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VIBRATIONS OF CIRCULAR CYLINDRICAL SHELLS UNDER
SPACE-VARYING INITIAL STRESSES AND BODY FORCES

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate School
of The Ohio State University

by

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1970

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Studies in Mathematics: Professor S. Drobot
ACKNOWLEDGMENTS

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Thanks are also due Miss Lindsay Hauser and Miss Patricia Murdock for their painstaking typing effort and to Mr. Jeffrey Rippel for his assistance in programming for the computer.

Free usage of the computer facilities was provided by the Computer Center of The Ohio State University.

Finally, to my wife Shashi, I owe my deep gratitude for her unfailing patience and understanding during the many cheerless moments of this undertaking.
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Values of

$(\gamma)$ Obtained with Tangential Inertias Neglected
$(\gamma)$ Obtained with Tangential Inertias Included

For Various Lobar Modes and Lengths of the Shell
Number of Terms Retained in Equation (54) = 3, $h/a = 0.001$, $\nu = 0.3$, $n_X^o = n_X^1 = n_S^o = 0$

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Table 17

Values of

$(\gamma)$ Obtained with Tangential Inertias Neglected
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Values of the Frequency Parameter
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Values of the Frequency Parameter
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CHAPTER I
INTRODUCTION

A large number of papers have appeared in the literature in recent years dealing with the free vibrations of shells, probably on account of the wide-spread applications that shells have found in aeronautics and astronautics, and in civil, mechanical, naval and other types of engineering structures. An extremely exhaustive compilation and digest of these is now under preparation (reference 1). Therein it appears that prestressed circular cylindrical shells have come under fairly intensive study. A survey of the scope of these papers dealing with the vibrations of prestressed shells appears in Chapter II.

But attention in the cases surveyed has largely been confined to studies where the initial or prestresses are constant. Time-varying stresses lead to problems in dynamic stability and are not considered in the present work. However, a spatial variation in the prestress may arise in a plate or a shell due to a variety of loadings, those due to gravity or, more generally, body forces (references 2, 3) or thermal loads (references 4, 5, 6, 7) being perhaps the simplest to realize. For instance, non-uniform hoop compression in the cylindrical shell arises when it is differentially heated along its length or in the case of non-uniform geometric constraints against thermal expansion.
Sucn cases of prestress give rise to additional complication causing non-constant coefficients in the coupled eighth order system of partial differential equations.

The differing equations being advanced by authors as defining the "true" behavior of shells is a further cause for bewilderment and needs a numerical clarification of their ranges of applicability from an engineering standpoint. Studies in this area have almost exclusively been limited to cases of no initial stress (References 8 and 9).

The scope of this investigation is to obtain solutions to some practical cases of free vibrations of circular cylindrical shells under non-uniform prestress using the Donnell theory on account of its applicability and simplicity. A critical examination has been made of the effect of the various parameters of the shell (thickness, length, bending stiffness, and restraint, neglect or retention of tangential inertia and orthotropy) on the vibrational frequencies. A numerical comparison of the Donnell theory with some more exact theories is limited to one specific problem. The Galerkin method has usually been employed in the analysis.

Chapter II contains a survey of available literature on the free vibrations of cylindrical shells under prestress. In Chapter III the theory and the method of analysis for the vibration problem in cylindrical shells is developed. The case of a cylindrical shell under gravity loading along its axis is treated in Chapter IV. The Donnell equation obtained with the neglect of the
in-plane inertias has generally been used. Two types of boundary conditions (viz., clamped-clamped and freely supported-freely supported ends) have been considered. In Chapter V the case of the circumferential stress varying in the axial direction is considered. The Donnell theory is compared with the Sanders and Washizu theories in this case. Chapter VI treats the case of the circumferential stress varying in the axial direction in addition to constant axial forces.
CHAPTER II
A SURVEY OF LITERATURE

A survey of literature on the free vibrations of elastic, complete circular cylindrical shells under initial loading is laid out in Tables 1 through 7. The papers have been grouped according to the types of initial stresses present and in the following order:
1. Shell subjected to uniform axial stress only.
2. Shell subjected to uniform circumferential stress only.
3. Shell subjected to a combination of uniform axial and circumferential stresses arising due to internal pressure only.
4. Shell subjected to an arbitrary combination of uniform axial and circumferential stresses.
5. Shell subjected to uniform torsional stress.
6. Shell subjected to an arbitrary prescribed combination of uniform axial, circumferential and torsional stresses.
7. Shell subjected to other types of loadings.

Roughly, the above classification coincides with the historical sequence of first attempts at obtaining solutions. Within each table the references have been alphabetically arranged according to the name of the author. Each reference has been further categorized by:
a) the type of end conditions assumed
b) the theory/equations used in the analysis
c) the numerical method of analysis
d) the type of material
e) the retention or neglect of in-plane or tangential inertias in the equations of motion
f) the modal distribution studied numerically

A majority of the papers deal with shells with freely-supported ends which are decidedly the simplest to deal with. It is also interesting to note that virtually no solutions exist for shells with one or both ends free. Solutions for shells with discontinuous supports are also non-existent.

A large number of shell theories have been used in the analyses but the Donnell equations with the Batdorf modification have been the most widely used ones. Some discrepancies were occasionally found between the equation actually used and the theory referred to by the reference and these have been pointed out.

The material of the shell is typified by either its being isotropic, orthotropic or anisotropic. For orthotropic materials the lines of orthotropy are always taken coincident with the principal axes of the shell. Orthotropy arising due to closely spaced stiffening elements is designated "structural orthotropy".

The subdivision regarding the modal distribution studied is somewhat a loose one. The letters m and n indicate the modal numbers in the axial and circumferential directions respectively (m being
the number of half-waves in the axial direction and \( n \) being the number of waves in the circumferential direction) and, when followed by an arrow only, signify that results have been obtained extensively.
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<th>Type of Analysis</th>
<th>Type of Material</th>
<th>Tangential Inertia</th>
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<td>Bozich (ref. 10)</td>
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<td>Exact</td>
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<td>General</td>
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<tr>
<td>Brogan (ref. 11)</td>
<td>Simply-supported, $w = \frac{\partial^2 w}{\partial x^2} = 0$</td>
<td>Donnell</td>
<td>Exact; only $w$ displacement possible i.e., $w = w(x,t)$</td>
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<tr>
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<td>Simply-supported ends $w = \frac{\partial^2 w}{\partial x^2} = 0$</td>
<td>Donnell</td>
<td>Exact; Axisymmetric</td>
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<td>$m \rightarrow n = 0$</td>
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<tr>
<td>Ivan'yuta and Finkel'shteyn (ref. 13)</td>
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<td></td>
<td>( n = 1 \rightarrow 15 )</td>
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<td>Nikulin (ref. 17)</td>
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<td>Donnell</td>
<td>Exact; Yu's approximation ( (\lambda/a)^2 ) neglected in comparison with ( 1 )</td>
<td>Isotropic</td>
<td>Neglected</td>
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<td>Galerkin (1 x 1) Axisymmetric</td>
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<td>Type of Material</td>
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<td>Modal Distribution</td>
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<tr>
<td>Bleich and Baron</td>
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<td>Membrane theory</td>
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<tr>
<td>Koval</td>
<td>a) Simply supported ends in w</td>
<td>Donnell (Batdorf)</td>
<td>Exact (Yu's approximation of $\left(\frac{\lambda}{a}\right)^2 \ll 1$)</td>
<td>Isotropic</td>
<td>Neglected</td>
<td>General n</td>
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<td>b) Clamped at both ends</td>
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<td>Kukudzhanov</td>
<td>Freely supported</td>
<td>Authors (refers to Nikulin) uncoupled 8th order equation in w</td>
<td>Exact</td>
<td>Isotropic</td>
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<td>General n</td>
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<tr>
<td>Lyons, Russell and Herrmann</td>
<td>A shell reinforced with periodically spaced rigid ribs or rings which are immoveable (lateral motion and rotation = 0)</td>
<td>Authors (claim it is Herrmann– Armenakas theory but it is not; initial stress terms agree with Flugge except equation 1)</td>
<td>Exact; No Results</td>
<td>Isotropic</td>
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<td>(ref. 24)</td>
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<td>Cottis (ref. 25)</td>
<td>Simply supported</td>
<td>Donnell-Batdorf</td>
<td>Green's method</td>
<td>Isotropic</td>
<td>Neglected</td>
<td>General</td>
</tr>
<tr>
<td>Coupy (ref. 26)</td>
<td>Freely supported at both ends</td>
<td>Author's (coincides with Timoshenko except in $L_{23}$ (eq. 25))</td>
<td>Exact (Yu's approximation)</td>
<td>Isotropic</td>
<td>Included; Predominantly longitudinal, tangential and radial modes investigated</td>
<td>General</td>
</tr>
<tr>
<td>DiGiovanni and Dugundji (ref. 27)</td>
<td>Freely supported at both ends</td>
<td>Washizu</td>
<td>Exact</td>
<td>Orthotropic</td>
<td>Included</td>
<td>General</td>
</tr>
<tr>
<td>Fung, Sechler and Kaplan (ref. 28)</td>
<td>Freely supported at both ends</td>
<td>a) Timoshenko, b) Donnell-Batdorf</td>
<td>Exact</td>
<td>Isotropic</td>
<td>a) Included, b) Neglected</td>
<td>General</td>
</tr>
<tr>
<td>Fung (ref. 29)</td>
<td>Freely supported at both ends</td>
<td>Timoshenko</td>
<td>Exact</td>
<td>Isotropic</td>
<td>a) Included, b) Neglected</td>
<td>General $n = 2 \rightarrow 10$ $m = 1, 2, 3$</td>
</tr>
<tr>
<td>Gottenberg (ref. 30)</td>
<td>Free - free</td>
<td>Experimental</td>
<td>Experimental</td>
<td>Isotropic</td>
<td>-</td>
<td>$m = 1 \rightarrow 7$ $n = 0 \rightarrow 20$</td>
</tr>
<tr>
<td>Author(s)</td>
<td>Type of End Support</td>
<td>Theory Used</td>
<td>Type of Analysis</td>
<td>Type of Material</td>
<td>Tangential Inertia</td>
<td>Modal Distribution</td>
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<tr>
<td>Greenspon (ref. 31)</td>
<td>Freely supported at both ends</td>
<td>Flügge</td>
<td>Exact</td>
<td>Isotropic</td>
<td>Included</td>
<td>General</td>
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<tr>
<td>Ivan'yuta and Finkel'shteyn (ref. 32)</td>
<td>a)Infinitely long b)Freely supported at both ends</td>
<td>1. Vlasov 2. Donnell (equation 26)</td>
<td>i)Exact ii)Galerkin applied to (3 x 3) Marguerre's equation in two variables</td>
<td>Isotropic</td>
<td>a)Included b)Neglected</td>
<td>General</td>
</tr>
<tr>
<td>Livanov (ref. 33)</td>
<td>Freely supported at both ends</td>
<td>Mushtari-Vlasov (unstressed part coincides with Donnell)</td>
<td>Exact Axisymmetric, infinitely long cylinder; higher modes also investigated</td>
<td>Isotropic</td>
<td>Included</td>
<td>General</td>
</tr>
<tr>
<td>Miserentino and Vosteen (ref. 34)</td>
<td>Clamped at both ends</td>
<td>Experimental</td>
<td>Results compared with those obtained from Marguerre's theory</td>
<td>Isotropic</td>
<td>-</td>
<td>General</td>
</tr>
<tr>
<td>Author(s)</td>
<td>Type of End Support</td>
<td>Theory Used</td>
<td>Type of Analysis</td>
<td>Type of Material</td>
<td>Tangential Inertia</td>
<td>Modal Distribution</td>
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<tr>
<td>Öry, Hornung and Fahlbusch</td>
<td>All types</td>
<td>Force-displacement relations</td>
<td>Matrix formulation</td>
<td>Orthotropic</td>
<td>Included</td>
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<tr>
<td></td>
<td></td>
<td>model that of Flügge and Schnell</td>
<td>longitudinal bending stiffness</td>
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<td></td>
<td></td>
<td></td>
<td>neglected. No numerical results.</td>
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<td>(5 x 5) matrices formed</td>
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<tr>
<td>Rakhimov</td>
<td>i) Freely supported at both</td>
<td>Nikulin</td>
<td>Exact</td>
<td>Structurally</td>
<td>Included</td>
<td>General n</td>
</tr>
<tr>
<td>(ref. 36)</td>
<td>ends</td>
<td></td>
<td></td>
<td>Orthotropic</td>
<td></td>
<td></td>
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<tr>
<td></td>
<td>ii) Freely supported-rigidly</td>
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<tr>
<td></td>
<td>supported</td>
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<tr>
<td>Reissner</td>
<td>Freely supported at both</td>
<td>Membrane theory</td>
<td>Exact</td>
<td>Isotropic</td>
<td>Included</td>
<td>General</td>
</tr>
<tr>
<td>(ref. 37)</td>
<td>ends</td>
<td>(resembles Donnell's with</td>
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<tr>
<td></td>
<td></td>
<td>bending stiffness = 0 and</td>
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<td></td>
<td></td>
<td>addition of N^w in equation 3</td>
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<td>Salame and liber (ref. 38)</td>
<td>Freely supported at both</td>
<td>Donnell</td>
<td>Exact</td>
<td>Isotropic</td>
<td>Included</td>
<td>n = 0</td>
</tr>
<tr>
<td></td>
<td>ends</td>
<td></td>
<td>Axisymmetric</td>
<td></td>
<td></td>
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<td>Author(s)</td>
<td>Type of End Support</td>
<td>Theory Used</td>
<td>Type of Analysis</td>
<td>Type of Material</td>
<td>Tangential Inertia</td>
<td>Modal Distribution</td>
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<tr>
<td>Serbin (ref. 39)</td>
<td>Simply-supported ends</td>
<td>Membrane theory</td>
<td>Rayleigh method</td>
<td>Isotropic</td>
<td>Included</td>
<td>lowest frequency in nearly inextensional mode</td>
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<tr>
<td>Watts (ref. 40)</td>
<td>Experimental</td>
<td>-</td>
<td>-</td>
<td>-</td>
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<tr>
<td>Author(s)</td>
<td>Type of End Support</td>
<td>Theory Used</td>
<td>Type of Analysis</td>
<td>Type of Material</td>
<td>Tangential Inertias</td>
<td>Modal Distribution</td>
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<tr>
<td>Armenakas</td>
<td>Freely supported at both ends</td>
<td>Hermann-Armenakas</td>
<td>Exact</td>
<td>Isotropic</td>
<td>Included</td>
<td>General $n = 1 \rightarrow 8$</td>
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<tr>
<td>Breslavskii</td>
<td>Freely supported at both ends</td>
<td>Mushtari-Vlasov</td>
<td>Exact</td>
<td>Isotropic</td>
<td>Neglected (also $k\Omega$, $N_x\Omega$ and $N_s\Omega$)</td>
<td>General $n = 2,3,4,6$</td>
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<tr>
<td>Cooper</td>
<td>Freely supported at both ends</td>
<td>Sanders</td>
<td>Exact (numerical)</td>
<td>Isotropic</td>
<td>Included</td>
<td>-</td>
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<tr>
<td>Fung, Sechler and Kaplan</td>
<td>Freely supported at both ends</td>
<td>Experimental</td>
<td>Results compared against Reissner's</td>
<td>Aluminum</td>
<td>-</td>
<td>(good correlation with theory)</td>
</tr>
<tr>
<td>Greenspon</td>
<td>Freely supported at both ends</td>
<td>Flügge</td>
<td>Exact</td>
<td>Orthotropic</td>
<td>Included</td>
<td>General $m = 1 \rightarrow 7$</td>
</tr>
<tr>
<td>Greenspon</td>
<td>Freely supported at both ends</td>
<td>Flügge</td>
<td>Exact</td>
<td>Anisotropic</td>
<td>Included</td>
<td>-</td>
</tr>
<tr>
<td>Hermann and Shaw</td>
<td>Freely supported at both ends</td>
<td>Hermann-Armenakas</td>
<td>Exact</td>
<td>Isotropic</td>
<td>Included</td>
<td>General $n = 2 \rightarrow 7$</td>
</tr>
<tr>
<td>Koval</td>
<td>Freely supported at both ends</td>
<td>Donnell-Reissner</td>
<td>Energy method: forming energy function, choosing modes and writing Lagrange's equations</td>
<td>Isotropic</td>
<td>Neglected</td>
<td>General $n = 3 \rightarrow 14$, $m = 1/2 \rightarrow 2$</td>
</tr>
<tr>
<td>Author(s)</td>
<td>Type of End Support</td>
<td>Theory Used</td>
<td>Type of Analysis</td>
<td>Type of Material</td>
<td>Tangential Inertias</td>
<td>Modal Distribution</td>
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<tr>
<td>Lipovskii and Takarenko</td>
<td>Simply supported ends (in w)</td>
<td>Reissner shallow shell theory</td>
<td>Galerkin (3 x 3)</td>
<td>Isotropic</td>
<td>Neglected</td>
<td>General (some lower modes)</td>
</tr>
<tr>
<td>McElman, Mikulas and Stein</td>
<td>Freely supported at both ends</td>
<td>Donnell</td>
<td>Exact</td>
<td>Structural Orthotropy</td>
<td>Neglected</td>
<td>General; ( m = 1,2 ) n (\to)</td>
</tr>
<tr>
<td>Mixson (ref. 51)</td>
<td>Freely supported at both ends</td>
<td>Donnell</td>
<td>Exact (1 and 2 term) (3 x 3) and (6 x 6) determinants</td>
<td>Isotropic</td>
<td>Neglected</td>
<td>General</td>
</tr>
<tr>
<td>Modi (ref. 52)</td>
<td>Freely supported at both ends</td>
<td>Timoshenko</td>
<td>Exact</td>
<td>Isotropic</td>
<td>Neglected</td>
<td>General</td>
</tr>
<tr>
<td>Mugnier and Schroeter</td>
<td>Freely supported at both ends</td>
<td>Authors (stressing terms resemble Timoshenko's; some similarity to Flügge; a peculiar term in the inertia expression later neglected)</td>
<td>Exact</td>
<td>Isotropic</td>
<td>Included</td>
<td>General</td>
</tr>
<tr>
<td>Olson and Fung (ref. 54)</td>
<td>Freely supported at both ends</td>
<td>Reissner shallow shell</td>
<td>Galerkin</td>
<td>Isotropic</td>
<td>Neglected</td>
<td>General</td>
</tr>
<tr>
<td>Öğn and Herrmann (ref. 55)</td>
<td>a) Freely supported at both ends b) Clamped at both ends</td>
<td>Batdorf-Donnell</td>
<td>Galerkin (1 x 1)</td>
<td>Isotropic</td>
<td>Neglected</td>
<td>General; large n</td>
</tr>
<tr>
<td>Author(s)</td>
<td>Type of End Support</td>
<td>Theory Used</td>
<td>Type of Analysis</td>
<td>Type of Material</td>
<td>Tangential Inertia</td>
<td>Modal Distribution</td>
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<tr>
<td>Prokopev (ref. 56)</td>
<td>simply supported at both ends</td>
<td>Author's (appears close to Flügge's but term missing in equation 3)</td>
<td>Exact; Axisymmetric</td>
<td>Orthotropic</td>
<td>Neglected</td>
<td>m →</td>
</tr>
<tr>
<td>Reissner (ref. 57)</td>
<td>freely supported at both ends</td>
<td>Reissner shallow shell</td>
<td>Exact</td>
<td>Isotropic</td>
<td>Neglected</td>
<td>m = n →</td>
</tr>
<tr>
<td>Seggelke (ref. 58)</td>
<td>a) freely supported at both ends b) clamped at both ends</td>
<td>Flügge</td>
<td>Galerkin</td>
<td>Isotropic</td>
<td>Included</td>
<td>General</td>
</tr>
<tr>
<td>Shkenev (ref. 59)</td>
<td>freely supported at both ends</td>
<td>Vlasov shallow shell</td>
<td>Galerkin (2 x 2)</td>
<td>Isotropic</td>
<td>Neglected</td>
<td>General</td>
</tr>
<tr>
<td>Voss (ref. 60)</td>
<td>a) freely supported at both ends b) simply supported at both ends</td>
<td>a) Goldenviezer b) Reissner shallow shell</td>
<td>a) Exact b) Galerkin (2 x 2)</td>
<td>Isotropic</td>
<td>Neglected</td>
<td>General</td>
</tr>
<tr>
<td>Author(s)</td>
<td>Type of End Support</td>
<td>Theory Used</td>
<td>Type of Analysis</td>
<td>Type of Material</td>
<td>Tangential or In-Plane Inertias</td>
<td>Modal Distribution</td>
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<tr>
<td>Koval (ref. 61)</td>
<td>a) Clamped-clamped&lt;br&gt;b) Freely supported-freely supported</td>
<td>Donnell</td>
<td>i) Exact; Yu's approximation i.e.: $\frac{\Delta}{\gamma}$ neglected&lt;br&gt;ii) Galerkin (trigonometric functions)&lt;br&gt;(2 x 2) (3 x 3) determinants.&lt;br&gt;iii) Energy Method (in expression for strain energy and kinetic energy, modes assumed and Lagrange equations written with assumed modes as generalized coordinates)&lt;br&gt;iv) Experimental</td>
<td>Isotropic</td>
<td>Neglected</td>
<td>General</td>
</tr>
<tr>
<td>Koval and Cranch (ref. 62)</td>
<td>a) Clamped-clamped&lt;br&gt;b) Freely supported-freely supported&lt;br&gt;c) Simply supported with restraints at both ends</td>
<td>Derived by author; uncoupled 8th order equation in $w$</td>
<td>Exact</td>
<td>1) Isotropic&lt;br&gt;2) Orthotropic (structural)</td>
<td>a) Included&lt;br&gt;b) Neglected</td>
<td>General</td>
</tr>
<tr>
<td>Nikulin (ref. 63)</td>
<td>a) Clamped-clamped&lt;br&gt;b) Freely supported-freely supported&lt;br&gt;c) Simply supported with restraints at both ends</td>
<td>Derived by author; uncoupled 8th order equation in $w$</td>
<td>Exact</td>
<td>1) Isotropic&lt;br&gt;2) Orthotropic (structural)</td>
<td>a) Included&lt;br&gt;b) Neglected</td>
<td>General</td>
</tr>
<tr>
<td>Author(s)</td>
<td>Type of End Support</td>
<td>Theory Used</td>
<td>Type of Analysis</td>
<td>Type of Material</td>
<td>Tangential Inertia</td>
<td>Modal Distribution</td>
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</tbody>
</table>
| Gontkevich (ref. 64) | a) Freely supported at both ends  
b) Other types of supports (integrals to be evaluated) | Love's with Donnell and Marquayre simplifications | No Numerical Results; reference to Nikulin's (ref. 16 and 63) papers | Isotropic | Included | - - |
| Nikulin (ref. 16) | a) Freely supported at both ends  
b) Clamped at both ends  
c) Simply supported with constraints | Authors (Uncoupled 8th order equation in w) | Exact | i) Isotropic  
ii) Structural Orthotropy | a) Included  
b) Neglected | - - |
| Ogibalov (ref. 65) | Freely supported at both ends | Donnell | Exact (axisymmetric and 3-dimensional cases) No results for prestressed shell | Isotropic | a) Included  
b) Neglected | - - |
<p>| Prokopev (ref. 66) | Freely supported at both ends | Donnell | Exact | Isotropic | Included | General |</p>
<table>
<thead>
<tr>
<th>Author(s)</th>
<th>Type of Loading</th>
<th>Type of End Support</th>
<th>Theory Used</th>
<th>Type of Analysis</th>
<th>Type of Material</th>
<th>Tangential Inertia</th>
<th>Modal Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Armenakas and Herrmann (ref. 67)</td>
<td>Hydrostatic pressure or centrally directed pressure, uniform bending moment and radial shear (induced by circumferential surface shear tractions)</td>
<td>Plane strain i.e.: $u = 0$</td>
<td>Authors' and</td>
<td>Exact</td>
<td>Isotropic</td>
<td>Included</td>
<td>General Circumferential $n = 2,3,6$</td>
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<tr>
<td>Buckens (ref. 68)</td>
<td>Uniform circumferential and torsional stress and a thermal moment arising out of non-uniform heating and axial restraint</td>
<td>Freely Supported</td>
<td>i. Timoshenko</td>
<td>Exact</td>
<td>Isotropic</td>
<td>Included</td>
<td>General</td>
</tr>
<tr>
<td>Author(s)</td>
<td>Type of Loading</td>
<td>Type of End Support</td>
<td>Theory Used</td>
<td>Type of Analysis</td>
<td>Type of Material</td>
<td>Tangential Inertia</td>
<td>Modal Distribution</td>
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<tr>
<td>Bushnell (ref. 69)</td>
<td>Constant axial stress and a pressure proportional to the displacement due to a Winkler type of core</td>
<td>Elastic, massive ring at one end and freely supported at the other</td>
<td>Donnell</td>
<td>Exact; Axisymmetric</td>
<td>Isotropic</td>
<td>Included</td>
<td>m →</td>
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<tr>
<td>Herrmann-Aménakas (ref. 70)</td>
<td>Bending and twisting moments, initial transverse shears and membrane forces with or without the inclusion of shear deformations and rotary inertia</td>
<td>Development of theory by authors</td>
<td>- - - - -</td>
<td>Isotropic</td>
<td>Included</td>
<td>- - -</td>
<td></td>
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<tr>
<td>Kessel and Schlack (ref. 71)</td>
<td>Gyroscopic forces produced by steady spinning around the axis and a simultaneous steady precession about a rotation axis</td>
<td>Freely supported</td>
<td>Reissner's shallow shell theory</td>
<td>Exact</td>
<td>Isotropic</td>
<td>Ignored</td>
<td>n →</td>
</tr>
<tr>
<td>Author(s)</td>
<td>Type of Loading</td>
<td>Type of End Support</td>
<td>Theory Used</td>
<td>Type of Analysis</td>
<td>Type of Material</td>
<td>Tangential Inertia</td>
<td>Modal Distribution</td>
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<tr>
<td>Levy and Jewell (ref. 72)</td>
<td>Internal pressure, radial loads and attached mass</td>
<td>Freely Supported</td>
<td>Experimental</td>
<td>- - -</td>
<td>-</td>
<td>With Stiffeners</td>
<td>- - -</td>
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<tr>
<td>Ong and Herrmann (ref. 55)</td>
<td>Thermal load along a longitudinal strip</td>
<td>a)freely supported at both ends b)clamped at both ends</td>
<td>Batdorf-Donnell</td>
<td>Galerkin</td>
<td>Isotropic</td>
<td>Neglected</td>
<td>General</td>
</tr>
<tr>
<td>Mizoguchi (ref. 73)</td>
<td>Rotating shell (constant angular velocity)</td>
<td>Radial</td>
<td>Authors</td>
<td>Exact</td>
<td>Isotropic</td>
<td>Included</td>
<td>- - -</td>
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<tr>
<td>Saleme and Liber (ref. 38)</td>
<td>Due to the shell partially filled with liquid (matching condition used at interface)</td>
<td>Freely Supported</td>
<td>Donnell</td>
<td>Exact; Axisymmetric</td>
<td>Isotropic</td>
<td>Included</td>
<td>- - -</td>
</tr>
<tr>
<td>Seggelke (ref. 74)</td>
<td>Uniform axial stress and axial stress arising due to a bending moment.</td>
<td>Freely Supported</td>
<td>Marguerre's (4 x 4)</td>
<td>Galerkin</td>
<td>Isotropic</td>
<td>Neglected</td>
<td>n -</td>
</tr>
<tr>
<td>Author(s)</td>
<td>Type of Loading</td>
<td>Type of End Support</td>
<td>Theory Used</td>
<td>Type of Analysis</td>
<td>Type of Material</td>
<td>Tangential Inertia</td>
<td>Modal Distribution</td>
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<tr>
<td>Weingarten</td>
<td>Internal pressure and a bending moment</td>
<td>Freely Supported</td>
<td>Batdorf</td>
<td>Galerkin (18 x 18) (25 x 25)</td>
<td>Isotropic</td>
<td>Neglected</td>
<td>$m = 1, 2, 3$ and $n = 0$</td>
</tr>
</tbody>
</table>
CHAPTER III
DEVELOPMENT OF THEORY FOR A VIBRATING CYLINDRICAL SHELL

The usual assumptions which govern the various commonly used shell theories are:

1. The shell is composed of a homogeneous, Hookean continuum.
2. The shell geometry is defined by its two dimensional middle surface which is imbedded in a three dimensional Euclidean space.
3. The ratio of the thickness of the shell to the minimum principal radius of curvature is smaller than unity by an order of magnitude. (For a circular cylindrical shell the least radius of curvature is the radius of the cross-section).
4. The displacements caused by loads are small compared to the thickness.
5. The unit elongations, shears and rotations associated with the displacements are also small compared with unity.

In addition the Kirchhoff assumption is usually made which states that the straight transverse fibers of the shell which were normal to the middle surface remain straight and normal during deformation and do not undergo any stretching or contraction.

Rotary inertia which arises in vibratory motion is usually neglected.
During vibration the internal stresses in the shell consist of the initial stress $\sigma^i$ and the vibratory stress $\sigma$. Since the shell under the action of the prestress alone is assumed to be in equilibrium, the potential corresponding to this state is taken as the datum. The initially stressed state is assumed to be membrane and the associated deformation is usually neglected. Thus there is no interaction between the prestress deformation and the additional vibratory stress. The prestress distribution within the shell is the solution of the static problem with the prescribed end conditions. The initial stress arising due to thermal loads is calculated by substituting the solution obtained for the temperature distribution from the heat conduction equation into the set of differential equations for the shell deformation, finding the displacement field from these for the static problem and then applying Hooke's Law.

The deformation of the shell is defined by $u$, $v$, and $w$, which represent the axial, circumferential and radial displacements and are functions of time and of the axial and circumferential coordinates $x$ and $s$ of the middle surface.

Stretching ($H$) and bending ($D$) stiffness matrices comprise the resistance of the shell against deformation. Retention of both results in a "bending theory" while only the stretching stiffness is taken into account in a "membrane theory" and only the bending stiffness in an "inextensional theory". Both the Donnell and Sanders theories dealt with herein are of the bending type.
The coordinate system chosen and the dimensions of the shell are shown in Figure 1. Non-dimensionalization of the length-like quantities is accomplished by defining the parameters

\[ k = \frac{h^2}{12\alpha^2} \]
\[ L = \frac{\ell}{\alpha} \]  

Similarly the displacements and the coordinates of the shell are also non-dimensionalized with respect to the radius while the dimensionless inertia which involves the time coordinate is defined as

\[ Y = \rho h\omega^2\alpha^2/\mu \]  

where \( \rho \) is the mass density of the material of the shell, \( \omega \) the natural frequency of vibration of the shell, and \( \mu \) the stretching stiffness.

We now go through a well known variational procedure to generalize Sanders' theory for cylindrical shells to include prestress, where the initial stress varies spatially. The procedure, which may be used repeatedly to derive any shell theory, has its roots in the principle of virtual work.

The total potential energy \( U \) of the system is the sum of the energies due to the internal strain energy arising out of the vibratory stresses \( \sigma \), and the initial stress \( \sigma^i \), and the work done by the body force \( \hat{B} \) and the edge forces. If the components of \( \sigma \), \( \sigma^i \) and \( \hat{B} \), referred to the chosen orthogonal set of coordinates \( x \) and \( s \) in Figure 1, are \( \sigma_x \), \( \sigma_s \), \( \tau_{xs} \), and \( \sigma^i_x \), \( \sigma^i_s \), \( \tau^i_{xs} \) and \( B_x \), \( B_s \), \( B_z \), respectively, then
Fig. 1 Coordinate System
\[ U = \frac{1}{2} \int_{\text{volume}} \left( \sigma_x \varepsilon_x + \sigma_\theta \varepsilon_\theta + \tau_{x\theta} \gamma_{x\theta} \right) \, dx \, ds \, dz \]

\[ + \int_{\text{volume}} \left( \tilde{\sigma}_x^i \varepsilon_x^i + \tilde{\sigma}_\theta^i \varepsilon_\theta^i + \tilde{\tau}_{x\theta}^i \gamma_{x\theta}^i \right) \, dx \, ds \, dz \]

\[ - \int_{\text{volume}} \left( B_x u + B_\theta v + B_z w \right) \, dx \, ds \, dz \]  

where \( \tilde{\sigma}_x^i, \tilde{\sigma}_\theta^i, \text{and } \tilde{\tau}_{x\theta}^i \) are the axial, circumferential and torsional prestresses respectively, and \( \varepsilon_x, \varepsilon_\theta, \text{and } \gamma_{x\theta} \) are the strain components. Note that a static bending moment is easily accounted for by assuming a cosine variation in \( \tilde{\sigma}_x^i \) along the circumference while internal pressure usually induces

\[ \tilde{\sigma}_x^i = \frac{\sigma_\theta^i}{2} = \frac{p a}{2h} \]  

where \( p \) is the pressure.

The kinetic energy of deformation of the shell is

\[ T = \frac{\rho}{2} \int_{\text{volume}} \left( \frac{\partial u}{\partial t} \right)^2 + \left( \frac{\partial v}{\partial t} \right)^2 + \left( \frac{\partial w}{\partial t} \right)^2 \, dx \, ds \, dz \]  

where \( t \) denotes time.

The vibratory stresses are related to the vibratory strains via Hooke's Law.

For an isotropic material
where $E$ is the Young's modulus of the material and $\nu$ is the Poisson's ratio.

Due to the assumption of the preservation of the normal element, the displacements are linearly distributed through the thickness of the shell. Thus, the displacement components may be represented by the following relationships:

\[
\begin{align*}
\sigma_x &= \frac{E}{1-\nu^2} \left( \epsilon_x + \nu \epsilon_s \right) \\
\sigma_s &= \frac{E}{1-\nu^2} \left( \epsilon_s + \nu \epsilon_x \right) \\
\tau_{xs} &= \frac{E}{2(1+\nu)} \gamma_{xs}
\end{align*}
\]

(6)

where $E$ is the Young's modulus of the material and $\nu$ is the Poisson's ratio.

Due to the assumption of the preservation of the normal element, the displacements are linearly distributed through the thickness of the shell. Thus, the displacement components may be represented by the following relationships:

\[
\begin{align*}
U(x,s,z) &= u(x,s) - z u'(x,s) \\
V(x,s,z) &= v(x,s) - z v'(x,s) \\
W(x,s,z) &= w(x,s)
\end{align*}
\]

(7)

where $u$, $v$, and $w$ represent the components of the displacement vector on the mid-surface which is the reference surface and the primes denote differentiation with respect to $z$. Thus, the quantities $u'$ and $v'$ represent the rotations of tangents to the surface oriented along $x$ and $s$ respectively. The third of equation (7) implies the condition of no thickness stretch, i.e.: the strain normal to the surface equals zero.
Thus, the strain of an element at a distance \( z \) from the middle surface consists of the stretching of the middle surface and that due to rotation of the element. Accordingly,

\[
\begin{align*}
\varepsilon_x &= e_x - z \chi_x, \\
\varepsilon_s &= e_s - z \chi_s, \\
\gamma_{xs} &= e_{xs} - z \chi_{xs}
\end{align*}
\]

(8)

where \( \varepsilon \) denotes the strains, the \( e \) represents stretching and shear of the mid-surface and \( \chi \) the changes in curvature. It must be noted that the \( e \) and \( \chi \) are not functions of \( z \).

Because the initial stresses may be large, it is necessary to use the second order strain-displacement relations in the second order strain-displacement relations in the second integral of equation (3) while retaining the linear terms only in the first, in order to maintain a homogeneity in the orders of magnitude. Since the initial stresses are assumed uniform through the thickness, it is sufficient to retain only linear terms in the expressions for the changes of curvature.

The second-order strain-displacement relations according to Sanders' theory (ref. 76) are

\[
\begin{align*}
\varepsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 + \frac{1}{8} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial s} \right)^2 \\
\varepsilon_s &= \frac{\partial v}{\partial s} - w + \frac{1}{2} \left( \frac{\partial w}{\partial s} + v \right)^2 + \frac{1}{8} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial s} \right)^2 \\
\varepsilon_{xs} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial s} + \frac{\partial w}{\partial x} \left( \frac{\partial w}{\partial s} + v \right)
\end{align*}
\]

(9)
The corresponding equations for some other theories, which are well-documented and available (ref. 27, 61, 70) are listed in Table 8 and 9.

Using equations (3) through (10) and integrating across the thickness, we have

\[
U = \frac{1}{2} H \int \int \left[ \left( \frac{\partial u}{\partial x} \right)^2 + 2y \frac{\partial u}{\partial x} \left( \frac{\partial v}{\partial s} - w \right) + \left( \frac{\partial v}{\partial s} - w \right)^2 + \frac{1}{2} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial s} \right)^2 \right] dx ds
\]

\[+ \frac{1}{2} D \int \int \left[ \left( \frac{\partial w}{\partial x} \right)^2 + 2y \frac{\partial w}{\partial x} \left( \frac{\partial w}{\partial s} + \frac{\partial v}{\partial s} \right) + \left( \frac{\partial w}{\partial s} + \frac{\partial v}{\partial s} \right)^2 \right]
\]

\[+ \frac{1}{2} \left( 1 - \nu \right) \left( \frac{\partial w}{\partial x \partial s} + \frac{3}{4} \frac{\partial w}{\partial x} - \frac{3}{4} \frac{\partial w}{\partial s} \right) \right] dx ds
\]

\[+ \frac{h}{2} \int \int \left[ \frac{\partial i}{\partial x} \left\{ \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 + \frac{1}{8} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial s} \right)^2 \right\} + \frac{\partial i}{\partial s} \left\{ \frac{\partial v}{\partial s} - w \right\}
\]

\[+ \frac{1}{2} \left( \nu + \frac{\partial w}{\partial s} \right)^2 + \frac{1}{8} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial s} \right)^2 \right] \right] dx ds
\]

\[+ \int \int \left( B_x - u + B_s - v + B_z - w \right) dx ds
\]

\[- \int (N_x u + N_s v) \bigg|_{x=L}^{x=0}
\]

\[- \int (N_x u + N_s v) \bigg|_{x=0}
\]

where \( N_x, N_s \) and \( N_{xs} \) are the edge resultants.
Table 8: Expressions for second order non-dimensionalized strains corresponding to equation (9)

<table>
<thead>
<tr>
<th>Theory/Author</th>
<th>$e_x$</th>
<th>$e_s$</th>
<th>$e_{xs}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Donnell</td>
<td>$\frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2$</td>
<td>$\frac{\partial v}{\partial s} - w + \frac{1}{2} \left( \frac{\partial w}{\partial s} \right)^2$</td>
<td>$\frac{\partial u}{\partial s} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial s}$</td>
</tr>
<tr>
<td>Herrmann-Armenakas*</td>
<td>$\frac{\partial u}{\partial x} + \frac{1}{2} \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial x} \right)^2 \right]$</td>
<td>$\frac{1}{1-\varepsilon} \left( \frac{\partial v}{\partial s} - w \right) + \frac{1}{2(1-\varepsilon)^2} \left[ \left( \frac{\partial u}{\partial s} \right)^2 + \left( \frac{\partial v}{\partial s} \right)^2 \right] + \left( v + \frac{\partial w}{\partial s} \right) \frac{\partial w}{\partial s}$</td>
<td>$\frac{1}{1-\varepsilon} \left( \frac{\partial u}{\partial s} + \frac{\partial v}{\partial x} + \frac{1}{2} \left[ \frac{\partial u}{\partial s} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial s} \frac{\partial v}{\partial x} \right] + \left( v + \frac{\partial w}{\partial s} \right) \frac{\partial w}{\partial x} \right)$</td>
</tr>
<tr>
<td>Washizu</td>
<td>$\frac{\partial u}{\partial x} + \frac{1}{2} \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial x} \right)^2 \right]$</td>
<td>$\frac{\partial v}{\partial s} - w + \frac{1}{2} \left[ \left( \frac{\partial u}{\partial s} \right)^2 + \left( \frac{\partial v}{\partial s} - w \right)^2 \right] + \left( \frac{\partial w}{\partial s} + v \right)^2$</td>
<td>$\frac{\partial u}{\partial s} + \frac{\partial v}{\partial x} + \left[ \frac{\partial u}{\partial s} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial s} \frac{\partial v}{\partial x} \right]$ + $\left( v + \frac{\partial w}{\partial s} \right) \frac{\partial w}{\partial x}$</td>
</tr>
</tbody>
</table>

*This theory in its general form takes into account shear deformations and rotary inertia.*
Table 9: Expressions for first order non-dimensionalized curvatures corresponding to equation (10)

<table>
<thead>
<tr>
<th>Theory/Author</th>
<th>$X_x$</th>
<th>$X_s$</th>
<th>$X_{xs}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Donnell</td>
<td>$\frac{\partial^2 w}{\partial x^2}$</td>
<td>$\frac{\partial^2 w}{\partial s^2}$</td>
<td>$2 \frac{\partial^2 w}{\partial x \partial s}$</td>
</tr>
<tr>
<td>Herrmann-Amenakas</td>
<td>$\frac{\partial^2 w}{\partial x^2}$</td>
<td>$\left(\frac{\partial^2 w}{\partial s^2} + \frac{\partial v}{\partial s}\right)$</td>
<td>$2 \frac{\partial^2 w}{\partial x \partial s} + \frac{\partial v}{\partial x}$</td>
</tr>
<tr>
<td>Washizu</td>
<td>$\frac{\partial^2 w}{\partial x^2}$</td>
<td>$\left(\frac{\partial^2 w}{\partial s^2} + \frac{\partial v}{\partial s}\right)$</td>
<td>$2 \frac{\partial^2 w}{\partial x \partial s} + \frac{\partial v}{\partial x}$</td>
</tr>
</tbody>
</table>

*The second order curvatures will have to be used in the presence of initial bending and twisting moments.
where the body and surface forces acting on the shell have been replaced by statically equivalent forces \( \tilde{F}_x, \tilde{F}_s \) and \( \tilde{F}_z \) respectively acting on the reference surfaces. It has also been assumed that the initial stresses do not vary across the thickness and

\[
H = \frac{Eh}{(1-v^2)}
\]

\[
D = \frac{Eh^3}{12(1-v^2)}
\]

Application of Hamilton's principle which states that

\[
\int_0^T \delta (T-U) \, dt = 0
\]

provides the equations of motion of the shell. Thus, the variation of \( T-U \) with respect to \( u, v \) and \( w \) is successively set equal to zero. With the neglect of rotary inertia the variation in \( u \) designated \( \eta \), in the absence of \( \tau_{xS} \), yields

\[
H \int_0^L \int_0^{2\pi} \left[ \frac{\partial u}{\partial x} \frac{\partial \eta}{\partial x} + \nu \frac{\partial \eta}{\partial x} \left( \frac{\partial v}{\partial s} - w \right) + \frac{1-\nu}{2} \frac{\partial \eta}{\partial s} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial s} \right) \right] \, dx \, ds
\]

\[
+ D \int_0^L \int_0^{2\pi} - \frac{1-\nu}{2} \frac{\partial \eta}{\partial s} \left( \frac{\partial^2 v}{\partial x \partial s} + \frac{3}{4} \frac{\partial v}{\partial x} - \frac{1}{4} \frac{\partial u}{\partial s} \right) \, dx \, ds
\]

\[
+ H \int_0^L \int_0^{2\pi} \left[ \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} \frac{\partial \eta}{\partial x} - \frac{\partial \eta}{\partial x} \left( \frac{\partial v}{\partial s} - \frac{\partial v}{\partial s} \right) - \frac{\partial \eta}{\partial s} \frac{\partial \eta}{\partial s} - \frac{\partial u}{\partial s} \right] \frac{\partial^2 v}{\partial x \partial s} \right] \, dx \, ds
\]

\[
- \rho h \int_0^L \int_0^{2\pi} \frac{\partial u}{\partial t} \frac{\partial \eta}{\partial t} \, dx \, ds
\]

\[
- \int_0^{2\pi} \left[ \tilde{N}_x \eta \right]_{x=0}^{x=L} \, ds
\]

\[
= 0
\]

(14)
where the limits of integration have been set for a complete shell of non-dimensional length $L$.

Since the body in initially stressed state is in equilibrium, integration by parts yields:

$$
H \int_0^L \int_0^{2\pi} \left[ \frac{\partial^2 u}{\partial x^2} + \nu \left( \frac{\partial^2 v}{\partial x \partial s} - \frac{\partial w}{\partial x} \right) + \frac{1 - \nu}{2} \left( \frac{\partial^2 v}{\partial x \partial s} + \frac{\partial^2 u}{\partial s^2} \right) \right] \eta \, dx \, ds
$$

$$
+ D \int_0^L \int_0^{2\pi} \left[ - \frac{1 - \nu}{2} \left( \frac{\partial^3 w}{\partial x \partial s^2} + \frac{3}{4} \frac{\partial^2 v}{\partial x \partial s} - \frac{1}{4} \frac{\partial^2 u}{\partial s^2} \right) \right] \eta \, dx \, ds
$$

$$
- h \int_0^L \int_0^{2\pi} \left[ \frac{\partial}{\partial s} \left\{ \frac{\partial_i x}{4} + \frac{\partial_i s}{4} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial s} \right) \right\} \right] \eta \, dx \, ds
$$

$$
+ \int_0^{2\pi} \left[ \bar{N}_x - q_x h - H \left\{ \frac{\partial u}{\partial x} + \nu \left( \frac{\partial u}{\partial s} - w \right) \right\} \right] \eta \left|_{x=0} \right. \, ds
$$

$$
+ \int_0^{2\pi} \left[ - H \frac{1 - \nu}{2} \left\{ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial s} \right\} \eta \right]_{S=0}^{S=2\pi} \, dx
$$

$$
+ D \int_0^L \int_0^{2\pi} \left\{ \frac{\partial_i x}{4} + \frac{\partial_i s}{4} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial s} \right) \right\} \eta \left|_{S=0} \right. \, dx
$$

$$
+ h \int_0^L \left\{ \frac{\partial_i x}{4} + \frac{\partial_i s}{4} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial s} \right) \right\} \eta \left|_{S=0} \right. \, dx \, ds
$$

$$
\eta \left|_{S=0} \right. \, ds
$$

(15)

Since $\eta$ is an arbitrary variation, the sum of the coefficients of $\eta$ must separately be equal to zero in the surface and line integrals. This yields the first of the three equilibrium equations as
\[
\frac{\partial^2 u}{\partial x^2} + \frac{1 - \nu}{2} \frac{\partial^2 u}{\partial s^2} + \frac{1 + \nu}{2} \frac{\partial^2 w}{\partial x \partial s} - \nu \frac{\partial w}{\partial x} - k \left[ \frac{1 - \nu}{2} \left( \frac{\partial^2 w}{\partial x \partial s^2} + 3 \frac{\partial^2 y}{\partial x \partial s} - \frac{1}{2} \frac{\partial^2 y}{\partial s^2} \right) \right] \]
\[
- \frac{\partial}{\partial s} \left[ \frac{\sigma_{xx}^i + \sigma_{ss}^i}{4} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial s} \right) \right] = \frac{\rho h \alpha^2}{\rho^2} (1 - \nu^2) \frac{\partial^2 u}{\partial t^2}
\]  
\tag{16}

where the initial stresses $\sigma_{xx}^i$, $\sigma_{ss}^i$ and $\tau_{xs}^i$ are non-dimensionalized by $\frac{E}{(1-\nu^2)}$ to yield $\sigma_{xx}^i$, $\sigma_{ss}^i$ and $\tau_{xs}^i$ respectively and $k = \frac{h^2}{12a^2}$.

In this process we have also derived one set of boundary conditions as

\[
\left[ - \bar{N}_x + \sigma_{xx}^i h + \frac{1}{h} \left( \frac{\partial u}{\partial x} + \nu (\frac{\partial v}{\partial s} - w) \right) \right] = 0
\]
\[
\left. u \right|_{x=0} = 0, \quad \left. u \right|_{x=L} = 0
\]
\[
\left. \bar{N}_x = 0 \right|_{x=0, L}
\]
\tag{17}

where $N_x$, $N_s$, $N_{xs}$, $M_x$, $M_s$, $M_{xs}$, $Q_x$ and $Q_s$ are the stress resultants defined by

\[
N_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x \, dz, \quad N_s = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_s \, dz, \quad N_{xs} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xs} \, dz,
\]
\[
M_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xz} \, dz, \quad M_s = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{zs} \, dz, \quad M_{xs} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xs} \, dz,
\]
\[
Q_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x \, dz, \quad Q_s = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_s \, dz, \quad Q_{xs} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xs} \, dz
\]
\tag{18}

The remaining line integrals in equation (15) provide the boundary conditions on the edges parallel to the generator but for a complete shell they ensure the periodicity in the circumferential direction.
Proceeding in a similar manner,

\[
\left[ \frac{1+\nu}{2} \frac{\partial^2}{\partial x \partial s} - \frac{\varepsilon_k (1-\nu)}{8} \frac{\partial^2}{\partial s^2} - \frac{\partial}{\partial x} \left( \frac{\sigma_x^i + \sigma_s^i}{4} \frac{\partial}{\partial s} \right) \right] u \\
+ \left[ \frac{1-\nu}{2} \frac{\partial^2}{\partial s^2} + \frac{k}{\varepsilon} \frac{\partial^2}{\partial s^2} + \frac{\varepsilon_k (1-\nu)}{8} \frac{\partial^2}{\partial x \partial s} - \frac{\varepsilon_h a^2}{H} \frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial x} \left( \frac{\sigma_x^i + \sigma_s^i}{4} \frac{\partial}{\partial s} \right) \right] v \\
+ \left[ -\frac{\partial}{\partial s} + \frac{3-\nu}{2} \frac{\partial^3}{\partial x^2 \partial s} + \frac{k}{\varepsilon} \frac{\partial^3}{\partial s^3} - \sigma_s^i \frac{\partial}{\partial s} \right] w = 0
\]

(19)

and the remaining boundary conditions are:

\[ N_x s + \frac{3}{2} M_{xs} + H \left( \sigma_x^i + \sigma_s^i \right) \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial s} \right) = 0 \quad \text{or} \quad v = 0 \]
\[ M_x = 0 \quad \text{or} \quad \frac{\partial w}{\partial x} = 0 \]
\[ \frac{\partial M_{xs}}{\partial x} + 2 \frac{\partial M_{xs}}{\partial s} - \sigma_s^i H \frac{\partial w}{\partial x} = 0 \quad \text{or} \quad w = 0 \]

(20)

In a like manner, the boundary conditions and differential equations obtained from the Donnell theory are, respectively:

\[ -\bar{N}_x + H \sigma_x^i + N_x = 0 \quad \text{or} \quad u = 0 \]
\[ N_{xs} = 0 \quad \text{or} \quad v = 0 \]
\[ M_x = 0 \quad \text{or} \quad \frac{\partial w}{\partial x} = 0 \]
\[ H \sigma_x^i \frac{\partial w}{\partial x} + Q_x + M_{xs} = 0 \quad \text{or} \quad w = 0 \quad \text{at} \quad x = 0, L \]

(21)

and

\[
\left[ \frac{\partial^2}{\partial x^2} + \frac{1-\nu}{2} \frac{\partial^2}{\partial s^2} - \frac{\varepsilon_h a^2}{H} \frac{\partial^2}{\partial t^2} \right] u + \left[ \frac{1+\nu}{2} \frac{\partial^2}{\partial x \partial s} \right] v + \left[ \frac{\partial}{\partial x} \right] w = 0 \\
\left[ \frac{1+\nu}{2} \frac{\partial^2}{\partial x \partial s} \right] u + \left[ \frac{1-\nu}{2} \frac{\partial^2}{\partial s^2} + \frac{\varepsilon_h a^2}{H} \frac{\partial^2}{\partial t^2} \right] v + \left[ \frac{\partial}{\partial s} \right] w = 0 \\
\left[ -\nu \frac{\partial}{\partial x} \right] u + \left[ \frac{\partial}{\partial s} \right] v + \left[ 1 + k \frac{\partial^2}{\partial s^2} + \frac{\varepsilon_h a^2}{H} \frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial x} \left( \sigma_x^i \frac{\partial}{\partial x} \right) - \frac{\partial}{\partial s} \left( \sigma_s^i \frac{\partial}{\partial s} \right) \right] w = 0
\]

(22)
Denoting the set of differential equations in the operator form

\[
L (u, v, w, t) = \begin{pmatrix}
L_{11} & L_{12} & L_{13} \\
L_{21} & L_{22} & L_{23} \\
L_{31} & L_{32} & L_{33}
\end{pmatrix}
\begin{pmatrix}
u \\
\end{pmatrix} = 0
\]

and

\[
L'(u, v, w, t) = L'(u, v, w, t) + k F (u, v, w, t) + G (\sigma', u, v, w, t)
\]

where

\[
L' = \begin{pmatrix}
\frac{\partial^2}{\partial x^2} + \frac{l \nu}{2} \frac{\partial^2}{\partial t^2} - \frac{\nu}{\nu^2} \frac{\partial^2}{\partial s^2} & \frac{1 + \nu}{2} \frac{\partial^2}{\partial x^2} & -\nu \frac{\partial}{\partial t} \\
\frac{1 + \nu}{2} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial s^2} - \frac{\nu}{\nu^2} \frac{\partial^2}{\partial t^2} & -\frac{\partial}{\partial s} & l + k \gamma + \frac{\nu \lambda}{\lambda^2} \\
- \nu \frac{\partial}{\partial t} & -\frac{\partial}{\partial s} & \frac{1 + \nu}{2} \frac{\partial^2}{\partial x^2}
\end{pmatrix}
\begin{pmatrix}
u \\
\end{pmatrix}
\]

the equations of motion for Donnell, Herrmann-Armenakas, Washizu and Sanders theories may be conveniently written because they differ from one another only in the form of the F and G operators, while the L' operators are identical to equation (25). The various forms for F and G for the above mentioned theories are listed in Tables 10 and 11.
Table 10: Expressions for $F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$ in equation (24) for various theories

<table>
<thead>
<tr>
<th></th>
<th>Donnell</th>
<th>Sanders</th>
<th>Herrmann - Arménakas</th>
<th>Washizu</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_1$</td>
<td>$0$</td>
<td>$1 - \gamma \frac{\partial^3 u}{\partial s^3} - 3(1-\gamma)\frac{\partial^3 v}{\partial x^2 \partial s} - \frac{1-\gamma}{2} \frac{\partial^3 w}{\partial x \partial s^2}$</td>
<td>$1 - \gamma \frac{\partial^3 u}{\partial s^3} + \frac{3}{2} \frac{\partial^3 w}{\partial x \partial s^2}$</td>
<td>$0$</td>
</tr>
<tr>
<td>$F_2$</td>
<td>$0$</td>
<td>$-\frac{3}{2} (1-\gamma) \frac{\partial^2 u}{\partial x \partial s} + \frac{9}{2} (1-\gamma) \frac{\partial^2 v}{\partial x^2}$</td>
<td>$3(1-\gamma) \frac{\partial^3 v}{\partial x^2} + \frac{3}{2} \frac{\partial^3 w}{\partial x^2 \partial s}$</td>
<td>$2(1-\gamma) \frac{\partial^3 v}{\partial x^2} + \frac{3}{2} \frac{\partial^3 w}{\partial x^2 \partial s}$</td>
</tr>
<tr>
<td>$F_3$</td>
<td>$0$</td>
<td>$-\frac{1}{2} (1-\gamma) \frac{\partial^2 u}{\partial x \partial s^2} + \frac{3}{2} \frac{\partial^2 v}{\partial x^2 \partial s}$</td>
<td>$\frac{3}{2} \frac{\partial^3 u}{\partial x^2 \partial s^2} - \frac{1-\gamma}{2} \frac{\partial^3 w}{\partial x^2 \partial s^2}$</td>
<td>$2(1-\gamma) \frac{\partial^3 u}{\partial x^2 \partial s} + \frac{3}{2} \frac{\partial^3 w}{\partial x^2 \partial s}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$+ W - 2 \frac{\partial u}{\partial s^2}$</td>
<td></td>
</tr>
</tbody>
</table>
Table 11: Expressions for $g = \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix}$ in equation (24) for various theories

($\sigma^i_s$ and $\tau^i_{xs}$ are assumed constant in the s direction)

<table>
<thead>
<tr>
<th></th>
<th>Donnell</th>
<th>Sanders</th>
<th>Herrmann – Armenakas</th>
<th>Washizu</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_1$</td>
<td>0</td>
<td>$\frac{\partial}{\partial s} \left( \sigma^i_s \right) \Delta u - \frac{\partial}{\partial s} \left( \sigma^i_s \right) \Delta v$ + $\frac{\partial^2}{\partial s^2} \left( \sigma^i_s \Delta v \right)$</td>
<td>$\frac{\partial}{\partial x} \left( \sigma^i_x \Delta u \right) + \frac{\partial}{\partial x} \left( \tau^i_x \Delta u \right)$ + $\frac{\partial}{\partial s} \left( \tau^i_s \Delta u \right)$ + $\frac{\partial}{\partial x} \left( \tau^i_x \Delta u \right)$</td>
<td>$\frac{\partial}{\partial x} \left( \sigma^i_x \Delta u \right) + \frac{\partial}{\partial x} \left( \tau^i_x \Delta u \right)$ + $\frac{\partial}{\partial s} \left( \tau^i_s \Delta u \right)$ + $\frac{\partial}{\partial x} \left( \tau^i_x \Delta u \right)$</td>
</tr>
<tr>
<td>$G_2$</td>
<td>0</td>
<td>$\frac{\partial}{\partial x} \left( \sigma^i_x \Delta v \right)$ - $\sigma^i_s \Delta w$ + $\frac{\partial}{\partial s} \left( \sigma^i_s \Delta w \right)$</td>
<td>$\frac{\partial}{\partial x} \left( \sigma^i_x \Delta v \right)$ + $\sigma^i_s \left( \Delta v + \frac{\partial^2 w}{\partial s^2} \right)$ - $\frac{\partial}{\partial s} \left( \tau^i_s \Delta w \right)$</td>
<td>$\frac{\partial}{\partial x} \left( \sigma^i_x \Delta v \right)$ + $\sigma^i_s \left( \Delta v + \frac{\partial^2 w}{\partial s^2} \right)$ - $\frac{\partial}{\partial s} \left( \tau^i_s \Delta w \right)$</td>
</tr>
<tr>
<td>$G_3$</td>
<td>$-\frac{\partial}{\partial x} \left( \sigma^i_x \Delta w \right)$ - $\sigma^i_s \Delta w$ + $\frac{\partial}{\partial s} \left( \sigma^i_s \Delta w \right)$</td>
<td>$\frac{\partial}{\partial x} \left( \sigma^i_x \Delta w \right)$ - $\tau^i_s \left( \Delta v + \frac{\partial^2 w}{\partial s^2} \right)$</td>
<td>$-\frac{\partial}{\partial x} \left( \sigma^i_x \Delta w \right)$ - $\tau^i_s \left( \Delta v + \frac{\partial^2 w}{\partial s^2} \right)$</td>
<td>$-\frac{\partial}{\partial x} \left( \sigma^i_x \Delta w \right)$ - $\tau^i_s \left( \Delta v + \frac{\partial^2 w}{\partial s^2} \right)$</td>
</tr>
</tbody>
</table>
A usual approximation consists of the neglect of the inertial terms in the expression for $L'_{11}$ and $L'_{22}$ in equation (25) which arise on account of in-plane motion of the shell. This neglect has been shown to the untenable when the number of circumferential waves of the vibration mode is small. This point is further examined in Chapter IV.

Turning our attention to the $L'$ and $F$ operators, it is noteworthy that each of the theories listed in Tables 10 and 11 are symmetric -- the result of the application of the principle of virtual work. This has essentially been pointed out earlier by Koiter (ref. 77). Some theories e.g. the Timoshenko theory, are deliberately omitted from the present consideration because of their non-conformity to the symmetry principle.

Dictated perhaps by the expediency of being the simplest, the most often used theory in the literature has been Donnell's. This theory was also derived independently in the Soviet Union by Mushtari (ref. 73) and has been extensively applied by Vlasov (ref. 79) and his students. Essentially a quasi-shallow analysis, the theory has been discussed at length with reference to stability problems by Batdorf (ref. 80), Kempner (ref. 81), Hoff (ref. 82) and others while references 9 and 83 deal with the dynamic aspect. Comparative studies with the more accurate Flugge equations (ref. 9 and 82) reveal that Donnell theory suffers most when there is a large unsupported length in the shell. The largest error occurs for the infinite cylinder, in which case the buckling load according to the Donnell
theory is in error by about 33 per cent (ref. 82). As was seen in Chapter II, the vibrational aspect of the Donnell theory has also received considerable attention. A rather comprehensive evaluation of the Donnell theory in predicting the dynamic behavior of cylindrical shells has been carried out in reference 84, although the study is restricted to cases of shells under no initial stress. The role of boundary conditions, tangential inertias and the circumferential modal number has been carefully studied and compared against Flugge’s theory.

The reason for the simplicity of the Donnell theory lies in the ease in uncoupling the equations especially when the tangential inertias are neglected. This leads to the eighth order equation in $w$ alone.

$$\nabla^4 \left[ -\frac{\partial}{\partial x} \left( \frac{\partial^2 w}{\partial x^2} \right) - \frac{\partial}{\partial s} \left( \frac{\partial^2 w}{\partial s^2} \right) \right] + k \nabla^2 w + \nu \frac{\partial^4 w}{\partial x^4} - \nabla^4 \left( \frac{Eh}{H} \frac{\partial^2 w}{\partial t^2} \right) = 0$$

where

$$\nabla^4 \equiv \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial s^2} \right)^2$$

Batdorf (ref. 80) suggested the alternate form

$$-\frac{\partial}{\partial x} \left( \frac{\partial^2 w}{\partial x^2} \right) - \frac{\partial}{\partial s} \left( \frac{\partial^2 w}{\partial s^2} \right) + \frac{1}{k} \nabla^4 w + (1-\nu) \frac{\partial^4 w}{\partial x^4} - \frac{Eh}{H} \frac{\partial^2 w}{\partial t^2} = 0$$

where

$$\nabla^{-4}, \nabla^4 w \equiv w$$

(27)

although any numerical advantage accruing thereby appears unsubstantiated (ref. 4). Moreover a basic limitation in using equation (27) seems to have gone completely unnoticed resulting in its erroneous use in literature in some instances. This point is further discussed in Chapters 5 and 6.
In the general case uncoupling of the differential equations may be accomplished according to the following scheme, provided the multiplication of two operators is a commutative operation, which is the case if the initial stresses are uniform. The first two equations obtained in expanding equation (23) are:

\[
\begin{align*}
L_{11} u + L_{12} v + L_{13} w &= 0 \\
L_{21} u + L_{22} v + L_{23} w &= 0
\end{align*}
\]  

(28)

Premultiplying the first by \(L_{22}\) and the second by \(L_{12}\) and subtracting we have

\[
\left( L_{11} L_{22} - L_{12} L_{21} \right) u = \left( L_{23} L_{12} - L_{13} L_{22} \right) w
\]  

(29)

Similarly, on eliminating \(u\) we get

\[
\left( L_{11} L_{22} - L_{12} L_{21} \right) u = \left( L_{13} L_{21} - L_{23} L_{11} \right) w
\]  

(30)

Using the equation (29 and 30) in

\[
\begin{align*}
L_{31} u + L_{32} v + L_{33} w &= 0
\end{align*}
\]  

(31)

we arrive at

\[
\left[ L_{31} \left( L_{12} L_{23} - L_{13} L_{22} \right) + L_{32} \left( L_{13} L_{21} - L_{23} L_{11} \right) + L_{33} \left( L_{11} L_{22} - L_{12} L_{21} \right) \right] w = 0
\]  

(32)

If the \(L\) matrix is symmetric we have

\[
\begin{align*}
\left[ L_{11} L_{22} L_{33} + 2 L_{12} L_{23} L_{31} - L_{12} L_{33}^2 - L_{22} L_{31}^2 - L_{33} L_{12}^2 \right] w &= 0
\end{align*}
\]  

(33)

But if the initial stresses are spatially varying, the operators in the \(L\) matrix usually do not commute with each other and in such cases obtaining an uncoupled equation becomes impossible. As far as the author is aware of, it is only the Donnell theory which allows uncoupling in these cases. The difficulties in solution which arise when uncoupling is not possible is discussed later in this Chapter.
The boundary conditions of the shell, for which an excellent discussion is contained in reference (85) also need some examination. The conditions arise in the same manner as equations (17, 20 and 21). The form of the boundary conditions are by no means universal for all the theories and have to be derived for each case in the same manner as the differential equations. It is also important to ensure that proper constraints are imposed on the shell in order that it may withstand the given prestress distribution. Further consideration of end conditions is deferred to the succeeding Chapters.

For free vibration the solution to the set of equations

\[
L.(u, v, w, t) = 0
\]  

(34)

must be harmonic in time. Thus,

\[
\begin{align*}
u(x,s,t) &= U(x,s) e^{i\omega t} \\
v(x,s,t) &= V(x,s) e^{i\omega t} \\
w(x,s,t) &= W(x,s) e^{i\omega t}
\end{align*}
\]

(35)

The primary consideration, of course, is the determination of \(\omega\) (or its non-dimensional equivalent), while the modal distributions \(U, V, W\) are also important from a practical point of view. The latter can be further separated into their axial and circumferential parts. Due to the requirement of periodicity of solution in a complete shell,
\[ U(x,s) = \sum_{m} \sum_{n} A_{mn}(x) e^{ins} \]
\[ V(x,s) = \sum_{m} \sum_{n} B_{mn}(x) e^{ins} \]
\[ W(x,s) = \sum_{m} \sum_{n} C_{mn}(x) e^{ins} \] (36)

where \( A_{mn}, B_{mn}, \) and \( C_{mn} \) are complex and are subject to the satisfaction of the differential equations and the end conditions.

Corresponding to each mode, i.e., each value of \( m, n \) in equation (36), the homogeneous set of differential equations yields a characteristic equation which is cubic in \( \omega^2 \) providing three characteristic frequencies. However, the frequency equation will be of the first degree in \( \omega^2 \) if the inplane inertias are ignored, since, the neglect implies the remaining two frequencies to be infinitely large. The lowest of the frequencies is associated with a motion which is usually predominantly radial while the remaining two arise primarily due to the extensional stiffness \( H \) of the shell and hence, belong to a higher realm of the frequency spectrum.

Axisymmetric vibration arises when \( n = 0 \) in equation (36), i.e., \( u, v, w \), are functions of \( x \) alone. Beam-type vibration arises when \( n = 1 \) while a mode corresponding to \( n > 2 \) is called a "breathing" or "lobar" mode (see Figure 2). Similarly the plane-strain or ring-like oscillations are characterized by no variation along the \( x \)-direction.
CIRCUMFERENTIAL NODAL PATTERN

AXIAL NODAL PATTERN

NODAL ARRANGEMENT FOR $n=3, m=4$

Fig. 2 Nodal Patterns
The qualitative role of the initial stress $\sigma_x^i$ in drastically altering the lowest of the three frequencies associated with each modal distribution has been long known. Also, the relatively insignificant change in the other two frequencies due to prestress has been pointed out by various authors. But the combinatorial capacity of prestresses on frequency characteristics of the shell is less clearly understood. For instance Langhaar (ref. 86) predicts that the departure from the nonlinear terms contained in the Donnell theory is a minor detail. But it is precisely the nonlinear terms which account for the terms multiplying the initial stresses in the differential equations. In this context it is noteworthy that according to the Donnell theory (eq. 22), internal pressure plays no part in axisymmetric oscillation of the shell. Reference 35 on the other hand interprets internal pressure in the shell as equivalent to circumferential bending stiffness. Interestingly, reference 68 views the relative influence of the prestress on the frequency as:

$$\omega^2 = \omega_0^2 \left(1 - \frac{\sigma_x^i}{(\sigma_x)_{cr}} - \frac{\sigma_y^i}{(\sigma_y)_{cr}} \right)$$  \hspace{1cm} (37)

where $(\sigma_x)_{critical}$ and $(\sigma_y)_{critical}$ are the values of the prestress which individually could cause the shell to buckle. It is also claimed that the relation would reasonably hold even when the initial stress field is nonuniform. This statement is scrutinized in the succeeding Chapters.
For determining the frequency parameter and the mode shapes the most often used method are: (a) exact, and (b) Ritz-Galerkin. In the exact method when the initial stresses do not vary along s the displacements are assumed as:

\[ u = \sum_{n=1}^{n} A_n^{(1)} e^{\frac{s}{2}} \cos ns e^{i\omega t} \]
\[ v = \sum_{n=1}^{n} A_n^{(2)} e^{\frac{s}{2}} \sin ns e^{i\omega t} \]
\[ w = \sum_{n=1}^{n} A_n^{(3)} e^{\frac{s}{2}} \cos ns e^{i\omega t} \]  \( (38) \)

where the \( \lambda \)'s are to be chosen such that the equation (38) satisfies the boundary conditions of the shell and the \( n \)'s are consecutive integers. Equation (38) is then substituted into the differential equation and the coefficient of each \( e^{\frac{s}{2}} \cos ns e^{i\omega t} \) in set \( \sin ns \) equal to zero. Thus, to each value of \( n \) there corresponds a set of three homogeneous equations. If the initial stresses are uniformly distributed in the shell, each set of equations will have only four unknowns viz., \( A_n^{(1)} \), \( A_n^{(2)} \), \( A_n^{(3)} \) and \( \omega \) and there is no coupling among circumferential modes in the resulting equations. For a non-trivial solution the determinant of the matrix of coefficients of \( A_n^{(1)} \), \( A_n^{(2)} \), and \( A_n^{(3)} \) is set equal to zero, the expansion of which yields a cubic in \( \omega^2 \).

If the initial stresses vary in the circumferential direction, they may be expanded in a Fourier series. Following the same procedure as before and after rewriting the products of trigonometric functions
which arise as sums and differences, the coefficient of the term
\[ e^{\lambda x} e^{i\omega t} \left( \cos \frac{n\pi}{L} \right) \] is set equal to zero for each value of \( n \). As before, three equations arise for every value of \( n \). However, each set of the three equations will, in general, have the entire range of \( A \)'s and \( \omega \) as unknowns. Hence, a non-trivial solution would involve the equating of the sum of an infinite number of third order determinants to zero. An approximate numerical solution may be obtained by truncating the series in equation (38) thus, limiting the number of determinants. Each \( \omega \) thus determined will be a function of more than one value of \( n \).

The determination of \( \lambda \) in equation (38) is difficult for most combinations of end conditions. Substituting equation (38) into the four boundary conditions at each end of the shell yields eight homogeneous equations which, on expanding, give a fourth degree polynomial in \( \lambda^2 \). Unfortunately, the eight roots, all of them complex, cannot be easily determined.

Only in the case of the shell "freely supported" at both ends (shear diaphragms) are the \( \lambda^2 \) easy to determine.

The two above mentioned difficulties compound to render the exact method of little use for the varying initial stress problem.

The Ritz-Galerkin procedure derives its logic from the calculus of variations and hence, offers the usual simplicity of energy methods. A detailed discussion of its relative merits vis-a-vis other well-known approximate methods appears in reference 87.
Briefly, the Galerkin procedure for finding a solution to the differential equation

\[ L(w) = 0 \]  

(39)

and subject to certain boundary conditions, consists of choosing a set of functions

\[ w = \sum C^k w^k \]  

(40)

where the \( C^k \)'s are unknown coefficients and \( w^k \) is a set of functions which is consistent with the constraints in the problem. The coefficients are determined by substituting equation (40) into equation (39) and orthogonalizing the resulting equation with respect to each member of the chosen set, the integration being performed over the domain of equation (39). Naturally, the choice of an orthogonal set offers great computational advantage. For a numerical evaluation of a solution, it becomes necessary to truncate the series.

The Ritz method is based upon the principle of minimum potential energy. It has at times been quoted as completely equivalent to the Galerkin method. The limitation of the equivalence is discussed in reference 87 but three points are particularly noteworthy. 1) If the Ritz method is admissible to the problem then an equivalent Galerkin procedure can also be formulated. 2) The Galerkin method is applicable even when no extremum problem exists. 3) Whenever the two methods are completely equivalent, it is simpler to use the Galerkin method. Accordingly, we confine our attention to the Galerkin method.
The Batdorf equation (eq. 27) is of the same class of equations as equation (39) and hence, the problem of finding a solution in such a case is straightforward. As has been demonstrated in plate problems, beam functions would probably serve excellently for the set in equation (40) for shells also. However, circular functions are usually adequate substitutes and relatively easy to handle algebraically. Hence, they will be chosen wherever appropriate.

If the differential equation is a set of three equations which do not admit uncoupling in the three dependent variables, such as

\[
\begin{pmatrix}
L_{11} & L_{12} & L_{13} \\
L_{21} & L_{22} & L_{23} \\
L_{31} & L_{32} & L_{33}
\end{pmatrix}
\begin{pmatrix}
u \\
v \\
w
\end{pmatrix}
= \begin{pmatrix}
o \\
o \\
o
\end{pmatrix}
\]  (41)

the Galerkin method will have to be modified. Three sets of functions, respectively, for \( u \), \( v \), and \( w \) have to be chosen in such a case and the orthogonalization performed according to the scheme:

\[
\int_s \int_s \left[ \sum_k A_k u_k + \sum_k B_k v_k + \sum_k C_k w_k \right] \delta u \, dx \, ds = 0
\]

\[
\int_s \int_s \left[ \sum_k A_k u_k + \sum_k B_k v_k + \sum_k C_k w_k \right] \delta v \, dx \, ds = 0
\]

\[
\int_s \int_s \left[ \sum_k A_k u_k + \sum_k B_k v_k + \sum_k C_k w_k \right] \delta w \, dx \, ds = 0
\]

(42)
where $A^k u_k$, $B^k v_k$, $C^k w_k$ are the chosen set of functions for $u$, $v$, and $w$ respectively, and where on substituting for the variations $\delta u$, $\delta v$ and $\delta w$ we get

$$
\int \int [ L_1 (\sum_k A^k u_k) + L_2 (\sum_k B^k v_k) + L_3 (\sum_k C^k w_k)] \ u_p \ dx \ ds = 0
$$

$$
\int \int [ L_1 (\sum_k A^k u_k) + L_2 (\sum_k B^k v_k) + L_3 (\sum_k C^k w_k)] \ v_p \ dx \ ds = 0
$$

$$
\int \int [ L_1 (\sum_k A^k u_k) + L_2 (\sum_k B^k v_k) + L_3 (\sum_k C^k w_k)] \ w_p \ dx \ ds = 0
$$

(43)

where $p = 1, 2, 3, \ldots, \ldots$

It may be verified that the logic of this modification follows directly from the principle of stationary value of energy, (see reference 10). If $n$ terms are retained in the chosen set of functions, the secular determinant will be of $(3n)$ th order.
CHAPTER IV

Shell Under Linearly Varying Axial Prestress

As a first example, the problem of a pressurized cylindrical shell under a linearly varying axial prestress is considered. Such a variation simulates the body-forces induced in the shell by gravity when the axis of the shell is in the vertical direction. Hence, the distribution of the initial stresses is assumed to be:

\[
\begin{align*}
\sigma_x &= n_x^0 + \frac{x}{L} n_x^1 \\
\sigma_r &= n_r^0 \\
\tau_{rs} &= 0
\end{align*}
\]  

(44)

where \(n_x^0, n_x^1\) and \(n_r^0\) are constants.

The shell is analyzed on the basis of the Donnell equations (22) generalized to include the effect of orthotropy although all numerical results presented herein are for an isotropic material. An uncoupled equation in the radial displacement \(w\) obtained by following the scheme indicated by equation (33) and using equation (25) is:
\[- a^2 \left[ \frac{5}{L} \frac{\partial^5 W}{\partial x^5} + (n_0^+ x L) \frac{\partial^6 W}{\partial x^6} \right] - (a + a_1^2 - a_2^2) \left[ \frac{3}{L} \frac{\partial^3 W}{\partial x^3} + (n_0^+ x L) \frac{\partial^4 W}{\partial x^4} \right] \]

\[- a^2 \left[ \frac{n_x^+}{L} \frac{\partial^5 W}{\partial x^5} + (n_0^+ x L) \frac{\partial^6 W}{\partial x^6} \right] - \gamma (1 + a_2^2) \left[ \frac{3}{L} \frac{\partial^3 W}{\partial x^3} + (n_0^+ x L) \frac{\partial^4 W}{\partial x^4} \right] \]

\[- \gamma (a + a_2^2) \left[ \frac{n_x^+}{L} \frac{\partial^5 W}{\partial x^5} + (n_0^+ x L) \frac{\partial^6 W}{\partial x^6} \right] - \gamma^2 \left[ \frac{n_x^+}{L} \frac{\partial^3 W}{\partial x^3} + (n_0^+ x L) \frac{\partial^4 W}{\partial x^4} \right] \]

\[- n_0^+ \left[ a^2 \frac{\partial^5 W}{\partial x^5} + (a + a_2^2 - a_3^2) \frac{\partial^6 W}{\partial x^6} + a a_5 \frac{\partial^6 W}{\partial x^6} + \gamma (1 + a_2^2) \frac{\partial^5 W}{\partial x^5} + \gamma (a + a_2^2) \frac{\partial^6 W}{\partial x^6} \right] + \gamma^2 (a_1 - \gamma) W + \gamma^2 (a_1 - \gamma) - \gamma^2 a_1^2 \frac{\partial^2 W}{\partial s^2} \]

\[+ \gamma^2 (a_1 - \gamma) + \gamma^2 (a_1 - \gamma) - \gamma^2 a_1^2 \frac{\partial^2 W}{\partial s^2} + a_1^2 \frac{\partial^4 W}{\partial s^4} + \gamma^2 a_1^2 \frac{\partial^2 W}{\partial s^2} \]

\[+ \gamma^2 (a_1 - \gamma) + \gamma^2 (a_1 - \gamma) - \gamma^2 a_1^2 \frac{\partial^2 W}{\partial s^2} + a_1^2 \frac{\partial^4 W}{\partial s^4} + \gamma^2 a_1^2 \frac{\partial^2 W}{\partial s^2} \]

\[+ \gamma \left[ a_2 (a + a_2) + a_2 (1 + a_2) \right] \frac{\partial^6 W}{\partial x^6} + \gamma \left[ a_2 (a + a_2) + a_2 (1 + a_2) \right] \frac{\partial^6 W}{\partial x^6} + a a_3 \frac{\partial^6 W}{\partial x^6} \]

\[+ \left[ a_2 (a + a_2) + a_2 (1 + a_2) \right] \frac{\partial^6 W}{\partial x^6} + \left[ a_2 (a + a_2) + a_2 (1 + a_2) \right] \frac{\partial^6 W}{\partial x^6} + \gamma \left[ a_2 (a + a_2) + a_2 (1 + a_2) \right] \frac{\partial^6 W}{\partial x^6} \]

\[+ \gamma \left[ a_2 (a + a_2) + a_2 (1 + a_2) \right] \frac{\partial^6 W}{\partial x^6} + \gamma \left[ a_2 (a + a_2) + a_2 (1 + a_2) \right] \frac{\partial^6 W}{\partial x^6} + \gamma \left[ a_2 (a + a_2) + a_2 (1 + a_2) \right] \frac{\partial^6 W}{\partial x^6} \]

\[+ \gamma \left[ a_2 (a + a_2) + a_2 (1 + a_2) \right] \frac{\partial^6 W}{\partial x^6} + \gamma \left[ a_2 (a + a_2) + a_2 (1 + a_2) \right] \frac{\partial^6 W}{\partial x^6} + \gamma \left[ a_2 (a + a_2) + a_2 (1 + a_2) \right] \frac{\partial^6 W}{\partial x^6} \]

\[+ \gamma \left[ a_2 (a + a_2) + a_2 (1 + a_2) \right] \frac{\partial^6 W}{\partial x^6} + \gamma \left[ a_2 (a + a_2) + a_2 (1 + a_2) \right] \frac{\partial^6 W}{\partial x^6} + \gamma \left[ a_2 (a + a_2) + a_2 (1 + a_2) \right] \frac{\partial^6 W}{\partial x^6} \]

\[+ \gamma \left[ a_2 (a + a_2) + a_2 (1 + a_2) \right] \frac{\partial^6 W}{\partial x^6} + \gamma \left[ a_2 (a + a_2) + a_2 (1 + a_2) \right] \frac{\partial^6 W}{\partial x^6} + \gamma \left[ a_2 (a + a_2) + a_2 (1 + a_2) \right] \frac{\partial^6 W}{\partial x^6} \]

\[= 0 \quad (45)\]
where $E_x$, $E_s$, $\nu_{xs}$, $\nu_{sx}$ and $G$ are the elastic constants associated with the lines of orthotropy which were assumed to coincide with the $x$ and $s$ directions respectively, and:

\[
\begin{align*}
H_x &= E_x h / (1 - \nu_{xs} \nu_{sx}) \\
H_s &= E_s h / (1 - \nu_{xs} \nu_{sx}) \\
H_k &= G h / (1 - \nu_{xs} \nu_{sx}) \\
D_x &= E_x h^3 / 12 (1 - \nu_{xs} \nu_{sx}) \\
D_s &= E_s h^3 / 12 (1 - \nu_{xs} \nu_{sx}) \\
D_k &= G h^3 / 12 (1 - \nu_{xs} \nu_{sx}) \\
\sigma_x^i &= \frac{\sigma_x^{i}}{E_x} (1 - \nu_{xs} \nu_{sx}) / E_x \\
\sigma_s^i &= \frac{\sigma_s^{i}}{E_s} (1 - \nu_{xs} \nu_{sx}) / E_x \\
\alpha_1 &= H_s / H_x \\
\alpha_2 &= H_k / H_x \\
\alpha_3 &= D_x / H_x \\
\alpha_4 &= D_k / H_x \\
\alpha_6 &= \nu_{xs} \alpha_1 + \alpha_2 \\
\alpha_s &= D_s / H_x \\
\alpha_7 &= (\nu_{sx} \alpha_3 + 4 \alpha_4 + \nu_{xs} \alpha_s) \\
\gamma &= \rho \omega^2 \alpha^2 (1 - \nu_{xs} \nu_{sx}) / E_x
\end{align*}
\]
The effect of closely spaced ribs or stiffeners may be taken into account as indicated in reference 45.

**Simply-Supported Ends:**

Seeking a solution for $w$ through the use of Galerkin method, the deflected shape is assumed as:

$$W(x,s) = \sum_{m=1,2,...} \sum_{n=0,1,2,...} A_m \sin \lambda_x \cos \lambda_y n s, \quad \lambda = \frac{m\pi}{L}$$

which is consistent with the assumed simply-supported end conditions for the shell, viz:

$$W = \frac{\partial^2 W}{\partial x^2} = 0 \quad @ \quad x = 0, L$$

(48)

Two more conditions, those involving the degree of restraint in the axial and circumferential directions at the ends of the shell are also to be satisfied by equation (47). But the choice in equation (47) automatically imposes certain restraints which may or may not coincide with the actual boundary conditions. For instance, in the case of isotropy and the neglect of the in-plane inertias, since

$$u(x,s,t) = U(x,s) e^{i\omega t} = V^{-4} \left[ \sqrt{V} \left( \frac{3}{2} \frac{\partial^3 W}{\partial x^3} - \frac{3}{2} \frac{\partial^3 W}{\partial x \partial s^2} \right) e^{i\omega t} \right]$$

$$v(x,s,t) = V(x,s) e^{i\omega t} = V^{-4} \left[ (2+\nu) \frac{\partial^3 W}{\partial x^2 \partial s} + \frac{3}{2} \frac{\partial^3 W}{\partial s^3} \right] e^{i\omega t}$$

where

$$V^{-4} \cdot V^4 W = W$$

(49)

equation (47) implies

$$N_x = V = 0 \quad @ \quad x = 0, L$$

(50)
Substituting equation (47) into equation (45) and successively orthogonalizing against each coefficient of \( A_m \) in equation (47) results in the system of equations:

\[
\begin{align*}
(-\gamma^2 + d_1 \gamma^2 + e_1 \gamma + f_1)A_1 + \left(h_1 \gamma^2 + i_1 \gamma + k_1 \right)A_2 + \left(l_1 \gamma^2 + m_1 \gamma + n_1 \right)A_3 + \ldots &= 0 \\
(-\gamma^2 + d_2 \gamma^2 + e_2 \gamma + f_2)A_1 + \left(h_2 \gamma^2 + i_2 \gamma + k_2 \right)A_2 + \left(l_2 \gamma^2 + m_2 \gamma + n_2 \right)A_3 + \ldots &= 0 \\
(-\gamma^2 + d_3 \gamma^2 + e_3 \gamma + f_3)A_1 + \left(h_3 \gamma^2 + i_3 \gamma + k_3 \right)A_2 + \left(l_3 \gamma^2 + m_3 \gamma + n_3 \right)A_3 + \ldots &= 0 \\
\ldots &\ldots \\
\ldots &\ldots \\
\end{align*}
\]

(51)

where \( d_1, e_1, \ldots, n_3 \) are constants and depend on the shell parameters. The algebraic expressions for these constants are given in Appendix A.

To obtain a non-trivial solution the determinant of the matrix of coefficients of \( A_1, A_2, \ldots \) in equation (51) was set equal to zero. It has been shown in reference 61 that the third order truncated determinant yields an adequately accurate result for the frequency parameter. Consequently, the maximum matrix size was limited to the third order in the present case, although a sample of convergence of results for higher orders is demonstrated for the case of the clamped-clamped cylinder later in this Chapter. The determinant was expanded (see Appendix B) and the roots of the resulting polynomial were evaluated by the computer using a FORTRAN polynomial root-finding sub-routine. Some of the roots are extraneous to the problem, but
the lowest provides the frequency parameter for the predominantly radial mode.

From statics, the prestress (see Fig. 3a) arising solely due to a gravitational force, when the shell is supported identically at the two ends, gives $n^c_X = -n^{l/2}_X$ in equation (44). The destabilizing role of such an initial stress distribution may be seen in Table 12. In the table the values of the frequency parameter have been given for various values of $n^{l}_X$ for $\frac{h}{a}$ and $L (=l/a)$ equal to 0.001 and 4 respectively, and in the absence of $n^o_S$. The reduction in the frequencies due to initial stress is also depicted in Figure 4. In this plot the ratio of the frequency parameter for the stressed and unstressed case is plotted against the amplitude of the initial stress $n^{l}_X$ for various values of the circumferential wave number.
Values of $\gamma$ for various numbers of circumferential waves $n$ (breathing modes) and amplitudes of varying axial stress ($n_x^1 = -2n_s^0$)

Simply-supported ends; isotropic case
Matrix size in equation (51) = $[3 \times 3]$

\[ h = 0.001 \]
\[ \frac{a}{L} = 4 \]
\[ v = 0.3 \]
\[ n_s^0 = 0 \]

<table>
<thead>
<tr>
<th>$n$</th>
<th>$n_x^1$</th>
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</thead>
<tbody>
<tr>
<td>0</td>
<td>0.012697*</td>
</tr>
<tr>
<td>1</td>
<td>0.00004</td>
</tr>
<tr>
<td>2</td>
<td>0.012697</td>
</tr>
<tr>
<td>3</td>
<td>0.003350</td>
</tr>
<tr>
<td>4</td>
<td>0.0011982</td>
</tr>
<tr>
<td>5</td>
<td>0.0005571*</td>
</tr>
<tr>
<td>6</td>
<td>0.00055915</td>
</tr>
<tr>
<td>7</td>
<td>0.00035971</td>
</tr>
<tr>
<td>8</td>
<td>0.00033874</td>
</tr>
<tr>
<td>9</td>
<td>0.00041470*</td>
</tr>
<tr>
<td>10</td>
<td>0.00042414</td>
</tr>
<tr>
<td>11</td>
<td>0.00059959</td>
</tr>
<tr>
<td>12</td>
<td>0.00086906</td>
</tr>
</tbody>
</table>

* values for the $[2 \times 2]$ determinant.
Figure 3. Axial stress distribution along the length of the shell
a) Assumed in obtaining results given in Tables 13 and 20
b) Assumed in obtaining results in Tables 14 and 15
Fig. 4: Variation of the ratios of frequency parameter \((\gamma/\gamma_0)\) of initially stressed and unstressed shell with initial stress \(n_s^0\) for various circumferential wave numbers: Freely-supported ends.

\[
\frac{h}{a} = 0.001 \quad L = 4 \quad v = 0.3 \quad n_s^0 = 0
\]
But from Table 12 it is seen that the destabilizing effect due to gravity increases sharply with initial stress. As is to be expected destabilization is most pronounced at the lowest frequencies. In rocket casings for weaponry where the structural design is far less conservative than for a space vehicle and the g loading much higher, the effect of the initial stress on the vibrational frequencies could be to reduce the lower ones drastically.

In addition, the support conditions in a launch structure approach the more flexible simply-supported-free case as against the simply-supported-simply supported case considered in Table 12. In such a case the distribution of the axial prestress will be of the type (see Fig. 3b).

\[ \sigma_x = \frac{x}{L} \eta_x \]  \hspace{1cm} (52)

In Table 13 the frequency parameter for a shell simply-supported at both ends and subjected to the axial prestress given by equation (52) is presented. Figure 5 illustrates the same effect with the ratio of the frequency for the stressed and unstressed case being plotted against \( n_x \). If the maximum amplitude of axial stress in Figures (3a) and (3b) are to be equal, the values of \( n_x \) in Table 13 corresponds to \( \frac{n_x}{2} \) in Table 12. A comparison of the two tables shows a further reduction in the frequencies under an axial stress distribution given by equation (52) where the entire length of the shell is in compression as against the distribution given by equation (44). Table 14 and Figure 6 demonstrate the effect of
TABLE 13: Values of $\gamma$ for various numbers of circumferential waves $n$ (lobar modes) and amplitudes of varying axial stress ($n^o_x = 0$)

Simply-supported ends; isotropic case

Matrix size in equation (51) = [ 3 x 3]

\[ h = 0.001 \quad L = 4 \quad v = 0.3 \quad n^o_s = 0 \]

<table>
<thead>
<tr>
<th>$n$</th>
<th>$n^o_x$</th>
<th>$-0.0002$</th>
<th>$-0.004$</th>
<th>$-0.006$</th>
<th>$-0.008$</th>
<th>$-0.010$</th>
</tr>
</thead>
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<tr>
<td>2</td>
<td>0.012697</td>
<td>0.012649</td>
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</tr>
<tr>
<td>4</td>
<td>0.0011982</td>
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<td>0.00096431</td>
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<td>0.00055915</td>
<td>0.00049955</td>
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<td>0.00029894</td>
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<td>0.00010443</td>
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</tr>
<tr>
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<td>0.00027676</td>
<td>0.00021096</td>
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<td>Buckled</td>
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<td>0.00036059</td>
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<td>0.00027227</td>
<td>&quot;  &quot;</td>
</tr>
</tbody>
</table>
Fig. 5: Variation of the ratio of frequency parameter \(\frac{\gamma}{\gamma_0}\) of initially stressed and unstressed shell with initial stress for various circumferential wave numbers: Freely-supported ends.
TABLE 14: Values of $\gamma$ for various numbers of circumferential waves $n$ (breathing modes) and amplitudes of varying axial stress ($n_{x} = 0$)

Simply-supported ends; isotropic case

Matrix size in equation (51) = [ 3 x 3]

\[
\begin{align*}
\frac{h}{a} &= 0.001 & L &= 4 & v &= 0.3 & n_{x} = 0.00004
\end{align*}
\]

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>-0.0002</th>
<th>-0.0004</th>
<th>-0.0006</th>
<th>-0.0008</th>
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<td>0.0039875</td>
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</table>
Fig. 6: Variation of the ratio of frequency parameter ($\gamma/\gamma_0$) of (axially) initially stressed and (axially) unstressed, pressurized shell with initial stress for various circumferential wave numbers: Freely-supported ends.
internal pressure acting on the shell in conjunction with an axial stress of the type given by equation (44). The conversion of the non-dimensional circumferential stress into pressure is effected by multiplying the former by the factor \( \frac{E}{1 - \nu^2} (\frac{a}{h}) \).

In order to study the effect of the neglect of tangential inertias in the differential equations, solutions were also obtained for the Donnell equation for the isotropic case. The latter equation in the absence of torsional prestress is:

\[
\nabla^4 \left[ -\frac{\partial}{\partial x} \left( \sigma^i \frac{\partial W}{\partial x} \right) - \frac{\partial^2}{\partial z^2} \frac{\partial W}{\partial z} + (1-\nu^2) \nabla^2 \frac{\partial W}{\partial x^2} + k \nabla^4 W - \gamma W \right] = 0
\]

(53)

As before the radial displacement for the simply-supported ends is assumed as

\[
\omega(x,s,t) = W(x,s) e^{i\omega t} = \left( \sum_{n_1=1,2,...} \sum_{n_2=1,2,...} \sum_{m=1} A_m \sin \lambda_m x \cos n \right) e^{i\omega t}
\]

(54)

Solutions for equation (53) using equation (54) were obtained in the same manner as for equation (45) particularized to the case of isotropy. The ratios of the frequency parameters obtained for equation (53) and equation (45) have been tabulated in Tables 15, 16, 17 and plotted in Figures 7, 8, and 9 respectively. In Table 15 the ratios have been calculated for various numbers of circumferential waves and non-dimensional lengths of the shell with the thickness parameter being held constant and the initial stresses zero. Table 16 shows the same effect with the thickness of the shell being varied instead of the length.

The variation of the ratio for various values of uniform axial stress is shown in Table 17.
TABLE 15: Values of (γ) obtained with tangential inertias neglected (γ) obtained with tangential inertias included

For various lobular modes and lengths of the shell

Number of terms retained in equation (54) = 3

h/a = 0.001, v = 0.3, n_x^0 = n_x^1 = n_s^0 = 0

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<th>n</th>
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Fig. 7: Variation of the ratio of frequency parameter \( \frac{\gamma_n}{\gamma_1} \) obtained with the neglect and inclusion of tangential inertias for various lengths of the shell and circumferential wave numbers: Freely-supported ends.

\[
\frac{h}{a} = 0.01 \\
v = 0.3 \\
n_x^e = n_x^1 = n_x^2 = 0
\]
TABLE 16: Values of (γ) obtained with tangential inertias neglected
(γ) obtained with tangential inertias included

For various lobar modes and thicknesses of the shell

Number of terms retained in equation (54) = 3

L = 4, ν = 0.3, n_x = n_y = n_z = 0

<table>
<thead>
<tr>
<th>n</th>
<th>thickness (= h/a)</th>
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<th>.005</th>
<th>.01</th>
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Fig. 8: Variation of the ratio of frequency parameter \( \frac{\gamma_n}{\gamma_i} \) obtained with the neglect and inclusion of tangential inertias for various thicknesses of the shell and circumferential wave numbers: Freely-supported ends.

\begin{align*}
L &= 4 \\
v &= 0.3 \\
n_x^0 &= n_x^1 = n_x^2 = 0
\end{align*}
TABLE 17: Values of \( (\gamma) \) obtained with tangential inertia neglected
\( (\gamma) \) obtained with tangential inertia included

For various lobar modes and uniform axial prestresses on the shell

Numbers of terms retained in equation (54) = 3

\[ L = 4, \; v = 0.3, \; h/a = .001, \; n_x^1 = n_s^0 = 0 \]

<table>
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<th>( n_x^0 ), Axial prestress (negative sign indicates compression)</th>
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<td>1.0286</td>
</tr>
<tr>
<td>7</td>
<td>1.0208</td>
</tr>
<tr>
<td>8</td>
<td>1.0159</td>
</tr>
<tr>
<td>9</td>
<td>1.0125</td>
</tr>
<tr>
<td>10</td>
<td>1.0101</td>
</tr>
</tbody>
</table>
Fig. 9: Variation of the ratio of frequency parameter \( \frac{\gamma_n}{\gamma_i} \) obtained with the neglect and inclusion of tangential inertias for various values of axial prestress and circumferential wave numbers: Freely-supported ends.

\[ L = 4 \]
\[ \frac{h}{a} = 0.001 \]
\[ \nu = 0.3 \]
\[ n_X^l = n_S^o = 0 \]
The results in Tables 15 and 16 seem to be in complete accord with the discussion contained in Reference 84. In addition the level of initial stress appears to cause no changes in the ratios computed for each value of \( n \). In this respect the role of initial stress seems to be identical to that of the thickness parameter. The greater sensitivity of the frequencies to tangential inertias for low values of \( n \) is general among the tables given. However, as has been pointed out earlier, the Donnell theory itself suffers in its accuracy for such values.

**Clamped-Clamped Ends:**

For a shell clamped at both ends the appropriate boundary conditions involving the radial displacement are

\[
W = \frac{\partial W}{\partial x} = 0 \quad \forall \quad x = 0, L
\]

(55)

The deflection function for \( W \) is assumed to be expressible by

\[
W = \sum_{n=1,3,5,\ldots} \sum_{n=2,4,6,\ldots} A_n (\cos \lambda_{n-1} x - \cos \lambda_n x) \cos \alpha x
\]

(56)

which satisfies equation (55) identically. However, the conditions involving \( u \) and \( v \), calculated in the same manner as earlier, are

\[
u = 0 \quad \text{or} \quad N_{xS} = 0 \quad \forall \quad x = 0, L
\]

(57)

**Solutions** were obtained for equation (53) using equation (56) and the Galerkin method. The secular determinant obtained thereby for the case of a uniform state of prestress is
where

\[
\begin{pmatrix}
2M_0 + M_2 - 3\gamma & -M_2 + \gamma & 0 & \cdots \\
-M_2 + \gamma & M_2 + M_4 - 2\gamma & -M_4 + \gamma & \cdots \\
0 & -M_4 + \gamma & M_4 + M_6 - 2\gamma & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix} \mathbf{n} = 0
\]

The eigenvalues for the determinant were evaluated by using an eigenvalue subroutine made available to the author through the courtesy of Mr. R. A. Rosanoff of North American Aviation, Inc. The subroutine, which converts the matrix to an upper Hessenberg form, calculated the roots in double precision arithmetic.

The convergence of the values of \( \gamma \) obtained for the above determinant with the circumferential prestress equal to zero is illustrated in Table 18 for various sizes of determinants formed by truncating the determinant in equation (58). The convergence study is mathematically acceptable because of the use of the Galerkin method and the series in equation (55) which may be shown to be a complete set.

The values of \( \gamma \) in Table 18 appear to converge quite satisfactorily.
The reduction in the frequency caused by a varying axial
prestress (eq. (44)) was also computed for the case of $n_x^o = -n_x^{1/2}$,
as was done earlier for the simply-supported case. These results
are shown in Table 19 for various amplitudes of the axial stress
and plotted in Figure 10 with the frequency non-dimensionalized
against that for an unstressed shell. Comparing Tables 12 and 19,
the destabilizing effect of the varying axial stress appears to
be less severe in the clamped-clamped case when compared to the
simply-supported-simply supported case. Consequently, it seems
reasonable to conclude that a more debilitating effect on the
frequencies due to body forces such as gravity results when the end
conditions are more flexible, like simply-supported-free.
TABLE 18: A sample of convergence of solution for a shell clamped at both ends: Isotropic Case

Equations used: Donnell
Method of solution: Galerkin

\[ L = 10, \; \frac{h}{a} = 0.005, \; v = 0.3, \; n_0^0 = 0.0004 \]

\[ n_0^0 = 0, \; n_s^s = 0 \]

Values of \( \gamma \) obtained by retaining various number of terms in equation (56) for various lobar modes \( n \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>Number of terms retained</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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</tr>
<tr>
<td>2</td>
<td>0.0025360</td>
</tr>
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<td>3</td>
<td>0.00076206</td>
</tr>
<tr>
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<td>0.00077074</td>
</tr>
<tr>
<td>5</td>
<td>0.0014418</td>
</tr>
<tr>
<td>6</td>
<td>0.0028082</td>
</tr>
<tr>
<td>7</td>
<td>0.0051011</td>
</tr>
<tr>
<td>8</td>
<td>0.0086326</td>
</tr>
</tbody>
</table>
TABLE 19: Values of $\gamma$ for various circumferential wave numbers $n$ (lobar modes) and amplitudes of varying axial stress

\[
(n_x^1 = -2n_x^0)
\]

Clamped ends; isotropic case

Number of terms retained in equation (56) = 3

\[
\begin{align*}
h/a &= 0.001 \\
L &= 4 \\
v &= 0.3 \\
n_x^0 &= 0
\end{align*}
\]

<table>
<thead>
<tr>
<th>$n$</th>
<th>$0$</th>
<th>.0004</th>
<th>.0008</th>
<th>.0012</th>
<th>.0016</th>
<th>.0020</th>
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<td>.00073274</td>
<td>.00068496</td>
</tr>
</tbody>
</table>
Fig. 10: Variation of the ratio of frequency parameter \((\gamma/\gamma_0)\) of initially stressed and unstressed shell with initial stress \(n_x^1\) for various circumferential wave numbers: Clamped ends.
Cylindrical Shell Under Circumferential Prestress Varying in the Axial Direction

A linearly varying circumferential prestress due to hydrostatic pressure in a cylindrical shell is a common occurrence. Non-uniform hoop compression which varies along the axis of the cylinder can also arise due to a thermal gradient along the axis or due to non-uniform geometrical constraints, like rings imposed along some sections against deformation due to temperature changes. In re-entry vehicles, for example, which are subjected to rapid heating, such a prestress could well cause a substantial loss of stability.

A remarkable feature of the few earlier investigations (refs. 4, 5, and 7) of such a class of problems is the exclusive and incorrect use of the Donnell equation as modified by Batdorf (eq. (30)).

Considering a general variation in the initial hoop stress along the axis, it may be represented in the form

\[ \sigma_s^i = \sigma_s^o + \sum_{r=1,2,...} \sigma_s^r \sin \lambda r^i x \]

\[ \lambda = \frac{r^i n_l}{l} \]  

(59)
Assuming that
\[ \sigma_x^i = \eta_x + c_r^i \]

substitution in the uncoupled Donnell equation in \( W \) obtained upon neglecting tangential inertias (eq. (26)) yields:

\[
V^4 \left[ - \eta_x \frac{\partial^2 W}{\partial x^2} - \left\{ \eta_x + \sum_{r=1}^\infty \eta_r \sin \lambda r \right\} \frac{\partial W}{\partial s^2} + (1 - \nu^2) V^4 \frac{\partial^2 W}{\partial x^4} + k V^4 W - \gamma W \right] = 0
\]

where

\[
\nabla^{-4} V^4 W = W
\]

If the ends of the shell are simply-supported, we may, as before, assume for each circumferential modal number \( n \)

\[
W = \cos ns \sum_{m=1}^\infty A_m \sin \lambda m x, \quad \lambda m = \frac{m \pi}{L}
\]

Substituting in the equation (60) we have

\[
\cos ns \sum_{m=1}^\infty A_m \left[ \lambda_m^2 \eta_x + (1 - \nu^2) \frac{\lambda_m^4}{\lambda_m^2 + \lambda_n^2} + k \left( \lambda_m^2 + \lambda_n^2 \right)^2 \sin \lambda m x \right]
\]

\[
+ \cos ns \sum_{m=1}^\infty A_m \sum_{r=1}^\infty n_r^2 \left( \eta_r \sin \lambda_r x \right) \left( \lambda_m^2 + \lambda_n^2 \right)^2 \sin \lambda m x
\]

\[
+ \cos ns \sum_{m=1}^\infty A_m \sum_{r=1}^\infty n_r^2 \left[ \left( \lambda_r^4 + 6 \lambda_r^2 \lambda_m^2 + 2 \lambda_r^2 n_r^2 \right) \sin \lambda_m x \sin \lambda m x
\]

\[-4 \lambda_r^2 \lambda_m^2 \left( \lambda_m^2 + \lambda_n^2 + n_r^2 \right) \cos \lambda_m x \cos \lambda m x
\]

\[
= 0
\]

Factoring \( \cos ns \) and combining terms
\[
\sum_{m=1,2,...} A_m \left( \lambda_m^2 + \eta^2 \right)^2 \left[ \left\{ \lambda_m^2 n_m^r + n_m^e \right\} + (1-\nu^2) \frac{\lambda_m^4}{(\lambda_m^2 + \eta^2)^2} + k(\lambda_m^2 + \eta^2)^2 \right] \sin \lambda_m x \\
+ \sum_{r=1,2,...} n_r^2 \frac{r_s^e}{2} \left( \frac{\lambda_m^2}{m-1} - \frac{\lambda_m^2}{m+1} \right) \\
+ \sum_{r=1,2,...} n_r^2 \frac{r_s^e}{2(\lambda_m^2 + \eta^2)^2} \left( \frac{\lambda_m^4}{m-1} + 6 \frac{\lambda_m^2}{m-1} + 2 \frac{\lambda_m^2}{m+1} \right) \left( \cos \frac{\lambda_m x}{m-1} - \cos \frac{\lambda_m x}{m+1} \right) \\
- \sum_{r=1,2,...} \frac{\lambda_m^2}{2(\lambda_m^2 + \eta^2)^2} \left( \frac{\lambda_m^2}{m-1} + \frac{\lambda_m^2}{m+1} \right) \left( \cos \frac{\lambda_m x}{m-1} + \cos \frac{\lambda_m x}{m+1} \right) \\
= 0
\]

In attempting to derive the preceding equation from equation (61), the earlier cited references have inadvertently failed to recognize that resulting in the absence of the third and last terms in equation (63) in their analyses. Actually the Batdorf modification of the Donnell equation cannot be used for varying prestress problems.

Following the Galerkin method as has been in the references cited (refs. 4, 5, 7) we obtain the following set of simultaneous equations:
As may be seen in equation (64), for a general distribution
do $\sigma_s^i$, the axial mode shape becomes irregular due to the coupling
do the modes brought about by the prestress. The number of axial
half waves in this case ceases to have any real meaning although
for low amplitudes of $\sigma_s$ the modal shape will be almost sinusoidal.

A more detailed analysis was carried out for the prestress distribution

\[
\sigma_x^i = \sigma_x^c, \quad \sigma_y^i = \sigma_y^c + \eta_s^i \sin \lambda x, \quad \tau_x^i = 0
\]  

Retaining five terms in the series (eq. (62)), the frequency parameter was evaluated for the Donnell equation in the absence of tangential inertia (eq. (61)) by using the Galerkin method.

The fifth order matrix determinant for the frequency is given in Appendix C. If the incorrect form of the Donnell equation is used as has been in reference (4), the matrix determinant undergoes a slight modification and this is also indicated in Appendix C.

The frequency parameters obtained from the correct and incorrect usage of the Donnell equation for $L = 4$, $h = 0.001$, $n_x = n_s = 0$ are given in tables 20 and 21 respectively for a range of prestress amplitudes $n_s^1$. The results apparently had converged to within 1% of the true value. It is seen that there is practically no difference between the two cases, but this, is a fortuitous circumstance. The ratio of the frequency parameter obtained for the stressed and unstressed case from table 20 has been plotted, against the amplitude $n_s^1$ of the prestress, in figure 11.

A measure of the importance of the omissions in earlier references can qualitatively be studied by looking at the term:
<table>
<thead>
<tr>
<th>Number of Circumfer. Waves, n</th>
<th>Amplitude of circumferential prestress, ( n_s ) (tension positive) (non-dimensionalized)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10^5</td>
</tr>
<tr>
<td>2</td>
<td>1.6282 x 10^2</td>
</tr>
<tr>
<td>3</td>
<td>3.8290 x 10^3</td>
</tr>
<tr>
<td>4</td>
<td>1.4934 x 10^3</td>
</tr>
<tr>
<td>5</td>
<td>7.9486 x 10^4</td>
</tr>
<tr>
<td>6</td>
<td>6.7559 x 10^4</td>
</tr>
<tr>
<td>7</td>
<td>7.6118 x 10^4</td>
</tr>
<tr>
<td>8</td>
<td>9.7218 x 10^4</td>
</tr>
<tr>
<td>9</td>
<td>1.2895 x 10^5</td>
</tr>
<tr>
<td>10</td>
<td>1.7146 x 10^5</td>
</tr>
</tbody>
</table>

**TABLE 20:** Values of the frequency parameters \( \gamma(=\omega^2 a^2 p (1-\nu^2))/E \) of a simply-supported (shear diaphragms) cylindrical shell under a sinusoidally varying circumferential prestress \( \sigma_s = n_s^1 \sin \lambda x \). Equations used: Donnell (excluding in-plane inertias) \( h/a = 1/1000 \). Method of solution: Galerkin-exact \( \nu = 0.3 \). Size of matrix: \( 5 \times 5 \). \( L = (= /a) \).
TABLE 21: Values of the frequency parameter \( \gamma(=\omega^2 a^2(1-\nu^2)/2) \) of a simply-supported (shear diaphragms) cylindrical shell under a sinusoidally varying circumferential prestress \( \sigma_z^1 = \sigma_z' \sin \lambda x \).

Equations used: Donnell-Batdorf
Method of solution: Galerkin
Size of matrix: \([5 \times 5]\)

<table>
<thead>
<tr>
<th>Number of Circumferential Waves, ( n )</th>
<th>Amplitude of circumferential prestress, ( n_z^1 ) (tension positive) (non-dimensionalized)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n )</td>
<td>( 10^5 )</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>2</td>
<td>1.628 \times 10^2</td>
</tr>
<tr>
<td>3</td>
<td>3.828 \times 10^3</td>
</tr>
<tr>
<td>4</td>
<td>1.413 \times 10^3</td>
</tr>
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<td>7.945 \times 10^4</td>
</tr>
<tr>
<td>6</td>
<td>6.753 \times 10^4</td>
</tr>
<tr>
<td>7</td>
<td>7.610 \times 10^4</td>
</tr>
<tr>
<td>8</td>
<td>9.720 \times 10^4</td>
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<tr>
<td>9</td>
<td>1.289 \times 10^5</td>
</tr>
<tr>
<td>10</td>
<td>1.714 \times 10^5</td>
</tr>
</tbody>
</table>
Fig. 11: Variation of the ratio of frequency parameter \( (\gamma/\gamma_0) \) of initially stressed and unstressed shell with initial stress \( n_s \) for various circumferential wave numbers: Freely-supported ends.
The error increases with the magnitudes of the second and third terms relative to the first. Thus for large numbers of circumferential waves $n$, the first term dominates and hence the disparity is small. However, when $n$ attains high values as would be the case when the circumferential stress is rapidly varying, the second term becomes significant although offset somewhat by the third. In this connection the following comment in reference 4 is interesting.

"An analysis of the buckling temperature of a clamped end cylinder is also given in (reference 6). The result of this analysis is indicated by the square symbol in figure 1 and is seen to be almost twice the value obtained in (reference 7).... The stress distribution and boundary conditions were the same in the two references; however, the eighth-order Donnell equation was used in (reference 6) whereas the fourth-order modified Donnell equation was used in (reference 7). A preliminary check of the results of (reference 6) indicates that there may be some numerical errors in the calculations; however, the correct numerical results obtained from the Donnell equation would still be significantly higher than those obtained in (reference 7). Batdorf (reference 83) indicated that the use of the Donnell equation in combination with a Galerkin solution, as was done in (reference 6), could lead to incorrect results for clamped cylinders. The discrepancy is possibly due to divergent series resulting from differentiating the deflection function 8 times .... it does appear that the Donnell equation gives incorrect results for clamped cylinders and that the modified equation be used."

This point will be further examined in the next chapter.

As a comparison the frequencies were also calculated starting from the Sanders and Washizu equations including tangential
inertias respectively with the prestress distribution assumed as in equation (59), and following the procedure outlined towards the end of Chapter III. For these cases the displacements were assumed as

\[
\begin{align*}
U &= \cos n s \ e^{i\omega t} \sum_{m} \lambda_m \cos \lambda_m x \\
V &= \sin n s \ e^{i\omega t} \sum_{m} B_m \sin \lambda_m x \\
W &= \cos n s \ e^{i\omega t} \sum_{m} C_m \sin \lambda_m x
\end{align*}
\]

(67)

which exactly satisfy the end conditions for a freely-supported shell. Five terms were retained in each of the summations in equation (67). Thus, the Galerkin method yielded fifteenth order matrix determinants for the evaluation of the frequencies. The algebraic form of the matricies are given in Appendicies D and E. The frequency parameter obtained for these cases have been tabulated in tables 22 and 23. The convergence of results again appeared satisfactory. The results in table 22 are also illustrated in figure 12.

A comparative examination of tables 22 and 23 reveals that the results from the Washizu and Sanders theories are practically coincidental with each other. This trend in the results was
### Table 22

Values of the frequency parameter $\gamma (= \omega^2 a^2 \rho (1 - \nu^2) / E)$ of a simply-supported (shear diaphragms) cylindrical shell under a sinusoidally varying circumferential prestress ($\sigma_i = n_i \sin \lambda x$)

<table>
<thead>
<tr>
<th>Number of Circumferential Waves, $n$</th>
<th>Amplitude of circumferential prestress, $n_i$ (tension positive) (non-dimensionalized)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$1.2711 \times 10^2$ $1.2704 \times 10^2$ $1.2699 \times 10^2$ $1.2696 \times 10^2$ $1.2695 \times 10^2$ $1.2694 \times 10^2$ $1.2694 \times 10^2$ $1.2694 \times 10^2$ $1.2694 \times 10^2$ $1.2694 \times 10^2$</td>
</tr>
<tr>
<td>3</td>
<td>$3.4030 \times 10^3$ $3.3759 \times 10^3$ $3.3597 \times 10^3$ $3.3545 \times 10^3$ $3.3488 \times 10^3$ $3.3436 \times 10^3$ $3.3384 \times 10^3$ $3.3223 \times 10^3$ $3.2951 \times 10^3$</td>
</tr>
<tr>
<td>4</td>
<td>$1.3073 \times 10^3$ $1.2518 \times 10^3$ $1.2178 \times 10^3$ $1.2071 \times 10^3$ $1.1959 \times 10^3$ $1.1843 \times 10^3$ $1.1731 \times 10^3$ $1.1394 \times 10^4$ $1.0837 \times 10^4$</td>
</tr>
<tr>
<td>5</td>
<td>$7.4227 \times 10^4$ $6.4874 \times 10^4$ $5.9252 \times 10^4$ $5.7385 \times 10^4$ $5.5517 \times 10^4$ $5.3594 \times 10^4$ $5.1667 \times 10^4$ $4.6113 \times 10^4$ $3.6729 \times 10^4$</td>
</tr>
<tr>
<td>6</td>
<td>$6.3408 \times 10^4$ $4.9423 \times 10^4$ $4.1021 \times 10^4$ $3.8209 \times 10^4$ $3.5397 \times 10^4$ $3.2616 \times 10^4$ $2.9668 \times 10^4$ $2.1333 \times 10^4$ $7.2996 \times 10^5$</td>
</tr>
<tr>
<td>7</td>
<td>$7.2093 \times 10^4$ $5.2605 \times 10^4$ $4.0889 \times 10^4$ $3.6941 \times 10^4$ $3.3084 \times 10^4$ $2.9160 \times 10^4$ $2.5177 \times 10^4$ $1.3520 \times 10^4$ $\text{Buckled}$</td>
</tr>
<tr>
<td>8</td>
<td>$9.2851 \times 10^4$ $6.7088 \times 10^4$ $5.1568 \times 10^4$ $4.6387 \times 10^4$ $4.1296 \times 10^4$ $3.6131 \times 10^4$ $3.0915 \times 10^4$ $1.5316 \times 10^4$ $\text{Buckled}$</td>
</tr>
<tr>
<td>9</td>
<td>$1.2427 \times 10^5$ $9.1577 \times 10^4$ $7.1837 \times 10^4$ $6.5237 \times 10^4$ $5.8628 \times 10^4$ $5.1915 \times 10^4$ $4.5268 \times 10^4$ $2.5329 \times 10^4$ $\text{Buckled}$</td>
</tr>
<tr>
<td>10</td>
<td>$1.6639 \times 10^5$ $1.2610 \times 10^5$ $1.0164 \times 10^5$ $9.3440 \times 10^4$ $8.5199 \times 10^4$ $7.6956 \times 10^4$ $6.8676 \times 10^4$ $4.3762 \times 10^4$ $1.8450 \times 10^5$</td>
</tr>
<tr>
<td>Number of Circumfer. Waves, n</td>
<td>Amplitude of circumferential prestress, $n_s^1$ (tension positive) (non-dimensionalized)</td>
</tr>
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<td>-------------------------------</td>
<td>------------------------------------------------</td>
</tr>
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</tr>
<tr>
<td>10</td>
<td>1.6631 $\times 10^5$</td>
</tr>
</tbody>
</table>
Fig. 12: Variation of the ratio frequency parameter \( (\gamma/\gamma_0) \) of initially stressed and unstressed shell with initial stress \( n_s^1 \) for various circumferential wave numbers: Freely-supported ends.
repeated for various lengths \((L = 1, 2, 3)\) and thickness \(h = 0.002, 0.005\) parameters. It may also be observed that the results from the Donnell theory neglecting tangential inertia deviate from the more accurate ones for small numbers of circumferential waves but rapidly approach the exact value for increasing values of \(n\). This again is in keeping with the discussion in reference 84.

The amplitude of circumferential stress \(n_1^1\) which causes the shell to buckle can also be easily determined by setting equal to zero in the matrices given in Appendices C, D, and E. The results so obtained for only the Washizu theory is given in table 24 and figure 13. One particular aspect which was numerically studied in some detail was the accuracy of the formula in equation (37) in predicting the frequency of the preloaded shell in terms of the frequency of the unloaded shell and the buckling load for the particular mode. Calculations were performed for length ratios \(= 4., 2., 1.\), thickness ratios \(= 0.001, 0.002, 0.005\) and for a range of prestress amplitudes and circumferential mode numbers, and compared against the frequencies obtained from the preceding analysis. In the worst case the formula gave a result 3% higher but in general was quite accurate. It appeared that the accuracy deteriorated slowly with increasing \(n\) and length ratios although the sign of the prestress did not cause any appreciable change.
TABLE 24: Values of the amplitude of buckling stress of a simply-supported (shear diaphragms) cylindrical shell under a sinusoidally varying circumferential prestress $\sigma_s^i = n_i \sin \lambda s$ for various values of the length parameter.

Equations used: Washizu $h = 1/1000$  
Method of solution: Galerkin $v = 0$  
Size of matrix: [15 x 15] $\alpha_x^i = 0$

<table>
<thead>
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<th>2</th>
<th>4</th>
</tr>
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</tr>
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<tr>
<td>18</td>
<td>$-3.6394 \times 10^5$</td>
<td>$-3.1009 \times 10^5$</td>
<td>$-2.8723 \times 10^5$</td>
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</tbody>
</table>
Fig. 13: Variation of the amplitude of buckling stress $n^1_s$ of a simply-supported cylindrical shell under circumferential stress $\sigma^l_S = n^l_S \sin \lambda x$ with circumferential modal number $n$ for various lengths $L$. $L = 1, 2, 4$

\[ \frac{h}{a} = 0.001 \]

\[ v = 0.3 \]

\[ n^c_X = n^c_S = 0 \]

Washizu Theory
CHAPTER VI

Cylindrical Shell Under Axial Prestress Varying
In the Circumferential Direction

The problem of a vibrating circular cylindrical shell under axial prestress which varies in the circumferential direction has received attention in literature (refs. 74, 74, 88, and 89) almost exclusively for the case arising due to a static bending moment. A more general problem, that of a shell which is heated along a narrow longitudinal strip of the cylinder, has been treated by Ong and Herrmann (ref. 55). A common deficiency among the solutions presented hitherto has been due to the incorrect use of the Batdorf modification of the Donnell equation with in-plane inertias neglected. This error is the one pointed out in the preceding chapter.

In the general case the variation in the axial prestress distribution may be represented by the Fourier series

\[ \sigma_x = \sigma_x^0 + \sum_{p=1,2,3,...} \sigma_x^p \cos ps \]  

(68)

Thus, for example, a combination of a uniform axial prestress and a static bending moment \( M \) gives rise to

\[ \tau_x = \tau_x^0 + \tau_x^1 \cos s \]  

(69)
where \( \sigma_x' = \frac{M}{\pi a^2 h} \)

A feature of these types of problems is that the frequency of modal vibration \( w \) is not identifiable by a single integer \( n \) corresponding to the number of circumferential waves but, in general, will depend upon all the harmonics \( n \). The coupling of the modes gives rise to an irregularity in the mode shape in the circumferential direction. This phenomenon is similar to what one obtains in the case of a shell with non-uniform support conditions at the ends.

An often quoted result from reference 75 is that for pure bending of a circular cylinder, based on small deflection theory, the maximum compressive stress is essentially the classical buckling stress for the cylinder in uniform compression. But this solution apart from being derived from an incorrect equation, is based on an extrapolation while the classical buckling stress in compression has been assumed to be that for an infinitely long cylinder.

To examine this point more closely, we substitute equation (68) into the Donnell equation obtained with the neglect of tangential inertias (eq. (25)) and assume the circumferential prestress to be constant and equal to \( n_o \). Thus we have
\[ \nabla^4 \left[ \sum_{\ell=1,2,\ldots} \lambda^\ell \cos \frac{\lambda^\ell}{n} \cos n \right] \frac{\partial^2 W}{\partial x^2} + \nabla^4 W + k \nabla^4 W + \gamma W \right] = 0 \quad (70) \]

where \( \nabla^4, \nabla^4 W \equiv W \)

Assuming as before that the ends of the shell are simply-supported and

\[ W = \sum_{m=1,2,\ldots} \sum_{n=0,1,2,\ldots} \lambda_{mn} \sin \frac{\lambda_mx}{n} \cos ns \quad (71) \]

It may be noted on substituting equation (71) into equation (70) that the modal frequency will be a function of a single value of \( m \). Thus the summation over \( m \) can be discarded and hence the subscript \( m \) for the coefficients \( \lambda \) in equation (71) will be dropped. We then have

\[
\sum_{n=0,1,2,\ldots} \left( \frac{\lambda^2 + n^2}{m^2} \right) ^2 \left[ \frac{\partial^2 \lambda}{\partial x^2} + \frac{\partial \lambda}{\partial n} \right] \lambda_n \cos ns \\
+ \sum_{n=0,1,2,\ldots} \sum_{p=1,2,3,\ldots} \left( \frac{\lambda^2 + n^2}{m^2} \right) \lambda_n \n \cos ps \cdot \lambda_n \cos ns \\
+ \sum_{n=0,1,2,\ldots} \sum_{p=1,2,3,\ldots} \eta_n \lambda_m \left( p^4 + \eta^2 + \lambda^2 \right) \cos ps \lambda_n \cos ns \\
- \sum_{n=0,1,2,\ldots} \sum_{p=1,2,3,\ldots} 4 \eta_n \lambda_m \left( p^3 n + pn^3 + pn \lambda^3 \right) \sin ps \lambda_n \sin ns = 0 \quad (72)
\]
Rewriting the products of the trigonometric functions by using a sum and difference formula, the coefficient of each component of the Fourier series \( \cos ns \) is collected and set equal to zero to yield a homogeneous set of simultaneous equations. As before the coefficient matrix of the column vector of \( A \) provides the secular determinant for the determination of the frequency parameter.

Denoting

\[ M_n \equiv n_n^c \lambda_m^2 + n^s n^2 + \left(1 - \nu^2\right) \frac{\lambda_m^4}{(\lambda_m^2 + n^2)^2} + k \left(\lambda_m^2 + n^2\right)^2 \]

\[ E_n \equiv (\lambda_m^2 + n^2)^2 \]

\[ F_n^{(1)} \equiv 1 + \varepsilon n^2 + 2 \lambda_m^2 \]

\[ G_n^{(1)} \equiv -4 \left(n + n^3 + n \lambda_m^2\right) \]

we have for the case of the prestress being a uniform axial load, pressure, and a pure applied moment
\[ A_n E_n (M_n - \gamma) + \frac{n x}{2} \left\{ \left( 1 + \delta_{in} \right) \lambda^2 \left( \varepsilon_{n-1}^0 + F_n^{(i)} + G_n^{(i)} \right) \right. \]
\[ + \lambda^2 \left( E_n + F_n^{(i)} - G_n^{(i)} \right) \}
= 0 \]

for \( n = 0, 1, 2, \)

where

\[ \delta_{ij} = \begin{cases} 
0 & \text{for } i \neq j \\
1 & \text{for } i = j 
\end{cases} \quad (74) \]

is the Kronecker delta.

The above set of equations reduce to the one given in reference 75 provided \( F_n^1 \) and \( G_n^1 \) are set identically equal to zero for all \( n \), while \( E_n \) is set equal to unity.

For a numerical evaluation of the frequency parameter, fifty terms were retained in the series given by equation (70) for each integral value of \( m \). Although the number of circumferential waves does not have any real meaning in the presence of the bending moment, by taking closely spaced values for the amplitude of the moment as in reference 75, each frequency can be identified with the spectral distribution for the unloaded cylinder. Table 25 presents the frequencies catalogued in this manner for a shell with \( m = 1, L = 4 \) and \( \frac{h}{a} = 0.001 \). These results are also shown in figure 14 with the ratio of the frequencies for the loaded and
TABLE 25: Values of the frequency parameter $\gamma = \frac{\omega^2}{\pi^2} \frac{a^2}{\rho(1-v^2)/E}$ for a circular cylindrical shell under a pure initial bending moment ($\frac{\sigma_x}{E} = \frac{n_{cr}^1}{n}$) for various amplitudes $n_{cr}^1$ non-dimensionalized against the axial buckling load in uniform compression, $n_{cr}$ $L = 4, v = 0.3, \ h/a = 0.001, \ Number \ of \ half \ axial \ waves \ m = 1$ $n_s^0 = 0$

<table>
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<tr>
<th>Number of Circumfer. Waves, n</th>
<th>$n_s^1/n_{cr}$</th>
<th>$n_{cr} = 0.560595 \times 10^3$</th>
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<tr>
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<td>$4.4868 \times 10^4$</td>
</tr>
<tr>
<td>9</td>
<td>$6.0709 \times 10^4$</td>
<td>$6.1253 \times 10^4$</td>
</tr>
</tbody>
</table>

100
Fig. 14: Variation of the ratio of frequency parameter \((\gamma/\gamma_0)\) of initially stressed and unstressed shell with initial stress \(n_x\) non-dimensionalized with respect to the uniform axial buckling stress \(n_{cr}\) for various circumferential wave numbers. (freely-supported)
unloaded shell plotted against the amplitude of the initial stress. The amplitude of the varying axial stress produced by the bending moment in the table has been non-dimensionalized with respect to the lowest axial buckling stress for the cylinder under uniform compression. This non-dimensionalizing quantity appears more meaningful than the one chosen in reference 75. Each of the values listed in the table had converged to at least five significant figures.

Extrapolating from the values in figure 14 appears to provide the lowest buckling stress \( \left( \frac{\sigma'_x}{\sigma_{cr}} \right) \) in the neighborhood of 1.3 which is the factor attributed to Flugge in reference 88. The variation in the result due to the incorrect use of Donnell equation as was done in reference 75 and 88 was fortuitously small for this case. This is in accordance with the discussion in Chapter V. The phenomena of some frequencies increasing in magnitude while some others decrease with an increase in the value of the bending moment.

For obtaining a further insight into the problem the case of the axial load varying in the manner

\[
\sigma'_x = \sigma_x^2 \cos 2s
\]  

(75)

was also studied numerically. The simultaneous equations corresponding to equation (74) are generated from
\[ A_n E_n \left( M_n - \gamma \right) \]
\[ + \frac{\lambda_n^2 n_x^2}{2} \left\{ \left( 1 + \frac{E_n}{2n} \right) A_{n-2} \left( E_{n-2} + F_{n-2}^{(2)} + G_{n-2}^{(2)} \right) \right. \]
\[ + \delta_{n} A_n \left( E_n + F_n^{(2)} - G_n^{(2)} \right) \]
\[ + A_{n+2} \left( E_{n+2} + F_{n+2}^{(2)} - G_{n+2}^{(2)} \right) \} = 0 \]
\[ \text{for} \quad n = 0, 1, 2, \ldots \]

where
\[ E_n = (\lambda_n^2 + n^2)^2 \quad (76) \]
\[ F_n^{(2)} = 16 + 24 n^2 + 2 \lambda_n^2 \]
\[ G_n^{(2)} = -4 \left( 8n + 2n^3 + 2n \lambda_n^2 \right) \]

The frequency parameter calculated from the set of equations given by equation (76) have been tabulated in table 26 and plotted in figure 15. Again fifty terms were retained in
TABLE 26: Values of the frequency parameter $\gamma = \omega^2 a^2 \left(1 - \nu^2\right)/E$ for a circular cylindrical shell under an axial prestress varying in the circumferential direction in the manner $c_x = n_x^2 \cos 2s$, for various amplitudes $n_x^2$, non-dimensionalized against the axial buckling load $n_{cr}$ of the cylinder under uniform compression.

$L = 4 \quad n_x^2 = 0$
$v = 0.3 \quad n_x^2 = 0.360595 \times 10^3$
$h/a = 0.001 \quad n_{cr} = 0.560595 \times 10^3$

<table>
<thead>
<tr>
<th>Number of Circumferential waves, $n$</th>
<th>$n_x^2/n_{cr}$</th>
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<tr>
<td>----------</td>
<td>----------------</td>
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<tr>
<td>0</td>
<td>9.1000 x 10^1</td>
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</table>
Fig. 15: Variation of the ratio of frequency parameters \((\gamma/\gamma_0)\) of initially stressed and unstressed shell with initial stress \(n_x^n\) nondimensionalized with respect to the uniform axial buckling stress \(n_{cr}\), for various circumferential wave numbers.
equation (71) during computations and the amplitude of the varying stress in equation (75) was non-dimensionalized with respect to the lowest buckling load of the cylinder, under uniform compression. It was noted that for this case the buckling stress was higher than was for the cylinder under the action of a pure bending moment. Again no definite value could be identified with the buckling stress because of a weak dependence on the shell parameters and the axial modal number. But most interestingly, it was found that in this case the correct analysis and in-correct formulation similar to the one in reference 55 consistently differed in the value of the frequency corresponding to \( n = 1 \). This difference although small was significantly more than was found in the case of the prestress being a pure bending moment. Hence a more rapidly varying circumferential prestress may be expected to cause a wider difference in the values for the frequency corresponding to a more sensitive value of the number of circumferential waves. On this account the results given in reference 55 appear dubious. Also, this effect probably explains the wide disparity reported between the results in references 4 and 5 as was reported in the comment from reference 4 which was quoted in the previous chapter.
In conclusion, it is well to re-emphasize the following observations, arrived at during the course of this investigation, in dealing with problems of vibrating circular cylinders under varying prestresses.

The differential equations of motion and the corresponding boundary conditions for the Donnell, Sanders and Washizu theories have been derived by a systematic variational procedure.

The Donnell theory, with the neglect of the terms due to the tangential inertias, is by far the simplest to handle from the point of view of algebra. The accuracy of this theory suffers vis-a-vis the more accurate ones for low values of the number of circumferential waves but in general provides a good engineering approximation for the value of the lowest frequency, especially so for short cylinders. The complete Donnell theory could also be uncoupled to yield a single eighth order equation in $W$ and yielded very good results provided again that the shell length was reasonably small. The Sanders, Washizu and other higher order theories differed very little among themselves, certainly so in the case of thin shells $\left( \frac{h}{a} < \frac{1}{100} \right)$. These theories too were
simple to use in conjunction with the Galerkin method.

The Galerkin method of solution is a convenient one from the point of view of algebraic calculations but the method is not fully programmable for the computer. The method is fairly accurate and in its generalized form the set of admissible functions chosen need not satisfy the boundary conditions, although this would lead to the necessity of additionally evaluating certain line integrals in forming the equations. However, as used herein the difficulty in choosing a function set increases rapidly when the boundary conditions involve high derivatives (e.g. free ends) or when the end conditions are not uniform. The exact method of solution is more straightforward but the size of the computations rapidly rises for an arbitrary distribution of initial stress.

The Batdorf modification, although erroneous for varying initial stress problems, appears to provide reasonable accurate answers if the stresses do not vary rapidly. However, it cannot be relied upon for an arbitrary distribution of prestresses.

The applicability of formula (37) appears to be quite good for engineering purposes. Although there is a great desirability for a more general type of interaction formula and minimum weight analysis criteria to take into account the shell parameters also, these are lacking. In their absence formula (37) can be quite useful for handbook design.
AREAS OF FUTURE RESEARCH

1. The analysis of shells with more general type of end conditions - in particular, very few solutions exist for shells with one or both ends free or for shells with elastically flexible supports.

2. The analysis of general shells of revolution under prestress and a relative evaluation of existing theories.

3. A detailed study of the orthogonality relationships for various problems of initial stresses and non-uniform boundary conditions.

4. An establishment of the range of validity of shell theories for forced vibration problems and their improvement.
Appendix A

Expressions for $d_1$, $e_1$, $\ldots$ in equation (51)

\[d_1 = [a_1 + (1 + a_2)\lambda^4 + (a_1 + a_2)n^2 + a_3\lambda^4 + a_1n^2\lambda^2 + a_4n^4] \]
\[+ n^2\lambda^2 + \frac{n^2}{2} \lambda^2\]

\[e_1 = [v_{xy} a_1^2 \lambda^2 - a_1(1 + a_2)\lambda^2 - a_1a_2n^2 - a_2\lambda^4 - (a_1 + a_2)n^2 \lambda^2 \]
\[+ a_6n^2 \lambda^2 - a_1a_2n^4 - a_3(1 + a_2)\lambda^6 - a_3(a_1 + a_2)n^2 \lambda^4 \]
\[- a_7(1 + a_2)n^2 \lambda^4 - a_1(a_1 + a_2)n^4 \lambda^2 - a_5(1 + a_2)n^4 \lambda^2 \]
\[- a_5(a_1 + a_2)n^6] - (n_X^o + \frac{n_X^n}{2})[(1 + a_2)\lambda^4 + (a_1 + a_2)n^2 \lambda^2] \]
\[- n_X^n[(1 + a_2)n^2 \lambda^2 + (a_1 + a_2)n^4] \]

\[f_1 = [a_1a_2 \lambda^4 - v_{xy}a_1^2a_2 \lambda^4 + (a_1(a_1 + a_2) + 2v_{xy}a_1^2a_2 - a_2 - v_{xy}a_2^3 \]
\[- a_1a_6)\lambda^2 n^2 + a_2a_3 \lambda^6 + \{a_3(a_1 + a_2 - a_2) + a_2a_7\}n^2 \lambda^6 \]
\[+ \{a_7(a_1 + a_2 - a_2) + a_2a_5 + a_1a_2a_3\}n^4 + \{a_5(a_1 + a_2 - a_2) \]
\[+ a_1a_2a_7)n^6 \lambda^2 + a_1a_2a_5n^8] + (n_X^o + \frac{n_X^n}{2})[a_2 \lambda^6 + (a_1 + a_2 - a_2)\lambda^4 n^2 \]
\[+ a_1a_2 \lambda^2 n^4] + n_X^n[a_2 \lambda^4 n^2 + (a_1 + a_2 - a_2)\lambda^2 n^4 + a_1a_2n^6] \]

\[h_1 = 2 \frac{n_X^n}{L^2} \left[ 2 \left( \frac{1}{3^2} - \frac{1}{1^2} \right) + 2 \left( 1 - \frac{1}{3} \right) \right] \]

\[i_1 = 2 \frac{n_X^n}{L^2} \left[ (3(1 + a_2)^2 \lambda^2 + (a_1 + a_2)n^2)(-1 + \frac{1}{3}) \right. \]
\[- \{(1 + a_2)^2 \lambda^2 + (a_1 + a_2)^2n^2\}(\frac{1}{3^2} - 1)] \]
\[ k_1 = 2 \frac{n^1_y}{L^2} \left\{ \left( a_2 2^6 \lambda^4_1 \right) \left( a_1 + a_2 - a_2^2 \right)^2 + \left( a_1 a_2 \right)^2 \lambda^2_1 + a_1 a_2 n^2 \left( \frac{1}{32} - 1 \right) \right\} \\
\left\{ 5a_2 \cdot 2^5 \lambda^4_1 + 3(a_1 + a_2^2 - a_2^6)^2 n^2 \lambda^2_1 + a_1 a_2 n^4 \right\} \left( -1 + \frac{1}{3} \right) \right\} \\
L_1 = 0 \\
m_1 = 0 \\
n_1 = 0 \\
\lambda = m \pi / L \\
m \\
d_2 = 2 \frac{n^1_x}{L^2} \left[ \frac{1}{3} - 1 - 1 + \frac{1}{3} \right] \\
e_2 = 2 \frac{n^1_x}{L^2} \left\{ \left( 3(1 + a_2) \lambda^2_2 \right) \left( a_1 + a_2 \right) n^2(1 + \frac{1}{3}) \right\} \\
\left\{ (1 + a_2)^2 + (a_1 + a_2^2) n^2 \left( \frac{1}{32} - 1 \right) \right\} \\
f_2 = 2 \frac{n^1_x}{L^2} \left\{ \left( a_2 \lambda^4_2 \right) \left( a_1 + a_2^2 - a_2^6 \right) \lambda^2_2 n^2 + a_3 a_2 n^4 \left( \frac{1}{32} - 1 \right) \right\} \\
\left\{ 5a_2 \lambda^4_2 + 3(a_1 + a_2^2 - a_2^6)^2 \lambda^2_2 n^2 + a_1 a_2 n^4 \right\} \left( 1 + \frac{1}{3} \right) \right\} \\
h_2 = \left[ a_1 + (1 + a_2) \lambda^2_2 \right. + (a_1 + a_2) n^2 + a_3 \lambda^4_2 + a_7 \lambda^2_2 n^2 + a_5 n^4 \right\} \\
+ \left( n^1_x + \frac{n^1_x}{2} \right) \lambda^2_2 + n^0_5 n^2 \]
\[ i_2 = \left[ v_{xy} a_1^2 \lambda_2^2 - a_1(1 + a_2)\lambda_2^2 - a_1 a_2 n^2 - a_2 \lambda_2^4 - (a_1 + a_2)\lambda_2^2 n^2 \right. \]
\[ + a_2^2 \lambda_2^2 n^2 - a_1 a_2 n^4 - a_3(1 + a_2)\lambda_6 - a_3(a_1 + a_2)\lambda_4 n^2 \]
\[ - a_7(1 + a_2)\lambda_4 n^2 - a_7(a_1 + a_2) + a_5(1 + a_2)\lambda_2^2 n^4 \]
\[ - a_5(a_1 + a_2)n^6 \left) - \left( n_x^5 + \frac{n_1^1}{2} \right) \left( (1 + a_2)\lambda_4^l + (a_1 + a_2)\lambda_2^2 n^2 \right) \right] \]
\[ n_5^0 [(1 + a_2)\lambda_2^2 n^2 + (a_1 + a_2)n^4] \]

\[ k_2 = \left[ a_1 a_2 \lambda_2^4 - v_{xy} a_1^2 a_2 \lambda_2^4 + \{a_1(a_1 + a_2^2) + 2v_{xy} a_1^2 a_6 - a_2 - v_{xy} a_3 \right. \]
\[ - a_1 a_6 \lambda_2^2 n^2 + a_2 a_3 \lambda_2^8 + \{a_3(a_1 + a_2^2 - a_2^2) + a_2 a_7 \lambda_2^6 n^2 \}
\[ + \{a_7(a_1 + a_2^2 - a_2^2) + a_2 a_5 + a_1 a_2 a_3 \lambda_2^4 n^4 + \{a_5(a_1 + a_2^2 - a_2^2) \}
\[ + a_1 a_2 a_7 \lambda_2^6 n^6 + a_1 a_2 a_5 n^8 \right) + \left( n_x^5 + \frac{n_1^1}{2} \right) [a_2 \lambda_2^6 + (a_1 + a_2^2 - a_2^2)n^2 \lambda_4^2 \]
\[ + a_1 a_2 \lambda_2^2 n^4 + n_5^0 [a_2 \lambda_2^4 n^2 + (a_1 + a_2^2 - a_2^2)\lambda_2^2 n^4 + a_1 a_2 n^6] \]

\[ L_2 = 2 \frac{n_1^1}{L}\left[3\left(\frac{1}{5^2} - 1\right) - 3\left(\frac{1}{5} - 1\right)\right] \]

\[ m_2 = 2 \frac{n_1^1}{L^2} [(3(1 + a_2)\lambda_3^3 + (a_1 a_2)\lambda_3 n^2)\frac{L}{\pi}(\frac{1}{5} - 1) - (1 + a_2)\lambda_3^4 \]
\[ + (a_1 + a_2)\lambda_3^2 n^2 \frac{L^2}{\pi^2}(\frac{1}{5} - 1) \]

\[ n_2 = 2 \frac{n_1^1}{L^2} [(a_2 \lambda_3^6 + (a_1 + a_2^2 - a_2)\lambda_4 n^2 + a_1 a_2 \lambda_3 n^4)\frac{L^2}{\pi^2}(\frac{1}{5} - 1) \]
\[ - 5a_2 \lambda_5^5 + 5(a_1 + a_2 - a_6)\lambda_3 n^2 + a_1 a_2 \lambda n^4)] \frac{L}{\pi}(\frac{1}{5} - 1) \]

\[ d_3 = 0 \]

\[ e_3 = 0 \]

\[ f_3 = 0 \]
\[ h_3 = 2 \frac{n^L}{L^2}[2^2\left(\frac{1}{5^2} - 1\right) - 2\left(\frac{1}{5} + 1\right)] \]

\[ i_3 = 2 \frac{n^L}{L^2}[\left(6(1 + a_2)\lambda^2 + 2(a_1 + a_2)n^2\right)\left(\frac{1}{5} + 1\right) - 4(1 + a_2)\lambda^2 + 4(a_1 + a_2)n^2 \left(\frac{1}{5} - 1\right)] \]

\[ k_3 = 2 \frac{n^L}{L^2}[4(a_2 \lambda^4 + (a_1 + a_2 - a_2)\lambda^2 n^2 + a_{12}n^4 \left(\frac{1}{5^2} - 1\right) - 10a_2 \lambda^6 + 6(a_1 + a_2 - a_2)\lambda^2 n^2 + 2a_{12}n^6 \left(\frac{1}{5} + 1\right)] \]

\[ L_3 = [a_1 + (1 + a_2)\lambda^2 + (a_1 + a_2)n^2 + a_3 \lambda^4 + a_7 \lambda^2 n^2 + a_{5}n^6] + \left(\frac{n^L}{n^L} + \frac{n^L}{2}\right)\lambda^2 + n^s n^2 \]

\[ m_3 = \left[\nu^2 a_2 a_3 \lambda^3 - a_1 (1 + a_2)\lambda^2 - a_2 n^2 - a_{12} \lambda^4 - (a_1 + a_2)\lambda^3 n^2 \right. \]

\[ + \ a_6 \lambda n^2 - a_1 a_2 n^4 - a_3 (1 + a_2)\lambda^6 - a_3 (a_1 + a_2)\lambda^4 n^2 - a_7 (1 + a_2)\lambda^4 n^2 \]

\[ \left. - (a_7(a_1 + a_2) + a_5(1 + a_2))\lambda^2 n^4 - a_5(a_1 + a_2)n^6\right] - (\frac{n^L}{n^L} + \frac{n^L}{2}][(1 + a_2)\lambda^4 + (1 + a_2)\lambda^2 n^2] \]

\[ - n^s[(1 + a_2)\lambda^2 n^2 + (a_1 + a_2)n^4] \]

\[ n_3 = [a_{12} \lambda^4 - \nu^2 a^2 a_3 \lambda^4 + (a_1 (a_1 + a_2) + 2\nu a^2 a_6 - a_2 - \nu a^3 \]

\[ - a_1 a_2^2)]\lambda^2 n^2 + a_2 a_3 \lambda^8 + [a_3(a_1 + a_2^2 - a_2^2) + a_2 a_7]\lambda^6 n^2 \]

\[ + (a_7(a_1 + a_2 - a_2) + a_2 a_5 + a_{12}a_{23})\lambda^4 n^4 + a_5(a_1 + a_2^2 - a_2^2) \]

\[ + a_1 a_2 a_7\lambda^2 n^6 + a_1 a_2 a_5 n^8] + (\frac{n^L}{n^L} + \frac{n^L}{2})[a_2 \lambda^6 + (a_1 + a_2^2 - a_2^2)\lambda^4 n^2 \]

\[ + a_{12} \lambda^2 n^4] + \frac{n_s}{n_s}[a_2 \lambda^4 n^2 + (a_1 + a_2^2 - a_2^2)\lambda^2 n^4 + a_{12}n^6] \]
Appendix B

Expansion of the matrix of coefficients of $A_1, A_2$, in equation (51)

with only the first three terms retained in equation (47).

\[- \gamma^9 + \gamma^8[d_1 + h_2 + i_3] + \gamma^7[e_1 + i_2 + m_3 - (d_1 + h_2)L_3] -
- (d_1h_2 - d_2h_1) + L_1h_3\]

\[+ \gamma^6[n_3 - m_3(d_1 + h_2) + i_3(d_1h_2 - d_1h_2 - e_1 + i_2) -
- (i_2d_1 + h_2e_1 - h_1e_2 - i_1d_2 - k_2 - f_1) + L_2i_3 + m_2h_3 - d_1L_3h_1\]

\[+ \gamma^5[-n_3(d_1 + h_2) + m_3(d_1h_2 - d_1h_2 - e_1 - i_2) + L_3(i_2d_1 - i_1d_2 + h_1e_1 -
- h_1e_2 - k_1 - f_1) - (d_2k_2 - d_2k_1 + f_1h_2 - f_2h_1 + e_1i_2 - e_1i_1) +
+ (L_2k_3 + n_2h_3 + m_3i_3) - d_1(L_2i_3 + mh_3) - e_1h_2\]

\[+ \gamma^4[n_3(d_1h_2 - d_2h_1 - e_1 - i_2) + m_3(i_2d_1 - i_1d_2 + e_1h_2 - e_2h_1 - k_2 - f_1) +
+ L_3(d_1k_2 - d_2k_1 + f_1h_2 - f_2h_1 + e_1i_2 + e_1i_1) - (e_1k_1 - e_1k_1) +
+f_1i_2 - f_2i_1) + m_2k_3 + n_2i_3 - d_1(L_2k_3 + n_2h_3 + m_3i_3) -
- e_1(L_2i_3 + mh_3) - f_1L_2h_3\]

\[+ \gamma^3[n_3(i_2d_1 + e_1h_2 - e_1h_2 - i_1d_2 - k_2 - f_1) + m_3(d_1k_2 - d_2k_1 + f_1h_2 -
- f_2h_1 + e_1i_2 - e_1i_1) - (f_1k_2 - f_2k_1) + L_3(e_1k_2 - e_2k_1 + f_1i_2 - f_2i_1) +
+ n_2k_3 - d_1(m_2k_3 + n_2i_3) - e_1(L_2k_3 + n_2h_3 + m_3i_3) - f_1(L_2i_3 + mh_3)]

\[+ \gamma^2[n_3(d_1k_2 - d_2k_1 + f_1h_2 - f_2h_1 + e_1i_2 - e_1i_1) + m_3(e_1k_2 - e_2k_1 +
+ f_1i_2 - f_2i_1) + L_3(f_1k_2 - f_2k_1) - d_1n_2k_3 - e_1(m_2k_3 + n_2i_3) -
- f_1(L_2k_3 + n_2h_3 + m_3i_3)]
\[+ \gamma[n_3(e_1k_2 - e_2k_1 + f_1i_2 - f_2i_1) + m_3(f_1k_2 - f_2k_1) - e_1n_2k_3 -
- f_1(m_2k_3 + n_2i_3)]
\[+ [n_3(f_1k_2 - f_2k_1) - f_1n_2k_3] = 0\]
APPENDIX C

The frequency determinant is formed from the matrix arranged in the following manner:

\[
\begin{pmatrix}
A_{1,1} - \gamma & A_{1,2} & A_{1,3} \\
A_{2,1} & A_{2,2} - \gamma & A_{2,3} \\
A_{3,1} & A_{3,2} & A_{3,3} - \gamma \\
& & \\
& & \\
& & \\
& & \\
\end{pmatrix}
\]

The matrix obtained from using the Donnell theory is:

Denoting

\[
F_i = (\lambda_i^2 + n_i^2)^2 + 7\lambda_i^4 + 2\lambda_i^2 n_i^2
\]

\[
F_p = (\lambda_p^2 + n_p^2)^2 + \lambda_p^4 + 6\lambda_p^2 \lambda_i^2 + 2\lambda_i^2 n_i^2
\]

\[
G_p = \lambda_i^3 \lambda_p + \lambda_p \lambda_i^3 + \lambda_i \lambda_p n_i^2
\]

\[
M_p = (\lambda_p^2 + n_p^2) \left[ (1 - \nu^2) \lambda_p^4 + k (\lambda_p^2 + n_p^2) + n_i \lambda_i \lambda_p + n_i^2 n_p^2 \right]
\]
\[
A_{1,1} = M_1 + n_s n_1^2 \frac{n_2}{\pi} \left( -\frac{8}{3} F_1 + \frac{16}{3} G_1 \right)
\]

\[
A_{1,2} = A_{1,4} = 0
\]

\[
A_{1,3} = n_s n_1^2 \frac{n_2}{\pi} \left( \frac{8}{15} F_3 - \frac{16}{5} G_3 \right)
\]

\[
A_{1,5} = n_s n_1^2 \frac{n_2}{\pi} \left( \frac{8}{105} F_5 - \frac{16}{21} G_5 \right)
\]

\[
A_{2,1} = A_{2,3} = A_{2,5} = 0
\]

\[
A_{2,2} = M_2 + n_s n_1^2 \frac{n_2}{\pi} \left( -\frac{32}{15} F_2 + \frac{32}{15} G_2 \right)
\]

\[
A_{2,4} = n_s n_1^2 \frac{n_2}{\pi} \left( \frac{64}{105} F_4 - \frac{64}{105} G_4 \right)
\]

\[
A_{3,1} = n_s n_1^2 \frac{n_2}{\pi} \left( \frac{8}{15} F_1 + \frac{112}{15} G_1 \right)
\]

\[
A_{3,2} = A_{3,4} = 0
\]

\[
A_{3,3} = M_3 + n_s n_1^2 \frac{n_2}{\pi} \left( -\frac{72}{35} F_3 + \frac{48}{35} G_3 \right)
\]

\[
A_{3,5} = n_s n_1^2 \frac{n_2}{\pi} \left( \frac{10}{63} F_5 - \frac{272}{63} G_5 \right)
\]

\[
A_{4,1} = A_{4,3} = A_{4,5} = 0
\]

\[
A_{4,2} = n_s n_1^2 \frac{n_2}{\pi} \left( \frac{64}{105} F_2 + \frac{70}{105} G_2 \right)
\]
The matrix determinant obtained from the incorrect use of the Batdorf equation for the stress distribution defined by equation (66) is the same as above if \( F_p \) and \( G_p \) are redefined as

\[
F_p \equiv (\lambda^2 + n^2)
\]

\[
G_p \equiv 0
\]
APPENDIX D

The frequency determinant is formed from the matrix arranged in the following manner:

\[
\begin{pmatrix}
A_{1,1} - \gamma & A_{1,2} & A_{1,3} \\
A_{2,1} & A_{2,2} - \gamma & A_{2,3} \\
A_{3,1} & A_{3,2} & A_{3,3} - \gamma
\end{pmatrix}
\]

The matrix obtained by using the Sanders Theory is symmetric.

Denoting

\[
U_1(m) = -\frac{\lambda^2}{m} - (1 - \nu)\frac{n^2}{2} - k(1 - \nu)\frac{n}{8} - n^0\frac{n^2}{4} - n^s\frac{n^2}{4}
\]

\[
V_1(m) = (1 + \nu)\lambda \frac{n}{m} - 3k(1 - \nu)\lambda \frac{n}{m} - n^0\frac{\lambda}{4} - n^s\frac{\lambda}{4}
\]

\[
W_1(m) = -\nu\lambda + k(1 - \nu)\frac{n^2}{m}
\]

\[
U_2(m) = V_1(m)
\]

\[
V_2(m) = -\frac{(1 - \nu)\lambda^2}{m} - \frac{n^2}{2} - kn^2 - \frac{9}{8} k(1 - \nu)\frac{\lambda^2}{m} - n^0\frac{\lambda^2}{4} - n^s\frac{\lambda^2}{4} - n(1 + \frac{\lambda}{m})
\]

\[
W_2(m) = n + (3 - \nu)k\frac{\lambda^2}{m} + kn^3 + n^s\frac{n}{m}
\]

\[
U_3(m) = W_1(m)
\]

\[
V_3(m) = W_2(m)
\]

\[
W_3(m) = -1 - k(\frac{\lambda^2}{m} + n^2)\frac{n^2}{2} - n^0\frac{\lambda^2}{4} - n^s\frac{n^2}{4}
\]
Denoting: $A_{m,n} = AD(m,n) + n^2 BD(m,n)$, $P = \pi$

\[
\begin{align*}
&AD(1,1) = U1(1.) \\
&AD(1,2) = V1(1.) \\
&AD(1,3) = W1(1.) \\
&AD(2,1) = AD(1,2) \\
&AD(2,2) = V2(1.) \\
&AD(2,3) = W2(1.) \\
&AD(3,1) = AD(1,3) \\
&AD(3,2) = AD(2,3) \\
&AD(3,3) = W3(1.) \\
&AD(4,4) = U1(2.) \\
&AD(4,5) = V1(2.) \\
&AD(4,6) = W1(2.) \\
&AD(5,4) = AD(4,5) \\
&AD(5,5) = V2(2.) \\
&AD(5,6) = W2(2.) \\
&AD(6,4) = AD(4,6) \\
&AD(6,5) = AD(5,6) \\
&AD(6,6) = W3(2.) \\
&AD(7,7) = U1(3.) \\
&AD(7,8) = V1(3.) \\
&AD(7,9) = W1(3.) \\
&AD(8,7) = AD(7,8) \\
&AD(8,8) = V2(3.) \\
&AD(8,9) = W2(3.) \\
&AD(9,7) = AD(7,9) \\
&AD(9,8) = AD(8,9) \\
&AD(9,9) = W3(3.) \\
&AD(10,10) = U1(4.) \\
&AD(10,11) = V1(4.) \\
&AD(10,12) = W1(4.) \\
&AD(11,10) = AD(10,11) \\
&AD(11,11) = V2(4.) \\
&AD(11,12) = W2(4.) \\
&AD(12,10) = AD(10,12) \\
&AD(12,11) = AD(11,12) \\
&AD(12,12) = W3(4.) \\
&AD(13,13) = U1(5.) \\
&AD(13,14) = V1(5.) \\
&AD(13,15) = W1(5.) \\
&AD(14,13) = AD(13,14) \\
&AD(14,14) = V2(5.) \\
&AD(14,15) = W2(5.) \\
&AD(15,13) = AD(13,15) \\
&AD(15,14) = AD(14,15)
\end{align*}
\]
\[ \text{AD}(15,15) = W3(5^*) \]

\[
\begin{align*}
\text{BD}(1,1) &= -N^*N/(3^*P) \\
\text{BD}(1,2) &= -N/(3^*L) \\
\text{BD}(1,7) &= -N^*N/(5^*P) \\
\text{BD}(1,8) &= 3^*N/(5^*L) \\
\text{BD}(1,13) &= -N^*N/(21^*P) \\
\text{BD}(1,14) &= 5^*N/(21^*L) \\
\text{BD}(2,1) &= \text{BD}(1,2) \\
\text{BD}(2,2) &= -P/(3^*L*L) - 8^*/(3^*P) \\
\text{BD}(2,3) &= 8^*N/(3^*P) \\
\text{BD}(2,7) &= -2^*N/L \\
\text{BD}(2,8) &= 6^*P/(L*L) + 8^*/(15^*P) \\
\text{BD}(2,9) &= 8^*N/(15^*P) \\
\text{BD}(2,13) &= 2^*N/(21^*L) - N/(21^*L) \\
\text{BD}(2,14) &= 10^*P/(21^*L*L) + 8^*/(105^*P) - 5^*P/(21^*L*L) \\
\text{BD}(2,15) &= -8^*N/(105^*P) \\
\text{BD}(3,2) &= \text{BD}(2,3) \\
\text{BD}(3,3) &= 6^*N^*N/(3^*P) \\
\text{BD}(3,8) &= -8^*N/(15^*P) \\
\text{BD}(3,9) &= 8^*N^*N/(15^*P) \\
\text{BD}(3,14) &= -8^*N/(105^*P) \\
\text{BD}(3,15) &= 8^*N^*N/(105^*P) \\
\text{BD}(4,4) &= -7^*N^*N/(15^*P) \\
\text{BD}(4,5) &= -14^*N/(15^*L) \\
\text{BD}(4,10) &= 19^*N^*N/(105^*P) \\
\text{BD}(4,11) &= 76^*N/(105^*L) \\
\text{BD}(5,4) &= \text{BD}(4,5) \\
\text{BD}(5,5) &= -28^*P/(15^*L*L) - 32^*/(15^*P) \\
\text{BD}(5,6) &= 32^*N/(15^*P) \\
\text{BD}(5,10) &= 38^*N/(105^*L) \\
\text{BD}(5,11) &= 152^*P/(105^*L*L) + 64^*/(105^*P) \\
\text{BD}(5,12) &= -64^*N/(105^*P) \\
\text{BD}(6,5) &= \text{BD}(5,6) \\
\text{BD}(6,6) &= -32^*N^*N/(15^*P) \\
\text{BD}(6,11) &= -64^*N^*N/(105^*P) \\
\text{BD}(6,12) &= 64^*N^*N/(105^*P) \\
\text{BD}(7,1) &= \text{BD}(1,7) \\
\text{BD}(7,2) &= \text{BD}(2,7) \\
\text{BD}(7,7) &= -17^*N^*N/(35^*P) \\
\text{BD}(7,8) &= -51^*N/(35^*L) \\
\text{BD}(7,13) &= 11^*N^*N/(63^*P) \\
\end{align*}
\]
\[
\begin{align*}
BD(7,14) &= 55 \cdot N/(63 \cdot L) \\
BD(8,1) &= BD(1,8) \\
BD(8,2) &= BD(2,8) \\
BD(8,3) &= BD(3,8) \\
BD(8,7) &= BD(7,8) \\
BD(8,8) &= -153 \cdot P/(35 \cdot L \cdot L) - 72 \cdot / (35 \cdot P) \\
BD(8,9) &= 72 \cdot N/(35 \cdot P) \\
BD(8,13) &= 33 \cdot N/(63 \cdot L) \\
BD(8,14) &= 250 \cdot P/(63 \cdot L \cdot L) + 40 \cdot / (63 \cdot P) - 85 \cdot P/(63 \cdot L \cdot L) \\
BD(8,15) &= -40 \cdot N/(63 \cdot P) \\
BD(9,2) &= BD(2,9) \\
BD(9,3) &= BD(3,9) \\
BD(9,8) &= BD(8,9) \\
BD(9,9) &= -72 \cdot N \cdot N/(35 \cdot P) \\
BD(9,14) &= -40 \cdot N/(63 \cdot P) \\
BD(9,15) &= 40 \cdot N \cdot N/(63 \cdot P) \\
BD(10,4) &= BD(4,10) \\
BD(10,5) &= BD(5,10) \\
BD(10,10) &= -31 \cdot N \cdot N/(63 \cdot P) \\
BD(10,11) &= -124 \cdot N/(63 \cdot L) \\
BD(11,4) &= BD(4,11) \\
BD(11,5) &= BD(5,11) \\
BD(11,6) &= BD(6,11) \\
BD(11,10) &= -124 \cdot N \cdot N/(63 \cdot L) \\
BD(11,11) &= -496 \cdot P/(63 \cdot L \cdot L) - 128 \cdot / (63 \cdot P) \\
BD(11,12) &= 128 \cdot N \cdot N/(63 \cdot P) \\
BD(12,5) &= BD(5,12) \\
BD(12,6) &= BD(6,12) \\
BD(12,11) &= BD(11,12) \\
BD(12,12) &= -128 \cdot N \cdot N/(63 \cdot P) \\
BD(13,1) &= BD(1,13) \\
BD(13,2) &= BD(2,13) \\
BD(13,7) &= BD(7,13) \\
BD(13,8) &= BD(8,13) \\
BD(13,13) &= -49 \cdot N \cdot N/(99 \cdot P) \\
BD(13,14) &= -249 \cdot N/(99 \cdot L) \\
BD(14,1) &= BD(1,14) \\
BD(14,2) &= BD(2,14) \\
BD(14,3) &= BD(3,14) \\
BD(14,7) &= BD(7,14) \\
BD(14,8) &= BD(8,14) \\
\end{align*}
\]
BD(14, 9) = BD(9, 14)
BD(14, 13) = BD(13, 14)
BD(14, 14) = 1225 \times P(99 \times L \times L) = 200 \times 7(99 \times P)
BD(14, 15) = 200 \times N/(99 \times P)

BD(15, 2) = BD(2, 15)
BD(15, 3) = BD(3, 15)
BD(15, 8) = BD(8, 15)
BD(15, 9) = BD(9, 15)
BD(15, 14) = BD(14, 15)
BD(15, 15) = -200 \times N \times N/(99 \times P)
APPENDIX E

The frequency determinant is formed from the matrix arranged in the following manner:

\[
\begin{pmatrix}
A_{1,1} - \gamma & A_{1,2} & A_{1,3} \\
A_{2,1} & A_{2,2} - \gamma & A_{2,3} \\
A_{3,1} & A_{3,2} & A_{3,3} - \gamma
\end{pmatrix}
\]

The matrix obtained by using the Washizu theory is symmetric.

Denoting

\[
U_1(m) = -\lambda^2 - (1 - \nu)\frac{n^2}{2} - n_s^o \lambda^2 - n_s^o n^2 \\
V_1(m) = (1 + \nu) \frac{\lambda}{m} \frac{n}{2} \\
W_1(m) = -\nu \frac{\lambda}{m} \\
U_2(m) = V_1(m) \\
V_2(m) = - (1 - \nu) \frac{\lambda^2}{m^2} - n^2 - 2k(1 - \nu)\lambda^2 - kn^2 \\
- n_s^o \lambda^2 - n_s^o (1 + n^2) \\
W_2(m) = n + k(2 - \nu)\lambda^2 n + kn^3 + 2nn_s^o \\
U_3(m) = W_1(m) \\
V_3(m) = W_2(m) \\
W_3(m) = -1 - k\left(\frac{\lambda^2}{m} + n^2\right)^2 - n_s^o \lambda^2 - n_s^o (1 + n^2)
\[ \begin{align*}
AD(1, 1) &= U1(1.1) \\
AD(1, 2) &= V1(1.2) \\
AD(2, 1) &= U2(1.2) \\
AD(2, 2) &= V2(1.3) \\
AD(2, 3) &= W2(1.3) \\
AD(3, 1) &= U3(1.4) \\
AD(3, 2) &= V3(1.4) \\
AD(3, 3) &= W3(1.4) \\
AD(4, 4) &= U1(2.1) \\
AD(4, 5) &= V1(2.2) \\
AD(4, 6) &= W1(2.2) \\
AD(5, 4) &= U2(2.1) \\
AD(5, 5) &= V2(2.2) \\
AD(5, 6) &= W2(2.2) \\
AD(6, 4) &= U3(2.1) \\
AD(6, 5) &= V3(2.2) \\
AD(6, 6) &= W3(2.2) \\
AD(7, 7) &= U1(3.1) \\
AD(7, 8) &= V1(3.2) \\
AD(7, 9) &= W1(3.2) \\
AD(8, 7) &= U2(3.1) \\
AD(8, 8) &= V2(3.2) \\
AD(8, 9) &= W2(3.2) \\
AD(9, 7) &= U3(3.1) \\
AD(9, 8) &= V3(3.2) \\
AD(9, 9) &= W3(3.2) \\
AD(10, 10) &= U1(4.1) \\
AD(10, 11) &= V1(4.2) \\
AD(10, 12) &= W1(4.2) \\
AD(11, 10) &= U2(4.1) \\
AD(11, 11) &= V2(4.2) \\
AD(11, 12) &= W2(4.2) \\
AD(12, 10) &= U3(4.1) \\
AD(12, 11) &= V3(4.2) \\
AD(12, 12) &= W3(4.2) \\
AD(13, 13) &= U1(5.1) \\
AD(13, 14) &= V1(5.2)
\end{align*} \]
<table>
<thead>
<tr>
<th>AD(13,15) = V1(5.)</th>
</tr>
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<tr>
<td>AD(14,13) = U2(5.)</td>
</tr>
<tr>
<td>AD(14,14) = V2(5.)</td>
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<td>AD(14,15) = W2(5.)</td>
</tr>
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<td>----------------------</td>
</tr>
<tr>
<td>AD(15,13) = U3(5.)</td>
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<td>AD(15,14) = V3(5.)</td>
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<tr>
<th>BD(1,1) = -4.<em>N</em>N/(3.*P)</th>
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<tr>
<td>BD(1,7) = 4.<em>N</em>N/(5.*P)</td>
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<tr>
<td>BD(1,13) = 4.<em>N</em>N/(21.*P)</td>
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<tr>
<th>BD(2,2) = -8.<em>(1.+N</em>N)/(3.*P)</th>
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</thead>
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<tr>
<td>BD(2,3) = 16.*N/(3.*P)</td>
</tr>
<tr>
<td>BD(2,8) = 8.<em>(1.+N</em>N)/(15.*P)</td>
</tr>
<tr>
<td>BD(2,9) = -16.*N/(15.*P)</td>
</tr>
<tr>
<td>BD(2,14) = 8.<em>(1.+N</em>N)/(105.*P)</td>
</tr>
<tr>
<td>BD(2,15) = -16.*N/(105.*P)</td>
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<th>BD(4,4) = -28.<em>N</em>N/(15.*P)</th>
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<tr>
<td>BD(4,10) = 76.<em>N</em>N/(105.*P)</td>
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<th>BD(5,5) = -32.<em>(1.+N</em>N)/(15.*P)</th>
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<td>BD(5,6) = 64.*N/(15.*P)</td>
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<tr>
<td>BD(5,11) = 64.<em>(1.+N</em>N)/(105.*P)</td>
</tr>
<tr>
<td>BD(5,12) = -128.*N/(105.*P)</td>
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<td>BD(6,11) = -128.*N/(105.*P)</td>
</tr>
<tr>
<td>BD(6,12) = 64.<em>(1.+N</em>N)/(105.*P)</td>
</tr>
</tbody>
</table>
\[ BD(7,1) = \frac{4* N*N}{5*P} \]
\[ BD(7,7) = -68* N*N/(35*P) \]
\[ BD(7,13) = 44* N*N/(63*P) \]

\[ BD(8,2) = 8*(1+N*N)/(15*P) \]
\[ BD(8,3) = -16* N/(15*P) \]
\[ BD(8,8) = -72* (1+N*N)/(35*P) \]
\[ BD(8,9) = 144* N/(35*P) \]
\[ BD(8,14) = 40* (1+N*N)/(63*P) \]
\[ BD(8,15) = -80* N/(63*P) \]

\[ BD(9,2) = -16* N/(15*P) \]
\[ BD(9,3) = 8* (1+N*N)/(15*P) \]
\[ BD(9,8) = 144* N/(35*P) \]
\[ BD(9,9) = -72* (1+N*N)/(35*P) \]
\[ BD(9,14) = -80* N/(63*P) \]
\[ BD(9,15) = 40* (1+N*N)/(63*P) \]

\[ BD(10,4) = 76* N*N/(105*P) \]
\[ BD(10,10) = -124* N*N/(63*P) \]

\[ BD(11,5) = 64* (1+N*N)/(105*P) \]
\[ BD(11,6) = -128* N/(105*P) \]
\[ BD(11,11) = -128* (1+N*N)/(63*P) \]
\[ BD(11,12) = 256* N/(63*P) \]

\[ BD(12,5) = -128* N/(105*P) \]
\[ BD(12,6) = 64* (1+N*N)/(105*P) \]
\[ BD(12,11) = 256* N/(63*P) \]
\[ BD(12,12) = -128* (1+N*N)/(63*P) \]

\[ BD(13,1) = 4* N*N/(21*P) \]
\[ BD(13,7) = 44* N*N/(63*P) \]
\[ BD(13,13) = -196* N*N/(99*P) \]

\[ BD(14,2) = 8* (1+N*N)/(105*P) \]
\[ BD(14,3) = -16* N/(105*P) \]
\[ BD(14,8) = 40* (1+N*N)/(63*P) \]
\[ BD(14,9) = -80* N/(63*P) \]
\[ BD(14,14) = -200* (1+N*N)/(99*P) \]
\[ BD(14,15) = 400* N/(99*P) \]
BD(15,2) = \frac{16 \cdot N}{105 \cdot P}

BD(15,3) = 8 \cdot \frac{1 + N \cdot N}{105 \cdot P}

BD(15,8) = -80 \cdot \frac{N}{63 \cdot P}

BD(15,9) = 40 \cdot \frac{1 + N \cdot N}{63 \cdot P}

BD(15,14) = 40 \cdot \frac{N}{99 \cdot P}

BD(15,15) = -200 \cdot \frac{1 + N \cdot N}{99 \cdot P}
REFERENCES


