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IN CALCULUS.

The Ohio State University, Ph.D., 1970
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FIRST COURSE IN CALCULUS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

Osiefield Anderson, B.S., M.A.

The Ohio State University
1970

Approved by

[Signature]
Adviser
College of Education
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This dissertation is dedicated to the memory of my mother and to my wife, Vestella.
VITA

July 17, 1928  Born, Pineview, Georgia

1957 . . . .  B.S., Mathematics Senior Award,
            The Fort Valley State College,
            Fort Valley, Georgia

1958 . . . .  M.S. Mathematics, The Atlanta University,
            Atlanta, Georgia

1958-1967 . . Instructor, Assistant Professor of
            Mathematics, Florida A. and M.
            University, Tallahassee, Florida

1967-1970 . . Teaching Associate in Mathematics,
            The Ohio State University,
            Columbus, Ohio

FIELDS OF STUDY

Studies in Mathematics Education  Professor Harold C.
            Trimble, Adviser

Studies in Mathematics

    Abstract Algebra
    Complex Analysis
    Foundations of Mathematics
    Geometry
    Number Theory
    Real Analysis
    Topology
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CHAPTER I

THE NATURE AND SIGNIFICANCE OF
THE PROBLEM

Introduction

The fact that mathematics is being created at an ever expanding rate and new applications of mathematics are being found in all fields of study are factors that helped to bring about significant changes in the mathematics curriculum at all levels.

If one looks in retrospect over the past 30 years, he will find that the kinds and amount of mathematics used in any particular field of thought have changed.

Some of the mathematics that is being used in many professions today was practically non-existent 30 years ago, for instance, game theory. This kind of change suggests to the writer the possibility that the mathematics that one learns for a profession at age 17 might be inadequate or even obsolete by age 47.

In order to cope with the mathematical changes and demands as reflected in the mathematics curriculum, a primary problem facing mathematics teachers and mathematics educators today is that of continuously finding teaching strategies and methodologies to make the mathematics taught
meaningful, functional, and accessible to the mathematics student. Such mathematics might provide a base for continued learning.

During the past 15 years, sustained efforts have been directed toward curriculum reform in mathematics at the elementary and secondary levels. Millions of dollars have been spent in experimental mathematics programs designed to find new and effective methods of teaching mathematics. The Department of Mathematics at The Ohio State University is continuously engaged in research to find efficient means to make the mathematics taught at Ohio State more meaningful and insightful to the students. In view of the pressing need for better methods of teaching mathematics, this study is designed to provide insight into the effectiveness of the use of counterexamples as a teaching strategy in teaching concepts in a first course in calculus.

A review of the literature shows that for a number of years, there has been an interest in the role of counterexamples in concept learning. However, the writer feels that the use of counterexamples at the elementary collegiate level is wanting.

The idea of counterexample is not new to the student of mathematics. For instance, the function

\[ f(x) = \begin{cases} 1, & \text{for } x \text{ rational} \\ 0, & \text{for } x \text{ irrational} \end{cases} \]
is a counterexample to the conjecture: "Every bounded function is Riemann integrable." Similarly, the greatest integer function

\[ f(x) = \lfloor x \rfloor \]

is a counterexample to the assertion: "Every function is continuous at some integer p." As a final example, let \( B = \{3, 7\} \) and \( C = \{\{3, 7\}, \{a, b\}\} \). One observes that \( 3 \in B \) and \( B \subseteq C \) but \( 3 \notin C \). Therefore, this is a counterexample to show that the relation indicated by "\( \subseteq \)" is not a transitive relation. In summary, the truth or falsity of a given statement in mathematics or another field may not be known. If one can find an example for which the statement is false, then such an example is called a counterexample to that statement.

**Motivation for the Study**

During the 1962-1963 school year, the writer was a student in the Department of Mathematics at The University of Washington, Seattle, Washington. At that time, he had the opportunity to take a course in real variables from Professor Edwin Hewitt. In the opinion of the writer, Professor Hewitt was a very effective teacher. His occasional use of counterexamples gave the writer greater insight into many concepts in real analysis. Consequently, the writer felt that counterexamples could possibly be used as an effective technique in teaching concepts in a first course in calculus. After returning to Florida A. and M.
University from The University of Washington, the writer was given the opportunity to teach a first course in calculus for three years. The opinions expressed in this section of this chapter are consequences of the writer's experiences as a graduate student at The University of Washington and of this particular teaching experience. This first course in calculus met four days weekly and carried four hours college credit. The topics covered were the usual basic topics treated in most first courses in calculus. During the three-year period of teaching this first course in calculus, the writer made frequent use of counterexamples and felt that the students who were more adept in exhibiting counterexamples to disprove given false conjectures developed a keener insight and an intuitive feeling as to the extent of the domain of validity of a given conjecture. These students seemed to the writer to have developed a feeling of discernment concerning the falsity of a given false conjecture although they often had some difficulty in providing a counterexample to disprove the conjecture.

It is the writer's contention that the student who can exhibit numerous counterexamples to show that boundedness of a sequence is not a sufficient condition for convergence of the sequence has a better mastery and command of the concept of convergence than the student who cannot exhibit such counterexamples.
The writer is of the opinion that the student can grasp the meaning of such concepts as the limit concept more easily if he is provided experiences involving false statements often assumed to be true about limits. For example, the graph

\[
\lim_{x \to p} f(x)
\]

is a simple pictoral illustration to show the student that \( \lim_{x \to p} f(x) \) does not necessarily exist. The writer feels that illustrative examples of this kind helped his students better understand the delta-epsilon definition of the limit and the fact that in the discussion of limit the concern is only with the behavior of the function in a delta-neighborhood of the point "p" and not at the point "p" itself.

In teaching a first course in calculus, most teachers no doubt have seen students assume that \( \sqrt{a+b} = \sqrt{a} + \sqrt{b} \) in order to integrate \( \sqrt{x^2+1} \, dx \). The writer feels that fewer students would make this mathematical error if they had evaluated several examples of the nature \( \sqrt{16+9} \) and \( \sqrt{16} + \sqrt{9} \) while studying the laws of radicals in algebra.

The writer feels that providing appropriate counterexamples and having students prove that certain conjectures
are false by the use of counterexamples will enhance their understanding of the concept under consideration and will make them more aware of the importance of the qualifications placed upon theorems and principles and of their domains of validity. Consequently, he feels that the students will be less likely to make hasty unwarranted mathematical generalizations. The simple example

\[ f(x) = |x| \]

should suffice to convince the student that, although one frequently works with functions that are both continuous and differentiable at a point "p", continuity of a function \( f \) at a point "p" does not imply differentiability of \( f \) at "p".

A typical question that the student encounters in mathematics and to which he must find an answer is of this nature: "Is the conjecture \( P \), which is of the form \( A \subseteq B \), true?" To determine the truth or falsity of \( A \subseteq B \), the student will show either that every \( x \) in \( A \) also belongs to \( B \) or that there exists an \( x \in A \) such that \( x \notin B \). For example, let "\( P \)" be the assertion: "Every bounded sequence converges." Then \( A \) consists of all bounded sequences and \( B \) consists of all convergent sequences. The sequence

\[ 1, -1, 1, -1, 1, \ldots \]

suffices to show that \( A \notin B \), since the sequence \( 1, -1, 1, -1, \ldots \) is an element of \( A \) but is not an element of \( B \). Similarly, let "\( P \)" be the assertion: "The intersection of any number of open sets is open." Then \( A \) is the set whose
elements are intersections of open sets and \( B \) is the set of all open sets. The example

\[
\bigcap_{n=1}^{\infty} \left( - \frac{1}{n}, \frac{1}{n} \right) = \left\{ 0 \right\}, \quad n = 1, 2, 3, 4, \ldots
\]

shows that \( A \not\subseteq B \), since \( \left\{ 0 \right\} \in A \) but \( \left\{ 0 \right\} \not\in B \). Therefore, the assertion "\( P \)" is false. As a final example, let "\( P \)" be the assertion: "Every function \( f \) continuous at a point \( p \) is differentiable at the point \( p \)." Then \( A \) is the set of all functions continuous at \( p \) and \( B \) is the set of all functions differentiable at \( p \). The counterexample

\[
f(x) = |x|
\]

is sufficient to show that the assertion "\( P \)" is false, since \( f \in A \) but \( f \not\in B \). Consequently, \( A \not\subseteq B \). The writer feels that students who make frequent use of counterexamples are more effective in finding answers to the typical questions encountered in mathematics.

It is the opinion of the writer that the nature of the theorems used in a first course in calculus or in any course in mathematics for that matter should excite the curiosity of the student of mathematics. For instance, consider the conditions of Rolle's theorem: If \( f \) is continuous on the closed interval \([a, b]\) and \( f' \) exists on the open interval \((a, b)\), and if \( f(a) = f(b) \), then there exists \( p \in (a, b) \) such that \( f'(p) = 0 \).

In discussing the theorems and principles in the first course in calculus taught at Florida A. and M.
University, the writer had the students investigate several conjectures formed by dropping certain conditions of known theorems. For example, let "P" be the assertion formed by dropping the condition $f(a) = f(b)$ in Rolle's theorem. Then, $f(x) = x^2$ on $[1, 2]$ is a counterexample to show that the assertion "P" is false, since $f'(x) = 2x$ and, consequently, $f'(x) = 0$ implies that $p = 0$. Thus, $p \notin (1, 2)$, since $0 \notin (1,2)$. The writer feels that exercises of this nature made his students more aware of each hypothesis of a theorem and, consequently, more aware of the extent of its applicability. It is the writer's opinion that experiences with many varied kinds of exercises in mathematics are essential to students of mathematics at all levels and particularly for the beginner. It can truly be said that, "Lots of good exercises are to the students of mathematics what Czerny is to the pianist."

The example-counterexample approach seems to be one effective method used in teaching the young child to learn about the world in which he finds himself. For instance, the young child learns the difference between a pen and a pencil or the difference between a watch and a clock, not by verbal definitions of these objects, but by being told: "This is a watch." "This is a clock." "No, this is not a watch; this is a clock," etc. Since this approach seems to have been an effective one in regard to this basic teaching-learning situation, the writer feels that it is reasonable
to ask whether such an approach could possibly prove to be rewarding and fruitful at more advanced levels of learning and understanding.

Statement of the Problem

The problem of this study was to determine the effect of the teaching strategy of an extensive treatment of counterexamples on learning concepts in a first course in calculus.

Procedure

The subjects of discourse for this study consisted of approximately 60 students enrolled in Mathematics 151 at The Ohio State University during the autumn quarter of the 1969-1970 academic year. On the first day of class, during the quarter of this study, all students scheduled to take Mathematics 151 reported to the Mathematics Building of Ohio State and were assigned to a class. In this limited sense, the subjects for this study were randomly assigned to two classes: One class designated as the experimental class or the E-group and the other as the control class or the C-group. The experimental class met at nine o'clock and the control class met at three o'clock. Both classes met five days weekly and were taught by the writer. The experimental treatment and the control treatment differed specifically in this manner: Whenever a concept or a theorem was introduced, the students in the experimental
class were exposed to several related examples and to several counterexamples, whereas, the students in the control class were exposed to related examples only. The writer attempted to keep the number of examples and counterexamples used with the experimental class about the same as the number of examples used with the control class. The text used with both classes was Calculus and Analytic Geometry, Second Edition, by Professor Robert C. Fisher and Professor Allen D. Ziebur, Prentice-Hall, Inc., Englewood Cliffs, New Jersey.

Since the students in the experimental class spent a considerable amount of time exhibiting and discussing counterexamples in regard to a given concept, the control class had more time to get involved in the kinds of classroom and homework activities that constituted approximately 90 per cent of the kinds of items encountered on the "mid-term" examinations and final examination administered. Consequently, this could have had an effect on the internal validity of the study. This and some other uncontrolled extraneous factors which might have affected the results of this study are discussion in the section on "Limitations."

Although complete randomization is the most adequate all-purpose assurance of equating groups, to further determine the comparability of the two classes, the writer computed a one-way analysis of variance on the students' ACT scores to mathematics. Since the loss of students was
approximately the same for both classes, the writer assumed that mortality did not affect the internal validity of the study. However, to substantiate this assumption, after the final loss of subjects, the writer computed a one-way analysis of variance on the scores that the remaining subjects of the study made on the pre-test that Mr. William Paul administered to all students enrolled in Mathematics 151 at Ohio State during the time of this study. Mr. Paul was conducting an experimental study which involved all students enrolled in Mathematics 151 at Ohio State during the time of this study.

Since most educators generally support the premise that attitudes affect achievement, the mathematics teacher and the mathematics educator must concern themselves with ways of improving attitudes. For instance, in his article: "Role of Attitude in Children's Failures and Successes," Paul A. Whitty states that, "In every subject area the efficiency of instruction will be heightened by the development of an instructional program which recognizes the significance of each child's attitude" (1, 422-423). In her article: "Emotional Blocks in Mathematics," Mary K. Tulock suggests that teaching strategies have a strong influence on the student's attitude toward that subject and that certain teaching techniques can help college students overcome severe emotional blocks (2, 572-576). To determine whether the two classes differed significantly in terms of
their attitudes toward mathematics in general, and most importantly, to determine whether or not the example-counterexample approach has any motivational merit, the writer administered the Donner-Shatkin "mathematics opinionnaire" to both classes before and after the treatments. This instrument was chosen because of its brevity and high reliability. A copy of the instrument is displayed in Appendix E.

The writer kept a daily record of the occurrences in both classes. Particular attention was given to the kinds of questions that the students asked, to the manner in which they gave answers to questions, to their opinions about mathematics, about the class, etc. Observable emotions and reactions of students were noted. In both classes, the writer attempted to instruct in such a manner as to elicit "good" questions from the students. The writer firmly believes that the student's ability to ask "good" questions about mathematics is indicative of his mathematical growth. Professor Arnold E. Ross, Chairman of the Department of Mathematics at Ohio State, supports this philosophy and made the following comment in this regard:

. . . A strong effort should be made to encourage the student to bring forth searching questions and to help the teacher to avoid the pedagogical sin of sins which consists of providing answers to questions which the student has not asked, or what is worse, is not even ready to ask at the time. It is very important to realize that the asking of good questions is as important for the growth of an individual or, for that matter, for a science as the answering of questions already posed. . . .
One readily admits that every discipline progresses through discovery on the esoteric heights of pure research. One is less prepared to admit that personal discovery is a vital ingredient of learning. . . . It is important to realize that creating an effective environment for learning means setting the stage for the telling flashes of awareness by providing the student with the kind of experience which leads to searching questions which, in turn, through the hazardous path of trial and error, lead to the desired understanding. . . . All too often we err by trying to disregard the vital underpinning of experience and by presenting to the novice the end result of some one else's successful struggle for understanding in the forlorn hope that this may provide an easy way out.

The duration of the study was one quarter (Autumn Quarter, 1969-1970). Four one-hour "midterm" examinations, a two-hour final examination, one 150-minute general examination, and a 90-minute generalization examination were administered to both classes. Copies of the examinations used in the study are displayed in the Appendixes.

**Null Hypotheses**

1. There will be no significant difference in the mathematical achievement of the two classes as measured by the four "midterm" examinations, by the 150-minute general examination, and by the final examination.

2. The means of the two groups on the generalization examination will not differ significantly at the five per cent level.
Statistical Design (Modified version):

\[ R \times 0_1 \]

\[ R \quad 0_2 \]

**Statistical Tests**

1. On the basis of the results of the analysis of variance computed on the students' ACT scores and on the basis of the results of the pre-test administered to the subjects, a one-way analysis of variance of the scores of each examination administered was sufficient to test the null hypotheses.

2. The writer computed a reliability coefficient of internal consistency for each examination used for the study by means of a Spearman-Product Moment.

In this study the writer's reference to the students' ACT scores will mean the students' ACT Mathematics raw scores.

For information concerning the statistical tests used in this study, the reader is referred to: **Basic Statistical Methods**, N. M. Downie and R. W. Heath, Harper and Row, Publishers, New York.

**Organization of the Dissertation**

In Chapter II the writer presents some literature relevant to the study. Chapter III provides some
information about the control class and the experimental class. In this chapter the writer discusses the nature of the treatments and provides some insight into the atmosphere of the two classes. Chapter IV is a presentation of the statistical analysis of the study. The results, conclusions, and limitations of the study are reported in this chapter. Chapter V, the final chapter, is an informal evaluation of the study based on the daily record kept by the writer of the occurrences in both classes, and of the students' comments, reactions, and opinions expressed both in and out of class.
CHAPTER II

REVIEW OF THE LITERATURE

Criteria of Selection

The writer has considered much of the literature of investigations that were published during the past 46 years in the areas of concept formation and mathematics education. Only those investigations concerning the "role of positive and negative instances" were considered in the area of concept formation. In the area of mathematics education, the writer addressed investigations concerning concept formation and experimental design. The sources of information of this review are: Dissertation abstracts, the educational periodicals which deal with mathematics education, and many bibliographies.

Nature of the Investigations

Most of these investigations related to counter-examples and concept formation are not specifically concerned with mathematics concepts per se but rather with concepts based on simple geometric designs. The first study reported in the review is based on Chinese characters. The final study reported in the review is the only one that is primarily mathematical in nature. It is the
study most related to the present one and, consequently, is of particular concern to the writer. In most of the studies of this review, the term "negative instance" is used in lieu of the term "counterexample."

**Reviews of Selected Investigations**

1. Kuo of the University of California conducted a study to determine the different modes of language response involved in the processes of inductive inference. By inductive inferences, he alluded to a series of language reactions to different stimulus settings or different groups of stimulus patterns when each of the latter have an element in common; it is provided that the language series terminates in a different organization or regrouping of words (conclusions, laws, or principles), and that this new organization, in turn, is such as might lead the subject to an immediate solution to the problem at hand. For example, suppose that the experimenter "E" asks the subject "S" to solve a Yerkes multiple choice problem. "E" requests that "S" make verbal responses. From trial to trial, "E" varies the number of keys (different stimulus patterns) but always keeps the right key in the center (constant stimulus-factor). At the outset, "S" will make a number of random guesses and, consequently, solves the problem only by accident. However, after a number of trials, "S" will discover the rule of the game: he will note and explain to "E" that the correct key is always in the center. Subsequent trials will
substantiate his conclusion. Thus, after having determined the rule of the game, "E" can readily give the solution to the problem rather than make random guesses. The materials used in the study consisted of 88 compound Chinese characters, their "radicals," "pseudo radicals," the "misleading characters," the "negative characters," and the "component cardboard." Seventy-five of the 88 compound Chinese characters are divided into five groups of 15 characters each. The characters of each group have a common radical. The radicals always appear on the left of the compound characters, and each has a definite meaning which can be determined from the common element in the meanings of the compound characters in which the radical appears. The five radicals signify: water, female, mouth, hand, and metal.

Definitions

1. Negative Instances and Misleading Characters.

Those compound characters which may tend to lead the subject to a false conclusion are called "misleading characters." There are two types of "negative instances"—those within the radical group (compound characters which share the same radical group (compound characters which share the same radical as the misleading characters) and those outside the radical group (compound characters which have no common structure with the misleading characters).
11. Pseudo Radicals. There are ten components, each of which may appear in two or more compound characters. However, such compound characters have no common relationship in meaning and are, therefore, called pseudo radicals.

iii. Component Cardboard. The components of the 88 compound characters were separately written on this board. Each character was non-uniformly written in two colors (black and green). The 88 compound characters were presented to the subjects in eight series.

The subjects of discourse consisted of 60 students (27 males and 33 females). Although most of the Ss had not taken any courses in psychology, a few were enrolled in an elementary course in psychology. The experimenter "E" told the Ss that the experiment was solely a memory test since one main object of the experiment was to study the process of inductive inference. The Ss were given the Chinese characters, with their English equivalent, and told to organize them in any manner and make any inference whatsoever. The results of the study showed that the more negative or misleading characters that were in the series, the less accurate was the conclusion made by the subjects (3, 247-293).

2. Smoke conducted two studies to compare the rapidity of concept learning from series composed of positive
instances only to those composed of positive and negative instances. In both studies, six concepts concerned with geometric designs were used. The series, each consisting of 16 cards, of "vec," "dax," "pog," "wez," "mob," and "zum" were used. These had geometrical meaning. For example, "sum": three straight line segments (red), two of which intersect the third, thereby trisecting it. For each concept, one of the series consisted of only positive instances and the other series consisted of eight positive instances and eight negative instances. A plus sign was drawn in India ink under each positive instance and a minus sign was drawn under each negative instance. The negative instances were drawn to appear very similar to those in the test series. They were drawn to satisfy all but one of the conditions essential to the concept to be determined, and each condition essential to the concept was violated by at least one figure.

The subjects were given a preliminary period of training to familiarize them with the procedure of the experiment. Each subject was given a pack of 16 cards and told that the drawing on each card was a "tov" and he was to determine what a drawing must be to be called a "tov." Each subject was further instructed to raise his hand as soon as he felt that he knew what a "tov" was. When a subject raised his hand to indicate that he had learned the concept, the experimenter gave him three tests to determine if he had, in fact, learned the concept. If he had not, the
experimenter defined it for him. Then the experimenter gave
the subject another pack of 16 cards and informed him that
some of the drawings were "gids" and some were not as in-
dicated by the plus and minus signs. The subjects was then
told to determine what a "gid" was and to raise his hand
when he thought that he knew. At that time, he was given
three tests. If he had not learned what a "gid" was, the
experimenter defined it for him.

The subjects of discourse consisted of 24 under-
graduates at The Ohio State University. Each subject re-
ported for two experimental periods, learning two concepts
the first period and four concepts the second period. Each
subject examined a series of drawings for each of the six
concepts. Half of the series contained only positive
instances and the other half contained both positive and
negative instances. The kind of series was alternated for
each subject. The variable of measurement was the time
that a subject spent examining the drawings and taking the
test. No significant difference was found in the rapidity
of concept learning for either type of series. However,
Smoke indicated that some subjects seemed to have found the
negative instances instructive (4, 34-40).

3. The second study conducted by Smoke was identical
to the first except that the subjects were presented all 16
drawings simultaneously. Similarly, the findings were the
same as in the first study. However, Smoke made the
following interesting observations:

Most of the subjects, however, did prefer to learn from both positive and negative instances. When asked at the conclusion of the experiment which method they preferred, twenty-three of the thirty subjects expressed a preference for learning from positive instances only, and one stated that he had no preference. There is a tendency for negative instances to discourage "snap judgment." When the subjects were learning from both positive and negative instances they tended to come to an initial wrong conclusion less readily, and to subsequent wrong conclusions less frequently, than they were learning from positive instances only. Thus, although negative instances may not make for rapidity in learning they tend to make for accuracy, especially in the case of difficult concepts. It appears that in so far as negative instances assist concept learning they do so largely because of the way in which they prevent the learner from coming to one or more erroneous conclusions while he is still in the midst of the learning process (5, 583-588).

4. Hovland, in his article "A Communication Analysis of Concept of Learning" discussed means of obtaining more reliable information concerning the "role of positive and negative instances" in concept formation. He proposed to analyze theoretically the amount of information required to specify a concept by positive or negative instances as one kind of instance may transmit more information than the other in defining a given concept. Hovland remarked that unless there exists some common understanding between the experimenter and the subject, it is difficult to determine what information is conveyed by each instance. Otherwise, factors not intended by the experimenter to be relevant may be selected as possibilities by the subject. Thus, subjects who are most adept in considering different possibilities
may be penalized. For example, suppose the subjects were presented with stimulus figures picturing squares, circles, or triangles. Then each may be one of three sizes—one inch, two inches, or three inches—and one of three colors—black, grey, or white. The concept to be learned could consist of a combination of one value of one dimension, say, the small one inch value of the size dimension with one value of another dimension, say, the circle from the shape category. Thus, the concept to be learned in this case would be a small circle. For any single concept, the third factor was irrelevant (in this example, shade). In this article, Hovland gives examples of conditions where two positive instances but 625 negative instances are required to define the concept and others where five positive instances but only two negative instances are required. He, therefore, concluded that the question asked in earlier concept-formation studies as to the relative effectiveness of positive and negative instances cannot be given a generalized categorical answer (6, 461-472).

5. Hovland and Weiss commented on the major weakness of the experiments conducted by previous experimenters of concept-formation. They remarked that it is very difficult to separate the effects of the two alternatives since the information conveyed by a positive or a negative instance cannot accurately be assessed in the absence of knowledge of the hypotheses considered by a given subject. They,
therefore, posed a situation where the subject knows the structure of the concept and the dimensions involved. Thus, the subjects can then determine how many of each type of instances are required, from a logical point of view to specify completely the characteristics of the concept to be defined.

The purpose of the experiment conducted by Hovland and Weiss was to compare the learning of concepts when the necessary information is conveyed by series of instances which are exclusively positive or exclusively negative and the effects of the order of presentation of mixed and negative instances. The subjects were instructed as to the meaning of the term "concept" exactly as it was to be used in the experiment. Before they were presented with the problem, they were told how many dimensions were to be considered, the number of possible values for each dimension, and the number of values which would be correct for each relevant dimension. The subjects were given some preliminary examples to solve for familiarization with the divergent manners of presenting the information necessary to define a concept.

In the first experiment, the dimensions were such that two positive instances or five or ten negative instances were sufficient to define the concept. The subjects for the experiment consisted of 12 Yale College students. Each subject reported for two one-hour sessions.
The percentage of subjects able to define the concept correctly when given all positive instances did significantly better than those who received all negative instances. The difference was significant at the .001 level (Chi square). However, there was no significant difference for other type series. Moreover, the data provided no suggestions that the series with positive instances presented first is superior to any other series.

The second experiment paralleled the first except that all series were presented to the subject at once. No significant difference was found except in the comparison of a number of subjects presented by an all positive or an all negative series, in which case, there was a consistent superiority for the all positive series (better at the .001 level--Chi square = 16.3, degrees of freedom = 3).

In experiment three, the general procedure employed was very similar to that of experiment two. The problems, however, were selected in such a manner that four instances were sufficient to completely determine the concept for a type of problem. Three types of series were used, involving four positives, four negatives or two positives and two negatives: (+++), (----), (++--). These three types of series made it possible to make an easy comparison of all positive, all negative, or mixed positive and negative. The subjects of discourse were 24 Yale College students none of whom, had taken part in either of the previous experiments.
Each subject was examined individually in a session which lasted about one hour. The findings showed that (++++) was significantly better than (++--) at the .001 level and (++--) better than (-----) at the .01 level. The experimenters made the following conclusions:

i. Mixed positive and negative instances are intermediate between all positive and all negative series in learning.

ii. When the negative instances are displayed simultaneously, the accuracy of concept attainment is higher than when they are presented successively.

iii. The correct concept is attained by a higher percentage of subjects when transmitted by all positive instances than by all negative instances.

From the results of this study, one may conclude, as did the experimenters, that all negative instances are found to be consistently inferior to all positive instances. At the same time, however, the results disprove the generalization that concepts cannot be learned from negative instances, since under appropriate conditions over half the subjects were able to arrive at the correct concept exclusively on the basis of negative instances (7, 174-182).

6. Donaldson conducted two studies to determine the role of counterexamples (positive and negative information in matching problems). The subjects of the study consisted of 19 children, ranging in age from 14 to 14 1/2 years. In
this study the subjects were required to construct a problem rather than solve a given problem in the sense commonly thought of in mathematics. All subjects had been involved in a more extensive study which involved the solution of a matching problem similar to the ones they would attempt to construct. The experimenter gave the subjects verbal instructions. Two types of problems were given:

Problem A

The subjects were asked to recall a particular problem that they had met previously. The verbal instructions were of this nature. Do you remember the problems about the five boys and the five schools? Instead of solving the problem, I want you to make up a similar problem. I will tell you what the answer is to be and you are to tell me the information what the children will have to be given to solve the problem. We will call the boys by their initials. Let's say their names are Arthur, Bob, Charles, Dick, and Edward; and we will just call the schools 1, 2, 3, 4, and 5. Now here is the answer--

A B C D E

4 2 1 3 5

That is, Arthur goes to school 4, Bob goes to school 2, and so on. Now you can give me information in two ways. Either you can say something like "Arthur does not go to school 2" (which we can write as A ≠ 2, for short). Pieces of information like A = 4 we will call positive. Pieces of information like A ≠ 2 we will call negative. Every positive
piece of information that you give will cost five points; every negative piece will cost one point. You have to try to give enough information for someone who does not know the answer to be able to find it out and you have to try to do it in such a way as to make the lowest possible score.

Problem B

Here are two boxes labelled A and B. The crosses in the boxes stand for the numbers 1, 2, 3, ..., 8. You are to find out which numbers are in A and which are in B, but you do not have to worry about the way in which the numbers are arranged inside the boxes. You will ask me for the pieces of information you think you will need. You may ask me for either a positive or a negative piece of information and you may tell me whether you want it to be about box A or about box B. If you ask me for a piece of positive information about box A, I will give it to you in the form: "one of the x's in A is 5" or whatever it may be. If you ask for negative information about box A, I will tell you--"none of the x's, in A is 7" or whatever it may be. Now each positive piece of information will cost you two points and each negative piece one point. You are to try to find the answer, making the lowest possible score.

In the second study, the subjects for the experiment consisted of 90 female junior students at a Scottish
teachers' training college. The subjects were tested as a group with problems A only, worded in more general terms. The subjects had no previous experiences with matching problems. The time allotted was 25 minutes. The findings of the study showed that in a matching situation, the subjects preferred positive information (8, 253-262).

The findings of the study mentioned above supported the findings reported by Bruner who found that, (1) the mere frequency of the positive and negative instances regardless to the order in which they are encountered governs the likelihood of encountering certain contingencies with respect to a given hypothesis under investigation. That is, one may encounter positive or negative instances and each of these being capable of verifying or disproving a conclusion that the subject has tentatively made concerning a correct concept, and (2) a general tendency is the inability or unwillingness of subjects to use efficiently information that is based on negative instances as negative instances require the transformation. That is, the subject is required to take information concerning what a concept is not and use it to infer what it is. Bruner said, that "It is as if information that results from 'in-the-head' transformation is distrusted perhaps through an appreciation of the possibility of the errors that can be made in such transformations (9, 72, 237)."
7. Freibergs and Tulving conducted an investigation to determine the effect of practice on utilization of information from positive and negative instances in concept identifications.

Twenty subjects were randomly selected from a class of students enrolled in a general course in psychology at the University of Toronto. Eighteen females and 2 males, ranging in age from 18 to 20, were chosen. The subjects were randomly divided into two groups. One group of subjects (P-group) worked with positive instances and the other group (N-group) worked with negative instances. Four positive or four negative instances were sufficient to determine a concept. The stimulus material consisted of 64 cards. One, two, three, or four geometric solid figures of a certain form and color were drawn on each card. The colors were blue, green, yellow, or red. Both groups were given 20 successive problems (trials) which involved concepts of the same general type. The particular concept assigned to a given subject on each trial was randomly drawn from a list of 48 concepts of the designated type within the limit of the stimulus material. For each concept, there were 36 cards in the deck of 64 that represented positive instances and 28 cards that represented negative instances of the concept. The four instances of a concept presented to the subject on each trial were randomly selected. Each subject was told whether he had positive or
negative instances and was told to determine the concept as quickly as he could. The variable of measurement was the time it took a subject to define a concept on any given trial. The maximum time allotted was 210 seconds. All subjects in the P-group identified all concepts correctly after the third trial. There were two major findings of the study: (1) A subject's ability to solve concept identification problems (ability to think?) is very greatly affected by practice. This is true for both positive and negative instances, and (2) although during the early stages of practice, the subjects seemed much less capable of assimilating information from negative than from positive instances; the difference was very small at the end of 20 trials (10, 101-106).

8. Huttenlocher made an investigation to determine whether concepts defined by all positive instances remained easier to learn than those defined by all negative instances on one-dimensional concepts with equivalent amounts of information per instance and whether performance on mixed positive and negative instances lies intermediate in difficulty between all positive and all negative series. The subjects of the study were 26 randomly chosen seventh grade boys from towns near Boston: Weston and Maynard. All concepts were presented in two instances, which were sufficient to specify the concept to be learned. Four categories of problems were addressed: (++) , (+-), (-+), and (--) .
Each subject was presented with 24 problems, six in each of the four categories. Each answer given by a subject was scored as right or wrong. Each problem was presented twice. The difference between the means was significant beyond the .001 level. The group of problems involving two negative instances (--) resulted in significantly poorer performance than those groups of problems involving at least one positive instance. Ordering of instances in mixed series had a significant effect. It is interesting to note that the best performance was obtained from the series beginning with a negative instance and terminating with a positive instance. Moreover, it was found that the use of negative instances to define concepts does not necessarily adversely affect problem solving efficiency (11, 35-42).

9. In an investigation conducted by Fryatt and Tulving an attempt was made to clarify the role that interproblem plays in utilization of information from positive and negative instances of concepts. Seventeen females and 23 males, ranging in age from 18 to 26 years, were randomly chosen from general psychology courses at the University of Toronto. The subjects were assigned to four groups. All subjects in all groups solved 24 concept identification problems which were divided into two series of 12 problems in order of presentation. All problems in a given series for a given group involved either three positive instances or one positive and two negative instances. The variable
of measurement was the time it took a subject to define a concept. It was found that (1) the subjects' performance in identifying concepts from mixed instances improved greatly with practice on problems of the same class, and that some improvement, although on a small scale, was noted for positive instances; (2) there was a pronounced interaction between practice type of instances used. The difference in performance of subjects working with positive and those working with mixed instances became progressively smaller with practice, and (3) there was positive transfer from practice with mixed instances to performance with positive instances, but no similar transfer from positive to mixed instances (12, 106-117).

10. Conant remarked that research findings seem to indicate that persons have considerable difficulty with disjunctive ("x or y or both") concepts and that this difficulty is a consequence of the necessity of utilizing information from negative instances for maximal efficient concept identification. Conant conducted a study to determine whether subjects given a limit amount of training can learn to effectively utilize negative instances in disjunctive problems; and whether efficiency in using negative instances only to solve a disjunction can be affected by varying the presentation order of negative and positive instances in a few training problems. The subjects of the study consisted of 81 undergraduates who were divided into two groups. The
subjects received three concept identification tasks, each of which had six successively presented instances of a disjunction involving two two-valued dimensions. For each instance, the subjects gave a concept hypothesis and a rating indicating their confidence in the hypothesis. Moreover, the subjects were asked on two occasions if they preferred the next instance to be presented to be positive or negative. The subjects were divided into six groups, each being characteristic of the order in which positive and negative instances were presented in the three problems:

1: - - - - + +
2: - - - + + +
3: + + + - - -
4: - + - + - +
5: + - + - + -
6: + + + + + +

After the three training problems, four array tests with disjunctive concepts were given: two arrays involved simultaneous presentation of all negative instances (N); two arrays consisted of all positive instances (P). The variable of measurement was the time it took a subject to solve a problem. The subjects were subdivided according to the order (NNPP, NPPN, or PNNP) in which they were given the positive and negative array tests. Groups 1 and 2 defined the concepts of the problem significantly faster than all other groups. A 3-way analysis of variance performed on the
speed scores of 54 of the 81 subjects indicated that array presentation order had a significant effect. It was also found the rank order of groups on array tests did not differ significantly from the predicted order: 1 - 2 - 3 - 4 - 5 - 6. Moreover, 94 per cent of the subjects solved the negative array tests after less training than ever reported in the literature. Additionally, it is of striking interest to note that there was a two to one ratio of subjects who preferred negative instances to those who preferred positive instances. The experimenter concluded that, "The difficulty of disjunctions has been overemphasized and that hypotheses which state that subjects cannot efficiently use negative instances and those that posit a cultural rigidified distaste for negative instances have been shown to be in error" (13, 6866).

11. Tavrow remarked that when using negative instances, it is necessary that the subject not see the values along the dimensions of the universe which are set before him, but it is necessary that he be aware of the remaining possibilities. He further remarked that there should be a very high correlation between the subject's degree of familiarity with the entire universe and the ease of remember what the values along each of the dimensions are which are not actually presented to him. Thus, the more familiar he is with the entire universe, the better he should be able to form concepts from negative instances
using that particular universe. Consequently, Tavrow conducted a study designed to investigate whether the improvement in performance using negative instances to form concepts is due only to previous practice in the use of negative instances as was claimed by some previous experimenters.

The subjects of the study were composed of 64 undergraduates of Columbia College. Four values along each of the two independent variables (practice with negative instances and familiarity with the universe) were manipulated. Each of the subjects received either 0, 5, 10, or 15 trials of practice with negative instances using one universe and 0, 5, 10, or 15 trials of becoming familiar with a second universe, forming concepts from positive instances. The dependent variable of the experiment was the subject's performance in forming concepts from negative instances using the latter universe. It was found that when the subjects formed concepts from the same universe using negative instances, their performance improved significantly with the number of trials. However, when the subjects were trained in the use of negative instances on one universe and were switched to a new universe, transfer of training was inhibited. There was some evidence that a U-shaped relationship exists between practice and performance, given this situation. The results seem to indicate that ability to form concepts from negative instances is a function not only, if at all, of the amount of previous practice in the
use of negative instances in general, but also of two other factors: (1) the amount of previous familiarity with the universe, and (2) the amount of practice in using the technique specific to the task of forming concepts from negative instances when the universe and the difficulty of the concept remain constant (14, 4857-4858).

12. The results of a study conducted by Haygood and Stevenson concerning the effects of varying proportion of positive instances in the stimulus sequence over a wide range (0.1, 0.2, 0.4, 0.6, and 0.8) indicated that the number of trials to a criterion generally increased as the proportion of positive instances decreased. The stimuli were geometric designs which varied along four ternary dimensions: color (red, yellow, or blue), size (large, medium, or small), shape (triangle, square, or hexagon), and number (one, two, or three). In each problem, two dimensions were relevant and two dimensions were irrelevant. The subjects of discourse consisted of 100 students from introductory courses in psychology at Kansas State University. The function for the number of errors was convex, however, reaching a maximum of 0.4 and decreasing toward the extremes. A thorough examination of the data disclosed that the subjects in the 0.1 and 0.2 categories were learning the probabilities and adopting a strategy of "when in doubt, say no" (15, 178-182).
13. Motivated by previous investigations, Bourne and Guy conducted a study to determine the role of positive and negative instances in learning conceptual rules. The subjects used in the study consisted of 216 undergraduates at the University of Colorado. The subjects solved attribute identification (rule given, stimulus attributes unknown) or rule learning (attributes given, rule unknown) problems on the basis of information provided by positive instances only, negative instances only, or a mixture of positive and negative instances. For different subjects, the conceptual rule was conjunctive, disjunctive, or conditional. The results of the study showed that in attribute identification, subjects performed best when the informational instances were selected from the smaller class, positive or negative, depending on the rule and worst when they were selected from the larger class. In rule learning, subjects performed best on all rules when the greatest variety of instances (mixture of positive and negative) was presented (16, 488-494).

14. Shumway conducted a study to determine the "role of counterexamples" in the development of mathematical concepts of eighth grade mathematics students. The subjects for the study were composed of 84 eighth grade mathematics students of Marshall-University High School. The subjects were randomly divided into four groups of approximately 22 students each, thus forming four classes. The four classes
were taught by two instructors. Each instructor taught two classes: one experimental class and one control class. The text used in all four classes was *Exploring Modern Mathematics, Book 2* by M. L. Keedy, R. E. Jameson, and P. L. Johnson. Whenever a principle or a concept was introduced to the experimental class, several appropriate counterexamples were used. No counterexamples were used in the control class. The students participated in suggesting counterexamples. The duration of the experiment was approximately three months. Significant differences in mean performance were found to favor the experimental class on tests involving generalizations. However, the null hypotheses that there is no significant difference in mean performance of specific and general attainment using certain standardized mathematics materials were not rejected (17, 4, 5, 93).

In the review of the literature, no studies were found that dealt with the use of counterexamples in a first course in calculus. In fact, the study by Shumway was the only study found in the review of the literature concerning the "role of counterexamples" in concept-formation that dealt with mathematics per se at any level. However, the review of the literature offers some evidence that does suggest that counterexamples play an important role in providing for more efficient learning in concept-formation.
It should be noted that the studies reviewed, except the one by Shumway, were psychological studies. These studies, including the Shumway study, dealt with concepts in mathematics of far less difficulty than the concepts encountered in a first course in calculus. Consequently, there may exist little transferability of the results and conclusions of these studies to the present one. However, the basic objectives of these studies and the basic objectives of this study have much in common. In this sense, these studies can be considered as being relevant to the present one.

The writer found the third study reviewed (the second study conducted by Smoke) to be of particular interest to him. The observations made by Smoke are to a great extent parallel to the observations reported by the writer in Chapter V (Informal Evaluation).
CHAPTER III

THE COURSE

Mathematics 151 at The Ohio State University is a five-hour credit, freshman level, first course in calculus. By the course, the writer alludes to the work of the two sections of Mathematics 151 whose students were the subjects of this study.


During the quarter of this study (Autumn Quarter, 1969-1970), 36 sections of Mathematics 151 were offered under the supervision of Professor Robert C. Fisher. The enrollment of students in all sections of Mathematics 151 during the quarter of this study was approximately 1100. Four one-hour "midterm" examinations and a two-hour final examination were administered to all students enrolled in Mathematics 151 during the time of this study. The "midterm" examinations were given every two weeks (every other Wednesday). The examinations were multiple choice. The two
sections of Mathematics 151 involved in this study were taught by the writer. One section was designated as the experimental class or the E-group and the other section was designated as the control class or the C-group.

The Control Class

The control class consisted of 30 students (11 females and 19 males). The class met five days weekly at 3 P.M.

The first meeting of the control class was devoted primarily to an orientation of the students to the structure of the class. At this time, the writer attempted to create an environment for effective learning.

The teaching method used with the control class was basically of this nature: Whenever a concept was introduced, the students were exposed to several illustrative examples. For instance, after the statement of, and a graphical interpretation of Rolle's theorem were presented to the students, several functions were used to illustrate applications of the theorem. Classroom participation was elicited and encouraged by the writer. Since the method of instruction used with the students in the control class was basically familiar to them, they were able immediately to get involved in classroom discussions and homework exercises. Many students in this group displayed a high level of mastery of techniques in finding solutions to problems. Rarely was there a problem assigned to which no one in the
group was able to find a solution. The nature of most questions asked by the students in the control class was to obtain information pertinent to finding a solution to a particular problem. However, they still did well with problems that called for a relationship between two concepts. Most students in the C-group expressed the belief that if they were able to solve all the homework problems, then they would be prepared to do well on the "midterm" examinations. A few students in this group attempted to solve all problems in the sections of the text that were to be covered on any "midterm" examination. It was gratifying to the writer to be part of the classroom activities of this group.

The Experimental Class

The experimental class was composed of 32 students (6 females and 26 males). The experimental class met five days weekly at 9 A.M.

The first two meetings of the experimental class constituted a period of orientation of the students to the structure of the class and to the example-counterexample approach. The writer commented on the role of counterexamples in mathematics and how counterexamples would be used with this group. During this period of orientation, an attempt was made to set the stage for effective learning.

The method of instruction used with the experimental class was as follows: Whenever a concept was introduced, the students were exposed to several related examples and to
several counterexamples. For example, in discussing the concept of continuity of a function $f$ at a point $p$, several examples of functions continuous at $p$ and several examples of functions not continuous at $p$ were exhibited. Moreover, both true and false assertions related to continuity were proposed for the students to analyze, thus providing an opportunity to exhibit counterexamples to the false assertions. The students participated in exhibiting examples and counterexamples in classroom discussions. Although most of the students in the E-group had little or no experiences with formal mathematical proof, it appeared to the writer that they understood that in an infinite mathematical system, considering a finite number of cases does not constitute a proof; whereas, only one counterexample is sufficient to disprove a false conjecture. The writer pointed out that even in a finite mathematical system, a proof involving specific examples would consist of examining all possible cases. For instance, the writer pointed out that if there were 100 baskets on a laws and one wanted to prove the assertion: "each basket contains an apple," observing that 99 of the 100 baskets contain an apple is not sufficient to conclude that the assertion is true; whereas, on the other hand, observing that one basket does not contain an apple is sufficient to disprove the assertion. Moreover, the writer pointed out how laborous this kind of proof can be even when possible. For instance,
in a miniature mathematical system "S" consisting of 10 elements, to prove by cases that a given binary operation on "S" is commutative requires considering 45 cases.

The class agreed that it seemed reasonable to assume that the more difficulty one has in exhibiting a counterexample in attempting to disprove a conjecture, the greater the likelihood that the conjecture is true. Moreover, the class agreed that usually it is easier for one to disprove a conjecture that he knows be false than it is to prove a conjecture is true when he knows that it is true. For example, most of the students could readily prove that \( \{ x \mid x^2 < 1 \} = [-1, 1] \) is false for real numbers \( x \) by simply choosing a number, say 1, which is an element of \([-1, 1]\) but which is not an element of \( \{ x \mid x^2 < 1 \} \). However, very few students had any notion how to prove \( \{ x \mid x^2 < 1 \} \subseteq [-1,1] \).

After discussing the first two assignments involving examples and counterexamples, the writer was able to detect a sense of scrutiny that the students began to employ when discussing a given problem. This fact was evidenced by their hesitancy in answering questions concerning the truth or falsity of a given conjecture and the manner in which they gave answers to such questions. At the beginning of the course, many students raised their hands immediately when the writer asked for a show of hands, indicating they thought expressions such as: \( a < b \) implies that \( a^2 < b^2 \); or \( a^2 = b^2 \) implies that \( a = b \); or \( |a| < |b| \) implies that \( a < b \).
for real numbers \(a\) and \(b\) were true (the writer explained to the students that the universe of numbers considered in the course would be the set of all real numbers unless specifically stated otherwise). However, after a few class meetings, before giving answers to questions of this nature, the students began to analyze the statement under consideration and asked such questions as: Are \(a\) and \(b\) both positive? Can \(a\) and \(b\) have opposite signs? In a discussion of functions, the writer challenged the class with examples like:

1. I have three boxes. Together they contain all of the functions. In box number I, I have all the even functions, and in box number II, I have all the odd functions. I claim that box number III is empty. Who wants to challenge me? Who agrees with me? In any case, justify the position you take.

2. You have shown by use of counterexamples that sometimes \(f(g(x)) \neq g(f(x))\). However, I claim that although \(f(g(x))\) is not in general the same as \(g(f(x))\), if the functions \(f\) and \(g\) have the same domain, then \(f(g(x))\) is always defined. Challenge me! The writer was delighted to have one student, out of the many students who challenged him, to give this counterexample: Let \(f(x) = x + 1\) and \(g(x) = x^2\). Let the domain for \(f\) and \(g\) be the set \(\{1, 3, 4\}\). Then \(f(g(x))\) is not defined since \(g(x)\) must
belong to the domain of \( f \) in order that \( f(g(x)) \) be defined, but \( g(3) \) and \( g(4) \) are not in the domain of \( f \).

The E-Group vs the C-Group

The experimental class and the control class were characterized on the basis of methods of instruction. The method of instruction employed in teaching concepts to the E-group was that of exposing the students to several related examples and to several counterexamples after the introduction of each concept; whereas, in the case of the C-group, illustrative examples were used after the introduction of a concept, but no counterexamples. To illustrate more specifically the difference between the experimental treatment and the control treatment, consider the following statements which might be proposed as part of a classroom dialogue or as a written exercise for the E-group:

Determine which of the following statements are true and which are false. If you say a statement is true, give a logical argument to justify your claim; if you say that a statement is false, give the correct answer to the statement or exhibit a counterexample to justify your claim.

1. \( \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = 0 \Rightarrow \frac{f(x) - f(a)}{x - a} = 0 \).

2. \( \lim_{x \to a} f(x) = \lim_{x \to b} g(x) \Leftrightarrow a = b \) or \( f(x) = g(x) \).

3. \( f(a) = L \Rightarrow \lim_{x \to a} f(x) = L \).

4. \( f(a) \) exists \( \Rightarrow \lim_{x \to a} f(x) \) exists.
5. \( \lim_{x \to 2} \frac{x^2 - 4}{x - 2} \) does not exist.

6. \( \lim_{x \to 0} \frac{1}{x} = 0 \).

7. \( \lim_{x \to a} f(x) = \lim_{x \to b} b(x) \implies a = b \).

8. \( \lim_{x \to a} f(x) = L \implies f(x) \) is continuous at \( x = a \).

9. Does Rolle's theorem apply to \( f(x) = x - x^3 \) on \([-1, 1]\)? If so, find the \( m \) such that \( f'(m) = 0 \) and, if not, explain.

A student might show that (3) is false by use of the function

\[
f(x) = \begin{cases} 
2, & \text{if } x > 2 \\
3, & \text{if } x < 2 \\
5, & \text{if } x = 2 
\end{cases}
\]

and consider \( \lim_{x \to 2} f(x) \). Here, \( f(2) = 5 \) but the \( \lim_{x \to 2} f(x) \) does not exist, since \( \lim_{x \uparrow 2} f(x) = 3 \) and \( \lim_{x \downarrow 2} f(x) = 2 \).

In the control class, the dialogue would be like the following:

Evaluate the following limits:

1. \( \lim_{x \to 2} (x^2 - 4)/(x - 2) \).

2. \( \lim_{x \to 5} (2x - 5) \).

3. \( \lim_{x \to 3} (x - 4) \).

4. \( \lim_{x \to 3} \lfloor x \rfloor \).

5. Given \( f(x) = x - x^3 \), find \( m \) that satisfies Rolle's theorem

6. \( \lim_{x \to 0} (\sin x)/x \).

7. \( \lim_{x \to a} (x - a) \).

8. \( \lim_{x \to 0} (1/x) \).
In the E-group, several students frequently asked such questions as: Suppose \( \lim_{x \to a} f(x) \) does not exist. Can one assume that \( f'(a) \) does not exist? Most students in the E-group were able to give an analytical answer to questions of this nature. Only one question of this nature was raised by the C-group during the entire quarter. It is the writer's opinion that these kinds of questions imply a healthy kind of mathematical insight and indicate mathematical growth on the part of the student. In answering the foregoing question, the writer replied, "Suppose I claim that \( \lim_{x \to a} f(x) \) does not exist and yet \( f \) is differentiable at \( x = a \)." One student in the E-group reasoned as follows: "If \( f'(x) \) exists, then \( f \) is continuous at \( x = a \). But \( f \) continuous at \( x = a \) implies that: (1) \( f \) is defined at \( x = a \), (2) the \( \lim_{x \to a} f(x) \) exists, and (3) \( \lim_{x \to a} f(x) = f(a) \). Therefore, by (2), \( \lim_{x \to a} f(x) \) exists, and the supposition is false."

Before the third "midterm" examination was given, Professor Fisher and the writer were interested in ascertaining what immediate effect, if any, the use of counterexamples had on the E-group in regard to learning and applying Rolle's theorem and the Mean Value theorem. Consequently, the third "midterm" examination included the items below which were designed specifically to determine whether or not the performance of the E-group and the performance of
the C-group would differ on these particular items and, if so, to what extent.

Let \( f \) be the absolute value function. Which of the following statements concerning the hypothesis and conclusion of Rolle's Theorem is true for the interval \([-1,1]\)?

a. Hypotheses true but conclusion false.
b. Hypotheses false but conclusion true.
c. Hypotheses false and conclusion false.
d. Hypotheses true and conclusion true.
e. None of the preceding is true.

Let \( f(x) = \| x \| x \). Which of the following statements concerning the Mean Value Theorem (Theorem of the Mean) is true for the interval \([0,3/2]\)?

a. Hypotheses true but conclusion false.
b. Hypotheses false but conclusion true.
c. Hypotheses false and conclusion false.
d. Hypotheses true and conclusion true.
e. None of the preceding is true.

Suppose \( F(x) = x^2 \) if \( x \leq 1 \) and \( F(x) = 2x-1 \) if \( 1 < x \).

Which one of the following statements is false?

a. \( F \) is continuous in \([0,2]\)
b. \( F \) is differentiable in \([0,2]\)
c. \( F' \) is continuous in \([0,2]\)
d. \( F' \) is differentiable in \([0,2]\)
e. \([0,2] \subseteq F([0,2])\)
20. Suppose \( F(x) = x^2 \) if \( x \leq 1 \) and \( F(x) = 2x - 1 \) if \( 1 < x \). Which one of the following statements is true?

a. \( F''(1) = 2 \)

b. \( \lim_{x \to 1} F''(x) = 2 \)

c. \( F''(1) = 0 \)

d. \( \lim_{x \to 1} F''(x) = 0 \)

e. None of the preceding statements is true.

As the writer predicted, scores on these items appeared to favor the students in the E-group. The means for the two groups on these items were: \( \bar{X}_e = 54.60 \), \( \bar{X}_c = 52.07 \). However, the computed \( t \) (\( t_{49} = 0.88 \)) indicated that this difference between the two means was not significant at the 5 per cent level.

The writer assigned the problems below as a written assignment to be analyzed and discussed in the E-group.

1. A function for which the Mean Value theorem does not apply.

\[ f(x) = \frac{(x - 1)}{(x - 2)}, \quad x \in [1, 2]. \]

2. A function for which Rolle's theorem does not apply.

\[ f(x) = 2x^2 - 4x, \quad x \in [0, 2]. \]

3. Let \( I = \{ x : \quad x \in \mathbb{Q} \text{ and } x \in [0, 2] \} \). Let

\[ f(x) = 4x - x^3, \quad x \in I. \]

Then, \( f \) is continuous on \( I \) and \( f'(x) \) exists on \( I \). Moreover, \( f(a) = f(b) = 0 \). But \( f'(c) = 0 \) implies that \( c = \pm \frac{\sqrt{3}}{3} \notin \mathbb{Q} \). Thus, there exists no
c ∈ Q such that f'(c) = 0. Therefore, Rolle's theorem does not apply to this function.

4. Let I = \{ x: x ∈ Q and x ∈ [0, 5] \} and let
\[ f(x) = \begin{cases} 0, & 0 \leq x \leq 3 \\ 1, & 3 < x \leq 5 \end{cases} \]
Discuss the assertion: "f'(x) ≠ 0 on [a, b] implies that f is constant on [a, b]."

5. Let I = \{ x: x ∈ Q and x ∈ [0, 2] \}. Let
\[ f(x) = 4x - x^3, \ x ∈ I. \]
Discuss the maximum and minimum values of f on the interval I.

6. Let f(x) = x^2 and I = \{ x: x ∈ Q and x ∈ [1, 2] \}.
Observe that f(1) = 1, f(2) = 4, and f(1) < 2 < f(2).
Discuss Wierstrass' Intermediate Theorem for f.

7. Let f(x) = 4x - x^3, I = \{ x: x ∈ Q and x ∈ [0, 2] \}.
Discuss the continuity and differentiability of f on I. Does Rolle's theorem apply to f on I? Explain.
The students in the C-group were assigned an equal number of related problems involving the application of these theorems.

In both groups, an attempt was made through classroom discussions and homework exercises to arouse the students' curiosity and, consequently, to evoke "good" questions from them in regard to the subject matter. For instance, as a prelude to the discussion of the limit concept, the writer challenged the students in both groups
with Zeno's paradox of the mythical race between Achilles and the tortoise, stated somewhat in this manner. "Suppose Achilles can run 10 times as fast as the tortoise. If the tortoise begins the race with a 100-yard head start, then Achilles can never overtake the tortoise." Zeno argued as follows: While Achilles is covering the 100 yards between his and the tortoise's starting points, the tortoise moves ahead 10 yards; while Achilles races over this 10 yards, the tortoise plods on one yard, and is still one yard ahead of him; when Achilles has covered this one yard, the tortoise is still one-tenth of a yard ahead. So, by dividing the distance run by Achilles into intervals of 100, 10, 1, 1/10, 1/100, and so on, yards, Zeno argued that Achilles would never catch the tortoise. In the discussion of this paradox, the students in both groups determined that the fact that an infinite set of distances could add up to a finite total distance was the "stumbling block" involved in this paradox.

The class period for each group on every other Tuesday was used to review for the "midterm" examination which was given the next day. Moreover, questions concerning items on the "midterm" examinations were discussed in each group's class period of the next day following the "midterm" examination. Homework was assigned daily to both groups and students in both groups took turns in discussing particular problems of interest at the board.
Summary

In summary, the experimental variable in this study was the method of instruction used with the E-group. The experimental treatment and the control treatment differed specifically in this manner: Whenever a concept was introduced in the experimental class, the students were exposed to several related examples and to several counterexamples; whereas, when a concept was introduced in the control class, the students were exposed to several related examples but to no counterexamples. The writer emphasized and elicited student participation in both groups. Moreover, an attempt was made to provide experiences for the students in both groups which would stimulate them to ask "good" questions. As beginners, the writer pointed out that the students should (1) give free rein to their intuition, (2) make use of analogies to obtain results for the truth of which they had strong physical evidence, (3) then realize that plausibility is not proof, and to subject themselves to the discipline of a logical chain of reasoning so that their conclusions depend, not on intuitions, or feelings about physical situations, but on underlying assumptions, definitions, and proved properties of real numbers. Homework assignments were made daily and constituted the basis for classroom dialogues.
CHAPTER IV

STATISTICAL ANALYSIS

During this study the writer administered the following tests and examinations to the subjects of the study: A pre-attitude test, a post-attitude test, four one-hour "midterm" examinations, a 150-minute general examination, a 90-minute generalization examination, and a two-hour final examination.

Professor Robert C. Fisher, Department of Mathematics, The Ohio State University, constructed the four one-hour "midterm" examinations and the two-hour final examination. The 150-minute general examination and the 90-minute generalization examination were constructed by the writer. The examinations constructed by Professor Fisher were administered to all students enrolled in Mathematics 151 at The Ohio State University during the time of this study. The examinations constructed by the writer were administered only to the subjects of the study.

The writer computed a reliability coefficient to measure the internal consistency of each examination used for the study; these were Spearman-Product Moment coefficients, based upon an odd-even split and the usual correction for length of the test.
Since the loss of students was approximately the same for both groups, the writer assumed that mortality did not affect the internal validity of the study. However, to substantiate this assumption, after the final loss of subjects from both groups, the writer computed a one-way analysis of variance of the scores that the remaining subjects of the study made on the pre-test that Mr. William Paul administered to all students enrolled in Mathematics 151 at The Ohio State University during the time of the study. Mr. Paul was conducting an experimental study which involved all students enrolled in Mathematics 151 at The Ohio State University during the time of this study.

The essence of this chapter is a presentation of the statistical results of all tests and examinations used in the study. The statistical hypothesis tested for each examination given to the subjects of the study was

\[ H_0 : \mu_e = \mu_c \] (there is no significant difference between the means of the two groups). The statement "\( F \) is not significant at the 5 per cent level" will indicate that the null hypothesis \( H_0 : \mu_e = \mu_c \) was not rejected at the 5 per cent level and, consequently, the two groups came from the same population. In regard to the classroom examinations given, accepting \( H_0 : \mu_e = \mu_c \) will indicate that the two methods of teaching are equal, that is, they have equal effect on learning as measured by the classroom examinations and with respect to the statistical test computed.
The writer computed an F-test of equivalence of the variances of the two groups for each examination to determine whether or not a one-way analysis of variance was applicable to test the null hypothesis of no significant difference between the means of the two groups.

The statistical analysis of the tests and examinations given to the subjects of the study are summarized in the tables below.

In Table 1, the writer summarized the reliability coefficients that were computed to measure the internal consistency of each examination used for the study.

**TABLE 1**

**SPERMAN PRODUCT-MOMENTS OF RELIABILITY COEFFICIENT**

<table>
<thead>
<tr>
<th>Test</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>Generalization Exam</th>
<th>Final Exam</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>.58</td>
<td>.57</td>
<td>.67</td>
<td>.56</td>
<td>.65</td>
<td>.67</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>.73</td>
</tr>
</tbody>
</table>

The writer summarized the statistical results of the ACT scores in determining the comparability of the two groups in Tables 2 and 3.
From Table 2, one observes that the mean of the scores on the ACT test for the C-group was slightly higher than the mean for the E-group on the scores of the ACT test. To determine whether or not this difference between the means was significant, a one-way analysis of variance was computed. The results of the analysis of variance are summarized in Table 3.

**TABLE 2**

MEANS, MEDIANs, AND STANDARD DEVIATIONS ON ACT SCORES

<table>
<thead>
<tr>
<th></th>
<th>E-Group</th>
<th>C-Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>Median</td>
<td>29.00</td>
<td>30.00</td>
</tr>
<tr>
<td>Mean</td>
<td>27.92</td>
<td>29.33</td>
</tr>
<tr>
<td>S.D.</td>
<td>3.82</td>
<td>4.10</td>
</tr>
</tbody>
</table>

**TABLE 3**

ANALYSIS OF VARIANCE ON ACT SCORES

\( H_0: \mu_e = \mu_c \)

Level: \( \alpha = 0.05 \)

Critical region: \( F > 4.11 \)

<table>
<thead>
<tr>
<th>Source of Variation</th>
<th>Sum of Squares</th>
<th>Degrees of Freedom</th>
<th>Mean Square</th>
<th>Computed F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatments</td>
<td>33.43</td>
<td>1</td>
<td>33.43</td>
<td>.91</td>
</tr>
<tr>
<td>Error</td>
<td>1,692.06</td>
<td>46</td>
<td>36.78</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>1,725.49</td>
<td>47</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
From the results summarized in Table 3, one observes that the computed \( F \) is not significant at the 5 per cent level, indicating that the two groups were equivalent at the beginning of the study.

The results of the statistical analysis of the scores on the pre-test constructed by Mr. William Paul are summarized in Tables 4 and 5.

**TABLE 4**

**MEANS, MEDIANs, AND STANDARD DEVIATIONS OF SCORES ON PRE-TEST**

<table>
<thead>
<tr>
<th></th>
<th>E-Group</th>
<th>C-Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>Median</td>
<td>30.00</td>
<td>30.00</td>
</tr>
<tr>
<td>Mean</td>
<td>25.00</td>
<td>30.00</td>
</tr>
<tr>
<td>S.D.</td>
<td>13.93</td>
<td>13.26</td>
</tr>
</tbody>
</table>

From Table 4, one observes that the mean for the C-group on the pre-test is somewhat larger than the mean for the E-group on the pre-test. To ascertain whether or not this difference between the two means on the pre-test was significant, the writer computed a one-way analysis of variance on the scores of the pre-test. These results are summarized in Table 5.
TABLE 5
ANALYSIS OF VARIANCE FOR PRE-TEST

\( H_0: \mu_e = \mu_c \)

Level: \( \alpha = 0.05 \)

Critical region: \( F > 4.04 \)

<table>
<thead>
<tr>
<th>Source of Variation</th>
<th>Sum of Squares</th>
<th>Degrees of Freedom</th>
<th>Mean Square</th>
<th>Computed F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Column means</td>
<td>3.19</td>
<td>1</td>
<td>3.19</td>
<td>1.76</td>
</tr>
<tr>
<td>Error</td>
<td>94.50</td>
<td>49</td>
<td>1.83</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>97.69</td>
<td>50</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

From 5, one observes that \( F \) is not significant at the 5 per cent level. Therefore, the null hypothesis of no significant difference between the two means was not rejected. Consequently, the writer assumed that the number of subjects did not affect the equivalence of the two groups.

The writer summarized the statistical results of the pre-scores on the attitude test in Tables 6 and 7.

TABLE 6
MEANS, MEDIANs, AND STANDARD DEVIATIONS OF PRE-SCORES ON ATTITUDE TEST

<table>
<thead>
<tr>
<th></th>
<th>E-Group</th>
<th>C-Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>Median</td>
<td>68.00</td>
<td>67.50</td>
</tr>
<tr>
<td>Mean</td>
<td>66.60</td>
<td>67.70</td>
</tr>
<tr>
<td>S.D.</td>
<td>4.30</td>
<td>4.50</td>
</tr>
</tbody>
</table>
One observes from Table 6 that the means for the two groups of the pre-scores on the attitude test differ very slightly. However, a one-way analysis of variance was computed to determine whether or not this difference was significant. The results of the analysis of variance are summarized in Table 7.

**TABLE 7**

**ANALYSIS OF VARIANCE OF PRE-SCORES ON ATTITUDE TEST**

\[ H_0: \mu_e = \mu_c \]

Level: \( \alpha = 0.05 \)

Critical region: \( F > 4.00 \)

<table>
<thead>
<tr>
<th>Source of Variation</th>
<th>Sum of Squares</th>
<th>Degrees of Freedom</th>
<th>Mean Square</th>
<th>Computed ( F )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Column means</td>
<td>8.95</td>
<td>1</td>
<td>8.95</td>
<td>0.41</td>
</tr>
<tr>
<td>Error</td>
<td>1,309.02</td>
<td>60</td>
<td>21.81</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>1,317.97</td>
<td>61</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

From Table 7, one observes that the computed \( F \) is not significant at the 5 per cent level. Therefore, the writer assumed that the attitudes of the two groups were statistically equivalent at the beginning of the study.

The results of the statistical analysis of the post-scores on the attitude test are summarized in Tables 8 and 9.
TABLE 8
MEANS, MEDIANS, AND STANDARD DEVIATIONS
OF POST-SCORES ON ATTITUDE TEST

<table>
<thead>
<tr>
<th></th>
<th>E-Group</th>
<th>C-Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>Median</td>
<td>68.00</td>
<td>68.00</td>
</tr>
<tr>
<td>Mean</td>
<td>68.35</td>
<td>68.04</td>
</tr>
<tr>
<td>S.D.</td>
<td>4.56</td>
<td>6.39</td>
</tr>
</tbody>
</table>

One observes from the results of Table 8 that the means for the post-scores of the two groups differed by .31. This very small difference between the means was assumed not to be significant by the writer. However, to substantiate this assumption, the writer computed a one way analysis of variance to test for significance of the means. These results are summarized in Table 9.

TABLE 9
ANALYSIS OF VARIANCE OF POST-SCORES ON ATTITUDE TEST

\[ H_0: \mu_e = \mu_c \]

\[ \text{Level: } \alpha = 0.05 \]

\[ \text{Critical region: } F > 4.04 \]

<table>
<thead>
<tr>
<th>Source of Variation</th>
<th>Sum of Squares</th>
<th>Degrees of Freedom</th>
<th>Mean Square</th>
<th>Computed F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Column means</td>
<td>1.10</td>
<td>1</td>
<td>1.10</td>
<td>0.07</td>
</tr>
<tr>
<td>Error</td>
<td>696.86</td>
<td>47</td>
<td>14.83</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>696.86</td>
<td>48</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
From Table 9, one observes that the computed F is not significant at the 5 per cent level. Therefore, the writer concluded that the treatments did not produce any measurable difference in the attitude of the two groups.

In Tables 10 through 22, the writer summarized the statistical results of the "midterm" examinations, of the general examination, of the generalization examination, and of the final examination. For each examination given, the writer summarized the medians, means, and standard deviations for both groups. Then, for all examinations except the generalization examination, the statistical results of the analysis of variance computed to test for significance of the means are summarized. The variance for the two groups were found to be significantly different at the 5 per cent level on the generalization examination. Therefore, a t-test was computed to test for significance of the difference of the means on this test.

**TABLE 10**

**MEANS, MEDIANS, AND STANDARD DEVIATIONS OF SCORES ON TEST 1**

<table>
<thead>
<tr>
<th></th>
<th>E-Group</th>
<th>C-Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>Median</td>
<td>65.00</td>
<td>65.00</td>
</tr>
<tr>
<td>Mean</td>
<td>60.81</td>
<td>63.39</td>
</tr>
<tr>
<td>S.D.</td>
<td>17.86</td>
<td>16.97</td>
</tr>
</tbody>
</table>
From Table 10, one observes that the mean for the CO group on the scores of test 1 was slightly larger than the mean for the E-group on this test. A one-way analysis of variance was computed to test for significance of the two means on test 1. The results of the analysis of variance are summarized in Table 11.

**TABLE 11**

**ANALYSIS OF VARIANCE OF SCORES ON TABLE 1**

\[ H_0: \mu_e = \mu_c \]

Level: \( \alpha = 0.05 \)

Critical region: \( F > 4.04 \)

<table>
<thead>
<tr>
<th>Source of Variation</th>
<th>Sum of Squares</th>
<th>Degrees of Freedom</th>
<th>Mean Square</th>
<th>Computed F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Column means</td>
<td>98</td>
<td>1</td>
<td>98</td>
<td>0.32</td>
</tr>
<tr>
<td>Error</td>
<td>17,385</td>
<td>57</td>
<td>305</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>17,483</td>
<td>58</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

From Table 11, the computed \( F \) of .32 indicates that the null hypothesis of no significant difference between the means of the two groups on test 1 was not rejected at the 5 per cent level. Therefore, the writer assumed that the methods of instruction had not produced any measurable difference between the two groups at the time of the first "midterm" examination.
TABLE 12
MEANS, MEDIANS, AND STANDARD DEVIATIONS
OF SCORES ON TEST 2

<table>
<thead>
<tr>
<th>E-Group</th>
<th>C-Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>Medians</td>
<td>40.00</td>
</tr>
<tr>
<td>Means</td>
<td>53.97</td>
</tr>
<tr>
<td>S.D.</td>
<td>20.46</td>
</tr>
</tbody>
</table>

From Table 12, one observes that there was a considerable difference between the means of the two groups on Test 2. The writer computed a one-way analysis of variance to determine whether or not this difference between the means was significant. The results of the analysis of variance are summarized in Table 13.

TABLE 13
ANALYSIS OF VARIANCE OF SCORES ON TEST 2

H_0: μ_e = μ_c

Level: α = 0.05

Critical region: F > 4.04

<table>
<thead>
<tr>
<th>Source of Variation</th>
<th>Sum of Squares</th>
<th>Degrees of Freedom</th>
<th>Mean Square</th>
<th>Computed F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Column means</td>
<td>747</td>
<td>1</td>
<td>747.00</td>
<td>2.24</td>
</tr>
<tr>
<td>Error</td>
<td>17,697</td>
<td>53</td>
<td>333.91</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>18,444</td>
<td>54</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

One observes from Table 13 that the computed F is not significant at the 5 per cent level. Therefore, the null
hypothesis of no significant difference between the means of the two groups on test 2 is not rejected. Consequently, the writer assumed that the two groups came from the same population.

TABLE 14

MEANS, MEDIANS, AND STANDARD DEVIATIONS OF SCORES ON TEST 3

<table>
<thead>
<tr>
<th></th>
<th>E-Group</th>
<th>C-Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>Median</td>
<td>56.00</td>
<td>55.00</td>
</tr>
<tr>
<td>Mean</td>
<td>57.90</td>
<td>57.60</td>
</tr>
<tr>
<td>S.D.</td>
<td>16.27</td>
<td>15.73</td>
</tr>
</tbody>
</table>

From Table 14, one observes that the difference between the mean of the E-group and the mean of the C-group on test 3 is 0.30. Although there is only a very small difference between the means, the writer computed a one-way analysis of variance to determine whether or not the difference between the means was significant. The results of the analysis of variance are summarized in Table 15.
TABLE 15

ANALYSIS OF VARIANCE OF SCORES ON TEST 3

\( H_0: \mu_e = \mu_c \)

Level: \( \alpha = 0.05 \)

Critical region: \( F > 4.03 \)

<table>
<thead>
<tr>
<th>Source of Variation</th>
<th>Sum of Squares</th>
<th>Degrees of Freedom</th>
<th>Mean Square</th>
<th>Computed F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Column means</td>
<td>739</td>
<td>1</td>
<td>739.00</td>
<td>2.72</td>
</tr>
<tr>
<td>Error</td>
<td>13,093</td>
<td>49</td>
<td>267.20</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>13,832</td>
<td>50</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

From Table 15, the computed \( F \) of 2.72 indicates that the null hypothesis of no significant difference between the means was not rejected at the 5 per cent level. Therefore, the writer concluded that the two methods of instruction had not produced measurable difference between the two groups at the time of Test 3.

TABLE 16

MEANS, MEDIANs, AND STANDARD DEVIATIONS OF SCORES ON TEST 4

<table>
<thead>
<tr>
<th></th>
<th>E-Group</th>
<th>C-Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>Median</td>
<td>50.00</td>
<td>60.00</td>
</tr>
<tr>
<td>Mean</td>
<td>47.65</td>
<td>55.80</td>
</tr>
<tr>
<td>S.D.</td>
<td>13.46</td>
<td>12.46</td>
</tr>
</tbody>
</table>

From the results summarized in Table 16, one observes that the mean for the C-group on test 4 is considerably
larger than the mean for the E-group on test 4. A one-way analysis of variance was computed to determine whether or not the difference between the means was significant. The result of the analysis of variance are summarized in Table 17.

### TABLE 17

**ANALYSIS OF VARIANCE OF SCORES ON TABLE 4**

<table>
<thead>
<tr>
<th>Source of Variation</th>
<th>Sum of Squares</th>
<th>Degrees of Freedom</th>
<th>Mean Square</th>
<th>Computed F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Column means</td>
<td>25.56</td>
<td>1</td>
<td>25.56</td>
<td>3.64</td>
</tr>
<tr>
<td>Error</td>
<td>343.93</td>
<td>49</td>
<td>7.02</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>369.49</td>
<td>50</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

From Table 17, one observes that the computed F is not significant at the 5 per cent level. Thus, the null hypothesis of no significant difference between the means on test 4 was not rejected at the 5 per cent level.

### TABLE 18

**MEANS, MEDIANs, AND STANDARD DEVIATIONS OF SCORES ON GENERAL EXAMINATION**

<table>
<thead>
<tr>
<th></th>
<th>E-Group</th>
<th>C-Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>Median</td>
<td>32.76</td>
<td>34.48</td>
</tr>
<tr>
<td>Mean</td>
<td>31.43</td>
<td>35.98</td>
</tr>
<tr>
<td>S.D.</td>
<td>17.22</td>
<td>16.86</td>
</tr>
</tbody>
</table>
From the results summarized in Table 18, one observes that the mean for the C-group on the General Examination was somewhat larger than the mean for the E-group on this examination. The writer computed a one-way analysis of variance to determine whether or not the difference between the means on this examination was significant. The results of the analysis of variance are summarized in Table 19.

**TABLE 19**

**ANALYSIS OF VARIANCE OF SCORES ON GENERAL EXAMINATION**

$H_0: \mu_e = \mu_c$

Level: $\alpha = 0.05$

Critical region: $F > 4.04$

<table>
<thead>
<tr>
<th>Source of Variation</th>
<th>Sum of Squares</th>
<th>Degrees of Freedom</th>
<th>Mean Square</th>
<th>Computed F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Column means</td>
<td>21.29</td>
<td>1</td>
<td>21.29</td>
<td>.77</td>
</tr>
<tr>
<td>Error</td>
<td>1,298.26</td>
<td>47</td>
<td>27.69</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>1,319.55</td>
<td>48</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

One observes from Table 19 that the computed $F$ is not significant at the 5 per cent level. Therefore, the null hypothesis of no significant difference between the means of the two groups on the general examination was not rejected at the 5 per cent level. Consequently, the writer concluded that the two groups came from the same population.
TABLE 20
MEANS, MEDIANS, AND STANDARD DEVIATIONS OF SCORES ON GENERALIZATION EXAMINATION

<table>
<thead>
<tr>
<th></th>
<th>E- Group</th>
<th>C-Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>Median</td>
<td>62.50</td>
<td>60.00</td>
</tr>
<tr>
<td>Mean</td>
<td>62.50</td>
<td>60.65</td>
</tr>
<tr>
<td>S.D.</td>
<td>15.08</td>
<td>10.03</td>
</tr>
</tbody>
</table>

The F-test for equivalence of the two variances for the generalization examination was found to be 2.26. The critical region, using 25 degrees of freedom for the variance of the E-group and 22 degrees of freedom for the C-group, was found to be 2.02. Therefore, the null hypothesis of no significant difference between the two variances was rejected at the 5 per cent level. Consequently, a t-test for small samples was used to test the difference between the two means.

\[
t_{.05} = \left( \frac{s_{\bar{x}_e}^2 (t_e) + s_{\bar{x}_c}^2 (t_c)}{s_{\bar{x}_e}^2 + s_{\bar{x}_c}^2} \right) / \left[ s_{\bar{x}_e}^2 + s_{\bar{x}_c}^2 \right]
\]

\(t_e\) is the 5 per cent value for \(t\) at 25 degrees of freedom and \(t_c\) is the 5 per cent value for \(t\) at 22 degrees of freedom.

\[s_{\bar{x}_e} = 227.41/25 = 9.10\]
\[s_{\bar{x}_c} = 100.60/22 = 4.57\]
\[t_e = 2.060, t_c = 2.074, \text{ and } t_{.05} = 2.065.\]

The computed \(t\) for this data is: \(t = .50\). Therefore, the null hypothesis is not rejected at the 5 per cent level.
TABLE 21
MEANS, MEDIANS, AND STANDARD DEVIATIONS
OF SCORES ON FINAL EXAMINATION

<table>
<thead>
<tr>
<th></th>
<th>E-Group</th>
<th>C-Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>Median</td>
<td>60.00</td>
<td>67.50</td>
</tr>
<tr>
<td>Mean</td>
<td>55.90</td>
<td>64.34</td>
</tr>
<tr>
<td>S.D.</td>
<td>16.30</td>
<td>15.90</td>
</tr>
</tbody>
</table>

One observes from Table 21 that the means of the two groups differed considerably on the scores of the final examination. The writer computed a one-way analysis of variance to determine whether or not this difference was significant. The results of the analysis of variance are summarized in Table 22.

TABLE 22
ANALYSIS OF VARIANCE OF FINAL EXAMINATION

\[ H_0: \mu_e = \mu_c \]

\[ \alpha = 0.05 \]

Critical region: \[ F > 4.06 \]

<table>
<thead>
<tr>
<th>Source of Variation</th>
<th>Sum of Squares</th>
<th>Degrees of Freedom</th>
<th>Mean Square</th>
<th>Computed F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Column means</td>
<td>56.78</td>
<td>1</td>
<td>56.78</td>
<td>1.26</td>
</tr>
<tr>
<td>Error</td>
<td>2,072.20</td>
<td>46</td>
<td>45.05</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>2,128.98</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

From Table 22, one observes that the computed \( F \) is not significant at the 5 per cent level. Therefore, the
null hypothesis of no significant difference between the means of the two groups on the final examination is not rejected at the 5 per cent level. Consequently, the writer concluded that the two methods of instruction did not produce any measurable difference during the course of the study.

Summary of Statistical Evaluation

The writer computed a one-way analysis of variance for each examination administered to the subjects of the study to determine whether or not there was a significant difference between the means of the two groups on any given examination. However, in no instance was the null hypothesis of no significant difference between the means of the two groups rejected.

Although the null hypothesis of no significant difference between the means of the two groups on the scores of the generalization examination was not rejected at the 5 per cent level, the variance of the two groups for this examination were found to be significantly different at the 5 per cent level.

Although the mean for the experimental class was slightly larger on the post-attitude test than the mean for the control group on the post-attitude test, the statistical results summarized in Table 6 and Table 7 provide no evidence to indicate that this difference was due to the experimental treatment.
Conclusions

The objectives of this study were: (1) to compare the learning of concepts in a first course in calculus when an extensive treatment of counterexamples is used to that of learning concepts in a first course in calculus when no counterexamples are used, and (2) to determine whether or not the example-counterexample approach to teaching concepts in a first course in calculus was motivational merit.

Remarks

1. On the basis of the statistical analysis of the data obtained from the "midterm" examinations, from the general examination, from the generalization examination, and from the final examination, neither method was found to be statistically better than the other.

2. There was no statistical evidence to support or deny the assumption that counterexamples make for motivation in teaching concepts in a first course in calculus.

Limitations

It should be noted that the results and conclusions concerning the problem are relative to the following extraneous variables involved in the study.

1. The possibility that the "midterm" examinations, the general examination, and the final examination did not accurately measure possible changes in mathematical achievement of the subjects.
2. The possibility that the results and conclusions might have been different if the E-group had spent more time in actual problem-solving activity and less time in exhibiting and discussing counterexamples.

3. The possibility that the results and conclusions might have been different had the E-group met at 3 P.M. instead of 9 A.M. and the C-group at 9 A.M. instead of 3 P.M.; and the possibility that the writer, himself, had a more favorable influence on one group than on the other group.

Although there was no statistical evidence to indicate that the use of counterexamples enhances the learning of concepts in a first course in calculus, the writer feels that the use of counterexamples did produce certain results that should be noted. These results, however, are not statistical results and are discussed in Chapter V.
CHAPTER V

INFORMAL EVALUATION

Chapter IV was concerned only with that aspect of evaluation that resulted from the statistical analysis of the several sets of data obtained from the tests administered to the subjects of the study. However, this chapter is strictly informal, and the writer does not claim nor intend to imply, in any way, that the contents of this chapter have any statistical basis whatsoever. This chapter has its foundation in the writer's observations of the reactions of the students, in their oral and written assignments, and their comments.

Students' Performance

In classroom discussions and on homework exercises, the writer noted that the students in the experimental class did better than the students in the control class with problems of the following nature:

1. Given that \( f(x) = \begin{cases} 
1, & x < -1 \\
-1, & -1 \leq x \leq 1 \\
x - 1, & x > 1
\end{cases} \)

Find the value(s) of \( x \) at which: (a) \( f \) is discontinuous, (b) \( f'(x) \) does not exist, and (c) \( f'(x) = 1 \).

3. Given that \( f(x) = |x|x^3 \). Determine which is false:
   (a) \( f \) is differentiable on \((-1, 1)\), (b) \( f' \) is differentiable at \( x = 0 \), (c) \( f'' \) is differentiable on \((-1, 1)\), (d) \( f'' \) is continuous on \((-1, 1)\), and (e) none.

4. Find \( y' \), given that \( y'' = 6x \) and that the function has a critical point at \((2, -16)\).

When most students in both groups gave correct answers to questions or correct solutions to problems of this nature, in most instances the students in the experimental class gave the correct answers or arrived at the correct solutions more quickly than the students in the control class.

The students in the control class did consistently better with problems that could be solved by memorizing a technique, for example, finding derivatives of functions and polynomials. The writer feels that this was possibly due to the fact that the students in the C-group spent more time actually engaged in problem-solving activity than the students in the E-group, who spent a considerable amount of time exhibiting and discussing counterexamples.

The students in the E-group were more analytical and critical in classroom dialogues than the students in the C-group. The students in the E-group frequently made such
comments as: "It might appear at first that . . . but after thinking about it, you can see that this is not the case, but rather. . . ." "Unless you think about it, you might make the mistake of assuming that. . . ." This kind of analysis was never expressed by the students in the C-group.

Most students in the experimental class stated that their analyzing examples for which a given rule was correctly or incorrectly applied was helpful to them in developing accuracy in applying the rule. For instance, they felt that problems of this nature were helpful to them in regard to computing derivatives: Let $f(x) = \sin 3x$, $g(x) = (x^2 - 2)^3$, $h(x) = (2 - x)^3$, $p(x) = (x^2 - 2x - 1)^4$, and $q(x) = \cos^3 x$. Determine which of the following are correct and which are false. If you say that an expression is false, explain why you think it is false:

- $f'(x) = \cos 3x$, $g'(x) = 3(x^2 - 2)^2 (2x)$, $h(x) = 3(2 - x)^2$, $p'(x) = 4(x^2 - 2x - 1)(2x - 2)$, and $q'(x) = 3\cos^2 x$. The writer feels that exercises of this nature made the students in the E-group more aware of the typical kinds of errors frequently made by students on tests in a first course in calculus and, consequently, more cautious of them.

It was of interest to the writer to note that the students in the experimental group seemed to have recalled more precisely the basic definitions and theorems somewhat longer than the students in the control group, for example, precise definitions of $\lim_{x \to a} f(x) = L$, $\lim_{x \to a} f(x) = \infty$, }
\[
\lim_{x \to \infty} f(x) = L, \text{ and the statements of Rolle's theorem and the Mean Value Theorem. This fact was evidenced during each review before a test, and in general classroom discussions which called for definitions and theorems previously discussed.}
\]

**Students' Interest**

Although there was no statistical evidence to indicate that the use of counterexamples promoted the students' interest, the writer observed a definite difference in the interest of the two groups in regard to their exercises. The students in the experimental group displayed much more interest and enthusiasm concerning their exercises than the students in the control group in regard to their exercises. Frequently, the students in the E-group requested and elected to turn in exercises that were assigned to be discussed in class. Occasionally, the students in the E-group requested that the writer make additional written assignments involving counterexamples. This kind of gesture was not made by the students in the control group.

**Some Comments of Students in the E-Group**

In informal discussions with the students in the experimental class, they commented favorably about the example-counterexample approach. They made comments of this nature: "You see how things might not work and this helps you to understand what's going on." "The method is most
helpful; it helps me to see what's really happening and I understand the concept being discussed much better." "That counterexample tactic is a very helpful aid to understanding." "The method starts you to thinking and makes you not assume too much." "The example-counterexample method of attack gets you more involved than just examples only, and in the end you learn more." "Sometimes I'm confused about a concept and when you hit us with one of those counterexamples, the notion comes through." "Working with counterexamples keeps me on my guard and makes me think." "The exercises involving counterexamples are somewhat like solving a puzzle and are more interesting than the usual type problems (problems not involving counterexamples)." "Exercises dealing with examples and counterexamples get the students more involved in the game than the problems not involving counterexamples. The students are in a way forced to be participants rather than spectators. Exercises not involving counterexamples give the students much more of a chance to play the game from the sideline."

Summary

Since the performance of the students in the E-group and the performance of the students in the C-group differed fairly consistently on certain specific kinds of problems like those mentioned above, the writer is inclined to feel that the extent of the use of counterexamples should depend on the objective of the course. If the objective of the
course is more one of teaching for technique mastery or immediate learning in a reasonably short period of time, the writer feels that an extensive treatment of counterexamples should not be used. For example, during the 1964-1965 academic year, the Department of Industrial Education at Florida A. and M. University, Tallahassee, Florida offered an interdepartmental one-quarter course in calculus for its majors. The basic objective of the course was to teach the students for a mastery of fundamental techniques of differentiation and integration. In a course of this nature, the writer would not recommend an extensive treatment of counterexamples. However, on the other hand, in a regular one-year calculus sequence with the objective being for greater depth and understanding of concepts, the writer feels that an extensive treatment of counterexamples can be effective and should be used.

To determine whether or not the writer's hypothesis is valid concerning the situation most conducive to maximum effectiveness of the use of counterexamples requires further investigation in this area.

The writer is committed to experimentation as the decision court for settling disputes regarding educational practices, and as the only means of verifying educational improvements. Consequently, the writer realizes that his information evaluation of this study is not conclusive. It does, however, give rise to some hypotheses for educational investigation.
1. The inequality $|2x - 3| \geq 7$ is equivalent to:
   (a) $-2 \leq x \leq 5$  (b) $5 \leq x \leq -2$  (c) $x \geq 5$ or $x \leq -2$
   (d) $3/2 \leq x \leq 5$  (e) None.

2. The set $\{x \mid [x] \geq 0\}$ can be described using interval notation as:
   (a) $(-\infty, -1) \cup (1, \infty)$  (b) $(-1, 1)$  (c) $[0, 1)$  (d) $(-1, 0]$  (e) None.

3. The set $\{t \mid -2 > |3t - 4|\}$ can be described using interval notation as:
   (a) $(2/3, 2)$  (b) $(-\infty, -2/3) \cup (2, \infty)$  (c) $(-\infty, -2/3)$
   (d) $(-2, 3)$  (e) None.

4. The sketch of $2 + y > 3$ is represented by:

   (a)  
   (b)  
   (c)  
   (d)  
   (e) None.
5. The graph represents

\[ y = \sin x \]

(a) \( \{(x, y) \mid x = \sin y\} \)  
(b) \( \{(t, \sin t) \mid t \in [0, \pi]\} \)  
(c) \( \{(t, \sin t) \mid t \in [0, \pi/2]\} \)  
(d) \( \{(t, \sin 2t) \mid t \in [0, \pi/2]\} \)  
(e) None.

6. The number \( \cos \frac{2\pi}{3} \) is equal to the number

(a) \( \cos \frac{\pi}{3} \)  
(b) \( \sin \frac{\pi}{3} \)  
(c) \( \tan^2 \left( \frac{\pi}{4} \right) \)  
(d) \( -\sin^2 \left( \frac{\pi}{4} \right) \)  
(e) None.

7. The sketch represents the graph of:

(a) \( xy = 1 \)  
(b) \( (|x| - 1)(|y| - 1) = 0 \)  
(c) \( (|x| - 1)(|y| - 1) = 0 \)  
(d) \( (x^2 - 1)(y^2 - 1) = 1 \)  
(e) None.

8. The point \( 1/4 \) of the way along the line segment from \((10,1)\) to \((2, -3)\) is

(a) \((8, 0)\)  
(b) \((4, -2)\)  
(c) \((6, -1)\)  
(d) \((-6, 1)\)  
(e) None.

9. The distance between the points \((u, u + v)\) and \((v, u - v)\) is

(a) \( u + v \)  
(b) \( \sqrt{3u^2 + 3v^2} \)  
(c) \( \sqrt{u^2 - 2uv + 5v^2} \)  
(d) \(|u|\)  
(e) None.

10. The set \( \{x \mid |2 - \frac{1}{x}| < 3\} \) can be described using interval notation as:

(a) \((-2, 3)\)  
(b) \((-\infty, -1) \cup (1/5, \infty)\)  
(c) \((-1, 1/5)\)  
(d) \((-\infty, -2) \cup (3, \infty)\)  
(c) None.

11. Which of the following is true:

(a) \( \| \sin 3 \| > 1 \)  
(b) \( \cos 0 < \sin 3 \)  
(c) \( \sin 3 > \cos 4 \)  
(d) \( \tan 3 > \sin 1 \)  
(e) None.
12. Which of the following describes a function
(a) \{(x,y) \mid y^2 + x^2 = 4\} \hspace{1cm} (b) \{(x,y) \mid y-4 = 2 + 3x^2\}
(c) \{(x,y) \mid \cos y = \sin x\} \hspace{1cm} (d) \{(x,y) \mid |x| = |y|\} \hspace{1cm} (e) None.

13. A function \(f\) is called an odd function if, for each \(x\) in its domain, 
\(f(-x) = -f(x)\). In which of the following cases is \(f\) an odd function
(a) \(f(x) = \cos x\) \hspace{1cm} (b) \(f(x) = -\sin x\) \hspace{1cm} (c) \(f(x) = x \sin x\)
(d) \(f(x) = x - \sin x^2\) \hspace{1cm} (e) None.

14. \(\sec^2 x + \csc^2 x\) is equivalent to
(a) \(\sec^2 x \csc^2 x\) \hspace{1cm} (b) \(\cos^2 x + \cot^2 x\) \hspace{1cm} (c) \(\sin x \tan x\)
(d) \(\sin^2 x + \tan^2 x\) \hspace{1cm} (e) None.

15. If \(f(x) = x^2 + 3\) and \(g(x) = 2x - 5\) then \(f(g(3))\) equals
(a) 0 \hspace{1cm} (b) 4 \hspace{1cm} (c) 12 \hspace{1cm} (d) 19 \hspace{1cm} (e) None.

16. The "understood" domain of the function defined by \(f(x) = \sqrt{3x - 9}\) is the interval
(a) \((-\infty, -3]\) \hspace{1cm} (b) \([3, \infty)\) \hspace{1cm} (c) \([\sqrt{3}, 3]\) \hspace{1cm} (d) \((3, \infty)\) \hspace{1cm} (e) None.

17. If \(f(x) = (x-1)^2 + \sqrt{x}\), then \(f((x+1)^2)\) equals
(a) \(x^4 + 4x^2 + x + 1\) \hspace{1cm} (b) \(x^2(x+2)^2 + |x+1|\) \hspace{1cm} (c) \(x^2 + |x + 1|\) \hspace{1cm} (d) \(x^2 + 2x + 3\) \hspace{1cm} (e) None.

18. The equation that represents the set of points 2 units from the point 
\((-1,3)\) is
(a) \(x^2 + 2x + y^2 - 6y + 6 = 0\) \hspace{1cm} (b) \((x-1)^2 + (y+3)^2 = 2\)
(c) \(x^2 - 2x + y^2 + 6y + 6 = 0\) \hspace{1cm} (d) \((x-1)^2 + (y+3)^2 = 4\) \hspace{1cm} (e) None.

19. For \(f(x) = x^2\), \(A = [-2,1]\), and \(B = [0,4]\), which of the following is false:
(a) \(f(A \cap B) = A \cap B\) \hspace{1cm} (b) \(f(A \cup B) = B\) \hspace{1cm} (c) \(B \subseteq f(B)\) \hspace{1cm} (d) \(f(A) \cap B\) \hspace{1cm} (e) None.

20. If \(f(x) = \frac{-2}{x}\) and \(A = (-4,0) \cup (2,3)\), then \(f(A)\) equals
(a) \((0,8) \cup (-\infty,-6)\) \hspace{1cm} (b) \([-2,-2/3) \cup (0,2)\) \hspace{1cm} (c) \((2,0) \cup (-1,-3/2)\)
(d) \((2/3,2) \cup [-2,0)\) \hspace{1cm} (e) None.
Write your name above; do not detach. Block in your name, section number, and form (A or B) on the answer sheet. Indicate the one best response to each question on the answer sheet. Your score is the number correct. "None" means "None of the above."

1. The line graph has for its equation

   \[
   (a) \ 2y + x = 1 \quad (b) \ 2y - x = 1 \quad (c) \ 2y + x = 2 \quad (d) \ 2y - x = 2 \\
   (e) \ None
   \]

2. The product of the slope and y-intercept of the line whose equation is \(4x - 2y + 5 = 0\) is

   \[
   (a) \ 2/5 \quad (b) \ 5/4 \quad (c) \ 5/2 \quad (d) \ 5 \quad (e) \ None
   \]

3. Suppose \(f\) and \(g\) are linear functions. Which of the following formulas for \(h(x)\) does not define a linear function \(h\)?

   \[
   (a) \ f(x) + g(x) \quad (b) \ 3f(x) + 2g(x) \quad (c) \ 3f(x) + 2 \quad (d) \ f(g(x)) \\
   (e) \ None
   \]

4. The equation of the line through \((a,b)\) which is perpendicular to the line \(Ax + By + C = 0\) is

   \[
   (a) \ Ay - Bx + Ba - Ab = 0 \quad (b) \ Ay - Bx + Bb - Aa = 0 \\
   (c) \ By - Ax - Ba + Ab = 0 \quad (d) \ By - Ax - Bb + Aa = 0 \quad (e) \ None
   \]

Questions 5 through 8 involve the function \(F\) defined as follows:

   \[
   F(x) = \begin{cases} 
   1 & \text{if } x \leq -2 \\
   x + 2 & \text{if } -2 < x \leq -1 \\
   |x| & \text{if } -1 < x
   \end{cases}
   \]

5. The function \(F\) is discontinuous for the following values of \(x\):

   \[
   (a) \ -2 \quad (b) \ -2 \text{ and } -1 \quad (c) \ -1 \text{ and } 0 \quad (d) \ -2, -1, \text{ and } 0 \quad (e) \ None
   \]
6. \( \lim_{x \to 1} F(x) \) is
   (a) -1 (b) 0 (c) 1 (d) does not exist (e) None.

7. The values of \( x \) for which \( F'(x) \) does not exist are
   (a) -2 (b) -2 and -1 (c) -1 and 0 (d) -2, -1, and 0 (e) None.

8. \( \{x \mid F'(x) = 1\} \) is
   (a) \( (\infty, -2) \) (b) \( (-2, -1) \) (c) \( (0, \infty) \) (d) \( (-2, -1) \cup (0, \infty) \) (e) None.

9. \( \lim_{x \to 1} \frac{1 - \sqrt{x}}{1 - x} \) is
   (a) 0 (b) \( \frac{1}{2} \) (c) 1 (d) does not exist (e) None.

10. \( \lim_{x \to 0} \frac{(3 + x)^{1/3} - 3^{-1}}{x} \) is
    (a) -1 (b) 0 (c) 1 (d) does not exist (e) None.

11. Let \( f(x) = \sqrt{2x + 1} \). Then using some algebra we find that
    \[
    \frac{f(x+h) - f(x)}{h} = 2 \left( \sqrt{2x + 2h + 1} + \frac{1}{\sqrt{2x + 1}} \right)^{-1}
    \]
    Thus \( f'(x) \) equals
    (a) \( \frac{1}{6} \) (b) \( \frac{1}{3} \) (c) 6 (d) 12 (e) None.

12. The derivative of \( \sqrt[3]{x} \) equals
    (a) \( \frac{1}{4} \) (b) 1 (c) \( \frac{\sqrt[3]{x}}{4x} \) (d) \( \sqrt[3]{x^3} \) (e) None.

13. The equation of the tangent line to \( y = \frac{3}{\sqrt{x}} \) at the point where \( x = 8 \) is
    (a) \( 12y - x = 16 \) (b) \( 12y + x = 32 \) (c) \( 3x + 4y = 32 \)
    (d) \( 3y + 4x = 16 \) (e) None.

14. The equation of the normal line (perpendicular to the tangent line) to
    \( y = x^3 \) at the point where \( x = 2 \) is
    (a) \( 12x - y = 16 \) (b) \( 12x + y = 32 \) (c) \( 12y - x = 94 \)
    (d) \( 12y + x = 98 \) (e) None.

15. A cube has volume \( V \). Let \( x \) denote the length of an edge and \( A \) the area of one face. Then the rate of change of volume with respect to edge length is
    (a) \( 3x \) (b) \( 3A \) (c) \( 3xA \) (d) \( 3A^2 \) (e) None.
16. A particle moves along an s-scale according to the equation \( s = t^2 - \pi t \).
   The velocity of the particle at \( t = 1 \) is
   (a) 0  (b) 1 - \( \pi \)  (c) 1  (d) 2-\( \pi \)  (e) None.

17. Let \( f(x) = |x| + \frac{|x|}{x} \). Which of the following statements (a,b,c, or d) is false?
   (a) \( \lim_{x \to 0} f(x) = 1 \)  (b) \( \lim_{x \to 0} f(x) = -1 \)  (c) \( \lim_{x \to 1} f(x) = 2 \)
   (d) \( \lim_{x \to 0} f(x) \) does not exist  (e) None of the above are false.

18. Let \( g(x) = \frac{1}{|x|} \). The definition of the derivative can be used to find
   that \( g'(-1) \) equals (a) -1  (b) 0  (c) 1/2  (d) 1  (e) None.

19. If \( f(x) = x^3 \) and \( g(y) = \sqrt{y} \), then \( f'(2) + \frac{1}{g'(4)} \) equals
   (a) 6  (b) 14  (c) 16  (d) 52  (e) None.

20. Which of the following graphs best represents the function \( G \) such that
    \( G(2) = G'(1) = 0, \ G(0) = G'(0), \) and \( G'(2) < G'(3) \).

   (a) \hspace{1cm} (b) \hspace{1cm} (c) \hspace{1cm} (d) \hspace{1cm} (e)
1. Let $M$ be the slope of the tangent line to the graph of the curve $y = \sin \pi x$ at the point $(1,0)$. Then (a) $-4 \leq M < -3$  (b) $-3 \leq M < -2$  (c) $-2 \leq M < -1$  (d) $-1 \leq M < 0$  (e) $0 \leq M$

2. $\frac{d}{dx} \sin g(x)$ equals  
(a) $\cos g'(x)$  
(b) $\cos g(x)$  
(c) $g'(x) \cos g(x)$  
(d) $g'(x) \sin g(x)$  
(e) None

3. If $y = \frac{x^2}{1+x^3}$, then $y'$ equals  
(a) $x(2+5x^3)/(1+x^3)^2$  
(b) $x(2+5x^3)/(1+x^6)$  
(c) $x(2-x^3)/(1+x^6)$  
(d) $x(2-x^3)/(1+x^3)^2$  
(e) None

4. If $y = 3\sqrt{1-x^3}$, then $y'$ equals  
(a) $-1$  
(b) $-1/y^2$  
(c) $-x^2/y^2$  
(d) $-3x^2/y$  
(e) None

5. If $y = \sqrt{2x^2+1}$, then $y'$ equals  
(a) $x/\sqrt{2x^2+1}$  
(b) $-x/\sqrt{2x^2+1}$  
(c) $2x/\sqrt{2x^2+1}$  
(d) $x/\sqrt{2x^2+1}$  
(e) None

6. If $f(x) = \csc \pi x$, then $f'(1/2)$ equals  
(a) $-\pi$  
(b) $-1$  
(c) 0  
(d) $\pi$  
(e) None

7. Let $f(u) = (3\sqrt{u}-1)^{-1}$. Then $f'(8)$ equals  
(a) $-1$  
(b) $-1/3$  
(c) $-1/4$  
(d) $-1/12$  
(e) None

8. Let $g(t) = (t^2-1)/(t^2+1)$. Then $g'(1)$ equals  
(a) 1  
(b) 2  
(c) $1/2$  
(d) 4  
(e) None
9. If \( g(t) = (1+t^2)^{-5} \), then \( g''(0) \) equals (a) 0 (b) -1 (c) -5 (d) -10 (e) -20

10. \( \lim \frac{\sin 2h}{h} \) is (a) 0 (b) 1 (c) 2 (d) 1/2 (e) None

11. If \( y = \tan^2 x \), then \( y' \) equals (a) \( \tan x \sec x \) (b) \( 2 \tan x \sec x \) (c) \( 2 \tan x \sec^2 x \) (d) \( 2 \tan^2 x \sec^2 x \) (e) None

12. If \( G(t) = \cos \left(\frac{2\pi t}{t^2}\right) \), then \( G'(\sqrt{3}) \) equals (a) \( 2\pi/3 \) (b) \( 4\pi/3\sqrt{3} \) (c) \( -4\pi/3\sqrt{3} \) (d) -\( 2\pi/3 \) (e) None

13. Let \( f \) be the absolute value function. Which of the following statements concerning the hypothesis and conclusion of Rolle's Theorem is true for the interval \([-1,1]\)?
   (a) Hypotheses true but conclusion false
   (b) Hypotheses false but conclusion true
   (c) Hypotheses false and conclusion false
   (d) Hypotheses true and conclusion true
   (e) None of the preceding is true.

14. If \( y = x^3 (1-x)^4 \), then \( y' \) equals (a) \( x^2(1-x)^3(3+x) \) (b) \( 12x^2(1-x)^3 \) (c) \( x^2(1-x)^3(3-7x) \) (d) \( -12x^2(1-x)^3 \) (e) None

15. \( D_x f(f'(x)) \) is equal to (a) \( f'(f'(x)) \) (b) \( (f'(x))^2 \) (c) \( f''(x) \) (d) \( f'(f'(x))f''(x) \) (e) None
16. Let \( f(x) = \left\lfloor x \right\rfloor x \). Which of the following statements concerning the Mean Value Theorem (Theorem of the Mean) is true for the interval \([0, 3/2]\).

(a) Hypotheses true but conclusion false
(b) Hypotheses false but conclusion true
(c) Hypotheses false and conclusion false
(d) Hypotheses true and conclusion true.
(e) None of the preceding is true.

17. Suppose \( F(x) = x^2 \) if \( x < 1 \) and \( F(x) = 2x - 1 \) if \( 1 < x \). Which one of the following statements is false.

(a) \( F \) is continuous in \([0, 2]\)
(b) \( F \) is differentiable in \([0, 2]\)
(c) \( F' \) is continuous in \([0, 2]\)
(d) \( F' \) is differentiable in \([0, 2]\)
(e) \([0, 2] \subset F ([0, 2])\)

18. Suppose \( F(x) = x^2 \) if \( x < 1 \) and \( F(x) = 2x - 1 \) if \( 1 < x \). Which one of the following statements is true.

(a) \( F''(1) = 2 \)
(b) \( \lim_{x \to 1} F''(x) = 2 \)
(c) \( F'''(1) = 0 \)
(d) \( \lim_{x \to 1} F'''(x) = 0 \)
(e) None of the preceding statements is true.

19. Let \( y = 3 \sin 2x \) and \( x = 2 \cos 3x \). Then \( y'' = z'' \) if

(a) \( y = z \)
(b) \( 2y = 3z \)
(c) \( 4y = 9z \)
(d) \( 3y = 3z \)
(e) None

20. Let \( f(x) = \sin x \). There is a number \( m \) in the open interval \((0, \pi/6)\) such that \( f'(m) \) is equal to

(a) \( \pi/3 \)
(b) \( 3/\pi \)
(c) 1
(d) \( \sqrt{3}/2 \)
(e) None
In the first four questions $F(x) = 2x^3 - 3x^2 + 2$.

1. $F$ is increasing for numbers in the set
   (a) $(0,1)$  (b) $(0,\infty)$  (c) $(-\infty,\infty)$  (d) $(1/2,\infty)$  (e) $(-\infty,0)\cup(1,\infty)$

2. The graph of $F$ is concave up for $x$ in the interval
   (a) $(1/2,\infty)$  (b) $(0,1/2)$  (c) $(-\infty,1/2)$  (d) $(0,1)$  (e) $(2,\infty)$

3. A (relative) maximum point on the graph of $F$ is
   (a) $(1,1)$  (b) $(1,2)$  (c) $(0,2)$  (d) $(0,0)$  (e) none

4. A (relative) minimum point on the graph of $F$ is
   (a) $(1,1)$  (b) $(1,2)$  (c) $(0,2)$  (d) $(0,0)$  (e) none
Questions 5 and 6 are about a body moving along a number scale. Its displacement $s$ from the origin at time $t$ is given by the equation $s = \sin(\pi t/2)$.

5. In which of the following time intervals is the displacement negative and the velocity positive?
   (a) (0,1)  (b) (1,2)  (c) (2,3)  (d) (3,4)  (e) none

6. In which of the following time intervals is the velocity positive and the acceleration negative?
   (a) (0,1)  (b) (1,2)  (c) (2,3)  (d) (3,4)  (e) none

7. A part of the graph of $y = x - \frac{1}{x}$ is
   (a)  
   (b)  
   (c)  
   (d)  
   (e)
8. The graph of $G$ is shown at the right. In which interval is $f'(x) < 0$ and $f''(x) < 0$?

(a) (0,1)  (b) (0,2)  (c) (1,2)  (d) (1,3)  (e) (2,3)

9. A part of the graph of the equation $y = x\sqrt{4 - x^2}$ is:

(a)  
(b)  
(c)  
(d)  
(e)  

10. A triangle with two equal sides of length 2 units is drawn so as to make the area a maximum. That maximum area is

(a) 1  (b) 2  (c) 3  (d) $\sqrt{2}$  (e) $2\sqrt{2}$

11. Repeat your answer to question 10.
12. An open trough is to be made in the shape of a half-cylinder whose vertical ends are halves of circular disk. The volume of the trough is to be $V$ units. Let $R$ be the radius of the ends, and $L$ the length of the trough. The expression to be minimized in order to find the radius of the trough of least surface material is

(a) $RL + \pi R^2$  
(b) $2\pi R + LR^2$  
(c) $\pi R^2 + \frac{V}{R}$  
(d) $\pi R^2 + \frac{2V}{R}$  
(e) $2\pi R^2 + \frac{V}{R}$

13. A 13 foot ladder leans against the wall of a house. Someone pulls the base of the ladder away from the house at the rate of 2 feet/sec. The speed of the top of the ladder sliding down the wall when the top is 12 feet from the ground is how many ft/sec?

(a) $2\frac{4}{5}$  
(b) $\frac{5}{6}$  
(c) $\frac{5}{12}$  
(d) 1  
(e) $\frac{12}{5}$
14. Suppose \( y = x^2(1 - 2x) \), and that \( x \) increases at a constant rate of 2 units per second. Which of the following statements is true?
(a) \( y \) never decreases  
(b) \( y \) always decreases  
(c) if \( x > 0 \), then \( y \) decreases  
(d) if \( x < 1/2 \), then \( y \) increases  
(e) if \( x > 1/3 \), then \( y \) decreases

15. A box with a square base and no top is to be constructed to hold a fixed volume \( V \). The material to be used to construct the bottom of the box costs twice as much as the material to be used to make the sides. The cost of the box is least if the length of a side of the base equals
(a) the height  
(b) twice the height  
(c) one half the height  
(d) one fourth the height  
(e) none

16. Repeat your answer to question 15.
17. Which of the following numbers is the X-coordinate of a maximum point on the graph of \( y = x - \cos x \) ?
(a) \( \pi/2 \)  (b) \( \pi \)  (c) \( 3\pi/2 \)  (d) \( 2\pi \)  (e) none

In Questions 18 and 19 use the approximation for \( f(x + h) \) in terms of \( f(x) \) and \( f'(x) \).

18. Which of the following numbers is the best approximation to \( \sqrt[3]{3} \) ?
(a) \( 9/4 \)  (b) \( 13/6 \)  (c) \( 25/12 \)  (d) \( 33/16 \)  (e) \( 49/24 \)

19. Which of the following numbers is the best approximation to \( \cos 6 \) ?
(a) 1.1  (b) 1  (c) .9  (d) .8  (e) .7

20. Using a single application of Newton's Method and an initial guess of zero, an approximation to a solution of the equation \( 2x^3 + 2x - 1 = 0 \) is
(a) .4  (b) -.4  (c) .5  (d) -.5  (e) 2
APPENDIX B
Write your name and instructor above. Black in your name, section number, and form A on your answer sheet. Indicate the one best response to each question. Your score is the number correct.

1. Which of the following describes a function?
   (a) \( \{(x,y)|x^2 = y^2\}\)  (b) \( \{(x,y)|\sin y = \cos x\}\)  (c) \( \{(x,y)|xy = 1\}\)
   (d) \( \{(x,y)|x^2 = |y|\}\)  (e) none

2. The "understood" domain of the function defined by \( f(x) = \sqrt{\frac{x-1}{x^3}} \) is
   (a) \((0,\infty)\)  (b) \([1,\infty)\)  (c) \((-\infty,-1]\cup(0,\infty)\)  (d) \((-\infty,0)\cup[1,\infty)\)  (e) \((-\infty,\infty)\)

3. The equation of the circle with radius 2, center \((1,-3)\) is
   (a) \((x+1)^2 + (y-3)^2 = 2\)  (b) \((x+1)^2 + (y-3)^2 = 4\)  (c) \(x^2 + y^2 = 4\)
   (d) \(x^2 - 2x + y^2 + 6y + 6 = 0\)  (e) \(x^2 + 2x + y^2 + 6y + 10 = 0\)

4. The figure at the right shows part of the graph of
   \( f \)
   (a) \( y = \cos x \)  (b) \( y = \sin x \)  (c) \( y = \cos^2 x \)
   (d) \( y = \sin^2 x \)  (e) \( y = 1 - \sin x \)

5. Suppose the graph of \( f \) is the parabola \( y = 1 - x^2 \). If \( A = [0,1] \) and \( B = [-1,1] \), which of the following is true?
   (a) \( B \subseteq A \)  (b) \( B \subseteq f(A) \)  (c) \( f(A) \subseteq B \)  (d) \( A \cup B \subseteq f(A) \)  (e) none
6. If \( f(x) = 2x^3 - 3x^2 \) and \( A = [0, 2] \) then \( f(A) \) equals
   (a) \([0, 4]\)  (b) \([0, 2]\)  (c) \([-1, 4]\)  (d) \([-2, 4]\)  (e) none

Questions 7 through 10 involve the function \( F \) defined as follows:
   if \( x \leq 0 \), then \( F(x) = |x| \); if \( x > 0 \) then \( F(x) = |x - 1| \).

7. The function \( F \) is discontinuous for \( x \) equal to
   (a) \(-1\)  (b) \(0\)  (c) \(1\)  (d) \(0\) and \(1\)  (e) none

8. \( \lim_{x \to 0} F(x) \) is
   (a) \(-1\)  (b) \(0\)  (c) \(1\)  (d) does not exist  (e) none

9. \( \{x | F'(x) \text{ does not exist} \} \) is
   (a) \((-1)\)  (b) \([0]\)  (c) \([1]\)  (d) \([0,1]\)  (e) none

10. \( \{x | F'(x) = -1\} \) is
    (a) \((-\infty, 0)\)  (b) \((0, 1]\)  (c) \((-\infty, 1)\)  (d) \((-\infty, 0) \cup (0, 1]\)  (e) none
11. \( \lim_{x \to 1} \frac{1 - |x|}{1 + x} \) is (a) -1 (b) 0 (c) 1 (d) does not exist (e) none

12. Suppose that \( \frac{f(x+h) - f(x)}{h} = \sqrt[3]{x+h} - \sqrt[3]{x} \). Then \( f'(4) \) equals
   (a) 0 (b) 2 (c) 1/2 (d) 1/4 (e) none

13. The equation of the tangent line to the parabola \( y = x^2 + x \) at the point \((-1,0)\) is (a) \( x + y = 1 \) (b) \( x + y + 1 = 0 \) (c) \( x - y + 1 = 0 \) (d) \( y - x + 1 = 0 \) (e) none

14. The slope of the normal line to the curve \( y = \sin 2x + \cos x \) at the point \((\pi, -1)\) is (a) 1 (b) -1 (c) 1/2 (d) -1/2 (e) none

15. A particle moves along an s-scale according to the equation \( s = \sin 2\pi t^2 \). The velocity of the particle at \( t = 2 \) is
   (a) 0 (b) \( 2\pi \) (c) 4 (d) \( 8\pi \) (e) none
16. Let \( f(x) = \frac{|x|}{\frac{x}{x^2}} \). Which of the following statements is false?

(a) \( \lim_{x \to 0} f(x) = 1 \)  
(b) \( \lim_{x \to 0} f(x) = -1 \)  
(c) \( f'(1) = 1 \)  
(d) \( f'(-1) = -1 \)  
(e) \( f'(0) = 0 \)

17. If \( f(x) = \sqrt{2x^2 - 1} \), then \( f'(1) \) equals

(a) 0  
(b) 1  
(c) 2  
(d) 4  
(e) none

18. If \( f(x) = \sec(\pi x) \), then \( f'(1/4) \) equals

(a) \( \sqrt{2} \)  
(b) \( \pi \)  
(c) \( \pi/\sqrt{2} \)  
(d) \( \pi\sqrt{2} \)  
(e) none

19. If \( y = (\sqrt{x} - 1)^{1/2} \) then

(a) \( 2y(y^2 + 1)y' = 1 \)  
(b) \( 2y(y^2 + 1) = y' \)  
(c) \( y'(y^2 + 1) = y^{-1} \)  
(d) \( 4y(y^2 + 1)y' = 1 \)  
(e) \( 4(y^2 + 1) = yy' \)

20. Suppose \( G'(0) = G'(2) \), \( G'(1) \leq G'(3) \), and suppose \( G''(0) \) and \( G''(2) \) have opposite signs. Which of the following graphs best represent \( G \)?

(a)  
(b)  
(c)  
(d)  
(e)  
(f)  
(g)  
(h)  
(i)  
(j)  
(k)  
(l)  
(m)  
(n)  
(o)  
(p)  
(q)  
(r)  
(s)  
(t)  
(u)  
(v)  
(w)  
(x)  
(y)  
(z)
21. \( \frac{d}{dx} \sqrt{g(x)} \) equals
   (a) \( \frac{1}{2} \sqrt{g(x)} \)
   (b) \( \sqrt{g'(x)} \)
   (c) \( g'(x) \sqrt{g(x)} \)
   (d) \( \frac{g'(x)}{2\sqrt{g(x)}} \)
   (e) none

22. If \( x^2 + y^3 = 1 \), then \( y' \) equals
   (a) \( -\frac{2x}{3y^2} \)
   (b) \( \frac{3\sqrt{1-x^2}}{3y^2} \)
   (c) \( -\frac{3y^2}{2x} \)
   (d) \( \frac{1-3y^2}{2x} \)
   (e) none

23. If \( g(t) = \frac{t}{\sqrt{t^2-1}} \), then \( g''(2) \) equals
   (a) \( \frac{2\sqrt{3}}{9} \)
   (b) \( \frac{5\sqrt{3}}{9} \)
   (c) \( -\frac{\sqrt{3}}{27} \)
   (d) \( -\frac{5\sqrt{3}}{9} \)
   (e) none

24. \( \lim_{h \to 0} \left[ \frac{\sin 2h}{h} - \frac{\sin h^2}{h} \right] \) is
   (a) 0
   (b) 1
   (c) 2
   (d) does not exist
   (e) none

25. A relative maximum point on the graph of \( y = x^2(x+2)^2 \) is
   (a) \((-1,1)\)
   (b) \( (0,0) \)
   (c) \((1,9)\)
   (d) \((2,64)\)
   (e) none
26. The graph of \( y = x^4 + 2x^3 - 12x^2 \) is concave up in the interval
   (a) \( (-\infty, -2) \) (b) \( (-2, -1) \) (c) \( (-1, 0) \) (d) \( (0, 1) \) (e) none

27. The graph of \( G \) is shown at the right. Which of the following is the longest interval in which the product \( f(x)f'(x)f''(x) \) is positive?
   (a) \( (0, 1) \) (b) \( (0, 2) \) (c) \( (1, 3) \) (d) \( (1, 4) \) (e) \( (2, 4) \)

28. The hyperbola \( 4(x-1)^2 - y^2 = 4 \) has one asymptote with slope
   (a) \( 1/4 \) (b) \( 1/2 \) (c) \( 2 \) (d) \( 4 \) (e) none

29. The x-coordinate of one focus of the ellipse \( 4x^2 + 9y^2 = 36 \) is
   (a) \( 2 \) (b) \( 3 \) (c) \( 4 \) (d) \( 9 \) (e) none

30. The vertex of the parabola \( y^2 - 4y - x + 5 = 0 \) is
   (a) \( (0, 5) \) (b) \( (1, 2) \) (c) \( (2, 1) \) (d) \( (5, 0) \) (e) none
31. The graph of $2x^2 - 3x + 3y^2 - 2y - 7 = 0$ is a(an)
   (a) circle  (b) ellipse  (c) hyperbola  (d) parabola  (e) none

32. The eccentricity of the conic $7x^2 - 6x + 19 = 5y^2 + 8y + 13$ is
   (a) -1  (b) 0  (c) $\sqrt{7/12}$  (d) 1  (e) $\sqrt{12/7}$

33. An ellipse with center (0,1) and semi-diameters of 2 and 1 has for its equation $x + By^2 + Cy = 0$. The quotient $B/C$ equals
   (a) $-1/2$  (b) $1/2$  (c) 2  (d) -2  (e) none

34. A box with a square base and top is to hold a fixed volume $v$. The material used for the bottom costs twice as much as the material used for the sides, and the material for the top costs half as much as the material for the sides. The cost of the box is least if the length of a side of the base equals
   (a) $4/5$ the height  (b) the height  (c) $5/4$ the height
   (d) $3/2$ the height  (e) none

35. Repeat your answer to number 34.
36. A 13 foot ladder leans against a wall. The base of the ladder is pulled away from the wall at the rate of 3 ft./sec. The speed in ft./sec. of the top of the ladder sliding down the wall when the top is 12 feet from the ground is

(a) 36/5 (b) 5/4 (c) 5/8 (d) 3/2 (e) 18/5

37. The area of a segment of a circle (a pie-shaped piece) of radius \( r \) and angle \( \theta \) radians is \( r^2 \theta /2 \). The length of a circular arc is \( r\theta \). If the perimeter of the segment is fixed at \( P \) units, then the radian measure of the angle which maximizes the area of the sector is

(a) \( \pi/4 \) (b) \( \pi/2 \) (c) 1 (d) 2 (e) none

38. Repeat your answer to number 37.

The next two questions deal with a function \( F \) with domain \([-c,c]\) such that \( F(0) = 0 \), and with the following statements:

I. \( \lim_{x \to 0} F(x) = 0 \)  
II. \( \lim_{x \to 0} \frac{F(x)}{x} = 0 \)  
III. \( F \) is continuous at 0  
IV. \( F'(0) = \frac{F(c)-F(-c)}{2c} \)  
V. \( F \) is differentiable at 0

39. Which of the following statements is true?

(a) I implies II  (b) I implies III  (c) II implies IV  
(d) III implies IV  (e) none

40. Which of the following statements is true?

(a) V implies IV  (b) V implies II  (c) III implies II  
(d) II implies I  (e) none
APPENDIX C
1. The maximum slope of \( y = -x^3 + 3x^2 + 9x - 27 \) is ____________________________

2. According to the law of the means what value must the slope of \( y = \frac{1}{x} \) assume in \((2,5)\)? Ans.

3. Find one equation of the tangent to the curve \( \begin{cases} x = t^3 - 4 \\ y = 2t^2 + 1 \end{cases} \) at the point where \( t = 2 \). Ans.

4. If \( 2x = t + \sqrt{t^2+1} \) and \( 3y = t - \sqrt{t^2-1} \), what is the value of \( y \) when \( x = 3 \)? Ans.

5. Let \( y = \ln x \). \( \frac{d^2y}{dx^2} = \)

6. \( \lim_{x \to 0} \frac{2 \tan^2 x}{1 - \cos^2 x} = \)

7. Given \( y = \frac{x \cos x}{1 + e^{-x}} \). \( y'(0) = \)

8. Given \( f(x) = \frac{x-1}{x+1} \). In terms of \( f(x) \), \( f(2x) = \)

9. Find the equation of curve given that \( y'' = -4 \) and the curve has a slope of 2 at \((2,1)\). Ans.

10. Find the equation of the curve given that \( y'' = 6x \) and the curve has a critical point at \((2,-16)\). Ans.

11. Find a point of inflection of \( y = ax^3 + bx^2 + cx + d \). Ans.

12. Determine \( a, b, \) and \( c \) so that \( y = ax^3 + bx^2 + cx \) may have a point of inflection at \((1,2)\) and a slope of \(-1\) at this point.

13. Find a critical point of \( y = ax^2 + bx + c \). Ans.
14. In problem (13) above, determine for what \( "a" \) is the pt. a maximum or a minimum.

Ans. max. for \[ \quad \] ; min. for \[ \quad \]

15. Given that \( y = ax^3 + bx^2 + cx + d \). Use \( b^2 - 3ac \) to determine the number of critical points \( y = ax^3 + bx^2 + cx + d \) has. Ans.

16. Determine \( a, b, c, \) and \( d \) so that the curve \( y = ax^3 + bx^2 + cx + d \) may have a critical point at \((-2,4)\) and a point of inflection at \((1,-10)\).

17. A boy rolls a ball up a sloping street. After \( t \) seconds its displacement \( s(\text{feet}) \) from the starting point is given by \( s = 48t - 4t^2 \). Find (a) when the ball is rolling up the street and when down, (b) how far up the street it rolls, (c) the speed when it passes the boy on the way down, and (d) its acceleration.

Ans. (a) \[ \quad \], (b) \[ \quad \]

(c) \[ \quad \], (d) \[ \quad \]

18. By definition, \( f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} \). Therefore, \( f''(a) = \)

19. \( \lim_{x \to 0} \frac{1}{\csc 4x} = \)

20. Given that \( f(x) = \lceil x \rceil x^2 \), which of the following statements concerning the theorem of the mean for the interval \([-1,1]\) is true?

(a) both true (b) Hyp. false but concl. true
(c) Hyp. true; concl. false (d) both false (e) none of these

21. Given that

\[
\begin{align*}
f(x) = \begin{cases} 
(x+1)^3, & x < -1 \\
0, & -1 \leq x \leq 1 \\
(x-1)^3, & x > 1 
\end{cases}
\end{align*}
\]

which of the following statements is true concerning the hypotheses and conclusion of the theorem of the mean for the interval \([-2,2]\)?

(a) both false (b) both true (c) Hyp. true; concl. false
(d) Hyp. false and concl. true (e) none of these
22. Let \( f(x) \) be defined as in 21. Then

(a) \( f \) is increasing on \([-2,0]\)  
(b) \( f'(3) \) is differentiable at \( x = 1 \)  
(c) \( \lim_{x \to 1} f''(x) = 6 \)  
(d) \( \lim_{x \to -1} f''(x) = 0 \)  
(e) \( \lim_{x \to 0} f''(x) = 6 \)

23. Let \( f(x) \) be defined as in 21. Then \( f' \) on \((0,2)\) is

(a) increasing  
(b) decreasing  
(c) concave up  
(d) concave down  
(e) none of these

24. Given that \( f(x) = \lfloor x \rfloor x^3 \). Which is false?

(a) \( f \) is diff. on \((-1,1)\)  
(b) \( f' \) is diff. at \( x = 0 \)  
(c) \( f'' \) is cont. on \((-1,1)\)  
(d) \( f'' \) is diff. on \((-1,1)\)  
(e) none of these  

25. \( f(x) = \frac{x-2}{(x+1)^2} \). \( f \) is increasing on

(a) \([1,3]\)  
(b) \((-\infty,1)\)  
(c) \([3,\infty)\)  
(d) \((1,3)\)  
(e) none of these

26. \( \lim_{x \to 0} (\lfloor 1-x \rfloor + \lfloor x+1 \rfloor) = \)

27. \( \lim_{x \to 1} \frac{1-\lfloor x \rfloor}{1+x} = \)

28. \( \lim_{x \to 0} \frac{x}{\lfloor x \rfloor} = \)

29. \( \lim_{x \to 0} \frac{x}{x} = \)
1. $f$ cont at $x = a \Rightarrow f'$ is cont at $x = a$. T F

2. Since $A$ open and $B$ open $\Rightarrow A \cap B$ is open, we conclude that if $A_i$ is open for $i \in \mathbb{Z}_+$, $\bigcap A_i$ is open. T F

3. Let $f$ be defined on $(a, b)$ and $c \in (a, b)$. Then $f'(c) \neq 0 = f(c)$ is neither the maximum nor the minimum value of $f$ on $(a, b)$. T F

4. $f$ not increasing on $[a, b] \Rightarrow f$ is decreasing on $[a, b]$. T F

5. $f$ not concave up on $[a, b] \Rightarrow f$ concave down on $[a, b]$. T F

6. Let $C$ be a collection of sets. Then if each set in $C$ is countable then the union of all sets in $C$ is a countable set. T F

7. $f''$ cont at $x = a = f'$ is cont at $x = a$. T F

8. $f'''$ diff at $x = a = f$ is cont at $x = a$. T F

9. $\lim_{x \to o} \frac{f(x)}{g(x)}$ exists and $\lim_{x \to o} g(x) = 0 = \lim_{x \to o} \frac{f(x)}{g(x)} = 0$. T F

10. Let $f'(x) = g'(x)$ on $[a, b]$. Then for $c \in [a, b], f(c) > 0 \Rightarrow g(c) > 0$. T F

11. $f''(a) = 0 \Rightarrow f$ has a pt. of inflection at $x = a$. T F

12. $(f+g)$ continuous on $[a, b] = f$ and $g$ are continuous on $[a, b]$. T F

13. $f'$ defined at $x - a = f$ defined at $x = a$. T F

14. $f''$ diff. on $[a, b] = f$ is cont on $[a, b]$. T F

15. $\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = 0 \Rightarrow \frac{f'(x) - f(a)}{x - a} = 0$. T F

16. $f$ cont on $[a, b] = f'$ is cont. on $[a, b]$. T F

17. $f'$ differentiable on $[a, b] = f''$ is differentiable on $[a, b]$. T F

18. $f'''$ differentiable on $[a, b] = f''$ is differentiable on $[a, b]$. T F

19. Let $n$ denote the number of sides of a polygon. It is true that as $n$ increases, the sum of the interior angles also increases. Take this fact and indicate what happens to the sum of the exterior angles as $n$ increases. Ans.

20. $f'''(a)$ exists $\Rightarrow \lim f''(x) = x \to a$ T F
21. \( \lim_{x \to \infty} \frac{\sin x}{x} \)
   (a) 0  (b) 1  (c) \( \infty \)  (d) None

22. \( f \) continuous at \( x = a \) implies that \( f \) is differentiable at \( x = a \).  T  F

23. \( f'(a) = 0 \) implies that \( f \) has a maximum value or a minimum value at \( x = a \).  T  F

24. \( f \) not continuous at \( x = a \) implies that \( f \) is not differentiable at \( x = a \).  T  F

25. Which of the following statements necessarily follows from "some calculus problems are not difficult"?
   (a) Some difficult problems are calculus problems.
   (b) Some difficult problems are not calculus problems.
   (c) Some calculus problems are difficult.
   (d) None.

26. \( f \) maximum at \( x = a \) implies that \( f'(a) = 0 \).  T  F

27. Since the terms of the sequence: 1, 1/2, 1/3, 1/4, ... tend to zero as \( n \) becomes infinite, the sum of the terms of the sequence is:
   (a) 0  (b) less than some positive number \( K \)  (c) some unique number \( N \)  (d) greater than any number \( K \).

28. Let \( I_n \) \((n = 1, 2, 3, 4, ...)\) be an interval in \( R \) such that
   \( I_1 \supset I_2 \supset I_3 \supset I_4 \supset \cdots \supset I_n \supset \cdots \). Then
   \( \cap_{n=1}^{\infty} I_n = I_1 \cap I_2 \cap I_3 \cap \cdots \cap I_n \cap \cdots = \emptyset \).  T  F

29. \( f \) not diff. at \( x = a \) implies that \( f \) not cont. at \( x = a \).  T  F

30. \( f'' \) cont. at \( x = a \) implies that \( f' \) is cont. at \( x = a \).  T  F
### MATHEMATICS OPINIONNAIRE

**Directions:** Each statement below expresses a feeling which a particular person has toward mathematics. You are asked to express the extent to which you personally agree or disagree with the opinion stated, on a 5-point scale: **SA** (Strongly Agree), **A** (Agree), **U** (Undecided), **D** (Disagree), and **SD** (Strongly Disagree). Fill in the circle at the right indicating the extent of your agreement with the feeling expressed.

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<tbody>
<tr>
<td>1</td>
<td>I feel at ease with mathematics.</td>
<td><strong>SA</strong></td>
<td><strong>A</strong></td>
<td><strong>U</strong></td>
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<td>2</td>
<td>When I hear the word mathematics, I have a distinct feeling of dislike.</td>
<td><strong>SA</strong></td>
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<td>3</td>
<td>I do not feel sure of myself in mathematics.</td>
<td><strong>SA</strong></td>
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<td>4</td>
<td>Mathematics is a subject I feel I can sink my teeth into.</td>
<td><strong>SA</strong></td>
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<td>5</td>
<td>Mathematics makes me feel uncomfortable, uneasy, irritable and impatient.</td>
<td><strong>SA</strong></td>
<td><strong>A</strong></td>
<td><strong>U</strong></td>
<td><strong>D</strong></td>
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<td>6</td>
<td>Mathematics is something which I enjoy doing a great deal.</td>
<td><strong>SA</strong></td>
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<td><strong>D</strong></td>
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<td>7</td>
<td>Mathematics is fascinating and fun for me.</td>
<td><strong>SA</strong></td>
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<td><strong>U</strong></td>
<td><strong>D</strong></td>
<td><strong>SD</strong></td>
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<td>8</td>
<td>I enjoy the challenge of mathematics problems.</td>
<td><strong>SA</strong></td>
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<td><strong>U</strong></td>
<td><strong>D</strong></td>
<td><strong>SD</strong></td>
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<td>9</td>
<td>I feel under a great strain in a mathematics class.</td>
<td><strong>SA</strong></td>
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<td><strong>U</strong></td>
<td><strong>D</strong></td>
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<td>10</td>
<td>I approach mathematics with a feeling of hesitation.</td>
<td><strong>SA</strong></td>
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<td>11</td>
<td>Mathematics is stimulating to me.</td>
<td><strong>SA</strong></td>
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<td>12</td>
<td>Mathematics is my most dreaded subject.</td>
<td><strong>SA</strong></td>
<td><strong>A</strong></td>
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<td><strong>D</strong></td>
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<td>13</td>
<td>I have a definite favorable reaction to math: it's enjoyable.</td>
<td><strong>SA</strong></td>
<td><strong>A</strong></td>
<td><strong>U</strong></td>
<td><strong>D</strong></td>
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<td>14</td>
<td>Working with mathematics is fun.</td>
<td><strong>SA</strong></td>
<td><strong>A</strong></td>
<td><strong>U</strong></td>
<td><strong>D</strong></td>
<td><strong>SD</strong></td>
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<td>15</td>
<td>It scares me to have to take mathematics.</td>
<td><strong>SA</strong></td>
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<td>16</td>
<td>At present, I would rate my general attitude toward math as favorable.</td>
<td><strong>SA</strong></td>
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<td><strong>D</strong></td>
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<td>17</td>
<td>Mathematics is very interesting to me.</td>
<td><strong>SA</strong></td>
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<td>18</td>
<td>When I approach my mathematics work, I experience a sense of fear of not being able to do it.</td>
<td><strong>SA</strong></td>
<td><strong>A</strong></td>
<td><strong>U</strong></td>
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<td>19</td>
<td>I have a feeling of insecurity when attempting mathematics.</td>
<td><strong>SA</strong></td>
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<td>20</td>
<td>Mathematics is a subject in school which I have liked and enjoyed studying.</td>
<td><strong>SA</strong></td>
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<td>21</td>
<td>The feeling I have toward math is a positive feeling.</td>
<td><strong>SA</strong></td>
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<td>22</td>
<td>Math makes me feel as though I'm lost in a jungle and can't find my way out.</td>
<td><strong>SA</strong></td>
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BIBLIOGRAPHY


