B-CONVEXITY IN BANACH SPACES

DISSERTATION

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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACKNOWLEDGMENTS</td>
<td>ii</td>
</tr>
<tr>
<td>VITA</td>
<td>iii</td>
</tr>
<tr>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>I. PRELIMINARIES</td>
<td>5</td>
</tr>
<tr>
<td>II. THE PROPERTY OPPOSITE TO B-CONVEXITY</td>
<td>17</td>
</tr>
<tr>
<td>III. B-CONVEXITY AND BASES</td>
<td>35</td>
</tr>
<tr>
<td>IV. P-CONVEXITY AND OTHER PROPERTIES RELATED TO B-CONVEXITY</td>
<td>50</td>
</tr>
<tr>
<td>V. SOME UNSOLVED PROBLEMS</td>
<td>65</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>74</td>
</tr>
</tbody>
</table>
INTRODUCTION

The notion of a B-convex Banach space was introduced in [2] by A. Beck as a characterization of those Banach spaces X having the property that a certain strong law of large numbers holds for X-valued random variables. B-convexity is a generalization of uniform convexity, as will be apparent from the definition, and is, by virtue of the characterization of Beck, invariant under equivalent renorming.

**Definition:** A Banach space X is said to be B-convex if there is some positive integer k, and some $\epsilon > 0$ so that for any $x_1, \ldots, x_k$; $||x_i|| \leq 1$, $i=1, \ldots, k$ there is a set of signs $\xi_1, \ldots, \xi_k$ so that $||\sum_{i=1}^{k} \xi_i x_i|| \leq k(1 - \epsilon)$.

The definition of a uniformly convex space [5] can be phrased:

**Definition:** A Banach space X is said to be uniformly convex if for each number $\epsilon > 0$ there is a number $\delta > 0$ so that if $||x_1|| \leq 1$, $||x_2|| \leq 1$ then either

$$\frac{1}{2} ||x_1 + x_2|| \leq 1 - \delta \text{ or } ||x_1 - x_2|| \leq \epsilon.$$ 

A lengthy discussion of the strong law of large numbers is contained in [3] in which the following question is raised: Can every B-convex space be renormed to be uniformly convex? This question is still open.

Further study of B-convexity was done by R. C. James [14,15,17] and D. P. Giesy [12] from a functional analytic point of
view. Giesy showed that B-convex spaces have many of the properties of reflexive spaces. James conjectured that all B-convex spaces are reflexive, and proved that in case $k = 2$ the conjecture is true. Both James and Giesy showed that the conjecture also holds for B-convex spaces having an unconditional basis. Later, C. A. Kottman [20,21] showed that another subclass of B-convex spaces are reflexive; these are said to be F-convex. Examples are known of spaces which are reflexive but not B-convex.

Chapter I below lists some of the known facts about B-convexity and extends one of them (Proposition 1.20). The chapter includes several examples of Banach spaces, most of which are from the literature, illustrating some of the relationships between B-convexity and other properties.

Chapter II takes up the property opposite to B-convexity. A characterization of this property by Giesy is weakened, and a family of four new characterizations is presented. A technique used by James in [14] is extended in a complicated way to give a condition sufficient for non-B-convexity. It is shown that a strengthened version of this condition resembles one of the family of four conditions referred to above and thereby is sufficient for non-B-convexity in a much simpler way. These sufficient conditions hold for $c_0$, $\ell_1$, and any space containing either of them. It is not known whether these sufficient conditions are also necessary.

Chapter III takes up questions involving B-convexity and the theory of Schauder bases. A. Pelczynski [26] has proved that a Banach space is reflexive if every subspace having a Schauder basis
is reflexive. The main theorem for this chapter is an analogous result for B-convex spaces: A space is B-convex if every subspace having a Schauder basis is B-convex. Also in this chapter a different proof is given of the known result that a B-convex space with unconditional basis is reflexive, using one of the family of four characterizations of non-B-convexity given in Chapter II, and using the unconditional basis in an elementary way.

Chapter IV investigates three properties related to B-convexity. Kottman \([20,21]\) defined a space to be P-convex if 

\[ P(n,X) < \frac{1}{2} \]

for some \( n \), where \( P(n,X) \) is the supremum of numbers \( r \) so that there is a set of \( n \) pairwise-disjoint closed balls of radius \( r \) in the closed unit ball of \( X \). P-convexity is a generalization of uniform convexity. Kottman showed that all P-convex spaces are also B-convex and reflexive. It is not known whether there is a space which is B-convex but not P-convex. (Some examples of Kottman are given in Chapter V to illustrate the nature of this problem.) It is known that the direct sum of two B-convex spaces is B-convex. The analogous problem for P-convex spaces seems more difficult, partly because it is not known whether P-convexity is preserved under equivalent renorming. A partial solution to the direct sum problem, two special cases, is given using a theorem of combinatorics. One of these special cases is critical to the proof of a result similar to the main theorem of Chapter III: A space is P-convex if each subspace having a Schauder decomposition into finite dimensional subspaces is P-convex.

Another generalization of uniform convexity is the Banach-Saks property. A space is said to have the Banach-Saks property if
every bounded sequence of elements of the space has a subsequence whose \((C,1)\) means converge strongly. An example is given of a space, shown by Nishiura and Waterman \([24]\) to have the Banach-Saks property, which is not B-convex. The converse question to this remains open and is of interest since it has been shown by Nishiura and Waterman \([24]\) that all Banach-Saks spaces are reflexive.

The last part of Chapter IV shows that a property proved in \([30]\) for spaces termed "near convex" actually holds for all B-convex spaces and all reflexive spaces.

Chapter V lists some unsolved problems and remarks.
CHAPTER I
PRELIMINARIES

The symbol $X$ will always represent a real Banach space. The closed unit ball of $X$, \( \{x : ||x|| \leq 1\} \), will be denoted $U(X)$. We denote the closed span of \( \{x_i\}_{i=1}^n \), that is, the smallest closed subspace containing these elements, by \( [x_i]_{i=1}^n \). We will write \( \{x_i\} \) and \( [x_i] \) instead of \( \{x_i\}_{i=1}^\infty \) and \( [x_i]_{i=1}^\infty \). For \( 1 \leq p < \infty \), $\ell_p^n$ will denote the Banach space of $n$-tuples with norm given by \( ||(x_1, \ldots, x_n)|| = (\sum_{i=1}^n |x_i|^p)^{1/p} \). $\ell^n_\infty$ is the Banach space of $n$-tuples with norm given by \( ||(x_1, \ldots, x_n)|| = \max_i |x_i| \). $\ell_p^n$ and $\ell_\infty^n$ are the usual sequence spaces denoted by these symbols.

1.1 Definition: Let $k$ be a positive integer and $\varepsilon$ a positive number. $X$ is said to be $k,\varepsilon$-convex if for any \( \{x_1, \ldots, x_k\} \subset U(X) \) there is some choice of signs $\xi_1, \ldots, \xi_k$ so that \( ||\sum_{i=1}^k \xi_i x_i|| \leq k (1-\varepsilon) \). $X$ is said to be $\mathcal{B}$-convex if it is $k,\varepsilon$-convex for some choice of $k$ and $\varepsilon$.

The terminology in the definition is that of Giesy [12]. James [14] has defined a space to be uniformly non-$\ell_1^n$ if it is $n,\varepsilon$-convex for some $\varepsilon$. A space which is uniformly non-$\ell_2^n$ was said to be uniformly non-square.\(^1\)

Geometrically, we can say that if $X$ is $2,\varepsilon$-convex then the largest "square" which can be inscribed in a plane cross section.

\(^1\)In James' terminology the subscript and superscript on $\ell$ are reversed from our convention.
of \( U(X) \) has side length, in terms of the norm of \( X \), less than \( 2 - \varepsilon \).

Listed below are a number of facts about \( E \)-convexity, most of which will be referred to later. The first group of facts can be proved by elementary applications of the definition. Except for fact 1.6, they were proved by Giesy [12].

1.2: If a space is \( k,\varepsilon \)-convex and \( \delta < \varepsilon \) then it is \( k,\delta \)-convex.

1.3: If a space is \( k,\varepsilon \)-convex then every subspace is \( k,\varepsilon \)-convex.

1.4: Uniformly convex spaces are \( 2,\varepsilon \)-convex for some \( \varepsilon \).

1.5: If the dimension of \( X \) is less than \( k \), \( X \) is \( k,\frac{1}{k} \)-convex.

Fact 1.2 shows that the number \( \varepsilon \) can be made smaller. It is also true that the number \( k \) can be made larger if \( \varepsilon \) is appropriately changed. A simple result in this direction is

1.6: If \( X \) is \( k,\varepsilon \)-convex then it is \( k+1,\varepsilon \frac{k+1}{k+1} \)-convex.

Proof: Given \( \{ x_1, \ldots, x_{k+1} \} \subseteq U(X) \) there is a set of signs \( \xi_1, \ldots, \xi_k \) so that

\[
\left| \xi_1 x_1 + \cdots + \xi_k x_k + x_{k+1} \right| \leq k (1 - \varepsilon)
\]

from which

\[
\left| \xi_1 x_1 + \cdots + \xi_k x_k + x_{k+1} \right| \leq k(1 - \varepsilon) + 1 = (k+1) \left[ 1 - \varepsilon - \frac{\varepsilon}{k+1} \right]
\]

A key fact is that by making \( k \) large enough, we can provide an \( \varepsilon \) nearly one:

1.7: If \( X \) is \( k,\varepsilon \)-convex and \( n \) is any positive integer, for any \( \{ x_1, \ldots, x_{k^n} \} \subseteq U(X) \) there is some choice of signs \( \{ \xi_1, \ldots, \xi_{k^n} \} \) so that

\[
\left| \xi_1 x_1 + \cdots + \xi_{k^n} x_{k^n} \right| \leq k^n (1 - \varepsilon)^n.
\]
If we let $e_n = 1 - (1-e)^n$ we have that $X$ is $k_n,e_n$-convex and $\lim_{n \to \infty} e_n = 1$. The proof of 1.7 is obtained by inductively applying the definition of $k, e$-convex and is contained in Giesy's proof of our Lemma 1.9 ([12], Lemma 1.4).

We now list several deeper results about B-convexity. Except for Proposition 1.20, which is new, and except where noted otherwise, these were proved by Giesy.

1.8 Definition: For a space $X$ we define the sequence $a_k(X), k \geq 2$, by

$$a_k(X) = \sup \{ \min \{ \frac{1}{k} \sum_{i=1}^{k} ||x_i|| : \frac{\sum_{i=1}^{k} \xi_i x_i}{\xi_i = \pm 1} \in U(X) \} \}.$$ 

From the definitions, for any $X$, $a_k(X) \leq 1$. If $X$ is B-convex there is some $k$ so that $a_k(X) < 1$. If $X$ is not B-convex, $a_k(X) = 1$ for all $k$. Fact 1.7 shows that if $X$ is $k,e$-convex, $a_k(X) \leq (1-e)^n$.

Further, Giesy proves the following.

1.9 Lemma: If $X$ is B-convex, $\lim_{k \to \infty} a_k(X) = 0$.

As noted, if $X$ is not B-convex the limit is 1.

The following lemma is a useful characterization of non-B-convexity giving insight to James' term "uniformly non-$\ell_n$".

Giesy proved this by a geometrical method.

1.10 Lemma: $X$ is not B-convex if and only if for every $k \geq 2, 1 > \epsilon > 0$ there are $x_1, \ldots, x_k \subseteq U(X)$ so that

$$(1 - \epsilon) \sum_{i=1}^{k} |\alpha_i| \leq \sum_{i=1}^{k} |\alpha_i x_i| \leq \sum_{i=1}^{k} |\alpha_i|.$$ 

If we define an $\epsilon$-isometry to be an isomorphism $T$ so that $||T|| \leq 1 + \epsilon$ and $||T^{-1}|| \leq \frac{1}{1 - \epsilon}$, then the above lemma shows that
A space is not $B$-convex if and only if for arbitrarily large $n$ and small $\epsilon$ there is an $\epsilon$ isometry from some subspace to $\ell_1^n$.

Several theorems of Giesy on the preservation of $B$-convexity under various operations are listed below.

1.11 Theorem: $X$ is $k$, $\epsilon$-convex if and only if $X^{**}$ is $k$, $\epsilon$-convex.

1.12 Theorem: $X$ is $B$-convex if and only if $X^*$ is $B$-convex.

In this theorem the $k$ and $\epsilon$ for $X$ and $X^*$ may be different.

1.13 Theorem: If $X$ is $B$-convex, and $Y$ is isomorphic to $X$, then $Y$ is also $B$-convex.

1.14 Theorem: If $Z$ is a closed subspace of $X$, then $X$ is $B$-convex if and only if $Z$ and $X/Z$ are $B$-convex.

1.15 Theorem: If $X$ is the linear span of two of its closed subspaces $Y$ and $Z$, then $X$ is $B$-convex if and only if $Y$ and $Z$ are $B$-convex.

1.16 Theorem: $X$ is $B$-convex if and only if each separable closed subspace of $X$ is $B$-convex.

As was mentioned in the introduction, James has conjectured that all $B$-convex spaces are reflexive. Theorem 1.11 above shows that if $X$ is $B$-convex, $X^{**}$ is not only $B$-convex, but is so with the same $k$ and $\epsilon$. Theorems 1.12 through 1.16 remain valid if the words "$B$-convexity" are replaced with the word "reflexive". As has been mentioned before, the following have been proved.

1.17 Theorem: (James) If $X$ is 2, $\epsilon$-convex it is reflexive.

1.18 Theorem: (James and Giesy) If $X$ is $B$-convex and has an unconditional basis, it is reflexive.

The subcase $P$-convex will be discussed in Chapter IV.

Kottman [21] has proved that all $P$-convex spaces are reflexive.
Let $S$ be an index set, $X$ a Banach space of real valued functions of $S$, $\{X_s\}_{s \in S}$ a family of Banach spaces. As in [6], page 31, or [25], we define the product space $P \times X_s$ to be the space of all functions $x$ on $S$ so that

i) $x_s \in X_s$ for all $s \in S$

ii) The real valued function $\zeta$ defined on $S$ by $\zeta(s) = |x_s|$ is in $X$.

If $X$ satisfies the condition

iii) If $\xi \in X$, and $\eta$ is a real valued function on $S$ so that

$$|\eta(s)| \leq |\zeta(s)| \quad \text{for all } s \in S,$$

then $P \times X_s$ is a Banach space.

Giesy has shown

1.19 Theorem: Let $S$, $X$, $\{X_s\}$ be as above. If $X$ is also uniformly convex and $X_s$ is $k_s, \epsilon_s$-convex where $\sup k_s < \infty$, $\inf \epsilon_s > 0$, then $P \times X_s$ is $B$-convex.

Fact 1.7 listed before can be written in a form for sequences in a $k, \epsilon$-convex space.

1.20 Proposition: If $X$ is $k, \epsilon$-convex and $\{x_n\} \in U(X)$ then there is a sequence of signs $\{\sigma_n\}$ so that

$$\lim_{n \to \infty} \frac{\sigma_1 x_1 + \cdots + \sigma_{k^n-1} x_{k^n-1}}{k^n} = 0,$$

the proof given is similar to Giesy's proof of our Fact 1.7 ([12], Lemma 1.4).
Proof: We construct signs in groups \( \{\sigma_1, \ldots, \sigma_k\} \), \( \{\sigma_{k+1}, \ldots, \sigma_{k+2}\} \), \( \ldots \), \( \{\sigma_{n+1}, \ldots, \sigma_{n+1}\} \), by induction as follows.

Let

\[
y_1^1 = \frac{1}{k} (\xi_1^1 x_1 + \ldots + \xi_k^1 x_k), \quad y_2^1 = \frac{1}{k} (\xi_{k+1}^1 x_{k+1} + \ldots + \xi_{2k}^1 x_{2k})
\]

and so on where the signs \( \xi_1^1 \) are chosen so that

\[
|y_j^1| \leq 1 - \epsilon \text{ for all } j.
\]

Let \( \sigma_1 = \xi_1^1, \ldots, \sigma_k = \xi_k^1 \).

For the second induction step let

\[
y_1^2 = \frac{1}{k} (\xi_1^2 x_1 + \ldots + \xi_k^2 x_k), \quad y_2^2 = \frac{1}{k} (\xi_{k+1}^2 x_{k+1} + \ldots + \xi_{2k}^2 x_{2k})
\]

and so on. Since \( |\frac{1}{1-\epsilon} y_j^1| \leq 1 \) for each \( j \), the signs can be chosen so that \( |\frac{1}{1-\epsilon} y_j^2| \leq 1 - \epsilon \), so that \( |y_j^2| \leq (1 - \epsilon)^2 \). Further they can be chosen so that \( \xi_1^2 = +1 \).

We can write

\[
y_1^2 = \frac{1}{k^2} (\xi_1^2 \xi_1^1 x_1 + \ldots + \xi_k^2 \xi_k^1 x_k + \xi_{k+1}^2 \xi_{k+1}^1 x_{k+1} + \ldots + \xi_{2k}^2 \xi_{2k}^1 x_{2k})
\]

Since \( \xi_1^2 = +1 \), the signs associated with \( x_1, \ldots, x_k \) are \( \sigma_1, \ldots, \sigma_k \).

Let \( \sigma_{k+1}, \ldots, \sigma_{2k} \) be the signs associated with \( x_{k+1}, \ldots, x_{2k} \).

Continue the construction with \( \xi_i^1 = +1 \) for all \( i \), and

\[
|y_i^1| = |\frac{1}{k^i} (\sigma_1 x_1 + \ldots + \sigma_i x_i)| \leq (1 - \epsilon)^i
\]

which gives the desired result.

We close Chapter I with a listing of examples, most of which
are from the literature. The first group are of spaces which are B-convex. The first two were mentioned above and are included here for completeness.

a) All finite dimensional spaces are B-convex.
b) All uniformly convex spaces are 2, \( e \)-convex for some \( e \).
c) A space which is 2, \( e \)-convex but neither finite dimensional nor uniformly convex was described by Giesy. We give here a geometrical description of that example. For a more complete proof and other properties of a generalization of this see Giesy [12], page 142.

We construct a sequence of spaces \( X_n \) which are all 2, \( e \)-convex for the same \( e \), hence \( X = \bigoplus X_n \) is 2, \( e \)-convex. Referring to the definition as given on page 1, we show \( X \) is not uniformly convex by showing there is \( \eta > 0 \) so that for any \( \delta > 0 \) there are \( x, y \in U(X) \) so that both \( \frac{1}{2} \|x + y\| \geq 1 - \delta \) and \( \|x - y\| \geq \eta \). Let \( X_n \) be a sequence of two dimensional normed linear spaces so that, if \( C \) is the Euclidean unit circle and \( H \) is a regular hexagon inscribed in the circle,

\[
C \supset U(X_1) \supset U(X_2) \supset \ldots \supset H
\]

and \( \cap_{n=1}^{\infty} U(X_n) = H \).

Letting \( \| \cdot \|_n \) be the norm in \( X_n \); \( \| \cdot \|_C \) be the Euclidean norm, and \( \| \cdot \|_H \) be the hexagonal norm, \( \lim_{n \to \infty} \|x\|_n = \|x\|_H \) for all \( x \).

Take \( \varepsilon \) so that \( \frac{2}{3} < 1 - \varepsilon < 1 \). We show \( X_n \) is 2, \( e \)-convex by showing that if a square is inscribed in \( U(X_n) \), its side length is less than or equal to \( 2/\sqrt{3} \) in \( \| \cdot \|_n \). If a square is inscribed in \( U(X_n) \), it is inside \( C \) so its side length must be less than or equal to \( \sqrt{2} \) in \( \| \cdot \|_C \). Since \( U(X_n) \) is outside \( H \), and so outside a circle
of radius $\sqrt{3}/2$, we have $\|\cdot\|_n \leq \frac{2}{\sqrt{3}}\|\cdot\|_C$, so that the length of the
side of the square must be less than or equal to $2\sqrt{2/3}$ in $\|\cdot\|_n$.

To find $\eta$, choose any $x \neq y$ on one face of the hexagon. Since $U(X_n) \subset H$ we have $\|x\|_n, \|y\|_n \leq 1$ for all $n$. Since $C \supset U(X_n)$,
$\|x - y\|_n \geq \|x - y\|_C$; let $\eta = \|x - y\|_C$. Since $\lim_{n \to \infty} \|x + y\|_n = \|x + y\|_H = 2$, we have for $n$ sufficiently large, $\|x + y\|_n \geq 2 - 2\eta$.
Thus, regarding $x$ and $y$ in $X_n$, which is a subspace of $X$, we have the
required inequalities.

Remark: No space is known which is 2, $\varepsilon$-convex but not
isomorphic to any uniformly convex space. James [15] has conjectured
that any space isomorphic to a 2, $\varepsilon$-convex space is also isomorphic
to a uniformly convex space.

d) A space which is B-convex but neither 2, $\varepsilon$-convex for
any $\varepsilon$ nor finite dimensional is $X = P_{\ell_2} X_n$ where $X_n$ is $\ell_1^2$. Each $X_n$
is 3, 1/3-convex since it is 2 dimensional, so $X$ is 3, 1/3-convex.
But $X$ has a subspace which is not 2, $\varepsilon$-convex for any $\varepsilon$, namely $X_n$
for any $n$, so is not 2, $\varepsilon$-convex.

e) The spaces $\ell_{p,\lambda}$ defined by Lindenstrauss and Pelczynski
in [22] are B-convex if $1 < p < \infty$. For $1 \leq p \leq \infty$, $X$ is said to be an
$\ell_{p,\lambda}$ space if every finite dimensional subspace $B$ is contained in
another finite dimensional subspace $E$ so that $d(E, \ell_p^n) \leq \lambda$, where $n$ is
the dimension of $E$, and $d(X,Y)$ is defined to be the infimum of
$\|T\|\cdot\|T^{-1}\|$ for all linear bounded invertible mappings $T$ from $X$ into
$Y$.

That these spaces are B-convex for $1 < p < \infty$ follows directly
from Theorem 7.1 of [22], which shows that if $X$ is $\ell_{p,\lambda}$, then there
is a measure \( \mu \) and a complemented subspace of \( L_p(\mu) \) which is isomorphic to \( X \). It also can be seen by the following elementary proof. We first note that, since \( \ell^n_p \subset \ell_p \), there is a \( \delta \) (depending on \( p \)) so that \( \ell^n_p \) is 2,\( \delta \)-convex for all \( n \). Let \( X \) be \( \ell_p^\lambda \). Choose \( m, \varepsilon; 0<\varepsilon<1; \) so that 
\( \lambda(1-\delta)^m \leq 1-\varepsilon \). We will show \( X \) is \( k,\varepsilon \)-convex, where \( k = 2^m \).

Let \( \{x_i^k\}_{i=1}^k \subset U(X) \). Then there is a subspace \( E \supset [x_i^k]_{i=1}^k \) and there is an isomorphism \( T \) from \( E \) into \( \ell^n_p \) so that \( ||T|| \cdot ||T^{-1}|| \leq \lambda \). Without loss of generality, \( ||T|| = 1 \) and \( ||T^{-1}|| \leq \lambda \). Therefore 
\( \{Tx_i^k\}_{i=1}^k \subset U(\ell^n_p) \), so by Fact 1.7, there is a choice of signs \( \xi, \ldots, \xi_k \) so that 
\[
|| \sum_{i=1}^k \xi_i^{k}Tx_i^k || \leq k(1-\delta)^m,
\]
and
\[
|| \sum_{i=1}^k \xi_i^{k}x_i^k || \leq \lambda k(1-\delta)^m \leq k(1-\varepsilon).
\]

The second group of examples are not \( B \)-convex.

f) \( \ell^n_1 \) is not \( B \)-convex. Let \( x_1, \ldots, x_k \) be the first \( k \) usual norm 1 basis vectors. Then for any choice of signs
\[
|| \sum_{i=1}^k \xi_i^{k}x_i^k || = \sum_{i=1}^k ||\xi_i^{k}x_i^k || = k
\]
so that \( \ell^n_1 \) is not \( k,\varepsilon \)-convex for any \( \varepsilon \).

g) \( c_0 \) is not \( B \)-convex. Let \( e_i \) be the \( i \)th usual basis vector and let
Clearly $|x_i| = 1$, $i = 1, \ldots, k$. For any choice of signs $\{\xi_i\}_{i=1}^k$ there is some $j$, $1 \leq j \leq 2^k$, so that the sign on $e_j$ in the $x_i$ sum is $\pm 1$ for each $i$. Therefore

$$\sum_{i=1}^{k} \xi_i x_i \geq \sum_{i=1}^{k} e_j = k$$

so that $c_0$ is not $k, \varepsilon$-convex for any $\varepsilon$.

h) A space which is reflexive but not $B$-convex is $\ell_2 \times \ell_1^n$. Day [7] has shown that the product $P_{\ell_p} X_i$ is reflexive if each $X_i$ is reflexive, so that this space is reflexive. Since it contains isometric copies of $\ell_1^n$ for each $n$, it is not $B$-convex.

James [14] and Giesy [12] have both given other examples of spaces which are reflexive but not $2, \varepsilon$-convex, resp., $B$-convex.

i) A space which is locally uniformly convex but not $B$-convex
was given by Giesy. (A space is locally uniformly convex if for each
\(x, ||x|| = 1\), and each \(\varepsilon > 0\) there is a number \(\delta > 0\) depending on \(\varepsilon\) and \(x\) so that if \(||y|| = 1\) then either \(||x-y|| \leq \varepsilon\) or \(1/2||x+y|| \leq 1 - \delta.\)

The space \(P_{p_2}^{n_1}(\ell_p^{n_1})\) where \(p_1\| p_1\) and \(n_1\| n_1\). Lovaglia [23] has shown that this product of locally uniformly convex spaces is locally uniformly convex. Since the spaces \(\ell_p^{n_1}\) become arbitrarily good approximations of arbitrarily large dimensional \(\ell_1^n\), the space is not B-convex by Lemma 1.10.

j) Giesy and James have proved that the example of James in
[16] of a space which is isometric to its second dual but not reflexive
is also not B-convex. The proof of Giesy, which is rather complicated,
is described in [12].

k) All infinite dimensional \(\ell_1,\lambda\) spaces (see example f above
for definition) are not B-convex since Lindenstrauss and Pelczynski
have shown ([22] Prop. 7.3) that infinite dimensional \(\ell_1,\lambda\) spaces have
a subspace isomorphic to \(\ell_1^n\).

Remark: Because of the use of \(\ell_1^n\) in the definition of
\(\ell_1,\lambda\) and in Giesy's characterization of non-B-convexity (lemma 1.10),
one might ask whether all non-B-convex spaces are \(\ell_1,\lambda\). The answer
to this is no, as is shown by example.

\(\ell_1 \oplus \ell_2\) is not B-convex and not \(\ell_1,\lambda\). The space is not
B-convex since it contains \(\ell_1\). Suppose it is \(\ell_1,\lambda\). Since the property
\(\ell_1,\lambda\) for some \(\lambda\) is invariant under isomorphism we may assume \(\ell_2\) is a
subspace of \(\ell_1 \oplus \ell_2\). Take \(B=\ell_2\) so that \(\dim(B) = m\). There is a sub-
space \(E \supset B\) and an isomorphism \(T: \ell_1^n \rightarrow E\), for some \(n\), having
\(||T|| \cdot ||T^{-1}|| \leq \lambda\), and without loss of generality we may assume
Let the images in $E$ of the $\ell^1_1$ basis vectors be $x_1, \ldots, x_n$. These form a basis for $E$ and satisfy the inequality

$$\frac{1}{\lambda} \sum_{i=1}^{n} |a_i| \leq \left\| \sum_{i=1}^{n} a_i x_i \right\| \leq \sum_{i=1}^{n} |a_i|,$$

for all $\{a_i\}_{i=1}^{n}$. Since $B$ is an $m$ dimensional subspace of $\ell^2_2$ it is isometric to $\ell^2_m$ and we may let $e_1, \ldots, e_m$ be the usual $\ell^2_m$ basis vectors. If we write

$$e_i = \sum_{j=1}^{m} a_{ij} x_j, \quad i = 1, \ldots, m,$$

we have, by the above inequality,

$$1 = \left\| e_i \right\| = \left\| \sum_{j=1}^{n} a_{ij} x_j \right\| \leq \sum_{j=1}^{n} |a_{ij}|.$$

Therefore, using the inequality again,

$$\sqrt{m} = \left\| \sum_{i=1}^{m} e_i \right\| = \left\| \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_j \right\| \geq \frac{1}{\lambda} \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}| \geq \frac{m}{\lambda}.$$

Since this must be true for all $m$, we have a contradiction.
CHAPTER II

THE PROPERTY OPPOSITE TO B-CONVEXITY

In this chapter we give five conditions equivalent to non-B-convexity and two conditions sufficient for non-B-convexity.

As stated in Lemma 1.10, Giesy has shown that a space is non-B-convex if and only if for every positive integer \( n \geq 2 \) and every \( 1 > \epsilon > 0 \) there is a subspace \( \epsilon \)-isometric to \( \ell_1^n \). We weaken this condition to get

2.1 Proposition: A space \( X \) is not B-convex if and only if there is some \( 1 > \epsilon > 0 \) so that for every \( k \geq 2 \) there are \( x_1, \ldots, x_k \subset U(X) \) satisfying

\[
(1-\epsilon) \sum_{i=1}^{k} |\alpha_i| \leq \left| \sum_{i=1}^{k} \alpha_i x_i \right| \leq \sum_{i=1}^{k} |\alpha_i| ,
\]

for all scalars \( \alpha_1, \ldots, \alpha_k \).

Proof: If \( X \) is not B-convex by Lemma 1.10 this condition holds. To prove the converse suppose the condition holds but \( X \) is B-convex.

Then there is \( k \) and \( \delta > \epsilon \) so that \( X \) is \( k, \delta \)-convex by Lemma 1.7 or 1.9. If \( x_1, \ldots, x_k \) are as asserted above, there is some choice of signs \( \xi_1, \ldots, \xi_k \) so that

\[
k(1-\delta) \geq \left| \sum_{i=1}^{k} \xi_i x_i \right| \geq (1-\epsilon) \sum_{i=1}^{k} |\xi_i| = k(1-\epsilon) ,
\]

giving \( \delta \leq \epsilon \) which is a contradiction.
This proposition shows that if for some $\varepsilon$ there is an $\varepsilon$-isometric image of every $\ell_1^n$ in $X$, then for any $\varepsilon$, $n$ there is an $\varepsilon$-isometric image of $\ell_1^n$ in $X$. James [14] proved a similar lemma for $\ell_1$: If $X$ contains an isomorphic image of $\ell_1$ (hence an $\varepsilon$-isometric image of $\ell_1$ for some $\varepsilon$), then for any $\varepsilon$, $X$ contains an $\varepsilon$-isometric image of $\ell_1$. The proof of James’ lemma is quite different.

By using the result in Chapter I we get a family of conditions equivalent to non-B-convexity which involve both points and functionals.

2.2 Proposition: The following four conditions are equivalent to non-B-convexity:

1) For some $0 < \varepsilon < 1$, $0 < C$, $p$ a positive integer, and any $k \geq 2$, $U(X)$ contains $x_1, \ldots, x_k$ so that for any choice of signs $\xi_1, \ldots, \xi_k$, there is a functional $f$, $||f|| \leq C(\ln k)^p$, such that $f(\xi_i x_i) > \varepsilon$, $i = 1, \ldots, k$.

1') For some $0 < \varepsilon < 1$, and any $k \geq 2$, $U(X)$ contains $x_1, \ldots, x_k$ so that for any choice of signs $\xi_1, \ldots, \xi_k$, there is a functional $f$, $||f|| \leq 1$, such that $f(\xi_i x_i) > \varepsilon$, $i = 1, \ldots, k$.

2) For some $0 < \varepsilon < 1$, $0 < C$, $p$ a positive integer, and for any $k \geq 2$, $U(X^*)$ contains $f_1, \ldots, f_k$ so that for any choice of signs $\xi_1, \ldots, \xi_k$, there is an $x \in X$, $||x|| \leq C(\ln k)^p$, such that $f_i(\xi_i x) > \varepsilon$, $i = 1, \ldots, k$.

2') For some $0 < \varepsilon < 1$, and any $k \geq 2$, $U(X^*)$ contains $f_1, \ldots, f_k$ so that for any choice of signs $\xi_1, \ldots, \xi_k$, there is an $x \in X$, $||x|| \leq 1$, such that $f_i(\xi_i x) > \varepsilon$, $i = 1, \ldots, k$. 
Remark: Conditions (1) and (2) are weakenings of Conditions (1') and (2'). Conditions (2) and (2') are duals of Conditions (1) and (1'). Condition (2') will be used later in this chapter.

Condition (1') will be used as a lemma in Theorem 3.10.

Proof: We will show: X not B-convex → (1') → (1) → X not B-convex, and (1') → (2') → (2) → X not B-convex.

First let δ be any number so that 0 < δ < 1 and choose

ε < 1 - δ. Since X is not B-convex, by Lemma 1.10 we have (for any k) \( x_1, \ldots, x_k \subseteq U(X) \) so that \((1-δ) \sum_{i=1}^{k} |a_i| < \sum_{i=1}^{k} a_i x_i \) for all \( a_1, \ldots, a_k \).

Therefore, \( x_1, \ldots, x_k \) are linearly independent and there are \( f_1, \ldots, f_k \), functionals defined on \([x_i]_1 \) so that \( f_i(x_j) = δ_{i,j} \), the Kronecker delta. For a given set of signs \( ε_1, \ldots, ε_k \) let \( f \) be an extension to \( X \) without increase of norm of the functional \((1-δ) \sum_{i=1}^{k} ε_i f_i \). Then

\[
||f|| = \sup \{ ||f(x)|| : x \in U([x_i]_1) \}.
\]

If \( x \in U([x_i]_1) \), we have \( x = \sum_{i=1}^{k} a_i x_i \) for some \( a_1, \ldots, a_k \) so that \((1-δ) \sum_{i=1}^{k} |a_i| < 1 \). For such \( x \)

\[
|f(x)| = (1-δ) \left( \sum_{j=1}^{k} ε_j f_j \right) \left( \sum_{i=1}^{k} a_i x_i \right) ≤ (1-δ) \sum_{i=1}^{k} |a_i| < 1,
\]

so \( ||f|| ≤ 1 \).

To see that (1') implies (1) let \( C = \frac{1}{\ln 2}, \ p = 1 \). Then for \( k \geq 2 \), \( ||f|| = \frac{C (\ln k)^p}{\ln 2} \).

To see that (1) implies that X is not B-convex, suppose X is \( n, δ \)-convex for some \( n, δ \). With the C, p of (1), \( \lim_{m \to \infty} (1-δ)^m C (\ln n)^p = 0 \).

Therefore with \( ε \) of (1), m can be chosen so that \((1-δ)^m C (\ln n)^p < ε \).
Let \( x_1, \ldots, x_m \) be as given by (1). By Lemma 1.7 there is some set of signs \( \xi_1, \ldots, \xi_m \) so that
\[
|| \sum_{i=1}^{n} \xi_i x_i || < n^m (1-\delta)^m .
\]

Therefore, for the functional \( f \) asserted by (1),
\[
n^m < f \left( \sum_{i=1}^{n} \xi_i x_i \right) \leq ||f|| \sum_{i=1}^{n} ||\xi_i x_i|| \leq C(n^m)^p \cdot n^m (1-\delta)^m < n^m ,
\]
which is a contradiction.

We now have (1') is equivalent to non-B-convexity. Using this we show (1') implies (2'). If \( X \) satisfies (1') then it is not B-convex; by Theorem 1.12 \( X^* \) is not B-convex so (1') holds for \( X^* \).

Therefore, for some \( 0 < \varepsilon < 1 \) and any \( k \geq 2 \), \( U(X^*) \) contains \( f_1, \ldots, f_k \) so that for any choice of signs \( \xi_1, \ldots, \xi_k \) there is \( \varphi \in X^{**} \), \( ||\varphi|| \leq 1 \) such that \( \varphi(f_i) > \varepsilon, i = 1, \ldots, k \).

The set \( \{ \psi \in X^{**} : |\psi(f_1) - \varphi(f_1)| < \frac{\varepsilon}{2}, i = 1, \ldots, k \} \)
is an \( X^* \) neighborhood of \( \varphi \) in \( X^{**} \). Since the usual imbedding of \( U(X) \) in \( U(X^{**}) \) is dense in the \( X^* \) topology of \( X^{**} \), there is some \( x \in U(X) \) so that its image is in this neighborhood, i.e.,
\[
|\xi_1 f_1(x) - \varphi(\xi_1 f_1)| < \frac{\varepsilon}{2}, i = 1, \ldots, k,
\]
therefore \( f_1(\xi_1 x) > \frac{\varepsilon}{2}, i = 1, \ldots, k \).

(2) We use the terminology of Dunford and Schwartz, [10].
To see that (2') implies (2) let \( C = \frac{1}{\ln 2}, \ p = 1 \).

To see that (2) implies non-B-convexity note that if \( X \) has property (2) then \( X^* \) has property (1) by letting the required functional in \( X^{**} \) be the natural imbedding of the point \( X \) asserted by (2). Thus \( X^* \) is not B-convex and by Theorem 1.12 \( X \) is not B-convex.

In the proof that \( 2, \varepsilon \)-convex spaces are reflexive, James defines for a Banach space \( X \) a sequence of numbers \( K_n \). He shows that if \( X \) is not reflexive then \( K_n \leq 2n \), and in that case \( X \) cannot be \( 2, \varepsilon \)-convex. The next result is an extension of the methods of the second part of James' result to show essentially that if \( K_n \) is a bounded sequence, then \( X \) cannot be B-convex. For reasons associated with Condition (3) below, we change slightly the definition of \( K_n \) and call the redefined sequence \( K'_n \).

After the result discussed above, we will present another condition, to be called Condition (3), which is a strengthening of the condition \( K_n' \) bounded. Condition (3) will trivially imply Condition (2'), and so will imply non-B-convexity in a much simpler way.

### 2.3 Definition: For \( \{f_i\}_{i=1}^{\infty} \subset U(X^*), \ p_1, \ldots, p_{2n} \) an increasing set of positive integers, let

\[
S(p_1, \ldots, p_{2n}; \{f_i\}) = \{x: (-1)^{i-1}f_k(x) \geq \frac{3}{4} \text{ for } p_{2i-1} \leq k \leq p_{2i}, i=1, \ldots, n\},
\]

\[
s_{p_1, \ldots, p_{2n}}[f_i] = \inf \{||x||: x \in S(p_1, \ldots, p_{2n}, \{f_i\})\}.
\]
(If $S(p_1, \ldots, p_{2n}; [f_1]) = \emptyset$ then let $s_{p_1, \ldots, p_{2n}}[f_1] = \infty$)

$$K(n, [f_1]) = \lim_{p_1 \to \infty} \ldots \lim_{p_{2n} \to \infty} s_{p_1, \ldots, p_{2n}}[f_1]$$

and

$$K_n' = \inf \{K(n, [f_1]) : [f_1] \subset U(X^*)\}.$$

James definition of $K_n$ is the same except that he lets

$$S(p_1, \ldots, p_{2n}; [f_1]) = \{x : \sum_{i=1}^{n} \frac{1}{4} (-1)^{i-1} f_k(x) \leq 1 \text{ for } p_{2i-1} \leq k \leq p_{2i}, i = 1, \ldots, n\},$$

and requires $||f||_1 = 1$, $i = 1, 2, \ldots$.

Clearly $K_n' \leq K_n$.

2.4 Lemma: For any $X$, $n$, $K_n' \leq K_{n+1}$.

Proof: From the definitions, for any $p_1, \ldots, p_{2n+2}; [f_1]$,

$$s_{p_1, \ldots, p_{2n}; [f_1]} \leq s_{p_1, \ldots, p_{2n+2}; [f_1]}.$$

Taking 2n limits, for any $p_1, p_2$ we have

$$\lim_{p_1 \to \infty} \lim_{p_{2n} \to \infty} s_{p_1, \ldots, p_{2n}; [f_1]} \leq \lim_{p_3 \to \infty} \lim_{p_{2n+2} \to \infty} s_{p_1, \ldots, p_{2n+2}; [f_1]},$$

so that $K(n, [f_1]) \leq K(n+1, [f_1])$; $K_n' \leq K_{n+1}$.
2.5 Theorem: Let \( K'_n \) be defined for a space \( X \) as in Definition 2.3. If there is a number \( M \) so that \( K'_n \leq M \) for all \( n \), then \( X \) is not \( B \)convex.

Proof: Given \( k, \delta \), we will show that \( X \) is not \( k, \delta \)-convex by showing there are \( x_1, \ldots, x_k \subseteq U(X) \) so that for any choice of signs \( \varepsilon_1, \ldots, \varepsilon_k \) we have

\[
\left| \sum_{i=1}^{k} \varepsilon_i x_i \right| > k(1-\delta) .
\]

Choose \( m \) so that

\[
\frac{K'_m}{K'_{3 \cdot 2^k m}} > 1 - \frac{\delta}{3} .
\] (2.5-1)

This is possible since the sequence \( K'_n \) is bounded and monotone.

Choose \( \mu \) so that

\[
0 < \mu < \frac{\delta (K'_m)^2}{3K'_{2^k m}} .
\] (2.5-2)

Choose \( \{f_i\}_{i=1}^{\infty} \subseteq U(X^*) \) so that

\[
K'_{3 \cdot 2^k m} + \mu > K(3 \cdot 2^k m, \{f_i\}) .
\] (2.5-3)

Choose \( \varepsilon \) so that

\[
0 < \varepsilon < \frac{\delta (K'_m)^2}{3(K'_{2^k m} + K'_m)} .
\] (2.5-4)

Using these four inequalities we obtain...
\[
\frac{K(2m, \{f_1\}) - \epsilon}{K(3 \cdot 2^k m, \{f_1\}) + \epsilon} \geq \frac{K_{2m} - \epsilon}{K(3 \cdot 2^k m, \{f_1\}) + \epsilon}
\]

(by inequality 2.5-3)

\[
> \frac{K'_{2m} - \epsilon}{K' \cdot 3 \cdot 2^k m} + \mu + \epsilon
\]

\[
= \frac{K'_{2m}}{K' \cdot 3 \cdot 2^k m} + \frac{K' \cdot 3 \cdot 2^k m}{K' \cdot 3 \cdot 2^k m} + \frac{eK'_{2m} - K'_{2m} \cdot 3 \cdot 2^k m}{K' \cdot 3 \cdot 2^k m} - \frac{K_{2m} - \epsilon}{K' \cdot 3 \cdot 2^k m}
\]

(by inequalities 2.5-1, 4 and 3)

\[
> 1 - \frac{\delta}{3} - \left( \frac{\delta}{3} \cdot \frac{K' \cdot 2^k m}{K' \cdot 3 \cdot 2^k m} \right) \left( \frac{K' \cdot 3 \cdot 2^k m + K'_{2m}}{K' \cdot 3 \cdot 2^k m} \right) - \left( \frac{\delta}{3} \cdot \frac{K' \cdot 3 \cdot 2^k m}{K' \cdot 3 \cdot 2^k m} \right) \left( \frac{K_{2m} - \epsilon}{K' \cdot 3 \cdot 2^k m} \right)
\]

We will choose \( k \) increasing sets of integers

\[
\{p_1^i, \ldots, p_{6 \cdot 2^k m}^i\}, \quad i = 1, \ldots, k,
\]

having the following properties:
(1) For each $i = 1, \ldots, k$ there is $u_i \in S(p_1^i, \ldots, p_{6 \cdot 2^k m}^i; [f_i])$ of minimal norm, that is, $|u_i| \leq K(3 \cdot 2^{k m}, [f_i]) + \epsilon$.

(2) For each choice of signs $\varepsilon_1, \ldots, \varepsilon_k$ there is an increasing set of integers

$$\{\sigma_1, \ldots, \sigma_{4m} \} \subset \{ p_j^i : j = 1, \ldots, 6 \cdot 2^{k m} ; i = 1, \ldots, k \},$$

depending on the choice of signs, so that

$$(2a) \frac{1}{k} \sum_{i=1}^{k} \varepsilon_i u_i \in S(\sigma_1, \ldots, \sigma_{4m}; [f_i]) ,$$

and so that

$$\text{(2b) any element of } S(\sigma_1, \ldots, \sigma_{4m}; [f_i]) \text{ will have norm greater than } K(2m, [f_i]) - \epsilon .$$

The procedure for choosing these integers will be described later. Having chosen the integers, let

$$x_i = \frac{1}{K(3 \cdot 2^{k m}, [f_i]) + \epsilon} u_i , \quad i = 1, \ldots, k .$$

By (1) above $||x_i|| \leq 1$. By (2) above, for any choice of signs we have

$$\frac{1}{k} \sum_{i=1}^{k} \varepsilon_i x_i \geq \frac{K(2m, [f_i]) - \epsilon}{K(3 \cdot 2^{k m}, [f_i]) + \epsilon} > 1 - \delta ,$$

which will prove the theorem.
We now describe the choice of the integers 
\[ \{p_j^i : j = 1, \ldots, 6 \cdot 2^k \cdot m ; i = 1, \ldots, k \} \]. They will be chosen in \( m \) blocks of \( 6 \cdot 2^k \cdot k \) integers each. The first of these blocks will be 
\[ \{p_j^i : j = 1, \ldots, 6 \cdot 2^k, i = 1, \ldots, k \} \]. The choice for this first block will be described. The other \( m-1 \) blocks of integers are chosen in the same way, choosing the integers of each block larger than those already chosen.

We now describe the choice of integers for the first block. There are \( 2^k \) choices of signs \( \varepsilon_1, \ldots, \varepsilon_k \). We will successively choose a set of integers for each choice of signs. The number of integers chosen for each choice of signs depends on the signs; \( 4k \) are chosen for each plus sign and \( 8k \) for each minus sign. Since \( \varepsilon_1 \) is plus for exactly half of the choices, and so on for \( \varepsilon_2, \ldots, \varepsilon_k \), we will choose altogether (in the first block) \( 6 \cdot k \cdot 2^k \) integers.

We first show in detail the choice of integers for the signs \( \varepsilon_1 = \ldots = \varepsilon_k = +1 \). In each step below, the chosen integer is to be larger than all previously chosen integers. Choose \( p_1 \) so that

\[
\lim_{r_2} \ldots \lim_{r} \left( \frac{s_1}{p_1}, \ldots, \frac{s_r}{r} \right)_{6 \cdot 2^k \cdot m} \leq k(3 \cdot 2^k \cdot m, \{\varepsilon_1\}) + \frac{\varepsilon}{6 \cdot 2^k \cdot m}
\]

(2.5-5)

(To simplify notation we omit "\( \lim \)" from the limit inf and "\( \{\varepsilon_1\} \)" from the index of \( s \).)

Choose \( p_i^1, i = 2, \ldots, k - 1 \) in increasing order so that 2.5-5 holds with \( p_1^1 \) replacing \( p_1 \). Choose \( p_k^1 \) so that 2.5-5 holds
with $p_1^k$ replacing $p_1^1$, and also so that

$$\lim_{r_2} \cdots \lim_{r_{4m}} \left( \frac{s_{p_1^k r_2, \ldots, r_{4m}}}{r_2 \cdots r_{4m}} \right) \geq K(2m, \lfloor f_1 \rfloor) - \frac{\varepsilon}{4m} \quad \quad \quad (2.5-6)$$

Choose $p_2^1$ so that

$$\lim_{r_3} \cdots \lim_{r_{4m}} \left( \frac{s_{p_1^1 p_2^1 r_3, \ldots, r_{4m}}}{r_3 \cdots r_{4m}} \right) \leq K(3 \cdot 2^{k-1}, \lfloor f_1 \rfloor) + \frac{2\varepsilon}{6 \cdot 2^{k-1}} \quad \quad \quad (2.5-7)$$

and so that

$$\lim_{r_3} \cdots \lim_{r_{4m}} \left( \frac{s_{p_1^1 p_2^1 r_3, \ldots, r_{4m}}}{r_3 \cdots r_{4m}} \right) \geq K(2m, \lfloor f_1 \rfloor) - \frac{\varepsilon}{4m} \quad \quad \quad (2.5-8)$$

Choose $p_2^i, i = 2, \ldots, k$ so that inequality 2.5-7 is satisfied with $p_1^i, p_2^i$, replacing $p_1^1, p_2^1$. Choose $p_3^i, i = 1, \ldots, k-1$, so that

$$\lim_{r_4} \cdots \lim_{r_{4m}} \left( \frac{s_{p_1^1 p_2^1 p_3^i r_4, \ldots, r_{4m}}}{r_4 \cdots r_{4m}} \right) \leq K(3 \cdot 2^{k-1}, \lfloor f_1 \rfloor) + \frac{3\varepsilon}{6 \cdot 2^{k-1}} \quad \quad \quad (2.5-9)$$

Choose $p_3^k$ so that inequality 2.5-9 holds and also so that

$$\lim_{r_4} \cdots \lim_{r_{4m}} \left( \frac{s_{p_1^1 p_2^1 p_3^k r_4, \ldots, r_{4m}}}{r_4 \cdots r_{4m}} \right) \geq K(2m, \lfloor f_1 \rfloor) - \frac{3\varepsilon}{4m} \quad \quad \quad (2.5-10)$$
Choose $p_4$ so that

\[
\lim_{r_5} \ldots \lim_{r_{6m}} \left( s_{p_1, p_2, p_3, p_4, r_5, \ldots, r_{6m}} \right) \leq K(3 \cdot 2^k \cdot m, \llbracket f \rrbracket) + \frac{4\varepsilon}{6 \cdot 2^k \cdot m},
\]

(2.5-11)

and so that

\[
\lim_{r_5} \ldots \lim_{r_{4m}} \left( s_{p_1, p_2, p_3, p_4, r_5, \ldots, r_{4m}} \right) \leq K(2m, \llbracket f \rrbracket) - \frac{4\varepsilon}{4m}.
\]

(2.5-12)

Choose $p_i^i$, $i = 2, \ldots, k$ so that inequality 2.5-11 is satisfied if $p_1, \ldots, p_4^i$ replaces $p_1^1, \ldots, p_4^1$.

This completes the step for $\xi_1 = \ldots = \xi_k = +1$.

Inequalities 2.5-5, 2.5-7, 2.5-9, and 2.5-11 are chosen so that (1) will hold. To see how (2) will hold let

\[
\sigma_1 = p_1^k, \quad \sigma_2 = p_2^1, \quad \sigma_3 = p_3^k, \quad \sigma_4 = p_4^1.
\]

Inequalities 2.5-6, 2.5-8, 2.5-10, and 2.5-12 will provide (2b).

We will see that (2a) follows from the order of choosing used. The order of choosing can be illustrated by the diagram below.
Illustration of choice of integers $p_j^i, 1 = 1, \ldots, k; j = 1, \ldots, 4$

for all plus signs.

In this diagram, $a$ to the right of $b$ means $a > b$.

To see why (2a) will hold (after completing all $m$ blocks), note that if

$$u_1 \in S(p_1^i, \ldots, p_4^i, r_5, \ldots, r_{6 \cdot 2^k m}, \{f_i\}),$$

then $f_j(u_1) \geq \frac{3}{4}$ for $p_1^i \leq j \leq p_2^i$ and $-f_j(u_1) \geq \frac{3}{4}$ for $p_3^i \leq j \leq p_4^i$.

Hence if $\sigma_1 = p_1^k \leq j \leq p_2^1 = \sigma_2$ we have $f_j(u_1) \geq \frac{3}{4}$ for all $i$

and $f_j \left( \frac{1}{k} \sum_{i=1}^{k} U_i \right) \geq \frac{3}{4}$. Also if $\sigma_3 = p_3^k \leq j \leq p_4^1 = \sigma_4$,

$-f_j(U_1) \geq \frac{3}{4}$ for all $i$ and $-f_k \left( \frac{1}{k} \sum_{i=1}^{k} U_i \right) \geq \frac{3}{4}$.

Thus $\frac{1}{k} \sum_{i=1}^{k} u_i \in S(\sigma_1, \ldots, \sigma_4, \{f_i\})$. 
For choices of signs containing one or more minus signs the construction is modified so that (2a) will hold with the new choice of signs. The construction is tedious and the same ideas are used as above, so only the modifications required by the minus signs will be described.

For illustration, suppose the second choice of signs

\[ \varepsilon_1, \ldots, \varepsilon_k \text{ has } \varepsilon_{i_0} = -1 \text{ and } \varepsilon_i = +1 \text{ for } i \neq i_0 \]

Choose integers in the order indicated in the diagram on the following page, satisfying inequalities similar to those above.

For this choice of signs let \( \sigma_1 = p_5 \), \( \sigma_2 = p_6 \), \( \sigma_3 = p_7 \), \( \sigma_4 = p_8 \). By using inequalities similar to those above, (1) and (2b) can be provided. To see why (2a) will hold, suppose that, for \( i \neq i_0 \), \( u_{i_0} \in S(r_1, \ldots, r_4, p_5, \ldots, p_8, r_9, \ldots, r \}_{6 \cdot 2^m} \)

\[ u_{i_0} \in S \text{ for some } r_1, \ldots, r_4, r_5, \ldots, r_8 \text{ and } \sigma_1 = p_5, \sigma_2 = p_6, \sigma_3 = p_7, \sigma_4 = p_8. \]

If \( p_5 \leq j \leq p_6 \) we have \( f_j(u_{i_0}) \geq 3/4 \) and if \( p_7 \leq j \leq p_8 \) we have \(-f_j(u_{i_0}) \geq 3/4\). On the other hand, if \( i = i_0 \) and \( u_{i_0} \in S(r_1, \ldots, r_4, p_5, \ldots, p_8, r_9, \ldots, r, \ldots, r_{6 \cdot 2^m}) \)

\[ f_j(u_{i_0}) \geq 3/4 \text{ and for } p_7 \leq j \leq p_8 \text{ we have } -f_j(u_{i_0}) = f_j(\xi_{i_0} u_{i_0}) \geq 3/4. \]

Therefore we have, if \( \sigma_1 \leq j \leq \sigma_2 \), \( f_j(\xi_{i_0} u_{i_0}) \geq 3/4 \) for all \( i \) and \( f_j(\varepsilon_{i_0} u_{i_0}) \geq 3/4 \).

If \( \sigma_3 \leq j \leq \sigma_4 \) we have \(-f_j(\xi_{i_0} u_{i_0}) \geq 3/4 \) for all \( i \) so \(-f_j(\varepsilon_{i_0} u_{i_0}) \geq 3/4 \).

Thus \( \frac{1}{k} \sum_{i=1}^{k} \varepsilon_i u_i \in S(\sigma_1, \ldots, \sigma_4; \{f_i\}) \).
Illustration of Choice of Integers

\[ p_j^i \text{ for } \xi_i = -1, \xi_i = +1 \text{ for } i \neq i_0 \]
The additional integers \( p_{11}, p_{12} \) are provided so that if \( p_o \geq p_8 \) and \( j \geq p_o \) than \( f_i(u_i) \geq 3/4 \) for all \( i \). Thus integers for other choices of sign may be chosen by the same method. This completes the proof of Theorem 2.5.

We now present Condition (3).

(3) For some \( 0 < \varepsilon < 1 \), \( U(X^*) \) contains a sequence \( \{f_i\}_{i=1}^{\infty} \) so that for any \( k \geq 1 \) and any choice of signs \( \xi_1, \ldots, \xi_k \) there is \( x \in X \), \( ||x|| \leq \varepsilon \), such that \( f_i(\xi_1 x_1^i) > \varepsilon, i = 1, \ldots, k \).

As discussed before Theorem 2.5, we will see that Condition (3) is a strengthening of \( K_n' \) bounded. It can also be seen that Condition (3) is an "infinite version" of Condition (2'), and so Condition (3) implies non-B-convexity in a much simpler way than the condition \( K_n' \) bounded.

2.6 Theorem: If \( X \) satisfies Condition (3), then it also satisfies Condition (2') and so is not B-convex. In addition \( K_n' \) is bounded.

Proof: The first statement is obvious by inspection and by Proposition 2.2.

To show the second it is sufficient to show that for \( \{f_i\} \) and any \( p_1, \ldots, p_{2n}, p_1, \ldots, p_{2n}, \{f_i\} \leq \frac{3}{4\varepsilon} \). This will imply that \( K_n' \leq \frac{3}{4\varepsilon} \). Given \( p_1, \ldots, p_{2n} \), choose \( \xi_1, i = 1, \ldots, p_{2n} \) so that if \( p_{2j-1} < i \leq p_{2j}, j=1, \ldots, n \) then \( \xi_i = (-1)^{j-1} \). Let \( x \) be as asserted by Condition (3). Let \( y = \frac{3}{4\varepsilon} x \). Then if \( p_{2j-1} < i \leq p_{2j}, j=1, \ldots, n \), then \((-1)^{j-1}f_i(y) > \frac{3}{4} \), so that \( y e S(p_1, \ldots, p_{2n}, \{f_i\}) \) and \( ||y|| \leq \frac{3}{4\varepsilon} \).
We do not know whether non-reflexivity implies $K_n'$ bounded or Condition (3). It would also be of interest to know whether $K_n'$ bounded or Condition (3) is equivalent to non-B-convexity. This would be true if Condition (2') implied Condition (3).

We do have, however, a large class of examples which satisfy Condition (3).

2.7 Proposition: The space $c_0$ satisfies Condition (3).

Proof: Take any $0 < \varepsilon < 1$. Let $(e^i, f^i)$ be the usual biorthogonal system in $c_0$, where $f^i(e^j) = \delta_{ij}$. Clearly $||f^i|| = 1$.

Given a set of signs $\xi_1, \ldots, \xi_k$, let

$$x = (\xi_1, \ldots, \xi_k, 0, \ldots) \in c_0.$$ 

Then $||x|| \leq 1$ and

$$f^i(\xi_i x_i) = 1 > \varepsilon \text{ for all } i = 1, \ldots, k.$$ 

2.8 Proposition: The space $l_1$ satisfies Condition (3).

Proof: Take any $0 < \varepsilon < 1$. Let $(e^i, g^i)$ be the usual biorthogonal system in $l_1$, where $g^i(e^j) = \delta_{ij}$. Let

$$f_1 = g_1 - g_2 + g_3 - g_4 + g_5 - \ldots$$

$$f_2 = g_1 + g_2 - g_3 - g_4 + g_5 + \ldots$$

$$f_3 = g_1 + g_2 + g_3 + g_4 - g_5 - \ldots$$

$\ldots$
Clearly \[ |f_i| = 1, \ i=1,2,\ldots. \]

For any choice of signs \( \xi_1,\ldots,\xi_k \) there is some \( j \) so that the sign on \( g_j \) in the sum for \( f_i \) is \( \xi_j \). Let \( x = e_j \). Then \( f_i(x) = \xi_i \), so \( f_i(x) = 1 > e \) for \( i=1,\ldots,k \).

2.9 Proposition: If \( Y \) satisfies Condition (3) and \( X \supset Y \) then \( X \) satisfies Condition (3).

**Proof:** Extend the asserted functionals in \( U(Y^*) \) to all of \( X \) by Hahn Banach theorem.

These propositions show any Banach space containing \( c_0 \) or \( \ell_1 \) satisfies Condition (3). All of these spaces are non-B-convex and non-reflexive (since they contain a non-B-convex space and a non-reflexive subspace).
CHAPTER III

B-CONVEXITY AND BASES

The main theorem of this chapter shows that $X$ is B-convex if every subspace having a Schauder basis is B-convex. Following this we include a different proof of the known fact that every B-convex space with unconditional basis is reflexive.

3.1 Definition: A sequence $\{x_i\} \subset X$ is said to be a (Schauder) basis for $X$ if for each $x_0 \in X$ there is a unique sequence of numbers $\{a_i\}$ so that

$$\lim_{n \to \infty} \left\| \sum_{i=1}^{n} a_i x_i - x_0 \right\| = 0 .$$

A sequence $\{x_i\}$ is said to be a basic sequence if it is a basis for its span $[x_i]$. We will use the following well known

3.2 Lemma: A sequence $\{x_i\}$ is a basis for $[x_i]$ if (and only if) there is some number $K$ so that for any integers $n$ and $q$ and any sequence of numbers $\{a_i\}$ we have

$$\left\| \sum_{i=1}^{n} a_i x_i \right\| \leq K \left\| \sum_{i=1}^{n+q} a_i x_i \right\| .$$

3.3 Remarks: We will prove the announced theorem, Theorem 3.9, by constructing a basic sequence in an arbitrary non-B-convex space in such a way that the span of this sequence is not B-convex.
The technique for construction of this basic sequence is an adaptation of the method of Day [8] and Gelbaum [11].

In the proof of Theorem 3.9 we shall need the following technical lemma.

3.4 Lemma: If \( \{\delta_i\} \) is a sequence of numbers, \( 1 > \delta_i \to 0 \), and \( p(m) \) is an increasing sequence of integers, then there are sequences \( \{\varepsilon_i\} \) and \( \{\eta_i\} \) so that

(1) \( \varepsilon_i \to 0 \)

(2) \( \eta_m \leq \delta_{p(m)} \)

(3) if \( 1 \leq n \leq p(1) \) we have

\[
\frac{1}{1-\varepsilon_1} \cdot (1 + \eta_1) \leq 1 + \delta_n
\]

(4) if \( p(m) < n \leq p(m+1) \) we have

\[
1 + \eta_m + \frac{1}{1-\varepsilon_{m+1}} \cdot (2 + \eta_m + \eta_{m+1}) \leq 3 + \delta_n
\]

Proof: Let \( \gamma_m = \inf \{ \delta_n : n = p(m) + 1, \ldots, p(m+1) \} \), \( m = 0, 1, \ldots \), where we let \( p(0) = 0 \).

Choose \( \varepsilon_i, i = 1, 2, \ldots \) positive decreasing to zero so that

\[
\varepsilon_1 \leq 1 - \frac{1}{\gamma_0} \cdot \frac{1}{1+\frac{\gamma_0}{4}}
\]

and
Then choose \( \eta_i, i=1,2,\ldots \), so that

\[
\varepsilon_m \leq \min \left[ \frac{1}{2}, 1 - \frac{1}{1 + \frac{\gamma_{m-1}}{6}} \right]. \tag{3.4-2}
\]

Then choose \( \eta_1, \) so that

\[
\eta_m \leq \min \left[ \frac{\gamma_{m-1}}{6}, \frac{\gamma_m}{9} \right]. \tag{3.4-3}
\]

Property (1) is clearly true. To see that property (2) holds, use inequality 3.4-3 to get

\[
\eta_m \leq \frac{\gamma_{m-1}}{6} \leq \frac{\delta_p(m)}{6} < \delta_p(m).
\]

To see that property (3) holds, note that

\[
1 - \varepsilon_1 > \frac{1}{1 + \frac{\gamma_0}{4}} \quad \text{by inequality 3.4-1, so that}
\]

\[
\frac{1}{1 - \varepsilon_1} < 1 + \frac{\gamma_0}{4}.
\]

By inequality 3.4-3, we have \( 1 + \eta_1 \leq 1 + \frac{\gamma_0}{6} < 1 + \frac{\gamma_0}{4} \),

so that, using the fact that \( \gamma_0 < 1 \),

\[
\frac{1}{1 - \varepsilon_1} (1 + \eta_1) < \left(1 + \frac{\gamma_0}{4}\right)^2 = 1 + \frac{\gamma_0}{2} + \frac{\gamma_0^2}{16} < 1 + \gamma_0 < 1 + \delta_n.
\]

For property (4), observe by inequality 3.4-2 that
$1 - \varepsilon_m \geq \frac{1}{2}$, so $\frac{1}{1 - \varepsilon_m} \leq 2$. Therefore, using inequality 3.4-3 we have

$$\eta_m \left(1 + \frac{1}{1 - \varepsilon_{m+1}}\right) \leq 3\eta_m \leq \frac{\gamma_m}{3}.$$  

Also by 3.4-3

$$\frac{\eta_{m+1}}{1 - \varepsilon_{m+1}} \leq 2\eta_{m+1} \leq \frac{\gamma_m}{1 + \delta}$$

and by 3.4-2, $1 - \varepsilon_{m+1} \geq \frac{1}{1 + \delta}$

so that

$$\frac{2}{1 - \varepsilon_{m+1}} \leq 2 + \frac{\gamma_m}{3}.$$  

Therefore

$$1 + \eta_m + \frac{1}{1 - \varepsilon_{m+1}} (2 + \eta_m + \eta_{m+1})$$

$$= 1 + \eta_m + \frac{\eta_m}{1 - \varepsilon_{m+1}} + \frac{\eta_{m+1}}{1 - \varepsilon_{m+1}} + \frac{2}{1 - \varepsilon_{m+1}}$$

$$\leq 1 + \frac{\gamma_m}{3} + \frac{\gamma_m}{3} + 2 + \frac{\gamma_m}{3} = 3 + \gamma_m \leq 3 + \delta_n.$$  

The following lemma is important to the Day-Gelbaum technique.

3.5 Lemma: If $X$ is finite dimensional, for any $\varepsilon > 0$ there are $\{f_i\}_{i=1}^n \subset U(X^*)$ such that for any $x$

$$||x|| \leq (1 + \varepsilon) \max \{f_i(x) : i=1,\ldots,n\}.$$
Proof: Let $S = \{ x : \| x \| = 1 \}$. For any $x \in S$ there is $f^*_x \in X^*$ so that $\| f^*_x \| = f^*_x (x) = 1$. The sets $S \cap f^*_x \left( \left( \frac{1}{1+\varepsilon} , \frac{1}{1-\varepsilon} \right) \right)$, $x \in S$, form a family of open (relative to $S$) sets covering $S$, which is compact, so that a finite subfamily of them cover $S$. Let the corresponding functionals be $\{ f^*_i \}_{i=1}^n$. Then for any $x \in X$ there is some $i$ so that

$$\frac{1}{\| x \|} x \in f^{-1}_i \left( \left( \frac{1}{1+\varepsilon} , \frac{1}{1-\varepsilon} \right) \right),$$

and so

$$f^*_i \left( \frac{1}{\| x \|} x \right) \geq \frac{1}{1+\varepsilon},$$

or

$$(1+\varepsilon) f^*_i (x) \geq \| x \|.$$

As usual, we have

3.6 Definition: The subspace $Y \subset X$ is said to be of finite codimension in $X$ if there is a finite dimensional subspace $Z$ so that $X = Y \oplus Z$.

We will need the following well known lemma. A proof for this lemma can be found in, for example, [11] Lemma 5a.

3.7 Lemma: If $\{ f^*_i \}_{i=1}^n \subset X^*$ and $Y = \bigcap_{i=1}^n f^{-1}_i (0)$ then $Y$ is a space of finite codimension.

A property of $B$-convexity needed for Theorem 3.9 is contained
in the following lemma, which is a simple corollary of a result of Giesy. In Chapter IV, Corollary 4.10, this property is shown to hold also for $P$-convexity by a quite different method.

**3.8 Lemma:** If $X$ is not $B$-convex, then all subspaces of finite codimension are not $B$-convex.

**Proof:** Let $X = Y \oplus Z$ where $Z$ is finite dimensional. If $Y$ is $B$-convex, by Theorem 1.15, $X$ is $B$-convex which is a contradiction.

We now present the main theorem: If all subspaces of $X$ having a basis are $B$-convex, then $X$ is $B$-convex. As noted earlier, this will be proved by showing any non-$B$-convex space has a basic sequence whose span is not $B$-convex. Certain additional facts about this basic sequence are revealed in the statement of the theorem.

**3.9 Theorem:** If $X$ is not $B$-convex, for any sequence of numbers $\{\delta_i\}, \delta_i \to 0$, and sequence of integers $\{k_i\}, k_i \to \infty$, there is a sequence $\{\varepsilon_i\}, \varepsilon_i \to 0$, a sequence of integers $p(m)$, and a sequence of vectors $\{x_i\} \subset U(X)$ so that:

1. The space $L = [x_i]_{i=1}^{\infty}$ is not $B$-convex, in particular, $\sum_{i=p(m)}^{k} |a_i| < \sum_{i=p(m-1)+1}^{\infty} |a_i|$, where $k \geq p(m)$ and $a_i x_i$ is an $\varepsilon$-isometric image of $\ell^1_m$.

1'. For each $m=1,2,\ldots$, the space $[x_i]_{i=1}^{p(m)+1}$ is an $\varepsilon$-isometric image of $\ell^1_m$ and an $\varepsilon$-isometric image of $x_i$.

and

2. $\{x_i\}$ is a basis for $L$, in particular,
(2') For any \( \{a_i\} \), \( n \), \( q \)

\[
\left\| \sum_{i=1}^{n} a_i x_i \right\| \leq (3 + \delta_n) \left\| \sum_{i=1}^{n+q} a_i x_i \right\|
\]

and if \( n = p(m) \), \( m=1,2,... \)

\[
\left\| \sum_{i=1}^{n} a_i x_i \right\| \leq (1 + \delta_n) \left\| \sum_{i=1}^{n+q} a_i x_i \right\|
\]

Proof: (1') implies (1) by Lemma 1.10 and (2') implies (2) by Lemma 3.2. Therefore it is sufficient to construct \( \{x_i\} \) so that (1') and (2') hold.

For \( m=1,2,... \) let \( p(m) = \sum_{i=1}^{m} k_i \) and let \( p(o) = o \). Let \( \eta_i, \varepsilon_i \) be as in Lemma 3.4. We will construct \( \{x_i\}_{i=p(m-1)+1}^{p(m)} \) by induction on \( m \). We denote \( \{x_i\}_{i=1}^{p(m)} \) by \( L_{p(m)} \). For each \( m \) we will choose \( \{x_i\}_{i=p(m-1)+1}^{p(m)} \) from a previously constructed subspace so that (1') is satisfied. We then will construct a subspace \( \Lambda_m \) so that \( \Lambda_m \cap L_{p(m)} = o \) and the projection \( P_{m} : \Lambda_m \oplus L_{p(m)} \rightarrow L_{p(m)} \) satisfies \( \left\| P_{m} \right\| \leq 1 + \eta_m \).

The \( x_i \) chosen in subsequent induction steps will be taken from \( \Lambda_m \).

Following the induction we will show that (2') is satisfied.

Let \( m = 1 \). Since \( X \) is not B-convex, we can choose \( k_1 = p(1) \)

\( \{x_i\}_{i=1}^{p(1)} \subset U(X) \) satisfying (1'). To construct \( \Lambda_1 \) choose

\( \{f_i\}_{i=1}^{q(1)} \subset U(L_{p(1)}) \) by Lemma 3.5 and extend them to \( X \) without increase of norm so that if \( x \in L_{p(1)} \),

\[
\left\| x \right\| \leq (1 + \eta_1) \max \{ f_i(x) : i=1,...,q(1) \}
\]
Let $\Lambda_1 = \bigcap_{i=1}^{\infty} f_i^{-1}(\{\cdot\})$. By Lemma 3.7, $\Lambda_1$ is of finite codimension.

$L_p(1) \cap \Lambda_1 = \emptyset$, since if $x \in L_p(1)$, $x \neq \emptyset$, then by the above inequality there is some $i$ so that $f_i(x) > 0$. Therefore there is a projection $P_1 : L_p(1) \otimes \Lambda_1 \to L_p(1)^*$. To see that $\|P_1\| \leq 1 + \eta_1$ we have for any $x \in L_p(1)^*$, $\lambda \in \Lambda_1$, and some $i = 1, \ldots, q(1)$, $\|P(x + \lambda)\| = \|x\| \leq (1 + \eta_1) f_i(x)$ 

$= (1 + \eta_1) f_i(x + \lambda) \leq (1 + \eta_1) \|x + \lambda\|$. 

Now suppose we have $\{x_i\}_{i=1}^{(m-1)}$ satisfying (1'),

$\|P_n\| \leq 1 + \eta_n$ where $P_n : \Lambda_n \oplus L_p(n) \to L_p(n)^*$. Since $\Lambda_m$ is of finite codimension, by Lemma 3.8 it is not B-convex so that there are $\{x_i\}_{i=q(1)+1}^{(m-1)+1} \subset U(X^*)$ and for $n = 1, 2, \ldots, m-1, \Lambda_n = \bigcap_{i=1}^{q(n)} f_i^{-1}(\{\cdot\})$, and

$\|P_m\| \leq 1 + \eta_m$.

To show (2') holds we first observe, for any $a_i, i=1, 2, \ldots$,

$$\|\sum_{i=1}^{p(m)} a_i x_i\| \leq (1 + \eta_m) \|\sum_{i=1}^{p(m)+q} a_i x_i\|$$

for any $q = 1, 2, \ldots$, and

since $\|\sum_{i=1}^{p(m)+q} a_i x_i\| \leq \sum_{i=1}^{p(m)+q} |a_i| |x_i|$ and $\|P_m\| \leq 1 + \eta_m$.

Further, we observe

B. If $p(m-1) < n \leq p(m)$ then

$$\|\sum_{i=p(m-1)+1}^{n} a_i x_i\| \leq \frac{1}{1 - \epsilon_n} \|\sum_{i=p(m-1)+1}^{p(m)} a_i x_i\|$$
since
\[ \sum_{i=p(m-1)+1}^{n} a_i x_i \leq \sum_{i=p(m-1)+1}^{n} |a_i| \leq \sum_{i=p(m-1)+1}^{p(m)} |a_i| \]

\[ \leq \frac{1}{1-\varepsilon_m} \sum_{i=p(m-1)+1}^{p(m)} a_i x_i \]

using the inequalities of (1').

We now prove (2') in four cases using A and B.

Case 1: 
\[ 1 \leq n < n+q \leq p(1) \]

Using B, with \( m=1 \), and Lemma 3.4-(3), we obtain
\[ \sum_{i=1}^{n} a_i x_i \leq \frac{1}{1-\varepsilon_1} \sum_{i=1}^{n+q} a_i x_i \leq (1-\delta_n) \sum_{i=1}^{n+q} a_i x_i. \]

Case 2: 
\[ 1 \leq n \leq p(1) < n+q. \]

Using B, A, where \( m=1 \), and Lemma 3.4-(3), we obtain
\[ \sum_{i=1}^{n} a_i x_i \leq \frac{1}{1-\varepsilon_1} \sum_{i=1}^{p(1)} a_i x_i \leq \frac{1}{1-\varepsilon_1} (1+\delta_n) \sum_{i=1}^{n+q} a_i x_i \]
\[ \leq (1+\delta_n) \sum_{i=1}^{n+q} a_i x_i. \]

Case 3: 
There is \( m \) so that \( p(m) < n < n+q < p(m+1) \).

Using A, and B with \( m \) replaced with \( m+1 \), we obtain
\[
\|
\sum_{i=1}^{n} a_i x_i \| \leq \|
\sum_{i=1}^{p(m)} a_i x_i \| + \|
\sum_{i=p(m)+1}^{n} a_i x_i \|
\]
\[
\leq (1+\eta_m) \|
\sum_{i=1}^{n} a_i x_i \| + \frac{1}{1-\epsilon} \|
\sum_{i=p(m)+1}^{n} a_i x_i \|
\]
\[
\leq (1+\eta_m) \|
\sum_{i=1}^{n} a_i x_i \| + \frac{1}{1-\epsilon} \|
\sum_{i=p(m)+1}^{n} a_i x_i \|
\]
\[
\leq (1+\eta_m) \|
\sum_{i=1}^{n} a_i x_i \| + \frac{1}{1-\epsilon} \|
\sum_{i=p(m)+1}^{n} a_i x_i \|\left(\|
\sum_{i=1}^{n} a_i x_i \| + \|
\sum_{i=1}^{n} a_i x_i \|\right)
\]

and using A and Lemma 3.4-(4) we continue

\[
\leq \left[\frac{1+\eta_m + \frac{1}{1-\epsilon}}{1-\epsilon} (1+1+\eta_m)\right] \|
\sum_{i=1}^{n} a_i x_i \|
\]
\[
\leq (3+\delta_n) \|
\sum_{i=1}^{n+q} a_i x_i \|
\]

\textbf{Case 4:} There is } m \text{ so that } p(m) < n < p(m+1) < n+q.

\text{Using A, and B with } m+1 \text{ replacing } m, \text{ we obtain}

\[
\|
\sum_{i=1}^{n} a_i x_i \| \leq \|
\sum_{i=1}^{p(m)} a_i x_i \| + \|
\sum_{i=p(m)+1}^{n} a_i x_i \|
\]
\[
\leq (1+\eta_m) \|
\sum_{i=1}^{n} a_i x_i \| + \frac{1}{1-\epsilon} \|
\sum_{i=p(m)+1}^{n} a_i x_i \|
\]
\[
\leq (1+\eta_m) \|
\sum_{i=1}^{n} a_i x_i \| + \frac{1}{1-\epsilon} \|
\sum_{i=p(m)+1}^{n} a_i x_i \|\left(\|
\sum_{i=1}^{n+q} a_i x_i \| + \|
\sum_{i=1}^{p(m)} a_i x_i \|\right),
\]
and using $A$ directly and with $m+1$ replacing $m$ we continue

$$\leq \left[ 1 + \eta_{m+1} \right] \left( 1 + \eta_{m+1} \right) \sum_{i=1}^{n+q} a_i x_i,$$

and finally by Lemma 3.4-(4),

$$\leq (3+\delta_n) \left( 3+6n \right) \sum_{i=1}^{n+q} a_i x_i,$$

proving the first part of (2'). The second part of (2') holds by using directly $A$ and Lemma 3.4-(2). This completes the proof of Theorem 3.9.

We now present the second theorem about B-convexity and basis.

**3.10 Theorem**: If $X$ has an unconditional basis and is B-convex, then it is reflexive.

This theorem has been proved by both James [14] and Giesy [12]. Both of these proofs use the fact, from James [16], that a non-reflexive space with an unconditional basis contains an isomorphic copy of either $c_0$ or $\ell_1$. Giesy uses the facts that $c_0$ and $\ell_1$ are not B-convex, that B-convexity is preserved by isomorphism, and that every subspace of a B-convex space is B-convex. James works directly from the B-convex definition using the lemma mentioned in Chapter II, namely, that the existence of an isomorphic copy of $\ell_1$ implies, for any $\epsilon$, the existence of an $\epsilon$-isometric copy of $\ell_1$, and a similar lemma for $c_0$.

The advantage of the proof presented here is that the unconditional property of the basis is used directly without invoking the theorem quoted above.
3.11 Definition: A basis \( \{x_i\} \) of \( X \) is said to be an unconditional basis if the convergence in Definition 3.1 is unconditional, that is, the limit is zero for any rearrangement of terms \( a_i x_i \).

\( \{x_i\} \) is said to be **boundedly complete** if \( \sum_{i=1}^{\infty} a_i x_i \) converges for each \( \{a_i\} \) such that \( \sup_n \left\| \sum_{i=1}^{n} a_i x_i \right\| < \infty \).

\( \{x_i\} \) is said to be **shrinking** if for each \( f \in X^* \), \( \lim_{n \to \infty} \|f\|_n = 0 \), where \( \|f\|_n \) is the norm of \( f \) on \( x_i \).

The following lemma contains the facts of basis theory we will use.

3.12 Lemma: Let \( \{x_i\} \) be a basis for \( X \) and \( \{f_i\} \) the associated biorthogonal sequence of functionals. Then

1. If \( X \) is not reflexive, either \( \{x_i\} \) is not shrinking or not boundedly complete,

2. If \( \{x_i\} \) is shrinking, then \( \{f_i\} \) is a boundedly complete basis of \( X^* \),

3. If \( \{x_i\} \) is shrinking and \( X \) is equivalently renormed, then \( \{x_i\} \) is shrinking in the new norm,

4. If \( \{x_i\} \) is not shrinking, there is \( \delta > 0 \), \( f \in X^* \), \( \{p(n)\} \), an increasing sequence of integers, and \( \{a_i\} \) so that if \( y_n = \sum_{i=p(n)+1}^{p(n+1)} a_i x_i \) then

\[
\|f\| = \|y_n\| = 1 \quad \text{and} \quad f(y_n) > \delta \quad \text{for} \quad n=1, 2, \ldots
\]

Proof: Statement (1) is Theorem 1 of James [16]. Statement (2) is Theorem 3 of the same paper.

To prove Statement (3) let the new norm \( \|\cdot\| \) be related to the old norm by \( k \|x\| \leq \|\cdot\| \leq K \|x\| \) for all \( x \in X \) then
\[ \{ x : ||x|| \leq 1 \} \supset \{ x : ||x|| \leq k \} \text{ so} \]

\[ ||f||_n = \sup \{|f(x)| : x \in [x_i]_{i=n}^{\infty}, ||x|| \leq 1 \} \]
\[ \geq \sup \{|f(x)| : x \in [x_i]_{i=n}^{\infty}, ||x|| \leq k \} \]
\[ = k \sup \{|f(x)| : x \in [x_i]_{i=n}^{\infty}, ||x|| \leq 1 \} = k ||f||_n \]

from which (3) follows directly.

To prove (4), since \([x_i]\) is not shrinking there is \(f\) so that \(\lim ||f||_n \neq 0\) and \(||f|| = 1\). Therefore there is \(p(1), p(2), \ldots, p(n)\) so that \(||z_1|| \leq 1 + \frac{\varepsilon}{4}\) and \(f(z_1) > \frac{\varepsilon}{4}\). By the same method there are \(z_2, z_3, \ldots\) with these properties and \(y_n = \frac{1}{||z_n||} z_n\) will satisfy (4).

The following lemma contains the facts about unconditional bases we will need.

3.13 Lemma: If \(X\) has unconditional basis \([x_i]\) with biorthogonal functionals \([f_i]\), we have

1. \(X\) can be equivalently renormed so that every rearrangement of \([x_i]\) is a monotone basis for \(X\). (A basis is said to be monotone if the \(K\) of Lemma 3.2 is 1.)

2. If \([x_i]\) is shrinking, \([f_i]\) is an unconditional basis for \(X^*\).

Proof: Statement (1) is Theorem IV-4-1 of Day [6] and statement (2) is Theorem 3c of James [16]. The proof of both of these theorems are based on elementary facts of unconditional basis theory.
The construction for the announced theorem is primarily in the following

**3.14 Lemma:** If $X$ has an unconditional, non-shrinking basis $\{x_i\}$, it is not B-convex.

**Proof:** Renorm $X$ so that every rearrangement of $\{x_i\}$ is a monotone basis by Lemma 3.13-(1). By Lemma 3.12-(3), $\{x_i\}$ is non-shrinking in the new norm so there is $f$, $y_n$, $p(n)$, as in Lemma 3.12-(4). We will construct, for any $k \geq 2$ and any choice of signs $\xi_1, \ldots, \xi_k$, a functional satisfying Proposition 2.2-(1'), namely, for this choice of signs there is a functional $g$ so that $||g|| \leq 1$ and $g(\xi_n y_n) > \frac{\delta}{3}$, $n = 1, \ldots, k$. By that proposition, this will show $X$ is not B-convex in the new norm, and consequently not in the original norm.

Given the signs $\xi_1, \ldots, \xi_k$, let $g$ be defined by

$$g(x_i) = \begin{cases} 
\frac{1}{3} \xi_n f(x_i) & \text{if } i = p(n)+1, \ldots, p(n+1), n=1, \ldots, k \\
\frac{1}{3} f(x_i) & \text{for other } i \\
p(n+1)
\end{cases}$$

Then $g(\xi_n y_n) = g(\xi_n \sum_{i=p(n)+1}^{p(n+1)} a_i x_i)$

$$= \frac{1}{3} f \left( \sum_{i=p(n)+1}^{p(n+1)} a_i x_i \right) > \frac{\delta}{3}$$

To show $||g|| \leq 1$ let

$$P = \{ i : p(n) < i \leq p(n+1) \text{ for some } n \text{ so that } \xi_n = +1 \}$$

$$N = \{ i : p(n) < i \leq p(n+1) \text{ for some } n \text{ so that } \xi_n = -1 \}$$

$$I = \{ i : i \notin P \cup N \}$$
Take any x so that $||x|| \leq 1$. Write $x = \sum_{i=1}^{\infty} a_i x_i$ and let

$$y = \sum_{i \in \mathcal{P} \cup \mathcal{I}} a_i x_i - \sum_{i \in \mathcal{N}} a_i x_i + \sum_{i \in \mathcal{I}} a_i x_i.$$ 

Since the rearrangement of $\{x_i\}$, $\{x_i\}_{i \in \mathcal{N}}$, $\{x_i\}_{i \in \mathcal{P} \cup \mathcal{I}}$ is a monotone basis, we have

$$||\sum_{i \in \mathcal{N}} a_i x_i|| \leq ||x|| \leq 1 \quad \text{so that}$$

$$||y|| = ||x - 2 \sum_{i \in \mathcal{N}} a_i x_i|| \leq 3$$

and $$|g(x)| = \frac{1}{3} |f(y)| \leq \frac{1}{3} ||f|| ||y|| \leq 1.$$

**Proof of Theorem 3.10:** If X is not reflexive, $\{x_i\}$ is either non-shrinking or shrinking and non-boundedly-complete. If the former, the proof is complete by Lemma 3.14. If the latter, the biorthogonal functionals form an unconditional non-shrinking basis for $X^*$ (unconditional by Lemma 3.13-(2), since $\{x_i\}$ is shrinking, and non-shrinking by Lemma 3.12-(2) since $X^*$ is not reflexive). Thus by Lemma 3.14, $X^*$ is not B-convex, so neither is X.
CHAPTER IV
P-CONVEXITY AND OTHER PROPERTIES RELATED TO B-CONVEXITY

In this chapter we consider three properties of Banach spaces, and the relations between these properties and B-convexity.

The first of these properties, P-convexity, was introduced in [20, 21] by C. Kottman.

4.1 Definition: For a Banach space \(X\), let \(P(n,X)\) be the supremum of all numbers \(r\) so that there is a set of \(n\) pairwise-disjoint closed balls of radius \(r\) inside \(U(X)\). We say \(X\) is P-convex if \(P(n,X) < \frac{1}{2}\) for some \(n\).

Note that \(P(n,X) \leq \frac{1}{2}\) if \(n > 1\) and \(P(2,X) = \frac{1}{2}\) for all \(X\).

Another way to express the property P-convexity is found in

4.2 Lemma: \(P(n,X) < \frac{1}{2}\) holds if and only if there is some \(\epsilon, \delta < \epsilon < 2\), so that if \(\{x_i\}_{i=1}^n\) are distinct points in \(U(X)\), then there are \(i \neq j\) so that \(||x_i - x_j|| < 2 - \epsilon\).

This is essentially Kottman's Corollary II.4 and Remark II.5.

A set \(\{x_i\}_{i=1}^n\) so that \(||x_i - x_j|| \geq 2 - \epsilon\) for all \(i, j=1, \ldots, n, i \neq j\), is called a 2 - \(\epsilon\) separated set of order \(n\). \(P(n,X) < \frac{1}{2}\) means that for some \(\epsilon\) there is no 2 - \(\epsilon\) separated set of order \(n\) in \(U(X)\).

Using this lemma it is easy to see (Kottman's Theorem IV.24) that if a space is P-convex it is also B-convex; in particular, if \(P(n,X) < \frac{1}{2}\) then \(X\) is \(n, \epsilon\)-convex for some \(\epsilon > 0\).

To see this observe that if \(x_1, \ldots, x_n\) are distinct points in
U(X), by Lemma 4.2 there is δ, i, j so that \(|x_i - x_j| < 2 - δ\).

Hence

\[
\sum_{m=1}^{n} x_m - x_j | \leq \sum_{m=1}^{n} |x_m| + |x_i - x_j| < n - δ
\]

Note that the signs on \(x_m, m \neq i, j\), are in fact completely arbitrary, so that for many different choices of signs the required inequality holds, including a choice with only one minus sign and a choice with only one change of signs.

It is not known whether there is a space which is P-convex but not B-convex. This problem will be discussed further in Chapter V.

Kottman proved that all P-convex spaces are reflexive. The proof uses the device of James discussed in Chapter II to show that if \(K_n \leq 2n\) then the space cannot be P-convex.

To compare P-convexity with B-convexity, we will discuss analogs of facts about B-convexity which are listed in Chapter I.

Analog to 1.3: If \(P(n, X) < \frac{1}{2}\) and \(Y \subseteq X\) then \(P(n, Y) < \frac{1}{2}\).
This is obvious using Lemma 4.2.

Analog to 1.4: If \(X\) is uniformly convex, then \(P(3, X) < \frac{1}{2}\).
This is Kottman's Theorem IV.31.

Analog to 1.5: If \(X\) is finite dimensional it is P-convex.
This follows from the compactness of \(U(X)\). No relationship has been shown between \(n\) so that \(P(n, X) < \frac{1}{2}\) and the dimension of the space. The smallest \(n\) so that \(P(n, \xi^m) < \frac{1}{2}\) is greater than \(2^m\) since the \(2^m\) points of the form \((\xi_1, \xi_2, \ldots, \xi_m)\), where \(\xi_i\) take on all possible
combinations of +1 and -1, form a 2 separated set of order n. It is not known whether there is an m dimensional space X having $P(2^m+1, X) = \frac{1}{2}$.

**Analog to 1.6:** $P(n+1, X) \leq P(n, X)$ for any X.

**Analog to 1.7 and 1.9:** The P-convex analog to these properties would be: If X is P-convex then $\lim_{n \to \infty} P(n, X) = 0$. While this is true for finite dimensional spaces (our Corollary 4.8), it is false in general.

For example, in $\ell_2$ let $x_i = \frac{\sqrt{2}}{\sqrt{2}+1} e_i$ where $e_i$ is the $i$th usual basis vector. For any $n, \varepsilon$ the balls

$$B_i = \{ x \in \ell_2 : \|x - x_i\| \leq \frac{1}{\sqrt{2}+1} - \varepsilon \}, i = 1, \ldots, n,$$

are disjoint (since $\|x_i - x_j\| = \frac{2}{\sqrt{2}+1} \varepsilon$) and twice the radius is $\frac{2}{\sqrt{2}+1} - 2\varepsilon$) and in $U(X)$ (since if $x \in B_i$ then

$$\|x\| \leq \|x - x_i\| + \|x_i\| = \frac{1}{\sqrt{2}+1} - \varepsilon + \frac{\sqrt{2}}{\sqrt{2}+1} = 1 - \varepsilon$$

so that for any n

$$P(n, \ell_2) \geq \frac{1}{\sqrt{2}+1}$$

**Analog to 1.10:** No $\varepsilon$-isometric imbedding property is known to characterize P-convex spaces.

**Analog to 1.11:** $P(n, X) = P(n, X^{**})$.

This holds if X is P-convex since in that case X is reflexive.

If X is not P-convex, $P(n, X) = 1$, and using the canonical imbedding of X in $X^{**}$ and the Analog to 1.3 $P(n, X^{**}) = 1$. 
Analog to 1.12: The property dual to P-convexity has been described by Kottman. However, it is not known whether this dual property is equivalent to P-convexity.

Analog to 1.13: It is not known whether P-convexity is preserved under isomorphism.

Analog to 1.14: It is not known whether P-convexity of Z and X/Z implies P-convexity of X.

Analog to 1.15: We will present two theorems (4.6 and 4.9) showing under certain conditions if $X = Y \oplus Z$ and $Y$ and $Z$ are P-convex then $X$ is P-convex. The question is not solved in general, however.

Analog to 1.16: We will show (Proposition 4.11) that a space is P-convex if each separable subspace is P-convex.

We will also present an analog to our Theorem 3.9 for P-convexity.

The problem of proving that the direct sum of two P-convex spaces is P-convex seems more difficult than that for B-convex spaces. We will prove this for the following special cases: If $X = Y \oplus Z$, and either

1. The norming of $X$ is $\| (y,z) \| = \max \{ \| y \|, \| z \| \}$,
2. $Y$ is finite dimensional and $Y$ and $Z$ are subspaces of $X$.

These results are based on a theorem of combinatorics proved in 1930 by Ramsey [27]. Ramsey's theorem can be stated as follows.

4.3 Theorem (Ramsey): Let $p$, $q$, and $r$ be integers so that $p, q, r > 1$. Then there is a number $n(p,q,r)$ having the following property. Let $S$ be a set having $n(p,q,r)$ or more elements. Let the
family of all r-subsets of S (where an r-subset is a subset having r elements) be divided into two disjoint families, $\alpha$ and $\beta$. Then either

1. There is $A \subset S$, a subset with $p$ elements, so that any r-subset of $A$ is in $\alpha$, or

2. There is $B \subset S$, a subset of q element, so that any r-subset of $B$ is in $\beta$.

We use this theorem to prove the following lemma.

4.4 Lemma: Let $A$ and $B$ be sets, $P_A$ a property which a pair of points $(a_i, a_j)$ in $A$ may have, and $P_B$ a property on pairs of points $(b_i, b_j)$ in $B$. Suppose there is an integer $N_A$ so that if $a_1, \ldots, a_n, n \geq N_A$, are distinct points of $A$, then there is $i, j$ so that $(a_i, a_j)$ has $P_A$, and there is $N_B$ with the corresponding property for $B$. Then there is an integer $N_{AB}$ so that if $n \geq N_{AB}$, $a_1, \ldots, a_n$ distinct points of $A$, $b_1, \ldots, b_n$ distinct points of $B$, then there is $i, j$ so that both $(a_i, a_j)$ has $P_A$ and $(b_i, b_j)$ has $P_B$. (By "pair" we mean unordered pair.)

Proof: Let $N_0 = \max (N_A, N_B)$ and let $N_{AB} = n(N_0, N_0, 2)$ from Ramsey's theorem. For $n \geq N_{AB}$ let $S = \{1, \ldots, n\}$. Given $\{a_i\}_{i=1}^n$, let

$$\alpha = \{(i, j) : (a_i, a_j) \text{ does not have } P_A\}$$

$$\beta = \{(i, j) : (b_i, b_j) \text{ does not have } P_B; (i, j) \not\in \alpha\}.$$ 

Now suppose there is no $i, j$ as asserted in the lemma. Then $\alpha \cup \beta$ is the set of all pairs of elements of $S$. Also $\alpha \cap \beta = \emptyset$, so Ramsey's theorem applies. If Conclusion 1 holds, there is $\{i_n\}_{n=1}^{n_0} \subset S$ so that each $(i_n, i_n) \in \alpha$. Thus $\{a_i\}_{i=1}^{n_0}$ is a set of $N_A$ or more points, no
A pair of which has $P_A$, which is a contradiction.

Lemma 4.4 will be incorporated into the following lemma for ordered pairs $(a,b) \in A \times B$

4.5 Lemma: Let $A,B,P_A,P_B,N_A,N_B,N_{AB}$ be as in Lemma 4.4 with the additional property that a pair having the same first and second elements of $a$, $(a_i,a_i)$, always has $P_A$, and the corresponding property for $B$. Then if $\{(a_i,b_i)\}_{i=1}^n$ is a set of distinct pairs from $A \times B$, (i.e., any two pairs differ in the first or second entries, or both) and $n \geq N_{AB}N_A N_B$, then there is $i,j$ so that both $(a_i,a_j)$ has $P_A$ and $(b_i,b_j)$ has $P_B$.

Proof: Let $a_1,a_2,\ldots,a^r_A$ be the distinct values of $\{a_i\}_{i=1}^n$ and write the sets

$$\{(a_i,b_i) : a_i = a^1\}, \{(a_i,b_i) : a_i = a^2\}, \ldots, \{(a_i,b_i) : a_i = a^r_A\}.$$

If one of these sets, say the $K^{th}$, has $N_B$ or more pairs, then for these pairs $\{b_i : a_i = a^K\}$ are distinct and so there is $i,j$ so that $(b_i,b_j)$ has $P_B$. By hypothesis $(a_i,a_j) = (a^K,a^K)$ has $P_A$ so the conclusion of the lemma holds. Otherwise each of the sets have less than $N_B$ pairs, so that the total number of pairs in all of the sets is $n < N_B r_A$. Since $N_{AB} N_A N_B \leq n$, we have $r_A > N_{AB} N_A$. By choosing one pair from each of the sets, we get a family of pairs $\{(a_i^n,b_i^n)\}_{i=1}^{r_A}$ having distinct first elements, i.e., if $n \neq m$ then $a_i^n \neq a_i^m$. Now let $b_1^n, b_2^n, \ldots, b_{r_B}^n$ be the distinct values of $\{b_i^n\}_{i=1}^{r_A}$ and write the sets

$$\{(a_i^n,b_i^n) : b_i^n = b_1^n\}, \ldots, \{(a_i^n,b_i^n) : b_i^n = b_{r_B}^n\}.$$
If any one of these sets has \( N_A \) or more elements, say the \( K^{th} \), then 
\[ \{a_i^n : b_i^n = b^K\} \] are distinct and there is \( j, k \) so that \((a^j_i, a^k_i)\) has \( P_A \) and \((b^j_i, b^k_i) = (b^K_i, b^K)\) has \( P_B \). Since \((a^j_i, b^j_i)\) and \((a^k_i, b^k_i)\) in the original set of pairs \( \{(a_i^n, b_i^n)\}_{i=1}^n \) the conclusion of the lemma holds. 
Otherwise each of the sets have less than \( N_A \) pairs, so that the total number of pairs in all the sets is \( r_A < r_B N_A \). Since we showed 
\[ N_{AB}N_A < r_A \] we have \( r_B > N_{AB} \). Take one pair from each of the sets to get 
\[ \{(a_j^j, b_j^j)\}_{j=1}^{r_B} \], a subset of \( \{(a_i^n, b_i^n)\}_{i=1}^n \), so that if \( j \neq k \) then \( a_j^j \neq a_k^k \) and \( b_j^j \neq b_k^k \). Thus \( \{a_j^j\}_{j=1}^{r_B} \) and \( \{b_j^j\}_{j=1}^{r_B} \) are distinct points of \( A \) and \( B \). Applying Lemma 4.4 to these pairs concludes the proof.

**4.6 Theorem:** Let \( Y \oplus Z \) be the direct sum of two \( P \)-convex Banach spaces normed by 
\[ \| (y, z) \| = \max (\|y\|, \|z\|) \]. Then \( Y \oplus Z \) is \( P \)-convex.

**Proof:** By Lemma 4.2, there is \( n_Y, \varepsilon_Y \) so that if \( \{y_i\}_{i=1}^n \) are distinct points in \( U(Y) \) and \( n \geq n_Y \) we have some \( i, j \) so that 
\[ \|y_i - y_j\| < 2 - \varepsilon_Y. \] Similarly there is \( n_Z, \varepsilon_Z \) with this property for points in \( U(Z) \). Let \( A = U(Y) \). Say \( (y_i, y_j) \) has \( P_A \) if 
\[ \|y_i - y_j\| < 2 - \varepsilon, \] where 
\[ \varepsilon = \min(\varepsilon_Y, \varepsilon_Z). \] Let \( N_A = N_Y \). Similarly let \( B = U(Z) \) and define \( P_B \) and \( N_B \). Let \( \{(y_i, z_i)\}_{i=1}^n \) be distinct pairs in \( U(Y + Z) \), \( n \geq N_{AB}N_AN_B \). Then \( \{y_i\}_{i=1}^n \subset A \), \( \{z_i\}_{i=1}^n \subset B \). By Lemma 4.5 there is \( i, j \) so that 
\[ (y_i, y_j) \text{ has } P_A; \text{ i.e., } \|y_i - y_j\| < 2 - \varepsilon. \]
and 
\[(z_i, z_j) \text{ has } P_B; \text{ i.e., } \|(z_i - z_j)\| < 2 - \epsilon\]

and thus
\[\|(y_i, z_i) - (y_j, z_j)\| = \|(y_i - y_j, z_i - z_j)\| < 2 - \epsilon.\]

To prove the second direct sum theorem we need

4.7 Lemma: If \(X\) is finite dimensional and \(\epsilon > 0\) there is \(N\) so that if \(\{x_i\}_{i=1}^{N}\) are distinct points in \(U(X)\), there is \(i, j\) such that 
\[\|(x_i - x_j)\| < \epsilon.\]

Proof: For each \(y \in U(X)\), let \(S_y = \{x : \|(x - y)\| < \frac{\epsilon}{2}\}\).

The family \(\{S_y : y \in U(X)\}\) covers \(U(X)\), which is compact, so we may find 
\(\{y_i\}_{i=1}^{N-1}\) so that \(\{S_{y_i} : i = 1, \ldots, N-1\}\) cover \(U(X)\). Then if \(\{x_i\}_{i=1}^{N} \subset U(X)\) there is some \(i, j \leq N, i \neq j\), so that \(x_i, x_j \in S_{y_i}\), and 
\[\|(x_i - x_j)\| < \epsilon.\]

We note that this trivially provides a partial analog to

B-convex facts 1.7 and 1.9:

4.8 Corollary: If \(X\) is finite dimensional, \(\lim_{n \to \infty} P(n, X) = 0\).

4.9 Theorem: Let \(Y\) and \(Z\) be subspaces of \(X\) so that \(X = Y \oplus Z\).

If \(Y\) is finite dimensional and \(Z\) is \(P\)-convex then \(X\) is \(P\)-convex.

Proof: Since \(Z\) is \(P\)-convex there is \(n_Z, \delta\) so that if \(\{z_i\}_{i=1}^{n}\) are distinct points in \(U(Z)\) and \(n \geq n_Z\) then there is \(i, j\) such that 
\[\|(z_i - z_j)\| < 2 - \delta.\]

Since \(Y\) is finite dimensional, by Lemma 4.7 there is \(n_Y\) so that if \(\{y_i\}_{i=1}^{n}\) are distinct points in \(U(Y)\) and \(n \geq n_Y\) then there is \(i, j\) such that 
\[\|(y_i - y_j)\| < \frac{\delta}{2}.\]

Let \(U(Y) = A, \text{ say } (y, y)\) has \(P_A\) if 
\[\|(y_i - y_j)\| < \frac{\delta}{2},\]

and let \(N_A = n_y.\) Let \(U(Z) = B, \text{ say } (z_i, z_j)\) has \(P_B\) if 
\[\|(z_i - z_j)\| < 2 - \delta,\]

and let \(N_B = n_z.\) Let \(\{y_i + z_i\}_{i=1}^{n}\) be distinct
pairs in $U(X)$, $n \geq N_{A\hat{a}N_B}$. Then \{y_i\}_{i=1}^n \subset A$ and \{z_j\}_{j=1}^n \subset B$.

By Lemma 4.5 there is $i,j$ so that

$$(y_i, y_j) \text{ has } P_A; \text{ ie } ||y_i-y_j|| < \frac{\delta}{2},$$

and $(z_i, z_j)$ has $P_B; \text{ ie } ||z_i-z_j|| < 2 - \delta$.

Thus

$$||\langle y_i + z_i \rangle - \langle y_j + z_j \rangle|| \leq ||y_i - y_j|| + ||z_i - z_j|| < 2 - \frac{\delta}{2}.$$

4.10 Corollary: If $X$ is not $P$-convex, and $Y$ is a subspace of $X$ of finite codimension, then $Y$ is not $P$-convex.

Proof: Write $X = Y \oplus Z$ where $Z$ is finite dimensional and use Theorem 4.9.

In analog to B-convex Fact 1.16 we prove

4.11 Proposition: $X$ is $P$-convex if each of its separable subspaces is $P$-convex.

Proof: If $X$ is not $P$-convex, choose a sequence $\epsilon_n \to 0$ of positive numbers less than 2. For each $n$, by Lemma 4.2, there is $x_1^n, \ldots, x_n^n \subset U(X)$ which is a $2-\epsilon_n$ separated set. Let $Y_n = \{x_1^n, \ldots, x_n^n\}$ and let $Y = \{Y_1, Y_2, \ldots\}$. $Y$ is clearly separable. It is also not $P$-convex, since for any $n, \epsilon$ we can select $m$ so that $\epsilon_m < \epsilon$, $m \geq n$, making $\{x_1^m, \ldots, x_n^m\}$ a $2-\epsilon$ separated set of order $n$ in $U(Y)$.

Improving upon this proposition, we have in analog to Theorem 3.9 for B-convex spaces, the fact that a space is $P$-convex if each subspace having a Schauder decomposition into finite dimensional subspaces is $P$-convex.
We will use the following

4.12 Definition: A sequence \([M_i]\) of closed subspaces of a Banach space \(E\) is a Schauder decomposition of \([M_i]\) if every element \(u\) of \([M_i]\) has a unique, norm-convergent expansion \(u = \sum_{i=1}^{\infty} u_i\), where \(u_i \in M_i\) for \(i=1,2,\ldots\).

Grinblyum [13] has characterized Schauder decompositions as follows.

4.13 Lemma: A sequence \([M_i]\) of closed subspaces of \(E\) is a Schauder decomposition of \([M_i]\) if and only if there is a constant \(K\) such that for all integers \(m,n\) and all sequences \([u_i]\) with \(u_i \in M_i\) we have

\[
\sum_{i=1}^{n+m} u_i \leq K \sum_{i=1}^{n} u_i
\]

We are now ready for the analog to Theorem 3.9, namely, a space is P-convex if every subspace having a Schauder decomposition into finite dimensional subspaces is P-convex.

4.14 Theorem: If \(X\) is not P-convex, for any sequences of numbers \(\{\delta_i\}, \{\varepsilon_i\}; 1 > \delta_i \to 0, 1 > \varepsilon_i \to 0\), and any sequence of integers \(\{k_i\}\), \(k_i \to \infty\), there is a sequence of integers \(p(m)\) and a sequence of vectors \([x_i] \subset U(X)\) so that

1) The space \(L = [x_i]\) is not P-convex, in particular

1') For each \(m=1,2,\ldots\) there is a set of distinct points \(\{y_{ij}\}_{j=1}^{k_m} \subset U([x_i]_{i=p(m-1)+1}^{p(m)})\) so that

\[
|y_i^m - y_j^m| > 2 - \varepsilon_m \quad \text{for} \quad i \neq j, \quad i,j=1,\ldots,k_m
\]

and

2) \(L\) has a Schauder decomposition into finite dimensional subspaces,
in particular

\[(2')\] Letting \(L^m = [x_i]_{i=p(m-1)+1}^{p(m)}\), for any \(n,q, \{u_i\}\)

where \(u_i \in L^i\), \(i=1,2,\ldots\), we have

\[\left\| \sum_{i=1}^{n} u_i \right\| \leq (1+\delta_n) \left\| \sum_{i=1}^{n+q} u_i \right\|.\]

**Proof:** \((1')\) implies \((1)\) by Lemma 4.2. \((2')\) implies \((2)\) by Lemma 4.13. We will prove \((1')\) and \((2')\). As in the proof of Theorem 3.9, we will construct \(\{x_i\}\) in blocks \([x_i]_{i=p(m-1)+1}^{p(m)}\) by induction on \(m\). Denote \([x_i]_{i=1}^{p(m)}\) by \(L_m\). For each \(m\) we will choose \(y_j^{m}\) and \(x_i^{p(m)}\) from a previously constructed subspace so that they satisfy \((1')\). We will then construct a subspace of finite codimension in \(X, \Lambda_m\), so that \(\Lambda_m \cap L_m = \emptyset\) and so that the projection \(P_m : L_m + \Lambda_m \to L_m\) has norm less than or equal to \(1 + \delta_m\). The \(\{x_i\}\), \(\{y_j\}\) chosen in subsequent induction steps are taken in \(\Lambda_m\).

Let \(m=1\). Since \(X\) is not \(P\)-convex, by Lemma 4.2 there is a set of distinct points \(\{y_j^1\}_{j=1}^{k_1} \subset U(X)\) so that \((1')\) is satisfied for \(m=1\). Let \(\{x_i^1\}_{i=1}^q\) be a linearly independent set spanning \(L_1 = \{y_j^1\}_{j=1}^{k_1}\).

To construct \(\Lambda_1\), choose \(\{f_i^1\}_{i=1}^q \subset U(L_1^p)\) by Lemma 3.5 and extend them to \(X\) so that if \(x \in L_1\), \(\|x\| \leq (1+\delta_1) \max_{i=1,\ldots,q(1)} f_i^1(x)\).

Define \(\Lambda_1 = \bigcap_{i=1}^{k_1} f_i^1^{-1}(o)\). Exactly as in Theorem 3.9, we have \(P_1 : L_1 + \Lambda_1 \to L_1\) and \(\|P_1\| \leq 1 + \delta_1\).

The induction step is similar to that of 3.9. Since \(\Lambda_{m-1}\) is of finite codimension, it is not \(P\)-convex by Corollary 4.10, and so contains \(\{y_j^m\}_{j=1}^{k_m}\) satisfying \((1')\). Taking \(\{x_i^m\}_{i=p(m-1)+1}^{p(m)}\) to be a linearly independent set spanning \([y_j^m]_{j=1}^{k_m}\), the induction step is
completed exactly as in the B-convex proof.

To see that (2') is satisfied, observe that, for 
\( u_i \in L^i, \ i=1,2,... \) and any \( n,q \),

\[
\frac{n+q}{n} \sum_{i=1}^{n} u_i = \sum_{i=1}^{n} u_i \quad \text{and} \quad ||P_n|| \leq 1 + \delta_n.
\]

The second property to be discussed in connection with B-convexity is known as the Banach-Saks property. A space is said to have the Banach-Saks property if each bounded sequence of elements has a subsequence whose \((C,1)\) means converge strongly. Banach and Saks [1] proved that this property holds in \( L_p (0,1) \) and \( \ell_p (p>1) \). The proof of Kakutani [18], with the fact that all uniformly convex spaces are reflexive, shows that all uniformly convex spaces have the Banach-Saks property. Nishiura and Waterman have proved recently in [24] that all spaces having the Banach-Saks property are reflexive. It is not known whether there are any reflexive spaces which do not have the Banach-Saks property.

It would be interesting to know whether B-convexity implies the Banach-Saks property since this latter property implies reflexivity. This is not known. However, the converse question is known to be false.

4.15 Proposition: There is a space which is not B-convex but has the Banach-Saks property.

Proof: Nishiura and Waterman [24] have shown, for \( p > 1 \), that \( P \) has the Banach-Saks property. But this space is not B-convex since, as shown for \( c_0 \), example g of Chapter I, for any \( k, \varepsilon \) we can find \( x_1, ..., x_k \) violating the B-convexity inequality.
Nishiura and Waterman used the following property, which they called \((\ast)\):

\((\ast)\) There is \(\theta_\varepsilon(o,1)\) so that if \(\{x_n\} \subset U(X)\), \(x_n\) converges weakly to zero, there is \(i,j\) so that \(\|x_i + x_j\| < 2\varepsilon\).

Kakutani, [18], using an equivalent formulation of \((\ast)\), showed every uniformly convex space satisfies \((\ast)\) and that every reflexive space satisfying \((\ast)\) is Banach-Saks. It is interesting to compare \((\ast)\) with the following formulations of P-convexity and \(2,\varepsilon\)-convexity.

P-convexity: There is \(N\) and \(\theta_\varepsilon(o,1)\) so that if 

\[
[x_i]_{i=1}^N \subset U(X) \text{ there is } i,j \text{ so that } \\
\|x_i - x_j\| < \theta.
\]

\(2,\varepsilon\)-convexity: There is \(\theta_\varepsilon(o,1)\) so that if \(x, y \subset U(X)\)

either \(\|x - y\| < 2\varepsilon\) or \(\|x + y\| < 2\varepsilon\).

The known relationships between these properties are summarized in the following diagram:

All of these conditions are generalizations of uniform convexity.

All except B-convexity and \((\ast)\) are known to imply reflexivity.

The last property to be discussed was called "near convex" by D. R. Smart in [30]. As was noted in the review of this paper, [31], near-convexity is the same as \(2,\varepsilon\)-convexity. We say a series \(\sum x_n\) is absolutely bounded if there is a number \(N\) so that for any finite set of distinct integers \(\{n_i\}\) we have \(\|\sum x_n\| \leq N\).

Smart showed that if a space is \(2,\varepsilon\)-convex then each absolutely bounded series converges.
It is natural to ask whether this also holds for any B-convex space or for any reflexive space. We will show that the answer to both questions is yes.

4.16 Definition: \( \sum_{n=1}^{\infty} x_n \) is weakly unconditionally convergent if for any permutation of integers \( k_n \), the sequence \( \sum_{n=1}^{\infty} x_{k_n} \) converges weakly.

4.17 Lemma: In any Banach space, \( \sum_{n=1}^{\infty} x_n \) is absolutely bounded if and only if it is weakly unconditionally convergent.

Proof: Bessaga and Pelczynski ([4] Lemma 2) have shown that \( \sum_{n=1}^{\infty} x_n \) is weakly unconditionally convergent if and only if there is a constant \( C \) so that for every bounded real sequence \( \{t_n\} \)

\[
\sup_{n} \left\| \sum_{i=1}^{n} t_i x_i \right\| \leq C \sup_{i} |t_i|.
\]

Thus if \( \sum_{n=1}^{\infty} x_n \) is weakly unconditionally convergent it is absolutely bounded. To see the converse suppose \( \sum_{n=1}^{\infty} x_n \) is absolutely bounded with constant \( N \) but not weakly unconditionally convergent. Then there is a permutation \( k_n \) and a functional \( f \) so that \( \sum_{n=1}^{\infty} f(x_{k_n}) \) does not converge. Hence there are sequences \( p_m, q_m \) and a number \( \epsilon > 0 \) so that

\[
\sum_{n=p_m}^{q_m} f(x_{k_n}) > 0 \quad m = 1, 2, \ldots
\]

Without loss of generality

\[
\sum_{n=p_m}^{q_m} f(x_{k_n}) > 0 \quad m = 1, 2, \ldots
\]
(Otherwise, if positive for infinitely many \( m \) we may reindex to omit those \( n \) for which the sum is negative, or if positive for only finitely many \( m \), change the sign on \( f \) and reindex as above.)

For any \( k \), we have

\[
k \varepsilon < f \left( \sum_{m=1}^{k} \sum_{n=p_m}^{q_m} x_{k,n} \right) \leq ||f|| \left\| \sum_{m=1}^{k} \sum_{n=p_m}^{q_m} x_{k,n} \right\| \leq ||f||N ,
\]

since \( \sum_{n=1}^{\infty} x_n \) is absolutely bounded by \( N \), which is a contradiction if

\[
k > \frac{||f||N}{\varepsilon}.
\]

Theorem 5 of [4] states that the following are equivalent:

(1) There is a sequence in \( X \) which is weakly unconditionally convergent but not unconditionally convergent,

(2) \( X \) contains a subspace isomorphic to \( c_0 \).

This gives directly, with the lemma,

4.18 Theorem: The following are equivalent:

(1) Every absolutely bounded series in \( X \) converges unconditionally,

(2) \( X \) has no subspace isomorphic to \( c_0 \).

Since reflexive spaces and B-convex spaces have no subspace isomorphic to \( c_0 \), Smart's result generalizes as desired.

The idea of this proof is due to Prof. W. J. Davis.
CHAPTER V
SOME UNSOLVED PROBLEMS

In this chapter some unsolved problems related to the topics of this dissertation are discussed. Some of the problems have also been treated in earlier chapters. The first group of problems relate to B-convexity.

**Problem 1:** Is every B-convex space reflexive?

This was conjectured to be true by James in [14] and [15]. To support this conjecture there are the three subclasses of B-convex spaces known to be reflexive (2,ε-convex spaces, spaces with unconditional basis, and P-convex spaces) and the list of properties (Theorems 1-12 through 1-16, and Theorem 3-9) which remain true if the words "B-convex" are replaced with "reflexive".

In [3] Beck asked

**Problem 1a:** Is every B-convex space isomorphic to a uniformly convex space?

and in [12] Giesy asked

**Problem 1b:** Is every B-convex space isomorphic to a 2,ε-convex space?

If either of these problems could be answered yes, Problem 1 would be solved.

In [15] James conjectured that Problems 1a and 1b are equivalent:
**Problem 2**: If a Banach space is isomorphic to a $2, \varepsilon$-convex space, is it also isomorphic to a uniformly convex space?

The converse to Problem 2 is trivially true. James points out that if Problem 2 can be answered yes, then the existence of an isomorphism to a uniformly convex space would be equivalent to the existence of an isomorphism to a uniformly smooth space. This follows since uniform smoothness and uniform convexity are dual properties [9] while $2, \varepsilon$-convexity is self dual [17]. Further discussion of Problem 2 can be found in [17].

Other questions which would support the truth of Problem 1 are

**Problem 3**: If a space is B-convex and separable, is either the first or second dual separable?

**Problem 4**: Is every B-convex space weakly complete?

The most important question regarding P-convexity is

**Problem 5**: Is every B-convex space isomorphic to a P-convex space?

If the answer to Problem 5 is yes, Problem 1 is solved. Kottman, in some private communications, has given a set of examples which illustrate the difficulty of this problem by showing that there is no simple relationship between $k$ for which a space $X$ is $k, \varepsilon$-convex and $n$ so that $P(n, X) < \frac{1}{2}$. He provides, for each $n$, a finite dimensional space $B_n$ for which there is $\varepsilon_n$ so that $B_n$ is $2, \varepsilon_n$ convex, but having $P(n, B_n) = \frac{1}{2}$. However, any space containing all the $B_n$ is not B-convex. Kottman's examples and their properties are described in detail below.
5.1 Definition: Let $\mathbb{B}_n$ be the space of $n$-tuples with norm defined by

$$
\| (x_1, \ldots, x_n) \| = \max \left\{ \sum_{i=1}^{k} \frac{|x_{p_i}|}{2^{i-1}} : 1 \leq p_1 < \ldots < n, k=1, \ldots, n \right\} .
$$

For example, if $n = 2$ we have

$$
\| (x_1, x_2) \| = \max \left\{ |x_1|, |x_2|, |x_1| + \frac{|x_2|}{2} \right\} ,
$$

so that $\mathbb{U}(B_2)$ an hexagon.

5.2 Proposition: For each $n$ there is $\varepsilon_n$ so that $\mathbb{B}_n$ is $2, \varepsilon_n$-convex.

Proof: We first prove it is sufficient to show there is no $x, y \in \mathbb{U}(B_n)$ so that $\| x-y \| = \| x+y \| = 2$: If $\mathbb{B}_n$ is not $2, \varepsilon$-convex for any $\varepsilon$, for each $m$ we can find $x_m, y_m \in \mathbb{U}(B_n)$ so that for both signs,

$$
\| x_m \pm y_m \| > 2(1 - \frac{1}{m}) .
$$

Since $\mathbb{U}(B_n)$ is compact, there is $\{m_j\}; x, y \in \mathbb{U}(B_n)$, so that $x_{m_j} \to x$ and $y_{m_j} \to y$. For any $\delta > 0$ we can choose $j$ so that

$$
| |x_{m_j} - x| | < \frac{\delta}{3} , | |y_{m_j} - y| | < \frac{\delta}{3} , \text{ and } \frac{2}{m_j} < \frac{\delta}{3} ,
$$

giving,

$$
| |x+y| | > | |x_{m_j} \pm y_{m_j}| | - | |x_{m_j} - x| | - | |y_{m_j} - y| | > 2 - \delta ,
$$

which is a contradiction.

Now suppose there is $x, y \in \mathbb{U}(B_m)$ so that $\| x+y \| = 2$. Then there is $k, 1 \leq p_1 < \ldots < p_k \leq n$, so that

$$
2 = \| x+y \| = \sum_{i=1}^{k} \frac{|x_{p_i} + y_{p_i}|}{2^{i-1}} \leq \sum_{i=1}^{k} \frac{|x_{p_i}|}{2^{i-1}} \leq \sum_{i=1}^{k} \frac{|y_{p_i}|}{2^{i-1}} \leq \| x \| + \| y \| .
$$
Thus equality holds and $\text{sgn} (x_{p_i}) = \text{sgn} (y_{p_i})$ for all $i$ such that both $x_{p_i}$ and $y_{p_i}$ are nonzero. Also we have

$$||x|| = \sum_{i=1}^{k} \left| \frac{x_{p_i}}{2^{i-1}} \right|, \quad ||y|| = \sum_{i=1}^{k} \left| \frac{y_{p_i}}{2^{i-1}} \right|.$$ 

Now if there is $i'$ so that $x_{p_i'} = 0$, then $x_{p_j} = 0$ for all $j > i'$, for otherwise we have

$$||x|| = \frac{|x_{p_1}|}{2^0} + \frac{|x_{p_2}|}{2^1} + \ldots + \frac{|x_{p_k}|}{2^k} < \frac{|x_{p_1}|}{2^0} + \ldots + \frac{|x_{p_{i'-2}}|}{2^{i'-2}} + \frac{|x_{p_{i'+1}|}{2^{i'}} + \ldots + \frac{|x_{p_k}|}{2^{k-1}},$$

which is a contradiction. Therefore, there is $k', k''$ so that

$$||x|| = \sum_{i=1}^{k'} \frac{|x_{p_i}|}{2^{i-1}}, \quad ||y|| = \sum_{i=1}^{k''} \frac{|y_{p_i}|}{2^{i-1}},$$

and $x_{p_i} \neq 0, i=1,\ldots,k'$, $y_{p_i} \neq 0, i=1,\ldots,k''$,

and we can assume $k'' \leq k'$.

In the same way, using $||x-y||$, we can get $m, l \leq q_1 < \ldots < q_m \leq n$, so that

$$||y|| = \sum_{i=1}^{m} \frac{|y_{q_i}|}{2^{i-1}}.$$
Let \( y_{q_1} \neq 0, i=1, \ldots, m \), and if both \( x \) and \( y \) are nonzero, 
\[
\text{sgn}(x) = -\text{sgn}(y) \quad \text{and} \quad \{p_1, \ldots, p_{k''}\} \cap \{q_1, \ldots, q_m\} = \emptyset \quad \text{since if} \quad j \quad \text{is in this intersection both} \quad x_j \quad \text{and} \quad y_j \quad \text{are nonzero and} \quad \text{sgn}(x_j) = -\text{sgn}(y_j)
\]
But since the norm of \( y \) is achieved with both the sets 
\[
\{p_1, \ldots, p_{k''}\} \quad \text{and} \quad \{q_1, \ldots, q_m\},
\]
there is some \( j \) so that \( |y_j| = \max\{|y_i|: i=1, \ldots, n\} \neq 0 \) and \( j \) must be in \( \{p_1, \ldots, p_{k''}\} \cap \{q_1, \ldots, q_m\} \). (If not, \( k'' \) or \( m \) < \( n \) and the \( p_i \) or \( q_i \) can be rechosen to include \( j \) and thereby produce a larger sum.)

5.3 Proposition: \( P(n, B_n) = \frac{1}{2} \)

Proof: We will exhibit \( n \) distinct points \( x_1, \ldots, x_n \in B_n \) so that \( |x_i - x_j| = 2 \) for \( i \neq j \).

Let 
\[
x_1 = (1, 0, \ldots, 0)
\]
\[
x_2 = (-\frac{1}{2}, 1, 0, \ldots, 0)
\]
\[
x_3 = (-\frac{1}{2}, -\frac{1}{2}, 1, 0, \ldots, 0)
\]
\[\ldots\]
\[
x_n = (-\frac{1}{2}, \ldots, -\frac{1}{2}, 1)
\]
To see that \( x_i \in U(X) \), the largest sum \( \sum_{i=1}^{k} \frac{|x_{p_i}|}{2^{i-1}} \) is obtained with \( k = i \) and \( p_1, \ldots, p_k = 1, \ldots, i \), in which case the sum is
\[
\frac{1}{2} \sum_{i=1}^{k-1} \left( \frac{1}{2^{i-1}} \right) + \frac{1}{2^{k-1}} = \frac{1}{2} \sum_{i=0}^{k-2} \left( \frac{1}{2^i} \right) + \frac{1}{2^{k-1}} = 1 - \frac{1}{2^{k-1}} + \frac{1}{2^{k-1}} = 1,
\]
so that $||x_1|| = 1, i=1,\ldots,n$. To see that $||x_i-x_j|| = 2$, suppose $i < j$. Then the number in the $i^{th}$ place of $x_i - x_j$ is $\pm \frac{3}{2}$, in the $j^{th}$ place is 1. Therefore, letting $k = 2$, $p_1 = i$, $p_2 = j$

$$||x_i-x_j|| \geq \frac{3}{2} + \frac{1}{2} = 2.$$  

In view of Propositions 5.2 and 5.3, one might attempt to use
Theorem 1.19 of Giesy to show that $P_\epsilon B_n$ is 2,\,$\epsilon$-convex for some $\epsilon$ but not P-convex. However, the following proposition shows that this cannot be done.

5.4 Proposition: If $X$ is a space containing isometrically $B_n, n=1,2,\ldots$, then $X$ is not $B$-convex.

Proof: We first observe two elementary facts.

A. Let $a$ be any number, $m \leq n$. If $x$ is an $n$-tuple having $a$ in $m$ of the $n$ entries, $||x|| \geq 2|a| \left(1 - \frac{1}{2^m}\right)$.

This holds since we can let $p_1, p_2, \ldots, p_m$ be the indices of the entries which are $a$. Then

$$||x|| \geq |a| \sum_{i=1}^{m} \frac{1}{2^{i-1}} = |a| \sum_{i=0}^{m-1} \frac{1}{2^i} = 2|a| \left(1 - \frac{1}{2^m}\right).$$

B. Let $b$ be any number. For any choice of signs $\{\varepsilon_i\}_{i=1}^{n}$,

$$||(\varepsilon_1 b_1, \varepsilon_2 b_2, \ldots, \varepsilon_n b_n)|| < 2|b|.$$  

This fact is clear since the largest possible sum

$$\sum_{i=1}^{n} \frac{|b|}{2^{i-1}} = 2|b| \left(1 - \frac{1}{2^n}\right) < 2|b|.$$
We now construct, for any \( k, \varepsilon, \{x_i\}_{i=1}^k \subseteq \mathcal{U}(x) \) so that for any choice of signs \( \{\xi_i\}_{i=1}^k \) we have

\[
\left\| \sum_{i=1}^k \xi_i x_i \right\| > k(1-\varepsilon)
\]

Choose \( m \) so that \( \frac{1}{2^m} < \varepsilon \). Let \( n = m \cdot 2^k \). Let \( A \) be the \( m \)-tuple \((\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})\) and form the \( n \)-tuple

\[
x_1 = (A, -A, A, -A, \ldots, A)
\]

using \( 2^k \) \( m \)-tuples \( A \). Similarly let

\[
x_2 = (A, A, -A, -A, A, \ldots, A)
\]

\[
\vdots
\]

\[
x_k = (A, A, \ldots, A, -A, -A, \ldots, -A)
\]

in each case using \( 2^k \) \( m \)-tuples \( A \). The pattern of signs is the same as in example \( g \) of Chapter I, \( c_0 \). From Fact B we have \( ||x_1|| \leq 1 \). For any choice of signs, the sum \( \sum_{i=1}^k \xi_i x_i \) will be an \( n \)-tuple containing the \( m \)-tuple \( kA \), so by Fact A above

\[
\left\| \sum_{i=1}^k \xi_i x_i \right\| \geq k \left( 1 - \frac{1}{2^m} \right) > k(1-\varepsilon)
\]
As was pointed out in Chapter IV, many facts known true for B-convex spaces are not known for P-convex spaces. The truth of these would support the truth of Problem 5.

**Problem 6:** Is P-convexity self dual?

**Problem 7:** Is P-convexity preserved under isomorphism?

**Problem 8:** If $Z$ is a closed subspace of $X$ so that both $Z$ and $X/Z$ are P-convex, is $X$ P-convex?

**Problem 9:** If $Y$ and $Z$ are subspaces spanning $X$, does P-convexity of $Y$ and $Z$ imply P-convexity of $X$?

**Problem 10:** If every subspace of $X$ having a basis is P-convex, is $X$ P-convex?

Note that an attempt to extend the proof of Theorem 4.14 to obtain this result, would need a number $M$ and a family of projections from each $L^n$ onto spans of some of the basis elements of $L^n$ such that the norms of all the projections are less than $M$.

It is known that for a given finite dimensional space such projections can be found with norm less than or equal to the dimension of the space, for example [32], but we know of no bound independent of the dimension of the space.

**Problem 11:** Is there a number $N(d)$ so that

$$
\text{if dim } (X) = d \text{ and } n > N(d), \ P(n,X) < \frac{1}{2}
$$

As was pointed out in the discussion of the P-convex analog to 1.5, $N(d)$ would have to be at least $2^d$.

Related to this, we have
Problem 12: Is there a number $D(n)$ so that

$$\text{if } P(n,X) = \frac{1}{2} \text{ then } \dim(X) > D(n)?$$

The major problem regarding the Banach-Saks property and $B$-convexity is

Problem 13: Does every $B$-convex space have the Banach-Saks property?

If the answer to Problem 13 is yes, Problem 1 is solved.

The converse to Problem 13 is false, as was pointed out in Proposition 4.15.

Related to this we have

Problem 14: Is there a reflexive space which does not have the Banach-Saks property?

If the answer to Problem 14 is no, Problem 13 is equivalent to Problem 1.

Problem 14 was first raised by Sakai in [28]. In [19] Klee attempted an affirmative solution, but as was pointed out in [33], his example was not reflexive.


