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THE EFFECTS OF DUST AND LYMAN ALPHA RADIATION ON THE
DYNAMICAL EVOLUTION OF PLANETARY NEBULAE

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

William Stephen Kovach, B.S.

************

The Ohio State University
1970

Approved

[Signature]
Adviser
Department of Astronomy
Acknowledgments

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VITA

April 3, 1941  Born - Cleveland, Ohio

1963 . . . .  B.S., Case Institute of Technology, Cleveland, Ohio

1963-1965  .  Research Engineer, General Dynamics/ Astronautics, San Diego, California

1965-1969  .  Research Assistant to Dr. Eugene R. Capriotti in the Astronomy Department, The Ohio State University, Columbus, Ohio

1969-1970  .  Holder of a Dissertation Fellowship awarded by Ohio State University, Columbus, Ohio

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Introduction

This dissertation describes the construction of a number of dynamical models of planetary nebulae. The nebulae are idealized as being spherical with a hot central star at the center radiating as a black body. The H II region surrounding the central star is assumed to be at a uniform temperature. The H II region is, in turn, completely enclosed by a thin shell containing a homogeneous mixture of neutral hydrogen atoms and dust grains. Magnetic fields, such as those depicted by Gurzadyen (1962, 1969), have been neglected.

The H II region is optically thin to the stellar radiation. The only mass flow into the H II region is through the ionization front and no account is taken of a possible stellar wind. It is assumed that there is no dust in the H II region and that the dust which enters through the ionization front is quickly sputtered and evaporated. This assumption can be verified by the constructed models.

The ratio of the neutral hydrogen density to the dust density is assumed to be uniform throughout the H I region. The total density in the H I region, however, is allowed to vary as a function of position. The neutral hydrogen in the H I region is in thermal equilibrium with the grains. The dust is assumed to exist as spherical grains whose extinction properties are given by Mie theory. The H I region is so constructed that the net acceleration between Lagrangian mass points is very small. This construction allows us to consider the H I region as a solid body moving at a uniform velocity with respect to the central star. This procedure will, of course, not be applicable if condensations or large scale density fluctuations occur in the H I region.
The stellar radiation below the Lyman limit will be completely absorbed by the neutral hydrogen at the ionization front. The stellar radiation above this limit will pass into the H I region and be absorbed and scattered by dust grains. Since the H I region is optically thick in the Lyman continuum and Lyman lines, essentially one Lyman alpha photon will be produced for every Lyman continuum photon. The $L_\alpha$ photon will be unable to diffuse in physical space until it has diffused sufficiently far in frequency space, so that it is in the optically thin Lorentz wing. It was found that the primary mechanisms for achieving this diffusion in frequency space is the continued reflection of $L_\alpha$ photons after interactions with the expanding spherical enclosure.

Schematically, the wall of this enclosure, which in reality is the spherically ionization front, is considered to be a moving mirror which repeatedly reflects the photon at a lower frequency than it had previous to the reflection. (Cf. Kahn 1967).

Eventually the "mirror" will become optically thin to the $L_\alpha$ photon, which is now in the far red side of the line, and the $L_\alpha$ photon will be able to diffuse into the H I region. This red shifted $L_\alpha$ photon and the stellar radiation longward of the Lyman limit will be absorbed and scattered by the grains. The absorbed photon's energy will be redistributed and reradiated isotropically in the infra-red region of the spectrum. By treating the ionization front as a discontinuity, we are able to avoid doing the detailed transfer problem of the $L_\alpha$ and $L_c$ radiation.

Three forces were included in the equation of motion of the slab: The gas pressure, stellar radiative pressure and the Lyman alpha pressure.
All of the models were terminated when the gas pressure became larger than the $L_\alpha$ pressure. At this point in the nebula's evolution a more detailed solution to the dynamical problem must be used as was done by Mathews (1966), and Sofia and Hunter (1968).

No attempt has been made to include many of the cooling mechanisms in the H II region. Forbidden line radiation, arising from the metastable transitions, will certainly have an effect on the thermal equilibrium of the grains. Due to the high temperature in the H II region, such thermostatic mechanisms should be taken into account in more detailed models.
PART I

FORMULATION OF THE PROBLEM
CHAPTER I
GRAIN TEMPERATURE

There are three basic assumptions made in the determination of grain temperatures: (1) the grains are in radiative equilibrium with the incident radiation field, (2) they are spherical, and (3) they radiate like black bodies at a temperature $T_g$. The first assumption, the most important in determining the grain temperature, was first applied to interstellar particles by Eddington (1959) and in a more general form by Van de Hulst (1946). Subsequent investigations of the temperature of grains around and in such diverse objects as NML Cygnus by Krishna Swamy and Wickramasinghe (1968a), Cocoon Stars by Davidson (1969), H II regions by Krishna Swamy and O'Dell (1967) and Mathews (1967) and the interstellar medium by Krishna Swamy and Wickramasinghe (1968b), and Field (1969) have all explicitly made the assumptions that the grains had reached equilibrium with the radiation field.

The choice of spherical grains is mathematically advantageous. The pioneering work of Mie on the absorption and scattering coefficients of grains is applicable only to spherical particles. Observations of complex index of refractions as a function of wavelength, like those found by Taft and Phillipp (1965) for graphite, enable one to find the efficiency factors with the aid of Mie theory, as a function of grain radius and wavelength. If deviations from sphericity are taken into account the problem becomes one of extreme complexity. The absorption and scattering coefficients then become functions of the relative
orientations of the particle with respect to the incident light as shown by Greenberg (1968).

The assumption that the grains radiate as black bodies is intimately connected with the second assumption of sphericity. As soon as deviations from spherical symmetry occur in the grain, the source function contains a strong angular dependence, which not only depends on the grain's shape and the point under consideration, but also the primary chemical constituents in the grain.

From the above discussion one may immediately question the validity of the assumptions relative to the actual physics of interstellar grains. In order to determine the true shape, internal constitution and structure of the grain, detailed experimental analysis of nucleation and grain growth rates in a low density, low temperature environment must be performed. Even the rough molecular equilibrium calculations by Gilman (1969) indicate the grains will be composed of a montage of the more abundant elements. It can certainly be envisioned that low speed collisions of grain nuclei with atoms and molecules would build up a rather "lumpy" grain that has no or very little lattice binding - analogous to the clumping of wet snow flakes. A grain of this type would be constantly changing due to interactions of low energy light, cosmic rays and elastic and inelastic collisions with ions and other grains. The optical properties of such a configuration have yet to be thoroughly investigated. Thus it is highly probable under these conditions that there is no "definite" geometric shape associated with grain growth. If after formation, the grains are exposed to a medium of sufficiently high temperature, they may well become amorphous. Such
conditions, if short compared to the evaporative lifetime of the grain, would tend to construct spherical grains due to the grain's own surface tension. A Mira variable with surface temperature variation of over 1000°K in a time scale of 1 year would certainly satisfy these conditions since the lifetime of a metallic type grain in such an environment is of the order of 50-100 years. (Cf. Mathews 1967).

Donn (1967) has reported that experimental results on the albedo of grains of different shapes did not differ markedly from that of spherical grains of the same size. Recent work by Lefèvre (1970) appears, however, to arrive at the opposite conclusion. Lefèvre investigated the optical properties of iron, silica and carbon dust particles. He found that only silica grains, which turned out to be spherical, had the same optical properties as predicted by Mie theory. The optical properties of the other types of particles, whose geometry was definitely non-spherical, differed considerably from the Mie calculations.

Due to the lack of both theoretical understanding and experimental justification, we will continue, as have previous workers, to use the three assumptions given on page 5.

The radiation from the central star shortward of the Lyman limit is completely absorbed by the neutral hydrogen at the ionization front. Since the nebula is ionization bounded and thus optically thick in the Lyman continuum, Case B exists for the Lyman lines (Baker and Menzel, 1938). Eventually there will be one Lyman alpha photon created for each Lyman continuum photon emitted by the central star. As pointed out in Chapter III, these photons will diffuse in frequency space into
the red side of the Lyman alpha line. Once in the neutral damping wing
the photon is able to diffuse into the H I region. If the nebula is
devoid of dust the $\lambda_\alpha$ photon would simply scatter and trace out a
random walk in the H I region. The presence of dust in the H I region,
however, efficiently destroys some fraction of the Lyman alpha photons
by absorption. The energy of the absorbed Lyman alpha photon is re­
radiated isotropically by the grain into the infra-red region of the
spectrum.

A second source of energy to the grain is the radiation longward
of the Lyman continuum. The H I region would be optically thin to this
radiation if it didn't contain dust, since essentially all of the
electrons will be in the ground state of hydrogen. The presence of dust,
however, effectively absorbs some of this radiation and reradiates it
into the infra-red.

A third source of energy to the grains, particularly in the
outer layers of the nebulae, is self absorption of the infra-red
radiation arising from the inner region. Even though the H I region
will be optically thin to the infra-red radiation, the infra-red
radiation field and the dust grains will still be coupled. The grain is,
in general, a very poor absorber of infra-red radiation. Because we
are assuming that Kirchhoff's law holds for the grains, the corollary
follows that the grains will be poor emitters of infra-red radiation.
Thus, once an infra-red photon is absorbed, it will be trapped by the
grain until it can be reemitted. This process enables the grain to
slowly collect energy from the infra-red radiation field and corres­
pondingly heat up. As seen in Figure 1, the absorption efficiency of
the grain increases rapidly with increasing frequency. As the grain heats up, the peak of its Plank radiation curve moves to higher and higher frequencies. Eventually the grain will then come into radiative equilibrium with the infra-red radiation field. This thermostatic mechanism will tend to give the grains a relatively uniform temperature throughout the nebula.

Summing the sources of energy, the equation of radiative equilibrium is written as

\[ \int_0^\infty Q_{ABS}(IR,a) J_v(\tau_v^{IR}) dv + W \int_0^\infty Q_{ABS}(V,a) B_v^*(\tau_v^{V}) dv + \int_0^\infty Q_{ABS}(L\alpha,a) J_x(\tau_d) dx \]

\[ = \int_0^\infty Q_{ABS}(IR,a) B_v(T_g) dv \]  

(1)

where \( Q_{ABS}(IR,a), Q_{ABS}(V,a) \) and \( Q_{ABS}(L\alpha,a) \) are the absorption efficiencies for the infra-red, ultraviolet-visible and Lyman alpha region respectively, \( a \) is the grain radius, \( J_v(\tau_v^{IR}) \) is the mean intensity of the infra-red radiation at an optical depth of \( \tau_v^{IR} \), \( B_v^*(\tau_v^{V}) \) is the central star's continuum intensity at optical depth \( \tau_v^{V} \), \( W \) is the geometrical dilution factor, \( J_x(\tau_d) \) is the mean intensity of Lyman alpha radiation in the wing at dust optical depth \( \tau_d \) and \( B_v(T_g) \) is the black body intensity of the grain which is at a temperature \( T_g \).

The mean intensity of the infra-red radiation is calculated in the plane parallel approximation for a finite atmosphere. Measuring the infra-red optical depth from the outer edge of the nebula, the mean intensity is written as

\[ J_v(\tau_v^{IR}) = \frac{1}{2} B_v(T_o^{IR}) E_2(\tau_v^{IR} - \tau_v^{IR}) + \frac{1}{2} \int_0^{\tau_v^{IR}} B_v(\tau_v^{V}) E_1(\tau_v^{V} - \tau_v^{IR}) d\tau_v \]  

(2)
where $\tau^\text{IR}_0$ is the optical depth at the inner surface of the nebula. $E_1$ and $E_2$ are the first and second exponential integrals.

The intensity of the central star's radiation longward of the Lyman continuum frequency, $\nu_C$, at optical depth $\tau^\nu_V$ is given by

$$R^*_\nu(T^\nu_V) = B^*_\nu(T^*_\nu) e^{-\tau^v_V} \quad (3)$$

In the literature, an extinction efficiency, which is the sum of the absorption efficiency and the scattering efficiency is generally given. The relation between the absorption coefficient needed in equation (1) and the extinction efficiency is

$$Q^\text{ABS}(\nu,a) = Q^\text{EXT}(\nu,a) \left[ 1 - \gamma(\nu,a) \right] \quad (4)$$

where $\gamma(\nu,a)$ is the albedo of the grain at a frequency $\nu$ and a radius a. $Q^\text{EXT}(\nu,a)$ and $\gamma(\nu,a)$ are found from Mie theory as outlined by Van de Hulst (1957). $\gamma(\nu,a)$ for different carbon grain radii, according to Harris (1970), are shown in figure 4. Defining the dimensionless size parameter $x = 2\pi a/\lambda$, where $\lambda$ is the wavelength of the incident light, one can develop asymptotic expansions for $x \ll 1$ (i.e. infra-red region). This so-called Rayleigh approximation for dielectrics is given by

$$Q^\text{ABS}(\text{IR},a) = 4 \times \text{IM} \left[ \frac{\frac{m^2}{2} - \frac{1}{2}}{m^2 + 2} \right] \quad (5)$$
where \( m = n(\lambda) - ik(\lambda) \) is the complex index of refraction of the grain considered. Since the absorption coefficient over the width of the Lyman alpha line is relatively constant, we can take it out of the third integral in equation (1). The evaluation of the integrals are extremely difficult unless several assumptions are made. It was found that for all types of grains a good approximation to the absorption efficiency over a limited frequency interval can be given by

\[
Q(v, a) = Q(v_o, a) \left( \frac{v}{v_o} \right)^n
\]

where \( n \) is not necessarily an integer and the subscript has been dropped on \( Q_{\text{ABS}} \) with the understanding that an unsubscripted \( Q \) refers to \( Q_{\text{ABS}} \). By writing \( Q \) as a power function in frequency, the integrals involving the black body function, upon making the substitution \( y = hv/kT \), take on the general form

\[
\int_{y_1}^{\infty} \frac{y^{n+3}}{e^y - 1} \, dy
\]

For \( y_1 = 0 \), equation (7) is equal to \( \Gamma(n+4) \xi(n+4) \) where \( \Gamma \) represents the Gamma function and \( \xi \) represents the Riemann-Zeta function (See for example Landau and Lifshitz, 1958). For \( y_1 = 0 \) we can expand the denominator in a binomial expansion which always converges. The resulting integral is analytic and can be written most conveniently in the form

\[
\int_{y_1}^{\infty} \frac{y^{n+3}}{e^y - 1} \, dy = \sum_{i=1}^{\infty} \left[ \frac{e^{-iy_1}}{i} \sum_{j=1}^{N+3} \frac{\Gamma(n+4)}{\Gamma(n+5-j)} \frac{y_1^{n+4-j}}{i\cdot(j-1)} \right] + \frac{\Gamma(n+4)}{\Gamma(n-N+1)} \frac{1}{i\cdot N+3} \int_{y_1}^{\infty} y^{N-N} e^{-iy} \, dy
\]
where \( N \) is the truncated integer of \( n \). The integral on the right hand side of equation (8) is computed by a Laguerre quadrature after the suitable Laguerre transformation.

In Table (1) are listed the values of the exponent and normalization factor used in equation (6). The complex index of refraction for graphite given by Werner and Salpeter (1969) for the infra-red, was used in equation (5). The results of this calculation for arbitrary grain radius, measured in microns, is given in Figure (1). The straight lines represent the solution to equation (6). The determination of \( Q \) in the visible and ultraviolet is quite approximate. In Figure (2) the dotted lines are values of \( \log_e [Q(V,a)/a] \) for different grain radii for the visible and ultraviolet region. The data points were obtained from the Mie calculations of \( Q_{\text{EXT}} \) given by Wickramasinghe (1967) in his Appendix I. \( Q \) was found by using equation (4) and the data of Harris (1970). The straight lines in Figure (2) are the approximations to the dotted curves.

Due to the saw tooth pattern, the second integral in equation (1) is evaluated over three successive intervals. \( \log_e [Q(L_{\alpha},a)/a] \) is given in Figure 3 as a function of the grain radius, \( a \), in microns. This curve was found by interpolation in Wickramasinghe's Appendix I of \( Q_{\text{EXT}} \) at the \( L_{\alpha} \) frequency and again using equation (4) to find \( Q \). As the grain radius approaches zero, we are again able to use the Rayleigh approximation since \( x \ll 1 \). From the data of Taft and Philipp (1965), we are able to extend the graph for small values of \( a \). In the limit of \( a \rightarrow 0 \), \( Q(L_{\alpha},a)/a \rightarrow 21 \).
### TABLE I

**GRAPHITE (RADIUS IN MICRONS)**

#### INFRA-RED REGION $Q(\text{IR}, a)$

<table>
<thead>
<tr>
<th>$n$</th>
<th>FREQUENCY RANGE</th>
<th>$Q(a, v_0)/a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.519</td>
<td>$\nu \leq v_0 = 2.477 \times 10^{13}$</td>
<td></td>
</tr>
<tr>
<td>.1102</td>
<td>$\nu &gt; v_0$</td>
<td>.181</td>
</tr>
</tbody>
</table>

#### VISIBLE AND ULTRAVIOLET REGION $Q(\nu, a)$

<table>
<thead>
<tr>
<th>$n$</th>
<th>FREQUENCY RANGE</th>
<th>$Q(a, v_0)/a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.02</td>
<td>$0 \leq \nu \leq \nu_1 = 1.35 \times 10^{15}$</td>
<td>93.69</td>
</tr>
<tr>
<td>-3.83</td>
<td>$\nu_1 &lt; \nu \leq \nu_2 = 2 \times 10^{15}$</td>
<td>20.91</td>
</tr>
<tr>
<td>3.11</td>
<td>$\nu_2 &lt; \nu \leq \nu_c = 3.29 \times 10^{15}$</td>
<td>99.48</td>
</tr>
</tbody>
</table>

#### LYMAN ALPHA LINE $Q(L_{\alpha}, a)$

\[
\frac{Q(L_{\alpha}, a)}{a} = \begin{cases} 
\text{Exp} (87a + 3) & 0 \leq a \leq .016 \\
\text{Exp} (-14.9a + 4.6) & .016 \leq a \leq .11 
\end{cases}
\]
The evaluation of the first integral in equation (1) using equation (2) presents a slight problem due to the singularity of the \( E_1 \) function at the origin. Since the singularity is logarithmic, it was subtracted out so that equation (2) is written

\[
J_v(\tau_v^{\text{IR}}) = \frac{1}{2} B_v(\tau_v^{\text{IR}}) E_2(\tau_v^{\text{IR}} - \tau_v^{\text{IR}}) + \frac{1}{2} B_v(\tau_v^{\text{IR}})[2 - E_2(\tau_v^{\text{IR}}) \nonumber \\
- E_2(\tau_0^{\text{IR}} - \tau_v^{\text{IR}})] + \int_0^{\tau_v^{\text{IR}}} \frac{B_v(\tau_v) - B_v(\tau_v^{\text{IR}})}{E_2 |\tau_v^{\text{IR}} - \tau_v|} \, d\tau_v. \tag{9}
\]

The integral over optical depth was found to be adequately approximated by dividing the slab into a series of smaller slabs, evaluating the kernel at the geometric mean of each slab and summing. The infra-red optical depth of dust is related to the dust's optical depth at the Lyman alpha frequency by

\[
\tau_v^{\text{IR}} = \frac{Q(\text{IR}, a)}{Q(L^\alpha, a)} (\tau_d(R_1) - \tau_d) \tag{10}
\]

where \( \tau_v^{\text{IR}} \) is measured in the opposite direction of that of \( \tau_d \). Thus, at the outer edge of the nebula \( \tau_v^{\text{IR}} = 0 \) and \( \tau_d = \tau_d(R_1) \). The evaluation of the second integral in equation (1) as it stands is difficult due to the optical depth's strong dependency on frequency. This problem is not encountered directly in the third integral because it is explicitly taken into account in the determination of the mean intensity in equation (43). As has been our practice, all optical depths are related to the dust's
optical depth at the Lyman alpha line frequency. Thus, the visual and ultraviolet optical depth in equation (3) is written

$$\tau_v = \frac{Q(v, a)}{Q(L_{\alpha}, a)} \tau_d \tag{11}$$

Since $Q(v, a)$ does not vary by more than 50% over the short frequency intervals which we have chosen, we can eliminate the frequency dependence of $\tau_v$ by defining a Plank mean over each interval

$$< Q(v, a) >_i = \frac{\int_{v_{i-1}}^{v_i} Q(v, a) B_v(T_\star) dv}{\int_{v_{i-1}}^{v_i} B_v(T_\star) dv} \tag{12}$$

These integrals are easily calculated with the help of equation (8). The use of the central star's intensity at the surface of the slab in evaluating $< Q(v, a) >_i$ throughout the slab is basically incorrect. Due to the frequency dependence of $Q(v, a)$ the net radiation at a point in the slab becomes more or less energetic depending on whether the slope of $Q(v, a)$ vs. $v$ is negative or positive respectively. Correspondingly, $< Q(v, a) >_i$ will be smaller or larger depending on whether the slope of its frequency dependence is positive or negative respectively. Thus, equation (3) should be used in equation (12) for $B_v(T_\star)$. This, however, defeats the intended purpose of defining equation (12) for we again end up with exactly the integral in the numerator of equation (12) that we wish to evaluate. We have attempted to make judicious use of the flexibility in the fit of $\text{LOG}_e [Q(v, a)/a]$ to compensate for the energy selection effect on $< Q(v, a) >$. The coarseness of the function that we
are fitting in figure (2) certainly doesn't warrant a meticulous approach to the evaluation of the integral in equation (3).

The equation of radiative equilibrium which must be solved for the grain temperature, $T_g$, is

$$
T_g 4 \left\{ \begin{array}{c}
\int_{x_0}^{x_1} \Gamma(n_1+4) \xi(n_1+4) + \sum_{j=1}^{2} (-1)^j \int_{x_0}^{x_1} \sum_{k=1}^{\infty} \frac{e^{-kx_0}}{k} \\
N_{j+3} \sum_{i=1}^{\Gamma(n_j+4)} \frac{x_0^{n_j+5-1}}{\Gamma(n_j+5-1)} + \frac{\Gamma(n_j+4)}{\Gamma(n_j-N_j+1)_{k+3}} \int_{x_0}^{x_1} x_j^{n_j-N_j} e^{-kx} dx \end{array} \right\}
$$

$$
= f_1(T_g) + f_2(B_v(T^*)_v) + f_3(u) ,
$$

(13)

where

$$
f_1(T_g) = \frac{e^2 h^3}{4 \pi k^4 Q(v_o, a)} \left\{ \int_0^{\infty} Q(IR, a) B_v(\tau_o IR) E_2(\tau_o IR, IR) dv \\
+ \int_0^{\infty} Q(IR, a) B_v(\tau_v IR) \left[ 2 - E_2(\tau_v IR) - E_2(\tau_o IR, IR) \right] dv \\
+ \int_0^{\tau IR} \int_0^{\tau_v IR} Q(IR, a) B_v(\tau_v IR) E_1(\tau_v IR, \tau_v) dv_d \right\}.
$$

(14)

$$
f_2(B_v(T^*)_v) = \frac{1}{4} \left( \frac{R^*_v}{K} \right)^2 \frac{1}{Q(v_o, a)} \sum_{i=1}^{3} \exp \left[ - \frac{<Q,v,a>_i}{Q(L_o,a)} \right] \\
$$

$$
\int_{v_{i-1}}^{v_i} Q(v, a) B_v(T^*)_v dv
$$

(15)
The mean intensity of radiation has been replaced by the energy density according to equation (47) and \( x_q = \frac{h\nu_0}{kT_g} \). Values of \( \eta_1, \eta_2, Q(\nu_0, a) \) and \( Q(L_\alpha, a) \) are all given in table (1). \( Q(V, a) \) is found from equation (6) using the value of the parameters given in table (1) and \( < Q(V, a) >_I \) is found from equation (12). \( R_\star \) and \( R_1 \) are the radius of the star and inner radius of the nebula respectively. Due to the inclusion of self-absorption of the infra-red radiation, equation (13) must be solved iteratively. Initially a uniform temperature of 1000°K was assumed in order to evaluate \( f_1(T_g) \). Then equation (13) was solved by Newton-Raphson for the grain temperature at each point in the nebula. Once the run of temperature with depth was found, equation (14) was evaluated with the new temperatures. This was substituted into equation (13) and the procedure was repeated. This back substitution procedure was found to be fairly efficient. Temperature changes of less than 0.5% over the slab between two successive passes through the equations was found in three or four iterations. The Newton-Raphson was performed to an accuracy of 0.1% between two successive iterations.

We define a mean grain temperature as

\[
< T_g > = \frac{\int_0^{\tau_d(R_2)} \tau_g^4 \langle \tau_d \rangle d\tau_d}{\tau_d(R_2)}
\]  

(17)
It was assumed that the H II was uniformly filled by infra-red radiation that could be characterized by a temperature given by equation (17). Thus, in equation (14), the intensity of the infra-red radiation at $\tau_o^{\text{IR}}$ was assumed to be given by a temperature of $< T_g^4 >^{1/4}$. 
Observations of Seyfert galaxies by Pacholczyk and Weymann (1968), the galactic center by Becklin and Neugebauer (1968) and Aumann and Low (1970), peculiar stars by Stein et al. (1969), and Woolf and Ney (1969) and planetary nebula by Gillett, Low and Stein (1967) have revealed infra-red luminosities equal to or greater than those found from the visible and ultraviolet region. The explanation of thermalized dust radiating in the infra-red for all of these objects seems to be on tenous grounds, particularly for the objects of galactic sizes (see Burbidge and Stein, 1970). Low (1970) has even suggested that the infra-red radiation from galaxies arises from the continuous creation of matter in the nucleus. The infra-red spectrum of the planetary nebula NGC 7027 may, without straining current physics, be explained by thermal emission from particles. Krishna Swamy and O'Dell (1968) have been able to match fairly closely the observed infra-red spectrum of NGC 7027 by emission from graphite flakes through their treatment of the problem was quite schematic.

We have calculated the infra-red flux distribution from the nebula in the plane parallel approximation. Recently Huang (1969a,b) has formulated and solved the transfer problem of infra-red and optical radiation in a spherical dust envelope for all optical depths. It was found, however, that the results given by his more general spherical treatment did not differ significantly from that given by the slab approximation - especially when the width of the slab was small compared with its distance from the central star and when it was optically thin in the infra-red. This is, of course, the criteria for the slab approximation to be valid.
For a finite atmosphere with a total optical depth in the infra-red of $\tau_0^{\text{IR}}$, the emergent flux at frequency $v$ in the plane parallel approximation is given by

$$\pi F_v(\tau_o) = 2\pi I_v(\tau_o^{\text{IR}})E_3(\tau_o^{\text{IR}}) + 2\pi \int_0^{\tau_o^{\text{IR}}} B_v(t_v)E_2(t_v)dt_v$$ (13)

where $E_1(x)$ and $E_2(x)$ are the third and second exponential integrals.

$I_v(\tau_o^{\text{IR}})$ is the infra-red radiation incident on the inner surfaces of the slab. If the nebula is optically thin in the infra-red then $E_3(\tau_o^{\text{IR}}) \approx .5$ and $E_2(\tau_o) \approx 1$.

Rather than construct appropriate quadrature formulas for the evaluation of equation (13), it was found sufficiently accurate to divide the slab up into a series of smaller slabs of optical depth $\Delta \tau_d$. A geometric mean was used to evaluate the grain temperature and optical depth between two successive slabs. The integral in equation (13) then reduces to a simple sum over optical depth. The luminosity in the infra-red is found by integrating equation (13) over all frequencies.
GRAIN DESTRUCTION

Two methods of grain destruction were considered: Evaporation and sputtering by heavy ions. A third possibility for grain destruction, as pointed out by Dexter (1964) and Mathews (1967), is optical erosion. Since little is known about this effect, either experimentally or theoretically, it has been neglected.

Physically it is envisioned that the radiation field impinging on the grain located in the ionization front, can be divided into two parts: high and low energy. The high energy photon is able to photo-dissociate an electron from the grain. This process does not heat up the grain since the excess energy goes into kinetic energy of the electron. The low energy photon hasn't enough energy to overcome the work function of the grain and its energy goes into heating the grain. If the work function of the grain is large enough, the grain will eventually heat up to a point where it will start to liquify and evaporate. When the radiation field is large or the work function is small, the grain will tend to become positively charged due to the net loss of electrons. Since protons are the main source of sputtering, they will be repelled by the positively charged grain if the grains are spherical and the charge is distributed uniformly around them. This, of course, will be impossible if the grains are pure dielectrics. If there are some impurities in a dielectric grain, the electrons will be able to move along any axis in the lattice structure. For graphite, where electrons are only free to move between the hexagonal shaped layers, an impurity is needed for them to be able to migrate perpendicular to the layers so that
the resulting grain will be uniformly charged over the surface. The assumption of spherical grains will then tend to minimize the rate of grain destruction. Moving further away from the high energy radiation source, while holding the work function constant, will decrease the rate of electron loss from photoelectric ejection. Inasmuch as electrons, as well as protons, are colliding with the grain, a neutrally charged grain would be expected to exist at some point in the nebula. Still further out, where the radiation field is further diluted, a grain would be expected to build up a net negative charge which would enhance grain destruction.

Using the procedure outlined by Spitzer (1948) and Mathews (1967) we can estimate the charge on the grain determined both by photoelectric ejection and by absorption of electrons and photons which strike the surface. The steady state charge in the grains is found from the solution of

\[ \xi_p e^\gamma + \Theta = \xi_e \left( \frac{m_H}{m_e} \right)^{1/2} (1 - \gamma) \quad \gamma < 0 \]

\[ (1 + \gamma) \xi_p + \Theta = \xi_e \left( \frac{m_H}{m_e} \right)^{1/2} e^{-\gamma} \quad \gamma > 0 \]  

(15)

where \( m_H \) and \( m_e \) are the proton mass and electron mass respectively, and \( \gamma \) is determined by the electrostatic potential of the grain \( V \)

\[ \gamma = -\frac{eV}{kT_g} = \frac{Z g e^2}{a kT_g} \]  

(16)

\( Z_g \) is the grain charge and \( a \) is the grain radius. \( \gamma \) will be positive if the grain is negatively charged and negative if it is positively charged.
$\xi_e$ and $\xi_p$ are the respective probabilities that an electron or proton incident on the grain will adhere and add to the charge. $\Theta$ is the ratio of photoelectrons emitted per second to the number of protons striking an uncharged grain per second. The latter quantity is found by setting $\gamma = 0$ and $\xi_e = 1$.

$\Theta$ is given by

$$\Theta = 1.0906 \times 10^6 \frac{\varphi}{N_{\text{p}} T_{\text{p}}} \left( \frac{R_{\text{s}}}{R_{\text{l}}} \right)^2 T_{\star}^3 \sum_{i=1}^{\infty} \frac{e^{-x_{T}}}{i^3} \left[ 1 + 2 i x_{T} + 2 i x_{T} + 2 \right] $$

where

$$x_{T} = 4.8 \times 10^{-11} \nu \nu_{0} / T_{\star} - \gamma $$

$\varphi$ is the efficiency of photoelectric ejection. From the data of Taft and Phillipp (1965) on graphite, we see that both the complex and real part of the index of refraction go to zero at about 500 Å. This implies that the albedo of graphite goes to unity with a strong forward scattering phase function. Since $\varphi$ appears to be relatively uniform for a large variety of materials (Mathews 1967), we use $\varphi = 0.2$ electrons per photon as a working value. $h\nu_{0}$ is the ionization potential of the most abundant element making up the grain. Substituting equation (17) into equation (15) yields a transcendental equation in $\gamma$ which can be solved by Newton-Raphson iteration for given values of $\xi_e$ and $\xi_p$. Spitzer (1948) estimated for dielectrics that $\xi_e \approx 0.2$. For $\xi_p$ he suggests values near unity so, like Mathews (1967), we have chosen $\xi_p = 1$.

The rate at which the grain radius decreases due to sputtering by protons is given by Mathews (1969)
\[
\frac{da}{dt} = -\frac{2\eta s M_s N_p (kT_p)^{3/2}}{(2\pi m_p)^{1/2} \rho_g} F(\kappa, \gamma, \alpha),
\]
(19)

\[
F(\kappa, \gamma, \alpha) = \begin{cases} 
\frac{1}{2\beta} \left[ (\kappa+3) e^{\gamma-\alpha} - (\kappa+3) e^{-\kappa \alpha} \right] & \kappa > 1 \\
\frac{1}{2} [2 + \alpha] e^{-\alpha} & \kappa = 1 
\end{cases}
\] (20)

and

\[
\alpha = \frac{L_v}{\eta kT_p}, \quad \beta = \alpha(\kappa-1)
\] (21)

\(M_s\) is the mass ejected from the grain per incident proton and for graphite is 12 \(M_{\text{AMU}}\). \(s=2.5\) an empirical constant, \(\eta = 4M_s M_H/(M_s + M_H)^2\), \(L_v = 2.6 \times 10^{22}\) ev/gm = .47 ev the heat of sublimation of the grain and \(N_p, T_p\) is the number density and the kinetic temperature of protons respectively. \(\rho_g = 2.2\) the grains density and \(\kappa\) is a dimensionless number between 8 and 20.

The rate of change of the grain radius due to evaporation is given by (Landau and Lipshitz 1958)

\[
\frac{da}{dt} = \frac{1}{\rho_g} \left( \frac{M_c}{2\pi k} \right)^{1/2} \frac{P_{\text{SAT}}(T_g)}{T_g^{1/2}}
\] (22)

where \(M_c\) = mass of the evaporated atom, \(T_g\) = the grain temperature and \(P_{\text{SAT}}(T_g)\) is the saturated vapor pressure. Using the vapor pressure of graphite tabulated by Wickramasinghe (1967), we fit a polynomial to \(T_g\) in \(\log_{10} P_{\text{SAT}}(T_g)\).
\[ \log_{10} P_{\text{SAT}}(T_g) = 39.13 + 24.62 \times 10^{-3} T_g - 3.66 \times 10^{-6} T_g^2 \]

\[(T_g > 1500^\circ K) \tag{23} \]

If the grain temperature was less than 1500^\circ K we set \( \frac{da}{dt} \bigg|_V = 0 \). The total rate of change of the grain radius is then the sum of equations (19) and (22). The time scale for grain destruction is then defined in the usual way.

\[ t_d = \frac{a}{|\frac{da}{dt}|} \tag{24} \]
CHAPTER II
THE TRANSFER PROBLEM

The solution to the general intego-differential equation describing the transfer of a $L_\alpha$ photon, with incomplete redistribution, through a moving spherical medium has, as of yet, not been found. To espy some solution to the problem, approximations are usually made in the integro-differential equation which tends to negate most of the interesting physics of the diffusion process. Until relatively recently, only problems involving static plane parallel geometry with complete redistribution in the core of the line have been considered. The use of large computers have, however, allowed Monte Carlo calculations to be made on a more general form of the problem. Auer (1968), using the Monte Carlo method, considered a static plane parallel geometry with partial redistribution in the line. His paper is extremely interesting, not necessarily because of his results (which do not differ significantly from earlier treatments of the problem by other authors), but by reason of his use of an extremely powerful mathematical tool to solve a rather idealistic problem. Thus he implicitly shows that the use of the equation of transfer to solve more complicated problems is, in all likelihood, a fruitless endeavor.

To determine the role of $L_\alpha$ radiation in the overall dynamics of a nebula, one would like to have the solution to the above mentioned equation. Since this has not been possible, authors in the past have been forced into making many simplifying assumptions. It has been found over the last fifty years that the role of $L_\alpha$ radiation in the dynamics
of nebulae is a direct function of the assumptions made. Zanstra (1939, 1949), solving a simple equation of transfer, showed that $L_\alpha$ pressure would quickly destroy any nebula if one assumed that the $L_\alpha$ photons were scattered coherently. When Zanstra (1949) assumed that the $L_\alpha$ photon scattered with complete redistribution, he found that the $L_\alpha$ pressure was only slightly larger than the pressure due to the Lyman continuum. Actually the redistribution function lies between these two extremes. Unno (1952) attempted to solve the case of partial redistribution with pure Doppler broadening in the Eddington approximation. His method requires many steps in frequency space in order to cover the entire line. The apparent last word in calculations of this type are by Auer (1968) whose work was discussed earlier. All of the above mentioned work was done for a static configuration.

The more complex problem of line formation in a moving atmosphere was first discussed by McCrea and Mitra (1936). Much of the work in this area has been devoted to computing absorption-line profiles in plane parallel moving atmospheres. Rottenburg (1952), however, has calculated emission-line profiles in spherical expanding envelopes with an arbitrary velocity gradient of the type found around P Cygni and Be stars. In his treatment of the scattering problem, he implicitly assumes that all scattering is coherent in the atom's rest frame. The expansion of the nebula enables the scattered photon to diffuse in frequency space due to the differential velocity between two successive emissions. A very general treatment of the problem of moving atmospheres has recently been completed by Magnan (1968, 1970). He traces the history of 1000 photons by Monte Carlo methods in a spherical atmosphere moving with a
constant radial velocity. Magnan also assumed complete redistribution in the scattering and that there are no Lorentz wings but only a Doppler core. None of the above papers evaluate the work the scattered photon does on the expanding atmosphere before it is completely absorbed or escapes in the Doppler wing.

Kahn (1968), reorganizing the ineptness of the transfer equation in handling the overall problem, utilized a classic physics phenomenon: the reflection of light from a moving mirror. This effect, first discussed in detail by Plank (1959), enables the frequency of the incident photon to change, due to the Doppler effect, upon reflections from a moving surface. In this manner, Kahn is able to consider only those photons that have diffused far enough in frequency space so that they can then diffuse in physical space. Thus, rather than doing a detailed analysis of the transfer problem over the entire line to determine the $L_\alpha$ pressure, he needs only to consider the relatively simple problem of the radiation pressure of photons in the natural damping wing.

Consider a hot central star, radiating as a black body, embedded in an H II region which is in turn surrounded by an expanding shell of neutral gas with a slight admixture of dust. The radiation from the central star travels through the H II region unattenuated till it reaches the ionization front. The radiation shortward of the Lyman limit will ionize the neutral hydrogen and charge the grains while the radiation longward of the limit will pass through the front and be absorbed by the dust. (see discussion on page 8). Under the conditions of Baker and Menzel Case B, every Lyman continuum photon will eventually give rise a $L_\alpha$ photon. Nearly all $L_\alpha$ photons will be created
in or near to the center of the core of the line. Assuming that the H I region is moving at speed \( R \) with respect to the central star, an observer on the star would see a photon distribution whose mean frequency is red shifted from the rest frequency \( \nu_\alpha \) by an amount \( Rv_\alpha/c \). The photon, in the reference system of the moving front, will simply scatter in the core of the line, with partial redistribution in frequency, till it finds itself in the relatively optically thin violet or red wing. Those photons in the wings will have a higher probability of escaping the ionization front if they are directed back into the H II region, unless they are in the far wings where it is optically thin in any direction.

Consider the photon in the violet wing first. This photon is able to transverse the H II region unimpeded till it gets to the ionization front on the other side of the nebula. To an observer in this opposite front, the first front is moving at a velocity of approximately \( 2R \). A photon will be in, or slightly biased, to the red side of the line center in the second reference frame when, relative to the first reference frame, it is in the violet wing.

Thus the photon will be absorbed in the core and will undergo repeated scatterings in this second reference frame until it again ends up in the wing and escapes the front. By the same reasoning, the photon in the red wing in the first reference system will appear even redder in the second reference system. If it is absorbed in the second reference system, it will be re-emitted coherently, since it is in the wing of the line (Osterbrock, 1962). Again, unless they are in the far wing, the photons will escape the ionization front if they are scattered back into the H II region. Thus we have a net transfer of photons into the
red side of the resonance line with very few escaping on the violet
side of the line. Eventually the photon will be able to pass through
the ionization front and into the neutral H I region where it will be
absorbed or scattered by the dust. By summing over the entire nebula,
an observer at rest would see a L line with a slow fall off on the red
side and a truncated violet side. We have made no assumption about the
shape of the core, only that it has a finite width in frequency space
and a high optical depth within this frequency interval. The entire
process described above can be represented schematically as a moving
mirror from which the L photons are repeatedly reflected.

Measuring frequencies with respect to the rest frequency of the
line, $v_{\text{rest}}$, the rate of change of the frequency shift for a nebula of
radius $R$, expanding at a velocity of $\dot{R}$, is

$$\dot{x} = v_{\alpha} \frac{\dot{R}}{R}$$

(25)

The rate of change of the energy density of $L_{\alpha}$ due to this "red-shift"
mechanism is found to be

$$\frac{\partial u_{x}}{\partial t} = - v_{\alpha} \frac{\dot{R}}{R} \frac{\partial u_{x}}{\partial x}$$

(26)

Once the photon has ended up far enough in the wing it will start to
diffuse into the H I region as already mentioned. Let us define this
critical frequency when the photon is considered to be in the line as
$x_m$. In practice, $x_m$ is chosen to be four Doppler widths from the line
center. At frequencies less than $x_m$ the H II region will have a uniform
mean intensity. A sphere of radius $R$ will contain $4\pi R^3 u_{x} / 3$ ergs per
unit frequency interval. The rate of leakage of the photons out of
this sphere is seen to be
\[ \frac{4\pi}{3} R^3 \frac{\partial u_x}{\partial t} = -4\pi R^2 F_x(R) \]  
(27)

where \( F_x(R) \) is the flux passing through the surface \( 4\pi R^2 \). Equating equations (26) and (27), we have a relation between the flux into the H I region and the change in the energy density with respect to the frequency.

\[ \nu \frac{R}{R} \frac{\partial u_x}{\partial x} + \frac{3}{R} F_x(R) = 0 \]  
(28)

\( F_x(R) \) represents the flux in the Lorentz wing of the line and can be found by simply considering the equation of transfer in the wings of the line. We assume that the H I region is thin enough that we may write the transfer equations in the plane parallel approximation.

Taking account of the destruction of \( L_\alpha \) by dust and assuming that scattering in the wing is coherent, the equation of transfer becomes

\[ \frac{dI_x}{dr} = -a_x N_H I_x - a_d N_d I_x + \frac{a_x N_H}{2} \int_{-1}^{1} I_x \, d\mu \]  
(29)

where \( \mu = \cos \theta \), \( N_H \) and \( N_d \) are the number density of hydrogen atoms and dust particles per cubic centimeter respectively and \( a_x \) and \( a_d \) are the atomic absorption coefficients of hydrogen in the wing, and dust respectively. Since the damping constant for \( T \approx 10^4 \) \( a_x \) is so small for \( L_\alpha \) the atomic absorption coefficients can be given by

\[ a_x = \frac{a_0}{x^2} \]  
(30)
where \( a_o = 7.705 \times 10^5 \text{ cm}^2/\text{sec}^2 \).

Defining the usual moments for the mean intensity, flux and pressure

\[
J_x = \frac{1}{4\pi} \int \omega I_x d\omega, \quad F_x = \int \omega \mu I_x d\omega, \quad K_x = \frac{1}{4\pi} \int \omega \mu^2 I_x d\omega
\]  

(31)

the first moment of the transfer equation becomes

\[
\frac{dF_x}{dr} = -4\pi a o N_d J_x
\]  

(32)

and the second moment is

\[
\frac{dK_x}{dr} = -\frac{(a H N_H + a d N_d)}{4\pi} F_x
\]  

(33)

At this point it is advantageous to introduce an optical depth due to dust

\[
d\tau_d = a d N_d dr
\]  

(34)

Let us require that the dust and hydrogen density vary through the nebula such that their ratio is a constant. Define the variable \( \eta_x \), which is independent of \( r \), as

\[
\eta_x = \frac{N_H a_x}{N_d a_d}
\]  

(35)

Also introducing the standard Eddington approximation, \( 3K_x = J_x \), into equation (33), we now have
Differentiating equation (37) and substituting equation (36), we arrive at the second order differential equation in the mean intensity

\[ \frac{d^2 j_x}{d \tau_d^2} = 3(1 + \eta_x) j_x \] (38)

The solution of equation (38) is easily written down

\[ j_x(\tau_d) = A e^{\alpha(x) \tau_d} + B e^{-\alpha(x) \tau_d} \] (39)

where we have defined

\[ \alpha(x) = [3(1 + \eta_x)]^{1/2} \] (40)

To find the constants A and B in equation (39), we use the boundary conditions at the inner surface of the nebula ($\tau_d = 0$)

\[ u_x(0) = \frac{4\pi}{c} j_x(0) \] (41)

and at the outer surface of the nebula ($\tau_d = \tau_o$) the emergent radiation is independent of angle and there is no radiation incident on the outer
surface. These latter conditions lead to the outer boundary condition

$$F_x(\tau_o) = 2\pi J_x(\tau_o)$$

(42)

Substituting equations (41) and (42) into equation (39) and using equation (37) for $F_x(\tau_o)$, we end up with two equations to be solved for the unknowns $A$ and $B$. Solving for $A$ and $B$, equation (39) becomes

$$c u_x(\tau) = \frac{\alpha(x)(\tau_o - \tau_d) - \alpha(x)(\tau_o - \tau_d)}{e^{\alpha(x)\tau_o - \xi e}}$$

(43)

where $\alpha(x)$ is given by equation (40) and

$$\xi = \frac{[3(1 + \eta_x)]^{1/2} - 2}{[3(1 + \eta_x)]^{1/2} + 2}$$

(44)

The energy density at optical depth $\tau_d$ is found from the relation

$$u_x(\tau_d) = \frac{4\pi}{c} J_x(\tau_d)$$

(45)

and the flux is given by

$$F_x(\tau_d) = \frac{c u_x(\tau) - \alpha(x)(\tau_o - \tau_d) - \alpha(x)(\tau_o - \tau_d)}{e^{\alpha(x)\tau_o - \xi e} - \alpha(x)\tau_o}$$

(46)
Substituting equation (46) into equation (28), we find that the change in energy density as a function of frequency is given by

$$\frac{du_x}{dx} = -\frac{e}{v_x^2} \left( \frac{3}{1+n_x} \right)^{1/2} u_x(o) \left[ \frac{e^{\alpha(x)}(\tau_0 - \tau_d) + \xi e^{-\alpha(x)}(\tau_0 - \tau_d)}{e^{\alpha(x)}\tau_0 - \xi e^{-\alpha(x)}\tau_0} \right]$$

(47)

Rather than solve equation (47) for all $\tau_d$ we solve it only for the interior surface where $\tau_d = 0$. We then move into the coordinate system of the slab and use equations (43) and (45) to find $u_x(\tau_d)$. Equation (47) was solved by Simpson's rule.

To see if our equations reduce to those of Kahn (1968) who did not consider dust in the H I region, we let $N_d \to 0$. Then

$$\lim_{N_d \to 0} \alpha(x)(\tau_0 - \tau_d) = \left[ 3(1 + \frac{N_H a_x}{N_d a_d}) \right]^{1/2} \left[ \int_{R_1}^{R_f} a_{d} N_d \, dr - \int_{R_1}^{r} a_{d} N_d \, dr \right]$$

$$\approx \left[ \frac{3N_H a_x}{N_d a_d} \right]^{1/2} N_d a_d (R_f - r) \approx (3N_H a_x N_d a_d)^{1/2} \Delta R$$

The exponent then becomes for small arguments

$$e^{\alpha(x)}(\tau_0 - \tau_d) \approx 1 + (3N_H a_x N_d a_d)^{1/2} \Delta R$$

(48)

Substituting (48) into equation (46) and reducing we find that
which is identical to Kahn's equation (20). Using the same expansions in equation (47) we find upon some reduction that as \( N_d \to 0 \)

\[
\lim_{N_d \to o} \frac{du_x(N_d \to o)}{dx} = \frac{c u_x(o)x^2}{Rv_\alpha a o N_H (R_x - R_1)}
\]

this equation can be integrated directly. Assuming that there is no thermal motion, as does Kahn, so that \( x_m = o \), we have

\[
u_x \propto \exp \left( - \frac{c}{3v_\alpha R a o \sigma} \sigma x^3 \right)
\]

where \( \sigma = N_H(R_x - R_1) = \text{surface number density} \). This is the same as Kahn's equation (26) representing the frequency distribution of the trapped photons in the red side of the line.

Since equation (47) deals only with the wing of the \( L_\alpha \) line, we must find the non-zero lower limit to the frequency integral. Equating the absorption coefficient due to Doppler broadening to the natural broadening coefficient, we find the outer frequency root at 3.33 Doppler widths for a thermal velocity of 20 km./sec. Thus the lower limit to the frequency integral is chosen to be \( x_m = 1 \times 10^{12} \) Hertz.

Equation (47) has an analytic solution for small values of the frequency. Setting \( N_H / N_d = 10^{11} \) and \( a_d = 2 \times 10^{-11} \) corresponding to
a grain radius of $0.02\mu$ (see equation A5) we find that we must have \[ x^2 \ll 3 \times 10^{27} \] for us to write equation (52) in the asymptotic form,

\[
\frac{du_x(\tau_d)}{dx} = -u_x(o) A x e^{-\frac{B}{x}}
\]  

(49)

where

\[
A = \frac{3}{\nu \alpha R \left( \frac{N_H a_o}{N_d a_d} \right)^{1/2}}, \quad B = \left( \frac{3N_H a_o}{N_a a_d} \right)^{1/2} \tau_d
\]  

(50)

For $\tau_d = o$, $B = o$ and the solution to equation (49) is then found to be

\[
u_x(o) = u_x_m(o) e^{-\frac{A}{2} (x^2 - x_m^2)}
\]  

(51)

Substituting equation (51) into equation (49) we have

\[
\frac{du_x(\tau_d)}{dx} = -u_x(o) A x e^{-\frac{B}{x}} e^{-\frac{A}{2} (x^2 - x_m^2)}
\]  

(52)

By expanding $\exp(-Ax^2/2)$ in an infinite series, the integral in equation (52) becomes analytic. After some algebra, the solution to equation (52) is
Equation (53) was used to check the integration of equation (47) for \( \tau_d = 0 \) and the subsequent use of equation (45) for \( \tau_d = 0 \).

An interesting calculation is the amount of energy that the L\(\alpha \) photon gives to the expanding H I region before the photon is destroyed on a grain. Following Kahn (1968) the mean energy, and thus the mean frequency, a L\(\alpha \) photon has when it escapes either by diffusion or destruction by the dust is

\[
\overline{x} = \frac{\int_{x_m}^{\infty} x \cdot F_x(x) \, dx}{\int_{x_m}^{\infty} F_x(x) \, dx} \quad (54)
\]

The energy given to the H I region is then

\[
\Delta E = \frac{\overline{x}}{\nu_{\alpha}} \quad (55)
\]
CHAPTER III
DYNAMICS

The role of Lyman alpha radiation in the overall dynamics of nebulae has always been a point of contention. The pioneering work of Ambarzumian (1932) and Zanstra (1934) indicated that $L_\alpha$ pressure is the dominant driving mechanism in nebulae. They assumed, however, that the $L_\alpha$ photon scattered coherently throughout the line. Later work by Zanstra (1949) and Sobolev (1960), using a complete redistribution function, indicated that the $L_\alpha$ pressure was not much greater than that due to the pressure in the Lyman continuum. Subsequent work on the dynamics of H II regions and planetary nebulae, by a variety of authors, has neglected the effect of $L_\alpha$ pressure. Hjellming (1966) gives an excellent review of the work done up to 1966. Khromov (1964) attempted to evaluate the amount of momentum transferred to a neutral shell of hydrogen and helium gas surrounding an H II region by gas pressure, Lyman continuum pressure and $L_\alpha$ pressure. He uses a relatively crude approximation to determine the mean lifetime of a $L_\alpha$ photon in the nebula before it escapes. Due to the relatively low trapping times that he found, he concludes that $L_\alpha$ pressure is never a dominant mechanism in the dynamics of the nebula. Capriotti (1967) using a more exact treatment of the transfer problem than did Khromov, determined the average $L_\alpha$ radiation density in planetary nebulae. He found that the ratio of $L_\alpha$ pressure to gas pressure was of the order $1/2$ for objects that have radii of approximately .07 pc. (O'Dell, 1962). Kahn (1968) using the same type of model as proposed by Khromov (1964)
calculated that the Lyman alpha radiation field is the dominant force in driving early high density nebulae. He calculates that the radiation pressure of $L_{\alpha}$ will dominate until $P(L_{\alpha})/P(\text{gas}) \geq 1$, which occurs when $\dot{R} \approx 20 \text{ km./sec at } R \approx 0.08 \text{ pc}$. Kahn did not consider the dynamical effect of dust embedded in the H I region. In general dust destroys the $L_{\alpha}$ photon before it can diffuse through the H I which effectively decreases $x$ (equation 54), the mean value of $x$ with which a photon escapes. This in turn decreases the energy given to the expanding H I shell. The inclusion of dust weakens the role of $L_{\alpha}$ in the overall dynamical problem.

The possible mechanisms for ejecting the shell from the central star will not be discussed here. We wish only to say that some process, perhaps the absorption of resonance lines radiation described by Luce and Solomon (1970), is responsible for ejection approximately $1 - 1 \text{ M}_\odot$ from the star. It is assumed that the shell expands spherically about the central stars and rapidly cools into a homogeneous mixture of hydrogen and, if conditions are suitable, dust grains. The nebula quickly becomes ionization bounded with either a strong R- or D-type or a weak R- or D-type ionization front in the nomenclature of Kahn (1954). Density perturbations, which propagate at the sound speed in the shell, are assumed to have damped out by the time the shell is about $10^{-3} \text{ pc}$ from the central star. This assumption, of course, will not be true if the ionization front is either a weak D-type or a strong R-type. This must be checked by the models. The thin shell, which we will treat as a slab, is assumed to move as a unit. Thus the Lagrangian derivative of the velocity of mass points in the slab will be zero.
The general equation of motion of the fluid in the slab is

\[ \rho_T \frac{dv}{dt} = -\frac{dp}{dr} + f_B \] (56)

where \( \frac{dp}{dr} \) is the pressure gradient at point \( r \), \( f_B \) is the body force of the slab and \( \rho_T \) is the density of the slab and is the sum of the hydrogen and dust density. The body force can be thought of as a pressure gradient of the slab acting in the opposite direction of the external pressure gradient and is given by

\[ f_B = \frac{\partial}{\partial r} [ c_1^2(r) \rho_T^2(r) ] \] (57)

where \( c_1(r) \) is the velocity of sound in the H I region at point \( r \). The velocity of sound will be a function of position in the H I region since the grains and the gas will have reached thermal equilibrium. The relaxation time for this process is quite rapid for a high density medium as pointed out by Spitzer (1949) and Spitzer and Savedoff (1950). Under the conditions that the Lagrangian derivative is equal to zero, then equation (56) becomes

\[ \frac{\partial (c_1^2 \rho_T)}{\partial r} = \frac{\partial p}{\partial r} \] (58)

The pressure gradient is considered to be a sum of two terms, the volume force due to the \( L_\alpha \) radiation field and the stellar radiation field, evaluated at a point \( r \) in the slab. Integration of equation (58) allows us to construct a density profile such that the condition \( \frac{dv}{dt} = 0 \) is automatically satisfied. The density at
point $r$ is

$$c_1^2(r) \rho_T(r) = c_1^2(R_1) \rho_T(R_1) + \int_{R_1}^r \frac{\partial P}{\partial r} \, dr$$

(59)

The volume force due to the continuum radiation from the central star is

$$\frac{\partial P}{\partial r} = \frac{\rho_T W}{c} \int_0^\infty \kappa_v^c \pi B_v(r) \, dv$$

(60)

and from $L_\alpha$

$$\frac{\partial P}{\partial r} = \frac{\rho_T}{c} \int_{x_m}^\infty \kappa_v^\alpha F_\alpha(x) \, dx$$

(61)

where $W = \frac{1}{4} \left( \frac{R_*/R_1}{c} \right)^2$ is the dilution factor. $\kappa_v^c$ and $\kappa_v^\alpha$ contain the sum of the absorption coefficients of dust and hydrogen over the stellar continuum and Lyman alpha line respectively. Hydrogen will not absorb frequencies lower than the Lyman limit. A simple calculation shows that the ratio of the dust absorption coefficient to the hydrogen absorption coefficient is about .02 at the Lyman limit. We thus assume that hydrogen absorbs all the stellar radiation below the Lyman limit and the dust absorbs all the radiation above the Lyman limit.

$$\kappa_v^c = \kappa_H^c (\nu \geq \nu_c) + \kappa_d^c (\nu < \nu_c)$$

(62)
In the frequency range of the Lyman alpha line, the absorption coefficient is

$$\kappa_\nu^\alpha = \kappa_\nu^H + \kappa_\nu^\alpha$$

(63)

The sound speed in an ideal gas is

$$c_1^2 = \frac{kT}{m_H}$$

(64)

Equation (59) becomes

$$c_1^2(r)\rho_T(r) = c_1^2(R_1)\rho_T(R_1) + \frac{1}{c} \left( 1 + \frac{\rho_H}{\rho_d} \right) \int_{R_1}^r \left\{ W \int_0^\nu c \rho_d \kappa_d^\nu B_v(r) dv \right. \left. + \int_{x_m}^\infty \rho_d \kappa_H^\alpha + \rho_d \kappa_d^\alpha \right\} F_x(r) dx \right\} + \frac{W}{c} \int_{R_1}^r \int_{x_m}^\infty \rho_d \kappa_H^\alpha \tau_B \nu^c r_
u(r) dv dx$$

(65)

where we have taken the ratio of densities out of the integrals since it is assumed constant throughout the nebula. It is advantageous to convert to the optical depth of dust at $L_\alpha$ at this point and perform the optical depth integration first. Equation (65) is then written

$$c_1^2(\tau_d)\rho_T(\tau_d) = c_1^2(o)\rho_T(o) + \frac{1}{c} \left( \frac{\rho_H}{\rho_d} + 1 \right) \left\{ W \int_0^\nu c \int_\nu^\infty \nu^c B_v(o) e^{-\tau_v} dv \int_0^t \kappa_H^\alpha \int_0^\nu c \nu^c B_v(o) e^{-\tau_v} dv \right\}$$

$$+ \int_{x_m}^\infty \kappa_d^\alpha \int_{x_m}^\infty \kappa_d^\alpha \int_0^\tau d F_x(t_d) dt_d dx \right\}$$

$$+ \frac{W}{c} \left( \frac{\rho_d}{\rho_H} + 1 \right) \int_{x_m}^\infty \kappa_c^\alpha \int_\infty^\nu c \nu^c B_v(o) e^{-\tau_c} dt_c dv$$

(66)
where \( \tau_\nu \) is given by equation (11) and

\[
\tau_c = \frac{\rho_H \kappa_H^c}{\rho_d \kappa_d} \tau_d
\]  

(67)

\( \tau_d \) is given by equation (34). The evaluation of the second and third integrals in equation (66) is achieved with the help of equations (46), (41), and (43). We find

\[
\int_0^{\tau_d} P_x(t_d) dt_d = \frac{c}{3(1 + \eta_x)} [u_x(o) - u_x(\tau_d)]
\]  

(68)

where \( \eta_x \) is given by equation (35). In order to evaluate the integrals over frequency in equation (66) several simplifying approximations were made. The dependence of the optical depth on frequency in equations (69) and (68) is eliminated by taking Plankian averages over the appropriate interval. As mentioned in the discussion after equation (12), the first integral in equation (66) must be divided into three parts. Defining an average optical depth over each of the three intervals, we are then able to take the optical depth factor out of the integration over frequency. The resultant integral is then easily evaluated by the use of equation (8). The Plankian average of the Lyman continuum mass absorption coefficient of hydrogen is

\[
< \kappa_H^c > = \frac{\int \kappa_H^c B_\nu(T_x) dv}{\int B_\nu(T_x) dv}
\]  

(69)
From Allen (1963) we can write \( \kappa_c = \kappa_0 / \nu^3 \) where \( \kappa_0 = 1.498 \times 10^{53} \text{ cm}^2 / \text{gm} \).

With the help of equation (8), equation (69) is easily evaluated and found to be

\[
< \kappa_H^c > = \frac{h^3 \kappa_0}{(k T_\infty)^3} \sum_{i=1}^{\infty} \frac{e^{-ixc}}{i} \left[ x_c^3 + \frac{3}{2} x_c^2 + \frac{6}{3} x_c + \frac{6}{4} \right]
\]

where \( x_c = \nu / k T_\infty = 1.578 \times 10^5 / T_\infty \).

The evaluation of equation (68) over frequency is facilitated by considering the definition of \( \eta_x \) and using the condition that in general \( \rho_d \kappa^\alpha_d \ll \rho_H \kappa^\alpha_H \). Thus to a good approximation we can write

\[
\frac{1}{3} \frac{\kappa_H^\alpha}{\kappa_d^\alpha} \left( 1 + \frac{\rho_H \kappa_H^\alpha}{\rho_d \kappa_d^\alpha} \right) \approx \frac{c}{3} \frac{\rho_d}{\rho_H}
\]

The integral over frequency of the energy density is done by trapezoidal rule. Using the same order of approximation in the third integral in equation (66), results in the relation

\[
\frac{c}{3} \frac{\rho_d}{\rho_H} \kappa_d^\alpha \int_{x_m}^{\infty} \frac{1}{\kappa_H^\alpha} [u_x(a) - u_x(\tau_d)] \, dx
\]

Equation (66) then becomes
\[ c_1^2(\tau_d) \rho_T(\tau_d) = c_1^2(0) \rho_T(0) + \frac{W}{c} \left( \frac{\rho_H}{\rho_d} + 1 \right) \sum_{i=1}^{3} \left( 1 - \exp \left( - \frac{<Q(v,a)>_i}{Q_d(a)} \right) \right) \]

\[ \ldots \int_{v_i}^{v_{i+1}} B_v(T_\star) \, dv + \left( \frac{\rho_d}{\rho_H} + 1 \right) \left\{ 2 \left[ 1 - \exp \left( - \frac{<Q(v,a)>_i}{Q_d(a)} \right) \right] \right\} \]

where the unsubscripted variables imply integration over frequency.

Given the total mass of the H I region, \( M_T \), the total optical thickness of the nebula is

\[ \tau_0 = \frac{\kappa^\alpha_d M_T}{4\pi R_1^2 \left( 1 + \frac{\rho_H}{\rho_d} \right)} \]

where \( \kappa^\alpha_d \) is found from \((A6)\) in the Appendix.

Construction of a density profile given in equation (73) allows us to treat the dynamics of the H I region as a quasi-hydrostatic problem. The net effect of the radiation and gas pressure is felt only on the inner edge of the nebula. As the nebula evolves, \( \rho_T(\tau_d) \) will gradually change with time. We have assumed, however, that the density will change slowly and uniformly with time and thus there are no large scale density perturbations in the nebula.
The equation of motion of the HI region then becomes simply

\[ \frac{d(M_{\text{HII}} \dot{R})}{dt} = 4\pi R_{\text{HII}}^2 (P_{\text{GAS}} + P_{r} + P_{\text{LY\alpha}}) \]  

(75)

The gas pressure is given by

\[ P_{\text{GAS}} = 2 N_{H} k T_{\text{HII}} \]  

(76)

We assume that the stellar radiation is all in the radial direction so that the pressure on the perfectly absorbing slab is

\[ P_{r} = \frac{4M}{c} \int_{0}^{\infty} \pi B_{\nu}(T_{\star}) \, d\nu = W u(T_{\star}) \]  

(77)

\( W \) is the dilution factor and \( \frac{4}{c} \int_{0}^{\infty} \pi B_{\nu}(T_{\star}) \, d\nu = 7.564 \times 10^{-15} T_{\star}^{4} \text{(ERGS/CM)}^{3} \).

The Lyman alpha pressure is nearly isotropic over the hemisphere encompassing the HI region so that

\[ P_{\text{LY\alpha}} = \frac{u(0)}{3} \]  

(78)

If there is no stellar wind, then the only mass entering the HII region will be that which flows through the ionization front. If the star luminosity remains constant then the mass flow rate through the front is

\[ \dot{M}_{\text{HII}} = \frac{3}{2} \frac{\dot{R}}{R} M_{\text{HII}} \]  

(79)

Equation (75) becomes upon substituting equations (76-79)
Equation (80) is integrated by Adam's method (Hildebrand, 1968) to second order. This method, however, requires the value of equation (80) at two previous time points. Initially then, the procedure is started with Euler's methods and a first order Adams scheme. Writing equation (80) as

\[
\frac{d\dot{R}}{dt} = f(\dot{R}, t) \tag{81}
\]

the integral value of \( \dot{R} \) is

\[
\dot{R}_2 = \dot{R}_1 + \Delta t \left[ \frac{f_1}{3} \right]
\]

\[
\dot{R}_3 = \dot{R}_2 + \Delta t \left[ \frac{3}{2} f_2 - \frac{1}{2} f_1 \right] \tag{82}
\]

\[
\dot{R}_{k+1} = \dot{R}_k + \Delta t \left[ \frac{23}{12} f_k - \frac{4}{3} f_{k-1} + \frac{5}{12} f_{k-2} \right] \quad k \geq 3
\]

where the abbreviation \( f_k = f(\dot{R}_k, t_k) \) and \( \dot{R}_k = \dot{R}(t_k) \). The time step was taken

\[
\Delta t = \text{MAX} \left[ 100 \text{ yr.}, \frac{a}{|da/dt|} \right] \tag{83}
\]

where \( a/|da/dt| \) is found from equation (24).
The inner radius is found by

$$R(t_{k+1}) = R(t_k) + \dot{R}(t_{k+1}) \Delta t$$

and correspondingly the total mass of the $\text{HI}$ region

$$M_{\text{HI}}(t_{k+1}) = M_{\text{HI}}(t_k) - M(t_{k+1}) \Delta t$$
CHAPTER IV
Thermal Stabilities

The condensations that are observed throughout the universe have been the subject of considerable speculation. The globules in planetary nebulae are particularly disturbing in that they appear to have been formed in a hot, diffuse gas. Field (1965) developed a general theory of thermal instabilities in which he showed that the criteria used by previous authors was incorrect. Using Fields criteria, Sofia (1966) examined the thermal stability in planetary nebulae under a wide range of physical conditions. In all cases, he found that the thermal equilibrium was stable. Harrington (1968), reconsidered the problem again using model atmospheres for the central stars and calculating the ionization stratification in the planetary nebulae. He also had negative results.

Capriotti (1970) has investigated the stability of a gas-dust mixture in the neutral regions of young, ionization bounded planetary nebulae. These conditions are exactly the same as are being investigated in this dissertation. He found that the dust is able to act as a sink for radiation so that low temperatures are found in the neutral gas. The low temperature, coupled with the high density in the early phases of the nebula, enables the Jean's criteria for gravitational instabilities to be met. By reason of the interesting results of Capriotti, we have checked the Jean's instability criteria throughout the evolution of the nebulae.

The critical wavelength $\lambda_o$ for which a mass will go gravitationally unstable is given by Jeans (1929) as
Assuming that the pressure is related adiabatically to the density so that \( p \propto \rho^{\gamma} \) and that the pressure of the gas is given by

\[
p = N_H k T
\]

we can now write the Jean's length as

\[
\lambda_o^2 = \frac{\pi}{\gamma \rho} \left( \frac{dp}{d\rho} \right)
\]

(86)

\( \lambda_o^2 \) can be calculated at any point in the nebula. When \( \lambda_o \) becomes substantially less than the dimensions of the medium, a gravitationally unstable mass will occur. The free-fall time is given by

\[
t_{\lambda_o} = (G \rho_H)^{-1/2}
\]

(89)

where \( G = 6.668 \times 10^{-8} \text{ DYNES CM}^2/\text{GM}^2 \).
PART II

THE MODELS CONSTRUCTED
CHAPTER V
Defining Parameters of the Models

In order to construct a particular model, it is necessary to specify the following quantities:

1) The temperature and radius of the central star given by \( T_* \) and \( R_* \) respectively.
2) The mass of the nebula, \( M_{\text{HI}} \).
3) The inner radius of the nebula, \( R_I \).
4) The number density of the nebular gas, \( N_H \).
5) The radius of the grains, \( a \).
6) The abundance, relative to hydrogen, of the element making up the grain, \( A_c \).

The temperature and radius of the central star were assumed to remain constant during the evolution of the nebula. A representative temperature and radius of \( 10^5 \) °K (Capriotti and Kovach 1968) and \( 1R_0 \) — for Abell type objects (O'Dell, 1963) — respectively were used in all the models constructed.

The initial inner radius of the nebula was chosen to be \( 10^{15} \) cm. At this distance the central star can be treated as a point source and the width of the nebula will be less than 3% of the distance to the central star. Since this percentage decreases as the nebula expands, we are justified in treating it as a slab.

We have used NGC 7027 as the prototype object in choosing the other physical parameters. The mean density in the H II region was chosen to be \( 2 \times 10^4 \) particles/cm³. This value is midway between the values quoted for NGC 7027 by O'Dell and Terzian (1970). The electron temperature in this region, as given by Kaler (1970), is \( 1.5 \times 10^4 \) °K.
Using the solution to the steady one-dimensional ionization front, the compression across the ionization front can be determined in terms of the isothermal sound speeds in the H I and H II regions. The ionization fronts that we are considering tend toward D-critical type (Spitzer 1968). Under these conditions, the ratio of gas densities between the H I and H II region is approximately 100. Thus, to an order of magnitude approximation, the density in the inner edge of the H I region is $2 \times 10^6$ atoms/cm$^3$. This is approximately the value used by Kahn (1968) in his analysis, and is used as an initial value in all of our models.

The grain radius was chosen to be $0.02\mu$. This choice was based on several factors. We have assumed that the grains are composed almost entirely of carbon. Due to the abundance of carbon relative to hydrogen, this requirement puts a definite upper limit on the number of grains and their size. If one is willing to accept that a representative value of the dust to gas number density ratio is of the order of $10^{-11}$ and a maximum carbon to hydrogen abundance of the order of $10^{-4}$ is contained in the grains, then the resulting maximum grain radius for a pure carbon grain is of the order of $0.02\mu$. Most likely the grain will contain other metals, as pointed out by Krishna Swamy and O'Dell (1967), and will correspondingly have radii of the order of $1\mu$. A radius of $0.02\mu$ for graphite grains was found to give the best fit to the infra-red observations of Gillett, Low and Stein (1967) as determined by Krishna Swamy and O'Dell (1968).

The free parameter in the models were chosen to be the mass of the nebula and the abundance of carbon locked up in the grains.
Table 2 lists the defining parameters for the 5 models made. Using a grain radius of 0.02\(\mu\), the abundances given in Table 2 correspond to a dust grain to hydrogen number density ratio of \(1.1 \times 10^{-11}\) for models (1) and (3) and \(5.5 \times 10^{-13}\) for models (2), (4) and (5).

All the models were terminated when the gas pressure in the H II region became larger than the Lyman alpha pressure.
<table>
<thead>
<tr>
<th>MODEL</th>
<th>$M_{\text{HI}}$ (GM.)</th>
<th>$A_{\text{c}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1 \times 10^{33}$</td>
<td>$1 \times 10^{-4}$</td>
</tr>
<tr>
<td>2</td>
<td>$1 \times 10^{33}$</td>
<td>$5 \times 10^{-6}$</td>
</tr>
<tr>
<td>3</td>
<td>$2 \times 10^{32}$</td>
<td>$1 \times 10^{-4}$</td>
</tr>
<tr>
<td>4</td>
<td>$2 \times 10^{32}$</td>
<td>$5 \times 10^{-6}$</td>
</tr>
<tr>
<td>5</td>
<td>$1 \times 10^{32}$</td>
<td>$5 \times 10^{-6}$</td>
</tr>
</tbody>
</table>
CHAPTER VI
Results and Discussion

The life time of the grains vs. time is plotted in figure 9. Since equation (24) is a function only of the conditions in the H II region, which we have assumed to be the same for all models, only one curve is given. It appears that grains of the type that were considered could not exist in the H II region for more than 100 years. The implicit assumption - that $L_\alpha$ is not destroyed in the H II region by grains - made in the transfer chapter, is generally valid. The existence of grains in the H II regions, as inferred by Krishna Swamy and O'Dell (1967), may come about if they exist in condensations. The condensations, formed in the neutral gas-dust region, would be gradually eroded away in the H II region by the high energy radiation and hot gas. In this manner, grains could be found in all parts of the H II region. The grain may become quite stable, as pointed out by Mathews (1967), due to a large charge build up which would retard sputtering.

The logarithm of the total optical depth of the dust at the frequency of Lyman alpha is plotted against time in figure 5. The numbers at the end of each line correspond to the model number. All the models are optically thick initially and all but model No. 1 become optically thin when $P_{gas} > P_L$. Near the end of the evolution of models Nos. 2, 4 and 5, .5%, 40% and 60% of the incident $L_\alpha$ flux was able to diffuse to the outer edge of the H I region respectively. Essentially, none of the $L_\alpha$ radiation escaped during the time considered for models No. 1 and 3.
One negative result of this study appears to be the inability of Lα to accelerate a 0.5 M⊙ nebula containing dust up to speeds of 20 km./sec. The velocity of expansion as a function of time are plotted for the 5 models in figure 7. Model No. 3, which is a 0.1 M⊙ nebula with a high dust density, is able to achieve a velocity of only 11 km./sec. before Pgas > PL. Model No. 5 was constructed to see if a low mass-low dust density nebula could achieve a high velocity. This model was able to reach a velocity of 26 km./sec. in approximately 1200 years. Model No. 4, another low dust density model, was able to reach velocities of the order of 20 km./sec. in 1700 years. The value of x, given by equation (54), is plotted as a function of time for the 5 models in figure 8. x is about 10 times less for models No. 1 and 3 than the value of 1 x 10^13 given by Kahn (1968) for a pure hydrogen nebula. For models No. 2, 4 and 5 there is less than a factor of two difference. Since x is a measure of the work done by the Lα photon on expanding the nebula, we can see why x is so small. The dust in models No. 1 and 3, which are high dust density nebulae, destroys the Lα radiation before it can accelerate the H I region to any appreciable speed. Once the dust density is sufficiently depleted, as in models No. 2, 4 and 5, the Lα radiation is able to do more work, before it is absorbed by the dust, and accelerate the H I region to observable velocities in about 10^{11} sec. This is essentially the result found by Kahn (1968) for his pure hydrogen nebula. Thus, one must conclude that if there is any sink of the magnitude of dust for Lα radiation, Lα will not be an important contribution to the dynamics of nebulae.
The fourth root of equation (17) is plotted in figure 6 as a function of time. Figure 6 shows that for about 50% of the time period that we have considered the mean grain temperature for all models lies in the 200-300°K range. It is in this range of temperatures that previous workers such as Krishna Swamy and O'Dell (1968), have attempted to draw black body curves to fit the observational data. Their choice of a 200 °K black body curve to fit the data points of Gillett, Low and Stein (1967) is shown in figure 31 as the dotted line.

We will discuss each model in turn. Five basic plots are made for each of the five models. The grain temperature, hydrogen number density, Jean's length and Jean's time are all plotted as a function of the fractional physical depth into the nebula, \((R-R_i)/(R_F-R_i)\) where \(R_i\) and \(R_F\) are the radius of the inner and outer surface of the nebula. The infra-red flux distribution is plotted as a function of frequency and is normalized to the observations of Gillett, Low and Stein (1967) of NGC 7027 at \(\log_{10} v_o = 13.4\). The numbers labeling each curve in the five plots refers to the time elapsed in years since the start of the model.

Model No. 1. This model has a very large optical thickness during most of its evolution. The temperature profile for various times during the nebula evolution are given in figure 10. Due to the large optical thickness of the dust, most of the \(L_\alpha\) and stellar radiation is absorbed within 5% of the inner surface so that 95% of the nebula is heated by the infra-red radiation. The early infra-red flux profiles shown in figure 11 resemble that of the NML Cygnus type stars. At later epochs
the nebula becomes cooler and the infra-red flux becomes redder. During
the latter part of the nebula's evolution, the infra-red flux profile
is able to match the observational points of Gillett and al. (1967)
quite easily. The maximum flux goes from 9.5\mu to 19\mu in 3000 years.

The density profiles given in figure 12 are quite uniform over
most of the nebula. This is due to the relatively uniform temperature
over most of the nebula. Because the density is so large, the Jean's
length becomes much smaller than the dimensions of the nebula. A plot
of \((R-R_p)A_o\) is given in figure 13. Over most of it's evolution, the
nebula is unstable against forming condensations. In figure 14 is a
plot of the logarithm of the Jean's time for various epochs. We see,
however, that only for the first 2000 years is the contraction time less
than 100 years. Thus, even if the nebula is unstable at later times,
the contraction time is so long that the condensation would be easily
disrupted in the H II region.

Since \(\Delta R A_o\) is a measure of the ability to form condensation in
the amount of remaining nebula, a large value for this quantity in-
dicates that the nebula will tend to fragment into subsidiary conden-
sations at a distance apart of the order of \(A_o\). The low value for the
contraction time will allow these small condensations to become quite
stable before the H I region moves past them and they are exposed to
the destructive environment of the H II region. Assuming the validity
of this model, one would expect to see small condensations along the
inner edge of ionization bounded nebula. Of course, once a condensation
forms, the density of the medium will change. This will affect the
transfer of the radiation in the H I region and the dynamics to such an
extent that the model will be generally invalid beyond this time.

**Model No. 2.** Due to the lower optical depth in this model, the temperature profiles shown for various times in figure 15 do not change as abruptly as they did in model No. 1. For epochs greater than 2000 years, the temperatures are very uniform and vary less than 30% between the inner and outer edge. This, however, gives a greater variation in the infra-red flux distribution shown in figure 16. The maximums of the flux profile vary from 3.8\(\mu\) to 30\(\mu\) in about 3500 years. In this low density model, the infra-red flux distribution at about 1000 years matches the observational data of NGC 7027. It is interesting to note that due to the long fall off of the red side of the distribution, the flux at 7.5\(\mu\) is equal to that at 48\(\mu\).

The density profiles shown in figure 17, also vary considerably from those of model No. 1. Initially the density is very high and uniform over the H I region but in the latter stages of the nebula's evolution the density changes slowly between the inner and outer edge. Due to this slow density change, the nebula is stable against condensations for more than half its evolutionary life as shown in figure 18. Only during the early phases can condensations form. As shown in figure 19, however, the Jean's contraction time quickly rises so that at the 500 year epoch the contraction time becomes larger than 100 years.

**Model No. 3.** This is a high dust density, low total mass nebula. The temperature profiles shown in figure 20, show large variations due to the rapid change in optical depth during the nebula's evolution. The maximum infra-red flux, shown in figure 21, goes from 4.8\(\mu\) to 24\(\mu\) in about 2000 years. During the latter stages in the nebula's evolution,
the flux profiles are relatively constant. There does not appear to be a very good fit between the observational data and the computed profiles. The violet sides of the distribution are too steep relative to the observations indicating that there is an insufficient number of high temperature grains in this model in order to match the data.

The density profiles shown in figure 22 are quite similar to those of model No. 2 but with a faster temporal variation, the Jean's gravitational criteria is satisfied only in the first 500 years. The rapid increase in velocity of model No. 3, shown in figure 7, coupled with the long Jean's contraction times, given in figure 24, would probably inhibit the permanent formation of any condensations before they become immersed in the H II region.

Model No. 4. This low dust density, low total mass nebula is interesting because of the rapid rate in which the temperature profile becomes uniform over the width of the nebula as shown in figure 25. The infra-red flux distribution, shown in figure 26, changes quite rapidly with time with the maximum flux going from 3.8μ to 30μ in only 1700 years. The infra-red profile at 816 years matches the observed points of NGC 7027 quite well. The flux at 95μ is equal to that at 4.8μ indicating that the distribution is quite flat out into the far infra-red.

The density profiles change considerably over the width of the nebula as shown in figure 27. Only during the first several hundred years would a uniform density approximation be valid. This model is gravitationally unstable only during the initial phases. Even then, though, it is not as unstable as models No. 1 and 2. The Jean's
length and contraction time are shown in figures 28 and 29 respectively.

Model No. 5. This low dust density, very low total mass is interesting in the fact that the $L_\alpha$ radiation can accelerate it up to speeds near 30 km./sec. as shown in figure 7. Due to the very low optical depth of this model, the temperature profiles, given in figure 30, are quite uniform. In the latter stages of the evolution, the nebula is so optically thin that only near the outer edge is a significant amount of radiation absorbed. This is seen by the "turning down" of the temperature profiles for epochs greater than 800 years. Before this time, most of the $L_\alpha$ and ultraviolet radiation had been significantly absorbed in the nebula so that the infra-red radiation was heating the outer layers. Due to the lack of high temperature dust grains, the infra-red flux profiles, given in figure 31, do not match the observational points of NGC 7027. The dotted line is a 200 °K black body curve normalized to Gillett, Low and Stein's (1967) observations.

The density profiles shown in figure 12, are similar to those in model No. 4 only with a faster temporal variation. The Jean's length and time are shown in figures 33 and 34 respectively. This model is marginally unstable against condensations only during the first 100 years or so.

In conclusion, we would like to point out that the agreement between the infra-red flux distribution calculated in some of the models and the observations doesn't necessarily mean that the grains are spherical and made of graphite. The distributions are mainly the results of the grain's efficiency to radiate in the infra-red. In general, most types of grains will have the same general characteristics in their
absorption efficiency in the infra-red. We can say, however, that grains with an absorption efficiency like that in figure 1 in the infra-red, when heated by \( L_\alpha \) radiation, ultraviolet radiation from the central star and infra-red radiation, can quite easily explain the emission from NGC 7027.
APPENDIX I

Consider a sheet of atoms arranged in a hexagonal pattern. The number of atoms in a sheet of length $L$ and width $W$ is

$$N_a = 4(m-1) + 2(m+1)(n-1) + 6 \quad \text{(A1)}$$

where

$$m = \frac{L}{1} \quad \text{number of bond length 1 along length } L$$

$$n = \frac{W}{\frac{3}{2}1} \quad \text{number of 1 1/2 bond lengths in a width } W$$

Assume that a crystal is constructed so that the hexagonal sheets form planes with separations of $\Delta h$. Then for a sphere of radius $a$ we have that $L = W = 2a$ and $m = 2a/1$, $n = 4a/3$. Equation (A1) then becomes

$$N_a = \frac{4}{3}a \left( 5 + 4 \frac{a}{1} \right) \quad \text{(A2)}$$

By considering the sphere to be inscribed by a cube of side $2a$, we find the total number of atoms in this perfect crystal to be

$$N_T = \frac{4\pi}{9} \left( \frac{a}{1\Delta h} \right)^2 \left( 5 + 4 \frac{a}{1} \right) \quad \text{(A3)}$$

For graphite $1 = 1.421 \ \text{Å}$ and $\Delta h = 2.46 \ \text{Å}$. The total mass of the grain is given by

$$m_g = M_e N_T \quad \text{(A4)}$$
where $M_c$ is the mass of each atom. For carbon $M_c = 2 \times 10^{-23}$ gm.

The mean absorption coefficient of the grain is

$$\kappa_g = \frac{\pi a^2 Q_v(a)}{m_g}$$  \hspace{1cm} (A5)

where $Q_v(a)$ is the absorption efficiency. Using equations (A4) and (A3), we have for (A5)

$$\kappa_g = \frac{9 \Delta h Q_v(a)}{4 M_c \left( 5 + 4 \frac{a}{l} \right)}$$  \hspace{1cm} (A6)
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Fig. 1 - Absorption efficiency divided by the grain radius in microns for graphite in the infra-red region. Dotted curve is theoretically determined and solid curve is an empirical fit.
Fig. 2 - Absorption efficiency divided by the grain radius in microns for graphite in the visible-ultraviolet region. Dotted curves are from Mie calculations and solid curve is an empirical fit.
Fig. 3 - Absorption efficiency divided by the grain radius in microns for graphite at the Lyman alpha frequency. Dotted curve from Mie theory and solid curve is an empirical fit.
Fig. 4 - Albedo for graphite from Harris (1970). The grain radius in microns is given for each of the curves.
Fig. 5 - The optical depth due to dust at the frequency of Lyman alpha for each model during its evolution.
Fig. 6 - The fourth root of equation 17 for each model during its evolution.
Fig. 7 - The expansion velocity of each model during its evolution.
Fig. 8 - The mean frequency shift of H\alpha when it escapes the H I region for each model during it's evolution.
Fig. 9 - The life time of graphite grains in the H II regions.
Fig. 10 - The temperature of the H I region for model No. 1. The time in years labels each graph.
Fig. 11 - The emergent infra-red distribution for model No. 1. The time in years labels each graph.
Fig. 12 - The hydrogen number density for model No. 1. The time in years labels each graph.
Fig. 13 - Distance remaining to the outer edge of the H I region divided by the Jean's length for model No. 1. The time in years labels each graph.
Fig. 14 - The Jean's contraction time for model No. 1. The time in years labels each graph.
Fig. 15 - The temperature of the H I region for model No. 2. The time in years labels each graph.
Fig. 16 - The emergent infra-red flux distribution for model No. 2. The time in years labels each graph.
Fig. 17 - The hydrogen number density for model No. 2. The time in years labels each graph.
Fig. 18 - Distance remaining to the outer edge of the H I region divided by the Jean's length for model No. 2. The time in years labels each graph.
Fig. 19 - The Jean's contraction time for model No. 2. The time in years labels each graph.
Fig. 20 - The temperature of the H I region for model No. 3. The time in years labels each graph.
Fig. 21 - The emergent infra-red flux distribution for model No. 3. The time in years labels each graph.
Fig. 22 - The hydrogen number density for model No. 3. The time in years labels each graph.
Fig. 23 - Distance remaining to the outer edge of the H I region divided by the Jean's length for model No. 3. The time in years labels each graph.
Fig. 24 - The Jean's contraction time for model No. 3. The time in years labels each graph.
Fig. 25 - The temperature of the H I region for model No. 4. The time in years labels each graph.
Fig. 26 - The emergent infra-red flux distribution for model No. 4. The time in years labels each graph.
Fig. 27 - The hydrogen number density for model No. 4. The time in years labels each graph.
Fig. 28 - Distance remaining to the outer edge of the H I region divided by the Jean's length for model No. 4. The time in years labels each graph.

Fig. 29 - The Jean's contraction time for model No. 4. The time in years labels each graph.
Fig. 30 - The temperature of the H I region for model No. 5. The time in years labels each graph.
Fig. 31 - The emergent infra-red flux distribution for model No. 5. The time in years labels each graph. The dotted line is a 200°K black body curve.
Fig. 32 - The hydrogen number density for model No. 5. The time in years labels each graph.
Fig. 33 - Distance remaining to the outer edge of the H I region divided by the Jean's length for model No. 5. The time in years labels each graph.

Fig. 34 - The Jean's contraction time for model No. 5. The time in years labels each graph.