocity, $n = 0.5$</td>
</tr>
<tr>
<td>37</td>
<td>Radial Development of the Vertical Velocity, $n = 0.5$</td>
</tr>
<tr>
<td>38</td>
<td>Streamfunction, $\psi$, for $n = -1.0$</td>
</tr>
<tr>
<td>39</td>
<td>Streamfunction, $\psi$, for $n = 0$</td>
</tr>
<tr>
<td>40</td>
<td>Streamfunction, $\psi$, for $n = +0.5$</td>
</tr>
<tr>
<td>41</td>
<td>Streamfunction, $\psi$, for $n = +1.0$</td>
</tr>
<tr>
<td>42</td>
<td>Evaluation of $\chi_1(0)$</td>
</tr>
<tr>
<td>43</td>
<td>Inner Solution, $n = 0.5$, Radial Velocity</td>
</tr>
<tr>
<td>44</td>
<td>Inner Solution, $n = 0.5$, Tangential Velocity</td>
</tr>
<tr>
<td>45</td>
<td>Eigenvalues and Multiples at Various $n$</td>
</tr>
<tr>
<td>46</td>
<td>Radial Velocity in the Outer Variable, $Z, n = 0.5$</td>
</tr>
<tr>
<td>47</td>
<td>Tangential Velocity in the Outer Variable, $Z, n = 0.5$</td>
</tr>
<tr>
<td>48</td>
<td>Phase Plane, $r^n u$ and $r^n v$, $n = 0.5$</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>A</td>
<td>Constant defined on page 101</td>
</tr>
<tr>
<td>a</td>
<td>Radius of the disc</td>
</tr>
<tr>
<td>(a_j)</td>
<td>Element of coefficient matrix, defined on page 37</td>
</tr>
<tr>
<td>B</td>
<td>Constant defined on page 105</td>
</tr>
<tr>
<td>(b_j)</td>
<td>Element of coefficient matrix, defined on page 37</td>
</tr>
<tr>
<td>C</td>
<td>Function of (r) defined on page 105</td>
</tr>
<tr>
<td>(c_j)</td>
<td>Element of coefficient matrix, defined on page 37</td>
</tr>
<tr>
<td>D</td>
<td>Constant defined on page 110</td>
</tr>
<tr>
<td>F</td>
<td>Element of outer expansion, radial component</td>
</tr>
<tr>
<td>(G)</td>
<td>Matrix of radial velocity elements, defined on page 35</td>
</tr>
<tr>
<td>(f)</td>
<td>Scaled radial velocity component</td>
</tr>
<tr>
<td>G</td>
<td>Element of outer expansion, tangential component</td>
</tr>
<tr>
<td>(H)</td>
<td>Matrix of tangential velocity elements</td>
</tr>
<tr>
<td>(g)</td>
<td>Scaled tangential velocity component</td>
</tr>
<tr>
<td>H</td>
<td>Element of outer expansion, vertical component</td>
</tr>
<tr>
<td>h</td>
<td>Scaled vertical velocity component</td>
</tr>
<tr>
<td>L</td>
<td>Constant defined on page 125</td>
</tr>
<tr>
<td>N</td>
<td>Constant defined on page 125</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>n</td>
<td>Exponent of radius for general vortex, $v \propto 1/r^n$</td>
</tr>
<tr>
<td>P</td>
<td>Coefficient matrix in eigenvalue calculation</td>
</tr>
<tr>
<td>p</td>
<td>Pressure</td>
</tr>
<tr>
<td>q</td>
<td>General velocity component</td>
</tr>
<tr>
<td>R</td>
<td>Array defined on page 35</td>
</tr>
<tr>
<td>r</td>
<td>Radial cylindrical polar coordinate</td>
</tr>
<tr>
<td>S</td>
<td>Array defined on page 35</td>
</tr>
<tr>
<td>T</td>
<td>Coefficient matrix, defined on page 37</td>
</tr>
<tr>
<td>t</td>
<td>Time coordinate</td>
</tr>
<tr>
<td>u</td>
<td>Radial velocity component</td>
</tr>
<tr>
<td>v</td>
<td>Tangential velocity component</td>
</tr>
<tr>
<td>w</td>
<td>Vertical velocity component</td>
</tr>
<tr>
<td>x</td>
<td>Stretched radial coordinate</td>
</tr>
<tr>
<td>y</td>
<td>Stretched azimuthal coordinate</td>
</tr>
<tr>
<td>Z</td>
<td>Outer variable, $= z/r^\beta$</td>
</tr>
<tr>
<td>z</td>
<td>Azimuthal cylindrical polar coordinate</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Exponent in outer variable</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>Transformed eigenvector defined on page 102</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Inner tangential component</td>
</tr>
<tr>
<td>$\eta$</td>
<td>Similarity variable $= z/r^{\frac{n+1}{2}}$</td>
</tr>
<tr>
<td>$\Theta$</td>
<td>Cylindrical polar coordinate</td>
</tr>
<tr>
<td>$\Lambda$</td>
<td>Function of the streamfunction</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Eigenvalue</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Eigenvalue</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>$\nu$</td>
<td>Kenimatic viscosity</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Density</td>
</tr>
<tr>
<td>$\Sigma$</td>
<td>Function of the streamfunction</td>
</tr>
<tr>
<td>$\tau$</td>
<td>Scaled time coordinate</td>
</tr>
<tr>
<td>$\bar{\phi}$</td>
<td>&quot;Outer streamfunction&quot; defined on p. 120</td>
</tr>
<tr>
<td>$\Phi$</td>
<td>Inner radial component</td>
</tr>
<tr>
<td>$\Psi$</td>
<td>Streamfunction, defined on page 13</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>Constant of general vortex</td>
</tr>
</tbody>
</table>

**Superscripts**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^\wedge$</td>
<td>Dimensional quantities</td>
</tr>
<tr>
<td>$^\ast$</td>
<td>Intermediate time values in numerical calculation</td>
</tr>
<tr>
<td>$^- -$</td>
<td>Old time value in numerical calculation</td>
</tr>
</tbody>
</table>

**Subscripts**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^0$</td>
<td>Values at the disc, first term in expansion</td>
</tr>
<tr>
<td>$^{\infty}$</td>
<td>Position boundary condition at infinity applied</td>
</tr>
<tr>
<td>$^{i,j}$</td>
<td>Elements of difference scheme</td>
</tr>
<tr>
<td>$^{\lambda, \mu}$</td>
<td>Eigenvector</td>
</tr>
</tbody>
</table>
INTRODUCTION

Vortex flows have been a subject of continued interest, over the years, and of late there has been a resurgence of concern due to several recent proposals. Here, we will be concerned with that class of the generalized vortex with the tangential velocity proportional to the radius to a power, $r^{-n}$, and zero radial velocity. This class of vortex does not satisfy the Navier-Stokes equations, as pointed out by Mack¹, although it is observed in actual physical situations. The radial velocity, though seldom measured, is typically so small that its deletion is justified. The prime objective of this study is to analyze the boundary-layer development beneath a general power-law vortex over a finite disc. For a potential vortex, the boundary layer has been shown to develop a double-structured terminal form in the limit as $r \to 0$. The double structure consists of two layers. An inner layer occurs near the disc in which the fluid flow is predominantly radial. An outer layer develops above the inner layer in which the velocity returns in an inviscid manner to that prescribed by a potential vortex. We will endeavor, here, to examine in detail the development of a
multi-structured terminal form for the boundary layer in the limit as \( r \to 0 \) beneath a generalized vortex.

A vortex of infinite length has a rather long decay time on the order of the inverse of the kinematic viscosity, \( \propto -1 \). However, in practice, a vortex has a short decay time due to the effect of the end walls, as discussed by Greenspan.\(^2\) Thus a means of artificially maintaining the vortex is important. Lewellen\(^3\) studied a method of generating and maintaining a vortex in a conducting fluid by means of electric and magnetic fields. Kerrebrock and Megreblain\(^4\) were concerned with the use of vortex confinement of fuel for a gaseous fission rocket and Holman and Moore\(^5\) were interested in vortex flows as a means of stabilizing an electric arc in plasma generators. These proposals considered a potential vortex where the tangential velocity, \( v \), is inversely proportional to radius in accordance with the conservation of angular momentum. For non-zero radial flow, however, Holman and Moore found that their flow more nearly fit a distribution with \( n = 0.68 \). Shepherd and Lapple\(^6\) investigating the flow in cyclone dust collectors found that by empirical curve fit the flow in the inner spiral of the dust collector satisfied \( n = 0.50 \). They also replotted some earlier data collected by Prockat\(^7\) and deduced that on the average \( n = 0.70 \). Others, as reported by Mack,\(^1\) trying to maintain vortices in chambers by the tangential
Introduction of fluid along the walls, have measured distributions corresponding to $0.40 \leq n \leq 0.80$.

In many of the above cases, the agreement between the theoretical expectations and the measured results was disappointing. An outstanding example here is the Ranque-Hilsch tube which was thoroughly discussed by Anderson. One of the prime reasons for the disagreement is the development of an end-wall boundary layer which becomes a dominant influence on the flow.

The air-land interaction of the atmospheric cyclone is also of some interest, though the development and driving forces are not understood. A vortex flow of this scale has been investigated by Byers, Hughes and Ooyama. Byers cites observational data with the wind distribution fitted by $n$ near 0.5 for relatively small-diameter cyclones. Hughes used an empirical reduction of cyclone data from the Pacific and reported that $n = 0.62$ fits the center annulus. However, the region outside this was satisfied by $n = +1.0$, a potential vortex, and the region inside by a number less than $n = 0.62$. In a numerical study of a proposed model of a tropical cyclone, Ooyama reported that $n = 0.80$ fits his solution "very well" during the deepening stage, but $n$ decreases to 0.70 for the mature storm.

These constitute just a few of the varied instances where generalized power-law vortices have been reported.
Even though turbulence and compressibility play a role in the overall physical make up, an understanding of the simpler case of laminar-incompressible flow is worthwhile as a starting point, and many such analyses have been carried out.

Theoretical investigations of the vortex boundary layer can be broken into two classes, the infinite disc and the finite disc. Rott and Lewellen\(^ {12} \) present a complete and concise review of boundary layers in rotating flows, and Mack\(^ {1} \) offers an interesting history of the general vortex problem. For present purposes, the review starts with the related problem of a rotating disc in a quiescent fluid. Von Kármán was able to reduce the Navier-Stokes equations to a set of ordinary differential equations by the selection of the appropriate similarity variable. Due to the rotation, the fluid near the disc is flung outward by centrifugal force causing an inflow from the stationary fluid. This results in a similarity solution that can be considered the initiator of the flow field. Since there are no regions of radial inflow in the boundary layer, at any radial station, the solution is equally applicable to a finite disc, as any edge effects cannot be felt upstream.

Later, Bödewadt examined the problem of solid-body rotation, \( n = -1.0 \), over a stationary infinite disc. Here, the viscous action at the wall allows the pressure
gradient to produce a radial inflow near the disc, in contrast to Von Kármán's problem with a radial outflow. Bödewadt found that the tangential velocity overshoots that of the outer flow by about 25% and then oscillates about the outer value, again in contrast to Von Kármán's monotonic approach to the outer value. Correspondingly, the radial flow has a maximum inward velocity near the wall; then instead of decreasing smoothly to zero at the outer edge, a region of outflow from the axis is formed followed by oscillations about zero with decreasing amplitude. The generalization of this problem to a finite disc is more difficult than with Von Kármán's problem since now any edge effects will influence the solution over the disc. The subject of the applicability of Bödewadt's solution to the finite disc and the region of such applicability have been argued for many years, and is discussed by Moore.¹³

Taylor¹⁴ investigated the swirl atomizer which was idealized by a potential vortex, n = +1.0, along the axis of a cone. He applied Pohlhausen's method to solve for the boundary layer. Cooke¹⁵ reapplied Pohlhausen's method with two thickness parameters (one for each of the velocity components) to the swirl-atomizer problem with a result he considered more accurate than Taylor's.¹⁴ Later, King and Lewellen¹⁶ showed by integrating the similarity equations that a contradiction arose for the boundary layer of a
potential vortex, and that a physically acceptable solution could not be obtained. This is discussed in detail in Chapter II.

The classical similarity approach with its restrictive geometry, an infinite disc, fails to represent a physical situation. Schultz-Grunow in 1935 was first to consider the problem of a finite disc in a rotating flow. He used the momentum-integral method on the problem of a finite disc under a solid-body rotation. At the edge of the disc he found the boundary layer grows as \((a-r)^{1/4}\) as opposed to the 1/2-power growth of a laminar boundary layer on a flat plate. Stewartson's analysis of the boundary layer at the edge of the disc confirmed the Schultz-Grunow result. Later, Rogers and Lance evaluated higher order terms in a series expansion based on Stewartson's solution valid over the outer region of the disc. Their solution was approaching the Bödewadt similarity solution at about \(r = 0.4\), and thus they concluded that the Bödewadt solution was correct over the inner part of the disc.

Since Stewartson's profiles at the edge of the disc were independent of the pressure gradient, Mack used them as well as polynomial profiles in a momentum-integral study of the general-vortex boundary layer. He concentrated his attention on the radial mass flow. For the
case of the potential vortex, he found that the mass flow increases monotonically with decreasing radius. This was in contrast to the other cases he evaluated, for which the maximum inflow existed at a radial station other than the axis. He also found that the location of the maximum radial mass flow moved away from the axis as \( n \) was decreased from +1.0. Unfortunately, the momentum-integral method is an approximate technique with no means of assessing the error involved unless the results are compared to an exact solution. In addition, details of the flow field are not apparent.

King and Lewellen\(^{16}\) continued the study of the general-vortex problem with the classical similarity approach. While the restrictive geometry limits the applicability of the results, such an approach provides the details of the velocity profiles, which may be regarded as either initial or terminal profiles in a more general non-similar boundary layer. A result of King and Lewellen's\(^{19}\) study, important to this paper, was that they were unable to obtain a solution for \( n \geq 0.10 \) with their numerical technique. We shall show later that a double structure develops for the terminal boundary layer as \( r \to 0 \).

Anderson\(^8\) conducted a numerical solution of the end-wall boundary layer for the two cases of solid-body
rotation and a potential vortex for the incompressible and slightly compressible cases. The equations of motion were integrated numerically from the edge of the disc into a non-dimensional radius of 0.4. At that point his solid-body-rotation solution was approaching the Bödewadt-similarity solution. For the potential vortex, due to his choice of variables, the boundary-layer profiles showed little change in the tangential component up to this point and the radial component was smoothly approaching zero. These results were highly misleading, as was shown by Burggraf, Stewartson and Belcher who examined the boundary layer beneath a potential vortex by numerically integrating the equations of motion from the edge of the disc into a non-dimensional radius of 0.03. At this point a double-structure development of the boundary layer was obvious. The double structure consists of a viscous inner layer dominated by the radial flow and an inviscid outer layer in which the flow returns to a potential vortex. This will be discussed in detail in Chapter II. An asymptotic expansion for $r \rightarrow 0$ was proposed for the terminal form of the boundary layer; it was shown to be in excellent agreement with the numerical results obtained.

At this point it was decided to check for the possible existence of other double-structured boundary layers for more general vortex flows. A preliminary investigation
suggested the possibility that an inner layer dominated by the radial flow might be the correct form for a power-law vortex with \( n > 0.1217 \). The impact of this value is reinforced by recalling that King and Lewellen\(^{16}\) could not find a numerical solution of the similarity equations for \( n \geq 0.1 \). A thorough numerical study using the time-dependent boundary-layer equations has confirmed the assumption of a multi-layered structure. An inner solution for \( r \to 0 \) is constructed utilizing the classical similarity variable but assuming negligible tangential velocity (to first order). This asymptotic solution for the inner layer is confirmed by the numerical calculations. However, the outer region develops a much more complex structure than in the case of the potential vortex. The details of the analysis for both of these layers is deferred to Chapters V and VI.
CHAPTER I
EQUATIONS OF MOTION

The most logical place to start the study of the boundary layer beneath a vortex is with the equations of motion. Here, the equations will be properly scaled for the analysis that is to follow. Let \( \hat{\mathbf{r}}, \Theta, \hat{z} \) be a set of cylindrical polar coordinates with the origin \( O \) attached to the center of a disc and the axis \( Oz \) perpendicular to the disc. Suppose the fluid extended everywhere except for this stationary finite disc of radius \( a \) located at \( \hat{z} = 0 \) and everywhere except near this disc the fluid rotates about the \( Oz \) axis such that

\[
\hat{u} \cdot \Theta, \quad \hat{\nu} \cdot \frac{\Theta}{\hat{r}} \to 0 \quad \hat{w} 
\]

where \( \hat{u}, \hat{v}, \hat{w} \) are the velocity components corresponding to \( \hat{r}, \Theta, \hat{z} \) and \( 2\pi \alpha r^{1-n} \) is the circulation and thus \( \alpha \) has the units, feet\(^{1+n}\)/second. With the disc stationary, applying the no-slip condition results in

\[
\hat{u} \cdot \hat{v} = 0 \quad \text{for } \hat{\nu} = 0 \text{ on } \hat{r} \leq a \quad (1-2)
\]

Assuming that the incompatibility between (1-1) and (1-2) is resolved by an incompressible axisymmetric laminar...
boundary layer, then the equations of motion as given by Moore\textsuperscript{13} are

\begin{align}
\frac{\partial \hat{u}}{\partial t} + \hat{u} \frac{\partial \hat{u}}{\partial r} + \hat{V} \frac{\partial \hat{u}}{\partial \hat{z}} - \frac{\hat{u}^2}{\hat{r}} &= - \frac{1}{\hat{r}} \frac{\partial P}{\partial \hat{r}} + \mathcal{V} \frac{\partial \hat{u}}{\partial \hat{z}} \tag{1-3a} \\
\frac{\partial \hat{V}}{\partial t} + \hat{u} \frac{\partial \hat{V}}{\partial r} + \hat{V} \frac{\partial \hat{V}}{\partial \hat{z}} + \frac{\partial \hat{w}}{\partial \hat{r}} &= \mathcal{V} \frac{\partial \hat{V}}{\partial \hat{z}} \tag{1-3b} \\
\frac{\partial \hat{w}}{\partial t} + \hat{u} \frac{\partial \hat{w}}{\partial r} + \hat{V} \frac{\partial \hat{w}}{\partial \hat{z}} &= 0 \tag{1-3c}
\end{align}

and where the time-dependent terms have been retained for later use. The boundary conditions for the above set of equations are given by

\begin{align}
\text{all } \hat{t} ; \quad \hat{r} = 0 ; \quad \hat{u} = 0 \quad \hat{V} = 0 \quad \hat{w} = 0 \tag{1-4a} \\
\hat{t} = 0 ; \quad \hat{r} = 0 \quad \hat{u} = 0 \quad \hat{V} = \frac{\Omega}{\hat{r}^m} \tag{1-4b} \\
\hat{t} > 0 ; \quad \hat{r} > 0 \quad \hat{u} \to 0 \quad \hat{V} \to \frac{\Omega}{\hat{r}^m} \tag{1-4c}
\end{align}

The standard boundary-layer assumptions applied above include implicitly the result that the pressure is constant through the boundary layer and thus independent of $\hat{z}$. Therefore the pressure gradient may be evaluated from the radial momentum equation as $\hat{z} \to \infty$, which results in

\begin{equation}
\left. \frac{1}{\hat{r}} \frac{\partial P}{\partial \hat{r}} = \frac{\hat{w}^2}{\hat{r}} \right|_{\hat{z} \to \infty} = \frac{\Omega^2}{\hat{r}^{2m+1}} \tag{1-5}
\end{equation}
Non-dimensionalizing the variables as shown

\[ r = \frac{\tilde{r}}{\alpha} \quad \quad \quad u = \frac{\tilde{u}}{\alpha} \]

\[ \tilde{\eta} = \frac{1}{\alpha^2} \left[ \frac{\alpha^2}{\Omega^2} \right]^{\frac{1}{2}} \quad \quad \quad \eta = \frac{\tilde{\eta} \alpha^2}{\Omega} \]

\[ t = \tilde{t} \left\{ \frac{\alpha}{\Omega} \right\} \quad \quad \quad \omega = \tilde{\omega} \left[ \frac{2 \Omega}{\alpha^2} \right]^{\frac{1}{2}} \]

results in the non-dimensional form of the equations presented below.

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \omega \frac{\partial u}{\partial \theta} - \frac{\partial^2 u}{\partial \theta^2} = \frac{1}{r^2 \cos \phi} - \frac{\partial^2 u}{\partial r^2} \quad (1-7a) \]

\[ \frac{\partial \theta}{\partial t} + u \frac{\partial \theta}{\partial r} + \omega \frac{\partial \theta}{\partial \theta} + \frac{u \theta}{r} = \frac{\partial^2 \theta}{\partial \theta^2} \quad (1-7b) \]

\[ \frac{\partial u}{\partial t} + \frac{u}{r} + \frac{\partial \omega}{\partial \theta} = 0 \quad (1-7c) \]

and the boundary conditions become

all \( t \), \( \tilde{\eta} \leq 1 \); \( u = \eta = \omega = 0 \) \( (1-8a) \)

\( t = 0 \); \( \tilde{\eta} \leq 1 \); \( u = \omega = 0 \), \( \eta \rightarrow \frac{1}{r} \) \( (1-8b) \)

\( t > 0 \); all \( r \); \( u \rightarrow 0 \), \( \eta \rightarrow \frac{1}{r} \) \( (1-8c) \)
It should be noted that the non-dimensional variable \( t \), measures radians of rotation of the flow at the edge of the disc.

At this time it is appropriate to introduce for later use the stream function, \( \psi \), defined in the usual manner by

\[
\psi = \frac{1}{r} \frac{\partial \psi}{\partial \theta} ; \quad \omega = -\frac{1}{r} \frac{\partial \psi}{\partial r}
\]

such that continuity is satisfied identically. The stream-function was not used in the numerical computations but is convenient for the analysis that follows.

A new set of coordinates was introduced for the numerical calculations. The primary region of interest was near the axis where rapid changes with radius were expected. There was, however, some question about the existence of a solution at the axis. Goldshtik\(^2\) had rigorously proven that a solution for the infinite disc for \( n = +1.0 \) does not exist. This will be discussed further in the next chapter. Also, King and Lewellen were unable to solve the similarity equations for \( n > 0.1 \). Therefore the radial coordinate was transformed to

\[
x = -\ln r
\]

This pushed the singular region at \( r = 0 \) to infinity. Here the axis cannot be reached, but yet it was hoped that one would at least be able to approach it closely enough.
so that the structure at \( r = 0 \) would be manifest. The time coordinate was left unscaled.

The known similarity variables were indeed taken advantage of. This led to a combined transformation composed of the similarity variables proposed by Stewartson,\(^{17}\) applicable near the edge of the disc, and those applicable near the axis proposed by King and Lewellen.\(^{16}\) Thus the azimuthal coordinate became

\[
y^* = (1-r)^{i/2} r^{1/2} \bar{z} = (1-\varepsilon^{-})^{i/2} e^{(\frac{1/2}{2})^*} \bar{z}
\]  

(1-11)

The velocity components were also scaled to take advantage of the appropriate similarity forms, in the same manner as the vertical coordinate. Therefore set

\[
u = r^{-m} (1-r)^{i/2} f(x,y,t) = e^{m^*} (1-\varepsilon^{-})^{i/2} f(x,y,t)
\]  

(1-12a)

\[
u = r^{-m} \varphi(x,y,t) = e^{m^*} \varphi(x,y,t)
\]  

(1-12b)

\[
u = r^{-m} (1-r)^{i/2} \varphi(x,y,t) = e^{(\frac{1/2}{2})^*} (1-\varepsilon^{-})^{i/2} \varphi(x,y,t)
\]  

(1-12c)

and the equations of motion given in Equation (1-7) become

\[
f_{yy} - A f_{y} + \left( \frac{1}{2} \varepsilon - \frac{3}{2} m \varepsilon \right) y f_{y} + (1-\varepsilon^-) f_{x} = 1 + \frac{1}{2} \varepsilon + (m+\frac{1}{2} m \varepsilon) \right) f^2
\]  

(1-13a)

\[
= + f_{x} \{ e^{(1/2)} (1-\varepsilon^-) \}
\]
and from (1-8) the boundary conditions are

\[
\begin{align*}
    \text{all } t; & \quad \psi = 0, \quad f = 0, \quad g = 0, \quad h = 0 \\
    t=0; & \quad \psi = 0, \quad f = 0, \quad g = 1 \\
    t>0; & \quad \psi \to \infty, \quad f \to 0, \quad g \to 1
\end{align*}
\]

Before proceeding with the solution of these equations for the general vortex, a review of the solution for the potential vortex is appropriate.
CHAPTER II
REVIEW OF THE POTENTIAL VORTEX

The specific problem of the potential vortex has been reported by Burggraf, Stewartson and Belcher. However a concise review of the problem's development and results is appropriate here due to its underlying importance to this problem of the general power-law vortex.

The potential vortex was chosen since it along with solid-body rotation forms the fundamental rotational flows that are exact solutions of the Navier-Stokes equations. King and Lewellen, in addition to their numerical solution, also show that there is no solution of the boundary-layer similarity equations for the potential vortex that will satisfy the boundary conditions. By integrating the similarity equations, they showed that the streamfunction at the edge of the boundary layer would be physically acceptable (a real number) only if the tangential velocity overshot the outer flow value. However, for the potential vortex, the slope of the tangential velocity does not change sign and hence cannot overshoot the outer value, as required for a solution. Moreover Gol'dshtik had rigorously shown the non-existence of a solution of the
similarity form of the Navier-Stokes equations for Reynolds number greater than 3, for the potential vortex.

In order to circumvent this problem, a finite disc was considered and the radial coordinate stretched to allow a gradual approach to the axis (see Equation 1-10). It was hoped that a numerical solution could be obtained over the disc and continued close enough to the axis to discover the terminal form of the boundary-layer flow, if it exists.

The numerical investigation involved the transformed boundary-layer equations of Chapter I, but setting to zero the time-dependent right-hand side of Equations (1-13). Hence, only steady flow was considered. The numerical results were obtained by solving a finite-difference analog (Crank-Nicolson) of Equations (1-13) at x-stations, starting at the edge of the disc and progressing toward the center. This method proved extremely successful and the solution was obtained from $x = 0$ to $x = 3.5$, $(r = 1$ to $r = 0.03)$. Figures 1 and 2 show some of the results of this numerical solution for the radial and tangential velocities at various radial stations.

While the boundary-layer thickness was growing very rapidly, in terms of the $y$-variable, it was noticed that a region near the wall was developing almost totally independent of the outer flow. The wall shear was almost constant regardless of the thickness of the boundary layer.
Figure 1. Numerical Solution for the Potential Vortex, Radial Velocity
Figure 2. Numerical Solution for the Potential Vortex, Tangential Velocity
The most striking feature of this flow near the wall was that it was almost entirely radial, even though the boundary condition at the outer edge of the boundary layer specified only tangential flow.

To account for this behavior near the axis, an inner layer was proposed near the wall where the tangential component of velocity was dominated by the radial component and where the pressure force in Equation (1-7) was balanced by the viscous and inertia terms. The resulting equations, given in general in Chapter V, were in terms of the classical similarity variable, \( \eta = z/r \), as required by the force balance. These ordinary differential equations, now uncoupled, were solved numerically and the results were shown to be in excellent agreement with the numerical solution of the full boundary-layer equations as can be seen in Figures 3 and 4. In these figures, the curve marked LIMIT refers to the solution of the ordinary differential equations as does the curve referred to by \( rv = r^2 \eta \).

To allow the recovery from this inner region with its dominant radial flow to the boundary condition at the edge of the boundary layer, an inviscid outer layer was proposed in which the two velocity components were of equal importance. A consistent analysis was constructed on the premise of \( z \) being the outer variable and the observed result that as the inner region was left both \( ru \) and \( rv \)
Figure 3. Potential Vortex Inner Solution, Radial Velocity
Figure 4. Potential Vortex Inner Solution, Tangential Velocity
became independent of the radius (for $r \to 0$). The properties of the outer layer revealed by the asymptotic analysis showed excellent agreement with those of the numerical calculations. The outer solution and the approach to it are shown in Figures 5 and 6. Again the curve marked LIMIT denotes the terminal profile for $r \to 0$. An interesting feature of the outer region was the continual entrainment of fluid from the potential vortex above, from the edge of the disc into the center, implying ultimate eruption of the boundary-layer fluid at the center of the disc.

One of the numerical problems encountered in this work was the very rapid growth of the boundary-layer thickness relative to the $y$-variable mentioned above. This thickness was very important in computing the vertical-velocity component due to the continual entrainment of fluid. If the thickness was not sufficient a shear layer formed at the edge of the boundary layer causing the vertical velocity to plunge to zero and change sign. This indicated the expulsion of fluid from the boundary layer which is unrealistic. Whenever this occurred the calculation was stopped and restarted at a previous station where the thickness had been adequate and the calculation was repeated with the boundary-layer thickness increased.

It is important to point out that in physical variables, the boundary-layer thickness actually becomes
Figure 5. Potential Vortex Outer Solution, Radial Velocity

- \( ru = - F_0(z) \)
Figure 6. Potential Vortex Outer Solution, Tangential Velocity
constant. The apparent growth is due to the stretching of the y-variable as a function of r indicated by Equation (1-11).

With the terminal form of the potential vortex well established, the question naturally arose - Do other vortex boundary layers exhibit similar properties? The answer is affirmative and will be discussed in depth in later chapters.
CHAPTER III
NUMERICAL PROCEDURES

Several numerical techniques and methods were used in the solution of the equations presented in Chapter I. These included (1) a steady-flow forward-marching calculation, (2) a time-dependent explicit calculation and (3) a time-dependent implicit calculation. Method (1) was unsuccessful for boundary layers with oscillatory radial-velocity profiles. Method (2) was successful but required an excessively small time step for stability for some cases. Method (3) reduced the overall computing time due to larger allowable step size. Details of the methods and their characteristics are given in this chapter.

The initial numerical attempt was to follow the procedure that had been applied successfully in the case of the potential vortex. The finite-difference analog of Equation (1-13) at each x station was solved marching from the edge of the disc inward. Unfortunately this method proved to be non-convergent for n less than +1.0. While many values of n and various combinations of groupings to achieve convergence were tried, the most successful calculation in terms of largest x realized was for n = 0.90.
Here the program failed to converge in 500 iterations at $x = 0.375$. The profiles obtained showed no oscillations in $f$ at $x = 0.30$ but $g$ had started to develop a slight overshoot to 1.0001. However, the last iteration at $x = 0.375$ showed a region of reverse flow starting to develop. These regions of reverse flow signaled the demise of the present steady-flow forward-marching method which is stable for parabolic equations such as these in $x$ and $y$ if the radial flow is completely inward or outward. The reverse flow, however, produces instabilities. It should be noted that regions of reverse flow are to be expected, because as Stewartson\textsuperscript{22} has shown, for a finite disc there must be oscillations about zero in the radial velocity for $n$ not equal to $\pm 1.0$, for all $r < 1$.

The problem was then attacked numerically by considering the unsteady boundary-layer equations and looking for the steady solution as time goes to infinity. The equations of motion have been given in (1-12). The idea of a time-similarity variable, while esthetically pleasing, quickly leads to numerical difficulties. Using a composite similarity form of time variable as described in Chapter I for the azimuthal coordinate, the time coordinate would take the form

$$2 \cdot e^{((n)m)x} (1 - e^x)^{-\frac{1}{2}} t$$  \hspace{1cm} (3-1)
This modifies Equation (1-13) by making the coefficient of the time derivative

\[ 1 - \left( \frac{\mu(\mu)}{\frac{1}{2}} \right) \frac{1}{1 - \frac{m}{2}} e^{2x^2} t \]

Since there will exist regions of inflow and outflow at each x station, this coefficient is always in danger of becoming zero. This quickly leads to a numerical problem similar to moving along a characteristic which would terminate the calculation.

It is readily seen in Equation (1-13) that the time derivative is singular at both the edge and the center of the disc, indicating that the steady-state Stewartson profiles are established in zero time; i.e. these profiles are present at the edge when time starts. The idea of the profiles being generated in zero time is supported by the fact that at the edge of the disc the boundary layer has zero physical thickness.

This same argument would hold for the center of the disc when considering the similarity solutions which also have zero thickness at the center, but unfortunately for the case of interest the solutions at the center are not known. Since the method is to march in time until a steady solution is found, the equations being parabolic in time require all the exterior boundary conditions of the flow region to be specified for \( t \geq 0 \). As the center
can not be reached under the present transformation, a method of terminating the inward extent of the solution at \( r \) equal some \( r_0 \) less than 1 must be found. The region of flow to be studied numerically consists of an anulus above the disc. Due to the axisymmetric property of the flow, steady or unsteady, the \((x, y)\) region may be considered as a rectangle bounded by the disc at \( y = 0 \) from \( x = 0 \) at \( r = 1 \) to \( x_\infty \) at \( r = r_0 \), where \( r_0 \) is to be chosen sufficiently small that the terminal solution is well approximated. The outer end is bounded by the steady similarity profiled at \( x = 0 \) from \( y = 0 \) to \( y_\infty \), a large value of \( y \) picked to represent the outer boundary conditions at infinity. The true upper boundary condition at infinity is satisfied at \( y_\infty \). This leaves only the undefined boundary at \( x_\infty \). These requirements are shown schematically in Figure 7.

If the radial flow were entirely inward as was the case for the potential vortex this boundary would present no difficulty as one could use a linear extrapolation to obtain any needed values at downstream locations near \( x_\infty \), as was done by Skoglund, Cole and Staiano,\(^{25}\) for instance. However, their procedure leads to questionable results in this case since the flow is expected to be oscillatory in the radial component. The use of downwind differencing, an old meteorological technique, in the
31.

\[ y = y_\infty, \ x-t \ plane \]
Boundary conditions at infinity imposed for all \( t \).

\[ x = 0, \ y-t \ plane \]
Boundary conditions fixed as steady Stewartson profiles.

\[ x = x_\infty, \ y-t \ plane \]
Open end for \( u < 0 \)
Boundary condition to be found for \( u > 0 \)

\[ t = 0, \ x-y \ plane \]
Initial conditions specified.

\[ y = 0, \ x-t \ plane \]
No slip conditions imposed for all \( t \).

Figure 7. Schematic of Boundary Conditions
radial direction leads to very satisfactory results everywhere except at $x_\infty$. For regions of outflow from the axis this method breaks down as the function values required at stations further inward ($x > x_\infty$) are not known. To complete the boundary conditions the $x$-derivative was set equal to zero at $x_\infty$ in regions of reverse flow. This is correct in the limit $x \to \infty$ as for the case of self-similar solutions. If one extrapolates outside the region of solution so that downward differencing of centered differences can be used in $x$, erroneous information is introduced into the area being studied, unless the proper extrapolation has been determined. When the similarity solutions are appropriate at the axis then the proper form for large $x$ is $\frac{d}{dx} y = 0$. Likewise, when simple similarity is not appropriate, a new form dependent on the proper outer similarity variable $Z$, discussed in Chapter VI, would be proper. In effect this procedure is similar to what was adopted by Reyhner and Flügge-Lotz\textsuperscript{24} by setting $f \frac{\partial}{\partial x} y = 0$ in regions of reverse flow, since in those regions they found that $f$ was small. However, as will be seen this is not quite true for the present cases of interest.

The initial time profiles were taken to be the inviscid solution except at the edge of the disc. That is
Except for $x = 0$, this represents a "turning on" of viscosity at the wall at $t = 0$.

In the hope of keeping things as simple as possible the first computer programs were explicit in time. Initially a Lax-Wendroff routine was tried as suggested by Skoglund, Cole and Stainao. An order-of-magnitude analysis indicated that the second-order correction was negligible. Since the second-order terms accounted for about 60% of the computation time, they were eliminated from the difference equations, which then reduced to the explicit form: centered in $y$, downwind in $x$ and forward in $t$.

A new program was developed using an implicit numerical scheme when as a result of increasing $x_\infty$ to 3.0 for the main case of interest $n = 0.5$, the time step was restricted to too small a value to be practical. With $x_\infty = 2.0$ the time step had been 0.0001 during the first 500 steps, with $\Delta t = 0.001$ thereafter. When the region was increased to $x = 3.0$ the time step had to be cut back to 0.0001. While this step size was acceptable for watching the development taking place near $x = 3.0$ it was too slow to allow the changes to develop near $x = 2.0$ in a reasonable amount of computer time. The implicit
program though on the order of 3 times slower for one time step, allowed the time step to be increased by a factor of about 50 so that a net decrease of computer time by a factor of approximately 20 was realized in this case. It is interesting to note that when the implicit program was applied to other cases of n, (-1.0 and 0) that had been solved by the explicit routine previously, it appeared that the decrease in total computing time was a function of n. For instance, with n = -1.0 the total computing time was changed by a factor of only 2. The numerical results showed no appreciable change as a result of the new computation scheme.

The implicit program utilized an average between new time values and previous time values for the time and y-derivatives but used downwind differencing in x at the former time step. A new time solution was iterated at each x station, successively sweeping through the x stations.

In detail, the difference analog for the implicit program consisted of taking uniform steps of spacing Δy across the boundary layer denoted by y_{j+1} = y_j + Δy with j = 0, 1, ..., l, where y_0 = 0 and y_l = y_∞, the upper approximation to the edge of the boundary layer. The distance along a radial line on the disc is divided into uniform steps Δx such that x_{i+1} = x_i + Δx with
i = 0, 1, ..., k such that \( x_0 = 0 \), the edge of the disc and \( x_k = x_\infty \), the inner limit of penetration. We denote the new time value of the velocity components by \( q_{1,j} \) at the point \( x_i, y_j \); denote the known velocity component at the last time value by \( q^*_{1,j} \) at \( t = t - \Delta t \). The difference analog of the differential equations are written down for the intermediate time \( t^* = 1/2(t + \bar{t}) \) with \( q^*_{1,j} = 1/2(q_{1,j} + \bar{q}_{1,j}) \), except for the continuity equation which is solved for \( h \) at the new time value using the known values of \( f \).

The \( x \) derivatives are evaluated at the last time station by downwind differencing, that is if \( \bar{f}_{1,j} > 0 \) then
\[
\bar{q}_{x1,j} = (\bar{q}_{1+1,j} - \bar{q}_{1,j})/\Delta x
\]
and if \( \bar{f}_{1,j} < 0 \) then
\[
\bar{q}_{x1,j} = (\bar{q}_{1,j} - \bar{q}_{1-1,j})/\Delta x.
\]
If \( f = 0 \) the derivative is not evaluated since the \( x \) derivative appears only as \( f \frac{\partial}{\partial x} \).

As a result of computing the \( x \) derivative at the old time, the routine is only partially implicit.

In matrix notation the difference analog can be expressed as

\[
\{T\}\{\mathbf{q}^*\} = \{R\} \quad \text{if} \quad \{T\}\{\mathbf{y}^*\} = \{S\} \tag{3-2}
\]

where the \( \{T\} \) represents a tridiagonal square matrix whose elements consist of \( x_i, y_j, f^*_{i,j}, \) and \( h^*_{i,j} \); \( \{\mathbf{q}^*\} \) and \( \{\mathbf{y}^*\} \) are column matrices representing the \( f_{i,j} \) and \( g_{1,j} \) at the \( x \) station under consideration and \( \{R\} \) and \( \{S\} \) are the column matrices whose elements are both known and unknown values of the flow properties. The \( l \) unknowns at each \( x \) station
are solved for iteratively using a Gaussian-elimination routine with the latest iterative values used in \( \{T\} \), \( \{R\} \) and \( \{S\} \). To help cut computation time the coefficient matrix \( \{T\} \) was kept the same for both the \( f \) and the \( g \) equations, and using \( \{R\} \) and \( \{S\} \) to accumulate the terms that differ. The reason for evaluating the \( x \)-derivatives at the old time step was so that each \( x \)-station could be iterated to completion without sweeping the entire region before iteration at each \( x \)-station. This was done as a means of reducing the storage requirements. Since it was expected that different \( x \)-stations would require a different number of iterations it also reduced the computation time. After the convergence at any \( x \)-station the next station in order was taken until the entire field had been swept, then time was advanced and the procedure repeated until a steady solution was found. Allowing a notation of the form

\[
\{T\} = \begin{bmatrix}
  b_1 & c_1 & 0 & \cdots & 0 & \cdots & 0 \\
  a_2 & b_2 & c_2 & 0 & \cdots & 0 & \cdots \\
  0 & a_3 & b_3 & c_3 & 0 & \cdots & 0 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  0 & 0 & 0 & \cdots & a_j & b_j & c_j & 0 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  0 & 0 & 0 & \cdots & 0 & a_k & b_k & c_k & \cdots \\
  0 & 0 & 0 & \cdots & 0 & 0 & a_l & b_l & \cdots \\
  \end{bmatrix}
\]  

(3-3)
then

\[\begin{align*}
a_j &= -1 + \left\{ \left( \frac{\mu_m}{2} - \frac{3+2m}{2} e^{x_j} \right) \frac{\Delta y}{\Delta x} \right\} y_j f_{ij} - h_{ij} \right\} \frac{\Delta y}{\Delta x} \right\} \\
b_j &= 2 \left\{ 1 + e^{-(\mu_m) x_j (1-e^{x_j})} \frac{\Delta y}{\Delta x} \right\} \right\} \right\} \right\} \\
c_j &= -2 - a_j
\end{align*}\]

The terms for \( \{ \mathbf{R} \} \) and \( \{ \mathbf{S} \} \) are given by

\[\begin{align*}
\mathbf{R}_j &= -2 \left\{ \left( 1 + e^{-(\mu_m) x_j (1-e^{x_j})} \frac{\Delta y^2}{\Delta x^2} \right) \frac{\Delta y}{\Delta x} \right\} y_j f_{ij} + \tilde{f}_{ij} + \tilde{h}_{ij} \\
&+ \left\{ \tilde{f}_{ij} - \tilde{f}_{ij-1} \right\} \left\{ \left( \frac{\mu_m}{2} - \frac{3+2m}{2} e^{x_j} \right) \frac{\Delta y}{\Delta x} \right\} y_j f_{ij} - h_{ij} \right\} \frac{\Delta y}{\Delta x} \\
&+ 2 \Delta y^2 \left\{ \left( 1 + g_{ij}^2 \right) \frac{\Delta y}{\Delta x} + \frac{\Delta y}{\Delta x} f_{ij} (1-e^{x_j}) + (m - \frac{\Delta y}{\Delta x}) f_{ij} \right\} \right\} \right\} \right\} \\
\mathbf{S}_j &= -2 \left\{ \left( 1 + e^{-(\mu_m) x_j (1-e^{x_j})} \frac{\Delta y^2}{\Delta x^2} \right) \frac{\Delta y}{\Delta x} \right\} y_j f_{ij} + \tilde{g}_{ij} + \tilde{g}_{ij+1} \\
&+ \left\{ \tilde{g}_{ij+1} - \tilde{g}_{ij} \right\} \left\{ \left( \frac{\mu_m}{2} - \frac{3+2m}{2} e^{x_j} \right) \frac{\Delta y}{\Delta x} \right\} y_j f_{ij} - h_{ij} \right\} \frac{\Delta y}{\Delta x} \\
&+ 2 \Delta y^2 \left\{ \frac{\Delta y}{\Delta x} \left( \frac{\Delta y}{\Delta x} \right) g_{ij}^2 \left( 1 - m \right) g_{ij}^2 \left( 1 - e^{x_j} \right) f_{ij} \right\} \right\} \right\} \right\} \\
\end{align*}\]

The first and last rows of \( \{ \mathbf{T} \} \) and the elements of \( \{ \mathbf{R} \} \) and \( \{ \mathbf{S} \} \) are modified to account for the boundary conditions thus

\[\begin{align*}
b_1 &= 1, \quad c_1 = 0, \quad R_1 = 0, \quad S_1 = 0 \\
a_2 &= 0, \quad b_2 = 0, \quad R_2 = 0, \quad S_2 = 1
\end{align*}\]

Convergence was rapid when it occurred. That is, no more than 5 iterations were ever required with 2 or 3 being
typical. After each iteration the vertical velocity component was integrated to obtain the new profile.

As a check on the velocity profiles obtained the profiles were used in the steady momentum-integral equations. These equations can be easily obtained from Equation (1-13) by setting the right-hand side to zero and integrating the equations from 0 to infinity on \( y \). To eliminate the vertical component, \( h \), in the momentum equations, continuity is considered in the form

\[
\frac{dh}{d\theta} = \left( \frac{\mu m}{\pi} - \frac{3\pi m}{4} \right) y f_y dy + (1 - e^y) f_y dy - \left( \frac{m}{2} - \frac{3\pi m}{4} \right) f_y dy
\]  

(3-6)

The integration could be performed directly but if the equations are rearranged the process is simplified. Looking at typical terms, the following reduction was performed,

\[
\int_0^\infty f_y dy - f_y(\infty) = f_y(\infty) = -f_y(\infty)
\]

\[
-\int_0^\infty h f_y dy = -h f_y + \int_0^\infty f_y dy
\]

\[
\left( \frac{\mu m}{\pi} - \frac{3\pi m}{4} \right) \int_0^\infty f y dy + (1 - e^y) \int_0^\infty f y dy - \left( \frac{m}{2} - \frac{3\pi m}{4} \right) \int_0^\infty f_y dy
\]

these terms can be added to those remaining in the equations so that
The tangential equation is handled in a similar manner,

\[
\int_0^\infty g_{yy} dy = q_d^{(\infty)} - q_d^{(0)} = -q_d^{(0)}
\]

\[
-\int_0^\infty h_q dy = -h_q^{(\infty)} + \int_0^\infty g dh
\]

\[
= -h_\infty + \int_0^\infty \left\{ \left( \frac{1}{4} + \frac{3-2m}{2} \right) y^2 + (1-e^2) \right\} f dy
\]

and \( h_\infty \) is evaluated from integrating (3-6)

\[
\int_0^\infty d h = h_\infty - h_0 = h_\infty = \left( \frac{1}{4} \right) \int_0^\infty f dy + (1-e^2) \int_0^\infty f dy - (1-m \cdot \frac{3-2m}{2} e^2) \int_0^\infty f dy
\]

\[
= (\frac{1}{4} - \frac{3-2m}{2} e^2) \int_0^\infty f dy - (1-e^2) \int_0^\infty f dy - (1-m \cdot \frac{3-2m}{2} e^2) \int_0^\infty f dy
\]

\[
- h_\infty = \left\{ \frac{3-2m}{2} - \frac{3-2m}{2} e^2 \right\} \int_0^\infty f dy - (1-e^2) \int_0^\infty f dy
\]

The remaining terms in the tangential equation can be grouped with the terms resulting from the \( \int h_{shy} dy \) term
When this is used in Equation (1-13), the results for the wall shear are

\[
\int_{0}^{c} y f_{g_{y}} dy + \int_{0}^{c} y f_{q_{y}} dy = \int_{0}^{c} y (f_{q})_{y} dy = y f_{q} \bigg|_{0}^{c} - \int_{0}^{c} f_{q} dy
\]

\[
= - \int_{0}^{c} f_{q} dy
\]

\[
\int_{0}^{c} f_{q} dy + \int_{0}^{c} f_{q_{x}} dy = \frac{1}{\Delta x} \int_{0}^{c} f_{q} dy
\]

When this is used in Equation (1-13), the results for the wall shear are

\[
f_{y} (x, \omega) = \left\{ \frac{3 \omega}{2} (1 - m) - \frac{3 \omega - 2 \omega n}{4} e^{\omega} \int_{0}^{c} f_{y} dy + \frac{1}{\Delta x} \int_{0}^{c} f_{q_{y}} dy + \int_{0}^{c} (q_{y} - 1) dy \right\}
\]

\[
q_{y} (x, \omega) = \left\{ \frac{3 \omega}{2} - \frac{9 \omega - 2 \omega n}{4} e^{\omega} \int_{0}^{c} f_{y} dy - \frac{1}{\Delta x} \int_{0}^{c} f_{q_{y}} dy - \int_{0}^{c} f_{q_{y}} dy + \frac{1}{\Delta x} \int_{0}^{c} f_{q_{y}} dy + \int_{0}^{c} (q_{y} - 1) dy \right\}
\]

The integrals were calculated using a Simpson's 3/8 integration routine and were compared to the wall shear obtained directly from the finite-difference solution by using a quadratic approximation for the derivative

\[
f_{y} (x, \omega) = \left\{ -3 f_{y} + 4 f_{y_{x}} - f_{y_{x}} \right\} \Delta y
\]

\[
q_{y} (x, \omega) = \left\{ -3 q_{y} + 4 q_{y_{x}} - q_{y_{x}} \right\} \Delta y
\]
No special difficulties were encountered other than a rapidly increasing thickness, in $y$, of the boundary layer for $n = 0.5$, as long as the step sizes were kept compatible. As mentioned earlier the time-step size was changed quite frequently but the $\Delta x$ and $\Delta y$ were held constant in all the results reported below for both the explicit and implicit programs. The size of $\Delta x = 0.1$ and $\Delta y = 0.3$ were chosen as a compromise between accuracy and storage and computation time. The size of $\Delta x$ while equivalent to a large radial step of $\Delta r = 0.095$, near the edge of the disc, is reduced to $\Delta r = 0.005$ at $x = 3.0$. This is in agreement with the changes expected, coarse at the edge and fine near the center. $\Delta y$ was chosen as 0.3 because a larger value would have too few points spanning an oscillation. Experience with the potential vortex suggested that the maximum radial velocity would develop near the wall and $\Delta y = 0.3$ is about as large as can be taken without severely affecting the results. Also considered was the resulting size of the arrays involved and the time to compute. Typical of the computation time to advance 100 time steps for a flow region of $0 \leq x \leq 3$ and $0 \leq y \leq 39.9$ would be 2 minutes for the explicit program and 5.8 minutes for the implicit program as computed on an IBM 360/75 at the Ohio State University.
The problems solved numerically were for \( n = -1.0, 0, +1.0 \) and \(+0.5\). These values were chosen for various reasons. Solid-body rotation, \( n = -1.0 \), represents a flow problem with a well-known similarity solution and hence a good test of the computer programs developed. In addition, the regions of reverse flow allowed a verification of the boundary condition being imposed at \( x_\infty \). The similarity solution for \( n = 0 \) is given by King and Lewellen\(^16\) and represents the largest value of \( n \) for which they could obtain a self-similar solution. Also it is near the postulated change from similarity solution to multi-structured but is a known solution. The potential vortex, \( n = +1.0 \), represents another known solution, but now of a double-layered structure. It was chosen to see if the single-mesh-size time-dependent approach would indicate the development of the multi-layered terminal solution. The last case, that of \( n = 0.5 \) was chosen due to its median position between the proposed change at \( n = 0.1217 \) and the known solution at \( n = +1.0 \). Also it is near the experimentally determined range of vortex motion. The results of the numerical calculations are presented in the following chapter.
n = -1.0: Solid-Body Rotation

The case of n = -1.0, solid-body rotation, has long been of interest. As the similarity solution for the profiles at and possibly near the axis are well known and exhibit regions of reverse flow, the problem was ideal for checking the program and verifying the assumptions relating to the inward boundary conditions. The explicit program was utilized for a region bounded by 0 ≤ x ≤ 1.0, 1.0 ≥ r ≥ 0.3679 and 0 ≤ y ≤ 30.0 with a time stepsize Δt = 0.01. After 3000 steps the maximum time derivative was 0.015, reduced from a value of 26.1 at the first step. The agreement at x = 1.0 was very favorable with the similarity solution of Bödewadt's problem obtained by Browning and recorded by Schlichting. In order to test the end condition at x = 1.0 the region size was increased to 0 ≤ x ≤ 2.0, 1.0 ≥ r ≥ 0.145 with y∞ held constant at 30.0 and the problem was recalculated. After 3000 steps the maximum time derivative was 0.048, at x = 2.0. The new end position resulted in a maximum change of only 2% in the greatest reverse flow velocity, at x = 1.0.
Figures 8-11 show the time development of the profiles at $x = 1.0$ and $x = 2.0$. The results presented in all figures are for $x_\infty = 2.0$, unless otherwise noted. These figures not only show the rapid initial development of the profiles followed by a slow approach to the steady result, but some rather large overshoots and oscillations during the development. These are illustrated at $x = 1.0$ by the radial velocity, Figure 8 at $t = 2.0$ the maximum inward velocity is 0.882 compared to the maximum for the steady profile of 0.638, a 38% overshoot. The solution for $t = 2.0$ shows almost no reversed flow but at $t = 3.0$ the maximum reverse flow is 0.472, while the maximum for the steady result is only 0.132, a rather large 160% overshoot. Similar changes can be noted in the tangential component also. Yet for $t = 4.0$ at $x = 1.0$, the profiles are within a few percent of the steady solution. However, at $x = 2.0$ the profiles still exhibit large oscillations.

Due to these large oscillations the numerical solution was re-obtained using the implicit routine from $t = 0$. This was done in order to see if the oscillations were spurious results of the numerical calculation. Some results are given in Figure 8 to show the excellent agreement between the results of the two computer programs. The slight difference, here, can be explained by the fact that the implicit method actually starts at $t = 0$ a little faster than the explicit. It can affect more points in
Figure 8. Time Development of the Radial Velocity at x = 1.0 for n = -1.0
Figure 9. Time Development of the Tangential Velocity at $x = 2.0$ for $n = -1.0$
Figure 10. Time Development of the Radial Velocity at x = 2.0 for n = -1.0

\[ f = (1-r)^{-1/2} r^{-1} u \]
Figure 11. Time Development of the Tangential Velocity at $x = 2.0$ for $n = -1.0$
the y-direction at each time step. However, once a profile is established this effect disappears.

Since the large oscillations cannot be attributed to the numerical techniques the question arises as to whether the oscillations are physically realistic. In my opinion they are realistic. One can clearly picture for the case of solid-body rotation, a large container of fluid with a submerged disc all rotating together and then the disc suddenly being stopped at $t = 0$. Due to the no-slip condition at the wall, the pressure gradient causes a radial inflow to develop: this can be seen in Figures 8 and 10 where for $t = 1.0$ and 2.0 the radial flow is almost entirely inward. This cannot continue and, as can be seen, does not continue. A large radial outflow has developed by $t = 3.0$. The profiles now oscillate with time and slowly approach a steady state where all the forces are in equilibrium. Another indication of the solution being physically realistic is that the proper steady solution is obtained.

Figures 12-14 show the radial change in the steady profiles and the comparison with the similarity solution of Browning. An interesting result is the overshoot of the terminal solution and the approach to it from larger values. As suggested by many authors the similarity solution is applicable over the inner 40% of the disc. In Figure 14, the vertical component of velocity clearly
Figure 12. Radial Development of the Radial Velocity, \( n = -1.0 \)

\[ f = (1-r)^{-1/2} \]

\( \text{Browning}^{25} \)}
Figure 13. Radial Development of the Tangential Velocity, $n = -1.0$
Figure 14. Radial Development of the Vertical Velocity, $n = -1.0$
depicts not only the smooth transition from the Stewartson profile at the edge of the disc to the similarity solution near the axis but also the transition from inflow at the edge to complete outflow by \( x = 0.5, \ r = 0.607 \).

Table 1 compares the wall-shear result obtained by direct finite difference at the wall, Equations (3-9), with the momentum-integral Equations (3-8) using the profiles of Figures 12 and 13. The poor agreement in \( \tau_y(x,0) \) for the results indicated by Momentum Integral #1 is rather disheartening after the fine comparison of Figures 12 and 13. Momentum Integral #1 refers to the momentum-integral results obtained using the profiles after the 3000 step explicit calculation. Much of this discrepancy was remedied after the implicit program was developed. The results of the explicit program at a time of 30.0 were used as the starting values and the implicit program was continued another 150 steps with a \( \Delta t = 0.05 \). The result reduced the maximum time derivative to 0.0008. Even though the profiles of Figures 12 and 13 for \( y < 12 \) were unchanged by the continuation, the main result was the stabilizing of the profiles for \( y > 20 \). This provided an excellent agreement with the momentum-integral results tabulated in Table 1 under the heading Momentum Integral #2.

Figure 14 shows why the unsettled region near \( y = \) did not have a more substantial effect on the results of
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Figures 8-13. Over the inner 55% of the disc, the flow is entirely outward from the disc. Thus errors in flow properties are carried away from the disc by the flow.

The excellent agreement between the present solution and the Bödewadt results was very encouraging. This agreement, together with the fact that only small changes of the solution occurred when $x_\infty$ was changed from 1 to 2, emphasizes that for this case the correct downstream boundary condition at $x_\infty$ is being applied.

With this reassurance, we now turn to cases of more immediate interest.

n = 0: Tangential-Velocity Independent of Radius

The case of $n = 0$, the tangential-velocity being a constant independent of the radius, was chosen because the similarity solution is given by King and Lewellen\textsuperscript{16} and yet it is close to the expected change from self-similar solutions to those with a more complicated structure. The region considered was bounded by $0 \leq x \leq 2.0$ and $0 \leq y \leq 30.0$. The problem was started with a step size of 0.001 for $\Delta t$ but after 500 steps this was increased to 0.005 and the program continued a total of 3500 steps to a time of 15.5. The maximum time derivative at 15.5 was 0.006 at $x = 0.6$ and occurred in the slow-to-settle outer region. Later the results were continued by the implicit program but little change was noted.
The velocity development with time is shown in Figures 15-13 for \( x = 1.0 \) and \( x = 2.0 \). These exhibit a much faster approach to the steady solution than the case \( n = -1.0 \) discussed earlier. They also lack the large overshoots mentioned for \( n = -1.0 \) but do show moderate overshoots for the reverse flow regions.

Figures 19-21 present the change in profiles with regard to radial station. The agreement with the similarity solution for \( n = -1.0 \) is better than for \( n = 0 \). Though the results for increasing \( x \) appear to be approaching the similarity solution asymptotically. The self-similar solution for \( n = 0 \) appears to be applicable only very close to the axis and the region of influence is much less than that for solid-body rotation. In fact this region seems to be much less than the \( r = 0.15, (x = 2) \) presented. This also helps explain the strange behavior of the vertical velocity at \( x = 2.0 \) shown in Figure 22. The vertical component is obtained from the continuity equation utilizing the known radial component. Any errors in the radial-velocity component are accumulated in the vertical and magnified, making it the most sensitive component. Now since the region of domination by the similarity solution has apparently not been reached, the boundary condition imposed at \( x_{\infty} \) is incorrect. While any error in the radial velocity in Figure 19 is not obvious, the accumulated error in the vertical component
Figure 15. Time Development of the Radial Velocity at $x = 1.0$ for $n = 0$
Figure 16. Time Development of the Tangential Velocity at $x = 1.0$ for $n = 0$
Figure 17. Time Development of the Radial Velocity at $x = 2.0$ for $n = 0$
Figure 18. Time Development of the Tangential Velocity at \( x = 2.0 \) for \( n = 0 \).
Figure 19. Radial Development of the Radial Velocity, \( n = 0 \)

\[ f = (1-r)^{-1/2} u \]
Figure 20. Radial Development of the Tangential Velocity, $n = 0$
Figure 21. Radial Development of the Vertical Velocity, n = 0

\[ h = (1-r)^{1/4} r^{1/2} \]
in Figure 21 is apparent.

Since King and Lewellen show the similarity solution still oscillating noticeably at \( y = 40 \). The height of the flow region was increased from \( y_\infty = 30 \) to \( y_\infty = 45 \), and studied with the implicit routine with \( \Delta t = 0.01 \). The results show a smoother approach to the boundary condition at \( y_\infty \) and only change slightly the profiles given in Figures 15-21 in the outer regions \( y > 25 \) and \( x > 1.5 \). This might be partially explained by the fortuitous placement of the outer boundary condition near a node of the oscillation in \( u \). Also as shown by the vertical velocity in Figure 21, the flow is entirely outward over the inner part of the disc and thus the outer boundary condition does not exert a major influence. The comparison of wall shear by finite-difference and momentum-integral is shown in Table 2. Included in this table under the heading Momentum Integral #2 are the results with the edge placed at \( y_\infty = 45 \). The results for the edge at \( y_\infty = 30.0 \) are presented under the heading Momentum Integral #1. The agreement is surprisingly good between all three, when compared with the case \( n = -1 \).

\( n = +1.0: \) The Potential Vortex

The case of the potential vortex was chosen for two reasons. First to check the results given in Ref. 20 by a different method. Second, as a check on the accuracy of
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the unsteady program with its single mesh; in particular
to show the development of the inner structure as found
earlier. Since the radial-flow profiles do not oscillate,
there would be no downstream boundary condition to invoke.
However, in Ref. 20 the location of the upper boundary
condition had been found to be of utmost importance in
establishing even the qualitative behavior of the flow for
decreasing $r$.

The size of the flow region studied remained the same
as for the cases of $n$, -1.0 and 0, discussed previously
($x_\infty = 2.0$ and $y_\infty = 30.0$). The initial time step in the
explicit program was required to be cut once again to
0.0001 but after 1000 steps was increased to 0.0005 and
was continued at that value to a time of 1.35. It is
interesting to note that while the step size changed, the
number of time steps required seemed relatively constant
and not a function of $n$. At time equal 1.35 the maximum
time derivative was reduced to 0.07 at $x_\infty = 2.0$. The
time development appears considerably more rapid for
$n = + 1.0$ than for $n = -1.0$, or $n = 0$. However, one
should recall that the non-dimensional time variable is a
measure of the number of radians of rotation of the flow
at the edge of the disc. This helps explain the more
rapid development since the flow at a mean radius rotates
much faster for $n = + 1.0$ than for $n = -1.0$. 

The resulting velocity profiles are shown in Figures 22-25 for the development with time. This development is both very smooth and rapid compared to both \( n = -1.0 \) and 0. There were no velocity overshoots. The time derivative increased monotonically from the edge of the disc to \( x_\infty = 2.0 \) and decreased in a smooth manner throughout the computation as time increased. The case \( n = +1.0 \) appeared to be the best-behaved case attempted. Also indicated in Figures 22-25 as plotted points are a sampling of the corresponding results from the steady analysis reported by Burggraf, Stewartson and Belcher.\(^{20}\) The values of Ref. 20 are regarded as more accurate owing to the smaller mesh size utilized in that computation.

Figures 26-28 present the radial development of the profiles from the unsteady analysis and the resulting profiles appear similar to those shown in Figures 3 and 4 from Ref. 20 presented earlier in Chapter II. In Figure 28, the vertical velocity also illustrates a difficulty concerning the thickness of the boundary layer. The sudden change of the profiles between \( x = 1.5 \) and \( x = 2.0 \) with \( h(2.0, \infty) \) approaching zero was the first clue that the thickness was insufficient, as there is nothing to indicate any difficulties in Figures 26 and 27. Although this problem was ignored for \( n = 0 \), it is of real concern now since it occurs here in a region of inflow and hence will affect the downstream points in the flow field. In
Figure 22. Time Development of the Radial Velocity at $x = 1.0$ for $\eta = 1.0$. 

$$f = (1-r)^{-1/2}$$
Figure 23. Time Development of the Tangential Velocity at $x = 1.0$ for $n \neq 1.0$.
Figure 24. Time Development of the Radial Velocity at $x = 2.0$ for $n = +1.0$
Figure 25. Time Development of the Tangential Velocity at $x = 2.0$ for $n = +1.0$. 

- Steady Solution
- Steady Profile
  - $t = 1.35$
  - $t = 0.85$
  - $t = 0.35$
  - $t = 0$
Figure 26. Radial Development of the Radial Velocity, $n = + 1.0$
Figure 27. Radial Development of the Tangential Velocity, \( n = +1.0 \)
Figure 28. Radial Development of the Vertical Velocity, $n = +1.0$
the steady analysis this was remedied by an increase in
the thickness of the boundary layer being computed. Since
increasing the boundary-layer thickness was now restricted
by considerations of storage and computation time, and the
potential vortex computation is here only a check on the
accuracy of the time-dependent program, this deficiency
was not corrected for this problem. It is actually
encouraging that the unsteady computation produces the
same anomalous behavior for a similar condition of insuf­
"cient thickness.

To answer the question of whether an inner develop­
ment would be clearly detected, Figures 29 and 30 show the
unsteady and steady approach to the limit curves in the
inner. As was expected from Figures 25-28, the agreement
is excellent and not affected by the problems concerning
the thickness of the boundary layer discussed above.

Table 3 presents a comparison of the wall shear for
the finite-difference results and from the momentum-
integral equations, also included are the finite-differ­
ence results obtained from the steady analysis. From
the comparisons cited earlier between the results for the
inner from both the unsteady and steady analyses, the
excellent agreement was expected from the two finite-
difference methods. Similarly the poor agreement between
the momentum-integral and finite-difference results for
x > 1.5 was expected due to the insufficient thickness.
Figure 29. Potential Vortex Inner Solution Unsteady Results, Radial Velocity
Figure 30. Potential Vortex Inner Solution, Unsteady Results, Tangential Velocity
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n = 0.5: The Square-Root Vortex

The real case of interest for this paper was n in the range of postulated multi-structured boundary layer; n = 0.5 was chosen since it was medial. The structure of n = +1.0 was known. The limit value of n was 0.1217. If the critical n was approached too closely it was felt that the calculations would have to be carried to a very large value of x to expose the terminal structure. In other words the closer one is to the critical n the smaller the region of influence of the terminal structure and hence the closer one must come to the axis to illustrate the terminal solution. This reasoning is supported by the slow approach to the similarity solution for n = 0.

The initial attempt was made using the explicit program for a region covering 0 \( \leq x \leq 2.0 \), 0 \( \leq y \leq 39.9 \) with \( \Delta t = 0.0001 \). After 500 calculations the time step was increased to 0.001. After 3000 steps at a time of 2.55 the profiles were smooth and approximately steady, but it appeared that a closer approach to the axis would be required. In order to extend the region to \( x = 3.0 \), the time step had to be cut to 0.0001. However this required too much computer time and prompted the implicit routine formerly discussed. The time step was increased to 0.001 and the results continued to a time of 2.83, to achieve nearly steady profiles. As noted in an earlier
section, the non-dimensional time corresponds to the radians of rotation of the flow at the edge of the disc \((x = 0)\). A check of the profiles for the two cases, \(x_{\infty} = 2.0\) and \(x_{\infty} = 3.0\), showed that changes of 20% had occurred at \(x = 2.0\) in the maximum magnitude of the reversed flow velocity, which was directly affected by the artificial boundary condition being imposed on the reversed flow at \(x = 3.0\). The analytical development to be discussed in Chapter VI suggested that when simple similarity fails the appropriate outer variable is

\[
Z = \frac{3}{\sqrt{\lambda}} \quad \beta = \frac{\sqrt{\lambda}}{Z} \left\{ \frac{z}{z_{\infty} + \lambda} \right\}^2
\]

where \(\lambda\) is the smallest eigenvalue of the tangential momentum equation to be discussed in Chapter VI. For \(n = +1.0\) this reduces to the proper form, \(Z = z\), used in Ref. 20. It also reduces to the conventional similarity variable for \(\lambda = 0\). Using this variable the boundary condition on the reversed flow at \(x = x_{\infty}\) becomes \(\frac{2}{\sqrt{\lambda}} \left. \frac{q}{z} \right|_{Z = 0} = 0\) instead of \(\left. \frac{q}{z} \right|_{q = 0} = 0\). This change was incorporated and tended to improve the solution, as will be seen.

The time development of the velocity profiles is reported in Figures 31-34. Once again the development is both smooth and very quick with none of the severe oscillations exhibited by the \(n = -1.0\) solutions. The result of changing the location of \(x_{\infty}\) from 2.0 to 3.0 and
$f = (1-r)^{-1/2} r^{1/2} u$

Figure 31. Time Development of the Radial Velocity at $x = 1.0$ for $n = 0.5$
Figure 32. Time Development of the Tangential Velocity at $x = 1.0$ for $n = 0.5$
Figure 33. Time Development of the Radial Velocity at \( x = 2.0 \) for \( n = 0.5 \)

\[ f = (1-r)^{-1/2} r^{1/2} u \]
Figure 34. Time Development of the Tangential Velocity at \( x = 2.0 \) for \( n = 0.5 \)
continuing with the implicit program to a time of 2.83 is shown on these figures. Note that the profile change produced by the change of $x_\infty$ is significant only for $y$ sufficiently large that $u$ has become negative. Hence even though the minor changes at $x = 1.0$ could be attributed to the additional time steps the changes at $x = 2.0$ cannot and led to the establishment of a new boundary condition at $x_\infty: \frac{3}{\partial r^2} = 0$.

The radial development of the velocity profiles is given in Figures 35-37. Figures 35 and 36 show a development that bears many similarities to the profiles presented for $n = +1.0$. The expansion of the radial inflow region in Figure 35 is striking. Also shown on these figures are the changes attributed to the change in boundary condition at $x_\infty = 3.0$. The increase in the size and magnitude of the first reverse flow region in Figure 35 is prominent. However other important changes are the increased slope as $f$ changes sign, indicating a better fit of boundary condition to the problem and that the new boundary condition at $x_\infty = 3$ produces thicker profiles.

The new boundary condition also has a smoothing effect on the tangential component, Figure 36 and on the vertical velocity, Figure 37. An interesting difference between the vertical velocity for $n = 0.5$ and those presented for $n = -1.0$ and 0 is that now there is a region of vertical inflow near the disc for all radial stations in contrast.
Figure 35. Radial Development of the Radial Velocity, $n = 0.5$
Figure 36. Radial Development of the Tangential Velocity, $n = 0.5$
Figure 37. Radial Development of the Vertical Velocity, $n = 0.5$. 

$$h = (1-r)^{1/4} r^{3/4}$$
to the similarity solutions and the potential vortex for which there was an overall inflow.

The region of negative $h$ at $x = 3.0$, $26 < y < 31$, may appear questionable, but it cannot be discounted even though the vertical component accumulates the errors in the radial velocity, as pointed out earlier and hence is a more sensitive indicator of any difficulties.

Another problem obvious in Figures 35-37 is that the boundary-layer mesh was chosen too thin. It appears that the effect of this difficulty is restricted to the outer regions of the flow field since, as indicated by the plot of the vertical velocity, the flow is outward from the disc and hence tends to carry any effects away from the regions of interest. However the questionable results shown above the first reversed flow region at $x = 3.0$, $y > 26$, clearly indicate the need for a thicker mesh for $x > 2$.

The comparison of wall shear between the finite-difference and momentum-integral methods shown in Table 4 is reassuring. Considering the discrepancies in the thickness of the boundary layer just discussed, the agreement is excellent for $x < 2.0$ but very poor in the neighborhood of $x = 3.0$. This is no doubt greatly affected by the inward radial flow which at $x = 3.0$ accounts for almost a third of the flow field considered, which is
TABLE 4
Comparison of Wall Shear by Finite Difference and Momentum Integral

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exaggerated by the insufficient mesh thickness.

**Streamlines**

The discussion of the streamlines for the four cases of \( n \) solved numerically warrants a detailed discussion. The figures presented here consist of representative values of the streamfunction in the flow regions considered. Also shown on these figures are the loci of the radial velocity, \( u \), equal zero, thus breaking the figure into regions of radial inflow and radial outflow.

Figure 38 portrays the streamlines in the boundary layer beneath solid-body rotation, \( n = -1.0 \). This figure illustrates some of the results mentioned earlier of inflow near the edge of the disc and outflow over the rest of the disc as was seen from Figure 14 of the vertical velocity. The repetition of the same gentle fluctuations in the streamline pattern for all \( \nu \), for \( x \geq 1.0 \) would appear to be due to the influence of the similarity solution over that portion of the disc. The locations of \( u = 0 \) for the similarity solution (\( x = \infty \)) are shown at \( x = 2.0 \). We now know that the finite-disc solution is approaching very close to the similarity solution from the direct comparison of the velocity profiles in Figures 12 and 13. However the curves of \( u \) equal zero superimposed on the streamline pattern in Figure 38 are not coincident with lines of constant \( y \). Even though the terminal
positions have not been reached, it was anticipated that in regions dominated by similarity the two curves would coincide. Thus we would have to conclude that the streamline pattern is not a good indicator of where similarity is achieved.

Figure 39 shows the streamlines for \( n = 0 \). Here the oscillations in the streamlines have become more severe than for solid-body rotation. This would be expected as a result of the increased magnitude of the non-dimensional velocities, \( u \) and \( w \). Even though the lines of \( u = 0 \) fall upon lines of constant \( y \), as would be expected for similarity, the streamlines are poor indicators of this as discussed above. This is supported by the direct comparison of the velocity profiles in Figures 19 and 20 which indicates that the similarity solution has not been achieved. As was the case for \( n = -1.0 \) the flow over the plate is dominated by vertical outflow. From the vertical velocity given earlier in Figure 21 it is seen that this outflow exists from the disc upward for \( x > 0.6 \) \((r < 0.549)\).

Figure 40 shows the streamlines for the case of \( n = 0.5 \), and is indeed interesting. As was noted earlier, there is a larger region of inflow near the edge of the disc, than for the similarity cases, in agreement with the results of Mack\(^1\) and the inward portion of the disc is primarily vertical outflow. The small region of vertical
Numbers indicate values of $r v$

Figure 38. Streamfunction, $\psi$, for $n = -1.0$
Number indicate values of $rv$

Figure 39. Streamfunction, $\psi$, for $n = 0$
Figure 40. Streamfunction, $\psi$, for $n = +0.5$
Figure 41. Streamfunction, $\psi$, for $n = +1.0$
inflow, shown in Figure 37, near the surface of the disc means that the fluid from the edge is dominant near the disc. This also leads to the streamlines emerging sharply from the inner region. It is interesting to observe that the radius of curvature of the streamlines about the lowest curve of \( u = 0 \) seems to be increasing and the curves opening as \( x \) increases, while about the next \( u = 0 \) the curves are closing and collapsing. This effect is also evident in Figure 35 which shows a larger velocity gradient at the second node \( u = 0 \) than at the first node. A possible explanation of this is the development of a viscous layer at the second node, which will be discussed in Chapter VI. Also shown in Figure 40 are some lines of \( Z = \text{constant} \). While the lines of \( u = 0 \) are not in the same direction in the upper part of the graph. This may be due to residual unsteady flow for large \( y \). The lower lines of \( u = 0 \), which were the first to settle down to steady flow, are certainly close to falling along the lines of constant \( Z \). It is obvious that \( y \) is not the proper variable to describe the flow in the outer region.

Figure 41 presents the streamlines for the potential vortex. Since \( u = 0 \) at the disc and at the outer edge of the boundary layer, with no oscillations, there are no lines of \( u = 0 \) for \( n = +1.0 \). Another obvious difference is the continual vertical inflow, or entrainment of fluid, over the entire disc. This illustrates why the position
of the outer edge is so important for \( n = + 1.0 \) and is less important for other values of \( n \). Also shown in Figure 41 is a curve of constant \( Z = z \). This curve emphasizes the similarity of the outer streamlines and helps to illustrate why this was the proper outer variable for the boundary layer of the potential vortex \(^{20}\) and plays a role similar to that of \( y \) in the similarity cases. Unfortunately the outer region of the flow is just beginning to be apparent and while the line of constant \( z \) appears to indicate the ultimate trend of the streamlines, the agreement is still weak. In Ref. 20, it was found that the terminal structure was not obvious for \( x < 3.0 \).
As a direct consequence of the potential-vortex investigation, a check was made to see if the hypothesis of an inner region dominated by the radial velocity was valid for other vortex flows. The numerical results in the case of the potential vortex had indicated that near the wall the radial-velocity component was much larger than the tangential component. An asymptotic solution for $r \to 0$ constructed from this evidence led to a structure which showed excellent agreement with the numerical results. Anticipating a similar pattern, a solution was assumed of the form

$$r^n u = \Phi'(\eta) + \ldots$$

$$r^n \Phi = r^2 \eta \chi(\eta) + \ldots$$

$$\eta = \frac{3}{{r^{1/2} \lambda}}$$

so that the viscous, pressure, and inertia forces would balance in the near wall region. The coordinate $\eta$ is the classical similarity variable used by King and Lewellen and is appropriate for achieving the above balance in the region near the axis. There is also an implicit assumption in the above series representation that $\lambda$ be greater than 99.
than zero in order that \( v \ll u \) as \( r \to 0 \). Under the transformation from \( r, z \) to \( r, \eta \) the steady-state laminar-incompressible boundary-layer equations become

\[
\begin{align*}
\frac{d}{dr} \left( r^n \frac{d}{dr} \right) \left( r^n w_n \right) - \frac{Lu}{r} \frac{d}{dr} \left( r^n w_n \right) + \left( r^n w_n \right) \frac{d}{dr} \left( \frac{d}{dr} r^n \right) - \frac{r^n w_n}{r} - m \frac{r^n w_n}{r} &= 0 \\
\frac{d}{dr} \left( r^n \frac{d}{dr} \right) \left( r^n u_n \right) - \frac{Lu}{r} \frac{d}{dr} \left( r^n u_n \right) + \left( r^n u_n \right) \frac{d}{dr} \left( \frac{d}{dr} r^n \right) + \frac{r^n u_n}{r} &= 0
\end{align*}
\]

Substitution of the assumed form in Equations (4-1) into Equations (4-2) and collecting together the coefficients of like powers of \( r \) leads to an ordinary differential equation for the first order term in the series expansion for the radial velocity

\[
\Psi'''' + \left( \frac{3-m}{2} \right) \Psi' \Psi'' + m \Psi''^2 = 1
\]

with the boundary conditions from (2-8)

\[
\Psi(\infty) = 0, \quad \Psi'(\infty) = 0
\]

At the outer edge of this layer, while the velocity may approach a constant not necessarily zero, all derivatives of the velocity must be zero, and \( \Psi'' \) must go to zero faster than \( \Psi' \to \infty \). Thus the third condition is that

\[
\Psi' \to \frac{1}{\sqrt{m}} \quad \text{as} \quad \eta \to \infty
\]
The sign is a result of requiring a physically acceptable solution, that is in the absence of a tangential-velocity component the pressure gradient forces the flow towards the axis.

Equation (4-3) was solved numerically for various values of \( n \). Since a matching to an outer similarity solution is required later, the asymptotic form of (4-3) is desired. It can be obtained without difficulty as

\[
\frac{\Psi_o'}{\Psi_o} \sim -\frac{1}{n} + A \gamma^{-\frac{n}{2-n}} \tag{4-5}
\]

where \( A \) is a constant and has been estimated from the numerical solution of (4-3) as \( A = 0.384 \) for \( n = 1/2 \).

Substituting (4-1) into (4-2b) results in the ordinary differential equation for the tangential component of velocity

\[
\gamma'' + \left( \frac{3-n}{2} \right) \Psi_o' \gamma' - (1-n) \Psi_o \gamma' = - \chi \Psi_o' \gamma, \tag{4-6}
\]

\[
\gamma_i(\infty) = 0 \tag{4-7a}
\]

The boundary condition given in (4-7a) is a direct result of (1-2), applying the no-slip condition. The second boundary condition results from the need to suppress the exponential growth of \( \gamma \) as \( \gamma \to \infty \), which would preclude any possibility of matching to an outer solution. Substituting the asymptotic form for \( \Psi_o \) from (4-5) into (4-6)
Note that the standard solution is of exponential form. If the exponential growth of $Y$ is suppressed by the appropriate choice of $\lambda$, the resulting asymptotic form then exhibits algebraic growth as will be shown later. Transforming Equation (4-6) to the canonical form for an eigenvalue problem by

$$r = \eta \int_0^\eta \Phi_0 \, d\eta$$

leads to the equation for $\Gamma$

$$\Gamma'' - \left\{ \frac{\eta - \eta^2}{\eta^2} \Phi_0' + \frac{1}{\eta} \left( \frac{\eta - \eta^2}{\eta^2} \right)^2 \Phi_0^2 \right\} \Gamma = \lambda \Phi_0 \Gamma$$

The algebraic growth of $Y$ is counteracted by the exponential decay of $\Gamma$ so that the boundary conditions for (4-10) are given by

$$\Gamma(0) = \Gamma(\infty) = 0$$

and the problem is now a conventional eigenvalue problem.

The eigenvalues $\lambda_1$ were obtained by solving the finite difference analog of (4-6) numerically using a Givens-Householder method modified for slightly nonsymmetrical tridiagonal matrices by Mr. Richard E. Jenson of the Ohio State University. The difference form of Equation (4-10) is given in matrix notation by
the coefficient matrix \( \{ P \} \) is a tridiagonal square matrix of elements consisting of the \( \Psi_{o1} \) and \( \Psi'_{o1} \) obtained from the finite difference solution of (4-3), such that for

\[
\{ P \} = \begin{bmatrix}
  b_j & c_j & 0 & 0 & \cdots & 0 \\
  a_{j-1} & b_j & c_j & 0 & \cdots & 0 \\
  & a_{j-1} & b_j & c_j & \cdots & 0 \\
  & & \ddots & \ddots & \ddots & \ddots \\
  & & & a_{j-1} & b_j & c_j \\
  & & & & a_{j-1} & b_j \\
  & & & & & a_{j-1}
\end{bmatrix}
\]

the elements are given by

\[
a_j = \frac{1}{(\Delta y_j^2) \Psi_{o_j}'}
\]

\[
b_j = -\left\{ \frac{2}{\Delta y_j^2} + \frac{7-5n}{4} \Psi_{o_j}'' + \frac{1}{4} \left( \frac{3-\Delta}{2} \right)^2 \Psi_{o_j}^2 \right\} / \Psi_{o_j}'
\]

\[
c_j = \frac{1}{(\Delta y_j^2) \Psi_{o_j}'}
\]

\( j = 1, m \)

for the boundary conditions: \( \Gamma_0 = 0, \Gamma_{m+1} = 0 \). A sign change has been incorporated in the elements so that the numerical results will be for \( -\lambda \). This was done since the routine solved for the largest eigenvalues and only the lowest eigenvalues were of interest.

The results for the five lowest eigenvalues are shown in Table 5 for various \( n \). As mentioned earlier, the assumption of \( r^m v \ll r^n u \) implied that \( \lambda \) must be greater than zero. This condition restricts the allowable range
<table>
<thead>
<tr>
<th>$n$</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>$\lambda_3$</th>
<th>$\lambda_4$</th>
<th>$\lambda_5$</th>
</tr>
</thead>
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<td>6.6206</td>
<td>8.7052</td>
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<td>9.3789</td>
</tr>
<tr>
<td>0.40</td>
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<td>7.2063</td>
<td>9.5852</td>
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<td>7.4493</td>
<td>9.9620</td>
</tr>
<tr>
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<td>0.0000</td>
<td>2.4731</td>
<td>4.9910</td>
<td>7.5345</td>
<td>10.0957</td>
</tr>
</tbody>
</table>
of $n$. The value of $n$ for which $\lambda = 0$ acts as the critical point at which double-structured solutions no longer apply. Presumably simple similarity holds beyond that point and indeed such solutions were found by King and Lewellen$^{16}$ for $n < .1$.

The asymptotic form of the eigenvector $\chi_i$ of Equation (4-6) can be obtained in a standard manner without difficulty as

$$\chi_i \sim B \chi_i^{(0)} \eta^2 \left( \frac{\eta}{\beta} \right)$$

and exhibits the algebraic growth described earlier.

The eigenvectors, $\gamma_1$, were obtained numerically using a Runge-Kutta backward integration from a large value of $\eta$ to zero and iterating on the eigenvalue until the boundary conditions were satisfied. The constant $B$ can be estimated from the numerical solution of (4-6) and for $n = 0.5$ has been found as 1.896. The slope at the wall, $\gamma_1'(0)$, can be evaluated from the numerical solution of the partial differential equations. Values of the function

$$C(r) = r^2 \left. \frac{\partial (r^{n+1})}{\partial \tilde{\eta}} \right|_{\tilde{\eta} = 0}$$

are displayed in Figure 42 as a function of $x$ for $n = 0.5$. This function, which should approach $\gamma_1'(0)$ in the limit
\[ C(r) = r^{\lambda_0} \left. \frac{\partial}{\partial \eta} (r^\eta \nu) \right|_{\eta=0} \]

\[ \chi_1'(0) = \text{Limit } \frac{C(r)}{r=0} \]

Figure 42. Evaluation of $\chi_1'(0)$
as \( x \to \infty \), is seen to be almost constant. From this it is estimated that \( C(0) = 0.512 \) for \( n = 0.5 \) and likewise \( \gamma'(0) = 0.512 \).

Figures 43 and 44 show a comparison of the approach to the limit profiles exhibited by the numerical results for \( n = 1/2 \) for both the radial and tangential components. These results correspond to Figures 29 and 30 which show the results of the unsteady calculation for the potential vortex. It should be noted that while the agreement is perhaps superior for the case \( n = 1.0 \), the trend is unmistakeable for \( n = 0.5 \).

Continuing the expansion process for the radial component, take

\[
\Psi = \Psi_0(\eta) + r^\mu \Psi_\mu(\eta)
\]

where the subscript anticipates the result of some intermediate terms due to the coupling with the tangential component. The resulting ordinary differential equation for \( \Psi_\mu \) is

\[
\Psi_\mu'' + \left( \frac{3-\mu}{2} \right) \Psi_0' \Psi_\mu'' + (2n \mu) \Psi_0' \Psi_\mu' + \frac{1}{2} (s^2 + 2 \mu) \Psi_0 \Psi_\mu'' = 0
\]

which is also of the eigenvalue form for \( \mu \). Taking advantage of the fact that a simple solution of Equation (4-14) is given by \( \Psi_\mu = \Psi_0' \) we perform the transformation
Figure 43. Inner Solution, \( n = 0.5 \),
Radial Velocity

\[ r^{1/2}u = \Psi'_o(\eta) \]
Symbols are typical results of the eigenvector calculation with $\gamma_1(0) = C(r)$.

Figure 44. Inner Solution, $n = 0.5$, Tangential Velocity


\[ \nabla \mu = \nabla_0 y(y) \]

which leads to

\[ \nabla_0' y(y)''' + (3\nabla_0'' + \frac{3\lambda}{2}\nabla_0\nabla_0') y'' \]

\[ + (3\nabla_0'' + (3-m)\nabla_0\nabla_0'' + (2m-\lambda)\nabla_0^2) y' = 0 \]

(4-18)

and applying

\[ y' = e^{-\frac{1}{2} \int_0^\eta \frac{3\nabla_0'' + \frac{3\lambda}{2}\nabla_0\nabla_0'}{\nabla_0'} \, d\eta} \]

results in

\[ x'' + \left\{ -\frac{3\lambda}{2} \frac{\nabla_0''}{\nabla_0^2} + \frac{3\lambda-3}{4} \nabla_0' - \frac{3}{4} \frac{\nabla_0'}{\nabla_0^2} \right\} \]

\[ - \frac{3(m-\lambda)^2}{16} \nabla_0^2 + \frac{3}{2} \nabla_0' \}

x = \mu \frac{\nabla_0'}{\nabla_0} \]

(4-19)

which has been solved in the same manner as (4-10) and the five lowest values of the eigenvalue, \( \mu \), are tabulated in Table 6 for various values of \( n \). The resulting asymptotic form of \( \nabla_\mu \) for large \( \eta \) is

\[ \nabla_\mu' \sim D \nabla_0^{-2 \frac{3m-\lambda}{3m}} \]

(4-20)

The accuracy of the numerical solutions does not warrant an estimate for the value of \( D \). With the knowledge of the eigenvalues \( \lambda \) and \( \mu \), the resulting expansions for the velocity components can be properly ordered. Since the eigenvalues are functions for \( n \), care must be taken in the ordering. Figure 45 for the eigenvalues and their
<table>
<thead>
<tr>
<th>( n )</th>
<th>( \mu_1 )</th>
<th>( \mu_2 )</th>
<th>( \mu_3 )</th>
<th>( \mu_4 )</th>
<th>( \mu_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
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<td>4.8928</td>
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<td>4.5955</td>
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<td>0.1217</td>
<td>1.9029</td>
<td>4.3585</td>
<td>6.8621</td>
<td>9.3942</td>
<td>11.9461</td>
</tr>
</tbody>
</table>
Figure 45. Eigenvalues and Multiples at Various n.
multiples illustrates this. While for $n = +1$, the correct form of the expansion is

$$r^n u = \psi' + r^{2,1} \psi_2' + r^{3,2} \psi_3' + r^{4,3} \psi_4' + \cdots \quad (4.21)$$

However for $n = 0.5$ the correct form of the expansion is

$$r^1 u = \psi' + r^{2,1} \psi_2' + r^{3,2} \psi_3' + r^{4,3} \psi_4' + r^{5,4} \psi_5' + \cdots \quad (4.22)$$

This form is important not only to represent the solution properly in the inner but also since the matching with the outer solution must be performed in the correct order.
CHAPTER VI
OUTER SIMILARITY SOLUTION

There are at least two ways that the results of the investigation of the outer similarity can be developed and both yield some independent findings worthy of note. One approach uses the geometric coordinates as independent variables in the outer region; it is relatively straightforward while remaining as general in specifying coordinates and exponents as possible. This method suggests the similarity variable for the outer layer that was mentioned in Chapter III. Another more direct approach is to use the streamfunction as independent variable; many of the same parametric results are obtained, with the notable exception of the geometric similarity variable.

Beginning here with the former approach, the following expansion is proposed for the outer region.

\[ r^\alpha u = F_\alpha(Z) + \cdots \]
\[ r^\gamma \omega = G_\gamma(Z) + \cdots \]
\[ r^5 \omega = H_0(Z) + \cdots \]

where \( Z \) is the appropriate outer similarity variable and \( 11^4 \).
\( \alpha, \gamma, \delta, \) and \( \beta \) are all constants to be determined. The boundary conditions are given by

\[
\alpha \to \infty \quad F_0 \to 0, \quad G_0 \to +1
\]  

(5-2)

Transforming the equations of motion from the \( (r,z) \) system of (1-7) to the \( (r,Z) \) coordinate system results in their becoming

\[
\frac{\partial^2 u}{\partial r^2} - \alpha \frac{\partial}{\partial r} + \frac{\partial^2 u}{\partial Z^2} + \frac{\partial u}{\partial r} - \frac{\alpha^2}{r} = \frac{1}{r^2m+1} + \frac{1}{r^2} \frac{\partial u}{\partial Z^2}
\]  

(5-3a)

\[
\frac{\partial^2 u}{\partial r^2} - \alpha \frac{\partial}{\partial r} + \frac{\partial^2 u}{\partial Z^2} + \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial u}{\partial Z^2} = 0
\]  

(5-3b)

\[
\frac{\partial u}{\partial r} - \beta \frac{\partial^2 u}{\partial r \partial Z} + \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial u}{\partial Z} = 0
\]  

(5-3c)

Substitution of the above expansion into these equations and collecting the lowest order terms results in

\[
\begin{align*}
& r^{-2m-1} \left\{ \frac{1}{2} \right\} + r^{-2m-1} \left\{ -\alpha F_0^2 - \beta z F_0 G_0 \right\} + r^{-2m-1} \left\{ -G_0^2 \right\} \\
& \quad + r^{-2m-1} \left\{ H_0 G_0 \right\} + r^{-2m-1} \left\{ H_0 \right\} + \cdots = 0
\end{align*}
\]  

(5-4a)

\[
\begin{align*}
& r^{-\alpha-1} \left\{ -\gamma F_0 G_0 - \beta z G_0^2 + F_0 G_0 \right\} + r^{-\alpha-1} \left\{ H_0 G_0 \right\} \\
& \quad + r^{-\alpha-1} \left\{ H_0 \right\} + \cdots = 0
\end{align*}
\]  

(5-4b)

\[
\begin{align*}
& r^{-\alpha+1} \left\{ -\alpha F_0 + F_0 - \beta z F_0 G_0 \right\} + r^{-\alpha-1} \left\{ H_0 \right\} + \cdots = 0
\end{align*}
\]  

(5-4c)
At this point let us list the arguments required to allow the selection of the constants $\alpha$, $\gamma$, $\delta$ and $\beta$.

1. Satisfying the boundary condition on $F_0$ as $z \to \infty$ does not help in determining $\alpha$. However matching the inner solution for large $\eta$ with the outer solution as $Z \to 0$ requires that $r \propto u$ as $Z \to 0$ be compatible with $r^\eta u$ as $\eta \to \infty$ which from (4.5) is known to be $-1/\sqrt{m}$. Thus $\alpha = n$.

2. Satisfying the outer boundary condition on $r^\eta v$ as $Z \to \infty$ immediately requires that $\gamma = n$.

3. By matching the asymptotic form of the inner expansion of $r^\eta v$ for large $\eta$ to $G_0(Z)$ for small $Z$ means that in terms of the outer variable

$$r^\eta \sim A r^\alpha \eta^2 \left( \frac{z-n}{3-m} \right)$$

$$\lim_{Z \to 0} A Z^2 \left( \frac{z-n}{3-m} \right) \frac{1}{r} = \beta \left( \frac{z-n}{3-m} \right)$$

Since $G_0$ is function of $Z$ and not $r$, there results

$$\beta = \frac{1}{Z \left( \frac{1+n-2\lambda}{3-m} \right)}$$

(5-5)

It should be noted that this form of $\beta$ is proper in two limiting cases. For $\lambda = 0$, $\beta$ reduces to $\frac{n+1}{2}$, the correct form of the classical similarity variable. This limit is appropriate since if $\lambda = 0$ the similarity solution holds and there are
no distinct inner and outer regions. At the other end of the spectrum if \( n = +1.0 \) then \( \beta = 0 \) and the outer variable \( Z \) becomes \( z \) which agrees with the result obtained in Ref. 20. For \( n = 1/2 \), \( \beta = 0.281 \).

4. The only constant not yet accounted for is \( \delta \), which is determined by considering the continuity Equation (5-4a). The result of requiring that both terms be of the same order is that

\[ \delta = 1 + n - \beta \]

Use of these four results in the equations above results in

\[
(1-m) F^2 - G^2 - \beta Z F' G' + H G' = 0 \quad (5-6a)
\]

\[
(1-m) F G - \beta Z F' G' + H G' = 0 \quad (5-6b)
\]

\[
(1-m) F - \beta Z F' + H' = 0 \quad (5-6c)
\]

with the boundary conditions that

\[
F_0 \rightarrow 0, \quad G_0 \rightarrow 1, \quad H_0 \rightarrow \text{finite as } Z \rightarrow \infty \quad (5-7a)
\]

and to satisfy the matching with the inner that

\[
\infty Z \rightarrow 0; \quad F_0 \rightarrow -\frac{1}{\sqrt{m}}, \quad G_0 \rightarrow Z^{-\frac{1}{2} \left( \frac{n+1}{3-2n} \right)} \quad (5-7b)
\]
Note that no viscous terms in Equation (5-3) appear in Equations (5-6); these are deferred to higher order terms in the asymptotic expansion, just as for the potential vortex. Hence the flow in the outer region is inviscid to first order. Equations (5-6b) and (5-6c) can be rearranged and integrated once to yield

\[
\left\{ H_0 - \beta F_0 \right\}' = - \left( 1 - m \gamma \right) F_0
\]

(5-8a)

\[
G_0 = \text{Constant} \left\{ H_0 - \beta F_0 \right\}^{-\frac{1}{1-\gamma}}
\]

(5-8b)

Equation (5-6a) is more difficult but can be combined with Equation (5-6b) to allow a phase-plane analysis. This leads to the relation

\[
\frac{F_0'}{G_0'} = \frac{(1-m)F_0 G_0}{1-m F_0^2 - G_0^2}
\]

(5-9)

which may be integrated to obtain

\[
\varepsilon F_0^2 + G_0^2 - \frac{1}{m} \int G_0 \frac{2m}{1-m F_0^2} = \text{Constant}
\]

(5-10)

If the constant is evaluated from the matching conditions with the inner, \( F_0 \rightarrow 1/\sqrt{n} \) and \( G_0 \rightarrow 0 \), the result is

\[
\left\{ F_0^2 + G_0^2 - \frac{1}{m} \right\} = 0
\]

(5-11)

so that in the phase plane the path is a circle. Unfortunately along this path the outer boundary conditions are
never satisfied, except for the special case of \( n = +1.0 \), the potential vortex.

Thus, it appears that an infinitely thin shear layer changes the velocities from the wall values to those at the edge of the inner, \( F_0 = -1/\sqrt{n} \), \( G_0 = 0 \). Then, at least to first order, with negligible viscosity, the hodograph of the velocity vector would pass along a semi-circle from \( F_0 = -1/\sqrt{n} \), \( G_0 = 0 \) to \( F_0 = 0 \), \( G_0 = 1/\sqrt{n} \) and on to \( F_0 = +1/\sqrt{n} \), \( G_0 = 0 \). At this point the question arises; what path is now followed? There is no physical or numerical justification for \( G_0 < 0 \), which would eliminate continuing along the circle. On the other hand, there is some numerical justification as in Figures 33 and 35 for a more rapid return to \( F_0 = -1/\sqrt{n} \). This suggests that the appropriate path might be another viscous shear layer to return the velocities from \( F_0 = +1/\sqrt{n} \), \( G_0 = 0 \) to \( F_0 = -1/\sqrt{n} \), \( G_0 = 0 \). Then the process would be repeated cyclically. This will be discussed again later in this chapter.

Equations (5-9) and (5-11) can be combined and integrated so that the outer terminal profiles can be obtained explicitly in terms of \( Z \). To simplify the notation now and later, let us introduce an "outer streamfunction" \( \bar{\psi} \) defined as
then \( \nu, \bar{\Psi}_0 \) and \( \bar{\Phi} \) are related by

\[
\nu = r^{\frac{3-m}{2}} \bar{\Psi}_0(\eta) = r^{1-m_\beta} \bar{\Phi}(z) \tag{5-13}
\]

Comparing Equation (5-12) with Equation (5-8a) we see that

\[
\bar{\Phi} = -\frac{1}{1-m_\beta} \left( \mathcal{H}_o(z) - \beta z \bar{\Psi}_o \right) \tag{5-14}
\]

Using this result in Equation (5-8b) yields

\[
G_o = \mathcal{B}' \left\{ -\bar{\Phi} \right\}^{1-m_\beta} \tag{5-15}
\]

where \( \mathcal{B}' \) can be evaluated by matching \( G_o \) as \( z \to 0 \) to the inner solution in Equation (4-14) as \( \eta \to \infty \). Thus \( \mathcal{B}' \) is given as

\[
\mathcal{B}' = (m) \frac{\eta_\infty}{\mathcal{Y}_1(\eta_\infty)} \tag{5-16}
\]

For \( n = 0.5 \) it has the value 0.778. Now Equation (5-11) can be solved for \( F_0 \) in terms of \( \bar{\Phi} \) such that

\[
F_0 = \sqrt{\frac{1}{m} - \mathcal{B}'^2 \left\{ -\bar{\Phi} \right\}^{2(\frac{1-m_\beta}{1-m_\gamma})}} \tag{5-17}
\]

We can use Equation (5-12) with (5-17) and integrate to obtain

\[
\mathcal{Z} = \int \frac{d\bar{\Phi}}{\sqrt{\frac{1}{m} - \mathcal{B}'^2 \left\{ -\bar{\Phi} \right\}^{2(\frac{1-m_\beta}{1-m_\gamma})}}} \tag{5-18}
\]
Now $Z$ is known as a function of $\Phi$ and hence we know $F_0$ and $G_0$ as functions of $Z$. The lower limit of integration in Equation (5-18) was determined by matching $\Phi$ as $Z \to 0$ to $\Psi_0$ as $\eta \to \infty$, where it was found that

$$\Phi \sim \frac{1}{\mathcal{E}} Z$$

(5-19)

the upper limit is bounded by that value of $\Phi$ at which $F_0 = 0$, i.e. the radical in Equation (5-18) becomes zero. Then $(-\dot{\Phi})$ will decrease back to zero. Performing the transformation

$$\sigma = m \mathcal{B}^2 \left\{ -\dot{\Phi} \right\}_2^{2(\frac{2}{4})} = m G^2$$

(5-20)

on Equation (5-18) results in

$$Z = \frac{3-m}{1-m+\lambda} \left\{ \frac{1}{\mathcal{E}} \right\}_0^{\frac{1}{\mathcal{E}}} \left\{ \frac{2}{4} \right\}_0^{\frac{2}{4}} \left\{ \frac{3-m}{1-m+\lambda} \right\}_0^{\frac{3-m}{1-m+\lambda}} \left\{ \frac{1}{\mathcal{E}} \right\}_0^{\frac{1}{\mathcal{E}}} \frac{d\sigma}{\sqrt{1-\sigma^2}}$$

(5-21)

where the integral is the incomplete Beta-function $\mathcal{B} \left( \frac{3-m}{1-n+\lambda}, 1/2 \right)$. Pearson has computed these functions, Tables of the Incomplete Beta-Function, but the increments were too large for use here, so Equation (5-18) was numerically integrated. The results are presented in Table 7.

The limit profiles for the radial and tangential velocities for $n = 0.5$ are shown in Figures 46 and 47 with the results of the numerical solution of the partial differential equations. Only the first cycle of the terminal
### TABLE 7

The Terminal Structure of the Outer Flow
\( n = 0.5 \)

<table>
<thead>
<tr>
<th>-( \Phi )</th>
<th>( Z )</th>
<th>( F_0 )</th>
<th>( G_0 )</th>
</tr>
</thead>
<tbody>
<tr>
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Figure 46. Radial Velocity in the Outer Variable, \( Z, n = 0.5 \)
Figure 47. Tangential Velocity in the Outer Variable, $Z$, $n = 0.5$
profiles is shown. The outer edge of the first cycle is at $Z = 6.565$. The first cycle terminates at this point, since for larger $Z$, $\bar{y}$ becomes positive and thus $G_0$ would be complex. It is clear from Figure 46 that for $Z > 4$, $x \leq 3$, the flow is viscous-dominated.

It appears in Figures 46 and 47 that the size of the regions of inflow and outflow remain relatively constant in terms of the $Z$ variable. This is in contrast to the rapidly growing inner regions shown in Figure 34 for $y$, the variable used in the numerical computation. Indeed, the agreement of the numerical solutions and the limit curve in Figures 46 and 47 is very reassuring; showing that $Z$ is the proper outer variable, even though the profiles at $x = 3.0$ are still removed from the terminal profile. The breakdown in agreement for $Z > 5$ was to be expected and reaffirms the suggestion of a viscous shear layer mentioned earlier.

The realization that the viscous terms no longer dominate in the outer region allows the above results to be obtained in a different light. The Von Mises transformation from the $(r,z)$ coordinate system to the $(r,\psi)$ space, when applied to Equation (1-7), allows continuity to be satisfied identically and gives the following equations

$$\frac{\partial u}{\partial r} \bigg|_{\psi} - \frac{\partial^2 u}{\partial \psi^2} = -\frac{1}{r\bar{z}^m} + \bar{r} \bar{\omega} \frac{\partial^2 u}{\partial \psi^2}$$

(5-22a)
These Equations (5-22) are general except for the evaluation of the pressure gradient in terms of a power-law vortex. Neglecting the viscous terms allows the integral of (5-22b) to be written

$$\nu \frac{\partial \psi}{\partial r} \bigg|_\psi + \nu \frac{\partial \psi}{\partial r} = r \nu \frac{\partial \psi}{\partial \psi^2}$$  \hspace{1cm} (5-22b)

where $\psi$ is a function of the streamfunction to be determined. This result allows the physical interpretation that along a streamline the angular momentum is conserved. With this result known, it can be readily achieved in retrospect from (5-15) by considering $\Phi$ in terms of the streamfunction,

$$G_\phi = B' r^{(-m)} (-\psi)^{\frac{1}{1+n}}$$

or

$$r^{(-m)} G_\phi = r^{1-n} = B' (-\psi)^{\frac{1}{1+n}}$$  \hspace{1cm} (5-24)

But now the result is explained in physical terms.

The integral of Equation (5-22a) may also be written down, after neglecting the viscous terms. Using Equation (5-23), Equation (5-22a) becomes

$$\frac{\partial}{\partial r} \left[ \frac{\nu^2}{2} + \frac{\nu^2}{2} - \frac{1}{2m} \right] \bigg|_{\psi} = 0$$
or

\[ \left( u^2 + \sigma^2 - \frac{1}{n} r^{2n} \right) \psi = \Lambda (\psi) \quad (5-25) \]

where again \( \Lambda \) is an unknown function of the streamfunction as was \( \Sigma \). This result represents the conservation of total pressure, or Bernoulli's Equation. This also corresponds to Equation (5-10) obtained earlier but now has been given a clearer physical interpretation. Even though the outer similarity variable is not indicated by this approach, the results are more amenable to a physical explanation when dealing with streamlines.

The streamline figures cited earlier (Figures 38-41) have the value of \( \psi \) indicated along some of the streamlines. As expected for the self-similar solutions \( n = -1.0 \) (Figure 38) and \( n = 0 \) (Figure 39) the value of \( \psi \) is not constant along the streamlines since in these flows the viscous, inertial and pressure forces are all of equal importance. In Figure 41, for the potential vortex, the outer inviscid layer is just beginning to be apparent and the change is smaller along a streamline than for either of the above cases. In Figure 40 for \( n = 0.5 \), the value of \( \psi \) is almost constant along the outer streamlines, except near the regions \( u = 0 \). This is the result postulated earlier in this chapter from the phase-plane analysis. However, the slight viscous forces cause the path in the phase plane
to become a spiral about the outer boundary condition. This is shown in Figure 48, which is the hodograph of $r^n v$ versus $r^n u$ for the case of $n = 0.5$. The inviscid outer solution should be a semi-circle with its center at 0 and a radius of $\sqrt{2}$. The numerical solution suggests that this is approached as $x \to \infty$.

The deviation from the semi-circle can be estimated by evaluating the quantities that are conserved in the outer layer, $\Sigma$ and $\Lambda$, in the outer fringes of the viscous inner layer, using the results for the inner layer with $\eta$ large.

From Equation (4-5) we have for large $\eta$

$$r^n u \sim \Xi' = -\frac{1}{\sqrt{m}} + A \eta^{\frac{2-n}{3}}$$  \hspace{1cm} (4-5)

$A$ was evaluated in Chapter V as 0.384 for $n = 0.5$. Since

$$\chi = \int_0^\infty r u \, d\gamma = r^{\frac{n}{2}} \int_0^\gamma (r^n u) \, d\eta$$

which gives

$$\chi = -\frac{1}{\sqrt{m}} \, r^{\frac{2-n}{2}} \eta$$  \hspace{1cm} (5-26)

Combining Equations (5-26) and (4-5) yields

$$\chi^2 \sim \frac{r}{M} - L(-\psi)^{\frac{2-n}{3}}$$  \hspace{1cm} (5-27)

Utilizing the value, $A = 0.384$, $L$ is found to be 1.416 for $n = 0.5$. From Equation (5-24)
Figure 48. Phase Plane, $r^{nu}$ and $r^{nv}$, $n = 0.5$
\[ \mathcal{N} = -\left\{ r_{i} \right\} \mathcal{B}_{i} \frac{d}{d\left( \frac{r_{m}}{r_{m}^{*}} \right)} \]  

(5-28)

where the constant \( B' \) was discussed earlier. Combining Equations (5-27) and (5-28) results in

\[ u^{2} + \sigma^{2} - \frac{1}{m} r^{2m} \sim -\frac{\text{\( \frac{\text{d}(r_{m})}{\text{d}(r_{m})^{*}} \)}}{\text{\( \frac{\text{d}(r_{m}^{*})}{\text{d}(r_{m}^{*})} \)}} \]

or

\[ \int u^{2} + \sigma^{2} - \frac{1}{m} r^{2m} \sim N \left( r^{2} \right)^{\frac{1}{m+1}} \]

and finally

\[ \int (r^{n}u)^{2} + (r^{n}\sigma)^{2} - \frac{1}{m} \int (r^{n})^{\frac{2m}{m+1}} N \left( r^{2} \right)^{\frac{2m}{m+1}} \]

(5-29)

where \( N \) is given in terms of \( L \) and \( B' \)

\[ N = L B' \frac{m}{m+1} \]

Thus for \( n = 0.5 \), \( N \) has the value 1.032. Equation (5.29) holds in the outer fringes of the inner layer and also is the proper form in the outer region where \( r^n u = 0 \).

The results of using Equation (5-29) to obtain \( r^n u \) for a given \( r^n v \) are shown in Figure 48. The agreement with the numerical results is quite good for \( r^n u < 0 \), even in the inner layer provided \( \eta \) is not too small, and for small positive \( r^n u \). The agreement for larger \( r^n u \) rapidly deteriorates and the results appear to be dominated by the viscous effects in the second shear layer, where in the
terminal solution, \( r^h u \) returns to negative values. These considerations make any further comparisons with the numerical results unwarranted.

An attempt was made to investigate the shear layer. Since this is an area dominated by the radial flow, the equation describing the shear layer is the same as that used in the study of the inner similarity region, Equation (4-3). However, the boundary conditions have to be changed from those given in (4-4) to a set appropriate to the range minus infinity to plus infinity, as given below:

\[
\eta \to -\infty, \quad \Psi_0' \to +\frac{1}{\sqrt{m}}; \quad \eta = 0, \quad \Psi_0 = 0; \quad \eta \to +\infty, \quad \Psi_0' \to -\frac{1}{\sqrt{m}} \quad (5-20)
\]

A numerical solution was attempted involving the same program used in the solution of the inner for \( \eta > 0 \) and coupled to a shoot-and-miss routine using a Runge-Kutta forward integration for \( \eta < 0 \). A satisfactory solution was not obtained as all the solutions found indicated that the velocity overshoots the boundary condition as \( \eta \to -\infty \).

For sufficiently small \( r \), a very large number of cycles in the velocity oscillations will occur before the viscous decay is significant. Because of this slow decay of the limit cycle, the first thought is that the boundary layer for the \( n = 0.5 \) vortex is thicker than that obtained for the potential vortex. Unfortunately the exact structure of the outer is not known. From Figures 46 and 47
which show the velocity components in the outer variable $Z$, it appears that the thickness is either increasing slightly or holding constant; however, in terms of the non-dimensional physical variable $z$, it is actually decreasing. The increasing boundary-layer thickness in the $Z$ variable is the result of $r \rightarrow 0$ in $Z$. If this result is correct for the general vortex then this is another example in which the potential vortex is exceptional, since the axis is approached with a finite thickness and mass flow.

While the confirmation of the outer structure by the numerical results is not as pronounced as for the inner structure, the proposed form of the terminal flow is strongly suggested by the numerical results.
CONCLUSION

The boundary-layer development beneath a general vortex on a finite disc has always taken a subordinate position to the similarity studies, even though that which pertains to a finite disc is the more physically realizable. This is partially a result of the absence of a technique with the ability to handle regions of reverse flow. The self-similar solutions avoid this difficulty by seeking only the terminal boundary layer, which is a difficult numerical problem in itself. In most instances, the similarity solutions leave a few unanswered questions, one of applicability to the finite disc and also if applicable what is the region of applicability and influence.

The momentum-integral method can be used for the finite disc, however, this is an approximate method with no means of assessing the error involved other than by comparison with an exact solution. In addition this approach can leave interesting details of the flow field ambiguous or absent. For steady flow, numerical integration from the edge of the disc, while the most desirable method, is unstable in the reverse flow regions and eventually this will cause a failure in the numerical computation. In fact
the better the computation scheme the sooner the failure will occur for boundary layers with back flow.

The time-dependent approach is not directly affected by the regions of reverse flow in any adverse manner. It not only allows the details of the steady solution to be perceived, but also permits one to watch the development of that steady solution from a given set of initial conditions. Specifying the open-end boundary conditions can be difficult but with care a realistic condition can be found. The implicit method certainly proved superior; even though it takes more time to compute each step than the explicit method, the resulting increase in step size more than compensates. The steady-state solutions from both the implicit and explicit routines were virtually identical, but this author favors the former method.

The numerical results presented earlier show the laminar boundary layer on a finite disc beneath a generalized power-law vortex and its time development from an inviscid initial condition at $t = 0$ for the four cases of $n = -1.0, 0, +1.0$ and $0.5$. The case of solid-body rotation, $n = -1.0$, over a finite disc shows that the similarity solution of Browning for the Bödewadt problem not only gives the proper terminal behavior of the flow at the axis but actually dominates the solution for $r \leq 0.4$. While others have reached this conclusion their solutions did not extend inward as far as $r = 0.145$. This
large region of influence is contrasted by the solution for
n = 0 which indicates that the self-similar solution of
King and Lewellen\(^16\) is applicable only very close to the
axis. This is shown by the fact that even at \(r = 0.15\) the
similarity solution is being approached slowly and is not
yet applicable.

The prime objective for this study was to examine the
hypothesis of a multi-layered terminal boundary layer for
\(n > 0.1217\). The importance of this range of \(n, 0.1217 < n \leq 1.0\), is that it covers the entire spectrum of experi­
mentally measured vortices. The multi-structured boundary
layer originally postulated for the case of the potential
vortex has now been shown to be the proper form of the
terminator for the more general case.

These boundary layers for \(n > 0.1217\) consist of an
inner layer near the disc dominated by the radial flow.
The agreement between this inner layer and the numerical
results is extremely good, as shown in Figures 43 and 44.
This is in direct opposition to the fact that the external
boundary condition requires that the flow be entirely
tangential. The recovery from the inner layer to the
external boundary condition is accomplished in an almost
inviscid layer. In other words an inviscid outer solution
would require in the phase plane an infinite number of
passages through the semi-circular cycle \(F_0 = -1/\sqrt{n}\)
$G_0 = 0$ to $F_0 = 0$, $G_0 = 1/\sqrt{n}$ to $F_0 = +1/\sqrt{n}$, $G_0 = 0$ and then returning to $F_0 = -1/\sqrt{n}$, $G_0 = 0$ across an infinitesimal shear layer. The effect of being slightly viscous causes these cycles to depart slightly from a circular path and very slowly spiral into the outer boundary condition. While not as completely confirmed as for the inner region, this concept of an outer layer is strongly suggested by the agreement of the limit cycle and of the numerical solution (Figures 46, 47 and 48).

The potential vortex is now seen to be a special case. Since the outer boundary condition lies on the semi-circular phase-plane path, it is satisfied by the inviscid outer solution without oscillations in the velocity components. Another difference between the generalized vortex, $n > 0.1217$, and the potential vortex is the uniqueness determining the outer solution. There existed some freedom in the outer structure of the potential-vortex boundary layer as a result of the constant entrainment of fluid from the potential vortex into the boundary layer. It appears that this is not the case for $n$ in the range, $0.1217 < n \leq 1.0$, as now the outer is entirely specified by the inner. This is clearly shown in Figure 40 of the streamlines for $n = 0.5$. All the flow in the outer must pass first through the inner. It was this difference between the generalized vortex and the potential vortex which allowed the explicit determination of the first cycle of
the outer structure, shown in Figure 46 and 47 for \( n = 0.5 \).

The effect of the outflow from the boundary layer on the power-law vortex is surely small. Consider the two displacement thicknesses of the three-dimensional boundary layer

\[
\delta_1 = \int_0^\infty \frac{r^m \hat{u}}{\Omega} d\delta
\]

\[
\delta_2 = \int_0^\infty (1 - \frac{r^m \hat{u}}{\Omega}) d\delta
\]

Both thicknesses scale as \( \left( \frac{a(m-2)}{a} \right) \). Thus they are small and must be considered as higher order effects. However, it was pointed out earlier that the vortex flow \( v \propto r^{-n} \) does not satisfy the Navier-Stokes equations. Hence the ultimate justification for its use as an outer boundary condition rests on the experimental observations cited in the Introduction.

While some of the questions surrounding the problem of a boundary layer beneath a general vortex have been resolved, there are still some questions unanswered. Immediate extensions of the present work might include obtaining a numerical solution for \( n = 0.5 \) over the region \( 0 \leq x \leq 4.0 \) and \( 0 \leq y \leq 100 \). This would more clearly show the outer development and more important the proposed shear layer. Such a calculation was attempted but the storage requirements and computation time were too exorbitant even
with the implicit routine. A new procedure that allows an even larger step size than the implicit method would have to be developed. A Newton iteration routine has been suggested by P. Williams (Private Communication); with this method he was unable to find a step size large enough to cause divergence in a two-dimensional boundary layer problem.

Examination of the effects of compressibility could extend the inward limit of applicability. A slightly-compressible investigation of the type performed by Anderson would increase the inward extent towards the sonic line. A full-compressible boundary-layer analysis could extend the domain of applicability past the sonic line. Shapiro shows how the external vortex has a limiting minimum radius that corresponds to the maximum possible velocity and hence, zero pressure. For the potential vortex the minimum radius is $1/\sqrt{6}$ times the radius at which the fluid velocity achieves the speed of sound. However near this radius the assumptions inherent in the analysis, such as a continuum fluid, would break down.

Finally, the inclusion of turbulence would make the model more realistic for dealing with atmospheric problems. However as there is no generally accepted model of turbulence, the results of any analysis would be restricted to the model used.
BIBLIOGRAPHY


