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By

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* * * * *

The Ohio State University
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INTRODUCTION

The many-body problem was born sometime in the early fifties. Initially the concern was with isolated problems like quantum mechanical treatment of the superfluid helium, selective summation of terms in perturbation theory of nuclear matter where the two-body interaction contains a hard core and a microscopic theory of superconductivity. While deeper understanding of these isolated problems was coming through the pioneering works of physicists like Landau, Brueckner, Bandeen, Cooper and Schnieffer, a new attempt was being made to give a unified treatment of the many-body problem.

It was initiated by the papers of Matsubara\(^1\) and Goldstone\(^2\), who showed that it is possible to treat the problem by using the powerful techniques of quantum field theory. Then there was a host of papers published almost simultaneously that described the application of field theoretical methods to the systems of interacting fermions and bosons. The most popular method turned out to be the propagator formalism and the language of Feynman graphs. Using this method it is possible to treat seemingly unrelated problems with the same method. A complete bibliography of early papers is impossible and even unnecessary since excellent texts like the ones by Nozieres\(^3\) and Abrikosov, Gorkov and Dzyaloshinski\(^4\) have appeared.

We restrict ourselves to the many-fermion problem. The seemingly unrelated problems we consider are the Brueckner theory and the B.C.S.
theory. Brueckner showed that it is possible to obtain the ground state properties of a many fermion system interacting via a two body potential, even if it contains a repulsive core, by introducing a reaction matrix. The reaction matrix sums up all the forward scattering graphs and is finite even if the two body potential is not. It was soon found out by Gottfried\textsuperscript{5} and Emery\textsuperscript{6} that for attractive interaction the reaction matrix is singular and thus the Brueckner theory fails in this case. Later other workers modified Brueckner's theory by including the hole-hole interaction along with the particle-particle interaction but still the singularity prevailed. Thus even when the most important terms were summed up in the perturbation theory, the reaction matrix as a function of energy $E$ developed a pole for $E = 2\mu$ where $\mu$ is the Fermi energy. The field theoretical generalization of Brueckner's theory is the so called self-consistent ladder approximation (SCLA). This approximation is satisfactory at low densities and is the simplest apart from the Hartree-Fock method. It takes into account two body correlations. The function that corresponds to the reaction matrix in Brueckner's theory is the vertex function. Galitskii calls it the four-pole. The theory of SCLA has been worked out by many authors, notably, Prange and Klein\textsuperscript{7}, Galitskii\textsuperscript{8}, and Wild\textsuperscript{9}.

This vertex function or rather its analytic continuation into the complex energy plane has a pole on the imaginary axis for attractive interaction. It was soon realized that this implies that the ground state of the interacting fermi system is radically different from the one assumed. As we shall elaborate in Chapter 1, it turns out that
the ground state is similar to the one encountered in the B.C.S. theory of superconductivity.

The B.C.S. theory, so highly successful in explaining almost all the phenomena in metals and alloys, is a Hartree-Fock theory. Elementary excitations in this theory are stable quasi particles. Their lifetime in infinite, i.e., damping, is neglected. A field theoretical treatment is needed if one wants to take into account the damping and retardation effects. This was done by Nambu\textsuperscript{10} and Schrieffer and co-workers\textsuperscript{11}. Their work was concerned with electron phonon interaction and coulomb interaction in metals.

We would like to investigate the ladder approximation equations in the theory of superfluidity. The reason is that this is the simplest method of treating two-body correlations when the potential contains a hard core. Thus for a strongly interacting system like nuclear matter or liquid He\textsuperscript{3} the SCLA formalism is the most suitable. Like their counterpart in the normal system, the SCLA equations will turn out to be a closed system of equations which must be solved self-consistently. In this case, however, the equations involve 2 x 2 matrices rather than simple functions as in the normal case, so we have four times as many equations to solve. The problem is prohibitively difficult to solve in general.

We shall first make sure that the vertex function (which is a 2 x 2 matrix) does not have the Cooper singularity for attractive interaction (which we saw was the trouble with the normal state vertex function). This we shall do by formally solving the integral equation for the (matrix) vertex function with a separable potential. Then we
shall simplify the set of equations for the special case when the gap in the single particle spectrum is much smaller than the Fermi energy (assuming of course that the gap exists). In this case we can derive a gap equation for the superfluid Fermi system with arbitrary interaction.

The following is the plan of the study: Chapter 1 presents the ladder approximation equations for a normal system. While the derivation is heuristic, great care is taken to define the causal functions and their associated analytic functions. The nature of the singularity of the vertex function is carefully stated and its implication mentioned. Chapter 2 shows how to remedy the trouble by modifying the formalism. The new formalism is developed and a new set of self-consistent matrix equations is derived which generalizes the SCLA for the normal system. In Chapter 3 we discuss the important analytic properties of these matrix functions and derive dispersion relations for them. Chapter 4 is devoted to the discussion of the new SCLA equations. The integral equation for the vertex part is solved by introducing a separable potential. The behavior of the matrix vertex function for small total momentum is examined to see if the singularity of the "normal" vertex function still exists. Then having satisfied that it does not, the behavior of the various functions near and away from the "Fermi surface" is examined. It is shown how to simplify the new SCLA equations for the case when the region of deviation of the normal state equations is small or the gap is small. A new simplified gap equation is derived. Foundation of a superfluid fermi liquid theory is laid.
Finally in Chapter 5 we investigate the form of the spectral function near the fermi surface. This is done by studying the imaginary part of the diagonal element of the self-energy matrix in perturbation theory. The procedure is similar to that used by Luttinger for the normal system. An application of this result is made to the single particle tunneling experiment and some interesting consequences are pointed out. Finally a model of superfluid fermi liquid is suggested with a Landau criterion for critical velocity and a pair-breaking mechanism.
CHAPTER I

SELF-CONSISTENT LADDER APPROXIMATION (SCLA)
EQUATIONS IN A NORMAL SYSTEM

In this chapter we derive the self-consistent ladder approximation (SCLA) equations. This really serves as an introduction to the next chapter in two ways. Firstly we shall investigate the nature of the singularity of the vertex function and thus look for the precise nature of modification needed to obtain SCLA equations with non-singular solution. Secondly we shall fix our notation and describe the diagram structure, which will help in writing the modified equations in Chapter 2. Since this problem is treated in details in papers of Prange and Klein\(^1\), Galitski\(^2\) and Wild\(^3\) we shall give only a brief derivation.

We consider a system of spinless Fermions for which the Hamiltonian in second quantized notation

\[
H = \sum \frac{\mathbf{p}^2}{2m} a_\mathbf{p}^\dagger a_\mathbf{p} + \frac{i}{2} \sum \nabla \mathbf{p} \mathbf{F} \cdot \mathbf{F} a_\mathbf{p}^\dagger a_\mathbf{p} a_\mathbf{p} a_\mathbf{p} \tag{1.1}
\]

The operators \(a_\mathbf{p}\) and \(a_\mathbf{p}^\dagger\) satisfy the Fermi anticommutation relations

\[
\{a_\mathbf{p}, a_\mathbf{p}^\dagger\} = \delta_\mathbf{p} \delta_\mathbf{p}', \quad \{a_\mathbf{p}, a_\mathbf{p}\} = \{a_\mathbf{p}^\dagger, a_\mathbf{p}^\dagger\} = 0 \tag{1.2}
\]

This system is best studied by introducing the one- and two-particle Feynman propagators:

\[
G_1(\mathbf{p}, t_i, \mathbf{p}, t_f) = -i \langle 0 | a_\mathbf{p}^\dagger(\mathbf{p}, t_i) a_\mathbf{p}(\mathbf{p}, t_f) | 0 \rangle \tag{1.3}
\]

\[
G_2(\mathbf{p}_1, t_1, \mathbf{p}_2, t_2; \mathbf{p}_1', t_1', \mathbf{p}_2', t_2') = (-i)^2 \langle 0 | a_\mathbf{p}_1^\dagger(\mathbf{p}_1, t_1) a_\mathbf{p}_2(\mathbf{p}_2, t_2) a_\mathbf{p}_1(\mathbf{p}_1', t_1') a_\mathbf{p}_2'(\mathbf{p}_2', t_2') | 0 \rangle \tag{1.4}
\]
where \( |0\rangle \) is the ground state of the interacting system and represents a Heisenberg field operator. In the ladder approximation \( G_1 \) and \( G_2 \) satisfy the following equations:

\[
G_1(l,s') = G_1^0(l,s') + G_1^0(l,2')M(2,2')G_1(2,2')
\]

\[
G_2(l,2',2') = G_1(l,2')G_1(2,2') - G_1(l,2')G_1(2,2')
\]

\[
+ iG_1(l,2')G_1(2,2')\Gamma(3',4',3,4)G_2(3,4,2,2')
\]

Here the number 1 means \( \bar{t}_1, t_1 \) etc., and we have used Schwinger's notation: repeated number in a product implies integration and/or summation.

We can express the solution of (1.6) in terms of a vertex function \( \Gamma(1 2 3 4) \) such that

\[
\Gamma(3',4';3,4)G_2(3,4;1,2') = \Gamma(3',4';3,4)\{G_1(3,1')G_1(4,1')\}^A
\]

Then \( \Gamma \) satisfies the equation where \( \{ \}^A \) denotes antisymmetrized product

\[
\Gamma(l,2';2') = \Gamma(l,2';2') + i\Gamma(l,2';3)G_1(3,3')G_1(4,4')\Gamma(3',4';1,2')
\]

The self energy function \( M \) is defined by

\[
M(2,2')G_1(2,2') = \Gamma(3,3';2,3')G_2(2,3,3';3')
\]

and is given in terms of \( \Gamma \) by

\[
M(2,2') = \left[ \Gamma(3,3';2,3) - \Gamma(2,3';3,2) \right] G_1(3,3')
\]

Equations (1-5), (1-8) and (1-9) define the SCLA.

We now write these explicitly for the case of an infinite system for which translation invariance holds. We go over to the Fourier transform with respect to \( t \) and define the 4-vector \( \mathbf{p} \equiv \mathbf{p}_0 \) (or \( \mathbf{p}, \mathbf{p}_0 \))

Then

\[
G_n(p,p') = G_n(p)\delta^4(p-p')
\]

\[
M(p,p') = M(p)\delta^4(p-p')
\]
Further \( \mathcal{V}(\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4) = \mathcal{V}(\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4) \delta(\vec{r}_1 + \vec{r}_2 - \vec{r}_3 - \vec{r}_4) \)

Then equations (1-5), (1-8) and (1-9) become:

\[
G(p) = \frac{1}{p_0 - \frac{p^2}{2m} - M(p)}
\]

\[
\Gamma(p, p', q, p - q) = \mathcal{V}(p, p', p', p') + \frac{i}{2\pi} \int \mathcal{V}(p, p', q, p - q) \times G(q)G(p - q) \Gamma(q, q', q', q') \quad (1.8')
\]

where \( p = p_0 + p_\perp = p'_0 + p'_\perp \)

and finally

\[
M(p) = \frac{i}{\pi} \int dq q G(q) \left[ \Gamma(p, p', q) - \Gamma(p, q, p') \right] \quad (1.9')
\]

The functions \( G(p) \) and \( M(p) \) are causal functions, and are not analytic in the variable \( \omega \). They have, however, associated with them analytic functions. There exists a function \( G(p, z) \) which is analytic everywhere except on the real axis.

One defines the retarded and advanced functions \( G^r \) and \( G^a \) by

\[
G^r(p, \omega) = G(p, \omega + i\eta), \quad G^a(p, \omega) = G(p, \omega - i\eta)
\]

where \( \eta \) is a positive infinitesimal. \( G^r \) and \( G^a \) are the boundary values of \( G(p, z) \) when one approaches the real axis from above and below respectively. Then

\[
G(p, \omega) = \theta(\omega - \mu) G^r(p, \omega) + \theta(\mu - \omega) G^a(p, \omega)
\]

\( G^r(p, \omega) \) can be analytically continued in the upper half plane and similarly \( G^a(p, \omega) \) in the lower half plane.

Similarly, associated with \( M(p, \omega) \) there are analytic, retarded and advanced functions \( M(p, z) \), \( M^r(p, \omega) \) and \( M^a(p, \omega) \) which are related
to $M(p,\omega)$ in the same manner.

These properties can be derived from the spectral representation of the function $G(p)$ and the Dyson eqn. (1-5'). The vertex function is given by the integral eqn. (1-8') and we must now study its analytic properties. Let us rewrite eqn. (1-8') in the variables

$$p = p_i - p_\omega, \quad p' = p'_i - p'_\omega, \quad p = p_i + p_\omega = p'_i + p'_\omega$$

$$\Gamma(p, p', p) = \mathbf{V}(p, p', p) + \frac{i}{2\pi} \int d\omega \mathbf{U}(p, \omega, p) \times \frac{G(\omega) G(p-\omega) \Gamma(\bar{p}, \bar{p}', p)}{G(\omega) G(p-\omega)}$$

Here

$$p = \bar{p}, p_\omega$$
$$q = \bar{q}, q_\omega$$
$$p = \bar{p}, p_\omega$$

Because of assumed form of $V$, $\Gamma$ is independent of $p_\omega$ and $p'_\omega$. We can integrate

$$\int d\omega G(\omega) G(p-\omega) = R(q, p)$$

obtaining

$$R(q, p) = 2\pi i \int dE \left( \frac{\theta(E-\mu)}{p_\omega - E + i\eta} + \frac{\theta(\mu-E)}{p_\omega - E - i\eta} \right) f(p, E)$$

where

$$f(p, E) = -\sigma(E-\mu) \int dE' \frac{1}{E'} \frac{1}{(E'-E)^2}$$

$p$ is the single particle spectral function.

Then

$$\Gamma(p, p', p) = \mathbf{V}(p, p', p) - \int d\omega \mathbf{V}(p, \omega, p) R(q, p) \Gamma(q, p', p)$$

This defines the causal function $\Gamma$. If we now define

$$R(q, p, z) = \int \frac{f(p, E)}{z-E} dE$$

$R(z)$ is analytic everywhere except on the real axis. We can define

$$R^{r, a}(q, p, p_\omega) = R(q, p, p_\omega \pm i\eta)$$

Then we can define $\Gamma^r$ and $\Gamma^a$ using these $R$'s in the kernel.
Thus symbolically
\[
\Gamma = V - V R \Gamma \tag{1.8''}
\]
\[
\Gamma^{r,a} = V - V R^{r,a} \Gamma^{r,a} \tag{1.10}
\]
\[
\Gamma(z) = V - V R(z) \Gamma(z) \tag{1.11}
\]

We now have the SCLA equations and the properties of the functions entering these equations at hand. The analytic properties are dictated by quite general theorems while the SCLA depends on particular summation of diagrams. If the summation is correct the resulting function must obey the theorems. E.g., \(\Gamma^r(P_0)\) must be analytic in the upper half plane.

Now it turns out that for attractive interaction \(\Gamma^r(P_0)\) calculated in the SCLA has a pole on the imaginary axis in the upper-half plane, when energy is measured with respect to Fermi energy \(\mu\). This happens when \(P \approx 0\) and when both the particles described by propagators \(G(q)\) and \(G(p-q)\) in \(R^r(P)\) are near the fermi surface. Then the poles of the function \(G(q)\) and \(G(p-q)\) coincide making \(R^r(P)\) highly divergent.

If \(P \approx 0\) then for \(P_0 \approx 0\), \(\Gamma^r(P_0)\) has the form:
\[
\Gamma^r(P_0) = \frac{\lambda}{1 + \frac{\omega c}{2\pi} \left[ \ln \left( \frac{2\omega_c}{P_0} \right) + \frac{i\pi}{2} \right]} \tag{1.12}
\]
where \(\lambda\) = constant strength of interaction

\(\omega_c\) = a constant describing the range of interaction in momentum variable.

For attractive interaction \(\lambda \approx 0\) the denominator vanishes for
\[
P_0 = 2i\omega_c e^{-\frac{i\pi}{2}m\mu} = i\omega_c
\]
Thus \(\Gamma^r(P_0)\) has a pole on the imaginary axis which is absurd since \(\Gamma^r(P_0)\) has to be analytic in the upper half plane. Further since the
term in the square bracket blows up for $P_0 = 0$ the denominator can
vanish for arbitrary small value of $\lambda$. It can be shown that\textsuperscript{15} if
$\mathcal{P} \neq 0$ the pole occurs for

$$P_0 = i\omega_0 \left( 1 - \frac{\nu^3}{\omega^2 P^3} \right)$$

where $\nu$ is the velocity on the fermi surface\textsuperscript{4}. Thus the important
values of $P$ are near $P = 0$.

This exhibits the breakdown of SCLA for attractive interaction. If we worked with $\mathcal{G}(P)$ we would obtain a pole on the imaginary
axis in the lower half plane.

The existence of a pole in the vertex function implies that one
exists also in the two-particle propagator. Since the pole is on the
imaginary axis it exhibits the instability of the normal system
against the formation of bound states of two particles.\textsuperscript{16} Application of an arbitrary attractive interaction leads to rebuilding the
ground states to a new one in which bound states are formed. Since
the zero momentum bound states are favored, these act like condensed
bosons and any number of them can accumulate in the ground state. A
correct SCLA theory should therefore incorporate these from the very
beginning.
CHAPTER II
DERIVATION OF SCLA IN THE THEORY OF SUPERFLUIDITY

We saw in the last chapter that the ladder approximation breaks down for a system of Fermions with attractive interaction because of the presence of a pole in the vertex function. This implies the instability of the ground state of the non-interacting fermi system against the formation of bound pairs also called Cooper Pairs. A very small attractive interaction leads to a rebuilding of the ground state. The new ground state contains these Cooper pairs and is thus radically different from the one assumed. The Cooper pairs behave like bosons. Since it is the pairs with zero total momentum that are most favored we consider these only. This is the case when there is no net flow. These pairs being bosons condense on the level with zero momentum. Thus our entire formalism needs a revision. We must incorporate these bound pairs in the formalism. However, since they exist only virtually i.e., as a possibility in the intermediate state they must be eliminated from the formalism when calculations are done. They cannot be ignored however, since the zero-momentum state is macroscopically occupied. We must allow for the processes in which two particles with equal and opposite momenta can form a bound pair and also the reverse process: a bound pair breaks up into two particles with equal and opposite mementa. We do this by artificially abandoning particle number conservation. Thus we assume our system is in contact with a
particle bath. We use a zero-temperature grand canonical ensemble by using $H - \mu N$ instead of $H$ where $\mu$ is the chemical potential and $N$ is the particle number operator. Further we include the zero-momentum bosons in the Hamiltonian. Since these exist virtually right from the beginning we include them in the unperturbed Hamiltonian. This will modify the formalism greatly. In the end however these zero-momentum bosons have to vanish explicitly from the perturbation formalism. We will do this by letting the coupling multiplying the term denoting these bosons go to zero. Exactly at what stage this has to be done will be made clear later on. We cannot do it before developing the essential formalism, as we would get back the normal system.

Thus we choose the following Hamiltonian:

$$H = H_0 - \mu N + H' + H_1$$

where

$$H_0 = \sum_{\sigma} \frac{\hbar^2}{2m} \hat{\text{Op}}_\sigma \hat{\text{Op}}_{\sigma}$$

$$H' = \frac{1}{2} \sum_{\vec{p}} \left( \hat{\text{F}}(\vec{p}) \hat{\text{F}}^\dagger_{\vec{p}} \hat{\text{F}}^\dagger_{-\vec{p}} \hat{b} + \text{h.c.} \right)$$

$$H_1 = \frac{1}{2} \sum_{\vec{p}, \sigma' \sigma} \nabla_{\vec{p}} \left( \hat{\text{F}}_{\vec{p}, \sigma} \hat{\text{F}}_{\vec{p}, \sigma'} \hat{\text{F}}_{\vec{p}, \sigma'} \hat{\text{F}}_{\vec{p}, \sigma} \right) \hat{\text{Op}}_{\sigma} \hat{\text{Op}}_{\sigma'} \hat{\text{Op}}_{\sigma'} \hat{\text{Op}}_{\sigma}$$

Here $\text{Op}$, $\text{Op}^\dagger$ are the Fermion operators obeying the usual anti-commutation relations and $b$, $b^+$ are annihilation and creation operators for the fictitious zero-momentum boson. Since the condensed state is macroscopically occupied we can replace the operators $b$ and $b^+$ by the c-number $\sqrt{N_0}$ where $N_0$ is the number of condensed pairs. See (Abrikosov, Garkov, Dzyaloshinski) Ref.
Then
\[ H' = \frac{\sqrt{N_0}}{2} \sum_p (u(p) d_{p\up}^+ d_{p\down} + h.c.) \]  

(2.2)

The generalization of the Feynman-Dyson formalism to include these particle non-conserving terms is non-trivial although straightforward. The perturbation theory contains new diagrams because the expectation values like \( \langle 0 | a_{\up} a_{\down} | 0 \rangle \) and \( \langle 0 | a_{\up}^+ a_{\down}^+ | 0 \rangle \) are no longer zero. The perturbation formalism is adequately treated by Nozieres in the last chapter of his book (17). Since we are specifically concerned with the ladder approximation equations we focus our attention on the propagators, self energy operators and their perturbation expansion.

In what follows we shall briefly develop the perturbation formalism relevant for our purpose. We will see that it is possible to construct dressed propagators starting from the bare propagators by a generalization of the usual perturbation theory. After making an infinite summation we can let the coupling \( u(p) \) go to zero. Then we end up with the self-consistent Dyson Equation whose structure is formally similar to the one in the normal state theory but by necessity is a matrix equation. First, we construct bare propagators for our system since the unperturbed Hamiltonian contains the terms like \( d_{p\up}^+ d_{p\down} \) and \( a_{\up} a_{\down} \) we have along with the usual propagator:

\[ G_{11}^0 (\vec{p}, t) = -i \langle 0 | T (a_{\up}^+ (t) a_{\down} (0)) | 0 \rangle \]  

(2.3)

two more "anomalous" propagators

\[ G_{12}^0 (\vec{p}, t) = -i \langle 0 | T (a_{\up}^+ (t) d_{p\down} (0)) | 0 \rangle \]  

(2.4)

and

\[ G_{21}^0 (\vec{p}, t) = -i \langle 0 | T (d_{p\up} (t) a_{\down}^+ (0)) | 0 \rangle \]  

(2.5)
These anomalous propagators do not represent normal propagation processes but rather the amplitudes for destruction and creation of a pair of particles with opposite momenta and spins. Together with these propagators we introduce a fourth propagator:

$$G_{12}^0(\mathcal{F}, t) = -i \langle 0 | T (\mathcal{A}_{+}^0(\mathcal{F}) G_{-}^0(0)) | 0 \rangle$$

which represents the propagation of spin down holes.

In what follows we will not need the spin index explicitly and although we will often use it as a label, in later calculations it will not appear. This is because we assume that the interaction is spin-independent. In order to go from the bare propagators $G_{ij}^0$ to the dressed propagators $G_{ij}$ we introduce the self-energy matrix $\bar{M}$ so that the Dyson equation is satisfied:

$$G = G^0 + G^0 \bar{M} G$$

where $G, G^0, \bar{M}$ are 2 x 2 matrices. $\bar{M}$ can be described by a sum of irreducible graphs. To do this we introduce some diagrammatic representation. We associate with the operators $\mathcal{A}_+$ and $\mathcal{O}_+$ half a directed line according to Fig. 1.

$$\begin{align*}
\mathcal{A}_+ & \quad \mathcal{O}_+ \\
\downarrow & \quad \downarrow
\end{align*}$$

Figure 1 Graphical Representation of $\mathcal{A}_+, \mathcal{O}_+$

The bare matrix propagator has the following representation

\[
\begin{pmatrix}
G_{11}^0 & G_{12}^0 \\
G_{21}^0 & G_{22}^0
\end{pmatrix} = \begin{pmatrix}
\uparrow & \uparrow \\
\downarrow & \downarrow
\end{pmatrix}
\]

Figure 2 Representation of the Bare Matrix Propagator
We have four kinds of interaction $V$ vertices.

(a) \[\text{Figure 3 The V-Vertices}\]

Besides there are the two $U$ vertices

(b) \[\text{Figure 4 The U-Vertices}\]

We enumerate below some of the lowest order self energy diagrams:

(a) \[\text{Figure 5 Typical Self-Energy Diagrams}\]

Besides the self energy diagrams that are similar to the ones encountered in the normal state perturbation theory there exist two lowest order diagrams that do not exist in the normal state theory.
They are

\[ u + \frac{G_{12}^0}{1 - \Sigma} \quad \text{and} \quad u^* + \frac{\Sigma}{1 - \Sigma} \]

\( \text{Diagram for } \Sigma \)

\( \text{Diagram for } \Sigma^* \)

- Figure 6 Diagrams Not Existing in the Normal State

These are typical of the superconducting state and as we shall show later on lead to the B.C.S. theory. We can insert these self energy corrections in the bare propagators and get dressed propagators. This is done by using the Dyson equation. At this stage it is convenient to introduce matrix notation. We form a matrix unperturbed propagator \( G_{ij}^0 \) and a matrix self energy operator \( \Sigma_{ij} \)

\[ \Sigma_{ij} = \Sigma_{ij}^0 + \begin{pmatrix} 0 & \sigma \\ \sigma^* & 0 \end{pmatrix} \]

We denote the elements of \( G \) by a double line.

We still have to let the coupling \( u(\rho) \rightarrow 0 \). If we just take the limit, all the bare anomalous propagators go to zero and hence the dressed anomalous propagators also go to zero and we get the normal system. This can be seen by writing out e.g. the expression for \( G_{12} \) from the Dyson equation.

\[ G_{12} = G_{12}^0 + G_{12}^0 \Sigma_{11} \Sigma_{12} + G_{12}^0 \Sigma_{22} \Sigma_{21} \]

and so when \( G_{12}^0 = 0 \), \( \Sigma_{12} = \Sigma_{21} = 0 \) and \( G_{12} = 0 \).

So we first dress the propagators in the self energy expansions of Figs. 4(a), (b), (c), (d), and then let \( u(\rho) \rightarrow 0 \).

E.g. the graphs (b) looks like:

\[ \begin{array}{c}
\text{Figure 7 Self-Energy Diagrams with Dressed Propagators}
\end{array} \]
So we first make an infinite summation and then let $\mathcal{U}(\mathcal{T}) \to 0$. Then

$$G_{ij} = G_{ij}^0 \tilde{M}_{ij} G_{ij} + G_{ij}^0 \tilde{M}_{ij} G_{ij}$$

Thus although $G_{ij}^0$ has vanished $G_{ij}$ exists. The bare matrix propagator

is now diagonal

$$G_{\omega} = \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}$$

Further

$$\sigma = \begin{array}{ccc} G_{ij} \\ \end{array}$$

$$\omega$$

and

$$\sigma^* = \begin{array}{ccc} G_{ij} \\ \end{array}$$

Figure 8 Diagrams for $\sigma$ and $\sigma^*$ with Dressed Propagators

We can evaluate any diagram by giving a prescription which is the same as in the normal state theory, e.g., as given in the book by Nozieres (1964). The only difference is that we associate a propagator $G_{ij}$ with any line depending upon the direction of the arrowheads at both ends. For convenience we state the rules below:

1. Associate with each double line of four-momentum $\vec{p}$ a propagator $G_{ij}(\mathcal{T})$ being given by the directions of the arrowheads at the two ends.

2. Ensure by construction the conservation of $\vec{p}$ and $\omega$ at each vertex.

3. Associate with each dotted line which involves one particle scattering from state $\vec{p}_1$ to $\vec{p}_3$ and from $\vec{p}_4$ to $\vec{p}_2$ the factor $V(p_1 p_2 p_3 p_4)$

4. Multiply by a factor $(-i)^n(-)^L$ where $n$ = number of interactions and $L$ = number of closed fermion loops.

5. Integrate over all internal 4-momenta according to

$$\int \frac{d^4p_1}{(2\pi)^4} \frac{d^4p_2}{(2\pi)^4} \cdots \frac{d^4p_n}{(2\pi)^4}$$
We can now write explicitly the set of self consistent ladder equations. The first is the Dyson equation for the single-particle matrix propagator:

\[
G^{-1}(p) = G_0^{-1}(p) - \tilde{M}(p)
\]

or

\[
G(p) = \frac{1}{D(p)} \begin{pmatrix}
\rho - \epsilon_p - M_{11}(p) & - (M_{12}(p) + \sigma_p) \\
-(M_{21}(p) + \sigma_p) & \rho + \epsilon_p - M_{22}(p)
\end{pmatrix}
\]

(2.8)

Here

\[
\epsilon_p = \frac{p^2}{2m} - \mu
\]

\[
D(p) = \det[G(p)]
\]

We can cast this equation in another form:

Put

\[
M_{11}(p) - M_{22}(p) = 2 \chi(p)
\]

(2.9)

\[
M_{11}(p) + M_{22}(p) = 2 \rho (1 - \chi(p))
\]

(2.10)

Then

\[
G(p) = \frac{1}{\Delta(p)} \begin{pmatrix}
\rho + \tilde{\chi}(p) + \tilde{\epsilon}(p) & M_{11}(p) + \sigma_p \\
M_{21}(p) + \sigma_p & \rho + \tilde{\chi}(p) - \tilde{\epsilon}(p)
\end{pmatrix}
\]

where,

\[
\tilde{\epsilon}(p) = \epsilon_p + \chi(p) \quad \Delta(p) = \left[\rho + \chi(p)\right]^2 - \tilde{\chi}(p) - (M_{12}(p) + \sigma_p)(M_{21}(p) + \sigma_p)
\]
If we divide the numerator and denominator by $\frac{\epsilon(p)}{\epsilon(p)}$

$$G(p) = \frac{1}{\Delta'(p)} \begin{pmatrix} \frac{1}{\epsilon(p)}(\mu_0 + \hat{\epsilon}(p)) & \frac{1}{\epsilon(p)}(\mu_1 + \sigma_p) \\ \frac{1}{\epsilon(p)}(\mu_1 + \sigma_p^*) & \frac{1}{\epsilon(p)}(\mu_0 - \hat{\epsilon}(p)) \end{pmatrix}$$

where

$$\hat{\epsilon}(p) = \frac{\epsilon(p)}{\epsilon(p)}$$

$$\Delta'(p) = \mu_0 - \epsilon(p) = \mu_0 - \frac{1}{\epsilon(p)}(\epsilon(p) + \phi^2(p))$$

$$\phi^2(p) = (M_{12}(p) + \sigma_p)(M_{12}(p) + \sigma_p^*)$$

The above equation for $G(p)$ comes closest in form to the B.C.S. equation and shows what modifications are introduced in the general case.

Suppose the matrix $M(p) = 0$ then

$$G(p) = \frac{1}{\Delta'(p)} \begin{pmatrix} \mu_0 + \epsilon \tau & \sigma_p \\ \sigma_p^* & \mu_0 - \epsilon \tau \end{pmatrix}$$

where

$$\Delta'(p) = \mu_0 - \epsilon \tau^2 - |\sigma_p|^2$$

But these are the equations of the B.C.S. theory. The graph

$\sigma_p(\omega \omega \sigma_p^*)$, being the "Hartree-Fock" graph for $M_{12}(\alpha \tilde{M}_{21})$, alone leads to the B.C.S. theory which is a Hartree-Fock theory. The renormalization and damping effects are contained in $M(p)$ and we now pay attention to this part.

Having derived the Dyson equation we now proceed to derive the remaining equations of the SCLA. At this stage it is convenient to introduce new notation in the theory due to Nambu(18). Following
him we introduce a two-component spinor field

\[ \Psi_p^\dagger = \begin{pmatrix} a_{p\uparrow} \\ a_{\downarrow p} \end{pmatrix} \]  

and its Hermitian adjoint

\[ \Psi_p = (a_{p\uparrow}^\dagger, a_{\downarrow p}) \]  

\[ \left\{ \Psi_p, \Psi_p^\dagger \right\} = \delta_{pp'} \mathbf{1} \quad ; \quad \left\{ \Psi_p, \Psi_{p'} \right\} = \left\{ \Psi_p^\dagger, \Psi_{p'}^\dagger \right\} = 0 \]  

The Hamiltonian can be written in terms of the field \( \Psi_p \) if we introduce the four Pauli matrices:

\[ \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ; \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad ; \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad ; \quad \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]  

then,

\[ H = \sum_{pp'} \epsilon_{p'} \Psi_{p'}^\dagger \tau_3 \Psi_p + \frac{1}{4} \sum_{pp'pq} \psi(p,q,p',q') \Psi_{p'}^\dagger \tau_3 \Psi_p \Psi_{p'} \Psi_{p'}^\dagger \Psi_p \]  

(See Schrieffer (19).)

We can make the notation more elegant by introducing

\[ \overline{\Psi}_p = \Psi_p^\dagger \tau_3 \]  

then

\[ \left\{ \overline{\Psi}_p, \overline{\Psi}_{p'} \right\} = \delta_{pp'} \tau_3 \]  

and

\[ H = \sum_{pp'} \epsilon_{p'} \overline{\Psi}_{p'} \tau_3 \overline{\Psi}_p + \frac{1}{4} \sum_{pp'pq} \psi(p,q,p',q') \overline{\Psi}_{p'} \tau_3 \overline{\Psi}_p \overline{\Psi}_{p'} \overline{\Psi}_{p'} \overline{\Psi}_p \]  

We introduce the matrix one body propagator

\[ G_{ij}^{(t_1, t_2; t_1', t_2')} = (-i)(-1)^{\delta_{2j}^i} \langle 0 | T \left( \overline{\Psi}_{F_{t_1'}} \gamma_i \Psi_{F_{t_2'}} \gamma_j \Psi_{F_{t_2}} \right) | 0 \rangle \]  

(2.26)

The elements \( G_{ij}^{ij} \) are identical to the propagators \( G_{ij} \) introduced previously. Similarly we introduce a two-body propagator,

\[ G_{2}^{ijkl}(F_{t_1}, F_{t_2}, F_{t_1'}, F_{t_2'}) = (-i)^i (-1)^{\delta_{2 i}^j + \delta_{2 j}^i} \langle 0 | T \left( \overline{\Psi}_{F_{t_1'}} \gamma_i \Psi_{F_{t_2'}} \gamma_j \Psi_{F_{t_2}} \gamma_k \right) | 0 \rangle \]  

(2.27)

\( G_2 \) can be regarded as a 4 x 4 matrix, with rows labeled by ij, and
columns by kl. $G_{i}^{kl}$ satisfies the following equation

$$G_{i}^{kl}(1,1') = G_{i}^{kl}(1,1') + G_{i}^{kl}(1,2') T_{3} V(2'3;34) G_{i}^{kl}(3;41')$$  \hspace{1cm} (2.24)$$

In writing this equation we have adopted the notation of Chapter 1: repeated indices imply summation and/or integration. Here we have an additional summation over the matrix index $\beta$. Eqn. (2.24) generalizes the eqn. (1.5) of Chapter 1 to the superfluid case. It is now easy to generalize the rest of the equations in Chapter 1. In order to keep the arguments of all the functions the same as in Chapter 1, we rewrite eqn. (2.24) by changing the dummy variables:

$$G_{i}^{kl}(1,1') = G_{i}^{kl}(1,1') + G_{i}^{kl}(1,2') T_{3} V(2'3;34) G_{i}^{kl}(3;41')$$

The equation for the two-body propagator is

$$G_{i}^{kl}(2,1') = G_{i}^{kl}(2,1') - G_{i}^{kl}(2,1') G_{i}^{kl}(1,1')$$

$$+ i G_{i}^{kl}(1,1') G_{i}^{kl}(1,1') V(1';2') G_{i}^{kl}(2,1')$$

$$+ i G_{i}^{kl}(1,1') G_{i}^{kl}(1,1') V(1';2') G_{i}^{kl}(2,1')$$

$$+ i G_{i}^{kl}(1,1') G_{i}^{kl}(1,1') V(1';2') G_{i}^{kl}(2,1')$$

$$+ i G_{i}^{kl}(1,1') G_{i}^{kl}(1,1') V(1';2') G_{i}^{kl}(2,1')$$

$$+ i G_{i}^{kl}(1,1') G_{i}^{kl}(1,1') V(1';2') G_{i}^{kl}(2,1')$$

$$+ i G_{i}^{kl}(1,1') G_{i}^{kl}(1,1') V(1';2') G_{i}^{kl}(2,1')$$

We introduce a $4 \times 4$ matrix vertex function $\Gamma^{ijkl}$ defined by

$$\Gamma^{ijkl}(3';4') = \Gamma^{ijkl}(3';4') G_{i}^{kl}(3;41')$$

Then if we define the matrix self energy function $M_{ij}$ by

$$M_{ik}(1',2') G_{i}^{kl}(2,1') = i \Gamma(1',2';3) T_{3} G_{i}^{kl}(2,1')$$

we get

$$M_{ik}(1',2') G_{i}^{kl}(2,1') = \left( \Gamma(1',2';3) - \Gamma(1',2';3) G_{i}^{kl}(2,1') \right) k_{i}^{n}$$

or

$$M_{ik}(1',2') = \left( \Gamma(1',2';3) - \Gamma(1',2';3) \right) G_{i}^{kl}(3;41')$$

$$+ \Gamma(1',2';3) G_{i}^{kl}(3;41')$$

$$+ \Gamma(1',2';3) G_{i}^{kl}(3;41')$$

$$+ \Gamma(1',2';3) G_{i}^{kl}(3;41')$$

$$+ \Gamma(1',2';3) G_{i}^{kl}(3;41')$$

$$+ \Gamma(1',2';3) G_{i}^{kl}(3;41')$$

$$+ \Gamma(1',2';3) G_{i}^{kl}(3;41')$$

$$+ \Gamma(1',2';3) G_{i}^{kl}(3;41')$$
Equation (2.28) generalizes eqn. (1.9) to the superfluid case. It remains to write the integral equation for the vertex equation.

It is the matrix generalization of eqn. (1.8)

\[
\Gamma_{ijkl}^{(12 \rightarrow 1')2'} = i\sqrt{1(12 \rightarrow 1')} \delta_{ij} \delta_{kl} + i\sqrt{1(12 \rightarrow 34)} X G_{i}^{im}(33') G_{j}^{jn}(44') \Gamma_{mnl}^{(34 \rightarrow 3'4')2'2} \tag{2.29}
\]

Eqns. (2.24), (2.28) and (2.29) define the SCLA formalism. They are a complicated set of matrix equations and are extremely difficult to treat. We shall instead concentrate on a subset of the equations. As eqn. (2.28) shows each element \(M_{ikl}\) is given by a sum of four terms minus their exchange parts. We can represent these diagramatically.

We use the following convention:

A vertex part \(\Gamma_{ijkl}\) is denoted by a rectangle with short ends that carry momentum variables.

![Figure 9 The Matrix Vertex](image)

We associate the number 1 with the bottom \(i, j\) if the short lines are ingoing and number 2 with them if they are outgoing. We associate the number 1 with the top \(k, l\) if the short lines are outgoing and the number 2 if they are ingoing. There must be a continuity of direction of a vertical line as it crosses a corner.

Now out of the four terms (and their exchange parts) contributing to \(M_{ikl}\) we choose one (and its exchange part). We do this by setting \(\beta = i\) and \(\ell = k\). Then

\[
M_{ikl}(2'2) = \left[ \Gamma_{iikkk}^{(2'2')22} - \Gamma_{iiikk}^{(2'2')22} \right] G_{i}^{ik}(33') \tag{2.30}
\]

We can then suppress two of the superscripts for \(\Gamma\) and write

\[
\Gamma_{iikk} = \Gamma_{ikk}
\]
Then eqn. (2.30) can be written in a form that involves only 2 x 2 matrix notation

\[ M_{ik}(2'z_2) = \left[ \Gamma_{ik}^{2'z_2} - \Gamma_{ik}^{z_2} \right] G_{1}^{k'} G_{1}^{33'} \]  

(2.30)

We can represent diagrammatically the elements Mik as given by Eqn. (2.30)

\[ M_{11}(2'z_2) = \Gamma_{11} \]  

\[ M_{12}(2'z_2) = \Gamma_{12} \]  

\[ M_{21}(2'z_2) = \Gamma_{21} \]  

\[ M_{22}(2'z_2) = \Gamma_{22} \]

Figure 10 Diagrammatic Representation of Equation (2.30')

The integral equation that \( \Gamma^{ij} \) satisfies can be read out from eqn. 2.29 by setting \( i=j \), \( k=l \), \( m=n \)

\[ \Gamma_{ij}^{ij}(z_{1}, z_{1}') = iV(z_{1}z_{1}z_{1}') \delta_{ij} + iV(z_{1}z_{1}z_{4}) G_{1}^{(35)} G_{1}^{(44)} G_{1}^{(34)} \]  

(2.31)
Finally we write explicitly the set of SCLA equations for the case of translationally invariant infinite system. We go over to the Fourier transform with respect to $t$. We shall define the 4 vector

$$\mathbf{p} = (\mathbf{p}, \omega) \quad (\sigma \mathbf{p}, p_0)$$

Then

$$G_i(\mathbf{p}, t') = G_i(t) \delta^4(\mathbf{p} - \mathbf{p}')$$

$$M_i(t, t') = M(t) \delta^4(\mathbf{p} - \mathbf{p}')$$

Further

$$V(t) \left( \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4 \right) = V(t) \left( \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4 \right) \delta^4(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)$$

We at this stage define the variables:

$$\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2 = \mathbf{P}_1 + \mathbf{P}_2$$

as the total 4 momentum

$$\mathbf{P} = \mathbf{P}_1 - \mathbf{P}_2 = \mathbf{P}_1 - \mathbf{P}_2$$

as the relative 4-momenta.

Eqn. (2.31) written explicitly is:

$$\Gamma^{ij}(\mathbf{P}, \mathbf{p}_2, \mathbf{p}_1, \mathbf{p}_2) = i V(\mathbf{P}, \mathbf{p}_2, \mathbf{p}_1, \mathbf{p}_2) \delta_{ij}$$

$$+ i \int d^4 q \left( V(\mathbf{P}, \mathbf{p}_2, \mathbf{p}_1, \mathbf{p}_2) G_i^{(1)}(q) G_j^{(1)}(p-q) \Gamma_{ij}^{(1)}(q, \mathbf{p}_2, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_2, \mathbf{p}_2, \mathbf{p}_2) \right)$$

(2.31)

Eqn. (2.30') reads:

$$M_{ik}(\mathbf{p}) = \int d^4 q G_i^{(1)}(q) \left[ \Delta^{jk}(p_1, \mathbf{p}_2) - \Delta^{kj}(p_1, \mathbf{p}_2) \right]$$

(2.30'')

Eqns. (2.31'), (2.30'') together with eqn. (2.8) define the set of SCLA equations for the superfluid Fermi system.
CHAPTER III

PROPERTIES OF THE MATRIX PROPAGATOR, MATRIX
SELF ENERGY OPERATOR AND THE MATRIX VERTEX FUNCTION

In this chapter we study some symmetry and analytic properties of
the matrix propagator, the matrix self energy operator and the matrix vertex function. We first note from the defining eqn. (1.1)
\[ G_{11}(-\vec{p},-t) = -i\langle 0 | T\left(A_{\vec{p}}^+(t)A_{\vec{p}}(t)\right) | 0 \rangle \]
\[ = i\langle 0 | T\left(A_{\vec{p}}^+(t)A_{\vec{p}}(t)\right) | 0 \rangle \]
\[ = -G_{12}(-\vec{p},t) \tag{3.1} \]
Similarly we find
\[ G_{12}(-\vec{p},-t) = -G_{12}(-\vec{p},t) \tag{3.2} \]
\[ G_{21}(-\vec{p},-t) = -G_{21}(\vec{p},t) \tag{3.3} \]
If we take the Fourier transform as defined
\[ G_{ij}(\vec{p},\omega) = \int_{-\infty}^{\infty} e^{i\omega t} G_{ij}(\vec{p},t) \, dt \]
Then
\[ G_{11}(-\vec{p},-\omega) = -G_{12}(\vec{p},\omega) \tag{3.4} \]
\[ G_{12}(-\vec{p},-\omega) = -G_{12}(\vec{p},\omega) \tag{3.5} \]
\[ G_{21}(-\vec{p},-\omega) = -G_{21}(\vec{p},\omega) \tag{3.6} \]
We now look more closely at the propagators $G_{12}(p,t)$ and $G_{21}(p,t)$. These are the causal propagators. Let us write these with superscript C. Then
\[ G_{12}^{C}(\vec{p},t) = -i \left[ \Theta(t) \langle 0 | A_{\vec{p}}(t)A_{\vec{p}}(t) | 0 \rangle + \Theta(-t) \langle 0 | A_{\vec{p}}(t)A_{\vec{p}}(t) | 0 \rangle \right] \]
\[ = -i \left[ \Theta(t) G_{12}^{+}(\vec{p},t) + \Theta(-t) G_{12}^{-}(\vec{p},t) \right] \]
Similarly

\[ G_{12}^c(\vec{p}, t) = -i \left[ \Theta(t) G_{21}^+(\vec{p}, t) + \Theta(-t) G_{21}^-(\vec{p}, t) \right] \]

\[ G_{12}^+(\vec{p}, t) = <0| a_{\vec{p}}^+(t) a_{\vec{p}}(0)|0> \]

\[ G_{12}^-(\vec{p}, t) = <0| a_{\vec{p}}^-(t) a_{\vec{p}}^+(0)|0> \]

Now consider

\[ G_{11}^+(\vec{p}, t) = <0| a_{\vec{p}}^+(t) a_{\vec{p}}(0)|0> = G_{21}^+(\vec{p}, -t) \]

\[ G_{12}^-(\vec{p}, t) = <0| a_{\vec{p}}^-(t) a_{\vec{p}}^+(0)|0> = G_{21}^-(\vec{p}, -t) \]

\[ G_{12}^c(\vec{p}, t) = -i \left[ \Theta(t) G_{21}^+(\vec{p}, t) + \Theta(-t) G_{21}^-(\vec{p}, t) \right] \]

\[ G_{12}^c(\vec{p}, -t) = -i \left[ \Theta(-t) G_{21}^+(\vec{p}, t) + \Theta(t) G_{21}^-(\vec{p}, t) \right] \]

\[ = G_{21}^c(\vec{p}, t) \]

\[ \text{or} \quad G_{12}^c(\vec{p}, t) = G_{21}^c(\vec{p}, -t) \quad (3.7) \]

where \( G_{21}^c(\vec{p}, t) \) is the anti-causal propagator. Thus we see that \( G_{12}^c \neq G_{21}^c \) and the matrix \( G_{ij} \) is not hermitian. In the exceptional case however \( G_{12} \) can equal \( G_{21}^c \). We shall look to this case later.

We next derive the spectral representation for the matrix propagator \( G_{ij} \) which will help us in obtaining the analytic properties of \( G_{ij} \). It is convenient to introduce a new notation due to Nambu.

We define a two component spinner field operator

\[ b_{\vec{p}} = \begin{pmatrix} a_{\vec{p}}^+ \\ a_{\vec{p}}^+ \end{pmatrix} \quad (3.8) \]
The hermitian conjugate 
\[ b^\dagger_P = (A^\dagger_P, A_{-P}) \]  
(3.9)

with 
\[ \{ b_P, b_P^\dagger \} = \delta_{P,0}, \quad \{ b_P^\dagger b_P^\dagger, b_P^\dagger b_P^\dagger \} = 0 \]  
(3.10)

Then the matrix propagator can be written as
\[ G_{ij}(\vec{P}, t) = -i \langle 0 | T \left( b_P^\dagger (t) b_P^\dagger (0) \right) | 0 \rangle \]  
(3.11)

This is the causal propagator. We write it as:
\[ G_{ij}(\vec{P}, t) = -i \left[ \theta(t) G^+_{ij}(\vec{P}, t) + \theta(-t) G^-_{ij}(\vec{P}, t) \right] \]

where
\[ G^+_{ij}(\vec{P}, t) = \langle 0 | b_P^\dagger (t) b_P^\dagger (0) | 0 \rangle \]
\[ G^-_{ij}(\vec{P}, t) = \langle 0 | b_P^\dagger (0) b_P^\dagger (0) | 0 \rangle \]

Consider
\[ G^+_{ij}(\vec{P}, t) = \langle 0 | b_P^\dagger (t) b_P^\dagger (0) | 0 \rangle \]
\[ = \langle 0 | e^{i H t} b_P^\dagger (0) e^{-i H t} b_P^\dagger (0) | 0 \rangle \]

We now introduce a complete set of states \( |n\rangle \) that are eigenstates of the Hamiltonian, between \( b^\dagger \) and \( b^\dagger t \).
\[ G_{ij}(\vec{P}, t) = \sum_n \langle 0 | e^{i H t} b_P^\dagger (0) | n \rangle \langle n | e^{-i H t} b_P^\dagger (0) | 0 \rangle \]
\[ = \sum_n \langle 0 | b_P^\dagger (0) | n \rangle \langle n | b_P^\dagger (0) | 0 \rangle e^{-i (E_n - E_0) t} \]  
(3.13)

Here \( E_0 \) is the ground state energy \( |0\rangle = E_0 |0\rangle \) and \( E_n \) is the energy of the state \( |n\rangle \), \( H |n\rangle = E_n |n\rangle \)
Let $E_n - \varepsilon_0 = \overline{E}_n$, then define

$$
P^{+ \to \to}_{ij} (\vec{r}, \vec{e}) = \sum_n <0 | b^{+ \to \to}_{\vec{r}} | n > < m | b^{+ \to \to}_{\vec{r}} | 0 > \delta (E - \overline{E}_n) \quad (3.14)$$

So we can write

$$
G_{ij}^+ (\vec{r}, t) = \int_{-\infty}^{\infty} P^{+ \to \to}_{ij} (\vec{r}, E) e^{iE \cdot t} dE \quad (3.15)
$$

We treat $G_{ij}^- (\vec{r}, t)$ similarly

$$
P_{ij}^- (\vec{r}, E) = \sum_n <0 | b^{- \to \to}_{\vec{r}} | n > < m | b^{- \to \to}_{\vec{r}} | 0 > \delta (E + \overline{E}_n) \quad (3.16)$$

where

$$
\overline{E}_n = E_n - \varepsilon_0
$$

then

$$
G_{ij}^- (\vec{r}, t) = \int_{-\infty}^{\infty} P_{ij}^-(\vec{r}, E) e^{-iE \cdot t} dE \quad (3.17)
$$

Now

$$
G_{ij} (\vec{r}, \omega) = \int_{-\infty}^{\infty} G_{ij} (\vec{r}, t) e^{i\omega t} dt
$$

$$
= -i \left[ \int_{-\infty}^{\infty} dE P^{+ \to \to}_{ij} (\vec{r}, E) \int_{-\infty}^{\infty} dt e^{i\omega t - iEt} - \int_{-\infty}^{\infty} dE P_{ij}^- (\vec{r}, E) \int_{-\infty}^{\infty} dt e^{i\omega t - iEt} \right]
$$

$$
= -i \left[ \int_{-\infty}^{\infty} dE \frac{P^{+ \to \to}_{ij} (\vec{r}, E)}{\omega - E + i\eta} + \int_{-\infty}^{\infty} dE \frac{P_{ij}^- (\vec{r}, E)}{\omega - E - i\eta} \right]
$$

$$
= \int_{-\infty}^{\infty} dE \left[ \frac{P^{+ \to \to}_{ij} (\vec{r}, E)}{\omega - E + i\eta} + \frac{P_{ij}^- (\vec{r}, E)}{\omega - E - i\eta} \right] \quad (3.18)
$$
We can cast eqn. (3.18) in another form: consider the definition of 
\( \rho_{ij}^+(p,E) \).

\[
\rho_{ij}^+(p,E) = \sum_n \langle 0 | b_p^+ | n \rangle \langle n | b_p^- | 0 \rangle \delta (E - \varepsilon_n)
\]

In this equation \( \varepsilon_n = E_n - E_0 \). The state \( n \rangle \) contains one extra 'particle' specified by \( b_p^+ \). Let \( E_n^0 \) be the lowest state specified by \( n \), i.e., \( E_n^0 \) is the smallest value of the set \( E_n \). Then write

\[
\varepsilon_n = E_n - E_0 = E_n - E_n^0 + E_n^0 - E_0 = \omega_n + \mu
\]

where \( \mu \) is the chemical potential. Then it follows that

\( \rho_{ij}^+(p,E) = \infty \) if \( E < \mu \)

We can treat \( \rho_{ij}^-(p,E) \) analogously and get \( \rho_{ij}^-(p,E) = 0 \) if \( E > \mu \)

So we can define one function \( \rho_{ij}^1(p,E) \) such that

\[
\rho_{ij}^1(p,E) = \theta(E - \mu) \rho_{ij}^+ (p,E) + \theta(\mu - E) \rho_{ij}^- (p,E)
\]

Then

\[
G_{ij}(p,\omega) = \int_{-\infty}^{\infty} dE \rho_{ij}^1(p,E) \left[ \frac{\theta(E - \mu)}{\omega - E + i\eta} + \frac{\theta(\mu - E)}{\omega - E - i\eta} \right]
\]

This expression can be put in a slightly different form which is sometimes desirable. If we measure all energies relative to \( \mu \), we can define another variable \( \omega' = \omega - \mu \). Then

\[
G_{ij}(p,\omega' + \mu) = \int_{-\infty}^{\infty} dE \rho_{ij}^1(p,E) \left[ \frac{\theta(E - \mu)}{\omega' - (E - \mu) + i\eta} + \frac{\theta(\mu - E)}{\omega' - (E - \mu) - i\eta} \right]
\]

The writing \( \omega \) for \( \omega' \) and \( E' = E - \mu \) and again writing \( E \) for \( E' \)

\[
G_{ij}(p,\omega + \mu) = \int_{-\infty}^{\infty} dE \rho_{ij}^1(p,E) \left[ \frac{\theta(E)}{\omega - E + i\eta} + \frac{\theta(\mu - E)}{\omega - E - i\eta} \right]
\]
Eqn. (3.20) or its equivalent (3.21) above is the spectral representation of the matrix propagator and contains a wealth of information about the single particle excitation and the analytic properties of $G_{ij}$ as a function of $p_0$. First we consider some properties of $P_{ij}(\vec{p},E)$.

Consider

$$P_{11}^+(\vec{p},E) = \sum_n \langle 0|a_{\vec{p}n}^+\langle n|a_{\vec{p}10}^+\rangle \delta(E - \tilde{E}_n)$$

$$P_{22}(\vec{p},E) = \sum_m \langle 0|a_{\vec{p}m}^+\langle m|a_{\vec{p}10}^+\rangle \delta(E + \tilde{E}_n)$$

$$\therefore P_{11}^+(\vec{p},E) = P_{22}(\vec{p},-E)$$

Similarly

$$P_{11}^-(\vec{p},E) = P_{22}(-\vec{p},-E)$$

$$\therefore P_{11}^+(\vec{p},E) = P_{11}^-(\vec{p},E)$$

(3.22)

$$P_{12}^+(\vec{p},E) = \sum_n \langle 0|a_{\vec{p}n}^+\langle n|a_{\vec{p}10}^+\rangle \delta(E - \tilde{E}_n)$$

$$P_{21}^+(\vec{p},E) = \sum_n \langle 0|a_{\vec{p}n}^+\langle n|a_{\vec{p}10}^+\rangle \delta(E - \tilde{E}_n)$$

$$\therefore P_{12}^+(\vec{p},E) = P_{21}^+(\vec{p},E)$$

Similarly

$$P_{12}^-(\vec{p},E) = P_{21}^-(\vec{p},E)$$

(3.23)

Matrix $P_{ij}(\vec{p},E)$ is hermitian. Thus although $G_{ij}$ is not hermitian, $P_{ij}$ is. Next we prove a sum rule for $P_{ij}$. Consider the anticommutation $\{ b_{\vec{p}}(t) b^T_{\vec{p}'}(0) \}$. Take the vacuum expectation value of the anticommutation

$$\langle 0| \{ b_{\vec{p}}(t) b^T_{\vec{p}'}(0) \}|0\rangle$$

$$= G_{ij}^+(\vec{p},t) + G_{ij}^-(\vec{p},t)$$

$$= \int_{-\infty}^{\infty} \left[ P_{ij}^+(\vec{p},E) + P_{ij}^-(\vec{p},E) \right] e^{i\omega t} dE$$
but for \( t = 0 \),
\[
\{ \hat{b}^\dagger_{\vec{p}}, \hat{b}_{\vec{p}} \} = \delta_{ij} \, \delta_{\vec{p} \vec{p}'}
\]

\[
= \int_{-\infty}^{\infty} \tilde{p}_{ij}(\vec{p}, E)\, dE = \delta_{ij}
\]

(3.24)

This is the desired sum rule.

Eqn. (3.20) or its equivalent form (3.21) contains valuable information about the analytic properties of \( G_{ij}(\vec{p}, \omega) \) which we shall now extract. First note that eqn. (2.20) can be written as:
\[
G_{ij}(\vec{p}, \omega) = \int_{-\infty}^{\infty} \frac{\tilde{p}_{ij}(\vec{p}, E)}{\omega - E} \, dE - i\pi \, \tilde{p}_{ij}(\vec{p}, \omega) \sigma(\omega - \nu)
\]

(3.25)

Here we have used the identity
\[
\frac{1}{\omega + i\eta} = \frac{\rho}{\pi} - i\pi \delta(\omega)
\]

Since the matrix \( \tilde{p}_{ij} \) is hermitian, the first term on the r.h.s. of (2.25) is hermitian and the second term is anti-hermitian. Let \( G^h_{ij} \) and \( G^a_{ij} \) be the hermitian and anti-hermitian parts of \( G_{ij} \) so that
\[
G_{ij} = G^h_{ij} + i \, G^a_{ij}
\]

Then eqn. (2.25) implies:
\[
G^h_{ij}(\vec{p}, \omega) = \int_{-\infty}^{\infty} \frac{G^h_{ij}(\vec{p}, \omega') \sigma(\omega' - \nu)}{\omega - \omega'} \, d\omega'
\]

(3.26)

If the function \( \sigma(\omega - \nu) \) were absent from the integrand the relation (2.26) would be the matrix Hilbert transform relation for \( G^h_{ij} \) and \( G^a_{ij} \). Thus we see that \( G^h_{ij} \) is not an analytic function of \( \omega \). However define a matrix function
\[
\hat{G}(\vec{p}, \omega) = \int_{-\infty}^{\infty} \frac{\tilde{f}(\vec{p}, \omega)}{\omega - \omega'} \, d\omega
\]

(3.27)

where we denote the matrix now and later by a 'cap' rather than the subscripts \( i, j \). \( \hat{G}(\vec{p}, \omega) \) is analytic in the entire complex z plane.
except on the real axis. We now assume the \( \hat{p}(\tau, \omega) \) vanishes in a region \(-\Delta \leq \omega \leq \Delta\). In that case we have two branch cuts — one from \(-\infty\) to \(-\Delta\) and the other from \(\Delta\) to \(\infty\). This assumption will modify the analytic properties of the matrix self energy operator as we shall see. Now consider

\[
\hat{G}_n(\bar{p}, \omega + i\eta) = \int_{-\infty}^{\infty} \frac{\hat{p}(\bar{p}, E)}{\omega - E - i\eta} dE
\]

\[
= \Pi \int_{-\infty}^{\infty} \frac{\hat{p}(\bar{p}, E)}{\omega - E} - i\pi \hat{p}(\bar{p}, \omega) \tag{3.28}
\]

We call this \( \hat{p}_n \). \( \hat{G}^r(p, \omega) \) or the retarded propagator. It is the boundary value of the analytic propagator \( \hat{G}(\bar{p}, z) \) from the above real axis, and is analytic in the upper half plane. If we write

\[
\hat{G}^r = \hat{G}_1^r + i \hat{G}_2^r
\]

then

\[
\hat{p}(\bar{p}, \omega) = -\frac{i}{\Pi} \hat{G}_2^r(\bar{p}, \omega) \tag{3.29}
\]

Similarly we define the advanced propagator

\[
\hat{G}^a(\bar{p}, \omega) = \hat{G}(\bar{p}, \omega - i\eta) = \int_{-\infty}^{\infty} \frac{\hat{p}(\bar{p}, E)}{\omega - E - i\eta} dE \tag{3.30}
\]

Then

\[
\hat{G}(\bar{p}, \omega) = \Theta(\omega - \bar{p}) \hat{G}^r(p, \omega) + \Theta(\bar{p} - \omega) \hat{G}^a(p, \omega) \tag{3.31}
\]

We now study the analytic propagator \( \hat{G}(\bar{p}, z) \) further. First we investigate if \( \hat{G}(\bar{p}, z) \) has complex zeros. Note

\[
\hat{G}(\bar{p}, z) = \int_{-\infty}^{\infty} \frac{\hat{p}(\bar{p}, \omega)}{z - \omega} d\omega
\]

\[
\hat{G}^+(\bar{p}, z) = \int_{-\infty}^{\infty} \frac{\hat{p}^+(\bar{p}, \omega)}{z - \omega} d\omega = \int_{-\infty}^{\infty} \frac{\hat{p}(\bar{p}, \omega)}{z - \omega} \quad \text{since} \quad \hat{p} = \hat{p}^+
\]

\[
\hat{G}^+\tau(\bar{p}, z) = \hat{G}(\bar{p}, z^*) \tag{3.32}
\]
Let \( z = x + i y \)

\[
\hat{G}(\vec{p}, z) = \hat{G}(\vec{p}, x + iy) = \int_{-\omega}^{\omega} \hat{p}(\vec{p}, \omega) \left\{ \frac{1}{x+i\omega+i\gamma} - \frac{1}{x-i\omega+i\gamma} \right\} d\omega
\]

\[= -2i\gamma \int_{-\omega}^{\omega} \frac{\hat{p}(\vec{p}, \omega)}{(x-\omega)^2 + \gamma^2} d\omega \]

but

\[
\hat{G}(\vec{p}, z) - \hat{G}^+(\vec{p}, z) = 2i \hat{G}^+(\vec{p}, z)
\]

\[
\therefore \quad \hat{G}_z(\vec{p}, z) = -\gamma \int_{-\omega}^{\omega} \frac{\hat{p}(\vec{p}, \omega)}{(x-\omega)^2 + \gamma^2} d\omega \quad (3.33)
\]

\[
\therefore \hat{G}_z \neq 0 \text{ if } y \neq 0 \text{ because the integral is non-zero. } \hat{G}(\vec{p}, z)
\]

has no complex zeros. Next we ask does \( \hat{G}(\vec{p}, z) \) have real zeros. Because we assume that \( \hat{p}(\vec{p}, \omega) \) vanishes in a region \( -\Delta < \omega < \Delta \), \( \hat{G}(\vec{p}, z) \) has the forms

\[
\hat{G}(\vec{p}, z) = \int_{-\omega}^{\omega} \frac{\hat{p}(\vec{p}, \omega)}{x-\omega} d\omega + \int_{0}^{\omega} \frac{\hat{p}(\vec{p}, \omega)}{x-\omega} d\omega 
\]

\[
(3.34)
\]

Let \( z = x \) a real variable \( \exists -\Delta < x < \Delta \). Consider the matrix

\[
\int_{\Delta}^{\omega} \frac{\hat{p}(\vec{p}, \omega)}{x-\omega} = \hat{A} \quad \text{say}
\]

Minimum value of \( \omega \) is \( \Delta \) and \( x < \Delta \). Further \( \hat{p} \) is positive definite. 

\( \therefore \hat{A} \) is -ve definite. Similarly consider

\[
\hat{B} = \int_{-\omega}^{\omega} \frac{\hat{p}(\vec{p}, \omega)}{x-\omega} d\omega
\]

\( \omega \) is always -ve and its maximum value is -\( \Delta \). Minimum value of \( x \) is > -\( \Delta \). Therefore \( \hat{B} \) is +ve definite. So in the region \( -\Delta < x < \Delta \)

\( \hat{G} \) can vanish on the real axis.
Now consider the matrix \( \frac{d}{dx} \hat{G}(\vec{r}, x) \)

\[
\frac{d}{dx} \hat{G}(\vec{r}, x) = -\int_{-\infty}^{\infty} \frac{\hat{P}(\omega) d\omega}{(\omega - \vec{r})} - \int_{-\infty}^{\infty} \frac{\hat{P}(\omega) d\omega}{(\omega - \vec{r}^*)}
\]

\( \therefore \frac{d}{dx} \hat{G}(\vec{r}, x) \) is negative definite. Both \( \hat{G}(\vec{r}, x) \) and \( \frac{d}{dx} \hat{G}(\vec{r}, x) \) are continuous in the interval \(-\infty < x < \infty\). Therefore, we conclude that at best \( \hat{G} \) can have one zero on the real axis. Let \( \omega = \omega_0 \) be that point.

This result and the previous one that \( \hat{G}(\vec{r}, x) \) can have no complex zero will now be used to obtain the analytic properties and dispersion relations for the matrix \( M \). We will then obtain a spectral representation for \( M \).

First, we derive a relation between the antihermitian part \( G_2 \) of \( G \) and the antihermitian part \( M_2 \) of \( M \). We omit the arguments and write the matrix equation:

\[
\hat{G} = \hat{G}_1 + i \hat{G}_2 \\
\hat{M} = \hat{M}_1 + i \hat{M}_2 \\
\hat{G}^+ = \hat{G}_1 - i \hat{G}_2 \\
\hat{M}^+ = \hat{M}_1 - i \hat{M}_2
\]

\( \therefore \hat{G} - \hat{G}^+ = 2i \hat{G}_2 \\
\hat{M} - \hat{M}^+ = 2i \hat{M}_2 \quad (3.35) \\
\hat{M} - \hat{M}^+ = 2i \hat{M}_2 \quad (3.36)
\]

Now

\[
\hat{G}_1 = [(\omega - \varepsilon)1 - \hat{M}]^{-1} \\
\hat{G}_1^+ = [(\omega - \varepsilon)1 - \hat{M}^+]^{-1}
\]

\( \therefore \hat{G} - \hat{G}^+ = 2i \hat{G}_2 = [(\omega - \varepsilon)1 - \hat{M}]^{-1} - [(\omega - \varepsilon)1 - \hat{M}^+]^{-1} = [(\omega - \varepsilon)1 - \hat{M}_1 - i \hat{M}_2]^{-1} - [(\omega - \varepsilon)1 - \hat{M}_1 + i \hat{M}_2]^{-1}
\]
Now for any two non-singular matrices \( X, Y \), we have
\[
X^{-1} - Y^{-1} = X^{-1}(Y - X)Y^{-1}
\]

\[
\therefore 2i \hat{G}_2 = ((\omega - \epsilon)1 - \hat{M})(2i \hat{M}_1)((\omega - \epsilon)1 - \hat{M}^\dagger)
\]

\[
= \hat{G}_0 (2i \hat{M}_1) \hat{G}_0^\dagger
\]

\[
\therefore \hat{G}_2 = \hat{G}_0 \hat{M}_1 \hat{G}_0^\dagger
\]

Thus if \( G_2 = 0; M_2 = 0 \) and \( M \) is hermitian.

We are now prepared to study the analytic properties of \( M \). First note that \( \hat{G}_0^{-1} = \hat{G}_0^{-1} - \hat{M} \) and \( G^{-1} \) is singular if \( G \) has a zero.

Since \( G_0^{-1} \) is not singular there, \( M \) is singular if \( G \) has a zero.

Further in the region \( -\Delta < \omega < \Delta \) \( M \) is hermitian as shown by eqn. (3.38). Since in the region \( G \) can have a zero at \( \omega = \omega_0 \) we conclude that \( M \) can have a pole there. We call the residue matrix at the pole \( \hat{A} \) so \( \hat{M} \) has the form \( \frac{\hat{A}}{\omega - \omega_0} \) at the pole. Now consider \( \hat{M}(\hat{p}, z) \) defined by:
\[
\hat{G}_0^{-1}(\hat{p}, z) = \hat{G}_0^{-1}(\hat{p}, z) - \hat{M}(\hat{p}, z)
\]

Since \( \hat{G}_0(\hat{p}, z) \) has no complex zeros, \( M(\hat{p}, z) \) has no complex poles. Therefore \( M(\hat{p}, z) \) is analytic in the entire complex \( z \)-plane except for cuts along the real axis \( -\infty \) to \(-\Delta \) and \( \Delta \) to \( \infty \) and a possible pole at \( \omega = \omega_0 \) on the real axis.

We prove one more property of the matrix \( M(p, z) \) before deriving the dispersion relations. We have from eqn. (3.32)
\[
\hat{G}_0^+(\hat{p}, z) = \hat{G}_0(\hat{p}, z^*) \quad \text{and} \quad \hat{G}_0^+(\hat{p}, z) = \hat{G}_0(\hat{p}, z^*)
\]

\[
\therefore [\hat{G}_0^{-1}(\hat{p}, z)]^+ = [\hat{G}_0^{-1}(\hat{p}, z)]^+ - \hat{M}^+(\hat{p}, z)
\]

or
\[
\hat{G}_0^{-1}(\hat{p}, z^*) = \hat{G}_0(\hat{p}, z^*) - \hat{M}^+(\hat{p}, z)
\]
but
\[ \hat{G}_z^{-1}(\bar{p},z) = \hat{G}_z^{-1}(p,z) - M(p,z) \]

\[ \therefore \hat{M}_z(\bar{p},z) = \hat{M}(\bar{p},z) \tag{3.39} \]

In order to derive dispersion relations for \( M \) we need to know the asymptotic properties of \( M(p,z) \). Using this knowledge and the previously derived analytic properties of \( M(p,z) \) we now derive dispersion relations for \( M(p,z) \).

Assume
\[ \lim_{|\xi| \to \infty} \hat{M}(\bar{p},z) = \hat{M}(\bar{p}) \]

We apply Cauchy's theorem to the function: \( M(p,z) - M(p) \), obtaining:
\[ \hat{M}(\bar{p},z) - \hat{M}(p) = \frac{1}{i\pi \gamma} \int_{C_1} \left[ \frac{\hat{M}(\bar{p},\xi) - \hat{M}(p)}{\xi - \gamma} \right] d\xi \]
\[ = \frac{1}{i\pi \gamma} \int_{C_1} \left[ \frac{\hat{M}(\bar{p},\xi) - \hat{M}(p)}{\xi - \gamma} \right] d\xi - \frac{1}{i\pi \gamma} \int_{C_2} \left[ \frac{\hat{M}(\bar{p},\xi) - \hat{M}(p)}{\xi - \gamma} \right] d\xi \tag{3.40} \]

where the contours \( C_1 \) and \( C_2 \) are as shown. We let the radius of the contour \( C_2 \to 0 \) and that the semicircle parts of \( C_1 \to \infty \). The second term of (3.40) gives \( \frac{\hat{A}}{\xi \to \infty} \). The first term gives the contribution from the branch cuts in terms of the discontinuity across the branch
cuts. Thus

\[
\hat{M}(\hat{p}, z) - \hat{M}(\hat{p}) = \frac{\hat{A}}{z - \omega_0} + \frac{1}{2\pi i} \int_{-\Delta}^{\Delta} \frac{\hat{M}(\hat{p}, \omega - \eta) - \hat{M}(\hat{p}, \omega + \eta)}{\omega - z} d\omega \\
+ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\hat{M}_2(\hat{p}, \omega) - \hat{M}(\hat{p}, \omega)}{\omega - z} d\omega
\]  

(3.41)

Now according to eqn. (3.39) \( \hat{M}(\hat{p}, \omega - \eta) = \hat{M}(\hat{p}, \omega + \eta) \) and

\[
\hat{M} - \hat{M}^+ = 2i \hat{M}_2 \\
\therefore \hat{M}(\hat{p}, z) - \hat{M}(\hat{p}) = \frac{\hat{A}}{z - \omega_0} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\hat{M}_2(\hat{p}, \omega)}{\omega - z} d\omega
\]

or remembering that \( \hat{M}_{\pm}(\hat{p}, \omega) \) for \(-\Delta < \omega < \Delta\) we write

\[
\hat{M}(\hat{p}, z) = \hat{M}(\hat{p}) + \frac{\hat{A}}{z - \omega_0} - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\hat{M}_2(\hat{p}, \omega)}{\omega - z} d\omega
\]  

(3.42)

This is the dispersion relation for \( M(p, z) \). Define two matrix functions

\[
\hat{M}^*(\hat{p}, \omega) = \hat{M}(\hat{p}, \omega + \eta) , + \text{ sign for } \nu \text{ and } - \text{ sign for } a.
\]

Then

\[
\hat{M}^*(\hat{p}, \omega) = \hat{M}(\hat{p}) + \frac{\hat{A}}{\omega - \omega_0 + \eta} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\hat{M}_2(\hat{p}, \omega)}{\omega - \omega' + \eta} d\omega'
\]  

(3.43)

\[
\hat{M}^a(\hat{p}, \omega) = \hat{M}(\hat{p}) + \frac{\hat{A}}{\omega - \omega_0 - \eta} - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\hat{M}_2(\hat{p}, \omega)}{\omega - \omega' - \eta} d\omega'
\]  

(3.44)

Finally if we define

\[
\hat{M}(\hat{p}, \omega) = \theta(\omega) \hat{M}^*(\hat{p}, \omega) + \theta(-\omega) \hat{M}^a(\hat{p}, \omega)
\]  

(3.45)

we have

\[
\hat{M}(\hat{p}, \omega) = \hat{M}(\hat{p}) + \frac{\hat{A}}{\omega - \omega_0 + \eta \theta(\omega)} - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \hat{M}_2(\hat{p}, \omega') \left[ \frac{\theta(\omega')}{\omega - \omega' + \eta} + \frac{\theta(-\omega')}{\omega - \omega' - \eta} \right] d\omega'
\]  

(3.46)
This is the spectral representation for $\hat{\Pi}(\tilde{p}, \omega)$.

It remains to discuss the properties of the matrix vertex function. This is done by generalizing the corresponding equations for the normal case discussed in Chapter 1. Vertex function satisfies the eqn. (2.31') which we write in the variables: $\tilde{p}, \tilde{p}', \tilde{p}$

$$
\Gamma_{ij}(\tilde{p}, \tilde{p}', \tilde{p}) = V_i(\tilde{F}, \tilde{F}', \tilde{F})\delta_{ij} + \int d^4 q V_i(\tilde{p}, \tilde{q}, \tilde{p}) G_{ik}(q) G_{jk}(p-q) \Gamma_{kj}(\tilde{q}, \tilde{p} - \tilde{q}, \tilde{p}', \tilde{p})
$$

(3.47)

where

$$
G_{ik}(q) G_{jk}(p-q) = \begin{pmatrix}
\delta_{ii}(q) G_{ii}(p-q) & \delta_{ij}(q) G_{ij}(p-q) \\
\delta_{ij}(q) G_{ji}(p-q) & \delta_{jj}(q) G_{jj}(p-q)
\end{pmatrix}
$$

We integrate

$$
\int d^4 q G_{ik}(q) G_{jk}(p-q) = R_{ik}(\tilde{p}, \tilde{p})
$$

$$
R_{ik}(\tilde{p}, \tilde{p}) = 2\pi i \int dE \left( \frac{\delta(E - E)}{\tilde{p}_i - \tilde{p}_j - \tilde{p}} + \frac{\delta(-E)}{\tilde{p}_i - \tilde{p}_j + \tilde{p}} \right) f_{ik}(\tilde{p}, \tilde{p})
$$

(3.48)

where

$$
f_{ik}(\tilde{p}, \tilde{p}) = -\sigma(\tilde{E}) \int d\tilde{E} \tilde{p}_{ik}(\tilde{p}, \tilde{p}; \tilde{E} + \tilde{E}) R_{ik}(\tilde{p}, \tilde{p}; \tilde{E} - \tilde{E})
$$

$\tilde{p}_{ik}$ are the single particle spectral functions. No summation involved in the product. Then

$$
\Gamma_{ij}(\tilde{p}, \tilde{p}', \tilde{p}) = V_i(\tilde{F}, \tilde{F}', \tilde{F})\delta_{ij} - \int d^4 q V_i(\tilde{p}, \tilde{q}, \tilde{p}) R_{ik}(\tilde{q}, \tilde{p}) \Gamma_{kj}(\tilde{q}, \tilde{p} - \tilde{q}, \tilde{p}', \tilde{p})
$$

This defines $\hat{\Gamma}(\tilde{p}, p)$ the causal function.
We define

$$\hat{R}(q, P, \xi) = \int \frac{f(R, E)}{x-E} dE$$  \hspace{1cm} (3.49)$$

\(\hat{R}(\xi)\) is analytic in the complex \(z\)-plane except on the real axis. Define

$$\hat{R}^{r,\alpha}(q, P, p_0) = \hat{R}(q, P, p_0 + i\eta)$$  \hspace{1cm} (3.50)$$

+ sign for \(r\) and - sign for \(\alpha\). Then we can define \(\hat{\Gamma}^{r,\alpha}\) and \(\Gamma(\xi)\) using these \(R\)'s in the kernel of the integral equation symbolically

$$\hat{\Gamma} = \nabla - \nabla \hat{R} \hat{\Gamma}$$  \hspace{1cm} (3.51)$$

$$\hat{\Gamma}^{r,\alpha} = \nabla - \nabla \hat{R}^{r,\alpha} \hat{\Gamma}^{r,\alpha}$$  \hspace{1cm} (3.52)$$

$$\hat{\Gamma}(\xi) = \nabla - \nabla \hat{R}(\xi) \hat{\Gamma}(\xi)$$  \hspace{1cm} (3.53)$$

Thus we know the equations satisfied by the various matrix vertex functions: analytic, retarded, advanced and causal. These equations along with the properties derived will be used in the next chapter to study the behavior of the vertex function as a function of energy near the fermi surface.
CHAPTER IV
TREATMENT OF THE SCLA EQUATIONS

The set of self consistent ladder approximation equations derived in Chapter 2 will be studied in this chapter. The purpose of this investigation is to simplify the equations as much as possible so a numerical solution can be attempted. We shall first show that the superfluid vertex function and the self energy function are slowly varying functions near the Fermi surface. Then starting with the assumption that the gap is small compared to Fermi energy we shall break up the integration regions into two parts; one in which the effect of superfluidity is negligible and one in which it is not, the latter one being a small region. We shall carefully define these regions and then treat the diagonal and off-diagonal parts of the various functions separately. It will be found that the diagonal parts of the self energy can be calculated with the normal state functions if the regions are chosen properly. All the calculations are done with a separable potential. This allows the solution of the integral equation for the vertex function. This in turn simplifies the expression for the self energy function. No numerical work is done. However we present the final set of SCLA equations in a simplified form in which the approximations made are explicitly stated. Finally, we apply these equations to the superfluid Fermi liquid. We define the input parameters analogous to the Landau theory of the normal Fermi liquid and obtain
the gap equation. Damping effects neglected here will be studied in
the next chapter.

Consider the eqn. (2.31') for the vertex function:
\[
\Gamma_{(p_{1},p_{2},p_{1}',p_{2}')}(p_{1},p_{2},p_{1}',p_{2}') = i \mathcal{V}(\vec{p},\vec{p}_{2},\vec{p}_{1},\vec{p}_{1}') \delta_{ij} + i \int \mathcal{D}q \mathcal{V}(\vec{q},\vec{p}_{2},\vec{p}_{1},\vec{p}_{1}') \times
\]
\[
\mathcal{G}_{(q)}^{\imath}(q_{1}) \mathcal{G}_{(q)}^{\imath}(q_{2}) \Gamma_{(p_{1},p_{2},p_{1}',p_{2}')}^{\imath}(q_{1},q_{2},\vec{p}_{1},\vec{p}_{2},\vec{p}_{1}',\vec{p}_{2}')
\]

This is a matrix integral equation. We first note that we can do the
\[\mathcal{Z}\] integration. Let us write
\[
\int \mathcal{D}q \mathcal{G}_{(q)}^{\imath}(q_{1}) \mathcal{G}_{(q)}^{\imath}(q_{2}) = \mathcal{I}^{\imath}(q_{1},q_{2})
\]

From now on we drop all the subscripts 1 on the functions \(G\) since
2-particle propagators will not appear. We also lower the matrix in­
dices. We write the above integral equation as:
\[
\Gamma_{(p_{1},p_{2},p_{1}',p_{2}')}(p_{1},p_{2},p_{1}',p_{2}') = i \mathcal{V}(\vec{p},\vec{p}_{2},\vec{p}_{1},\vec{p}_{1}') \delta_{ij} + i \int \mathcal{D}q \mathcal{V}(\vec{q},\vec{p}_{2},\vec{p}_{1},\vec{p}_{1}') \times
\]
\[
\mathcal{I}^{\imath}(q_{1},q_{2}) \mathcal{F}_{(q_{1},q_{2})}^{\imath}(q_{1},q_{2},\vec{p}_{1},\vec{p}_{2},\vec{p}_{1}',\vec{p}_{2}')
\]

We solve this equation by introducing a separable potential first in­
troduced by Yamaguchi (20). See also W. Wild (21). We choose
\[
\mathcal{V}(\vec{p},\vec{p}_{2},\vec{p}_{1},\vec{p}_{1}') = \lambda \mathcal{V}(\frac{\vec{p}_{1} - \vec{p}}{2}) \mathcal{V}(\frac{\vec{p}_{1}' - \vec{p}}{2})
\]

Then eq. (4.2) reads if we suppress the matrix indices:
\[
\Gamma(p_{1},p_{2},p_{1}',p_{2}') = i \lambda \mathcal{V}(\frac{\vec{p}_{1} - \vec{p}}{2}) \mathcal{V}(\frac{\vec{p}_{1}' - \vec{p}}{2}) +
\]
\[
i \lambda \mathcal{V}(\frac{\vec{p}_{1} - \vec{p}}{2}) \int \mathcal{D}q \mathcal{V}(\frac{\vec{p}_{1}' - \vec{p}}{2}) \mathcal{I}^{\imath}(q_{1},q_{2}) \mathcal{F}_{(q_{1},q_{2})}^{\imath}(q_{1},q_{2},\vec{p}_{1},\vec{p}_{2},\vec{p}_{1}',\vec{p}_{2}')
\]

Matrix product \(\mathcal{I}^{\imath}\mathcal{F}\) is understood. We see that \(\vec{p},\vec{p}',\vec{p}_{1},\vec{p}_{1}'\) are the vari­
ables involved in this equation. We symbolically write the matrix
integral equation, representing the variables $x', x''$, by the symbols $x', x''$.

\[ \Gamma(x', x'') = i \lambda \psi(x') \psi(x'') + i \lambda \psi(x') \int dx'' \psi(x'') \Gamma(x', x'') \]  

(4.4)

Let

\[ A(x') = \int dx'' \psi(x'') \Gamma(x', x'') \]  

(4.5)

then

\[ \Gamma(x', x'') = i \lambda \psi(x') \psi(x'') + i \lambda \psi(x') A(x'') \]  

(4.6)

Putting (4.6) in (4.5):

\[ A(x') = \int dx'' \psi(x'') \Gamma(x', x'') \left[ i \lambda \psi(x') \psi(x'') + i \lambda \psi(x') A(x'') \right] \]

or

\[ \left[ 1 - i \lambda \int dx'' \psi(x'') \Gamma(x', x'') \right] A(x') = i \lambda \psi(x') \int dx'' \psi(x'') \Gamma(x', x'') \]

\[ \therefore A(x') = \left[ 1 - i \lambda \int dx'' \psi(x'') \Gamma(x', x'') \right]^{-1} i \lambda \psi(x') \int dx'' \psi(x'') \Gamma(x', x'') \]

Plugging this in eqn. (4.6) we get

\[ \Gamma(x', x'') = i \lambda \psi(x') \psi(x'') - \lambda^2 \psi(x') \psi(x'') \left[ 1 - i \lambda \int dx'' \psi(x'') \Gamma(x', x'') \right]^{-1} \int dx'' \psi(x'') \Gamma(x', x'') \]

\[ \therefore \left[ 1 - i \lambda \int dx'' \psi(x'') \Gamma(x', x'') \right] \Gamma(x', x'') = i \lambda \psi(x') \psi(x'') + \lambda^2 \psi(x') \psi(x'') \int dx'' \psi(x'') \Gamma(x', x'') \]

\[ \therefore \left[ 1 - i \lambda \int dx'' \psi(x'') \Gamma(x', x'') \right]^{-1} \int dx'' \psi(x'') \Gamma(x', x'') \]

(4.7)

Now comparing eqn. (4.4) with (4.2') we write the solution of eqn. (4.2') as:

\[ \Gamma_{ij} (\vec{r}_i, \vec{r}_j; \vec{r}_k, \vec{r}_l) = i \lambda \psi (\vec{r}_i, \vec{r}_j, \vec{r}_k, \vec{r}_l) \psi (\vec{r}_i, \vec{r}_j, \vec{r}_k, \vec{r}_l) \Gamma_{ij} (\vec{r}) \]  

(4.8)

where

\[ \Gamma (\vec{r}) = \left[ 1 - i \lambda \int d^3 \psi (\vec{r}, \vec{r'}) \Gamma (\vec{r}, \vec{r'}) \right]^{-1} \]  

(4.9)

\[ \Gamma_{ij} (\vec{r}) = \int d^3 \psi (\vec{r}, \vec{r'}) \frac{\partial^2}{\partial^2 \vec{r}} \left[ \frac{\partial}{\partial^2 \vec{r}} \right] \psi (\vec{r}, \vec{r'}) \Gamma_{ij} (\vec{r}, \vec{r'}) \]  

(4.10)
We see from the above expression $\eta_{ij}$ depends on the energy variable $P$ through the "reduced" matrix $\eta_{ij}(P)$. Any singularities of $\eta_{ij}$ in the energy variable are due to the singularities of $\eta_{ij}$. If the off-diagonal elements of $\eta_{ij}(P)$ are identically zero we get the normal $\Gamma$ and hence the normal $\Gamma$. In that case we have a Cooper singularity of $\eta(P)$ due to the vanishing of the denominator

$$| -i\lambda \eta(P) | \approx \lambda \omega.$$  

The important fact here is that $I(P)$ diverges logarithmically for $P = 0$ and therefore the pole occurs for an arbitrarily small attractive potential. This phenomenon in fact led us to modify the formalism and to the new vertex function $\eta_{ij}$. It is necessary therefore to examine if the Cooper singularity has vanished.

To see this we have to consider the behavior of the expression

$$f(\lambda, P) = \frac{\lambda}{\det(1 - i\lambda \eta(P))} = \frac{\lambda}{\det(1 + 2i\lambda R(P))}$$

where $\lambda = \frac{1}{2\pi c}$. 

We see that the determinant can vanish for $\lambda \omega$. If it does so for an arbitrarily small $\lambda$ the Cooper singularity exists; otherwise not.

We rewrite the expression (4.11) as

$$f(\lambda, P) = \frac{\lambda}{1 + 2\pi \lambda \text{Tr} R + 4\pi^2 \lambda^2 \det R}$$

$$= \frac{1}{\frac{1}{2} + 2\pi \lambda \text{Tr} R + 4\pi^2 \lambda \det R}$$

Thus as $\lambda \to 0$ we need consider only $\text{Tr} R$. This is because $\det R$ is expected to diverge logarithmically. It $\text{Tr} R$ diverges for some $P$ we would get the Cooper singularity. Thus the problem reduces to evaluating $\text{Tr} R$. By definition $R_{ij} = \frac{1}{2\pi c} \eta_{ij}$. 

Consider
\[ I_{ij}(\tau) = \int d^3q \, u^*(i \frac{\tau_{x} - \tau_{y}}{2}) C_{ij}(q) C_{ij}(\tau - q) \]

We can do the \( q \) integration and obtain in the manner we got in Chapter 3 (see eqn. 3.48)
\[ R_{ij}(\tau) = \frac{1}{2\pi i} I_{ij}(\tau) = \int d^3q \, u^*(i \frac{\tau_{x} - \tau_{y}}{2}) R_{ij}(q, \tau) \]  \hspace{1cm} (4.13)

where \( R_{ij}(q, \tau) \) is given by eqn. (3.48). We have
\[ R_{ij}(q, \tau) = \int d\varepsilon \left[ \frac{\theta(\varepsilon)}{p_0 - \varepsilon + i\eta} + \frac{\theta(-\varepsilon)}{p_0 - \varepsilon - i\eta} \right] R_{ij}(\varepsilon, \tau) \]  \hspace{1cm} (4.14)

where
\[ R_{ij}(\varepsilon, \tau) = -\sigma(\varepsilon) \int d^3q \, u^*(i \frac{\tau_{x} - \tau_{y}}{2}) \int d\varepsilon \, \frac{\theta(\varepsilon)}{p_0 - \varepsilon + i\eta} R_{ij}(\varepsilon + \tau, \tau) R_{ij}(\varepsilon, \tau) \]  \hspace{1cm} (4.15)

\( \tilde{R}_{ij}(\varepsilon, \tau) \) is the single particle matrix spectral function. \( \tilde{R}_{ij}(\varepsilon, \tau) \) as defined above leads to the definition of \( R_{ij}(q, \tau) \) which is analytic in the complex \( \tau \)-plane except on the real axis (see eqn. 3.49). If we use this \( R_{ij}(q, \tau) \) in eqn. (4.9) the resulting \( \tilde{R}_{ij}(q, \tau) \) is analytic in the \( \tau \)-plane except on the real axis. The evaluation of \( R_{ij}(q, \tau) \) requires the single particle spectral functions. We are interested in the case \( q = 0 \) and \( \tau \approx 0 \). In the \( q \) integration of eqn. (4.15) the region of interest is near \( q = \tau \). We use B.C.S. spectral functions in eqn. (4.15). This is justified because any deviation from the B.C.S. expression which occurs for large value of the argument \( q \) in \( \tilde{R}_{ij}(q, \tau) \) will not affect singular part of \( R_{ij}(\tau) \) for \( p \approx 0 \).

We have the B.C.S. expressions for the spectral functions:
\[ \tilde{R}_{ij}(q) = u_{ij}^2 \delta(q - \varepsilon_{F}) + v_{ij}^2 \delta(q + \varepsilon_{F}) \]  \hspace{1cm} (4.16)
\[ \tilde{R}_{ij}(q) = u_{ij}^2 \delta(q - \varepsilon_{F}) + v_{ij}^2 \delta(q + \varepsilon_{F}) \]
where
\[ u_k^2 = \frac{1}{2} \left( 1 + \frac{\xi_k}{E_k} \right), \quad V_k^2 = \frac{1}{2} \left( 1 - \frac{\xi_k}{E_k} \right), \quad E_k = \sqrt{\xi_k^2 + \sigma^2} \]

\( \sigma \) is the gap in the elementary excitation. Then a straight-forward calculation gives:
\[ M H = \left[ \frac{V_k}{\xi_k} \right]^2 \left[ \frac{1}{\xi_k^2 + \sigma^2} \right] = -\int_{\xi_k^2 + \sigma^2}^{\infty} \frac{\xi_k^2}{E_k^2} dx \]

We now evaluate these expressions using a shell-potential. Specifically we choose:
\[ u_k(q) = u^2 = \text{const.} \quad \text{for} \quad -\omega < E_k < \omega \]
\[ = 0 \quad \text{otherwise} \]

Then putting \( \xi_k^2 dq = \frac{1}{2} \xi_E^2 \frac{dE}{q} \) we get
\[ R_n(\xi) = 4\pi \alpha^2 \xi_E^2 \int_{-\omega}^{\omega} \frac{\xi_k^2}{\sqrt{\xi_k^2 + \sigma^2} (4\xi_k^2 + 4\sigma^2 - \xi^2)} dE_k \]

\[ R_{22}(\xi) = 4\pi \alpha^2 \xi_E^2 \int_{-\omega}^{\omega} \frac{-\xi_k^2}{\sqrt{\xi_k^2 + \sigma^2} (4\xi_k^2 + 4\sigma^2 - \xi^2)} dE_k \]

The first term in the integrand is zero, giving:
\[ R_n(\xi) = R_{22}(\xi) = 2\pi \alpha^2 \xi_E^2 \int_{-\omega}^{\omega} \frac{2\xi_k^2 + \sigma^2}{\sqrt{\xi_k^2 + \sigma^2} (4\xi_k^2 + 4\sigma^2 - \xi^2)} dE_k \]

The integral is evaluated by elementary method. We write the expression for \( \xi = i\gamma \) for \( \sigma \ll \omega \).

\[ R_n(i\gamma) = R_{22}(i\gamma) = \pi \alpha^2 \xi_E^2 \left[ \ln \left( \frac{\omega + \sqrt{\omega^2 + \sigma^2}}{-\omega + \sqrt{\omega^2 + \sigma^2}} \right) - \frac{\gamma^2}{\gamma \sqrt{\gamma^2 + 4\sigma^2}} \ln \left( \frac{\gamma + \sqrt{\gamma^2 + 4\sigma^2}}{-\gamma + \sqrt{-\gamma^2 + 4\sigma^2}} \right) \right] \]
We see that the expression does not blow up if $\sigma \neq 0$. If $\sigma = 0$ it gives the logarithmic divergence of the normal state theory. Thus the existence of non-zero $\sigma$ is essential in this case for otherwise we encounter the Cooper singularity.

We now turn our attention to the set of SCLA equations and see if we can introduce some approximations which will simplify the equations. First we rewrite eqn. (2.30") for the self-energy functions, using special form of $I^r$ resulting from use of separable potential

$$M_{ij}(\not p) = \int d^4q \, G_{ji}(q) \left[ 2 \Gamma_{ij}(\not p, \not q) - \Gamma_{ij}(\not q, \not p) \right]$$

If we use eqns. (4.5) and (4.9) for $\Gamma$ we get

$$\Gamma_{ij}(\not p, \not q) = \Gamma_{ij}(\not q, \not p) = i\lambda v^2(1 - \frac{\not p \cdot \not q}{\not p}) \Gamma_{ij}(\not p + \not q)$$

Inserting this in eqn. (2.30") above for $M_{ij}$ we get

$$M_{ij}(\not p) = i \int d^4q \, v^2(1 - \frac{\not p \cdot \not q}{\not p}) G_{ji}(q) \Gamma_{ij}(\not p + \not q)$$

(4.23)

In this form $M_{ij}(\not p)$ can be studied as a function of $\not p$ because the integrand is explicitly expressed in terms of $G_{ij}(q)$ and $\Gamma_{ij}(\not p \cdot \not q)$ for which expressions are available. We reproduce these for the sake of discussion:

$$G(\not p) = \left[ \not p^2 - \not p \cdot \not q - M(\not p) \right]^{-1}$$

(4.24)

$$\Gamma(\not p) = \left[ 1 - i\lambda \Gamma(\not p) \right]^{-1}$$

(4.25)

$$\Gamma(\not p) = \int d^4q \, v^2(1 - \frac{\not p \cdot \not q}{\not p}) G_{ji}(q) G_{j}(\not p + \not q)$$

(4.10)
Equations (4.23), (4.24), (4.9) and (4.10) form a closed set of equations which must be solved self-consistently. As matrix equations they have the same form as the corresponding equations in the normal state theory but now there are four times as many equations. If the off-diagonal elements of \( C, M, E, r \) are zero we get back to the set of equations for the normal state. The typical superfluid state modifications are introduced by the existence of the off-diagonal elements. The diagonal elements themselves differ from their counterparts in the normal state theory. Just how much they differ has to be found out by working on the SCLA equations. We make the basic assumption that \( M_i(\eta) \) and hence \( C_i(\eta) \) differ from the corresponding normal state functions only in a small region around the Fermi surface. By Fermi surface we henceforth mean \( |\vec{q}| = q_F \) and \( \omega = 0 \), where we measure all energies with respect to \( \mu \). Similarly \( \tau_i(\rho) \) is assumed to differ from the corresponding normal state functions in a small region around \( \rho = 0 \). We call these regions the "regions of distortion" for the corresponding functions. We assume they are of the same order of magnitude. We shall soon define these regions more specifically. It is within these regions that the off-diagonal elements have appreciable magnitude outside they are negligibly small. These assumptions are realistic and in fact are generalizations of the observed facts in nature. In B.C.S. theory the off-diagonal elements of the self energy matrix give the gap in the single-particle excitations. It is only in a region of the order of magnitude of the gap that the B.C.S. propagator differs from the normal state one. We carry over
this observation to the matrix functions and arrive at the assumptions just mentioned. Here we must remember that we are dealing with 4-vectors so we have separate regions for the energy variable and the momentum variable. In the various expressions encountered involving 4-integrations we have to separate the regions accordingly.

We define the regions of distortion as follows: we define $D_1$ as that region for which $q_0 \in (-\Delta_1, \Delta_1)$ and $|\vec{q}| \in (q_r - \Delta_1', q_r + \Delta_1')$ and assume that the functions $G_{ii}(\vec{q}, q_0)$ and $M_{ii}(\vec{q}, q_0)$ can be approximated by the normal state functions outside $D_1$. The off-diagonal parts $G_{ij}(q)$ and $M_{ij}(q)$ shall be assumed to vanish outside of $D_1$. Similarly we define the distortion region $D_2$ as that region for which $p_0 \in (-\Delta_2, \Delta_2)$ and $|\vec{p}| \in (0, \Delta_2')$ and assume that $\tau_{ii}(p)$ can be approximated by the normal state function outside $D_2$. Outside $D_2$ $\tau_{ij}$ shall be assumed to vanish. Writing these conditions explicitly:

\[
\begin{align*}
G_{ii}(\vec{q}, q_0) &= G^N(\vec{q}, q_0) \\
G_{12}(\vec{q}, q_0) &= -G^N(-\vec{q}, -q_0) \\
M_{12}(\vec{q}, q_0) &= M^N(\vec{q}, q_0) \\
M_{ii}(\vec{q}, q_0) &= -M^N(-\vec{q}, -q_0)
\end{align*}
\]  

\[
\begin{align*}
\tau_{ii}(\vec{p}, p_0) &= \tau^N(\vec{p}, p_0) \\
\tau_{12}(\vec{p}, p_0) &= \tau^N(-\vec{p}, -p_0)
\end{align*}
\]

With this separation of the regions we now write the various functions of the SCLA equations as:
\[ G_{ij}(q) = \delta G_{ij}(q) + \Delta G_{ij}(\delta q) \delta_{ij} \]
\[ M_{ij}(q) = \delta M_{ij}(q) + \Delta M_{ij}(\delta q) \delta_{ij} \]
\[ \tau_{ij}(p) = \delta \tau_{ij}(p) + \Delta \tau_{ij}(\delta p) \delta_{ij} \]
\[ \delta \Sigma_{ij}(p) = \delta \Sigma_{ij}(p) + \Delta \Sigma_{ij}(\delta p) \delta_{ij} \]

where the functions \( \delta G, \delta M, \delta \tau, \delta \Sigma \) are the values of the functions \( G, M, \tau, \Sigma \) respectively in the regions of distortion. The diagonal elements of \( G, \) etc., go over into the normal state functions \( G^N, \) etc., outside the regions whereas the off-diagonal elements rapidly fall off to zero. We assume that we have a knowledge of the normal state functions and focus our attention on the functions \( \delta G, \delta M, \delta \tau, \delta \Sigma \).

We first consider \( \delta M_{ij}(\delta p) \). We have from eqn. (4.23)
\[ \delta M_{ij}(\delta p) = \int d^3q \left( \frac{e^{-q^2/2}}{2\pi} \right) \tau_{ij}(p+q) G_{ij}(q), \delta p \in D, \]
where \( \delta p \in (-\Delta, \Delta) \quad ; \quad |\delta p| - \delta q \in (-\Delta, \Delta) \quad . \) The integration is over all values of \( q \). Even if \( q \) goes outside the region of distortion, \( p + q \) can still be within and the \( \tau_{ij}(\delta p+q) \) will be \( \delta \tau_{ij}(\delta p+q) \). We shall split off from the integral the contribution from the part in which both \( \tau_{ij} \) and \( G_{ij} \) are outside the regions of distortion. This can be done as follows: consider the variable \( \delta p \) first. If \( |\delta p| > |\Delta + \Delta| \) then we get contribution from normal state \( G \) and \( \tau \) in (4.23).

The case with the momentum integration is similar. We first draw the Fermi sphere of radius \( \delta q \). Then draw concentric spheres of radii \( \delta q + \Delta, \delta q + \Delta, \delta q + \Delta + \Delta \) and \( \delta q - \Delta, \delta q - \Delta, \delta q - (\Delta + \Delta) \).
Then we get contribution from normal $\tau$ and $G$ outside the sphere of radius $q_F + \Delta_1 + \Delta_2$ and inside the sphere $q_F - (\Delta_1 + \Delta_2)$. Let us call $\Delta_1 + \Delta_2 = \Delta$ and $\Delta_1 + \Delta_1 = \Delta'$. We write the expression (4.23) for $\delta M_{ij} (p)$ as follows:

$$\frac{1}{2} \delta M_{ij} (p) = \int d^3q \left[ -p^2 \left( \frac{\vec{p} \cdot \vec{q}}{q^2} \right) \right] T_{ij} (p+q) G_{ij} (q)$$

We see that we get the contribution from superfluid $\tau_{ij}$ and superfluid $G_{ij}$ only from the term

$$\int_{q_F - \Delta}^{q_F + \Delta} \int d^3q \int d^3q_0 \left[ \int_{q_F - \Delta}^{q_F + \Delta} d^3q \right] T_{ij} (p+q) G_{ij} (q)$$

Since we are interested in isolating the contribution from the normal functions we can do the $q_0$ integration which is easier to do and consider the three momentum integrals separately. That is, we write:

$$\frac{1}{2} \delta M_{ij} (p) = I_1 + I_2 + I_3$$

$$I_1 = \int d^3q_0 \int_{q_F - \Delta}^{q_F + \Delta} \int_0^\infty \int_0^\infty \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T_{ij} (p+q) G_{ij} (q)$$

$$I_2 = \int d^3q_0 \int_{q_F - \Delta}^{q_F + \Delta} \int_0^\infty \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T_{ij} (p+q) G_{ij} (q)$$

$$I_3 = \int d^3q \int_{q_F - \Delta}^{q_F + \Delta} d^3q_0 \int_0^\infty \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T_{ij} (p+q) G_{ij} (q)$$
From these expressions we shall show that the major contribution
to the diagonal elements comes from $I_1$ and $I_3$. For the off-diagonal
elements it comes from $I_2$. We first study the diagonal elements.
This can be done by inserting the normal state $\gamma$ and $G$ in $I_1$ and $I_3$.
$I_2$ involves the superfluid functions but since the momentum integration extends over a small region the contribution of $I_2$ is expected
to be small particularly since the superfluid $\gamma_j$ has no singularity
in that region as we saw. What we have to verify is that by inserting
the normal state functions we do not get arbitrarily large contribution
to $I_1$ and $I_3$. The reason for expecting this is that if $\Delta'\to 0$, $\gamma(\mathbf{p})$
becomes singular. For finite but small $\Delta'$ we can have a $\gamma(\mathbf{p})$ which is
"too large". So we must have some criterion as to the magnitude of $\Delta'$
which will result in $I_1$ and $I_3$ that will render $\delta M_{ij}(\mathbf{p})$ a slowly
varying function of $p$. In the following calculation we obtain this
for $\delta M_{ij}$ and establish that $I_1$ and $I_3$ are the important terms. Con­sider first $I_1$. We insert in the normal free particle $\gamma$ and $G$ and
do the $p_0$ integration obtaining

$$I_1(\mathbf{p}) = \int d\omega^q q^s d^q \gamma^{s}(\mathbf{F}^q) \times$$

$$\left[ \frac{1}{1 + \frac{\lambda \langle q^s \rangle}{2\pi \hbar} \left( \frac{4\omega^2}{V^3 - (p + \epsilon_q)^2} \right)} + \frac{p_0 + \epsilon_q}{V|F^q|} \ln \left| \frac{V|F^q| - (p_0 + \epsilon_q)}{V|F^q| + (p_0 + \epsilon_q)} + \frac{i\pi}{2} \right| \right]$$

(4.32)

where $\omega$, $\gamma$ are the constants of the shell potential used to evaluate
$\gamma$ and $V$ is the velocity on the Fermi surface $\frac{2F}{\hbar}$. If $\Delta' = 0$
the denominator contains terms that diverge logarithmically. Let us
look at these terms more closely. We write the expression in the
square bracket in (4.32) above as
The potentially divergent terms to consider are

\[
-\ln \left( V \sqrt{\mathbf{F}^{\mathbf{2}}} \right) \left( P_0 + \varepsilon_1 \right)^2 + \frac{P_0 + \varepsilon_1}{V \sqrt{\mathbf{F}^{\mathbf{2}}}} \ln \left( \frac{V \sqrt{\mathbf{F}^{\mathbf{2}}} - \left( P_0 + \varepsilon_1 \right)}{V \sqrt{\mathbf{F}^{\mathbf{2}}} + \left( P_0 + \varepsilon_1 \right)} \right) \quad (4.33)
\]

Put

\[
V \sqrt{\mathbf{F}^{\mathbf{2}}} - \left( P_0 + \varepsilon_1 \right) = \alpha
\]

\[
V \sqrt{\mathbf{F}^{\mathbf{2}}} + \left( P_0 + \varepsilon_1 \right) = \gamma
\]

\[
\therefore \quad P_0 + \varepsilon_1 = \frac{\alpha - \gamma}{2}
\]

\[
V \sqrt{\mathbf{F}^{\mathbf{2}}} = \frac{\alpha + \gamma}{2}
\]

Then write (4.33') above as

\[
-\ln \alpha + \frac{\alpha - \gamma}{2} \ln \left( \frac{\alpha}{\gamma} \right)
\]

\[
= \alpha \ln \alpha + \gamma \ln \gamma
\]

\[
\left( 4.33'' \right)
\]

Next let \( \alpha = \mathbf{r} \cos \Theta \), \( \gamma = \mathbf{r} \sin \Theta \). Then (4.33'') becomes

\[
\frac{\mathbf{r} \cos \Theta}{\mathbf{r} \sin \Theta + \cos \Theta} \left( \ln \mathbf{r} + \ln \mathbf{r} \cos \Theta \right) + \frac{\mathbf{r} \sin \Theta}{\mathbf{r} \sin \Theta + \cos \Theta} \left( \ln \mathbf{r} + \ln \mathbf{r} \sin \Theta \right)
\]

\[
= \ln \mathbf{r} + \frac{\cos \Theta \ln \cos \Theta + \sin \Theta \ln \sin \Theta}{\sin \Theta + \cos \Theta}
\]

where

\[
\mathbf{r}^2 = 2 \left( V \sqrt{\mathbf{F}^{\mathbf{2}}} \right) \left( P_0 + \varepsilon_1 \right)^2
\]

The second term is bounded so it is the term \( \ln \mathbf{r} \) which may diverge and is therefore of interest. In fact, we can write (4.33) as

\[
\frac{1}{1 + \frac{\lambda \mathbf{W}^{\mathbf{2}} \mathbf{E}^{\mathbf{2}}}{16 \pi^2} \left[ \ln \mathbf{r} + \text{finite terms} \right]}
\]

\[
= \frac{1}{1 - \frac{\lambda \mathbf{W}^{\mathbf{2}} \mathbf{E}^{\mathbf{2}}}{16 \pi^2} \left[ \ln \mathbf{r} + \text{finite terms} \right]} \quad \text{for} \quad \lambda < 0
\]
If $l_n$ becomes arbitrarily large, an arbitrarily small $\lambda$ can make the denominator vanish. However by letting $\Delta'$ be finite we can limit $\min r = \sqrt{2\pi} \Delta'$. So if we write

$$\frac{1}{\lambda l^2} = -\frac{1}{\ln \delta}, \quad \delta = e^{-16\pi^2/\lambda l^2}$$

the denominator becomes:

$$1 + \frac{\ln(\sqrt{2\pi} \Delta')}{\ln \delta} + \text{terms} \rightarrow 0 \text{ as } \lambda \rightarrow 0$$

Then we can always choose

$$\frac{\ln(\sqrt{2\pi} \Delta')}{\ln \delta} = \alpha, \quad \alpha \ll 1$$

and the denominator $\sim 1$ giving $\delta M_n(p)$ a slowly varying function. Thus we have for the width of the region

$$\Delta' = \frac{1}{\sqrt{2\pi}} \delta^{1/2} = \frac{1}{\sqrt{2\pi}} e^{-16\pi^2/\lambda l^2}$$

since $\Delta' \sim \Delta'$ and $\Delta = \Delta' + \Delta^\prime$ we have an estimate of the regions involved in the integrations.

We treat the term $I_3$ exactly similarly. The only difference is in the range of momentum integration. The excluded region is of the same magnitude as in $I_1$. Thus $I_1 + I_3$ give the main contribution to $\delta M_n(p)$ and if $\Delta'$ is chosen as shown above the resulting $\delta M_n(p)$ is slowly varying. Contribution of $I_2$ is expected to be small because of the smallness of the region of momentum integration. We can estimate the contribution of $I_2$:

$$I_2 = \int d\Omega \int_{t_f - \Delta'}^{t_f + \Delta'} \int d^2 \mathbf{q} u^2(\mathbf{q}) \int_{-\infty}^{\infty} d q_{1} \tau_n(p + q) G_n(q)$$

$$\approx \int d\Omega \cdot \frac{1}{2} \int d\mathbf{q} \left( \text{terms} \sim 1 \right) \approx \Delta$$
The region in which a slowly varying function varies appreciably is of the order $q_0 \sim \xi_F$ and $q_0 \sim 1/\Lambda$ \cite{22}. Therefore the error involved in neglecting $I_2$ is of the order $\Lambda/\mu$. Therefore to order $\Lambda/\mu$ we can replace $\delta M_{\mu}(\nu)$ by the normal state function provided in the calculation we omit the region of integration around the Fermi surface. The same applies to $\delta M_{\mu}(\nu)$.

We therefore introduce the important approximation in our theory: we neglect the effect of superfluidity on the diagonal elements of $\delta M$
and \( \Delta \), and replace them by the normal state expressions. This simplifies the formalism greatly. Besides obtaining simpler expressions for \( \Delta \) and \( \delta \) we now need consider only the off-diagonal elements for self-consistency. The expression for \( G_{ij} \) becomes

\[
G_{ij}(\mathbf{p}) = \frac{1}{D(\mathbf{p})} \begin{pmatrix}
\frac{p_0 + \epsilon_p + M(-\mathbf{p})}{\omega_0^2} & \frac{M_{12}(\mathbf{p}) + \sigma_p^*}{\omega_0^2} \\
\frac{M_{12}(\mathbf{p}) + \sigma_p}{\omega_0^2} & \frac{p_0 - \epsilon_p - M(\mathbf{p})}{\omega_0^2}
\end{pmatrix}
\]

(4.37)

The expression for \( T_{ij} \) becomes:

\[
T_{ij}(\mathbf{p}) = \frac{1}{q(\mathbf{p})} \begin{pmatrix}
1 - i\lambda I_{ij}(\mathbf{p}) & -i\lambda I_{ij}(\mathbf{p}) \\
\lambda I_{ij}(\mathbf{p}) & 1 + i\lambda I(\mathbf{p})
\end{pmatrix}
\]

(4.38)

As a final step in simplifying these SCLA equations we will rewrite the expressions for the off-diagonal elements of the self-energy so that the momentum integrations are around the Fermi surface. In our previous treatment we saw that the terms \( I_1 \) and \( I_3 \) contain regions far from the Fermi surface. Although \( I_1 \) and \( I_3 \) do not give major contributions to \( M_{ij} \) they cannot be neglected. We have a similar situation with the expressions for \( \sigma_p \) and \( \sigma_p^* \). We have the following expressions:

\[
M_{ij}(\mathbf{p}) = i \int d^3q \; \psi^*(\mathbf{p+q}) \psi^*_{ij}(\mathbf{p+q}) \psi_{ij}(\mathbf{q})
\]
Now according to eqns. (4.37) above we have

\[
G_{\text{txt}'}(\xi') = \int d\eta \, \psi(\xi') G_{12}(\eta)
\]

We see that the eqns. (4.39, 4.40, 4.41) form a set of coupled integral equations. We would like to rewrite these equations so that the region of momentum integration is the region \( q_F - \Delta \to q_F + \Delta \). This can be done in the special case of the pairing in the \( S \) state \( (l = 0) \). In this case the ground state is invariant under time reversal and we have

\[
G_{12} = G_{21} \quad ; \quad M_{12} = M_{21} \quad ; \quad \sigma = \sigma^* \]

and hence

\[
\tau_{12} = \tau_{21}
\]

Then eqns. (4.39, 4.40, 4.41) reduce to the single equation

\[
\tilde{M}_{12}(\eta) = \tilde{M}_{21}(\eta) = i \int d\eta' \, V(\eta', \eta) G_{12}(\eta') G_{21}(\eta) \tilde{M}_{12}(\eta) \]

where

\[
V(\eta, \eta') = \psi^2(|\tilde{E}_{\xi'}|) \left( 1 + \tau_{12}(\eta, \eta') \right)
\]

is the "effective interaction" and \( \tilde{M}_{12}(\eta) = M_{12}(\eta) + \sigma_{\eta} \)

In eqn. (4.44) let us assume \( \eta \in (-\Delta_1, \Delta_1) \) and \( |\tilde{E}_{\xi'}|-2F \in (-\Delta_1, \Delta_1) \)

then \( \tilde{M}_{12} = \tilde{M}_{12} \) in our previous notation. We shall rewrite eqn. (4.44) above so that \( |\tilde{E}_{\xi'}|-2F \in (-\Delta_1, \Delta_1) \). We use a method due to
Migdal (25). Let us write
\[ G_N(q)G_N(-q) = A(q) + B(q) \]
where
\[ A(q) = G_N(q)G_N(-q)\theta(q) \]
\[ B(q) = G_N(q)G_N(-q)[1 - \theta(q)] \]
\[ \theta(q) = \begin{cases} 1 & \text{if } |q| \leq q_F, \\ 0 & \text{otherwise} \end{cases} \]

We write eqn. (4.44) in the operator form:
\[ \vec{M} = i\vec{V}(A+B)\vec{M} \]

Then iterating,
\[ \vec{M} = iVAM + iVB( iVAM + iVB\vec{M} ) \]
\[ = iVAM + iVB( iVAM + iVB_{VAM} + iVB\vec{M} ) \]
\[ = iVAM + iVB( iVAM + iVB_{VAM} + iVB_{VAM} + \ldots ) \]
\[ = (iV + iVBV + iVB_{VAM} + \ldots ) A\vec{M} \]
\[ = VAM \]
where
\[ VAM = iV + iVBVAM \]

Thus we finally write the equation for \( \vec{M}_{12}(\gamma) \) as
\[ \vec{M}_{12}(\gamma) = \int_{\Omega_0}^{\infty} \int_{q_F}^{q_0} \int_{q_F}^{q_0} VAM(q)G_N(q)G_N(-q)\vec{M}_{12}(\gamma) \]

Eqns. (4.45), (4.46) above together with the eqn. (4.37) for form the set of SCLA equations in the final form for the case of pairing in \( S \) state. Thus they do not apply to liquid \( \text{He}_3 \) for which the pairing can occur in \( D \) state. A straightforward generalization of the above procedure is needed. The usefulness of the above equation
for $\delta\tilde{W}_m$ lies in the fact that both the variables $p$ and $q$ are near the Fermi surface. As a first approximation the function $\sqrt{A^i(p,q)}$ may be replaced by its value on the Fermi surface. It then depends only on the angle between $p$ and $q$ and plays a role analogous to the scattering amplitude introduced by Landau in the theory of normal Fermi liquid. In fact at this stage one can formulate a theory of superfluid fermi liquid by using the pole approximation for the single particle propagators. The parameters introduced into the theory are the effective mass and the gap parameter which must be determined self consistently.

The most important use of these SCLA equations is, however, in obtaining the form of the single particle spectral function near the fermi surface. This requires a numerical solution of these equations. The equations in the final form are easier to handle. It was the purpose of the investigation to simplify the original set of equations. We see now that for $\delta$ -state pairing this can be done. For the case of liquid He$^3$ a modification of the above method is needed.
CHAPTER V

BEHAVIOR OF THE IMAGINARY PART OF $M_{11}$: APPLICATIONS

In Chapter IV we set up a set of self-consistent equations that if solved would yield the single-particle spectral function. A numerical solution of the equations would require additional information about the nature of the specific physical system under consideration. It is desirable to know as much as possible about the form of the spectral function from general considerations. Besides shedding some light on the nature of the superfluidity of the Fermi system this will form an important step towards a numerical solution of the self-consistent equations as we will know the input to the closed system of the self consistent equations.

Specifically we investigate the form of the spectral function $\tilde{\rho}_{\omega}(\overline{r},\omega)$ for the small value of $\omega$. $\tilde{\rho}_{\omega}(\overline{r},\omega)$ is related to the imaginary part of the self-energy function $M_{11}(\overline{r},\omega)$. In Chapter II we saw that the self-energy function is given in the perturbation theory by a set of irreducible diagrams. We shall evaluate some of the low-order diagrams and examine the behavior of $I_{\omega} M_{11}(\overline{r},\omega)$ for small $\omega$.

The perturbation theory for superfluids is discussed in detail by Nozières (27). We summarize the important points that are relevant to the present problem. Purely for the notational convenience we assume the particles to be spinless and the pairing to be in $S$-state. The unperturbed state is then the B.C.S. ground state. The unperturbed
Hamiltonian is

\[ H_0 = \sum_{\mathbf{p}} \varepsilon_{\mathbf{p}} a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + \sigma \sum_{\mathbf{p}} (a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + h.c.) \]  

and the perturbation \( H_1 \) is given by

\[ H_1 = \frac{i}{\hbar} \sum_{\mathbf{p}} V(\mathbf{p}, \mathbf{r}, \mathbf{r}, \mathbf{r}) a_{\mathbf{p}}^\dagger a_{\mathbf{p}}^\dagger a_{\mathbf{p}} - \sigma \sum_{\mathbf{p}} (a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + h.c.) \]

One diagonalizes \( H_0 \) by the Bogoliubov Transformations and arrives at the description of the quasi-particles of the system. These form convenient intermediaries in the calculations. The vacuum of these quasi-particles is the unperturbed ground state of the system which is the BCS ground state. The unperturbed propagator is the 2 x 2 matrix whose elements are given by

\[ G_n(\mathbf{p}, t) = -i \left[ u_p^2 \Theta(t) e^{-iE_p t} - v_p^2 \Theta(-t) e^{iE_p t} \right] \]

\[ = G_{xx}(-p, t) \]

\[ G_{iz}(\mathbf{p}, t) = -i u_p v_p e^{iE_p t} \]

\[ = G_{iz}(\mathbf{p}, t) \]

where

\[ u_p^2 = \frac{1}{2} \left( 1 + \frac{E_p}{\varepsilon_0} \right) \quad v_p^2 = \frac{1}{2} \left( 1 - \frac{E_p}{\varepsilon_0} \right) \quad E_p^2 = \varepsilon_0^2 + \sigma^2 \]

The Fourier transforms of these propagators are

\[ G_n(\mathbf{p}, \omega) = u_p^2 \frac{1}{\omega - E_p + i\eta} + v_p^2 \frac{1}{\omega + E_p - i\eta} = -G_{xx}(\mathbf{p}, \omega) \]

\[ G_{iz}(\mathbf{p}, \omega) = u_p v_p \left[ \frac{1}{\omega - E_p + i\eta} - \frac{1}{\omega + E_p - i\eta} \right] = G_{iz}(\mathbf{p}, \omega) \]

The true ground state is generated by adiabatically switching on the perturbation \( H_1 \). The Dyson expansion is derived just as in the case of normal systems. Using a generalized form of Wick's theorem a
Feynman diagram prescription for each term in the perturbation expansion can be derived. As in the normal state case one has both, the time-ordered diagrams and the Feynman diagrams. The recipe for the evaluation of a Feynman diagram in the energy-momentum space is as following:

1. Associate with each V vertex a factor $\sqrt{(P_1 P_2 P_3 P_4)}$.
2. Associate with each internal line with 4-momentum $P$ a factor $\chi_i^a(\gamma_1)$, $i$, $j$ depending on the directions of arrowheads at the ends of the line.
3. Ensure the conservation of 4-momenta at all the vertices.
4. Integrate over all internal 4-momenta.
5. Add an over-all factor $(-1)^{p+1} \frac{(-1)^l}{r 2^{m+n}}$

where
- $p =$ number of vertices
- $r =$ number of permutations of vertices leaving the diagram invariant
- $m =$ number of pairs of equivalent lines
- $n =$ number of symmetric loops
- $l =$ number of closed loops

With this recipe we now consider low-order diagrams contributing to $M_{ll}(\gamma)$. The first-order diagram is the Hartree-Fock diagram and is energy-independent. It is not of interest and we go to the second order diagram. It is shown in the figure below:

Figure 12 Second Order Contribution to $M_{ll}$
The contribution of this diagram to $M_\text{h}(\vec{p}_\omega)$ can be immediately written by using the rules given above

\begin{align*}
M_\text{h}(\vec{p}_\omega) &= -\frac{i}{2} \int d\vec{p}_1 d\vec{p}_2 d\vec{p}_3 \left| V(\vec{p}_1, \vec{p}_2, \vec{p}_3) \right|^2 \delta(\vec{p}_1 - \vec{p}_2 + \vec{p}_3) \delta(\omega - \omega_1 + \omega_2 + \omega_3) \\
&= -\frac{i}{2} \int d\vec{p}_1 d\vec{p}_2 d\vec{p}_3 \left| V(\vec{p}_1, \vec{p}_2, \vec{p}_3) \right|^2 \left[ \frac{u^2_{\text{h}}}{\omega - E_\text{h} + i\eta} + \frac{v^2_{\text{h}}}{\omega + E_\text{h} - i\eta} \right] \delta(\omega - \omega_1 + \omega_2 + \omega_3).
\end{align*}

We can do the $\omega_3$ integration using the $\delta$-function. Then $\omega_1, \omega_2$ integration can be done by the residue method getting:

\begin{align*}
M_\text{h}(\vec{p}_\omega) &= 2\pi \int d\vec{p}_1 d\vec{p}_2 d\vec{p}_3 \left| V(\vec{p}_1, \vec{p}_2, \vec{p}_3) \right|^2 \left[ \frac{u^2_{\text{h}}}{\omega - E_\text{h} - E_\text{p}_3 + i\eta} + \frac{v^2_{\text{h}}}{\omega + E_\text{h} + E_\text{p}_3 - i\eta} \right] \\
&= 2\pi \int d\vec{p}_1 d\vec{p}_2 d\vec{p}_3 \left| V(\vec{p}_1, \vec{p}_2, \vec{p}_3) \right|^2 \left[ \frac{u^2_{\text{h}}}{\omega - E_\text{h} - E_\text{p}_3 + i\eta} + \frac{v^2_{\text{h}}}{\omega + E_\text{h} + E_\text{p}_3 - i\eta} \right]
\end{align*}

We evaluate the multiple integral for the special case of the shell potential. That is $V(\vec{p}, \vec{p}_\omega) = \text{const.}$ if $|M_\text{h}(\vec{p}_\omega)| \approx 2F$ and zero otherwise. We are interested in $\text{Im} M_\text{h}(\vec{p}_\omega)$ for small values of $\rho$. In the integration with respect to $\vec{p}_\omega$ we put

$$d\vec{p}_\omega = \vec{p}_\omega d\vec{p}_\omega = 2F \frac{d\vec{p}_\omega}{\omega} d\omega.$$

The contributions to $\text{Im} M_\text{h}(\vec{p}_\omega)$ from the two terms in the integrand are non-overlapping. Consider the first term. Its contribution to $\text{Im} M_\text{h}(\vec{p}_\omega)$ is

$$\text{Im} M_\text{h}(\vec{p}_\omega) \propto \int d\vec{p}_1 d\vec{p}_2 d\vec{p}_3 \frac{u^2_{\text{h}}}{\omega - E_\text{h} - E_\text{p}_3} \delta(\omega - E_\text{h} - E_\text{p}_3)$$

(5.9)
Since $E_{fc}$ is a positive quantity we note that, due to the $s$ function, the maximum value of $E_{fc}$ is $\omega$. Let the limits of integration of $dE_{fc}$ be from $-\omega$ to $\omega$. Putting in the expressions for $u_{fc}^2, \psi_{fc}^2$ in above we get

$$\Im M_{\mu} (\vec{r}, \omega) \propto \int dE_{fc} d\epsilon_{fc} d\epsilon_{p} \left( \frac{E_{p} + \epsilon_{p}}{E_{p} - \epsilon_{p}} \right) \left( \frac{E_{p} + \epsilon_{p}}{E_{p} - \epsilon_{p}} \right) \delta (\omega - E_{p}, -I_{pc} - E_{pc})$$

(5.10)

We change the variables of integration from $E_{fc}$ to $E_{pc}$. Since $E_{pc}^2 = E_{p}^2 + \sigma^2$, we have $dE_{pc} = \frac{\epsilon_{pc}}{E_{pc}} d\epsilon_{p}$. And

$$\Im M_{\mu} (\vec{r}) \propto \int \frac{d\epsilon_{pc}}{\epsilon_{pc}} \left( \frac{\epsilon_{pc}}{E_{pc}} \right) \delta (\omega - E_{pc})$$

(5.11)

We first observe that because of the $s$-function $\Im M_{\mu} (\vec{r}, \omega) \to 0$ if $\omega < 3\sigma$. If $\omega \gg \sigma$ we get the behavior in the normal case. The specific modifications due to superfluidity occur in the vicinity $\omega \approx 2\sigma$.

We shall evaluate the integral approximately in this region. First consider the integral

$$\int_{\sigma}^{\omega} dE_{pc} \frac{E_{pc}}{E_{pc}^2 - \sigma^2} \delta (\omega - E_{pc})$$

$$= \int_{\sigma}^{\omega} dE_{pc} \frac{E_{pc}}{E_{pc}^2 - \sigma^2} \delta (\omega - \frac{E_{pc}}{E_{pc} - \sigma})$$

We are interested in the values of $E_{pc}$ slightly greater than $\sigma$.

In that case we can write

$$E_{pc}^2 - \sigma^2 \approx \approx E_{pc} (E_{pc} - \sigma)$$

We change the variables $E_{pc} - \sigma \to \sigma x_i$. Then

$$\Im M_{\mu} (\vec{r}, \omega) \propto \int \frac{d\epsilon_{pc}}{\epsilon_{pc}} \left( \frac{1}{\epsilon_{pc}} \right) \delta (\omega - \sigma - \sigma \frac{\epsilon_{pc}}{\epsilon_{pc}^2})$$
If we use the representation of the $\delta$-function

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk$$

we get,

$$\text{Im} M_n(p, \omega) \propto \int_{-\infty}^{\infty} e^{iky} dk \left[ \int_{0}^{\infty} e^{-\frac{y}{\sigma x}} e^{ikx} dk \right]^3$$

where $y = \omega - 3\sigma$

$$\propto \int_{-\infty}^{\infty} e^{iky} \left[ \int_{0}^{\infty} \frac{e^{-t/(\sigma k^2)}}{(\sigma k^2)} dt \right]^3$$

where $k' = \epsilon + ik$, $\epsilon \to 0$

$$\propto \int_{-\infty}^{\infty} \frac{e^{iky}}{(ik + \epsilon)^{3/2}} \left[ \int_{0}^{\infty} e^{-t/(k^2 + \epsilon)} dt \right]^3$$

$$= \int_{-\infty}^{\infty} \frac{e^{iky}}{(ik + \epsilon)^{3/2}} \left[ \Gamma\left( \frac{1}{2}, k\sigma \right) \right]^3$$

(5.12)

where $\Gamma\left( \frac{1}{2}, k\sigma \right)$ is the incomplete Gamma function. We use the expansion of the incomplete Gamma function(28)

$$\epsilon^{-\alpha} \Gamma(a, z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n! (a+n)}$$

(5.13)

Therefore if $y$ is sufficiently small,

$$\Gamma\left( \frac{1}{2}, k\sigma \right) \approx 2(k\sigma)^{1/2} \left( 1 - \frac{k\sigma}{3} \right)$$

$$\left[ \Gamma\left( \frac{1}{2}, k\sigma \right) \right]^3 \approx 8(k\sigma)^{3/2} \left( 1 - \frac{2(k\sigma)^3}{27} - (k\sigma)^3 + (k\sigma)^3 \right)$$

$$\therefore \text{Im} M_n(p, \omega) \propto \int_{-\infty}^{\infty} \frac{e^{iky}}{(ik + \epsilon)^{3/2}} (k\sigma)^{3/2} dk$$

where we retained the lowest order term in $y$ that gives non-zero con-
tribution to \( \text{Im } M_{11}(p, \omega) \)

We change the variable \( iky = t \). Then

\[
\text{Im } M_{11}(p, \omega) \propto \gamma^{1/2} \int \mathcal{E}^+ t \, dt
\]

\[
\propto \gamma^{1/2} = (\omega - 3\sigma)^{1/2}
\]

Therefore we establish the result that in the limit \( \omega \to \omega_{\text{c}} \),

\[
\text{Im } M_{11}(p, \omega) = c (\omega - 3\sigma)^{1/2}
\]  \( (5.14) \)

Since \( \text{Im } M_{11}(p, \omega) \) is equal to zero for \( \omega \leq 3\sigma \) we can write

\[
\text{Im } M_{11}(p, \omega) = c \theta(\omega - 3\sigma) (\omega - 3\sigma)^{1/2}
\]  \( c5.14 \)

We can treat the second term of \( (5.8) \) similarly and obtain

\[
\text{Im } M_{11}(p, \omega) = c \theta(-3\sigma - \omega)(-\omega - 3\sigma)^{1/2}
\]  \( c5.15 \)

for \( \omega \leq \omega_0 \). This behavior is different from that obtained by Luttinger for the normal system\(^{(29)}\). It can be proved that the same expression could be derived by extending the upper limit of the integral in \( (5.11) \) to \( \infty \).

In order to establish the result generally we have to do two things. We must use full propagators instead of the B.C.S. propagators and consider all higher-order diagrams. The former problem is easier and follows Luttinger's treatment closely. If we use the full propagators we get an expression similar to \( (5.8) \) for \( M_{11}(p) \) except that we have true spectral functions instead of the \( \delta \)-functions characteristic of the B.C.S. theory. However for \( \omega \) near \( 3\sigma \) and \( p \) near Fermi surface the spectral function is a \( \delta \)-function with normalized
energy. Thus our previous reasoning that led to the final result is still valid.

We next investigate the case of higher order diagrams. The diagram of importance is the third order diagram and is shown in the figure below.

Figure 13:
Third Order Contribution to $M_{11}$

The contribution of this diagram to $I_n M_n(p, \omega)$ can be evaluated by the same rules to be

$$M_n^{(3)}(p, \omega) \propto \int d\tilde{p}_1 \cdots d\tilde{p}_5 \int d\omega_1 \cdots d\omega_5 \sqrt{(p_1 + i\eta)(p_2 + i\eta)(p_3 + i\eta)} \sqrt{(p_4 + i\eta)(p_5 + i\eta)} \times$$

$$\delta(p_1 + p_2 + p_3 - p_4) \delta(p_3 + p_4 - p_5) \delta(p_1 + p_2 + p_3 - p_5) \times$$

$$\delta(\omega_1 + \omega_2 + \omega_3 - \omega_4) \delta(\omega_3 + \omega_4 - \omega - \omega_5) \prod_{i=1}^{5} \left[ \frac{V_{p_i}}{\omega_i - \tilde{p}_i - i\eta} + \frac{V_{\tilde{p}_i}}{\omega_i - \tilde{p}_i - i\eta} \right]$$

(5.16)

The integrations can be done in a straightforward way. We evaluate the imaginary part in the same way as in the second order diagram by using the $i\eta$'s. We get in all a sum of six terms which involve integrations over $P_1 \ldots P_5$. A typical integrand contains a factor like

$$u_{p_1}^2 u_{p_2}^2 u_{p_3}^2 u_{p_4}^2 u_{p_5}^2 \delta(\omega - \frac{5}{\epsilon_i} E_p)$$

multiplied by potential functions. We evaluate this using the shell potential as in the previous case and doing the $\tilde{p}$ integrations in the same way. Each term gives a contribution proportional to

$$\int d\epsilon_{p_1} \cdots d\epsilon_{p_5} \prod_{i=1}^{5} \left( \frac{\epsilon_{p_i} + \tilde{p}_i}{\epsilon_{\tilde{p}_i}} \right) \left( \frac{\epsilon_{\tilde{p}_i} + \epsilon_{p_i}}{\epsilon_{p_i}} \right) \delta(\omega - \frac{5}{\epsilon_i} E_p)$$
We make appropriate change of variable as in the second order case and finally get a contribution

\[ \text{Im} M^{(2)}(\varphi, \omega) \propto \int \frac{d \xi}{i} \left( d \xi \xi \left( \xi \xi \right) \right) \]

This multiple integral can be evaluated as in the second order case obtaining

\[ \text{Im} M^{(3)}(\varphi, \omega) \propto \Theta(\omega - 3\sigma)(\omega - 5\sigma)^{3/2} \]  

(5.17)

If we go to the next higher order diagram we would arrive at the expression

\[ \text{Im} M^{(4)}(\varphi, \omega) \propto \Theta(\omega - 4\sigma)(\omega - 7\sigma)^{4/2} \]  

(5.18)

It is of interest to note that each diagram has a threshold below which its contribution is zero. Thus the second order diagram contributes above \( \omega = 3\sigma \), the third order above \( \omega = 5\sigma \) and so on. Further, we have here a bootstrap-like situation. The B.C.S. propagators determine the second order contribution. The third order is determined by the second order and so on. As a result \( \text{Im} M^{(n)}(\varphi, \omega) \) shows sudden jumps at \( 3\sigma, 5\sigma \), etc. Interesting consequences of this will now be considered.

We now write the expression for \( \text{Im} M^{(n)}(\varphi, \omega) \) in the form:

\[ \text{Im} M^{(n)}(\varphi, \omega) \approx c_1(\omega - 3\sigma)^{n/2} + c_2(\omega - 5\sigma)^{n/2} + \ldots \]  

(5.18)

\[ = \Gamma(\omega) \text{ for } \omega \geq 0. \]

Similar expression being for \( \omega < 0 \). We see that we have thresholds for 3-particle decay, 5-particle decay and so on. If \( \omega < 3\sigma \) there is no decay and \( \text{Im} M^{(n)}(\varphi, \omega) = 0 \). The implication of this for the superfluid hydrodynamics will be reviewed later on. We see that \( \text{Im} M^{(n)}(\varphi, \omega) \) is not a smooth function of \( \omega \) but has sudden jumps at \( 3\sigma, 5\sigma \), etc.
When $\omega \gg \sigma^-$, $\text{Im} \mathfrak{m}_\omega(\mathcal{F}_\omega) \sim \omega^-$ as we shall show later on. This is the normal behavior predicted by Luttinger. We first consider the important question: what is the effect of this behavior of $\text{Im} \mathfrak{m}_\omega(\mathcal{F}_\omega)$ on the physical systems and how can it be experimentally observed? Since $\text{Im} \mathfrak{m}_\omega(\mathcal{F}_\omega)$ is related to $\mathfrak{p}_\omega$ which is the single-particle spectral function, we have arrived at the behavior of $\mathfrak{p}_\omega$ for $\omega \gg \sigma^-$. Now $\mathfrak{p}_\omega$ in turn determines the density of states of superfluid system. A sensitive probe to study the density of states is the well-known tunneling experiment. Specifically, we consider single-particle tunneling between normal and superconducting metals.

![Figure 14 - The Tunneling Experiment](image)

A typical experimental arrangement is as shown above. The normal and superconducting metals are separated by a thin insulator which may be an oxide coating and which acts as a barrier. The experiment consists in determining the current $I$ as a function of the applied D.C. bias $V$. We define $V$ to be $(-e)$ times the actual applied voltage. No current flows from $N$ to $S$ until $V$ equals $\sigma$, the gap. Thereafter the current as a function of voltage depends on the density of states in the superconducting metal available for the injected electrons. A detailed theory of this experiment has been worked out and is discussed by Schrieffer\(^{(29)}\). We shall write down the final results which are
necessary for our purpose.

Let $I_{NS}$ and $I_{NN}$ be the currents for the normal to superconductor and normal to normal metal respectively. The quantity of interest is the experimentally determined conductance $g(\omega)$ given by

$$g(\omega) = \frac{dI_{NS}}{dV} \frac{dV}{dI_{NN}}$$  \hspace{1cm} (5.19)$$

It turns out that $g(\omega)$ is simply given by $g(\omega) = N(\omega)/N(0)$ where $N(\omega)$ is the density of states in the superconducting metal and $N(0)$ that in the normal metal. The general theory of tunneling yields the relation

$$N(\omega) = \sum \frac{1}{2\pi} \left| \text{Im} G_n(i\omega) \right|^2$$  \hspace{1cm} (5.20)$$

For $\omega \ll \Gamma$ we assume the following form of $G_n(i\omega)$

$$G_n(i\omega) = \frac{V_F^2}{\omega - E_p + i\Gamma(\omega)} + \frac{V_F^2}{\omega + E_p + i\Gamma(\omega)}$$  \hspace{1cm} (5.21)$$

where $\Gamma(\omega)$ is given by (5.18'). We convert the sum over $F$ into an integral over $E_p$ in the usual manner. The coherence factors $V_F^2, V_F^2$ drop out and we get for $N(\omega)$

$$N(\omega) = \frac{N(0)}{\pi} \int_0^\omega \frac{dE}{d\omega} \left[ \frac{\Gamma(\omega)}{(\omega - E)^2 + (\Gamma(\omega))^2} \right] dE$$

$$= \frac{N(0)}{\pi} \int_0^\omega \frac{dE}{d\omega} \left[ \frac{\Gamma(\omega)}{(\omega - E)^2 + (\Gamma(\omega))^2} \right] dE$$

$$= \frac{N(0)}{\pi} \int_0^\omega \frac{dE}{d\omega} \left[ \frac{\pi}{2} + \tan^{-1} \left( \frac{\omega - E}{\Gamma(\omega)} \right) \right]$$

$$\frac{N(\omega)}{N(0)} = \frac{\omega}{\pi (\omega - \sigma^2)^{1/2}} \left[ \frac{\pi}{2} + \tan^{-1} \left( \frac{\omega}{\Gamma(\omega)} \right) \right]$$  \hspace{1cm} (5.22)$$
If $\varphi(\omega)=0$ we get the expression for the B.C.S. case. The specific modifications due to the spread in the spectral function is contained in the second term in the square brackets. $\varphi(\omega)$ is not a smooth function of $\omega$ but goes as $(\omega-\epsilon)^{\frac{1}{2}}$ for $\omega$ slightly greater than $3\epsilon$. Then for $\omega > 5\epsilon$ we get a new contribution proportional to $(\omega-5\epsilon)^{\frac{3}{2}}$ and so on. We see that at $3\epsilon, 5\epsilon, 7\epsilon$...etc., $N(\omega)/\nu(\omega)$ should show sudden jumps. The same is reflected in the curve $g(V)$ versus $V$. Experimental curves for various materials exist. The one for the B.C.S. case has been plotted by many authors. We show below the B.C.S. curve and the modification introduced by the expression (5.21) above.

We see that the normal spectral function given by a $\varphi(\omega) \sim \omega^2$ does not modify the conductance-voltage curve too much. The superconducting spectral function shows bumps in the region above $3\epsilon$. These are not quoted in literature. The main reason seems to be that the early experimentalists were happy with the agreement with the B.C.S. theory and their curves extend up to about $3\epsilon$. When the anomalies
appeared in the curve for Pb, Schrieffer and his co-workers explained it as due to the phonon spectrum. They worked out in detail the theory of strong superconductors of which Pb is an example. We feel that a careful measurement of $\gamma(C\nu)$ for $\nu \gg 3\sigma$ should reveal the new structure. One should, of course, choose a weak superconductor where phonon spectrum is unimportant in the region of interest.

In passing it is worth noting the negative resistance characteristic of the $\gamma(C\nu) \rightarrow \nu$ curve in the region $\nu \approx 2\sigma$. Indeed the current-voltage curve has the shape shown in the figure below:

![The Current-Voltage Characteristic](image)

The region between $3\sigma$ and $5\sigma$ resembles a triode characteristic. Similar characteristics have been observed in tunneling between two superconductors (I. Giaever\textsuperscript{31}). Giaever mentions that amplifiers and oscillators have been constructed. In the present case it may be easier to construct a device as one of the metals is normal. In any case, the utility of such devices in liquid He temperatures may lead to useful applications of the above result.

Finally we look at the behavior of $\mathcal{M}_n(F,\omega)$ when $\omega$ is much larger than $\sigma$. We know that when $\omega$ is so large that in the integration with respect to $F_n$ in the expression for $\Im \mathcal{M}_n(F,\omega)$, $\sigma$ can be
neglected, we get the expression for the normal case, namely the Luttinger $\omega^2$ behavior. In order to see what happens in the intermediate region we consider again the integral

$$\text{Im} \, M_{1i}(\mathbf{p}, \omega) \propto \int \prod_i \left( dE_{E} \frac{E_p}{E_{E_F}} \right) \delta(\omega - E_{E_F})$$

Unlike the previous case we do not put $E_{E_F} + \sigma \approx E_{E_F}$. Again changing the variables $E_{E_F} - \sigma = \sigma \omega$

$$\text{Im} \, M_{1i}(\mathbf{p}, \omega) \propto \int \prod_i \left( \frac{1 + \omega}{\omega \xi(\omega + \xi)} \right) d\omega \, \delta(\omega - 3\sigma - \sigma \omega)$$

We are now guided by our previous result namely we can extend the upper limit of integration to $\infty$. Using the integral representation of the $S$-function,

$$\text{Im} \, M_{1i}(\mathbf{p}, \omega) \propto \int_{-\infty}^{\infty} e^{ik\xi} \Delta k \left[ \int_{-\infty}^{\infty} \frac{e^{-ikx}}{\sqrt{x(\omega + \xi)}} \right]^3$$

where $\gamma = \omega - 3\sigma$.

The integral in square brackets is known. We have

$$\int_{-\infty}^{\infty} \frac{1 + x}{\sqrt{x(\omega + \xi)}} e^{-ikx} dx = e^{ik} K_1(\xi + ik)$$

where $K_1(\xi)$ is the modified Bessel function and $\xi$ is a positive infinitesimal. We use the expansion of $K_1(\xi)$

$$K_1(\xi) = \left( \frac{\pi}{2\xi} \right)^{1/2} e^{-\xi} \left[ \sum_{k=0}^{\infty} \frac{(-1)^k (1,k) (2\xi)^{-k}}{k!} + O(\xi^{-k-1}) \right]$$

where $(1,k) = \frac{(4^{-1})(4^{-2}) \cdots (4^{-1-k+1})}{2^{1k} k!}$, $(1,0) = 1$.
If we retain the first two terms in the series, we obtain

\[
[\text{e}^{x_k(c_z)}]^{3} \propto \frac{1}{x_{2/3}} \left(1 + \frac{a}{x_{2/3}}\right)^{3}
\]

Then

\[
\text{Im} M_n(z, \omega) \propto \int_{-\infty}^{\infty} \text{e}^{iky} \frac{1}{k^{\nu_k}} \left(1 + \frac{c_{1}}{k} + \frac{c_{2}}{k^{2}} + \frac{c_{3}}{k^{3}}\right)dk
\]

We change the variable \(iky = \text{and obtain}\n
\[
\text{Im} M_n^{(\nu_k)}(z, \omega) \propto \gamma^{\nu_k} \int_{C} \text{e}^{t\left(\frac{1}{t^{\nu_k}} + \frac{c_{1} y}{t^{1/3}} + \frac{c_{2} y^{2}}{t^{1/3}} + \frac{c_{3} y^{3}}{t^{1/3}}\right)} dt
\]

where the contour \(C\) is as shown in the figure.

Figure 17
Contour for the integral in (5.26) and (5.27)

Noting the contour integral representation of the function \(\frac{1}{\Gamma(z)}\)

\[
\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{C} \text{e}^{t} t^{-z} dt
\]

we get

\[
\text{Im} M_{n}(z, \omega) \propto a(\omega - 3\sigma)^{1/2} + b(\omega - 3\sigma)^{3/2} + \ldots
\]

We see that when \(\omega\) increases, the square root behavior goes over into 3/2 power. To see what happens when \(\omega \gg 3\sigma\) we again consider the expression (5.23) for \(\text{Im} M_{n}(z, \omega)\). This time since \(\gamma \gg \sigma\) we consider \(K_{1}(\tau)\) for \(|\tau|\) very small. We have

\[
K_{1}(\tau) \sim \frac{1}{\tau} \quad \tau \to 0
\]
We then obtain
\[ \text{Im} \, M_{\parallel}^2(p, \omega) \propto (\omega - \omega_c)^2 \]
and since \( \omega_c \) we obtain the Luttinger \( \omega^2 \) behavior. We see that \( \text{Im} \, M_{\parallel}^2(p, \omega) \) has the following form:

\[ \text{Im} \, M_{\parallel}^2(p, \omega) \]

Figure 18 - Behaviour of \( M_{\parallel}^2(p, \omega) \) as a function of \( \omega \)

Our discussion about the behavior of \( \text{Im} \, M_{\parallel} \) has an important implication for the single-particle excitations in the superfluid Fermi liquid. We see that unlike the case of the normal Fermi liquid, there exist undamped single-particle excitations in the superfluid Fermi liquid. Only those quasi-particles with energy greater than \( \omega_c \) are damped. Further the damping is rather weak; i.e., if \( p \) is the momentum of the quasi-particle, the damping term \( \Gamma_p \) is proportional to \((\tilde{p} - p)^2\) where \( \frac{\tilde{p}^2}{2\mu} = \mu + \sqrt{\sigma} \). The point \( p' \) on the \( E(p) \) versus \( p \) curve is a singular point at which the decay into two excitations becomes possible. This has a simple hydrodynamic interpretation. Suppose there is a net flow of the superfluid with velocity \( \vec{v}_s \). In the rest frame the energy of the quasi-particle is \( E(p) + \vec{p} \cdot \vec{v}_s \). We know that if \( \vec{v}_s \) is sufficiently large it becomes energetically favorable to break a Cooper pair into two excitations. This occurs when \( \vec{v}_s \gamma \frac{\sigma}{\gamma_F} \) and gives the Landau critical velocity. This Landau critical velocity does not imply the des-
struction of superfluidity which is attained at a higher velocity. There is a higher critical velocity given by \( V_s = \frac{2\sigma}{\Gamma P} \) at which the damping of the quasi-particles sets in. This is the onset of gaplessness. The exact gapless state is arrived when \( \Gamma_P = \sigma \). Thus, if we calculate \( \Gamma_P \) using the expression for the imaginary part of \( M_H \) it is possible to evaluate experimentally the proportionality constant. The method for this is again a tunneling experiment with uniform current flowing parallel to the junction. Such an experiment is described by Fulde (34) and Claeson (35). It should be possible in the near future to verify these conclusions by studying the experimental data carefully.
CONCLUSION

In this paper we have studied the superfluid Fermi system with arbitrary interaction. In order to study the properties of such a system, it is necessary to go beyond the Hartree-Fock approximation. The simplest approach is the Self-Consistent Ladder Approximation (SCLA) and the investigation is mainly concerned with studying this method. We first point out in Chapter I the occurrence of singularities in the SCLA equations if one used the formalism of the Normal System. In Chapter II we show how to modify the formalism logically and set up the SCLA equations for the superfluid system. We show the self-energy diagrams considered, namely those for particle-particle interaction. The particle-hole diagrams have been neglected. The spin has been omitted.

After studying the properties of the various functions in Chapter III, we turn to the study of the SCLA equations in Chapter IV. We use the physical fact that the gap is much smaller than the Fermi energy to simplify the set of equations. Specifically we use the fact that superfluidity affects the various functions in small regions which we call regions of distortion. We make a study of these regions and estimate their magnitudes. This enables us to introduce some important approximations into the theory and simplify the SCLA equations. We finally reduce the self-consistent equations to a form in which the quantities of interest and the various integration ranges are restricted to the regions of distortion. These equations can be used to describe a
superfluid Fermi Liquid by introducing suitable phenomenology. As they stand, these equations are suitable for the investigation of the superfluidity of nuclear matter. The results of Chapter IV can be used for a numerical work.

An important question neglected in Chapter IV is about the imaginary parts of the various functions. In particular one is interested in the imaginary part of the diagonal element of the self-energy matrix as this determines the damping of quasi-particles. In Chapter V we take up this problem. We make a perturbation theoretical analysis of some low order self-energy diagrams and establish the behavior of

\[ \text{Im} M_0(\omega) \] as a function of \( \omega \). We find an important modification from that in the normal state theory. It is suggested that simple-particle tunneling experiments should reveal this modification. This result in fact suggests further work in this field. It is important to establish the behavior of the imaginary parts of the vertex function and the off-diagonal elements of the self-energy matrix, the later being connected with the gap function. It is only after one knows these behaviors that one can attempt a complete self-consistent solution of the Ladder Approximation Equations.
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