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STRESSES AND DEFORMATIONS IN MULTIPLY BIAS PNEUMATIC TIRES SUBJECT TO INFLATION PRESSURE LOADING

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in the Graduate School of The Ohio State University

By

HOWELL KEITH BREWER, B.M.E., M.S.

The Ohio State University

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Approved by

[Signature]
Adviser
Department of Engineering Mechanics
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VITA

September 24, 1937 . . . Born, Corydon, Indiana

1961. . . . . . . . B.M.E., Cleveland State University, Cleveland, Ohio

1962. . . . . . . . Politecnico di Torino, Italy

1963. . . . . . . . Research Engineer, B. F. Goodrich Research Center, Brecksville, Ohio

1967. . . . . . . . M. Sc., The Ohio State University, Columbus, Ohio

1968. . . . . . . . Research Engineer, Air Force Flight Dynamics Laboratory, Dayton, Ohio

PUBLICATIONS


FIELDS OF STUDY

Major Field: Engineering Mechanics

Studies in Plates and Shells. Professors Arthur Leissa and Peter Korda

Studies in Classical Dynamics. Professor Charles West

Studies in Applied Mathematics. Professor Drobot
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NOTATION

$A_1, A_2$ — Lame' parameters

$A_{ij}, B_{ij}, D_{ij}$ — submatrices in laminate constitutive equations

$C_{ijkl}, S_{ijkl}$ — stiffness and compliance tensors

$E_{ij}$ — Young's modulus

$G_{ij}$ — shear modulus

$K_\phi, K_\theta, K_{\phi\theta}$ — change in curvature and torsion

$L_f$ — force on wheel flange

$N$ — number of terms in least square approximation, number of components in a tensor

$M_\phi, M_\theta, M_{\phi\theta}$ — bending moments

$N_\phi, N_\theta, N_{\phi\theta}$ — membrane force resultants

$Q_\phi, Q_\theta$ — transverse shear force resultants

$P_f$ — tensile force in a single cord

$T_b$ — tensile force in bead hoop
\( R_{ij}, Q_{ij} \) — stiffness and compliance matrices of 2-dimensional composite

\( R \) — residual in least square approximation

\([T]\) — 6 x 6 matrix of variable coefficients

\([C]\) — 6 x 1 matrix of arbitrary constants

\([Y]\) — 6 x 6 matrix of complementary solutions

\([Z]\) — 6 x 1 matrix of particular solutions

\([P]\) — 6 x 1 matrix of non-homogeneous loading functions

\( a_0, a' \) — diagonal length of rhombus in unlifted and lifted tire resp.

\( a_k \) — coefficients in least square approximation of tire meridian

\( C_t \) — cord extension ratio

\( d \) — dimensionality of a space

\( d_b \) — diameter of bead wire

\( d_f \) — diameter of tire cord

\( h \) — thickness of tire carcass
\( h_k \) — Z-coordinate of \( k^{th} \) ply

\( k \) — rank of a tensor

\( \bar{k}, \bar{l}, \bar{m}, \bar{n} \) — elastic constants of composite in Hermans' equations

\( k_f, l_f, m_f, n_f \) — elastic constants of fiber in Hermans' equations

\( k_m, l_m, m_m, n_m \) — elastic constants of matrix in Hermans' equations

\( n \) — total number of cords in a single ply, number of experimental x-y coordinates used to define tire meridian

\( n_0, n' \) — cord density in unlifted and lifted tire resp.

\( n_b \) — number of wire strands in bead hoops

\( n_p \) — number of plies in tire

\( p_f, p_m, \bar{p} \) — fiber, matrix and composite properties resp.

\( p_n \) — number of pressure increments

\( q_n, q_\phi \) — external loads on shell

\( r \) — radius from axis of rotation to point on tire meridian

\( r_b \) — bead hoop radius
$r_m$ — radius to point of maximum width on tire meridian

$r_s$ — minimum radius of curvature

$r_{\phi}, r_\theta$ — principal radii of curvature

$S_\phi, S'$ — side length of rhombus in unlifted and lifted tire resp.

d$S_z$ — elemental arc length at a distance $z$ above neutral surface

$S_{\phi}, S_\theta$ — arc length along meridian and parallel resp.

$t$ — thickness of a single ply

$t_{ij}$ — transformation matrix

$u, v, w$ — displacements tangent to meridian, tangent to parallel and normal to surface resp.

$V_f, V_m$ — volume fraction of fiber and matrix resp.

$x_1, x_2, x_3$ — Cartesian coordinates

$x_i, y_i$ — experimental coordinates of tire meridian

$[y]$ — $6 \times 1$ matrix of fundamental variables

$z$ — shell thickness coordinate
\( a_1, a_2 \) — curvilinear coordinates of a surface

\( \rho_i, a_i \) — polar coordinates of tire meridian

\( \beta \) — cord angle in lifted tire

\( \gamma \) — cord angle in green tire

d\( \delta \), d\( \chi \) — central angles defined in Figure 34

\( \varepsilon_{ij} \) — strain tensor

\( \varepsilon_{\phi}, \varepsilon_{\theta}, \gamma_{\phi\theta} \) — strains in shell neutral surface

\( \xi \) — angle between tangent to meridian and axis of rotation

\( \eta \) — factor in Halpin-Tsai Formulae

\( \eta_{\phi\theta} \) — shear coupling ratio

\( \phi, \theta \) — curvilinear coordinates of tire surface

\( \mu_m, \mu_t, \mu \) — elastic constants in Hermans’ Equations

\( \nu_{ij} \) — Poisson’s Ratios

\( \xi \) — numerical factor in Halpin-Tsai Formulae
\( \sigma_{ij} \) — stress tensor

\( \sigma_b \) — stress in bead wire

\( \sigma_f \) — stress in a single tire cord

\( \tau_{\phi\theta} \) — interlaminar shear stress

\( \psi \) — orientation angle of fiber with respect to shell coordinates -- complement of tire cord angle \( \beta \)

\( \omega_\phi \) — rotation of shell element

\( \Gamma \) — length criterion in stepwise integration

\( \Phi \) — angle between normal to shell and the vertical axis
1. Introduction

1.1 Historical Review

In 1923 H. F. Schippel\textsuperscript{(1)} published what was probably the first attempt at stress analyzing pneumatic tires. His paper entitled “Fabric Stresses in Pneumatic Tires” was based on certain concepts borrowed from the theory of cylindrical pressure vessels. Today Schippel’s analytical work is regarded as mostly of historical interest, but the commentary that prefaces his paper is still worthy of note. The following is a brief excerpt:

Whether the result (improvements in tire design) is achieved by the process of experiment or by mathematical calculation, is not often clear. But there is no doubt that the latter method would often shorten the toil and expense of the former, by eliminating the unnecessary and condensing the necessary experiment, and thereby accelerate the conclusion.

Nearly fifty years have passed since Schippel’s work, yet it must be admitted that the analytical design of pneumatic tires remains an unattained goal. Despite this, tire engineers have continued to make vast improvements in their product, so much so, that the modern pneumatic tire is truly a remarkable structure. This, however, does not nullify the arguments cited by Schippel for an analytical approach to the problem.

It should not be inferred either that little significant progress has been made in the analytical area. On the contrary, during the last decade analytical tire mechanics has received a new impetus and has attracted the efforts of many researchers both here and abroad. The immense bibliography compiled by Frank and Hofferberth\textsuperscript{(2)} in their recent review of tire mechanics clearly indicates the magnitude of the effort that has been and is being put forth.

In the area of analytical stress analysis, the investigations thus far have dealt almost exclusively with axisymmetric loadings due to inflation pressure and centrifugal force. This is obviously because of the simplicity that results from eliminating any dependency on the circumferential coordinate.
The first rational attempt at predicting the shape and stress under inflation loading was probably that of Day and Purdy. Although this work was carried out in 1928, it was not published until quite recently. In this work, the equilibrium of a single isolated tire cord was considered. This allowed equations for cord tension and equilibrium shape to be derived. In 1955 Hofferberth published the first widely accepted theory of the inflated tire. In his analysis the tire is considered to be a membrane network of elastic cords, hinged at their crossover points and devoid of any elastic rubber matrix. Since the cords must carry the entire load the membrane forces have their vector resultant coincident with the cord direction at every point. From this requirement and the membrane equilibrium equations the membrane stresses can be calculated. However, to execute this calculation requires a priori knowledge of the principal radii of curvature of the inflated tire. Subsequently, Hofferberth developed an equation for predicting the inflated tire profile using the same mathematical model. But this too involved a knowledge of geometrical parameters which had to be measured from the inflated tire.

Hofferberth's equation for the tire equilibrium shape was in the form of a complicated hyperelliptic integral to which he gave no solution in his paper. Lauterbauch and Ames subsequently computed membrane stresses and equilibrium profiles by numerically integrating Hofferberth's equation on a digital computer. The recent works of Ames and Ames and Walston are essentially extensions of Hofferberth's approach.

Although the work cited above marked a major step forward in analytical tire mechanics, it nevertheless has several limitations. (1) The tire structure is considered to be a membrane; thus, the bending stiffness is ignored. Moreover, membrane theory does not allow a realistic determination of the boundary stress since in a tire, normal displacements are prevented at the bead areas. This condition is a violation of the assumptions of membrane theory. (2) The network theory does not account for the reinforcing effect of the rubber matrix. (3) The computations require as input data parameters which must be measured from the actual inflated tire.

In the early 1960's, S. K. Clark at the University of Michigan initiated a comprehensive study of tire mechanics and related problems, under the sponsorship of five leading U. S. tire companies. Clark has made significant contributions to the theory of
cord-rubber composites$^{(9, 10, 11)}$ and other tire problems.$^{(12, 13)}$ He has also conducted extensive research in the stress analysis of inflated tires. However, this latter work has not yet been published in the open literature.$^{(14)}$

1.2 Pneumatic Tire Construction

There are at the present time three fundamentally different types of pneumatic tire constructions: (1) bias tire, (2) belted-bias tire, (3) radial tire. They are distinguished by the presence or absence of additional reinforcing plies under the tread and by the cord path of the underlying carcass (Fig. 1). The radial and belted-bias tire constructions are comparatively recent innovations in the industry, but their superior wear resistance has already challenged the bias tire in automotive applications. Aircraft tires, however, must perform under an entirely different set of operating conditions, of which the high rotational speeds and large vertical deflections are the two most demanding. Whether or not workable radial or belted-bias aircraft tires can be made remains to be seen, but for the present and foreseeable future, bias tires will continue to be used exclusively, on aircraft.

The construction of a bias tire requires two basic steps: (1) the laying up of the green tire and (2) the lifting and curing of the green tire into its final toroidal form.

The green carcass is made by laying sheets or plies of uncured cord-rubber composite around a cylindrical building drum. Each ply is cut so that the reinforcing cords form a constant angle $\pm \gamma$ with the building drum circumferential line (Fig. 2). The angle $\gamma$ is known as the bias cutting angle. The green plies are laid down in an alternating fashion; that is, each ply of $+\gamma$ material is followed by one of $-\gamma$, and so on until the required number of plies has been put down. Thus, bias tires always contain an even number of plies.

In addition to the ply material, the green tire contains two or more bead rings and a final thick layer of uncured rubber. The bead rings are circular hoops wound with strands of high strength steel wire. Their function is to provide a rigid anchoring point for the ply fabric. The final rubber layer becomes a protective and wear resistant tread in the finished tire.

Once building of the green tire is completed, it is removed from the building drum and positioned on end in the lower half of the curing mold. As the mold halves close, a
pressurized rubber bladder expands inside the green tire cylinder. It is during this phase of manufacture that the cylindrical green tire is transformed or “lifted” into its final toroidal shape. When the lifting process is complete and the mold halves are closed, heat and pressure are applied to begin the curing cycle.

As the green tire undergoes deformation during lifting, both cord angle and cord density changes accompany the dimensional changes. These parameters greatly influence the elastic properties of the finished tire, and therefore it is necessary to develop expressions relating their initial and final values. The following development is similar to that of Gough. It will be assumed that:

(1) The cords hinge or swivel at their crossover points (pantographic action).

(2) All cords in the -γ plies have exactly the same geometrical shape of path; the same applies to the +γ plies except that their paths are of opposite hand.

(3) The cords are evenly spaced around the tire periphery.

Figure 3 shows a portion of the tire before and after lifting. We will focus our attention on the deformation of the rhombus formed by the intersecting cords. Let $s_0$, $a_0$, $\gamma$ be the side length, diagonal length and cord angle before forming and $s'$, $a'$, $\beta$ their corresponding values afterward. Then from the figure:

$$\cos \gamma = \frac{a_0}{2s_0}$$  \hspace{1cm} (1.1)

$$\cos \beta = \frac{a'}{2s'}$$  \hspace{1cm} (1.2)

We now note that the angles $d\theta$ subtended by both the deformed and undeformed diagonals are equal. This follows from the fact that the length of the diagonal is $1/n$ th of the circumference of both the deformed and undeformed parallel circles, $n$ being the total number of cords in a single ply. Since $n$ is constant, the subtended angles must be equal.
Thus we have the equality

\[ \frac{a'}{r} = \frac{a_0}{r_b} \]  

\[ (1.3) \]

where \( r \) and \( r_b \) are the radii from the axis of rotation, of the lifted and green tire respectively. Equations (1.1), (1.2) and (1.3) can be combined to give

\[ \frac{s' \cos \beta}{r} = \frac{s_0 \cos \gamma}{r_b} \]  

\[ (1.4) \]

or by defining the cord extension ratio \( c_t \) as

\[ c_t = \frac{s'}{s_0} \]  

\[ (1.5) \]

Equation (1.4) becomes

\[ \frac{\cos \gamma}{r_b} = c_t \frac{\cos \beta}{r} \]  

\[ (1.6) \]

Equation (1.6) is known as the "lift equation." Its accuracy depends upon how well the assumption of pantographic cord motion is realized in the lifted tire. Gough,\(^{(15)}\) in a detailed analysis of cord paths in tires, discusses deviations from the lift equation due to sliding of the crossover points. However, the differences are not great. This is also shown in Fig. 4, which compares the lift equation with experimental measurements.\(^{(16)}\)

The change in cord density of the lifted tire is easily calculated from the fact that the total number of cords intersecting a parallel circle on the green and lifted tire must be the same. Thus if \( n_0 \) and \( n' \) are the initial and final cord densities expressed as the number of cords per unit length perpendicular to the cord path, we have the equality

\[ 2\pi r_b n_0 \sin \gamma = 2\pi r n' \sin \beta \]  

\[ (1.7) \]
1.3 Statement of the Problem

It is proposed in this work to calculate the stresses and deformations in an inflated aircraft tire by considering it to be a layered, anisotropic, toroidal shell of revolution. This approach has the following salient features:

(1) The calculation requires only basic elastic properties of the cord and rubber, construction parameters, and geometrical shape of the uninflated tire as input data. No intermediate experimentation is required.

(2) The reinforcing effect of the rubber is taken into account by considering the ply to be a cord-rubber composite.

(3) The approach is capable of treating any general tire meridian shape and is therefore not limited to simple circular or elliptical cross sections.

(4) The bending rigidity of the tire shell will be taken into account. This will enable a more realistic determination of the boundary stresses in the area of the bead.

(5) The effect of large deformations will be accounted for by incrementally applying the load and adjusting the intermediate shell geometry.

The overall problem can be divided logically into four smaller sub-problems. These are illustrated diagrammatically in Fig. 5. The first step is to experimentally determine the basic elastic properties of the cord and rubber. These will serve as input data to micro-mechanics theory which will determine the overall sheet elastic moduli. Next the
laminate properties will be calculated from the sheet properties, orientation, and stacking sequence. These will be combined with classical linear shell theory to predict the response of the structure under load. Once the structural response is known, the process is reversed to compute stresses and strains at any point in the tire.

We will not concern ourselves here, specifically, with the problems attendant in experimentally defining the cord and rubber elastic properties. Instead we shall assume that these are given, and proceed with the task of determining the single ply moduli.
2. Some Basic Principles of Anisotropic Elasticity

2.1 Introduction

In this chapter we will examine briefly some of the basic concepts of the theory of anisotropic elasticity. The purpose is to provide the theoretical framework for discussing the micro-mechanics of cord-rubber composites and the elastic properties of two-dimensional orthotropic sheets, topics which will be treated in later chapters. Since the objective here is one of providing background information, the discussion must of necessity be very selective. The standard reference works of Hearmon,\(^{(17)}\) Lekhitnitskii,\(^{(18)}\) and Love\(^{(19)}\) are therefore recommended for detailed study. The development and notation in what follows is essentially that of Tsai\(^{(20)}\) and Hearmon.\(^{(17)}\)

2.2 Cartesian Tensors — Indicial Notation

The use of tensor analysis in formulating problems in elasticity has the advantage of notational simplicity and brevity. This is especially true when working in generalized coordinate systems or when dealing with anisotropic materials. Although curvilinear coordinates will be used later to define the neutral surface of the tire shell, they will not be used directly to specify its elastic properties. It will prove more advantageous to define these properties with reference to a local cartesian system which is coincident with the curvilinear system at every point. Thus, the mathematical complexity of general tensor theory is not required, and we need only consider a minimum number of concepts pertaining to cartesian tensors.

All geometrical and physical quantities are in reality tensors of various rank. A scalar quantity such as mass is a tensor of rank zero. Vectors such as velocity and acceleration are tensors of the first rank, and stress and strain are tensors of the second rank. The rank of a tensor determines the number of components that are required to define it in a certain dimensional space. Thus, the total number of components \(N\) of a tensor of rank \(k\) is
where \( d \) is the dimension of the space. A second rank tensor, for example, has 9 components in 3-dimensional space. The tensorial character of a quantity is indicated symbolically by the presence of subscripts. If such a subscript appears alone, i.e., it is not repeated, then it is referred to as a free subscript or free index. The number of such free indices indicates the rank of the tensor. \( \sigma_{ij} \) and \( C_{ijkl} \), for example, are second and fourth rank tensors respectively.

It is often convenient to arrange the components of a tensor in matrix form. This does not mean, however, that matrices and tensors are equivalent. The elements of a matrix may represent any arbitrary array of quantities. But if they represent components of a tensor, they must obey a certain set of rules which specify how they are to change under a coordinate transformation. These are known as the transformation equations.

For our purposes, a coordinate transformation will consist of a rotation of one cartesian reference frame with respect to another. The relation between the new and old coordinates, denoted by \( X'_i \) and \( X_i \) respectively, are given by

\[
\begin{bmatrix}
X'_1 \\
X'_2 \\
X'_3
\end{bmatrix} =
\begin{bmatrix}
t_{11} & t_{12} & t_{13} \\
t_{21} & t_{22} & t_{23} \\
t_{31} & t_{32} & t_{33}
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2 \\
X_3
\end{bmatrix}
\]  

(2.2)

where the \( t_{ij} \) are direction cosines between the \( i \)-axis in the new system and the \( j \)-axis in the old one. For example, if the coordinate transformation consisted of a counterclockwise rotation through an angle \( \psi \) about the 3-axis, then Equation (2.2) would be
where $m = \cos \psi$ and $n = \sin \psi$. The matrix $t_{ij}$ of direction cosines is known as the transformation matrix.

We now introduce two conventions of the subscripted or indicial notation as follows:

1. Range Convention: an index which occurs just once in a term, i.e., free index, takes on all the values $1, 2, \ldots, d$ where $d$ is the dimensionality of the space.

2. Summation Convention: an index which is repeated just twice in a term implies summation with respect to that index over the range $1, 2, \ldots, d$.

Because of these two conventions we may rewrite Equation (2.2) as,

$$X'_i = t_{ij} X_j$$

(2.4)

2.3 Transformation Equations

It was stated above that the components of a tensor must obey a certain transformation equation. The general form of this equation is

$$C'_{ij \ldots kl} = t_{im} t_{jn} \ldots t_{ko} t_{lp} C_{mn \ldots op}$$

(2.5)

where $C_{ij \ldots kl}$ are the components of the transformed tensor $C_{mn \ldots op}$ and $t_{ij}$ is the transformation matrix. For first, second, and fourth rank tensors, Equation (2.5) has the form
\[ c'_i = t_{im} c_m \] (2.6)
\[ c'_{ij} = t_{im} t_{jn} c_{mn} \] (2.7)
\[ c'_{ijkl} = t_{im} t_{jn} t_{ko} t_{lp} c_{mnop} \] (2.8)

It is instructive to carry out the details of a tensor transformation, using the indicial notation. Let us, for example, consider the second rank tensor of Equation (2.7). We will specify a coordinate transformation in 2-dimensional space which consists of a positive rotation through an angle \( \psi \). The transformation matrix is

\[
t_{ij} = \begin{bmatrix} m & n \\ -n & m \end{bmatrix}
\] (2.9)

We now proceed as follows:

Let \( i = 1, j = 1 \) or,

\[ c'_{11} = t_{1m} t_{1n} c_{mn} \]

Summing on \( m \) gives

\[ c'_{11} = t_{11} t_{1n} c_{1n} + t_{12} t_{1n} c_{2n} \]

Summing on \( n \) gives

\[ c'_{11} = t_{11} (t_{11} c_{11} + t_{12} c_{12}) + t_{12} (t_{11} c_{21} + t_{12} c_{22}) \] (2.10)

The other three components are obtained in a similar manner. The final results are:
Now when the components of the transfer matrix $t_{ij}$ are substituted into Equations (2.10) through (2.13), the results can be written in matrix form as

\[
\begin{bmatrix}
  \mathbf{C}_{11}' \\
  \mathbf{C}_{22}' \\
  \mathbf{C}_{12}' \\
  \mathbf{C}_{21}'
\end{bmatrix} =
\begin{bmatrix}
  m^2 & n^2 & mn & mn \\
  n^2 & m^2 & -mn & -mn \\
  -mn & mn & m^2 & -n^2 \\
  -mn & mn & -n^2 & m^2
\end{bmatrix}
\begin{bmatrix}
  \mathbf{C}_{11} \\
  \mathbf{C}_{22} \\
  \mathbf{C}_{12} \\
  \mathbf{C}_{21}
\end{bmatrix}
\]

If $\mathbf{C}_{ij}$ is a symmetric tensor, then

\[
\mathbf{C}_{ij} = \mathbf{C}_{ji}
\]

and Equation (2.14) simplifies further to

\[
\begin{bmatrix}
  \mathbf{C}_{11}' \\
  \mathbf{C}_{22}' \\
  \mathbf{C}_{12}' \\
  \mathbf{C}_{21}'
\end{bmatrix} =
\begin{bmatrix}
  m^2 & n^2 & 2mn \\
  n^2 & m^2 & -2mn \\
  -mn & mn & m^2-n^2
\end{bmatrix}
\begin{bmatrix}
  \mathbf{C}_{11} \\
  \mathbf{C}_{22} \\
  \mathbf{C}_{12}
\end{bmatrix}
\]

\[
(2.14)
\]

\[
(2.15)
\]

\[
(2.16)
\]
If $C_{ij}$ represents the stress tensor, then Equation (2.16) is recognized as the familiar plane stress transformation equations upon which Mohr’s circle is based.

In 3-dimensional space, the range of the indices is 1, 2, and 3. A rotation about the 3-axis by an angle $\psi$ gives the transformation matrix,

$$
t_{ij} = \begin{bmatrix}
m & n & 0 \\
-n & m & 0 \\
0 & 0 & 1
\end{bmatrix}
$$

(2.17)

By following the same procedure as above, the final transformed equations for a symmetric tensor of second rank in 3-dimensional space are

$$
\begin{bmatrix}
C'_{11} \\
C'_{22} \\
C'_{33} \\
C'_{23} \\
C'_{31} \\
C'_{12}
\end{bmatrix} =
\begin{bmatrix}
m^2 & n^2 & 0 & 0 & 0 & 2mn \\
n^2 & m^2 & 0 & 0 & 0 & -2mn \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & m & -n & 0 \\
0 & 0 & 0 & n & m & 0 \\
-mn & mn & 0 & 0 & 0 & m^2-n^2
\end{bmatrix}
\begin{bmatrix}
C_{11} \\
C_{22} \\
C_{33} \\
C_{23} \\
C_{31} \\
C_{12}
\end{bmatrix}
$$

(2.18)
2.4 Generalized Hooke's Law

In the theory of elasticity, the mathematical concepts of stress and strain are developed independently of each other and, moreover, independently of the type of elastic material involved. These concepts, therefore, remain the same whether one is dealing with an isotropic or anisotropic body. The consequence of this is that the equations of motion and the strain compatibility equations also carry over unaltered from isotropic to anisotropic elasticity. It is only in the relationship between stress and strain, i.e., Hooke's Law, that the differences come into play. The form of these differences and the way that they come about follow directly from certain symmetry properties of the stiffness and compliance tensors in Generalized Hooke's Law. This law is embodied in either of the equations,

\[ \sigma_{ij} = C_{ijkl} \varepsilon_{kl} \]  \hspace{1cm} (2.19)

\[ \varepsilon_{ij} = S_{ijkl} \sigma_{kl} \]  \hspace{1cm} (2.20)

where \( \sigma_{ij}, \varepsilon_{ij}, C_{ijkl}, \) and \( S_{ijkl} \) are the stress, strain, elastic stiffness, and elastic compliance tensors respectively. The stiffness and compliance tensors are of fourth rank and, therefore, in 3-dimensional space they contain 81 components each.

We will need to examine the way in which these components change under a coordinate transformation. It will be convenient, however, in the case of these fourth rank tensors, to introduce a modification in the indicial notation, which will be called the contracted indicial notation.

In the contracted indicial notation, the number of free indices is reduced from 4 to 2 but the range is expanded from 3 to 9. The number of components according to Equation (2.1) remains unchanged; e.g., \( 3^4 = 81 \) and \( 9^2 = 81 \).

The contracted notation is established by arbitrarily replacing the double-index system of the stress and strain tensors with a single index system as shown below.
\[ \begin{align*}
\sigma_{11} &= \sigma_1 \quad &\varepsilon_{11} &= \varepsilon_1 \\
\sigma_{22} &= \sigma_2 \quad &\varepsilon_{22} &= \varepsilon_2 \\
\sigma_{33} &= \sigma_3 \quad &\varepsilon_{33} &= \varepsilon_3 \\
\sigma_{23} &= \sigma_4 \quad &2\varepsilon_{23} &= \varepsilon_4 \\
\sigma_{31} &= \sigma_5 \quad &2\varepsilon_{31} &= \varepsilon_5 \\
\sigma_{12} &= \sigma_6 \quad &2\varepsilon_{12} &= \varepsilon_6 \\
\sigma_{32} &= \sigma_7 \quad &2\varepsilon_{32} &= \varepsilon_7 \\
\sigma_{13} &= \sigma_8 \quad &2\varepsilon_{13} &= \varepsilon_8 \\
\sigma_{21} &= \sigma_9 \quad &2\varepsilon_{21} &= \varepsilon_9
\end{align*} \]

(2.21)

Note that, in contracted notation, engineering strain is used instead of tensorial strain. Equations (2.19) and (2.20) may now be written

\[ \varepsilon_i = S_{ij} \sigma_j \quad \quad \quad (2.22) \]

\[ (i, j = 1, 2, \cdots, 9) \]

\[ \sigma_i = C_{ij} \varepsilon_j \quad \quad \quad (2.23) \]

It should also be noted that, in this notation, the range and summation conventions are retained, but some modifications in the interpretation must be made. First, the range of
free indices no longer corresponds to the dimensionality of the space. Second, the tensorial rank no longer corresponds to the number of free indices.

From the theory of elasticity it is known that the stress and strain tensors are symmetric, i.e.,

\( \sigma_{ij} = \sigma_{ji} \)  \hspace{1cm} (2.24)

\( \epsilon_{ij} = \epsilon_{ji} \)

This allows the range of the indices in Equations (2.22) and 2.23) to be reduced from 9 to 6. It also reduces the number of independent components of \( S_{ij} \) and \( C_{ij} \) from 81 to 36. \(^{(21)}\) Thus Equations (2.22) and (2.23) become

\[ \epsilon_i = S_{ij} \sigma_j \]  \hspace{1cm} (2.25) \quad (i, j = 1, 2, \cdots, 6)

\[ \sigma_i = C_{ij} \epsilon_j \]  \hspace{1cm} (2.26)

Finally, one further notational convention is introduced. The shear stresses and strains will also be denoted by \( \tau_{ij} \) and \( \gamma_{ij} \), \( (i \neq j) \). This will allow us, later on, to conform to the standard notation used in shell theory. These notational conventions are now summarized below.
It was noted that the compliance and stiffness tensors of Equations (2.25) and (2.26) have 36 independent components. This may be reduced further to 21 if $S_{ij}$ and $C_{ij}$ can be shown to be symmetric. With the assumption that stresses can be derived from an elastic potential function, the symmetry of $S_{ij}$ and $C_{ij}$ can be proven.\(^{(21)}\) No further reductions in the number of independent components can be achieved unless the material possesses some sort of elastic symmetry. Thus, Hooke's Law for a completely anisotropic material is

\[
\sigma_{11} = \sigma_1 \hspace{1cm} \varepsilon_{11} = \varepsilon_1
\]

\[
\sigma_{22} = \sigma_2 \hspace{1cm} \varepsilon_{22} = \varepsilon_2 \hspace{1cm} (2.27)
\]

\[
\sigma_{33} = \sigma_3 \hspace{1cm} \varepsilon_{33} = \varepsilon_3
\]

\[
\sigma_{23} = \sigma_4 = \tau_{23} \hspace{1cm} 2\varepsilon_{23} = \varepsilon_4 = \gamma_{23}
\]

\[
\sigma_{31} = \sigma_5 = \tau_{31} \hspace{1cm} 2\varepsilon_{31} = \varepsilon_5 = \gamma_{31}
\]

\[
\sigma_{12} = \sigma_6 = \tau_{12} \hspace{1cm} 2\varepsilon_{12} = \varepsilon_6 = \gamma_{12}
\]
where the symmetry of the $S_{ij}$ matrix is indicated by omitting terms appearing below the main diagonal. A similar equation can be written for the inverted form, involving the $C_{ij}$ matrix.

We have specified the conversion between the contracted and uncontracted notation for stress and strain by Equation (2.27). It now remains to develop a similar conversion between the stiffness and compliance tensors, $C_{ijkl}$ and $S_{ijkl}$, and their contracted form, $C_{ij}$ and $S_{ij}$. This does not follow directly from Equation (2.27) because correction factors are introduced due to the difference between engineering and tensorial strains. The proper conversion can be achieved by expanding Equations (2.19), (2.20), (2.25), and (2.26) and comparing the corresponding coefficients. For example, for $i = j = 1$ in Equation (2.20)

$$
\varepsilon_{11} = S_{1111} \sigma_{11} + S_{1122} \sigma_{22} + S_{1133} \sigma_{33} + (S_{1123} + S_{1132}) \sigma_{23}
$$

$$
+ (S_{1131} + S_{1113}) \sigma_{31} + (S_{1112} + S_{1121}) \sigma_{21}
$$

(2.29)
Expanding Equation (2.25),

\[ \varepsilon_1 = S_{11} \sigma_1 + S_{12} \sigma_2 + S_{13} \sigma_3 + S_{14} \sigma_4 + S_{15} \sigma_5 + S_{16} \sigma_6 \]  

(2.30)

By Comparing Equations (2.29) and (2.30) and by recalling the symmetry, \( S_{11kl} = S_{11lk} \), one obtains

\[
\begin{align*}
S_{1111} &= S_{11} \\
S_{1122} &= S_{12} \\
S_{1133} &= S_{13} \\
2S_{1123} &= S_{14} \\
2S_{1131} &= S_{15} \\
2S_{1112} &= S_{16}
\end{align*}
\]  

(2.31)

If this process is repeated, the following conversion equations can be developed for the components of the compliance matrix:

\[
\begin{align*}
S_{ijkl} &= S_{qr} \text{ for } q, r = 1, 2, 3 \\
2S_{ijkl} &= S_{qr} \text{ for } q = 1, 2, 3; r = 4, 5, 6 \\
4S_{ijkl} &= S_{qr} \text{ for } q, r = 4, 5, 6
\end{align*}
\]  

(2.32)
In a similar fashion, the conversion relations for the stiffness tensor can be established. Let \( i = j = 1 \) in Equation (2.19);

\[
\sigma_{11} = C_{1111} \varepsilon_{11} + C_{1122} \varepsilon_{22} + C_{1133} \varepsilon_{33} + (C_{1123} + C_{1132}) \varepsilon_{23} + (C_{1131} + C_{1113}) \varepsilon_{31} + (C_{1112} + C_{1121}) \varepsilon_{12}
\]  
(2.33)

Expanding Equation (2.26),

\[
\sigma_1 = C_{11} \varepsilon_1 + C_{12} \varepsilon_2 + C_{13} \varepsilon_3 + C_{14} \varepsilon_4 + C_{15} \varepsilon_5 + C_{16} \varepsilon_6
\]  
(2.34)

Again, by comparing terms and using the symmetry of the stiffness tensor, one obtains

\[
\begin{align*}
C_{1111} &= C_{11} \\
C_{1122} &= C_{12} \\
C_{1133} &= C_{13} \\
C_{1123} &= C_{14} \\
C_{1131} &= C_{15} \\
C_{1112} &= C_{16}
\end{align*}
\]  
(2.35)

By continuing this process for other indices, it can be shown, in general, that

\[
C_{ijkl} = C_{qr}
\]  
(2.36)
Thus, the conversion factor for the components of the stiffness tensor is unity, while those for the compliance tensor are given by Equation (2.32).

We now turn to the problem of establishing the transformation equations for the stiffness and compliance tensors under a coordinate rotation. As before, the rotation that is of interest to us is a rotation about the 3-axis through an angle $\psi$, whose transformation matrix $t_{ij}$ is given by Equation (2.17).

Since both $C_{ijkl}$ and $S_{ijkl}$ are fourth rank tensors, their transformation is governed by Equation (2.8). Note that, when carrying out this transformation, one must revert to the full indicial tensor notation, rather than use the contracted notation. This is because $C_{ij}$ and $S_{ij}$ are not really tensors as such, in that they do not obey the transformation Equation (2.5). Thus the transformed stiffness and compliance tensors are

\[ C_{ijkl} = t_{im} t_{jn} t_{ko} t_{lp} C_{mnop} \]

\[ S_{ijkl} = t_{im} t_{jn} t_{ko} t_{lp} S_{mnop} \]  

(2.37)

For example, if $i = j = k = l = 1$, then from Equation (2.37)

\[ S_{1111} = t_{1m} t_{1n} t_{1o} t_{1p} S_{mnop} \]

Summing on the four repeated indices, $m$, $n$, $o$, and $p$, and noting from the transformation matrix that $t_{13} = 0$, one finally obtains

\[ S_{1111} = t_{11} \left\{ t_{11} \left[ t_{11} (t_{11} S_{1111} + t_{12} S_{1112}) + t_{12} (t_{11} S_{1121} + t_{12} S_{1122}) \right] + t_{12} \left[ t_{11} (t_{11} S_{1211} + t_{12} S_{1212}) + t_{12} (t_{11} S_{1221} + t_{12} S_{1222}) \right] \right\} \]

\[ + t_{12} \left\{ t_{11} \left[ t_{11} (t_{11} S_{2111} + t_{12} S_{2112}) + t_{12} (t_{11} S_{2121} + t_{12} S_{2122}) \right] + t_{12} \left[ t_{11} (t_{11} S_{2211} + t_{12} S_{2212}) + t_{12} (t_{11} S_{2221} + t_{12} S_{2222}) \right] \right\} \]
Combining terms and substituting for the components of the transformation matrix $t_{ij}$ from Equation (2.17), one obtains

$$S'_{1111} = m^4 S_{1111} + m^3 n S_{1112} + m^3 n S_{1121} + m^2 n^2 S_{1122}$$

$$+ m^3 n S_{1211} + m^2 n^2 S_{1212} + m^2 n^2 S_{1221} + m n^3 S_{1222}$$

$$+ m^3 n S_{2111} + m^2 n^2 S_{2112} + m^2 n^2 S_{2121} + m n^3 S_{2122}$$

$$+ m^2 n^2 S_{2211} + m n^3 S_{2212} + m n^3 S_{2221} + n^4 S_{2222}$$

(2.38)

Introducing the conversion factors from Equation (2.31), this may be written in contracted notation as

$$S'_{11} = m^4 S_{11} + \frac{m^3 n}{2} S_{16} + \frac{m^3 n}{2} S_{16} + m^2 n^2 S_{12}$$

$$+ \frac{m^3 n}{2} S_{16} + \frac{m^2 n^2}{4} S_{66} + \frac{m^2 n^2}{4} S_{66} + \frac{m n^3}{2} S_{62}$$

$$+ \frac{m^3 n}{2} S_{16} + \frac{m^2 n^2}{4} S_{66} + \frac{m^2 n^2}{4} S_{66} + \frac{m n^3}{2} S_{62}$$

$$+ m^2 n^2 S_{21} + \frac{m n^3}{2} S_{26} + \frac{m n^3}{2} S_{26} + n^4 S_{22}$$

(2.39)

Since $S_{ij}$ is symmetric, Equation (2.39) can be further reduced to

$$S'_{11} = m^4 S_{11} + 2 m^2 n^2 S_{12} + 2 m^3 n S_{16} + n^4 S_{22}$$

$$+ 2 m n^3 S_{26} + m^2 n^2 S_{66}$$

(2.40)
By repeating this process, the other 20 components of the compliance matrix, and all of the components of the stiffness matrix, can be derived. The final results are shown in the following tabular form.

<table>
<thead>
<tr>
<th>( S'<em>{11}(C'</em>{11}) )</th>
<th>( S'<em>{12}(C'</em>{12}) )</th>
<th>( S'<em>{16}(2C'</em>{16}) )</th>
<th>( S'<em>{22}(2C'</em>{22}) )</th>
<th>( S'<em>{26}(2C'</em>{26}) )</th>
<th>( S'<em>{66}(4C'</em>{66}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m^4 )</td>
<td>( 2m^2n^2 )</td>
<td>( 2m^3n )</td>
<td>( n^4 )</td>
<td>( 2mn^3 )</td>
<td>( m^2n^2 )</td>
</tr>
<tr>
<td>( m^2n^2 )</td>
<td>( m^4 + n^4 )</td>
<td>( mn^3 )</td>
<td>( m^2n^2 )</td>
<td>( m^3n )</td>
<td>( -m^2n^2 )</td>
</tr>
<tr>
<td>( -2m^3n )</td>
<td>( 2m^3n )</td>
<td>( m^4 )</td>
<td>( 2mn^3 )</td>
<td>( 3m^2n^2 )</td>
<td>( m^3n )</td>
</tr>
<tr>
<td>( -2mn^3 )</td>
<td>( -3m^2n^2 )</td>
<td>( -n^4 )</td>
<td>( -mn^3 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( n^4 )</td>
<td>( 2m^2n^2 )</td>
<td>( -2mn^3 )</td>
<td>( m^4 )</td>
<td>( -2m^3n )</td>
<td>( m^2n^2 )</td>
</tr>
<tr>
<td>( -2mn^3 )</td>
<td>( 3m^2n^2 )</td>
<td>( -n^4 )</td>
<td>( -3m^2n^2 )</td>
<td>( -m^3n )</td>
<td></td>
</tr>
<tr>
<td>( 4m^2n^2 )</td>
<td>( -8m^2n^2 )</td>
<td>( 4mn^3 )</td>
<td>( 4m^2n^2 )</td>
<td>( 4m^3n )</td>
<td>( (m^2 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( -4m^3n )</td>
<td></td>
<td></td>
<td>( -n^2)^2 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( S'<em>{13}(C'</em>{13}) )</th>
<th>( S'<em>{23}(C'</em>{23}) )</th>
<th>( S'<em>{36}(2C'</em>{36}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m^2 )</td>
<td>( n^2 )</td>
<td>( mn )</td>
</tr>
<tr>
<td>( n^2 )</td>
<td>( m^2 )</td>
<td>( -mn )</td>
</tr>
<tr>
<td>( -2mn )</td>
<td>( 2mn )</td>
<td>( m^2n^2 )</td>
</tr>
</tbody>
</table>

(2.41)
\begin{align*}
\begin{array}{|c|c|c|c|}
\hline
S_{44}(C_{44}) & S_{45}(C_{45}) & S_{55}(C_{55}) \\
\hline
S'_{44}(C'_{44}) & m^2 & -2mn & n^2 \\
S'_{45}(C'_{45}) & mn & m^2-n^2 & -mn \\
S'_{55}(C'_{55}) & n^2 & 2mn & m^2 \\
\hline
\end{array}
\end{align*}

\begin{align*}
\begin{array}{|c|c|}
\hline
S_{34}(C_{34}) & S_{35}(C_{35}) \\
\hline
S'_{34}(C'_{34}) & m & -n \\
S'_{35}(C'_{35}) & n & m \\
\hline
\end{array}
\end{align*}

\begin{align*}
\begin{array}{|c|c|c|c|c|c|}
\hline
S_{14}(C_{14}) & S_{15}(C_{15}) & S_{24}(C_{24}) & S_{25}(C_{25}) & S_{46}(2C_{46}) & S_{56}(2C_{56}) \\
\hline
S'_{14}(C'_{14}) & m^3 & -m^2n & mn^2 & -n^3 & m^2n & -mn^2 \\
S'_{15}(C'_{15}) & m^2n & m^3 & n^3 & mn^2 & mn^2 & m^2n \\
S'_{24}(C'_{24}) & mn^2 & -n^3 & m^3 & -m^2n & -m^2n & mn^2 \\
S'_{25}(C'_{25}) & n^3 & mn^2 & m^2n & m^3 & -mn^2 & -m^2n \\
S'_{46}(2C_{46}) & -2m^2n & 2mn^2 & 2m^2n & -2mn^2 & m^3 & n^3 & -mn^2 & -m^2n \\
S'_{56}(2C_{56}) & -2mn^2 & -2m^2n & 2mn^2 & 2m^2n & m^2n & m^3 & -n^3 & -mn^2 \\
\hline
\end{array}
\end{align*}
\[ S_{33}'(C_{33}) = S_{33}(C_{33}) \]  \hspace{1cm} (2.46)

In the case of the stiffness matrix \( C_{ij} \), appropriate factors, shown in the column and row headings, must be properly incorporated; e.g.,

\[ C_{11}' = m^4 C_{11} + 2m^2 n^2 C_{12} + 4m^3 n C_{16} + n^4 C_{22} \]  \hspace{1cm} (2.47)
\[ + 4mn^3 C_{26} + 4m^2 n^2 C_{66} \]

2.5 Elastic Symmetries

It has been shown above that there are 21 independent components of the elastic stiffness and compliance tensors in a general anisotropic material. This number can be reduced further only if the material displays certain elastic symmetries which are characterized by coordinate transformations. For example, if in a \( X_1, X_2, X_3 \) coordinate system, the \( X_1-X_2 \) plane is a plane of elastic symmetry, then a coordinate transformation which consists of an inversion of the \( X_3 \) axis will leave the \( C_{ij} \) and \( S_{ij} \) matrices unchanged; i.e., \( C_{ij}' = C_{ij}, S_{ij}' = S_{ij} \). This inversion is described by the transformation matrix

\[
 t_{ij} = \begin{bmatrix}
 1 & 0 & 0 \\
 0 & 1 & 0 \\
 0 & 0 & -1
\end{bmatrix}
\]  \hspace{1cm} (2.48)

Applying this transformation to the stress and strain tensors according to Equation (2.7), one obtains
Now considering the first equation of Generalized Hooke's Law, written in the new and old coordinate systems,

\[
\sigma_1' = \sigma_1, \quad \epsilon_1' = \epsilon_1
\]
\[
\sigma_2' = \sigma_2, \quad \epsilon_2' = \epsilon_2
\]
\[
\sigma_3' = \sigma_3, \quad \epsilon_3' = \epsilon_3
\]

(2.49)
\[
\sigma_4' = -\sigma_4, \quad \epsilon_4' = -\epsilon_4
\]
\[
\sigma_5' = -\sigma_5, \quad \epsilon_5' = -\epsilon_5
\]
\[
\sigma_6' = \sigma_6, \quad \epsilon_6' = \epsilon_6
\]

Since, by Equation (2.49), \(\sigma_1' = \sigma_1\), we must have

\[
\sigma_1' = C_{1j} \epsilon_j' = C_{1j} \epsilon_j
\]

(2.50)
\[
\sigma_j = C_{1j} \epsilon_j
\]

(2.51)

By expanding this equation and taking into account the relation between \(\epsilon_j'\) and \(\epsilon_j\) given in Equation (2.49), it can be concluded that

\[
C_{14} = C_{15} = 0
\]

(2.53)
In the same way, it can be shown, by considering the other five equations in the Generalized Hooke's Law, that

\[ C_{24} = C_{25} = C_{34} = C_{35} = C_{64} = C_{65} = 0 \]  \hfill (2.54)

Thus, for a material possessing one plane of elastic symmetry, the number of components in the stiffness and compliance matrices is reduced to 13, as shown below.

\[
S_{ij} = \begin{bmatrix}
S_{11} & S_{12} & S_{13} & 0 & 0 & S_{16} \\
S_{22} & S_{23} & 0 & 0 & S_{26} \\
S_{33} & 0 & 0 & S_{36} \\
S_{44} & S_{45} & 0 \\
S_{55} & 0 \\
S_{66}
\end{bmatrix} \hfill (2.55)
\]

\[
C_{ij} = \begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\
C_{22} & C_{23} & 0 & 0 & C_{26} \\
C_{33} & 0 & 0 & C_{36} \\
C_{44} & C_{45} & 0 \\
C_{55} & 0 \\
C_{66}
\end{bmatrix} \hfill (2.56)
\]
If the $X_2 - X_3$ plane had been the plane of symmetry, then the $S_{ij}$ matrix would have been:

$$S_{ij} = \begin{bmatrix}
S_{11} & S_{12} & S_{13} & S_{14} & 0 & 0 \\
S_{22} & S_{23} & S_{24} & 0 & 0 \\
S_{33} & S_{34} & 0 & 0 \\
S_{44} & 0 & 0 \\
S_{55} & 0 \\
S_{66}
\end{bmatrix}$$

Further, if both the $X_1 - X_2$ and $X_2 - X_3$ planes are planes of symmetry, then the components of $S_{ij}$ must satisfy Equations (2.55) and (2.57) simultaneously. Such a material is called orthotropic. Its compliance matrix is

$$S_{ij} = \begin{bmatrix}
S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\
S_{22} & S_{23} & 0 & 0 & 0 \\
S_{33} & 0 & 0 & 0 \\
S_{44} & 0 & 0 \\
S_{55} & 0 \\
S_{66}
\end{bmatrix}$$

(2.58)
An orthotropic material has 9 independent components in its $S_{ij}$ and $C_{ij}$ matrices. It should be noted also that if a material has two orthogonal planes of elastic symmetry, it will automatically have symmetry with respect to the third.

A material that has a plane in which there is no preferred orientation with respect to its elastic properties is said to be transversely isotropic. Suppose $X_1 - X_2$ is such a plane; then a rotation about the $X_3$-axis will leave the elastic constants invariant. The 1 and 2 indices will be interchangeable in $S_{ij}$ and $C_{ij}$. Thus,

\begin{align*}
S_{11} &= S_{22}, \quad S_{13} = S_{23} \\
C_{11} &= C_{22}, \quad C_{13} = C_{23}
\end{align*}

The shear moduli between the $X_3$ direction and the isotropic $X_1 - X_2$ plane must also be equal.

\begin{align*}
S_{44} &= S_{55}, \quad C_{44} = C_{55}
\end{align*}

Equations (2.59) and (2.60) reduce the number of independent components in $S_{ij}$ and $C_{ij}$ to 6. A further reduction is possible by considering the fact that $S_{12}$ must remain invariant under a rotation about the $X_3$-axis. The transformed $S'_{12}$ is from Equation (2.41).

\begin{align*}
S'_{12} &= m^2 n^2 S_{11} + (m^4 + n^4) S_{12} + m^2 n^2 S_{22} - m^2 n^2 S_{66}
\end{align*}

Combining terms and using the identity

\begin{align*}
(m^4 + n^4) &= 1 - 2 m^2 n^2
\end{align*}

Equation (2.61) becomes

\begin{align*}
S'_{12} &= S_{12} + m^2 n^2 (S_{11} - 2 S_{12} + S_{22} - S_{66})
\end{align*}
Since $S_{12} = S_{12}$ and $S_{11} = S_{22}$, it follows that

$$S_{66} = 2(S_{11} - S_{12}) \quad (2.64)$$

and in a similar way, it can be shown that

$$C_{66} = \frac{C_{11} - C_{12}}{2} \quad (2.65)$$

Therefore, a transversely isotropic material relative to the $X_3$-axis has the following stiffness and compliance matrices.

\[
S_{ij} = \begin{bmatrix}
S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\
S_{12} & S_{11} & S_{13} & 0 & 0 & 0 \\
S_{13} & S_{13} & S_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & S_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & S_{44} & 0 \\
0 & 0 & 0 & 0 & 0 & 2(S_{11} - S_{12})
\end{bmatrix}
\]
If the isotropic plane is the 2-3 plane, then the compliance matrix becomes

\[
C_{ij} = \begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
C_{11} & C_{13} & 0 & 0 & 0 & 0 \\
C_{33} & 0 & 0 & 0 & 0 & 0 \\
C_{44} & 0 & 0 & 0 & 0 & 0 \\
\frac{(C_{11} - C_{12})}{2} & & & & & \\
\end{bmatrix}
\] 

(2.67)

\[
S_{ij} = \begin{bmatrix}
S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\
S_{22} & S_{23} & 0 & 0 & 0 & 0 \\
S_{22} & 0 & 0 & 0 & 0 & 0 \\
2(S_{22} - S_{23}) & 0 & 0 & 0 & 0 & 0 \\
S_{66} & 0 & & & & \\
S_{66} & & & & & \\
\end{bmatrix}
\] 

(2.68)
A material with no preferred orientation in any plane is called isotropic. It can be shown, by arguments similar to those used above, that an isotropic material has only two independent elastic constants. Its $S_{ij}$ and $C_{ij}$ matrices are

\[
S_{ij} = \begin{bmatrix}
S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\
S_{11} & S_{12} & 0 & 0 & 0 \\
S_{11} & 0 & 0 & 0 & 0 \\
2(S_{11} - S_{12}) & 0 & 0 & 0 \\
2(S_{11} - S_{12}) & 0 & 0 & 0 \\
2(S_{11} - S_{12}) & 0 & 0 & 0 \\
\end{bmatrix}
\]

(2.69)

\[
C_{ij} = \begin{bmatrix}
C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\
C_{11} & C_{12} & 0 & 0 & 0 \\
C_{11} & 0 & 0 & 0 & 0 \\
\frac{(C_{11} - C_{12})}{2} & 0 & 0 & 0 & 0 \\
\frac{(C_{11} - C_{12})}{2} & 0 & 0 & 0 & 0 \\
\frac{(C_{11} - C_{12})}{2} & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

(2.70)
2.6 Engineering Constants

The components of the compliance and stiffness matrices become more understandable when they are converted into ordinary engineering constants such as Young's Moduli, Poisson's ratios, and shear moduli. The relationship between the components of the compliance matrix and the engineering constants is easily established by assuming stress states of uniaxial tension and pure shear. From these one obtains

\[
\begin{align*}
S_{11} &= \frac{1}{E_{11}} \\
S_{22} &= \frac{1}{E_{22}} \\
S_{33} &= \frac{1}{E_{33}} \\
S_{12} &= -\nu_{12} = \frac{-\nu}{E_{11} E_{22}} \\
S_{23} &= -\nu_{23} = \frac{-\nu}{E_{22} E_{33}} \\
S_{31} &= -\nu_{31} = \frac{-\nu}{E_{33} E_{11}} \\
S_{66} &= \frac{1}{G_{12}} \\
S_{55} &= \frac{1}{G_{13}} \\
S_{44} &= \frac{1}{G_{23}}
\end{align*}
\]  

(2.71)
The stiffness components $C_{ij}$ can also be related to the engineering constants, but these relationships are more complicated and will not be listed here. They can be obtained easily by inverting the $S_{ij}$ matrix for the particular material involved.
3. Elastic Moduli of Cord-Rubber Composites

3.1 Introduction

A composite is a two-phase material consisting of high strength, high modulus fibers embedded in a relatively softer and lower strength matrix. The basic load carrying element is the fiber, while the matrix acts as a stress transfer medium. The fibers of a composite may be either continuous or discontinuous; in the latter case the term short fiber composite is used. A composite may have its fibers aligned in a parallel order or randomly distributed in all directions. In the cord-rubber composites used in tires, the cords are continuous and aligned. The matrix and fiber may be elastically isotropic or anisotropic or a combination of both. A tire ply is an example of the latter, in which a highly anisotropic textile cord is embedded in an isotropic rubber matrix.

A composite is thus a very heterogeneous material. As a consequence, even the simplest kinds of loadings produce very complicated internal stress patterns. The analysis of a composite structure under load would be a hopelessly complicated task if this heterogeneity were retained. To avoid this difficulty, a composite is generally characterized as macroscopically homogeneous, but anisotropic. With this simplification the structural analysis can then be carried out, using the ordinary methods of elasticity theory. However, one is still left with the problem of defining the anisotropy of the composite, i.e., determining its elastic constants. Since this anisotropy is brought about by the differing elastic properties of the fiber and matrix material as well as their geometrical arrangement, it is natural to expect the composite elastic constants to be functional relations of these variables. The means of establishing this functional relationship is embodied in a branch of elasticity theory which has come to be known as micro-mechanics. The word "micro" is intended to mean that it applies to problems whose dimensions are one order of magnitude less than those of the structure. For example, in micro-mechanics theory, the actual shape of the reinforcing fiber, as well as its spatial arrangement, enter into the analysis.
The principal aim of the present chapter is to show how micro-mechanics theory can be employed to predict the macroscopic elastic moduli of a cord-rubber composite. Since this theory requires the elastic properties of the cord and rubber as inputs, it will be necessary to establish these first.

3.2 Elastic Properties of Rubber

The properties of rubber will be considered first since it is elastically the simplest of the two materials. For the purposes of this work, the rubber matrix material will be characterized as one which:

a. is completely elastic

b. obeys Hooke’s Law (linearly elastic)

c. is isotropic

These assumptions require justification.

First, it is well known that all organic polymers such as rubber display viscoelastic behavior. This means that the stress-strain relationship for such materials involves time as an additional parameter. Experimentally this time dependency is exhibited by the different stress-strain curves that can be obtained for the same material, depending upon the time rate of load application. However, the time effect can be minimized if the load is applied very slowly or quasi-statically. The loading which is pertinent to the present problem is the application of inflation pressure. Since this occurs rather gradually in practice, it is reasonable to consider it as quasi-static loading. It is on this basis that the viscoelastic nature of the rubber will be ignored.

Figure 6 illustrates a typical stress-strain curve for a cross-linked or vulcanized rubber. Such a material is clearly not linearly elastic or Hookean over the entire strain range. But if we restrict our attention to the portion of the curve lying between 0% and 10% strain,
it can be seen that in this range the material is nearly linear. Since the strains in pneumatic tires due to inflation pressure rarely exceed 10%, the characterization of the rubber as linear is regarded as a justifiable approximation.

Finally, the isotropic nature of rubber on a macroscopic scale is an established experimental fact. As such, it has only two independent components in its stiffness matrix which, in terms of engineering constants, are Young's modulus $E_m$ and Poisson's ratio $\nu_m$. The shear modulus $G_m$ is related to $E_m$ and $\nu_m$ by

$$G_m = \frac{E_m}{2(1 + \nu_m)}$$  \hspace{1cm} (3.1)

and since Poisson's ratio for rubber is very nearly equal to 1/2, $G_m$ can be approximated by

$$G_m \approx \frac{E_m}{3}$$  \hspace{1cm} (3.2)

### 3.3 Elastic Properties of Textile Tire Cord

A textile tire cord resembles a miniature rope in its construction. It is made by twisting parallel fibers of a material such as nylon or rayon into a yarn. Then two or more of these yarns are twisted together to form a cord. A rope is made by twisting together several cords.

Because of this fibrous quality, a tire cord is obviously not a continuous material. It, in fact, contains a certain amount of void between its fibers. Therefore an idealization must be made before the equations of continuum mechanics can be applied to such a material. This idealization must be one which reflects the essential features of the elastic behavior of the cord. With this idea in mind, a tire cord will henceforth be characterized as a solid, homogeneous, transversely isotropic material. In addition, its geometrical cross section will be taken as circular with an area equal to that of the real cord.

The motivation for characterizing a tire cord as transversely isotropic can perhaps best be explained by considering the physical significance of each of the five independent
elastic constants. In terms of engineering constants, Hooke's Law, for a transversely isotropic cord, is (Equations 2.68 and 2.71)

\[
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\gamma_{23} \\
\gamma_{31} \\
\gamma_{12}
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{E_{f11}} & \frac{-\nu_{f12}}{E_{f11}} & \frac{-\nu_{f12}}{E_{f11}} & 0 & 0 & 0 \\
\frac{-\nu_{f12}}{E_{f11}} & \frac{1}{E_{f11}} & \frac{-\nu_{f23}}{E_{f22}} & 0 & 0 & 0 \\
\frac{-\nu_{f23}}{E_{f22}} & \frac{1}{E_{f22}} & \frac{1}{E_{f22}} & 0 & 0 & 0 \\
\frac{2(1 + \nu_{f23})}{E_{f22}} & \frac{1}{G_{f12}} & \frac{1}{G_{f12}} & 0 & 0 & \tau_{23} \\
\frac{1}{G_{f12}} & \frac{1}{G_{f12}} & \frac{1}{G_{f12}} & 0 & 0 & \tau_{31} \\
\frac{1}{G_{f12}} & \frac{1}{G_{f12}} & \frac{1}{G_{f12}} & \frac{1}{G_{f12}} & 0 & \tau_{12}
\end{bmatrix}
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\tau_{23} \\
\tau_{31} \\
\tau_{12}
\end{bmatrix}
\]

(3.3)

In Equation (3.3), the plane of isotropy is taken as the 2-3 plane with the 1-axis coinciding with the longitudinal axis of the cord (Figure 7). Also, the additional subscript \( f \) attached to all the constants indicates that they pertain to the fiber. The five independent constants are \( E_{f11}, E_{f12}, \nu_{f12}, \nu_{f23}, \) and \( G_{f12} \). Let us consider the physical meaning of these terms individually.

\( E_{f11} \) is the Young's modulus along the cord axis. It is the easiest of all the constants to measure experimentally since it requires only uniaxial loading along the cord. Figure 8 is
a typical stress-strain curve for a textile tire cord. Again, we invoke the arguments of quasi-static loading and small strains to characterize the cord as linearly elastic.

The Poisson's ratio $\nu_{f12}$ represents a contraction transverse to the cord as a result of a stress applied along its axis. This contraction is rather large for twisted tire cords. It is basically the result of a change in the helical twist angle which occurs as the cord is elongated axially.\(^{22}\) Available experimental data\(^{23}\) indicates that $\nu_{f12}$ is approximately equal to 0.7 for all twisted tire cords.

$G_{f12}$ is the shear modulus in the axial direction. The only available data\(^{22}\) indicates that this constant has a value of about 700 psi. The other shear modulus pertaining to shear in the transverse direction is not an independent constant, as shown in Equation (3.3).

Young's modulus in a direction perpendicular to the cord axis is represented by $E_{f22}$. Although this transverse modulus is considerably less than the longitudinal modulus $E_{f11}$, accurate measurements have not as yet been made of its value. However, in the case of highly anisotropic composites ($E_{f22} \gg E_m$) such as cord-reinforced rubber the overall composite modulus is very insensitive to the true value of $E_{f22}$. Hence accurate values for this constant are not really required. This will be demonstrated in a later section.

Finally, $\nu_{f23}$ is the Poisson's ratio which represents a contraction in the 3-direction caused by a stress in the 2-direction. There is no experimental data on this constant either, but fortunately it does not enter into the calculations involving 2-dimensional composites.

It is obvious that the anisotropic elastic properties of textile tire cord are not at all well defined. It is the author's hope that this work will stimulate interest in the problem and that more accurate values for the cord elastic properties will be available in the future.
3.4 Micro-Mechanics of Fiber Reinforced Composites

The micro-mechanics theory of composites is basically an attempt to determine a particular gross or average property of the composite in terms of the corresponding properties of the constituent materials. The dielectric constant, thermal conductivity, and elastic moduli are examples of such properties which have successfully been predicted thus far. The greatest amount of effort, however, has been expended on calculating the elastic moduli. The result is that today these constants can be fairly accurately computed using formulae that are rigorously founded on the principles of elasticity theory.

We will not attempt to review the literature dealing with elastic moduli of composites, since it is far too extensive to be summarized briefly. Instead, references 24 through 33 are simply cited as being representative of the present state of the art. For our own purposes we shall appeal to the work of Hermans. This work is attractive from several points of view: (1) it has the flexibility of allowing both the fiber and matrix to be transversely isotropic, (2) the results are obtained by the application of the theory of elasticity, and (3) the final equations are in closed form and are fairly simple.

Hermans' analysis is based on a cylindrical composite model which consists of a cylindrical fiber of radius $r_f$ (Figure 9) surrounded concentrically by a cylindrical shell of matrix whose outer radius is $r_m$. This composite model is embedded in an unbounded homogeneous medium which has the transversely isotropic properties of the composite as a whole. A certain set of external loadings are then applied to the unbounded medium at infinity and the resulting stresses and displacements are computed at the boundary $r_m$. These calculations are then repeated, this time for a solid cylinder of radius $r_m$ which has the same characteristics as the homogeneous medium. The stresses and deformations of the two cylinders are then equated, which allows the composite moduli to be solved for in terms of those of the fiber and matrix. For the details of these calculations, the reader is referred to the original paper.

A slightly modified form of Hooke's Law is used by Hermans, so that before presenting his results it will be necessary to list some notational conversions. They are as follows:
\[ k = \frac{E_{22}}{2(1 - \nu_{23} - 2\nu_{21}\nu_{12})} \]

\[ l = \frac{E_{22}\nu_{12}}{1 - \nu_{23} - 2\nu_{21}\nu_{12}} \]

\[ n = \frac{E_{22}(1 - \nu_{23})}{1 - \nu_{23} - 2\nu_{21}\nu_{12}} \] (3.4)

\[ m = \frac{E_{22}}{2(1 + \nu_{23})} \]

\[ \mu = G_{12} \]

where \( k, l, m, n, \) and \( \mu \) are Hermans' new constants. The inverted form of Equation (3.4) is

\[ E_{11} = \frac{nk - l^2}{k} \]

\[ E_{22} = \frac{4m(kn - l^2)}{(k + m)n - l^2} \] (3.5)

\[ \nu_{23} = \frac{(k - m)n - l^2}{(k + m)n - l^2} \]

\[ \nu_{12} = \frac{l}{2k} \]

\[ \nu_{21} = \frac{2ml}{(k + m)n - l^2} \]

\[ G_{12} = \mu \]
With these conversions established, we now present the equations derived by Hermans.

\[
\frac{1}{k} = \frac{k_m (k_f + m_m) V_m + k_f (k_m + m_m) V_f}{(k_f + m_m) V_m + (k_m + m_m) V_f}
\]  \hspace{1cm} (3.6)

\[
\frac{1}{m} = \frac{2V_f m_f (k_m + m_m) + 2V_m m_f m_m + V_m k_m (m_f + m_m)}{2V_f m_m (k_m + m_m) + 2V_m m_f m_m + V_m k_m (m_f + m_m)}
\]  \hspace{1cm} (3.7)

\[
\frac{1}{\mu} = \frac{(\mu_f + \mu_m) \mu_m V_m + 2\mu_f \mu_m V_f}{(\mu_f + \mu_m) V_m + 2\mu_m V_f}
\]  \hspace{1cm} (3.8)

\[
\begin{align*}
\frac{1}{k} - \frac{1}{k_f} &= \frac{1}{k} - \frac{1}{k_m} = \frac{1 - V_f l_f - V_m l_m}{1 - l_m} = \frac{1 - V_f n_f - V_m n_m}{n - V_f n_f - V_m n_m}
\end{align*}
\]  \hspace{1cm} (3.9)

In Equations (3.6) through (3.9) the subscripts \( f \) and \( m \) refer to the fiber and matrix respectively, while the barred letters refer to the composite as a whole. In addition, \( V_f \) and \( V_m \) are the volume fractions of fiber and matrix respectively. It should be noted also that once \( k \) is found, then \( l \) and \( n \) follow directly from the identity (3.9).

In the application of these equations, one starts with the measured engineering constants of the fiber and matrix. These are substituted into Equation (3.4), which defines Hermans' constants. The new constants of the fiber and matrix are then substituted into Equations (3.6) through (3.9), which give the composite constants. These can then be converted into engineering constants by Equation (3.5).

3.5 Approximation Formulae of Halpin and Tsai

The preceding equations of Hermans have been reduced to a very simple and useful approximate form by Halpin and Tsai.\(^{34}\) The details of this reduction are given in
Appendix I, while the equations themselves are as follows:

\[ E_{11} = E_{f11} V_f + E_m V_m \]  
\[ \nu_{12} = \nu_{f12} V_f + \nu_m V_m \]  
\[ \frac{\bar{p}}{p_m} \cong \frac{(1 + \xi \eta V_f)}{(1 - \eta V_f)} \]  

where

\[ \eta = \frac{(p_f/p_m - 1)}{(p_f/p_m + \xi)} \]  

In this formulation the quantities \( \bar{p}, p_f, p_m, \) and \( \xi \) are identified as:

\( \bar{p} \) = composite moduli, \( E_{22}, G_{12} \) or \( \nu_{23}; \)

\( p_f \) = corresponding fiber modulus \( E_{f22}, G_{f12}, \nu_{f23} \) respectively;

\( p_m \) = corresponding matrix modulus, \( E_m, G_m, \nu_m \) respectively;

\( \xi \) = numerical factor; \( \xi_E = 2 \) and \( \xi_G = 1 \) (see Appendix I)

Recalling the assumption that \( E_{f22} \gg E_m \) we can make the approximation that the factor \( \eta_E \) which pertains to the calculation of \( E_{22} \) is given by
\[ \eta_E \cong 1 \] \hspace{1cm} (3.14)

and hence the modulus \( E_{22} \) is approximately

\[ E_{22} \cong \frac{E_m (1 + \xi V_f)}{(1 - V_f)} \] \hspace{1cm} (3.15)

This is tantamount to assuming that for the purposes of calculating \( E_{22} \), the cord may be treated as a rigid inclusion. This indicates that \( E_{22} \) is quite insensitive to the true value of \( E_{f22} \), provided of course that the assumptions of Equation (3.14) are met.
4. Elastic Properties of Cord-Rubber Laminates

4.1 Two Dimensional Composites

In Chapter 2, the three dimensional form of generalized Hooke’s Law was set forth for a completely anisotropic material with 21 elastic constants. It was then specialized for orthotropic materials having 9 constants, transversely isotropic materials having 5 constants, and isotropic materials having only 2 constants. Hooke’s Law can be further simplified if all of the stress components associated with a certain coordinate direction vanish. This in effect reduces the problem to one of two dimensions.

The classical theory of thin elastic shells involves certain hypotheses which allow this reduction in dimension to be made. They are based on the so-called “preservation of the normal” and the “thinness criterion” of shells. While these assumptions will be discussed at an appropriate time in a later chapter, their effects will be considered here.

Let us, for example, assume that the stress components, associated with the 3-axis, vanish. Then we have

\[ \sigma_{33} = \tau_{23} = \tau_{31} = 0 \]  
(4.1a)

Or in terms of the contracted notation

\[ \sigma_3 = \sigma_4 = \sigma_5 = 0 \]  
(4.1b)

For an orthotropic material defined by Equation (2.58) this means that

\[ \gamma_{23} = \gamma_{31} = 0 \]  
(4.2a)

\[ \epsilon_3 = S_{13} \sigma_1 + S_{23} \sigma_2 \]

or for a transversely isotropic material defined by Equation (2.68)

\[ \gamma_{23} = \gamma_{31} = 0 \]  
(4.2b)

\[ \epsilon_3 = S_{12} \sigma_1 + S_{23} \sigma_2 \]

Since \( \epsilon_3 \) is not an independent component, in either case, it may be dropped from the relations.
Under these conditions, the generalized Hooke's Law for both orthotropic, Equation (2.58), and transversely isotropic, Equation (2.68), materials reduces to an identical form, i.e.,

\[
\begin{bmatrix}
\epsilon_1 \\
\epsilon_2 \\
\epsilon_6
\end{bmatrix}
= \begin{bmatrix}
S_{11} & S_{12} & 0 \\
S_{22} & 0 & \\
0 & S_{66}
\end{bmatrix}
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_6
\end{bmatrix}
\] (4.3)

or equivalently

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_6
\end{bmatrix}
= \begin{bmatrix}
C_{11} & C_{12} & 0 \\
C_{22} & 0 & \\
0 & C_{66}
\end{bmatrix}
\begin{bmatrix}
\epsilon_1 \\
\epsilon_2 \\
\epsilon_6
\end{bmatrix}
\] (4.4)

This two dimensional orthotropic material is depicted in Figure 10, where the (1, 2) — axes coincide with the principal directions of orthotropy. In terms of engineering constants, the components of the $S_{ij}$ and $C_{ij}$ matrices are

\[
S_{11} = \frac{1}{E_{11}}
\]

\[
S_{22} = \frac{1}{E_{22}}
\]

\[
S_{12} = \frac{-\nu_{12}}{E_{11}} = \frac{-\nu_{21}}{E_{22}}
\]

\[
S_{66} = \frac{1}{G_{12}}
\] (4.5)
\[ C_{11} = \frac{E_{11}}{(1 - \nu_{12} \nu_{21})} \]

\[ C_{22} = \frac{E_{22}}{(1 - \nu_{12} \nu_{21})} \]

\[ C_{12} = \frac{\nu_{12} E_{22}}{(1 - \nu_{12} \nu_{21})} = \frac{\nu_{21} E_{11}}{(1 - \nu_{12} \nu_{21})} \]  

\[ C_{66} = G_{12} \]

Notice that there are only four independent constants and that the Poisson's ratio \( \nu_{23} \) does not appear, as was indicated in Section (3.3).

Normally, the lamina principal axes do not coincide with the principal axes \((1,2)\) of orthotropy as shown in Figure 11. Hooke's Law, written with respect to the lamina coordinates \((\phi, \theta)\), then becomes more involved. This is because of the coupling that exists between normal stress and shear strain. Figure 12 illustrates that both shear deformation and lateral contraction accompany the extensional strain when the lamina is stressed in a direction which is not a principal direction of orthotropy. Thus Hooke's Law with respect to non-principal axes becomes

\[
\begin{bmatrix}
\varepsilon_{\phi} \\
\varepsilon_{\theta} \\
\gamma_{\phi\theta}
\end{bmatrix} =
\begin{bmatrix}
R_{11} & R_{12} & R_{16} \\
R_{22} & R_{26} & \\
R_{66}
\end{bmatrix}
\begin{bmatrix}
\sigma_{\phi} \\
\sigma_{\theta} \\
\tau_{\phi\theta}
\end{bmatrix}
\]  

(4.7)
Engineering constants can also be defined with respect to the \((\phi, \theta)\) coordinate axes by means of Equation (4.7). But now, in addition to the four normal constants, two additional constants appear.

\[
\begin{bmatrix}
\sigma_\phi \\
\sigma_\theta \\
\tau_{\phi\theta}
\end{bmatrix} =
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{16} \\
Q_{22} & Q_{26} & \\
& & Q_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_\phi \\
\varepsilon_\theta \\
\gamma_{\phi\theta}
\end{bmatrix}
\]

(4.8)

\[
R_{11} = \frac{1}{E_{\phi\phi}}
\]

\[
R_{22} = \frac{1}{E_{\theta\theta}}
\]

\[
R_{12} = \frac{-\nu_{\phi\theta}}{E_{\phi\phi}}
\]

(4.9)

\[
R_{66} = \frac{1}{G_{\phi\theta}}
\]

\[
R_{16} = \frac{-\eta_{\phi\theta}}{E_{\phi\phi}}
\]

\[
R_{26} = \frac{-\eta_{\theta\phi}}{E_{\theta\theta}}
\]
The quantities $\eta_{\psi\phi}$ and $\eta_{\phi\psi}$ are analogous to Poisson's ratios except that shear strains are involved. They are referred to as the shear coupling ratios.

The fact that the compliance and stiffness matrices are now fully populated does not mean that additional independent elastic constants have been introduced. There are still only 4 independent constants. All of the $R_{ij}$ and $Q_{ij}$ can be expressed in terms of these by means of the transformation equations developed in Chapter 2. Thus from Equation (2.41)

$$R_{11} = m^4 S_{11} + n^4 S_{22} + m^2 n^2 (2S_{12} + S_{66})$$

$$R_{12} = m^2 n^2 (S_{11} + S_{22} - S_{66}) + (m^4 + n^4) S_{12}$$

$$R_{16} = 2mn (n^2 S_{22} - m^2 S_{11}) + (m^3 n - mn^3) (2S_{12} + S_{66})$$

$$R_{22} = n^4 S_{11} + m^4 S_{22} + m^2 n^2 (2S_{12} + S_{66})$$

$$R_{26} = 2mn (m^2 S_{22} - n^2 S_{11}) + (mn^3 - m^3 n) (2S_{12} + S_{66})$$

$$R_{66} = 4m^2 n^2 (S_{11} + S_{22} - 2S_{12}) + (m^2 - n^2)^2 S_{66}$$

where we have introduced the notation $R_{ij} = S'_{ij}$.

The variation in the engineering constants with the angle $\psi$ is expressed by Equations (4.9) and (4.10). The effect is best illustrated by considering a specific cord-rubber lamina. For example, suppose we are given a sheet of thickness $t$ and cord density $n_0$ as shown in Figure 10. If the cord diameter is $d_f$, then the volume fraction of cord is

$$V_f = \frac{\pi}{4} d_f^2 \frac{n_0}{t}$$

(4.11)
and the volume fraction of rubber matrix is

\[ V_m = 1 - V_f \quad (4.12) \]

Given the cord and rubber elastic constants, the principal orthotropic moduli can be calculated next from Equations (3.10) through (3.15). For a typical aircraft tire ply we have

\[ t = 0.043 \text{ in.} \]
\[ n_0 = 26 \text{ cords/in.} \]
\[ d_f = 0.031 \text{ in.} \]
\[ E_{f11} = 1.56 \times 10^5 \text{ lb./in.}^2 \]
\[ G_{f12} = 700 \text{ lb./in.}^2 \]
\[ \nu_{f12} = 0.7 \]
\[ E_m = 0.45 \times 10^3 \text{ lb./in.}^2 \]
\[ \nu_m = 0.490 \]

The resulting principal moduli are

\[ E_{11} = 7.11 \times 10^4 \text{ lb./in.}^2 \]
\[ \nu_{12} = 0.588 \]
\[ E_{22} = 1.65 \times 10^3 \text{ lb./in.}^2 \]
\[ G_{12} = 275 \text{ lb./in.}^2 \]
Using these values and Equations (4.5), (4.9), and (4.10), the engineering constants in any direction can be found. The results are displayed in Figures 13 and 14.

It is seen that the moduli vary markedly with the orientation angle of the cords. It will be recalled from Chapter 1 that the cord angle in a bias aircraft tire varies continuously according to the lift equation (1.6). Hence the tire structure will exhibit elastic properties which vary with position.

4.2 Laminated Composites

In the last section, the elastic properties of a two dimensional cord-rubber composite were examined. When a number of such laminae are firmly bonded together they behave structurally as a single layer. The problem that confronts us now is how to describe the elastic behavior of this laminated composite structure as a whole. It will be shown that the single lamina properties form the basic building blocks, which, together with certain results of thin shell theory, serve to characterize the laminate.

The geometrical theory of shell deformation will be discussed in detail in Chapter 5. There it will be shown that the strain in a thin shell consists of two components, one which denotes the stretching of the surface and a second which varies linearly through the shell thickness and is associated with bending. If \((\phi, \theta)\) are the shell coordinates, then the total strain associated with these directions is given by

\[
\begin{bmatrix}
\varepsilon_{\phi} \\
\varepsilon_{\theta} \\
\gamma_{\phi\theta}
\end{bmatrix} = \begin{bmatrix}
\varepsilon_{\phi}^0 \\
\varepsilon_{\theta}^0 \\
\gamma_{\phi\theta}^0
\end{bmatrix} + z \begin{bmatrix}
K_{\phi} \\
K_{\theta} \\
K_{\phi\theta}
\end{bmatrix}
\]

(4.13)

where \(\varepsilon_{\phi}^0\), \(\varepsilon_{\theta}^0\) and \(\gamma_{\phi\theta}^0\) are the strains due to stretching, \(K_{\phi}\), \(K_{\theta}\), and \(K_{\phi\theta}\) are changes in curvature and torsion of the neutral surface and \(z\) is the thickness coordinate.
Both the in-plane strains and changes in curvature vary with the surface coordinates \((\phi, \theta)\) but they are independent of \(z\).

Suppose now that the shell structure consists of \(n_p\) layers (Figure 15), whose principal directions of orthotropy are in general not aligned with the curvilinear coordinates \((\phi, \theta)\) of the shell (Figure 16). Then Hooke’s Law for the \(k^{th}\) layer is denoted by

\[
\begin{bmatrix}
\sigma_\phi \\
\sigma_\theta \\
\tau_{\phi\theta}
k
\end{bmatrix} =
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{16} \\
Q_{21} & Q_{22} & Q_{26} \\
Q_{66}
k
\end{bmatrix}
\begin{bmatrix}
\varepsilon_\phi \\
\varepsilon_\theta \\
\gamma_{\phi\theta}
k
\end{bmatrix}
\]

(4.14)

The \(Q_{ij}\) are expressed in terms of the principal moduli \(C_{ij}\) by means of the transformation equation (2.41)

\[
Q_{11} = m^4 C_{11} + n^4 C_{22} + 2m^2n^2 (C_{12} + 2C_{66})
\]

\[
Q_{12} = m^2n^2 (C_{11} + C_{22} - 4C_{66}) + (m^4 + n^4) C_{12}
\]

\[
Q_{16} = m^3n (C_{12} + 2C_{66} - C_{11}) + mn^3 (C_{22} - C_{12} - 2C_{66})
\]

\[
Q_{22} = n^4 C_{11} + 2m^2n^2 (C_{12} + 2C_{66}) + m^4 C_{22}
\]

\[
Q_{26} = mn^3 (C_{12} + 2C_{66} - C_{11}) + m^3n (C_{22} - C_{12} - 2C_{66})
\]

\[
Q_{66} = m^2n^2 (C_{11} + C_{22} - 2C_{12}) + (m^2 - n^2)^2 C_{66}
\]

(4.15)

where we have made the notational substitution \(Q_{ij} = C'_{ij}\). The stresses in the \(k^{th}\) layer in terms of mid-plane strains, changes in curvature and \(z\) coordinate are
Equation (4.16) can now be used to calculate the force and moment resultants which act on a differential shell element as shown in Figure 17. Let \( N_\phi, N_\theta, \) and \( N_{\phi\theta} \) be the force resultants and \( M_\phi, M_\theta, M_{\phi\theta} \) be the moment resultants. Then by definition

\[
\begin{align*}
\begin{bmatrix}
N_\phi \\
N_\theta \\
N_{\phi\theta}
\end{bmatrix}
&= 
\begin{bmatrix}
\int_{-h/2}^{h/2} s_\phi \\
\int_{-h/2}^{h/2} s_\theta \\
\int_{-h/2}^{h/2} s_{\phi\theta}
\end{bmatrix} \\
\begin{bmatrix}
M_\phi \\
M_\theta \\
M_{\phi\theta}
\end{bmatrix}
&= 
\begin{bmatrix}
\int_{-h/2}^{h/2} z s_\phi \\
\int_{-h/2}^{h/2} z s_\theta \\
\int_{-h/2}^{h/2} z s_{\phi\theta}
\end{bmatrix}
\end{align*}
\]

Equations (4.17) and (4.18) define a system of forces and moments which, acting at the geometric middle surface, are statically equivalent to the stresses. Before this integration can be carried out, the stresses must be expressed as functions of the \( z \) coordinate. Within each individual layer, this is given by a relationship such as Equation (4.16). Thus the integration must be carried out piecemeal, and the force and moment resultants will be given as the sum of \( n_p \) simple integrals (Figure 15):
\[
\begin{align*}
\begin{bmatrix}
N_\phi \\
N_\theta \\
N_{\phi\theta}
\end{bmatrix} &= \sum_{k=1}^{n_p} \int h_{k+1} h_k \begin{bmatrix}
\sigma_\phi \\
\sigma_\theta \\
r_{\phi\theta}
\end{bmatrix} \, dz \\
M_\phi &= \sum_{k=1}^{n_p} \int h_{k+1} h_k \begin{bmatrix}
\sigma_\phi \\
\sigma_\theta \\
r_{\phi\theta}
\end{bmatrix} \, zdz \\
M_\theta &= \sum_{k=1}^{n_p} \int h_{k+1} h_k \begin{bmatrix}
\sigma_\phi \\
\sigma_\theta \\
r_{\phi\theta}
\end{bmatrix} \, zdz \\
M_{\phi\theta} &= \sum_{k=1}^{n_p} \int h_{k+1} h_k \begin{bmatrix}
\sigma_\phi \\
\sigma_\theta \\
r_{\phi\theta}
\end{bmatrix} \, zdz
\end{align*}
\]

By substituting for the stresses in the \(k^{th}\) layer from Equation (4.16) into Equations (4.19) and (4.20) and then carrying out the indicated integration and summation, it can be verified that

\[
\begin{align*}
\begin{bmatrix}
N_\phi \\
N_\theta \\
N_{\phi\theta}
\end{bmatrix} &= \begin{bmatrix}
A_{11} & A_{12} & A_{16} \\
A_{22} & A_{26} & \\
A_{66}
\end{bmatrix} \begin{bmatrix}
\varepsilon^\phi \\
\varepsilon^\theta \\
\gamma_{\phi\theta}
\end{bmatrix} + \begin{bmatrix}
B_{11} & B_{12} & B_{16} \\
B_{22} & B_{26} & \\
B_{66}
\end{bmatrix} \begin{bmatrix}
K^\phi \\
K^\theta \\
K_{\phi\theta}
\end{bmatrix} \\
M_\phi &= \begin{bmatrix}
B_{11} & B_{12} & B_{16} \\
B_{22} & B_{26} & \\
B_{66}
\end{bmatrix} \begin{bmatrix}
\varepsilon^\phi \\
\varepsilon^\theta \\
\gamma_{\phi\theta}
\end{bmatrix} + \begin{bmatrix}
D_{11} & D_{12} & D_{16} \\
D_{22} & D_{26} & \\
D_{66}
\end{bmatrix} \begin{bmatrix}
K^\phi \\
K^\theta \\
K_{\phi\theta}
\end{bmatrix}
\end{align*}
\]
where

\[ A_{ij} = \sum_{k=1}^{n_p} (Q_{ij})_k (h_{k+1} - h_k) \]  \hspace{1cm} (4.23)

\[ B_{ij} = \frac{1}{2} \sum_{k=1}^{n_p} (Q_{ij})_k (h_{k+1}^2 - h_k^2) \]  \hspace{1cm} (4.24)

\[ D_{ij} = \frac{1}{3} \sum_{k=1}^{n_p} (Q_{ij})_k (h_{k+1}^3 - h_k^3) \]  \hspace{1cm} (4.25)

For purposes of clarity, Equations (4.21) and (4.22) can be displayed as a single matrix equation.

\[
\begin{bmatrix}
N_\phi \\
N_\theta \\
N_{\phi\theta} \\
M_\phi \\
M_\theta \\
M_{\phi\theta}
\end{bmatrix}
=
\begin{bmatrix}
A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\
A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\
A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\
B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\
B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\
B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66}
\end{bmatrix}
\begin{bmatrix}
\epsilon_\phi \\
\epsilon_\theta \\
\gamma_{\phi\theta} \\
\phi \\
\theta \\
\phi\theta
\end{bmatrix}
\hspace{1cm} (4.26)

The resemblance of this equation to generalized Hooke's Law has prompted some authors to refer to it as the "laminate constitutive equation."
4.3 Simplification of the Laminate Constitutive Equation

It was mentioned in Chapter 1 that a bias tire carcass consists of an even number of plies laid up in a sequence of alternating cord angles. That is, the cord angle in any one ply is equal in value but opposite in sign to the cord angle in its adjacent plies. This arrangement allows for a considerable simplification in the laminate constitutive equations.

To demonstrate this, we first note from Equation (4.15) that the $Q_{ij}$ are either even or odd functions of the angle $\psi$, e.g.,

$$Q_{ij} (+ \psi) = Q_{ij} (- \psi) \quad (ij) = 11, 12, 22, \text{ or } 66 \quad (4.27)$$

$$Q_{ij} (+ \psi) = -Q_{ij} (- \psi) \quad (ij) = 16 \text{ or } 26 \quad (4.28)$$

Considering first the $A_{ij}$ defined by Equation (4.23), it is apparent that the $(h_k^+ + 1 - h_k^-)$ terms represent the ply thickness and as such are always positive. Since the $A_{ij}$ are sums of products of the thickness and the $(Q_{ij})_k$, they will always be positive if the $(Q_{ij})_k$ are positive. On the other hand, if some of the $(Q_{ij})_k$ are positive and others negative, then cancellations will occur. Specifically, it can be shown that if for every lamina of a $+\psi$ orientation, there is another lamina of the same orthotropic properties and thickness with a $-\psi$ orientation, then

$$A_{16} = A_{26} \equiv 0 \quad (4.29)$$

Similar arguments apply to the $D_{ij}$ of Equation (4.25), since the geometrical contribution $(h_k^+ + 1 - h_k^-)$ is always positive. Thus, if for every $+\psi$ layer at a given distance above the mid-plane there is an identical layer at the same distance below the mid-plane oriented at $-\psi$, then we also have

$$D_{16} = D_{26} \equiv 0 \quad (4.30)$$
Finally, the $B_{ij}$ given by Equation (4.24) involve the factors $(h^2_k + 1 - h^2_k)$. These are positive for layers above the mid-plane and negative for layers below. Hence, the $B_{ij}$ terms in which $(Q_{ij})_k$ is always positive will vanish identically, i.e.,

$$B_{11} = B_{12} = B_{22} = B_{66} = 0 \quad (4.31)$$

The laminate constitutive equation can now be written in the simpler form

$$\begin{bmatrix} N_\phi \\ N_\theta \\ N_{\phi\theta} \\ M_\phi \\ M_\theta \\ M_{\phi\theta} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 & 0 & B_{16} \\ A_{12} & A_{22} & 0 & 0 & 0 & B_{26} \\ 0 & 0 & A_{66} & B_{16} & B_{26} & 0 \\ 0 & 0 & B_{16} & D_{11} & D_{12} & 0 \\ 0 & 0 & B_{26} & D_{12} & D_{22} & 0 \\ B_{16} & B_{26} & 0 & 0 & 0 & D_{66} \end{bmatrix} \begin{bmatrix} e_\phi \\ e_\theta \\ \gamma_{\phi\theta} \\ K_\phi \\ K_\theta \\ K_{\phi\theta} \end{bmatrix} \quad (4.32)$$

One additional simplification can be made for laminates containing a large number of layers; aircraft and heavy truck tire carcasses fall into this category. For such multi-ply laminates the $B_{16}$ and $B_{26}$ terms become negligibly small, and can be dropped from Equation (4.32). The decreasing magnitude of $B_{16}$ and $B_{26}$ with the number of layers is illustrated graphically in Figure 18. With this approximation, the laminate constitutive equations take on a surprisingly simple form since all of the $B_{ij}$ are now zero. Thus Equation (4.32) can be decomposed into
These two equations have a very important physical meaning. They say, in effect, that a multi-ply laminate whose layers are of alternating angular sign behaves as if it were a single, homogeneous, orthotropic layer with its principal directions of orthotropy aligned with those of the laminate.

4.4 Significance of the $B_{16}$, $B_{26}$ Coupling Terms

Suppose now that instead of a multi-ply aircraft tire, we are dealing with a 2-ply automobile tire. Then the $B_{16}$ and $B_{26}$ terms could not be discarded, and the laminate constitutive equations are given by Equation (4.32). Let us examine the first of these equations separately.

$$N_\phi = A_{11} \varepsilon_\phi + A_{12} \varepsilon_\theta + B_{16} K_{\phi \theta}$$

This expression indicates that the in-plane force $N_\phi$ originates from twisting as well as from stretching of the neutral surface. This unusual behavior has been demonstrated experimentally\(^{(36)}\) for 2-ply cord-rubber laminates. Figure 19 illustrates a $\pm 30^\circ$ laminate.
subject to an $N_\phi$ loading. The loading device assures that there are no bending moments or shear forces applied to the upper or lower edges. Since the sides are free from stress, $N_\phi$ is the only non-zero resultant, i.e.,

$$N_\theta = N_{\phi\theta} = M_\phi = M_\theta = M_{\phi\theta} = 0$$

(4.36)

$$N_\phi \neq 0$$

The twisting of the laminate, due to the $B_16$ coupling term, is clearly visible. To prevent this warping requires the application of a twisting moment $M_{\phi\theta}$ of a specific magnitude. Its value can be determined by inverting Equation (4.32), setting $K_{\phi\theta}$ equal to zero, and solving for the required $M_{\phi\theta}$. This moment will, of course, result in additional stresses within the laminate.

This is apparently what occurs in 2-ply tires, since there is no visible twisting of the carcass when it is inflated. The boundary supports, i.e., the wheel, must supply the proper external loads to prevent warping, but by doing so they introduce additional stresses within the carcass material. The magnitude of these stresses and hence their significance is a point to be resolved by future investigations.

Finally, it should be noted that, in the case of 4-ply tires, the $B_{ij}$ coupling could be eliminated completely by laying the plies in a $+ - - +$ sequence instead of the $+ - + -$ sequence that is currently the practice. In the former arrangement, due to the symmetry through the thickness, the $B_{ij}$ vanish identically. But the $D_{16}$, $D_{26}$ terms then reappear, and there is coupling between bending moments and twist curvature.
5. The Tire As a Thin Elastic Shell of Revolution

5.1 Shell Geometry

The geometry of a shell is entirely defined by specifying the form of the middle surface and the thickness of the shell at any point. To describe the form of the middle surface it is necessary to present some of the important geometric properties of a surface. Proofs will not be given here but can be found in texts dealing with differential geometry.\(^{(37)}\)

The position of points on any smooth surface can be described in terms of two independent parameters \((a_1, a_2)\). If the range of these parameters is restricted so that every point on the surface corresponds to one and only one pair of values \((a_1, a_2)\), then these parameters constitute a curvilinear coordinate system for points on the surface.

Equations \(a_1 = \text{constant}\) and \(a_2 = \text{constant}\) represent families of curves on the surface (Fig. 20). These parametric curves are called coordinate lines. Thus, a surface can be completely described by a doubly infinite set of such parametric curves where the position of any point on the surface is determined by the values of \(a_1\) and \(a_2\). A simple illustration of this concept is the lines of latitude and longitude on a world globe which can be thought of as coordinate lines. In general, the parametric lines do not intersect at right angles. For our purposes, however, we will limit the discussion to those that do, i.e., orthogonal curvilinear coordinates.

The distance between two neighboring points on a surface can be related by the differential distance \(dS\). The square of the linear element \(dS\) of any curve traced on the surface is given by an expression of the form (Fig. 20)

\[
dS^2 = A_1^2 da_1^2 + A_2^2 da_2^2
\]

(5.1)

The coefficients \(A_1\) and \(A_2\) are, in general, functions of \(a_1\) and \(a_2\) and are sometimes referred to as the Lame' parameters. They can be interpreted geometrically as lengths of
linear elements along constant coordinate lines of the surface when the increment of one of the two independent variables has a unit value.

A particular set of orthogonal curvilinear coordinates is usually chosen which simplifies the equations of the surface. Such a system is that in which the two families of coordinate curves are simultaneously lines of curvature. A line of curvature is a curve on the surface which possesses the property that normals to the surface at consecutive points on the curve all intersect, thereby generating a developable surface or plane.

In the present application of shell theory we will be concerned with a particularly simple kind of surface, called a surface of revolution. Such a surface is obtained by rotating a plane curve about an axis lying in the plane of the curve. This curve is called a meridian and its plane is the meridian plane. The intersection of the surface with planes perpendicular to the axis of rotation are parallel circles and are called parallels.

For such surfaces, the lines of curvature are its meridians and parallels. Accordingly, a convenient selection of coordinates is the angle \( \phi \) (between the normal to the middle surface and the axis of rotation) and the angle \( \theta \), determining the position of a point on the corresponding parallel circle (Fig. 21).

The two principal radii of curvature are denoted by \( r_\phi \) and \( r_\theta \). \( r_\phi \) and \( r_\theta \) are the radii of curvature of the surface along the meridian and parallel curves respectively. The radius of curvature \( r_\theta \) will always be equal to the length of the intercept of the normal to the middle surface between the surface and the axis of rotation. This is because the normals from two different adjacent points on the same parallel circle intersect each other at the axis of rotation.

The element of arc of the meridian and parallel lines is given by

\[
dS_1 = r_\phi \, d\phi
\]

\[
dS_2 = r_\theta \sin \phi \, d\theta = r \, d\theta
\]
Thus the Lamé' parameters are

\[ A_1 = r_\phi \]

\[ A_2 = r_\theta \sin \phi = r \]

where \( a_1 = \phi \) and \( a_2 = \theta \)

A surface of revolution therefore is completely specified by its radii of curvature \( r_\phi \) and \( r_\theta \). Moreover, these will be functions of the \( \phi \) coordinate alone.

The particular surface of revolution that will occupy our attention now is the middle surface of a pneumatic tire carcass. A typical generating curve (meridian section) is shown in Figure 22. This curve is not a well defined mathematical function. In practice it is specified by giving the values of a discrete number of experimentally measured x-y coordinates. From this information one must then characterize the tire surface in terms of the \( \phi-\theta \) coordinates. This means determining the radii of curvature \( r_\phi \) and \( r_\theta \) as functions of \( \phi \).

At first sight, this appears to be a rather routine matter involving functional approximation. But in reality it is quite a troublesome problem. The source of the difficulty stems from the fact that the radii of curvature involve higher order derivatives of the function used to represent the experimental data. Since this data is somewhat uncertain, it is unwise to require the approximating function to duplicate it exactly. If this were done, the small oscillations that occur between the discrete data points would be greatly magnified in the first and second derivatives and would result in unreliable radii of curvature.

These difficulties can be minimized if, instead of forcing the approximating function to meet the data points exactly, we require it to be one which in all probability is the closest to the real function over the complete range. Such a function would in effect smooth out the oscillations about the real curve and thereby give greater accuracy to the computed derivatives. One technique for determining this smoothing function is the method of least squares.
In applying this technique, it will be convenient to resort to the polar coordinates \( \rho - \alpha \). These are referred to an x-y system in which the y-axis lies in the tire middle plane and the x-axis passes through the point of maximum width of the tire meridian (Fig. 23). From the figure

\[
x_i = \rho_i \sin \alpha_i \quad (i = 0, 1, \ldots n)
\]

\[
y_i = \rho_i \cos \alpha_i
\]

The inverse relations are

\[
\rho_i = (x_i^2 + y_i^2)^{1/2} \quad (i = 0, 1, \ldots n)
\]

\[
\alpha_i = \tan^{-1} \left( \frac{x_i}{y_i} \right)
\]

where \( i \) ranges over the complete set of \( n+1 \) discrete data points. We will now apply the method of least squares to approximate the meridian curve in polar coordinates. First, it is assumed that the smoothed curve \( \rho (\alpha) \) is of the form

\[
\rho(\alpha) = \sum_{k=0}^{N} a_k f_k(\alpha)
\]

where \( a_k \) are as yet undetermined constant coefficients and \( f_k(\alpha) \) is a set of arbitrary functions to be specified later. Then we define the residual \( R(\alpha_i) \) by the equation

\[
R(\alpha_i) = \rho(\alpha_i) - \sum_{k=0}^{N} a_k f_k(\alpha_i)
\]
The sum of the squared residuals over all of the \( n+1 \) discrete points is

\[
\sum_{i=0}^{n} R^2(a_i) = \sum_{i=0}^{n} \left[ \rho(a_i) - \sum_{k=0}^{N} a_k f_k(a_i) \right]^2
\]

Minimizing the aggregate squared residual with respect to the arbitrary coefficients leads to a set of \( N+1 \) normal equations.

\[
\frac{\partial}{\partial a_r} \left[ \sum_{i=0}^{n} R^2(a_i) \right] = \sum_{i=0}^{n} \left\{ \rho(a_i) f_r(a_i) - \sum_{k=0}^{N} a_k f_k(a_i) f_r(a_i) \right\} \equiv 0
\]

\((r = 0, 1, \ldots, N)\)

These are linear algebraic equations in the unknown \( a_k \) and can be inverted by ordinary equation solving routines.

A natural choice for the \( f_k(a) \) are the even trigonometric functions

\[
f_k(a) = \cos(ka) \quad (k = 0, 1, \ldots, N)
\]

since the tire meridian is symmetrical about the \( y \)-axis. Once these functions are chosen and the constants \( a_k \) are found, the meridian radius of curvature can be computed from

\[
r_\phi(a) = \left[ \frac{\rho^2 + (\rho')^2}{\rho^2 + 2(\rho')^2 - \rho \rho''} \right]^{3/2}
\]

\((5.14)\)
where the derivatives are given by

\[ \rho' = - \sum_{k=1}^{N} k a_k \sin (ka) \]  

(5.15)  

\[ \rho'' = - \sum_{k=1}^{N} k^2 a_k \cos (ka) \]  

(5.16)  

The radius of curvature \( r_\theta \) is expressed by

\[ r_\theta(a) = \frac{r}{\sin \phi} = \frac{(r_m + \rho \cos a)}{\sin \phi} \]  

(5.17)  

Both \( r_\phi \) and \( r_\theta \) are now given as functions of the polar angle \( \alpha \). It would be desirable to express them as explicit functions of the coordinate \( \phi \), but we shall see that the relationship between \( \phi \) and \( \alpha \) is such that this is not practical.

Referring to Fig. 24, the slope of a tangent at a point \((x,y)\) on the meridian is

\[ \tan \beta = \frac{dy}{dx} \]  

(5.18)  

and the slope of the corresponding normal is

\[ \tan \phi = - \frac{dx}{dy} \]  

(5.19)  

From Equations (5.5) and (5.6)

\[ x' = \rho' \sin \alpha + \rho \cos \alpha \]  

(5.20)  

\[ y' = \rho' \cos \alpha - \rho \sin \alpha \]  

(5.21)
Thus

$$\phi = \tan^{-1} \left[ \frac{\rho' \sin \alpha + \rho \cos \alpha}{\rho \sin \alpha - \rho' \cos \alpha} \right]$$  \hspace{1cm} (5.22)

To express $\alpha$ as an explicit function of $\phi$ would require the inversion of Equation (5.22), which is not possible except by numerical means. We shall return to this point when discussing the numerical integration of the shell equations in the next chapter.

Figures 25 and 26 illustrate the result of applying the above techniques to the meridian shape shown in Figure 22. Eight terms were used in the series of Equation (5.9). The resulting RMS error, based on 22 data points, is 0.011.

Before leaving the subject of shell geometry, an important geometric identity must be established for surfaces of revolution defined by $\phi-\theta$ coordinates. Referring to Fig. 27, it can be seen that

$$dr = r_\phi d\phi \cos \phi$$

Hence

$$\frac{dr}{d\phi} = r_\phi \cos \phi$$  \hspace{1cm} (5.23)

Equation (5.23) is one of the three Gauss-Codazzi conditions that must be satisfied in order for the combination of Lame parameters $A_1, A_2$, and radii of curvature $r_\phi, r_\theta$ to define a valid surface. The other two are satisfied identically.

5.2 **Basic Assumptions of Shell Theory**

The classical theory of thin elastic shells is founded on the following assumptions:

1. The shell is thin, i.e., $(h/r_\phi \ll 1)$ and $(h/r_\theta \ll 1)$
2. The displacements are small relative to the shell dimensions and the strains are small in comparison to unity

3. The transverse normal stress and the strain resulting from it is negligible

4. Normals to the reference surface of the shell remain normal to it and undergo no change in length during deformation.

These four assumptions are sufficient to develop a linearized theory in which the stresses and deformations at any point in the shell are determined from those of its neutral surface. In the following sections, the pertinent equations of thin shell theory will be developed by appealing to these assumptions.

5.3 Deformation of a Shell Element

In Section (4.3) it was shown that a multi-ply laminate behaves structurally as though it were a single homogeneous orthotropic layer. The effective principal directions of orthotropy were shown to lie along lines which bisect the angles formed by two crossing cords. In a bias tire, the meridians and parallels are the bisectors of these angles (Figure 28). Hence the principal directions of orthotropy coincide everywhere with the parametric lines of the shell. The consequence of this is that, in the case of axisymmetric loading, e.g., inflation pressure, the displacements in the $\theta$-direction must be identically zero. In addition, the stresses and deformations must be independent of the $\theta$-coordinate.

Let us consider the deformation of the shell element shown in Figure 29. Suppose for the present that the shell undergoes pure bending in the $\phi$ direction. There is no stretching of the neutral surface and the face BD simply rotates about its intersection with the neutral axis. According to assumption 4 of Section (5.2), normal segments such as AC and BD remain straight and normal to the middle surface as shown. Let the primed
quantities refer to the deformed element. Then the length of a line element located at a
distance \( z \) from the neutral surface before and after deformation is given by

\[
dS_z = (r_\phi + z) \, d\phi \\
\]

(5.24)

\[
dS'_z = (r'_\phi + z) \, d\phi' \\
\]

(5.25)

Hence the strain is

\[
e_\phi = \frac{(r'_\phi + z) \, d\phi' - (r_\phi + z) \, d\phi}{(r_\phi + z) \, d\phi} \\
\]

(5.26)

Since the neutral surface does not strain during pure bending, we have

\[
r'_\phi \, d\phi' = r_\phi \, d\phi \\
\]

(5.27)

By substituting for \( d\phi' \) into Equation (5.26) and simplifying, one obtains

\[
e_\phi = \frac{z}{(1 + z/r_\phi)} \left( \frac{1}{r'_\phi} - \frac{1}{r_\phi} \right) \\
\]

(5.28)

Now let the element undergo stretching deformation as well as bending. The face BC
is displaced parallel to its original position and is also rotated as shown in Figure 28-b. As
before, we compute the strain in an element located a distance \( z \) from the neutral surface.
Equation (5.26) remains unchanged, but Equation (5.27) becomes

\[
r'_\phi \, d\phi' = (1 + e_\phi^o) \, r_\phi \, d\phi \\
\]

(5.29)
Substituting for \( d\phi' \) in Equation (5.26), one obtains after some algebraic rearrangement

\[
e_{\phi} = \frac{1}{(1 + z/r_\phi)} \left[ e_{\phi}^0 + z \left( \frac{1 + e_{\phi}^0}{r_\phi} - \frac{1}{r_\phi} \right) \right]
\]  (5.30)

By assumption 1 of Section (5.2) we neglect the terms \( z/r_\phi \) in comparison to unity. The effect of the strain on the change in curvature will also be neglected by assumption 2. Hence

\[
e_{\phi} = e_{\phi}^0 + z K_\phi
\]  (5.31)

where \( K_\phi \) is the change in curvature. These arguments can be repeated for deformations in the \( \theta \) direction with the result

\[
e_{\theta} = e_{\theta}^0 + z K_\theta
\]  (5.32)

It can be shown\(^{38}\) that the shear strain is given by a similar equation

\[
\gamma_{\phi\theta} = \gamma_{\phi\theta}^0 + z K_{\phi\theta} \equiv 0
\]  (5.33)

But, because of the symmetry and homogeneity cited at the beginning of this section, the shear strain and torsion are zero.

The strains \( e_{\phi}^0 \), \( e_{\theta}^0 \) and changes in curvature \( K_\phi \), \( K_\theta \) must now be expressed in terms of the displacements of the middle surface. Let \( u \), \( v \), and \( w \) be the displacements tangent to meridian, tangent to the parallel, and normal to the surface respectively. It has been noted earlier that \( v \equiv 0 \), hence we need only consider \( u \) and \( w \). Figure 30-a illustrates an element of the shell meridian which has undergone a general displacement. Considering the \( w \) displacement by itself (Figure 30-b), the following expression gives the rotation \( \omega_\phi \) of the right side of the shell element.
The $\omega$ displacement by itself (Figure 30-c) produces a rotation of the right side equal to

$$\omega_\phi = \frac{u}{r_\phi} \quad (5.35)$$

Then the total rotation is given by the sum

$$\omega_\phi = \frac{1}{r_\phi} \left( u - \frac{dW}{d\phi} \right) \quad (5.36)$$

The change in curvature $K_\phi$ can now be calculated. From Figure 31

$$\phi' = d\phi + d\omega_\phi \quad (5.37)$$

Substituting for $d\phi'$ in Equation (5.29) gives

$$r_\phi' (d\phi + d\omega_\phi) = (1 + \epsilon_\phi^o) r_\phi d\phi \quad (5.38)$$

Dividing by $r_\phi' r_\phi d\phi$ and rearranging, one obtains

$$\frac{1 + \epsilon_\phi^o}{r_\phi'} - \frac{1}{r_\phi} = \frac{d\omega_\phi}{r_\phi d\phi} \quad (5.39)$$

Again, neglecting the effect of strain on the change in curvature

$$K_\phi = \frac{d\omega_\phi}{r_\phi d\phi} \quad (5.40)$$
There is no rotation in the \( \theta \)-direction because of symmetry. However, there is a change in curvature in this direction due to the rotation \( \omega_\phi \). Referring to Figure 32

\[
\begin{align*}
r' &= r_\theta' \sin \phi' = r_\theta' \sin (\phi + \omega_\phi) \\
\text{Also} \\
r' &= r + \Delta r = r_\theta' (\sin \phi + \omega_\phi \cos \phi)
\end{align*}
\]

using the approximation \( \cos \omega_\phi \approx 1, \sin \omega_\phi \approx \omega_\phi \), since the rotations are assumed small. Then the change in curvature \( K_\theta \) is

\[
K_\theta = \frac{1}{r_\theta'} - \frac{1}{r_\theta} = \frac{(\sin \phi + \omega_\phi \cos \phi)}{r + \Delta r} \frac{\sin \phi}{r} 
\]

Expanding this equation

\[
K_\theta = \frac{1}{r} \left( \frac{\omega_\phi \cos \phi - \frac{\Delta r}{r} \sin \phi}{1 + \frac{\Delta r}{r}} \right) 
\]

Dropping higher order terms, this simplifies to

\[
K_\theta = \frac{\omega_\phi \cos \phi}{r} 
\]

To develop an expression for the strain \( \varepsilon_\phi \) in the meridian direction, again consider the displacements \( u \) and \( w \) separately. The increase of the segment length due to the tangential displacement is equal to

\[
dS'_\phi = u + \frac{du}{d\phi} d\phi - u = \frac{du}{d\phi} d\phi 
\]
From Figure 33 the increase due to radial displacement is

\[ \text{d}S^\phi = w \text{d} \phi \quad (5.47) \]

Thus the total change in length equals

\[ \text{d}S^\phi = \frac{\text{d}u}{\text{d} \phi} \text{d} \phi + w \text{d} \phi \quad (5.48) \]

The strain is equal to the change in length divided by the initial length \( r^\phi \text{d} \phi \).

\[ \epsilon^\phi = \frac{1}{r^\phi} \left( \frac{\text{d}u}{\text{d} \phi} + w \right) \quad (5.49) \]

The strain \( \epsilon^\theta \) is equal to the radial deflection \( \Delta r \) divided by \( r \). From Figure 34,

\[ \epsilon^\theta = \frac{\Delta r}{r} = \frac{1}{r} (u \cos \phi + w \sin \phi) \quad (5.50) \]

5.4 Equilibrium of a Shell Element

In this section the equations of equilibrium will be developed for the differential shell element shown in Figure 35. Since the displacements are small by assumption 2 of Section 5.2, the equations will be written with respect to the undeformed geometry. A detailed sketch of the forces and moments acting on this element, as seen from a direction normal to each coordinate, is shown in Figure 36. Notice that the shear force resultant \( Q_\phi \) has now been introduced, despite that fact that in Chapter 4 the transverse shear stress was set equal to zero. This is the familiar inconsistency that also arises in classical beam and plate theory. These shear resultants are required for equilibrium, but they cannot be computed from a stress integral. Again, due to symmetry, there are no twisting moments \( M^\phi_\theta \) or transverse shear forces \( Q_\theta \).
Before writing the equations of equilibrium, we will need expressions for the central angles $d\delta$ and $d\chi$ defined in Figure 36. These can be developed by referring to Figure 35. The line segment AB, tangent to the surface and intersecting the $x_3$-axis is given by

$$AB = r_\theta \tan \phi$$

(5.51)

hence we have the equality for the elemental arc length $dS_\theta$

$$r_\theta \tan \phi \, d\delta = r_\theta \sin \phi \, d\theta$$

(5.52)

from which

$$d\delta = \cos \phi \, d\theta$$

(5.53)

In a similar manner the angle $d\chi$ is found from

$$r_\theta \, d\chi = r_\theta \sin \phi \, d\theta$$

(5.54)

or

$$d\chi = \sin \phi \, d\theta$$

(5.55)

Returning to Figure 36, and summing forces in the $\phi$-direction, we obtain

$$
\left( N_\phi + \frac{dN_\phi}{d\phi} \, d\phi \right) \left( r + \frac{dr}{d\phi} \, d\phi \right) \, d\theta - N_\phi \, r \, d\theta - N_\theta \, r_\phi \, d\phi \, d\theta \cos \phi \\
+ \left( Q_\phi + \frac{dQ_\phi}{d\phi} \, d\phi \right) \left( r + \frac{dr}{d\phi} \, d\phi \right) \frac{d\theta}{2} \, \frac{d\phi}{2} + Q_\phi \frac{rd\phi \, d\theta}{2}
$$

(5.56)

$$+ r \, r_\phi \, q_\phi \, d\phi \, d\theta = 0$$
where the approximation \( \cos (d \phi /2) \simeq 1 \) has been used. Expanding Equation (5.56), neglecting terms of higher order, and dividing by \( d \phi d \theta \) yields the first equilibrium equation.

\[
N_\phi \frac{dr}{d\phi} + r \frac{dN_\phi}{d\phi} - N_\theta r_\phi \cos \phi + Q_\phi r + r r_\phi q_\phi = 0 \tag{5.57}
\]

Summing forces in the z direction results in the second equilibrium equation.

\[
-N_\phi r \frac{d\phi}{2} - (N_\phi + dN_\phi \frac{d\phi}{d\phi}) \frac{(r + dr \frac{d\phi}{d\phi})}{(d\phi \frac{d\phi}{d\phi})} - N_\theta r_\phi \frac{d\phi}{d\phi} \frac{d\theta}{2} \sin \phi
\]

\[
+ \left( Q_\phi + \frac{dQ_\phi}{d\phi} \right) \left[ r + dr \frac{d\phi}{d\phi} \right] d\theta - Q_\phi \frac{rd\theta + q_n r r_\phi \frac{d\phi}{d\phi} \frac{d\theta}{2}}{d\phi} = 0 \tag{5.58}
\]

Again, expanding, dropping higher order terms, and dividing by \( d \phi d \theta \), yields

\[
Q_\phi \frac{dr}{d\phi} + r \frac{dQ_\phi}{d\phi} - N_\phi r - N_\theta r_\phi \sin \phi + r r_\phi q_n = 0 \tag{5.59}
\]

Finally, summing moments about the \( \theta \)-axis gives the third equilibrium equation.

\[
(M_\phi + dM_\phi \frac{d\phi}{d\phi}) \frac{(r + dr \frac{d\phi}{d\phi})}{(d\phi \frac{d\phi}{d\phi})} d\theta - M_\phi \frac{rd\theta}{d\phi} - M_\theta r_\phi \frac{cos \phi \frac{d\phi}{d\phi} \frac{d\theta}{2}}{d\phi}
\]

\[
- (Q_\phi + dQ_\phi \frac{d\phi}{d\phi}) \frac{(r + dr \frac{d\phi}{d\phi})}{(d\phi \frac{d\phi}{d\phi})} r_\phi d\phi d\theta = 0 \tag{5.60}
\]

which, after expanding, neglecting terms of higher order, and dividing by \( d \phi d \theta \), simplifies to
\[
\frac{r}{d\phi} \frac{dM_\phi}{d\phi} + \frac{M_\phi}{d\phi} \frac{dr}{d\phi} - M_\phi \frac{r_\phi}{d\phi} \cos \phi - O_\phi r_\phi = 0
\]

(5.61)

Summing forces in the \(\theta\)-directions and moments about the \(\phi\) and \(z\) axes leads to the other three trivial equilibrium equations.
6. Numerical Solution of the Tire Shell Equations

6.1 Fundamental Equations - Boundary Conditions

Previous chapters have been devoted to developing the equations which characterize the pneumatic tire as an elastic shell. In this chapter, these results will be brought together in a form suitable for numerical solution. Specifically, a system of six first-order ordinary differential equations will be derived in six fundamental variables, which when integrated numerically provide all the information needed to determine the state of stress in the tire. This technique was first developed by Goldberg and Bogdanoff\(^{39}\) and later refined by Kalnins\(^{40}\) and Kalnins and Lestingi.\(^{41}\) It is well summarized in the recent book by Kraus.\(^{38}\)

To begin, we select as the fundamental dependent variables, the following six quantities:

1. \(u\) - tangential displacement
2. \(\omega_\phi\) - rotation
3. \(Q_\phi\) - transverse shear force
4. \(M_\phi\) - bending moment
5. \(w\) - normal displacement
6. \(N_\phi\) - membrane force

Our aim is to develop a set of equations in which all other variables have been eliminated in favor of these six. Anticipating the need to replace the membrane stress \(N_\theta\) and bending moment \(M_\theta\) in the equilibrium equations we will first express these in terms of the fundamental variables.

Eliminating the membrane strain \(\epsilon_\phi^0\) from the first two of Equations (4.33), and solving for \(N_\theta\) gives

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Substituting for $\varepsilon_\theta^0$ in the above from Equation (5.50) one obtains

$$N_\theta = \frac{A_{12}}{A_{11}} N_\phi + \left( \frac{A_{11}A_{22} - A_{12}^2}{A_{11}} \right) \varepsilon_\theta^0$$

(6.1)

In a similar fashion, elimination of the change in curvature $K_\phi$ from the first two of Equations (4.34) and solving for $M_\theta$ yields

$$M_\theta = \frac{D_{12}}{D_{11}} M_\phi + \left( \frac{D_{11}D_{22} - D_{12}^2}{D_{11}} \right) K_\theta$$

(6.2)

Substituting for $K_\theta$ from Equation (5.45)

$$M_\theta = \frac{D_{12}}{D_{11}} M_\phi + \left( \frac{D_{11}D_{22} - D_{12}^2}{D_{11}} \right) \frac{\cos \phi}{r} \omega_\phi$$

(6.3)

We will also need expressions for the membrane strain $\varepsilon_\phi^0$ and change in curvature $K_\phi$ in terms of the fundamental variables. The former is obtained by solving the first of Equations (4.33) for $\varepsilon_\phi^0$ and then substituting for $\varepsilon_\theta^0$. The result is

$$\varepsilon_\phi^0 = \frac{N_\phi}{A_{11}} - \frac{A_{12}}{A_{11}} \left( \frac{u \cos \phi + w \sin \phi}{r} \right)$$

(6.4)

An expression for $K_\phi$ is obtained in a similar manner from the first of Equations (4.34). Solving for $K_\phi$ and substituting for $K_\theta$ from Equation (5.45) gives
The six fundamental equations can now be set down in order. The first is derived from the strain-displacement Equation (5.49). Rearranging this equation and substituting for $e_{\phi}^0$ from Equation (6.5), one obtains

$$K_{\phi} = \frac{M_{\phi}}{D_{11}} - \frac{D_{12} \cos \phi}{D_{11} r} \omega_{\phi} \tag{6.6}$$

The second equation is obtained by substituting for $K_{\phi}$ from Equation (6.6) into the curvature-rotation Equation (5.40)

$$\frac{d\omega_{\phi}}{r_{\phi} d\phi} = \frac{M_{\phi}}{D_{11}} - \left( \frac{D_{12} \cos \phi}{D_{11} r} \right) \omega_{\phi} \tag{6.8}$$

The third equation is derived from the second equilibrium equation (5.59). Making use of the Gauss Equation (5.23), substituting for $N_{\theta}$ from Equation (6.2) and dividing by $r r_{\phi}$, one finally obtains

$$\frac{dQ_{\phi}}{r_{\phi} d\phi} = \left( \frac{A_{12} \sin \phi}{A_{11} r} + \frac{1}{r_{\phi}} \right) N_{\phi} + \frac{\sin \phi \cos \phi}{r^2} \left( \frac{A_{11} A_{22} - A_{12}^2}{A_{11}} \right) u + \left( \frac{\sin \phi}{r} \right)^2 \left( \frac{A_{11} A_{22} - A_{12}^2}{A_{11}} \right) w \tag{6.9}$$

$$- \frac{\cos \phi}{r} \dot{Q}_{\phi} - q_n$$

The fourth equation is derived from the third equilibrium equation (5.61) in a similar way. Thus, using the Gauss condition, substituting for $M_{\theta}$ from Equation (6.4), and dividing by $r r_{\phi}$ yields
The fifth equation is simply the displacement-rotation Equation (5.36)

$$\frac{dM_\phi}{r_\phi d\phi} = \frac{\cos \phi}{r} \left( \frac{D_{12} - D_{11}}{D_{11}} \right) M_\phi + \left(\frac{\cos \phi}{r}\right)^2 \left( \frac{D_{11} D_{22} - D_{12}^2}{D_{11}} \right) \omega_\phi + Q_\phi$$

(6.10)

The sixth equation comes from the first equilibrium equation (5.57) by using the Gauss condition, substituting for $N_\phi$ from Equation (6.2), and dividing by $r_\phi^2$.

$$\frac{dN_\phi}{r_\phi d\phi} = \frac{\cos \phi}{r} \left( \frac{A_{12} - A_{11}}{A_{11}} \right) N_\phi + \left(\frac{\cos \phi}{r}\right)^2 \left( \frac{A_{11} A_{22} - A_{12}^2}{A_{11}} \right) u$$

$$+ \left(\frac{\sin \phi \cos \phi}{r^2}\right) \left( \frac{A_{11} A_{22} - A_{12}^2}{A_{11}} \right) w - \frac{Q_\phi}{r_\phi} - q_\phi$$

(6.12)

Some remarks concerning the form of the fundamental Equations (6.7) through (6.12) are in order. To begin with, they are linear, first order, ordinary differential equations in the single variable $\phi$. Secondly, they are coupled and have variable coefficients which reflect the variable geometry and elastic properties. Finally, it is observed that no other derivatives are involved, except those of the fundamental variables. This last fact makes this form of the shell equations particularly appealing from the standpoint of applying them to the tire problem. It means that only pointwise values of the elastic constants, thickness, radii of curvature, etc. are required in the numerical integration. Hence, the variability of these quantities is incorporated into the solution with relative ease.

To complete the formulation of the tire shell problem, a set of boundary conditions must be specified. Since the tire is symmetrical with respect to the wheel plane, we need only obtain a solution for an interval which consists of half the meridian arc length, i.e.,
between the tire bead and tire crown. At the bead, we shall specify a clamped support, while at the crown the boundary conditions are dictated by symmetry. Thus

\[ u = w = \omega_\phi \equiv 0 \]  

(6.13)

at the bead, and

\[ u = \omega_\phi = \Omega_\phi \equiv 0 \]  

(6.14)

at the crown. The clamped support at the bead is considered to be very close to reality since in practice there is a high degree of restraint due to interference fit between the tire bead and the wheel.

6.2 Stepwise Integration of the Fundamental Equations

The problem confronting us then is that of finding a solution to a system of first order differential equations of the form

\[
\frac{d}{dS_\phi} [y(S_\phi)] = [T(S_\phi)] [y(S_\phi)] + [P(S_\phi)]
\]  

(6.15)

subject to the simple boundary conditions

\[ [y(a)] = 0 \]  

(6.16)

\[ [y(b)] = 0 \]  

(6.17)

where \([y(S_\phi)]\) is a 6 \times 1 matrix of the fundamental variables, \([T(S_\phi)]\) is a 6 \times 6 matrix of variable coefficients and \([P(S_\phi)]\) is a 6 \times 1 matrix of non-homogeneous loading functions. For the purposes of developing the integration techniques, the independent variable is taken as the meridian arc length \(S_\phi\), but the choice of variables is clearly
immaterial. In the actual numerical calculations we shall revert to $\phi$ as the independent variable. The boundary conditions $[y(a)]$ and $[y(b)]$ are $3 \times 1$ matrices where $a$ and $b$ are the initial and terminal points on the meridian curve respectively.

The solution to Equation (6.15) can be written symbolically as

$$[y(S_\phi)] = [Y(S_\phi)] [C] + [Z(S_\phi)]$$  \hspace{1cm} (6.18)

where $[Y(S_\phi)]$ is a $6 \times 6$ matrix of complementary solutions, $[C]$ is a $6 \times 1$ matrix of arbitrary constants, and $[Z(S_\phi)]$ is a $6 \times 1$ matrix of particular solutions. $[Y(S_\phi)]$ and $[Z(S_\phi)]$ must therefore satisfy the equations

$$\frac{d}{dS_\phi} [Y(S_\phi)] = [T(S_\phi)] [Y(S_\phi)]$$  \hspace{1cm} (6.19)

$$\frac{d}{dS_\phi} [Z(S_\phi)] = [T(S_\phi)] [Z(S_\phi)] + [P(S_\phi)]$$  \hspace{1cm} (6.20)

There are several techniques available for numerically integrating a system of equations such as (6.15). The most common and perhaps simplest to apply is that of Runge-Kutta. However, these methods in general require that initial values be given for all the dependent variables, before the integration process can be started. Thus, some modifications must be made before such techniques can be applied to shell boundary value problems, in which conditions are specified both at the beginning and end of the interval.

One method of determining the solution functions $[Y(S_\phi)]$ and $[Z(S_\phi)]$ is to numerically integrate Equations (6.19) and (6.20) by assuming an arbitrary set of initial conditions. Special steps must then be taken to insure that the general solution (6.18) satisfies the real boundary conditions. A convenient set of initial conditions is

$$[Y(a)] = [I]$$  \hspace{1cm} (6.21)

$$[Z(a)] = 0$$  \hspace{1cm} (6.22)
where \([1]\) is the unit matrix. Substituting these values into the general solution, Equation (6.18), evaluated at \(S_\phi = a\), gives

\[
[y(a)] = [C]
\]

Hence the general solution at \(S_\phi = b\) is

\[
[y(b)] = [Y(b)] [y(a)] + [Z(b)]
\]

In this equation \([Y(b)]\) and \([Z(b)]\) are known from the numerical integration of Equations (6.19) and (6.20). \([y(b)]\) and \([y(a)]\) represent twelve undetermined constants, of which six are known from the boundary conditions (6.16) and (6.17). Thus the number of unknowns in Equation (6.24) equals remaining boundary values to be determined.

This method, in effect, replaces the single boundary value problem of Equations (6.15) - (6.17) with a set of seven initial value problems defined by Equations (6.19) - (6.22). These however are directly amenable to numerical integration by the Runge-Kutta technique.

Despite the fact that a much greater computational effort is required, it appears that the above method is well suited to our problem. In actual practice, however, this technique breaks down when applied to shells with long meridian arc lengths. The failure is characterized by a complete loss of accuracy after a certain critical length has been exceeded. This is not due to accumulative or round-off error in the numerical integration process. Rather, it is the result of subtracting very large and nearly equal numbers in Equation (6.24).

Kalnins\(^{(40)}\) explains the effect as follows: Suppose for simplicity that the deformations in an axisymmetric shell are caused purely by loadings of unit value along the edge \(S_\phi = a\). It is known that the solutions to the shell equations characteristically involve increasing and decreasing exponentials of the arc length. Moreover, these combine in such a way as to cause the stress resultants and deformations to decay rapidly along a meridian,
resulting in the well known edge effect. At a certain distance $S_\phi = b$ from the edge the stress resultants will be nearly zero. At this point the terms in Equation (6.24) will have the following relative magnitudes: $[y(b)]$ small, $[Y(b)]$ large, $[y(a)]$ unity, and $[z(b)] \equiv 0$ since we are dealing with the homogeneous case. Therefore, the only way for the matrix product $[Y(b)] [y(a)]$ to yield the small values of $[y(b)]$ is through subtraction of large terms of $[Y(b)]$ which are nearly equal. When this happens, a large number of significant figures is lost and the accuracy of the calculation is destroyed.

A useful length criterion for determining the point at which the accuracy of this method breaks down is given by

$$\Gamma S_\phi \leq 3$$

(6.25)

where

$$\Gamma^4 = \frac{3(1 - \nu_{\phi\phi} \nu_{\theta\phi})}{(r_s h)^2}$$

(6.26)

and $r_s$ is the minimum radius of curvature. For ordinary aircraft tires the critical arc length is generally between 2 and 4 inches. This of course is much smaller than the total meridian length; hence the indication is that the above scheme would fail.

This difficulty can be circumvented by breaking the shell up into a number of segments and applying the stepwise integration technique to each segment individually. The length of each segment is chosen such that it meets the criterion of Equation (6.25). In this way shells of much greater meridian length can be handled. Special measures must be taken however to assure that there is continuity in the fundamental variables between segments.

Let the shell be divided into $M$ segments for which $\Gamma S_\phi \leq 3$. Each segment is identified by $S_i$ and its meridian length extends from $s_i$ to $s_{i+1}$ as shown in Figure 37. The initial edge of the shell coincides with $S_\phi = s_1$ and the terminal edge with $S_\phi = s_{M+1}$. The
solution over the total interval $S_1 \leq S_{\phi} \leq S_{M+1}$ is by analogy to Equation (6.18)

$$[y(S_{\phi})] = [Y_i(S_{\phi})] [y(S_i)] + [Z_i(S_{\phi})]$$

(6.27)

where $S_{\phi}$ denotes any point in the interval, and $i$ identifies the particular segment in which the point lies. $[Y_i(S_{\phi})]$ and $[Z_i(S_{\phi})]$ denote matrices corresponding to $[Y(S_{\phi})]$ and $[Z(S_{\phi})]$ in each segment $S_i$. They are given by extension of Equations (6.19) through (6.22), i.e.,

$$\frac{d}{dS_{\phi}} [Y_i(S_{\phi})] = [T(S_{\phi})] [Y_i(S_{\phi})]$$

(6.28)

$$\frac{d}{dS_{\phi}} [Z_i(S_{\phi})] = [T(S_{\phi})] [Z_i(S_{\phi})] + [P(S_{\phi})]$$

(6.29)

$$[Y_i(s_i)] = [I]$$

(6.30)

$$[Z_i(s_i)] = 0$$

(6.31)

The requirement of continuity in the fundamental variables across the junction points of the segments is expressed from Equation (6.27) as

$$[y(s_{i+1})] = [Y_i(s_{i+1})] [y(s_i)] + [Z_i(s_{i+1})]$$

(6.32)

(i = 1, 2, ..., M)

Equation (6.32) involves $M+1$ unknown 6 x 1 matrices: $[y(s_i)]$, $i = 1, 2, \ldots, M+1$. However, the total number of unknowns is reduced by 6 since the boundary conditions at $s_i$ and $s_{M+1}$ are expressed in terms of the fundamental variables. Thus the system of matrix
equations (6.32) contain exactly 6M unknowns which is equal to the total number of
equations. Hence the problem is well set and in theory could be solved by any standard
equation solving routine.

There are certain numerical difficulties, however, associated with solving large
systems of equations. To avert these, Kalnins\(^{40}\) applies the method of Gaussian
elimination directly to the system of matrix equations (6.32). This is accomplished by first
partitioning the equations as follows:

\[
\begin{bmatrix}
Y_1(s_{i+1}) \\
Y_2(s_{i+1})
\end{bmatrix}
= \begin{bmatrix}
Y_{11}(s_{i+1}) & Y_{21}(s_{i+1}) \\
Y_{31}(s_{i+1}) & Y_{41}(s_{i+1})
\end{bmatrix}
\begin{bmatrix}
Y_1(s_i) \\
Y_2(s_i)
\end{bmatrix}
+ \begin{bmatrix}
Z_{11}(s_{i+1}) \\
Z_{21}(s_{i+1})
\end{bmatrix}
\]

\(i = 1, 2, \ldots, M\) (6.33)

so that each of the Equations (6.32) becomes a pair of equations which can be written as

\[
[Y_{11}(s_{i+1})] [Y_1(s_i)] + [Y_{21}(s_{i+1})] [Y_2(s_i)] - [Y_1(s_{i+1})] = -[Z_{11}(s_{i+1})]
\]

\[
[Y_{31}(s_{i+1})] [Y_1(s_i)] + [Y_{41}(s_{i+1})] [Y_2(s_i)] - [Y_2(s_{i+1})] = -[Z_{21}(s_{i+1})]
\]

\(i = 1, 2, \ldots, M\) (6.34)

Equation (6.34) represents a system of 2M linear matrix equations in which the known
coefficients \([Y_{ji}(s_{i+1})]\) are 3 x 3 square matrices, \([Z_{ji}(s_{i+1})]\) are 3 x 1 column matrices, and
\([y_j(s_i)]\) are 3 x 1 column matrices of unknowns.

It is presumed in what follows that elements of \([y_1(s_i)]\) are the 3 known boundary
conditions of the leading edge and \([y_2(s_{M+1})]\) are those of the final edge. If this is not
naturally the case, it can be brought about by a simple rearrangement of the rows of the
matrices in the first and last of Equations (6.34). With this accomplished, the method of Gaussian elimination is applied directly to the system of matrix equations (6.34). The technique is best illustrated by means of an example.

Suppose the shell is divided into three segments, i.e., \( M = 3 \). Then Equation (6.34) can be written in the expanded form

\[
\begin{bmatrix}
Y_{21}(s_2) & 0 & 0 & 0 & 0 & 0 \\
Y_{41}(s_2) & 0 & -1 & 0 & 0 & 0 \\
0 & Y_{12}(s_3) & Y_{22}(s_3) & -1 & 0 & 0 \\
0 & Y_{32}(s_3) & Y_{42}(s_3) & 0 & -1 & 0 \\
0 & 0 & 0 & Y_{13}(s_4) & Y_{23}(s_4) & -1 \\
0 & 0 & 0 & Y_{33}(s_4) & Y_{43}(s_4) & 0
\end{bmatrix}
\begin{bmatrix}
y_2(s_1) \\
y_1(s_2) \\
y_2(s_3) \\
y_1(s_3) \\
y_2(s_4) \\
y_1(s_4)
\end{bmatrix}
\]

\[
\begin{bmatrix}
-Z_{11}(s_2) - Y_{11}(s_2) & y_1(s_1) \\
-Z_{21}(s_2) - Y_{31}(s_2) & y_1(s_1) \\
-Z_{12}(s_3) \\
-Z_{22}(s_3) \\
-Z_{13}(s_4) \\
-Z_{23}(s_4) + y_2(s_4)
\end{bmatrix}
\]

(6.35)
Applying Gaussian elimination to this set of matrix equations leads to the triangularized system

\[
\left[
\begin{array}{cccc}
[E_1] & -[I] & 0 & 0 \\
[C_1] & -[I] & 0 & 0 \\
[E_2] & -[I] & 0 & 0 \\
[C_2] & -[I] & 0 & 0 \\
[E_3] & -[I] & 0 & 0 \\
[C_3] & -[I] & 0 & 0 \\
\end{array}
\right]
\left[
\begin{array}{c}
[y_2(s_1)] \\
[y_1(s_2)] \\
[y_2(s_2)] \\
[y_3(s_2)] \\
[y_2(s_3)] \\
[y_1(s_4)] \\
\end{array}
\right]
= \left[
\begin{array}{c}
[A_1] \\
[B_1] \\
[A_2] \\
[B_2] \\
[A_3] \\
[B_3] \\
\end{array}
\right]
\]

where \([E_i], [C_i]\) are 3 x 3 square matrices and \([A_i], [B_i]\) are 3 x 1 column matrices, which are defined in general by

\[
[E_1] = [Y_{21}]
\]

\[
[C_i] = [Y_{4i}] [E_i]^{-1} \quad (i = 1, 2, \ldots, M)
\]

\[
[E_i] = [Y_{2i}] + [Y_{1i}] [C_{i-1}]^{-1}
\]

\[
[C_i] = \left\{ [Y_{4i}] + [Y_{3i}] [C_{i-1}]^{-1} \right\} [E_i]^{-1}
\]

\[
-[A_1] = [Z_{11}] + [Y_{11}] [y_1(s_1)] \quad (i = 1)
\]

\[
-[B_1] = [Z_{21}] + [Y_{31}] [y_1(s_1)] + [Y_{41}] [E_1]^{-1} [A_1]
\]
\[-[A_i] = [Z_{1i}] + [Y_{1i}]\ [C_{i-1}]^{-1}\ [B_{i-1}] \]
\[-[B_i] = [Z_{2i}] + [Y_{3i}]\ [C_{i-1}]^{-1}\ [B_{i-1}] \quad (i = 2, 3, \ldots, M - 1) \tag{6.39}
\]
\[+[Y_{4i}] + [Y_{3i}]\ [C_{i-1}]^{-1}\ [E_{i}]^{-1}\ [A_i] \]
\[-[A_M] = [Z_{1M}] + [Y_{1M}]\ [C_{M-1}]^{-1}\ [B_{M-1}] \quad (i = M) \]
\[+[Y_{2M}] + [Y_{3M}]\ [C_{M-1}]^{-1}\ [E_{M}]^{-1}\ [A_M] \quad (6.40)\]

In the above equations \([Y_{ji}(s_{i+1})]\) and \([Z_{ji}(s_{i+1})]\) have been replaced by \([Y_{ji}]\) and \([Z_{ji}]\) respectively, for purposes of brevity.

The unknown matrices of Equation (6.36) are now obtained in succession, starting with the last and working forward. In general they are given by
\[+[y_1(s_{M+1})] = [C_M]^{-1}\ [B_M] \quad (i = M) \tag{6.41}\]
\[+[y_2(s_M)] = [E_M]^{-1}\ \left\{ [y_1(s_{M+1})] + [A_M] \right\} \]
\[+[y_1(s_{M-i+1})] = [C_{M-i}]^{-1}\ \left\{ [y_2(s_{M-i+1})] + [B_{M-i}] \right\} \]
\[+[y_2(s_{M-i})] = [E_{M-i}]^{-1}\ \left\{ [y_1(s_{M-i+1})] + [A_{M-i}] \right\} \quad (i = 2, 3, \ldots, M-1) \tag{6.42}\]
Once all the unknowns \( y(s_t) \) have been determined, then the values of the fundamental variables can be computed from Equation (6.27) for any value of \( S_{\phi} \) at which the solutions \( Y_1(S_{\phi}) \) and \( Z_1(S_{\phi}) \) are stored during the integration of the initial value problems (6.28) - (6.31).
6.3 Incremental Method of Solving Nonlinear Tire Problem

The fundamental equations presented in Section (6.1) are restricted by the assumption that the shell undergoes only small displacements. This assumption falls far short of being valid for an inflated pneumatic tire. It is not uncommon, for example, to have displacements at the crown of the tire, which are several times greater than the carcass thickness. It is clear therefore that the linear equations (6.7)-(6.12) are not directly applicable to the tire problem.

A more appropriate set of equations could be developed by reformulating the problem without imposing the restriction of small displacements. This would, of course, lead to a more involved set of nonlinear differential equations, which would be difficult to solve even numerically. In order to avoid the complications introduced by nonlinear shell theory, we shall attempt to approximate the nonlinear solution to the tire problem by a sequence of linearized solutions. The idea is illustrated graphically in Figure 38.

According to this scheme, the final deformed state of the tire shell is determined by passing incrementally through a sequence of equilibrium configurations, each of which is obtained from the previous one by the application of linear theory. It is expected that as the steps decrease in size and increase in number for a specified total load, the incremental solution will converge to the actual nonlinear solution.

To carry out this incremental solution requires repeating the following five basic steps for each increment:

(1) Specify the instantaneous shell configuration.

(2) Define the total instantaneous state of stress and deformation.

(3) Write a set of linear shell equations appropriate to the instantaneous configuration and state of stress.
(4) Apply an additional increment of load.

(5) Solve for the incremental stresses and deformations by the method of stepwise integration.

Let us now consider each of these steps in order.

It was noted previously that the neutral surface of the shell is completely defined by its principal radii of curvature \( r_\phi \) and \( r_\theta \) as functions of the coordinate \( \phi \). Recursion relations for each of these quantities can be developed which will allow us to pass from one configuration to the next. Let the subscript \( i \) refer to a particular state of the shell. Let us agree also that the shell passes from the \( i-1 \) st. to the \( i^{th} \) state by the application of the \( i^{th} \) increment of load. Then the deformed \( r_\phi \) is given by Equation (5.38) which can also be written in the expanded form

\[
(r_\phi)_{i} = (r_\phi)_{i-1} \left[ 1 + \epsilon^o_\phi \frac{d\omega_\phi}{d\phi} \right]_i \quad (6.43)
\]

The deformed \( r_\theta \) is obtained from the deformed \( r \) by applying Equation (5.3). The latter is given by Equation (5.42) which can also be written as

\[
r_i = r_{i-1} \left( 1 + \epsilon_\theta^o \right)_i \quad (6.44)
\]

The corresponding coordinate \( \phi \) is given by the sum

\[
\phi_i = \phi_{i-1} + \omega_\phi_i \quad (6.45)
\]

It is convenient here to set down also the recursion relations which define the deformed tire meridian with respect to the fixed \( x-y \) coordinate system. These are determined from the incremental \( u \) and \( w \) displacements (Figure 39) as follows:
In terms of the initial coordinates \((x_0, y_0)\) and incremental displacements we may also write the above as

\[
x_i = x_0 + \sum_{k=1}^{i} w_k \cos (\phi_{i-1}) - u_k \sin (\phi_{i-1})
\]

\[
y_i = y_0 + \sum_{k=1}^{i} w_k \sin (\phi_{i-1}) + u_k \cos (\phi_{i-1})
\]

In addition, let us also write relations which give the total accumulated \(u\) and \(w\) displacements as a result of \(i\) increments of load. These will be referred to the original undeformed meridian as shown in Figure 40. Here and in the sequel an asterisk (*) will denote total or cumulative values. Thus we have the recursion equations

\[
u_i^* = u_{i-1}^* + u_i \cos (\omega_\phi^*)_{i-1} + w_i \sin (\omega_\phi^*)_{i-1}
\]

\[
w_i^* = w_{i-1}^* - u_i \sin (\omega_\phi^*)_{i-1} + w_i \cos (\omega_\phi^*)_{i-1}
\]

where \(\omega_\phi^*\) represents the total rotation of an element and is given by the sum

\[
(\omega_\phi^*)_i = \sum_{k=1}^{i} (\omega_\phi)_k
\]

The total displacements in Equations (6.50) and (6.51) can also be expressed solely in terms of the incremental values. Thus
The above equations are sufficient to specify the shell configuration at any stage of the incrementing process. Let us now turn to the problem of defining the corresponding state of stress. To this end we examine the stresses on an element of the shell as it is deformed from the \( i \)-1 st. to the \( i \)th position. For simplicity we consider only the normal stress \( \sigma_\phi \) as shown in Figure 41.

Prior to the \( i \)th increment of load, the element is under an initial stress denoted by \( \{\sigma_\phi^*\}_{i-1} \). We suppose that this stress, which is generated by the previous sequence of loadings, is referred to the \( i \)-1 configuration, i.e., to the coordinates \( (\phi_{i-1}, \theta_{i-1}) \). The element is then deformed into its \( i \)th position by the \( i \)th load increment. This is accomplished by a rigid body displacement \( \bar{U}_i \), followed by a rigid rotation \( (\omega_\phi)_i \) and finally by the pure deformations \( (\epsilon_\phi)_i \) and \( (\epsilon_\theta)_i \).

Let the initial stress, referred now to the new coordinates \( (\phi_i, \theta_i) \) be denoted by \( \{\sigma_\phi^*\}_{i-1} \). The magnitude of \( \{\sigma_\theta^*\}_{i-1} \) differs from that of \( \{\sigma_\phi^*\}_{i-1} \) due to the change in area of the elemental face. This is expressed by the relation

\[
\{\sigma_\phi^*\}_{i-1} = \frac{\{\sigma_\phi^*\}_{i-1}}{1 + (\epsilon_\theta)_i}
\]  

(6.55)

The pure deformation gives rise to an additional or incremental component of stress. This stress is denoted by \( (\sigma_\phi)_i \) and \( (\sigma_\phi)_i \) when referred to the \( (\phi_i, \theta_i) \) and \( (\phi_{i-1}, \theta_{i-1}) \) coordinates respectively. If the incremental stress is computed on the basis of linear shell theory, it is referred to the original coordinates. Implicit in the linear theory, however, is the assumption that the difference between stresses referred to the original and deformed coordinates is indistinguishable, therefore we may write
The total stress \( (\sigma^*_\phi)_i \) on the deformed element, referred to the \((\phi_i, \theta_i)\) coordinates, is given by the sum

\[
(\sigma^*_\phi)_i = (\bar{\sigma}^*_\phi)_i - 1 + (\bar{\sigma}^*_\phi)_i
\]  

or

\[
(\sigma^*_\phi)_i = \frac{(\sigma^*_\phi)_i - 1}{[1 + (\epsilon^*_\phi)_i]} + (\sigma^*_\phi)_i
\]

Repeating these arguments for the \(\sigma_\theta\) stress leads to a similar result

\[
(\sigma^*_\theta)_i \approx \frac{(\sigma^*_\theta)_i - 1}{[1 + (\epsilon^*_\phi)_i]} + (\sigma^*_\theta)_i
\]

According to the assumptions of linear shell theory, the incremental strains are negligible compared to unity. Thus Equations (6.51) and (6.52) can be simplified to

\[
(\sigma^*_\phi)_i \approx (\sigma^*_\phi)_i - 1 + (\sigma^*_\phi)_i
\]

\[
(\sigma^*_\theta)_i \approx (\sigma^*_\theta)_i - 1 + (\sigma^*_\theta)_i
\]

An equivalent form of these two equations involving only the incremental stresses is obviously

\[
(\sigma^*_\phi)_i \approx \sum_{k=1}^{i} (\sigma^*_\phi)_k
\]
\[ (a_0^*)_i \approx \sum_{k=1}^{i} (a_0)_k \] (6.63)

Since the force and moment resultants are integrals of these stresses across the shell thickness, it follows that their total values at the end of \( i^{th} \) load increment are given by the expressions

\[
\begin{align*}
(N_\phi^*)_i & \approx (N_\phi^*)_{i-1} + (N_\phi)_i \\
(N_\theta^*)_i & \approx (N_\theta^*)_{i-1} + (N_\theta)_i \\
(M_\phi^*)_i & \approx (M_\phi^*)_{i-1} + (M_\phi)_i \\
(M_\theta^*)_i & \approx (M_\theta^*)_{i-1} + (M_\theta)_i
\end{align*}
\] (6.64)

Or equivalently

\[
\begin{pmatrix}
N_\phi^* \\
N_\theta^* \\
M_\phi^* \\
M_\theta^* \\
\end{pmatrix}_i \approx \sum_{k=1}^{i} 
\begin{pmatrix}
N_\phi \\
N_\theta \\
M_\phi \\
M_\theta \\
\end{pmatrix}_k
\] (6.65)

These resultants are also referred to the \((\phi_i, \theta_i)\) coordinates.

Let us turn next to the problem of defining the total state of strain in terms of the incremental values. Consider an element of the shell is three successive configurations denoted by \( i-1 \), \( i \) and \( i+1 \). In passing from the \( i-1 \)st. to the \( i^{th} \) position the elemental arc length \( dS_\phi \) becomes
Similarly in passing from the $i^{th}$ to the $i+1$ st. position

\[(dS_\phi)_{i+1} = dS_\phi [1 + (\epsilon_\phi)_i]\]  \hspace{1cm} (6.67)

The total strain resulting from these two deformations is by definition

\[\epsilon^*_\phi \bigg|_{i}^{i+1} = \frac{(dS_\phi)_{i+1} - (dS_\phi)_{i-1}}{(dS_\phi)_{i-1}} \hspace{1cm} (6.68)\]

Substituting from Equations (6.66) and (6.67) this becomes after canceling terms

\[\epsilon^*_\phi \bigg|_{i}^{i+1} = \frac{[1 + (\epsilon_\phi)_i] [1 + (\epsilon_\phi)_{i+1}] - 1}{1 - 1} \hspace{1cm} (6.69)\]

Clearly, by following the changes in elemental arc length in passing from the zero-th to the $i^{th}$ position, the total strain $(\epsilon^*_\phi)_i$ may be written as

\[(\epsilon^*_\phi)_i = \frac{(dS_\phi)_0 \left\{ \prod_{k=1}^{i} [1 + (\epsilon_\phi)_k] \right\} - (dS_\phi)_0}{(dS_\phi)_0} \hspace{1cm} (6.70)\]

or simply

\[(\epsilon^*_\phi)_i = \left\{ \prod_{k=1}^{i} [1 + (\epsilon_\phi)_k] \right\} - 1 \hspace{1cm} (6.71)\]

where $\prod_{k=1}^{i}$ represents a finite product of $i$ terms. Since the incremental strains are small quantities, their products are negligible compared to the incremental values themselves.
Neglecting these products in Equation (6.71) the total strain may be approximated by the sum

\[(\varepsilon_\phi^*)_i \approx \sum_{k=1}^{i} (\varepsilon_\phi)_k\]  

(6.72)

Repeating these arguments for changes in elemental arc length in the \(\theta\) direction leads to a similar result.

\[(\varepsilon_\theta^*)_i \approx \sum_{k=1}^{i} (\varepsilon_\theta)_k\]  

(6.73)

Having defined the instantaneous configuration and the instantaneous state of stress and strain, it is now possible to derive a set of equations which will govern the incremental behavior of the shell. In doing this we must, as before, deal with three basic sets of equations; (1) the kinematic equations relating strain, displacement rotation, and change in curvature, (2) the laminate constitutive equations which are derived from Hooke's Law, (3) the equations of static equilibrium.

The kinematic relations developed in Section 5.3 are founded on purely geometrical principles and Love's hypothesis that normals remain normal. They are therefore completely independent of the previous history of deformation of the shell and carry over unaltered to the incremental case. They are summarized below, again using the subscript \(i\) to denote a particular incremental value

\[\varepsilon_\phi = \varepsilon_\phi^0 + z K_\phi\]  

(6.74)

\[\varepsilon_\theta = \varepsilon_\theta^0 + z K_\theta\]  

(6.75)

\[\varepsilon_\phi^0 = \left( \frac{du}{r_\phi d\phi} + \frac{w}{r_\phi} \right)_i\]  

(6.76)

\[\varepsilon_\theta^0 = \left( \frac{u \cos \phi}{r} + \frac{w \sin \phi}{r} \right)_i\]  

(6.77)
(K_\phi)_i = \left( \frac{d\omega_\phi}{r_\phi d\phi} \right)_i \quad (6.78)

(K_\theta)_i = \left( \frac{\omega_\phi \cos \phi}{r} \right)_i \quad (6.79)

(\omega_\phi)_i = \left( \frac{u}{r_\phi} - \frac{dW}{r_\phi d\phi} \right)_i \quad (6.80)

(\gamma_{\phi\theta})_i = (K_{\phi\theta})_i = (\omega_\theta)_i \equiv 0 \quad (6.81)

Since the incremental stresses and incremental strains are both referred to the same instantaneous coordinate system, it may be assumed that they are related according to Hooke's Law.

\begin{align*}
\begin{bmatrix}
\epsilon_\phi \\
\epsilon_\theta \\
\gamma_{\phi\theta}
\end{bmatrix}_i &=
\begin{bmatrix}
R_{11} & R_{12} & R_{16} \\
R_{22} & R_{26} \\
R_{66}
\end{bmatrix}
\begin{bmatrix}
\sigma_\phi \\
\sigma_\theta \\
\tau_{\phi\theta}
\end{bmatrix}_i
\end{align*}

(6.82)

By repeating the derivations contained in Chapter 4 the incremental laminate constitutive equations can be developed from the incremental Hooke's Law (6.82). They are analogy to Equations (4.33) and 4.34),

\begin{align*}
\begin{bmatrix}
N_\phi \\
N_\theta \\
N_{\phi\theta}
\end{bmatrix}_i &=
\begin{bmatrix}
A_{11} & A_{12} & 0 \\
A_{22} & 0 \\
A_{66}
\end{bmatrix}
\begin{bmatrix}
\epsilon_\phi \\
\epsilon_\theta \\
\gamma_{\phi\theta}
\end{bmatrix}_i
\end{align*}

(6.83)
The equations of static equilibrium, unlike the kinematic equations, do not carry over unchanged to the incremental case. This is because, at any stage of the incrementing process, there are now initial stresses as well as incremental stresses acting on an element. Although the net stress resultants are given by the simple sum of the initial and incremental resultants, the proper equilibrium equations cannot be derived by simply substituting these values into the linear equilibrium equations (5.57), (5.59) and (5.61). The reason for this is that the linear form of the equilibrium equations implies that certain nonlinear terms involving stress resultants are small and may be dropped as higher order terms. Although this may be quite true for the incremental stress resultants, it is not at all true for the initial resultants since these may be orders of magnitude larger than the incremental values. Therefore, to derive the correct equilibrium equations for a shell under initial stress one must first deal with the nonlinear equations.

The nonlinear equations of equilibrium are written in terms of the net stress resultants. These are then replaced by the sum of the initial and incremental resultants. The resulting expressions are linearized with respect to the incremental resultants to give the final form of the equilibrium equations.\(^\text{(43)}\)

These points will become clear as the explicit equations are developed below.

A very general form of the nonlinear equilibrium equations has been derived by Sanders\(^\text{(44)}\) by means of energy principles. When the approximations of small strain and small but finite rotations are introduced these equations reduce to a form analogous to the equilibrium equations used in the von Kármán, large deflection theory of flat plates.

Under these approximations, the nonlinear equilibrium equations for an axisymmetrically loaded shell of revolution are

\[
\begin{bmatrix}
M_\phi \\
M_\theta \\
M_{\phi\theta} \\
\end{bmatrix}
= 
\begin{bmatrix}
D_{11} & D_{12} & 0 \\
0 & D_{22} & 0 \\
& & D_{66}
\end{bmatrix}
\begin{bmatrix}
K_{\phi} \\
K_\theta \\
K_{\phi\theta}
\end{bmatrix}
\]

\text{ (6.84)}
These equations, which are equivalent to Sanders Equations (91)-(93), have also been
derived by Kempner\(^{(45)}\). They are identical to the linear equations (5.57), (5.59) and (5.61)
except for the terms

\[-r \omega_\phi N_\phi\]

and

\[-\frac{d}{d\phi} (r \omega_\phi N_\phi)\]

The physical significance of these nonlinear terms can be seen by examining the stress
resultants on a differential element in its original and deformed positions (Figure 42). The
meridian stress resultants \(N_\phi\) and \(N_\phi + \frac{dN_\phi}{d\phi} d\phi\) acting on the deformed element are resolved
into components acting normal and tangential to the faces of the undeformed element. The
tangential components were considered in the derivation of the linear equation in Section
(5.4), while the normal components represent the nonlinear contribution. The normal
component acting on the right edge of the shell is equal to \(\omega_\phi N_\phi\). The component on the
left edge is

\[
\left( N_\phi + \frac{dN_\phi}{d\phi} d\phi \right) \left( \omega_\phi + \frac{d\omega_\phi}{d\phi} d\phi \right)
\]
Expanding and neglecting higher order terms, we find that the component on the left edge reduces to

\[ \omega_\phi N_\phi + \frac{d}{d\phi} \left( \omega_\phi N_\phi \right) d\phi \]

Resolving these stress resultants into the local \((\phi, z)\) coordinate directions, and summing forces in the \(\phi\)-direction, we find that the net force, denoted by \(f_\phi\), is

\[ f_\phi = - \left[ \omega_\phi N_\phi + \frac{d}{d\phi} \left( \omega_\phi N_\phi \right) \right] \frac{d\phi}{2} \left[ r + \frac{dr}{d\phi} d\phi \right] d\theta - \omega_\phi N_\phi \frac{d\phi}{2} r d\theta \quad (6.88) \]

Expanding and neglecting terms of higher order this reduces to

\[ f_\phi = -r \omega_\phi N_\phi d\phi d\theta \quad (6.89) \]

Similarly, the net force \(f_n\) in the \(z\)-direction is

\[ f_n = (\omega_\phi N_\phi) r d\theta - \left[ \omega_\phi N_\phi + \frac{d}{d\phi} \left( \omega_\phi N_\phi \right) d\phi \right] \left( r + \frac{dr}{d\phi} d\phi \right) d\theta \quad (6.90) \]

Again, expanding and neglecting higher order terms, this reduces to

\[ f_n = -\frac{d}{d\phi} \left( r \omega_\phi N_\phi \right) d\phi d\theta \quad (6.91) \]

If the force components \(f_\phi\) and \(f_n\) are added to Equations (5.56) and (5.58) respectively, one obtains after dividing by \(d\phi d\theta\), the nonlinear Equations (6.85) and (6.86).

The nonlinear equilibrium equations now provide the framework for developing a set of linear incremental equilibrium equations. This is accomplished by replacing the net stress resultants in Equations (6.85)-(6.87) by the sum of the initial and incremental resultants, from Equation (6.64). These equations are then linearized with respect to the
incremental resultants, i.e., terms involving products of the rotation and incremental stress resultants are dropped as higher order. In order to avoid a too lengthy notation in the resulting equations, we shall omit the subscript $i-1$ on the quantities $\phi$, $r_\phi$, $r$, $N_\phi^*$, $N_\theta^*$, $M_\phi^*$, $M_\theta^*$, $Q_\phi^*$, $q_\phi^*$, $q_n^*$, and the subscript $i$ on the quantities $\omega_\phi$, $N_\phi$, $N_\theta$, $M_\phi$, $M_\theta$, $Q_\phi$, $q_\phi$, $q_n$. The linearized incremental equilibrium equations may then be written as

\[
\begin{align*}
\frac{dN_\phi^*}{d\phi} + (N_\phi - N_\theta) r_\phi \cos \phi + Q_\phi^* r - r \omega_\phi N_\phi^* + rr_\phi (q_\phi^* + p_\phi^*) &= 0 \quad (6.92) \\
\frac{dQ_\phi^*}{d\phi} + Q_\phi^* r_\phi \cos \phi - N_\phi^* r - N_\theta r_\phi \sin \phi - \frac{d}{d\phi} (r \omega_\phi N_\phi^*) + rr_\phi (q_n^* + p_n^*) &= 0 \quad (6.93) \\
\frac{dM_\phi^*}{d\phi} + (M_\phi - M_\theta) r_\phi \cos \phi - Q_\phi^* r r_\phi + p_m^* &= 0 \quad (6.94)
\end{align*}
\]

where

\[
\begin{align*}
p_\phi^* &= \frac{dN_\phi^*}{r_\phi d\phi} + (N_\phi^* - N_\theta^*) \frac{\cos \phi}{r} + \frac{Q_\phi^*}{r_\phi} + q_\phi^* \\
p_n^* &= \frac{dQ_\phi^*}{r_\phi d\phi} + Q_\phi^* \frac{\cos \phi}{r} - \frac{N_\phi^*}{r_\phi} - N_\theta^* \frac{\sin \phi}{r} + q_n^* \\
p_m^* &= \frac{dM_\phi^*}{r_\phi d\phi} + (M_\phi^* - M_\theta^*) \frac{\cos \phi}{r} - Q_\phi^*
\end{align*}
\]

In these equations we have assumed that the transverse shear force $Q_\phi$ and the external loads $q_\phi$ and $q_n$ can be written, in analogy with Equation 6.64, as
The terms \( r\omega_\phi N_\phi^* \) and \( \frac{d}{d\phi} (r\omega_\phi N_\phi^*) \) now represent the influence of the initial stress. The latter term can be expanded as follows:

\[
\frac{d}{d\phi} \left( r\omega_\phi N_\phi^* \right) = \omega_\phi \left( r \frac{dN_\phi^*}{d\phi} + N_\phi^* r_\phi \cos \phi \right) + \left( rN_\phi^* \right) \frac{d\omega_\phi}{d\phi}
\]

Substituting for \( \frac{d\omega_\phi}{d\phi} \) from Equation (6.73) and using the definition (6.95) this becomes

\[
\frac{d}{d\phi} \left( r\omega_\phi N_\phi^* \right) = \omega_\phi \left[ rr_\phi (p_\phi^* - q_\phi^*) + \dot{N}_\theta^* r_\phi \cos \phi - \dot{Q}_\phi^* r \right] + rr_\phi N_\phi^* K_\phi
\]

In view of Equation (6.100), the second equilibrium equation (6.93) becomes

\[
rr_\phi N_\phi^* K_\phi + rr_\phi (q_n^* + p_n^*) = 0
\]

Let us consider now the significance of the quantities \( p_\phi^* \), \( p_n^* \), and \( p_m^* \) as defined by Equations (6.95)-(6.97). These terms reflect the fact that the accumulated stress resultants together with the accumulated loads do not exactly satisfy the equilibrium equations written with respect to the instantaneous coordinates. If they did, then these
terms would vanish identically. It could be argued that the accumulated stress resultants very nearly satisfy equilibrium with respect to instantaneous coordinates, and that therefore $p_\phi^*$, $p_n^*$ and $p_m^*$ are small and may be disregarded. However, it is a simple matter to retain these terms, and to treat them as fictitious loads which are required to maintain equilibrium.

Consider, for example, $p_\phi^*$ defined by Equation (6.95). In this expression all of the quantities are known at any stage of the incrementing process with the exception of the derivative. This can be computed in terms of the incremental derivatives by means of a recursion relation. Substituting for $(N\phi^*)_i-1$ from Equation (6.64)

$$\frac{d (N\phi^*)_{i-1}}{(dS\phi)_i} = \frac{d}{(dS\phi)_i} [(N\phi^*)_{i-2} + (N\phi)_{i-1}]$$

(6.102)

and noting that

$$\frac{d}{(dS\phi)_i} = \frac{1}{[1 + (\epsilon \phi^o)_{i-1}]} \frac{d}{(dS\phi)_{i-1}}$$

(6.103)

we may write

$$\frac{d(N\phi^*)_{i-1}}{(dS\phi)_i} = \frac{1}{[1 + (\epsilon \phi^o)_{i-1}]} \left[ \frac{d(N\phi^*)_{i-2}}{(dS\phi)_{i-1}} + \left( \frac{dN\phi}{dS\phi} \right)_{i-1} \right]$$

(6.104)

Neglecting incremental strains compared to unity, this becomes

$$\frac{d(N\phi^*)_{i-1}}{(dS\phi)_i} \approx \frac{d(N\phi^*)_{i-2}}{(dS\phi)_{i-1}} + \left( \frac{dN\phi}{dS\phi} \right)_{i-1}$$

(6.105)
The second term on the right hand side of the above equation is simply the (i-1) st. incremental derivative. Once the (i-1) st. incremental solution is known, this derivative is obtained by simple substitution into the differential equation (6.92). Similarly, the other derivatives contained in $p_n^*$ and $p_m^*$ are given by the recursion formulae

$$\frac{d}{(dS_\phi)_i} (Q_\phi^*)_{i-1} \approx \frac{d}{(dS_\phi)_{i-1}} (Q_\phi^*)_{i-2} + \left(\frac{dQ_\phi}{dS_\phi}\right)_{i-1}$$ (6.106)

$$\frac{d}{(dS_\phi)_i} (M_\phi^*)_{i-1} \approx \frac{d}{(dS_\phi)_{i-1}} (M_\phi^*)_{i-2} + \left(\frac{dM_\phi}{dS_\phi}\right)_{i-1}$$ (6.107)

By means of Equations (6.95)-(6.97) and Equations (6.105)-(6.107) the fictitious load terms may now be calculated at any particular point in the incrementing process.

It is now possible to write a set of six linear differential equations analogous to Equations (6.7)-(6.12) which govern the incremental deformation of the tire shell. Again, we shall omit the subscripts i-1 and i to avoid a cumbersome notation. The first and second equation carry over directly from Equation (6.7) and (6.8) since these are purely kinematic in nature.

$$\frac{d\mu}{r_\phi d\phi} = \frac{N_\phi}{A_{11}} \left(\frac{A_{12} \cos \phi}{A_{11}} \right) u - \left(\frac{A_{12} \sin \phi}{A_{11}} + \frac{1}{r_\phi}\right) w$$ (6.108)

$$\frac{d\omega_\phi}{r_\phi d\phi} = \frac{M_\phi}{D_{11}} \left(\frac{D_{12} \cos \phi}{D_{11}} \right) \omega_\phi$$ (6.109)

The third equation is derived from the incremental equilibrium Equation (6.101) by substituting for $N_\theta$ from Equation (6.1), $K_\phi$ from Equation (6.6) and dividing by $rr_\phi$. 
\[ \frac{dQ_\phi}{r_\phi d\phi} = \left( \frac{A_{12}}{A_{11}} \frac{\sin \phi}{r} + \frac{1}{r_\phi} \right) N_\phi + \frac{\sin \phi \cos \phi}{r_\phi} \left( \frac{A_{11} A_{22} - A_{12}^2}{A_{11}} \right) u \]
\[ + \left( \frac{\sin \phi}{r} \right)^2 \left( \frac{A_{11} A_{22} - A_{12}^2}{A_{11}} \right) w_0 \frac{\cos \phi}{r} \frac{N_\phi^*}{D_{11}} + \frac{N_\phi^*}{D_{11}} M_\phi \]  
\[ + \left( N_\theta - \frac{D_{12}}{D_{11}} N_\phi^* \right) \frac{\cos \phi}{r} \frac{Q_\phi^*}{r_\phi} + \frac{p_\phi^* - q_\phi^*}{r_\phi} \right) \omega_\phi - (q_n + p_n^*) \]  

The fourth equation is the same as Equation (6.10) since it is derived from the moment equilibrium equation (5.61) which is identical to (6.87).

\[ \frac{dM_\phi}{r_\phi d\phi} = \cos \phi \left( \frac{D_{12} - D_{11}}{D_{11}} \right) M_\phi + \left( \frac{\cos \phi}{r} \right)^2 \left( \frac{D_{11} D_{12} - D_{12}^2}{D_{11}} \right) \omega_\phi \]  
\[ + Q_\phi \]  

The fifth equation also carries over from Equation (6.11) since it is strictly kinematic.

\[ \frac{dw}{r_\phi d\phi} = \frac{u}{r_\phi} - \omega_\phi \]  

The sixth equation is derived from the incremental equilibrium Equation (6.92) by substituting for \( N_\theta \) from Equation (6.1) and dividing by \( r r_\phi \).

\[ \frac{dN_\phi}{r_\phi d\phi} = \cos \phi \left( \frac{A_{12} - A_{11}}{A_{11}} \right) N_\phi + \left( \frac{\cos \phi}{r} \right)^2 \left( \frac{A_{11} A_{22} - A_{12}^2}{A_{11}} \right) u \]
\[ + \left( \frac{\sin \phi \cos \phi}{r^2} \right) \left( \frac{A_{11} A_{22} - A_{12}^2}{A_{11}} \right) w_0 \frac{Q_\phi}{r_\phi} + \frac{N_\phi^*}{r_\phi} \omega_\phi \]
\[ - (q_\phi + p_\phi^*) \]  

(6.113)
The incremental boundary conditions that must be satisfied by the above six equations are identical to Equation (6.13) and (6.14).

It is apparent that the fundamental differential equations and boundary conditions are of the same form as Equations (6.15)-(6.17). Therefore, the method of stepwise integration, developed previously, is applicable to their solution.

6.4 Implementation of the Method of Stepwise Integration

To apply the method of stepwise integration, the tire meridian is first divided into a number of stations. For convenience these are chosen to coincide with the experimental data points shown in Figure 22. The stations are then arranged into groups such that they divide the meridian into a smaller number of segments. Each of these segments contains several stations, at which the solution is to be evaluated. The segment lengths must also meet the criterion of Equation (6.25).

It is recalled from Section 5.1 that the principal radii of curvature are expressed as functions of the polar angle $\alpha$ according to Equations (5.14) and (5.17). However, $\alpha$ cannot be explicitly expressed in terms of $\phi$ except by numerically inverting Equation (5.22). It is not efficient to perform this inversion for every increment of the independent variable as the integration proceeds. Therefore, as an alternative, the radii of curvature and the angle $\phi$ are computed at each station on the tire meridian prior to the integration. These are then stored and used to interpolate the intermediate values of $r_\phi$ and $r_\theta$ during the integration process.

There are several integration schemes that could be applied to solve the fundamental equations. The fourth-order Runge-Kutta method was selected because it is both accurate and easy to apply. Details of this technique can be found in texts dealing with numerical analysis.\(42\)

Figure 43 is a photograph showing a typical aircraft tire meridian section. It is seen that in some places the tread rubber is quite thick in comparison with the carcass. Despite this, the effects of the tread will be ignored and the tire will be considered to consist solely
of the carcass formed by its textile reinforced plies. This is a reasonable approximation since
the stiffness of the carcass is much greater than that of the rubber tread.

The photograph of Figure 43 also shows that the carcass thickness varies around the
tire, increasing near the area of the bead. This is due to the fact that the individual plies
wrap around the bead wire hoops and then extend back up the carcass for a short distance.
In the tire industry, this is known as the "turn-up." The effect of the turn-up is to increase
the carcass stiffness in the lower region of the tire. This can be taken into account by simply
allowing the carcass thickness to vary with position, under the assumption that the cords in
the turn-up carry an equal share of the total load. Strictly speaking, this is not true, since
they are not continuous. Therefore, the computed cord stresses will not be completely
accurate in this region. However, this is not a serious shortcoming because the highest cord
loads occur in the crown of the tire and not at the bead.

It was noted in a previous section that either the arc length $S_\phi$ or the angle $\phi$ could
be considered the independent variable in the fundamental equations. In carrying out the
numerical integration it is convenient to regard the angle $\Phi$ between the normal to the
neutral surface and the vertical as the independent variable (Figure 37). This permits the
integration to commence at $\Phi = 0$, which would not be the case if $\phi$ were the variable of
integration. This change in variable results only in minor modifications to the fundamental
equations since

$$\phi = \Phi + \pi/2 \quad (6.114)$$

and

$$d\phi = d\Phi \quad (6.115)$$

6.5 **Shear Stress Distribution**

Once the values of the fundamental variables have been found from the stepwise
integration, the incremental values of the stress and strain can be determined at any point in
the tire. The incremental values then combine according to Equations (6.60), (6.61), (6.72) and (6.73) to yield the total state of stress and deformation.

Considering the strains first, we have

\[
\{\varepsilon_\phi\}_i = \{\varepsilon_\phi^0\}_i + z \{K_\phi\}_i
\]

\[
\{\varepsilon_\theta\}_i = \{\varepsilon_\theta^0\}_i + z \{K_\theta\}_i
\]  

(6.116)

The membrane strains and changes in curvature are given directly in terms of the fundamental variables by Equations analogous to (5.45), (5.50), (6.5) and (6.6). Therefore the incremental strains are completely known.

We have assumed that the incremental strains are related to the incremental stresses by Hooke's Law

\[
\begin{bmatrix}
\sigma_\phi \\
\sigma_\theta \\
\tau_{\phi\theta}
\end{bmatrix}_i =
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{16} \\
Q_{22} & Q_{26} \\
0 & & 0
\end{bmatrix}_i
\begin{bmatrix}
\varepsilon_\phi \\
\varepsilon_\theta \\
K_{\phi\theta}
\end{bmatrix}_i
\]

(6.117)

where the shear strain \(\gamma_{\phi\theta}\)_i has been set equal to zero. According to Equation (4.33) the shear force resultant \(N_{\phi\theta}\) vanishes by virtue of the fact that the shear strain is zero. But clearly the shear stresses are non-zero by Equation (6.117). The stresses \(\tau_{\phi\theta}\)_i are in fact the shear stresses which are required to prevent the shear strain of the type depicted in Figure 12 from occurring. Let us examine the distribution of the incremental shear stress across the shell thickness. The stress in the \(k^{th}\) layer is

\[
\{\tau_{\phi\theta}\}_ik = (Q_{16})_k \{\varepsilon_\phi\}_i + (Q_{26})_k \{\varepsilon_\theta\}_i
\]

(6.118)
Substituting for the strains from Equation (6.116) and rearranging yields

\[ \tau_{\phi\theta} \mid_{ik} = \left[ (Q_{16})_k (\epsilon^o_{\phi})_i + (Q_{26})_k (\epsilon^o_{\theta})_i \right] \\
+ z \left[ (Q_{16})_k (K_{\phi})_i + (Q_{26})_k (K_{\theta})_i \right] \]

(6.119)

Thus the shear stress across any layer is the sum of a constant component plus a component which varies linearly across the thickness. According to Equation (4.28) the \( Q_{16} \) and \( Q_{26} \) terms are positive for \( +\psi \) orientation of the cord and negative for \(-\psi\) orientations. Thus, for example, a 4-ply laminate has the component shear stress distributions shown in Figure 44. An examination of this stress distribution also reveals that neither the shear force resultant \( (N_{\phi\theta})_i \) nor the twisting moment \( (M_{\phi\theta})_i \) is zero as has been assumed. This contradiction is to be expected however, since we are dealing here with a finite number of layers. It will be recalled that the \( B_{16}, B_{26} \) terms of Equation (4.32) which give rise to \( N_{\phi\theta} \) and \( M_{\phi\theta} \) vanish only as the number of layers becomes infinite. Our assumption that these terms are identically zero for a multiply laminate is the source of the contradiction. The extent to which \( N_{\phi\theta} \) and \( M_{\phi\theta} \) do not vanish will serve as a measure of the validity of this assumption.

Numerical values of these quantities will be presented in the next chapter for the specific case of a 6-ply aircraft tire.

The total shear stress due to \( i \) increments of load is given by the summation

\[ \tau_{\phi\theta}^{*} \mid_{ik} = \sum_{j=1}^{i} \left[ (Q_{16})_k (\epsilon^o_{\phi})_j + (Q_{26})_k (\epsilon^o_{\theta})_j \right] \\
+ z \sum_{j=1}^{i} \left[ (Q_{16})_k (K_{\phi})_j + (Q_{26})_k (K_{\theta})_j \right] \]

(6.120)

\[ h_k \leq z \leq h_{k+1} \]
6.6 **Cord Tension**

An approximate expression for the tension in a single cord can be derived by considering the strain in the cord direction of an individual lamina. Applying the transformation Equation (2.16) to the incremental strains of Equation (6.116) gives

\[(e^i)j = m^2(e^1)i + n^2(e^0)i\]

\[(e^2)i = n^2(e^1)i + m^2(e^0)i\] (6.121)

\[(\gamma_{12})i = -2mn[(e^1)i - (e^0)i]\]

The incremental normal stress \((\sigma^1)i\) in the axial direction of a cord embedded in a two-dimensional lamina is given by Hooke’s Law as

\[(\sigma^1)i = \frac{1}{(1-\nu_{f12}\nu_{f21})}[E_{f11}(\varepsilon^1f1)i + \nu_{f12}E_{f22}(\varepsilon^2f)i]\] (6.122)

Assuming that in the axial direction the cord and rubber behave elastically as springs in parallel we have from compatibility of strains

\[(e^1)i = (e^1f1)i = (e^m)i\] (6.123)

In the transverse direction, it has been noted previously that the cord acts essentially as a rigid inclusion in a soft rubber matrix. Therefore in this direction it may be assumed for the present purposes that the lamina strain is due entirely to the deformation of the rubber between cords, i.e.,
In addition, the product of the Poisson's ratios in Equation (6.122) can be neglected with respect to unity since

\[ E_{f22} \ll E_{f11} \]

and

\[ \nu_{f12} \nu_{f21} = (\nu_{f12})^2 \frac{E_{f22}}{E_{f11}} \]  

With these approximations, the cord stress is equivalent to uniaxial tension. Thus

\[
\sigma_{f1} \approx E_{f11} (\varepsilon_{1})_i
\]

\[
= E_{f11} [m^2 (\varepsilon_{\phi})_i + n^2 (\varepsilon_{\theta})_i]
\]

The incremental force \((P_f)_i\) in a single cord of area \(A_f\) is then

\[
(P_f)_i \approx A_f (\sigma_{f1})_i \]

\[
= A_f E_{f11} [m^2 (\varepsilon_{\phi})_i + n^2 (\varepsilon_{\theta})_i]
\]

The total cord tension after \(i\) load increments is given by the sum,

\[
(P_f^*)_i = \sum_{k=1}^{i} (P_f)_k
\]
6.7 **Bead Hoop Stress and Wheel Force**

The bead stress and wheel force can be computed from the force resultants at the fixed boundary of the tire shell (Figure 45). The net vertical component of the transverse shear force \( Q_\phi \) and membrane force \( N_\phi \) must be reacted by the bead wire hoops. If \( r_b \) is the bead hoop radius then the bead tension is given by

\[
T_b = r_b (N_\phi \cos \phi_b - Q_\phi \sin \phi_b)
\]  

(6.129)

where \( \phi_b \) is the value of \( \phi \) at the fixed edge of the shell. The bead wire stress is

\[
\sigma_b = \frac{4r_b}{\pi n_b d_b^2} (N_\phi \cos \phi_b - Q_\phi \sin \phi_b)
\]  

(6.130)

where \( d_b \) is the bead wire diameter, and \( n_b \) is the total number of strands.

The horizontal components of \( Q_\phi \) and \( N_\phi \) must be reacted by the wheel flange. The flange force per unit of circumferential length is therefore

\[
L_f = (N_\phi \sin \phi_b + Q_\phi \cos \phi_b)
\]  

(6.131)

The forces at the tire-wheel interface must be distributed in such a way that they produce a moment which will counter the \( M_\phi \) existing at the bead.
7. Sample Calculation and Experimental Verification

7.1 Inflation Pressure Loading Sequence

In this chapter, the tire shell theory and incremental solution technique developed in previous chapters will be demonstrated by means of a specific numerical example. The particular tire chosen for this purpose is the 32 x 8.8 Type VII aircraft tire shown in Figure 43. Detailed numerical results for this tire will be presented in the following section, while in this section we will take up the general problem of establishing the incremental loading sequence, i.e., the size and number of the pressure increments to be applied.

The basic criterion that governs the magnitude of a particular pressure increment is that the deflections it produces must be everywhere small compared to the shell thickness. This requirement is based on the assumptions of linear shell theory, which is used to compute the incremental deformations. As a practical matter in tire analysis, it is sufficient to restrict our attention to the normal displacement at the tire crown, since it is here that the maximum deflection occurs. Therefore we shall require that the value of any pressure increment be such that the resulting crown displacement is small relative to the carcass thickness, i.e.,

\[ |w_i| \ll h \text{ at } \Phi = 0 \]  \hspace{1cm} (7.1)

Although there is no absolute standard on the required relative smallness of the displacement in linear theory we shall, for the sake of definiteness, adopt the following general rule.

\[ |w_i| \leq \frac{h}{10} \text{ at } \Phi = 0 \]  \hspace{1cm} (7.2)
This criterion of course does not in itself establish the inflation pressure sequence, since $w_i$ is an unknown to be calculated. However it does determine whether a given load increment, however obtained, is acceptable or not.

The establishment of the loading sequence using the above criterion is therefore a trial and error process. It can be automated, however, within the computer program. Consider, for example, the plot of crown deflections versus inflation pressure for the 32 x 8.8 aircraft tire shown in Figure 46. A particular pressure increment in this sequence is determined by providing an initial guess which is then either accepted or rejected according to whether or not the criterion 7.2 is met. If the criterion is met, the program proceeds, by repeating the calculation for the next increment. If it fails, that result is rejected and the calculation is repeated with the value of the pressure increment adjusted according to the simple proportion.

\[
(q_n)^i_{\text{new}} = (q_n)^i_{\text{old}} \frac{h}{10 \left| w_i \right|_{\text{old}}} \quad (7.3)
\]

A means of providing an accurate guess for the trial pressure increment is also included in the computer program. This is based on an interpolation scheme which fits a second degree polynomial to the three previous points of the curve in Figure 46 in order to predict the next point.

Using the techniques outlined above, the following loading sequence was established for the 32 x 8.8 Type VII aircraft tire.
<table>
<thead>
<tr>
<th>Number (i)</th>
<th>((q_n)_i) PSI</th>
<th>((q_n)_i) PSI</th>
<th>(Total)</th>
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<td>0.60</td>
<td>0.60</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.80</td>
<td>1.40</td>
<td></td>
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<td>0.80</td>
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<td>3.30</td>
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<td></td>
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7.2 Numerical Results

A set of numerical results obtained by applying the analysis of the previous chapters to the 32 x 8.8 aircraft tire is presented in this section. The meridian profile of this tire is defined by the experimental x-y coordinates given in the first two columns of Table 1. The calculated values of the angle $\Phi$ and radius of curvature $r_\phi$, based on the least square curve fitting procedure, are listed in columns three and four respectively. The corresponding final values at 95 psi are given in the last four columns of Table 1.

The input data required for the numerical computation is as follows:

A. Cord Properties

1. $E_{f11} = 1.56 \times 10^5 \text{ lbs/in}^2$
2. $\nu_{f12} = 0.7$
3. $G_{f12} = 700 \text{ lbs/in}^2$
4. $d_f = 0.031 \text{ in.}$

B. Rubber Properties

1. $E_m = 450 \text{ lbs/in}^2$
2. $\nu_m = 0.49$
C. Tire Construction Parameters

1. \( n_p = 6 \text{ plies} \)

2. \( \gamma = 57^\circ \)

3. \( r_b = 9.15 \text{ in} \)

4. \( n_o = 26 \text{ cords/in} \)

5. \( t = 0.043 \text{ in} \)

6. x-y coordinates (Table 1)

The results of the numerical computation are exhibited in Tables 2 through 6. Table 2 contains the u-displacements, w-displacements and \( \omega_\phi \) rotations in columns one through three respectively. The input data is also listed in the upper portion of the table. Table 3 contains the membrane force resultants \( N_\phi, N_\theta, \) and \( N_{\phi\theta} \) in columns one through three respectively and the transverse shere force resultant \( Q_\phi \) in column four. The bending moment resultants \( M_\phi, M_\theta \) and \( M_{\phi\theta} \) are listed in Table 4. The membrane strains \( \varepsilon_\phi^0 \) and \( \varepsilon_\theta^0 \) are contained in columns one and two respectively of Table 5. The corresponding surface strains are listed in columns three and four. The cord tensions for the midply and outer ply are given in columns one and two respectively of Table 6.

These results are also displayed graphically in Figures 47 through 56. A comparison of the uninflated and inflated meridian profile is given in Figure 47. The deflection at the crown is slightly over 1/2 inch, which is about twice the carcass thickness at this point. The necessity of accounting for changing geometry is clear since this ratio of deflection to thickness is far beyond the assumptions of linear shell theory.
Figure 48 illustrates the variation of membrane force resultants. The meridian resultant $N_\phi$ is seen to be fairly constant while the circumferential resultant $N_\theta$ decreases markedly from the tire crown to the bead. These results are in general agreement with Ames and Lauterbauch. The membrane shear force resultant $N_{\phi\theta}$ is shown in Figure 49. According to our original assumption of zero $V$-displacements, this resultant should also be zero. As shown it is indeed small compared with the $N_\phi$ and $N_\theta$ resultants.

Figures 50 and 51 show the variation of the bending moments $M_\phi$ and $M_\theta$. The maximum moment occurs at the bead which corresponds to the clamped support. These curves also display the decaying oscillatory form which is typical of the shell edge effect. Notice, however, that the decay is relatively gradual in the case of the tire shell and that these quantities still have significant values even at points quite far removed from the bead. This raises doubts about the accuracy of membrane solutions applied to such multiply tires.

The twisting moment $M_{\phi\theta}$ is shown in Figure 52. This quantity, like $N_{\phi\theta}$, should be zero according to the assumption of vanishing $V$-displacements. The plot shows however that in the vicinity of the tire crown this moment is rather large compared with $M_\phi$ and $M_\theta$. Although this appears to cast some doubt on our basic assumption it should be pointed out that the 6-ply tire used in the sample calculation represents a rather extreme case as far as aircraft tires are concerned. In general, aircraft tires contain many more plies. As we have shown, this reduces the effect of the $B_{16}$, $B_{26}$ coupling terms in the shell constitutive equations which give rise to the $N_{\phi\theta}$ and $M_{\phi\theta}$ resultants. Moreover, it is unlikely, even in the present case, that the magnitude of $M_{\phi\theta}$ shown in Figure 52 would result in any appreciable $V$-displacement, shear strain, or torsion of the tire shell.

Figure 53 shows the transverse shear force resultant $Q_\phi$. Again the shell edge effect due to the clamped support is clearly visible.

The membrane and surface strains are displayed in Figures 54 and 55. The contribution to the circumferential strain $e_\phi$ due to bending is insignificant. This is to be expected since the radius of curvature $r_\theta$ changes only slightly as a result of deformation. In contrast, the bending deformation makes a significant contribution to the meridian strain.
$\varepsilon_\phi$, particularly at the crown ($\Phi = 0$ rad.) and sidewall region ($\Phi = 1.5$ rad.) The effect of bending on the cord tension is also quite noticeable as seen in Figure 56.

7.3 Tire Deflection and Strain Measurements

An experimental check on the preceding calculations was performed by comparing the shapes of the deformed and undeformed tire meridians. These were determined by means of sectioned plaster of paris casts made of the tire surface at 0 and 95 psi. The difference between the cast profiles represents the actual tire deflection due to inflation pressure. Figure 47 shows a comparison between the calculated deflection and the actual deflection from the plaster casts. In general, the agreement is seen to be quite good.

This method however does not provide a completely satisfactory verification, since the casts give only an envelope of the tire deflections. This means that points on the deformed profile cannot be related directly to the corresponding points on the undeformed profile, except of course at the bead and crown. A more rigorous test of the theoretical calculations is provided by a measurement of strain.

The nature of the tire structure makes strain measurements by conventional methods extremely difficult. Ordinary foil or wire gages must be ruled out because of their high modulus and low strain capability, while optical and photographic techniques are complicated by the geometry of the doubly curved tire surface.

A gage suitable for strain measurements on tires is available however from the Peekel Laboratories in Rotterdam. This gage is constructed by helically winding a length of strain gage wire around a stretched rubber thread (Figure 57). The ends of the wire are tied off after a prescribed number of turns and the lead wires are then attached. Removing the tension from the thread allows it to expand laterally, which in turn pre-tentions the wire helix. The entire assembly is then encapsulated in a soft rubber carrier. In operation, a tensile strain causes the thread to contract, thereby relieving the load in the wire. This
causes its resistance to decrease which then becomes a measure of the strain. The Peekel gage has a capability of 20% tensile strain and 15% compressive strain and being rubber itself, it does not reinforce the surface to which it is attached.

Figure 58 illustrates the Peekel rubber strain gages applied to the 32 x 8.8 aircraft tire. The tire tread was buffed away prior to installation of the gages so that they could be positioned on the surface of the outer ply. An ordinary SR-4 strain indicator was used for the measurements. A typical plot of strain versus pressure is shown in Figure 59.

The meridian and circumferential surface strains were each measured at four positions on the tire meridian. These were located at intervals of 2 inches in arc length from each other, starting at the crown. The experimental strains are compared with the calculated values in Figure 55.

Notice that at the crown of the tire, the agreement is fairly good while at the shoulder and sidewall there is something to be desired. Whether the disagreement is the result of inaccuracies in the theoretical calculations or in the experimental procedures is not completely known at this time. In all probability it involves both, meaning that additional refinements are in order. As for the theory, these could take the form of a more accurate lift equation (1.6) and a better means of determining the principal radii of curvature. The experimental technique could be greatly improved by subjecting each strain gage to a calibration on a cord-rubber composite prior to its installation on the tire.

* Peekel Laboratorium Voor Electronica N.V. Albasstraat 1, Rotterdam, The Netherlands
8. Closure

8.1 Summary and Conclusions

The inflation stresses and deformations in multi-ply pneumatic tires have been analyzed by treating the tire carcass as a laminated, anisotropic, toroidal shell of revolution. Unlike existing membrane solutions which are suitable for automotive use only, the present analysis incorporates the effects of bending stiffness, thereby extending its applicability to heavy truck and aircraft tires as well.

The anisotropy of the tire structure is accounted for by treating the plies as a two-phase composite material consisting of textile cords embedded in a rubber matrix. A simplified micro-mechanics theory, which is based on Hermans (26) self-consistent model, is used to characterize the macroscopic elastic moduli of the three-dimensional composite. In this theory the tire cord and rubber matrix are considered to be linearly elastic within the small strain range caused by inflation pressure. Additionally, the textile tire cord is characterized elastically as transversely isotropic while the rubber matrix is considered to be simply isotropic.

The thinness of the ply material allows it to be modeled as a two-dimensional orthotropic lamina. When a number of such laminae are firmly bonded together with alternating cord angle orientations they form the basic tire carcass structure. The overall behavior of this laminated structure is characterized by the laminate constitutive equations which relate force and moment resultants to in-plane strain and changes of curvature, by means of the matrices \( A_{ij}, B_{ij}, \) and \( D_{ij} \).

\[
\begin{bmatrix}
N \\ M
\end{bmatrix} = \begin{bmatrix}
A & B \\ B & D
\end{bmatrix} \begin{bmatrix}
\epsilon \\ \kappa
\end{bmatrix}
\]  

(8.1)
For the case of a multi-ply tire carcass it is shown that the laminate constitutive equations simplify to the form

\[
\begin{bmatrix}
N_\phi \\
N_\theta \\
N_{\phi\theta}
\end{bmatrix} = \begin{bmatrix}
A_{11} & A_{12} & 0 \\
A_{22} & 0 & \gamma^0 \\
0 & A_{66} & \eta^0
\end{bmatrix} \begin{bmatrix}
\varepsilon_\phi^0 \\
\varepsilon_\theta^0 \\
\gamma_{\phi\theta}^0
\end{bmatrix}
\] (8.2)

\[
\begin{bmatrix}
M_\phi \\
M_\theta \\
M_{\phi\theta}
\end{bmatrix} = \begin{bmatrix}
D_{11} & D_{12} & 0 \\
D_{22} & 0 & K_\phi \\
0 & D_{66} & K_{\phi\theta}
\end{bmatrix} \begin{bmatrix}
K_\phi \\
K_\theta \\
K_{\phi\theta}
\end{bmatrix}
\] (8.3)

Physically this implies that on the whole the carcass behaves as though it were a single equivalent orthotropic layer. Moreover, the principal directions of orthotropy are shown to coincide with the parametric lines of the tire shell. Because of this and the axial symmetry of the geometry and the loading, the circumferential displacements and resulting shear strains and twist curvature are taken to be zero in the tire shell equations.

The strain-displacement, equilibrium and laminate constitutive equations governing the tire shell are derived and cast into the form of six first order, ordinary differential equations with variable coefficients. The form of these equations allows the incorporation of variable elastic properties, radii of curvature, thickness, etc., with relative ease, and are therefore well suited to the problem.

A multi-segment, forward integration technique is used to solve these equations. This scheme transforms the two point shell boundary value problem into an equivalent set of initial value problems, which can be integrated numerically with a fourth order Runge-Kutta routine.
The finite displacements are taken into account by an incrementing process in which the total solution is built up as a sequence of linearized solutions. This is accomplished by applying the load in small increments and updating the shell coordinates, radii of curvature, and initial stress after deformation.

The analysis and solution technique are demonstrated by means of a numerical example. The tire chosen for this purpose is a 32 x 8.8 Type VII aircraft tire consisting of six nylon cord plies and having a rated inflation pressure of 95 psi. A complete set of numerical results are presented, including displacements, force and moment resultants, strains, and cord loadings. The numerical data indicates that bending stiffness can indeed have a significant effect on the inflated shape as well as individual cord loads.

The analysis is verified experimentally by comparing the inflated and uninflated meridian profiles. This is accomplished by means of sectioned plaster of paris casts taken on the actual tire. The experimental profile is shown to be in good agreement with the calculated profile. In addition, surface strain measurements were also made with special Peckel Rubber Strain Gages. However the agreement between experimental and calculated values is only fair. This is partially to be expected since strains are derivatives of displacements.

Finally, we should remark that the convergence of the incremental solution technique was investigated numerically and found to be quite satisfactory. This was accomplished by adjusting the criterion 7.2 such that different pressure loading sequences were required to reach the final inflation pressure of 95 psi. Three different loading sequences, consisting of 15, 23, and 32 pressure increments each, were applied. The results on the crown deflection are illustrated in Figure 46. This figure shows that for all practical purposes the curves superimpose, i.e., the incremental solution has converged to the nonlinear solution. This is also true of the other variables such as force and moment resultants, however the numerical data has been omitted for the sake of brevity.

8.2 Recommendations for Future Research

As always, in any complex engineering problem, we have been obliged to make a number of simplifying assumptions and approximations in order to render the problem...
amenable even to numerical solution. In this process, engineering judgement has played a key role in that we have attempted to retain only those features which we believe have a significant influence on the overall behavior of the tire structure. Unfortunately the decision to either retain or neglect a certain effect cannot always be made with any high degree of certitude, and ultimately a more refined analysis is required to settle the issue. In this section we shall outline a few of these so called “gray areas” in which it is quite possible that additional effort is necessary.

The most basic assumption of the present work is that the tire behaves according to the predictions of classical thin shell theory. There are, however, certain instances in practical tire design where the thinness criterion

\[ \frac{h}{r_\phi}, \frac{h}{r_\theta} \ll 1 \]

may not be realized. For example, on some very small high performance aircraft tires it is possible in the sidewall region to have a thickness to radius of curvature ratio as high as 0.3, i.e.,

\[ \frac{h}{r_\phi} = 0.3 \]

Moreover, even in normal aircraft tires, with multiple bead hoops, the build-up in thickness at the bead area due to the turn-up may be sufficient itself to violate the thinness criterion. In these instances one should resort to a higher order theory which relaxes the thinness requirement. A discussion of such theories is of course beyond the scope of this work, and we therefore merely cite the works of Flugge\(^{(46)}\), Bryne\(^{(47)}\), and Biezeno\(^{(48)}\) as being typical extensions of thin shell theory which might be applied in this case.

Another feature of the classical thin shell theory that may not always be achieved in actual practice is associated with the geometrical assumption that normals remain normal after deformation. This assumption leads to the neglecting of shear strains associated with
the transverse direction. It is possible in certain tire shell configurations and for certain pressure loadings that these transverse shear strains may no longer be neglected. This is particularly true at the tire bead area where large transverse shear forces are generated by the fixed support. To take these additional effects into account one must resort to a shear deformation theory for shells. One such theory, which again we merely reference, is that developed by Hildebrand.\(^{(49)}\)

Beyond the approximations inherent in the classical shell theory, we have in addition introduced into the laminate constitutive equations the important assumption that, for multiply laminates, the \(B_{16}, B_{26}\) coupling terms can be neglected as small quantities. This in turn allows for a considerable simplification in the tire shell equations since the \(v\)-displacements can be set equal to zero. However, it also raises a contradiction, because the shear stresses \(\tau_{\phi\theta}\) across the laminate thickness give rise to non-zero values for the membrane force \(N_{\phi\theta}\) and the twisting moment \(M_{\phi\theta}\). This contradiction would disappear if a more precise theory were used which retains the \(B_{16}, B_{26}\) terms. It is possible to formulate such a theory by simply admitting the possibility of \(v\)-displacements. These displacements result in shear strains, circumferential rotations, and twisting of the tire surface, all of which are compatible with the non-zero values of \(N_{\phi\theta}\) and \(M_{\phi\theta}\). The resulting theory would of course be more cumbersome since an eighth order system of equations would be involved as opposed to the sixth order of the present work. It appears unlikely that this more involved theory would significantly alter the results presented here, but the final word must await the more refined analysis.
FIGURE 1 THREE BASIC TIRE CONSTRUCTIONS

(A) BIAS  (B) RADIAL  (C) BELTED BIAS
FIGURE 2 GREEN TIRE
FIGURE 3 LIFTING OF GREEN TIRE
FIGURE 4 CORD ANGLE IN THE CURED TIRE—REF. 16
FIGURE 5 STEPS IN ANALYZING COMPOSITE STRUCTURE

(B) COMPOSITE

(C) LAMINATE

(A) CORD

(RUBBER

(D) TIRE STRUCTURE
FIGURE 6  FORCE-EXTENSION RATIO CURVE FOR A TYPICAL RUBBER  REF. 11
FIGURE 7 TRANSVERSELY ISOTROPIC TIRE CORD
FIGURE 8 STRESS-STRAIN CURVE FOR A TYPICAL TEXTILE FIBER REF. 11
FIGURE 9 HERMANS COMPOSITE MODEL
Figure 10: Two Dimensional Orthotropic Lamina

Figure 11: Lamina Axis Rotation
FIGURE 12 SHEAR COUPLING IN ORTHOTROPIC LAMINA
FIGURE 13 VARIATION OF ELASTIC MODULI
FIGURE 14 VARIATION OF ELASTIC MODULI
FIGURE 15 SECTION OF LAYERED SHELL
RELATIONSHIP BETWEEN SHELL COORDINATES AND PRINCIPAL DIRECTION OF ORTHOTROPY

FIGURE 16
(A) STRESS ACTING ON ELEMENT

(B) FORCE RESULTANTS

(C) MOMENT RESULTANTS

FIGURE 17  FORCE AND MOMENT RESULTANTS
$B_{ij} = \frac{2B_{ij}}{Q_{ij}}$,  \(i_j = 16, 26\)

**Figure 18 Variation of $B_{16}, B_{26}$**
Figure 19. Effect of B_{16} Coupling Term
FIGURE 20 CURVILINEAR COORDINATES
FIGURE 21 GEOMETRY OF A SURFACE OF REVOLUTION
FIGURE 23 APPROXIMATION OF TIRE MERIDIAN SHAPE
FIGURE 24 RELATION BETWEEN $\phi$ AND $\alpha$-COORDINATES
FIGURE 25 PHI VERSUS ALPHA
FIGURE 26 RADIUS OF CURVATURE $r_{\phi}$
FIGURE 27 GAUSS CONDITION FOR A SURFACE OF REVOLUTION
Figure 28 Principal Directions of Orthotropy
FIGURE 29 DEFORMATION OF SHELL ELEMENT
FIGURE 30 DISPLACEMENT OF A MERIDIAN ELEMENT
\[ d\varphi = d\varphi + d\omega_\varphi \]

**Figure 31** Change in curvature in \( \phi \)-direction

**Figure 32** Change in curvature in \( \theta \)-direction
FIGURE 33 \( \varepsilon_\varphi \) STRAIN DUE TO \( W \)-DISPLACEMENT

FIGURE 34 \( \varepsilon_\theta \) STRAIN DUE TO \( U \) AND \( W \)-DISPLACEMENTS
FIGURE 35 ISOLATED SHELL ELEMENT
FIGURE 36 FREE BODY DIAGRAM OF SHELL ELEMENT
FIGURE 37 STEPWISE INTEGRATION OF TIRE SHELL EQUATIONS
FIGURE 38 APPROXIMATION OF NONLINEAR SOLUTION

TIRE DEFLECTION

INFLATION PRESSURE

INCREMENTAL SOLUTION

NONLINEAR SOLUTION

i=1 i=2 i=3 i=4
RECTANGULAR COMPONENTS OF DISPLACEMENT

FIGURE 39
FIGURE 40 ACCUMULATED DISPLACEMENTS
FIGURE 41 ACCUMULATED STRESSES
EFFECT OF ROTATION ON EQUILIBRIUM

FIGURE 42
Figure 43. Meridian Section of 32 x 8.8 Aircraft Tire
**Figure 44** Shear Stress Distribution
FIGURE 45 TIRE BOUNDARY FORCES
FIGURE 46 CROWN DISPLACEMENT

- 15 increment sequence
- 23
- 32
32 x 8.8-6 PLY AIRCRAFT TIRE

95 PSI (CALC.)

EXPERIMENTAL

0 PSI

FIGURE 47 INFLATED MERIDIAN PROFILE
FIGURE 49 MEMBRANE SHEAR FORCE

$N_{\phi \theta}$

$\phi$ - RAD.
FIGURE 50: BENDING MOMENT $M_\phi$
Figure 51: Bending Moment $M$
FIGURE 52  TWIST MOMENT

$M_{\phi \theta}$

$\Phi - \text{RAD.}$

TWIST MOMENT IN. LB./IN.

0  5  10

0.4  0.8  1.2  1.6  2.0  2.4
FIGURE 53 TRANSVERSE SHEAR FORCE
FIGURE 54 TIRE MEMBRANE STRAIN
FIGURE 55 TIRE SURFACE STRAIN
FIGURE 56 TIRE CORD LOAD
Figure 57. Peekel Rubber Strain Gage
Figure 58. Strain Gages Applied to Aircraft Tire
FIGURE 59  $\varepsilon_\theta$ STRAIN AT CROWN

- STRAIN - PERCENT

- PRESS. PSI
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**TABLE 6**
Appendix I

The approximate formulae developed by Halpin and Tsai for predicting the elastic moduli of a fiber reinforced composite are based on the observation that the constants \( k, m, \) and \( \mu \) given by Hermans’ equations (3.7) – (3.9) can be arranged in the following generalized form:

\[
\frac{p}{p_m} = \frac{(1 + \xi \eta V_f)}{(1 - \eta V_f)} \tag{A1}
\]

where

\[
\eta = \frac{(p_f - 1)}{(p_m + \xi)} \tag{A2}
\]

\( p \) = a composite modulus \( k, m, \) or \( \mu \)

\( p_f \) = corresponding fiber modulus \( k_f, m_f, \mu_f \)

\( p_m \) = corresponding matrix modulus \( k_m, m_m, \mu_m \)

\( \xi \) = a measure of reinforcement

A physical interpretation of the \( \xi \)-factor can be obtained by considering the effect on the generalized equation (A1) when \( \xi \) takes on its extreme values. If for example \( \xi = 0 \), then

\[
\frac{p}{p_m} = \frac{1}{(1 - \eta V_f)} = \frac{1}{p_m \left( \frac{V_f}{p_f} + \frac{V_m}{p_m} \right)} \tag{A3}
\]
or
\[
\frac{1}{\bar{p}} = \frac{V_f}{p_f} + \frac{V_m}{p_m}
\]  
(A4)

This is the series connected model which gives a lower bound of a composite modulus. At the other extreme \(\xi = \infty\) we have from (A2)

\[
\eta = 0 \quad \text{(A5)}
\]

\[
\eta \xi = \left( \frac{p_f}{p_m} - 1 \right) \quad \text{(A6)}
\]

from which it follows that

\[
\bar{p} = p_f V_f + p_m V_m \quad \text{(A7)}
\]

This is the parallel model giving the upper bound on the composite moduli. Thus \(\xi\) is seen to be a measure of reinforcement covering the entire spectrum of moduli as it varies from zero to infinity. If this factor were known for a particular fiber geometry and spatial arrangement, then the composite moduli could be calculated directly from the generalized equation (A1).

From these considerations, Halpin and Tsai proposed that the engineering constants \(E_{ij}, \nu_{ij},\) and \(G_{ij}\) could themselves be predicted by a relation similar to (A1). In this new equation, \(\bar{p}, p_f,\) and \(p_m\) represent the engineering constants of the composite, fiber, and matrix respectively, instead of Hermans' constants. The \(\xi\) factor is determined such that the prediction of this new generalized formula agrees with the various existing elasticity solutions.

For the purposes of calculating \(E_{11}\) and \(\nu_{12}\) the \(\xi\)-factor is taken to be infinity, i.e., these constants are given quite accurately by the parallel model. In the case of \(E_{22}\) and
It was found that

\[ \xi_E = 2 \]  \hspace{1cm} \text{(A8)}

\[ \xi_G = 1 \]  \hspace{1cm} \text{(A9)}

for fibers with a circular cross section.
References


