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By

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INTRODUCTION

For Lebesgue outer measure on the real line, closed sets are measurable and sets which are translates of each other have the same measure. Thus in generalizing Lebesgue measure to abstract spaces we may consider two problems: first, the problem of constructing an outer measure on a space so that certain "nice" subsets of the space are measurable, and second, the problem of whether there exists a non-trivial outer measure on a space for which certain "congruent" subsets of the space have the same measure. A non-trivial measure is one for which there exist both "nice" sets of non-zero positive measure and "nice" sets of finite measure.

An early generalization of the first type is the construction of Hausdorff measure in metric spaces. By requiring that the outer measure measure \( m \) constructed on a metric space \( (X,d) \) have the additivity property that \( m(A \cup B) = m(A) + m(B) \) whenever \( d(A,B) > 0 \), we assure that closed subsets of \( X \) are \( m \)-measurable. Munroe\[10\] (numbers in [ ] refer to bibliography) is a general reference on this topic. Recently, Willmott\[17\] has given a construction for an outer measure \( u \) on a uniform space \( (X,\mathcal{U}) \) which has
the additivity property that \( u(A \cup B) = u(A) + u(B) \) whenever there is \( U \in \mathcal{U} \) such that \( U[A] \cap B = \emptyset \). This assures that sets \( A \subset X \) of the form \( A = \bigcap \{ U_n[A]: n \geq 1 \} \) for some sequence \( \{ U_n: n \geq 1 \} \subset \mathcal{U} \) for which \( U_{n+1} \cup U_{n+1} \subset U_n \), are \( u \)-measurable. Knowles\(^7\) has also done work along this line.

For general topological spaces, Rogers and Sion\(^12\) have given a construction of an outer measure \( r \) for which closed sets are \( r \)-measurable. (\( m, u, r \) are the author's notations).

Now all of these constructions proceed similarly.

We are given a family \( C \) of subsets of \( X \) such that \( \emptyset \in C \), and a non-negative, real valued function \( \varphi \) on \( C \) such that \( \varphi(\emptyset) = 0 \). We have some family \( \{ R_\alpha \} \) of conditions which apply to the members of \( C \). For each \( R_\alpha \) we define an outer measure \( \nu_\alpha \) on \( X \) by: for \( E \subset X \) let \( \nu_\alpha(E) = \inf \{ \Sigma \varphi(C_i): \{ C_i \} \subset C \) is countable, \( E \subset \cup \{ C_i \} \) and each \( C_i \) satisfies \( R_\alpha \} \). Then an outer measure \( \nu \) on \( X \) is defined by: for \( E \subset X \) let \( \nu(E) = \sup \{ \nu_\alpha(E): \alpha \}. By choosing the family \( \{ R_\alpha \} \) of conditions suitably, \( \nu \) is just \( m, u, \) or \( r \).

Since the construction procedure for all these outer measures is so similar, we ask whether there is a general construction for which they are special cases. The affirmative answer to this question is given in Chapter Five, using background theory developed in the first four chapters. Basically, we needed to answer two questions in order to obtain our general construction: first, what type of space should be used in our construction, and second,
what family \( \{ R_\alpha \} \) of conditions should be used? Proximity spaces provide the answer to the first question, and covering classes provide the answer to the second.

A proximity space \((X, \delta)\) is a space for which the "nearness" of sets is the primitive concept. Sets \( A, B \subset X \) are either "near" \((A \triangleleft B)\) or "far" \((A \triangleright B)\). Both pseudo-metric spaces and uniform spaces are proximity spaces, having canonical proximity relations \( \delta(d) \) and \( \delta(u) \) defined on them. In fact, the conditions on the sets \( A \) and \( B \) which yield the additivity property for the outer measures \( m \) and \( u \), are just the conditions that \( A \subseteq (d) B \) and \( A \supseteq (u) B \).

By using the idea of a semi-proximity space (a proximity space for which the triangle inequality is not assumed), we are able to include the Rogers-Sion construction in our theory. Thron[15] is a reference for proximity spaces.

Now given a semi-proximity space \((X, \delta)\), and an outer measure \( v \) on \( X \), we say \( v \) is a \( \delta \)-measure iff \( v \) has the additivity property that \( v(A \cup B) = v(A) + v(B) \) whenever \( A \subseteq (d) B \). Thus, \( m \) is a \( \delta(d) \)-measure and \( u \) is a \( \delta(u) \)-measure. We show that for a \( \delta \)-measure \( v \), the sets \( F \subset X \) of the form \( F = \cap \{ F_n : n \geq 1 \} \) for some sequence \( \{ F_n : n \geq 1 \} \) of subsets of \( X \) for which \( F_{n+1} \subseteq (X-F_n) \), are \( v \)-measurable. This result generalizes the corresponding result for the outer measures \( m, u, \) and \( r \).

Now in the Rogers-Sion construction, the condition \( R_\alpha \) is that the family \( \{ C_\alpha \} \) under consideration refine a certain
cover \( P_a \) of \( X \); that is, that for each \( C_i \) there is \( P \in P_a \) such that \( C_i \subseteq P \). Now the \( R_a \) conditions for the outer measures \( m \) and \( u \) can also be stated in terms of requiring that the family \( \{ C_i \} \) refine a certain cover of \( X \); namely, a cover of \( X \) consisting essentially of all spheres of a given radius. We denote the classes consisting of all such covers by \( P(d) \) and \( P(u) \), respectively. Engelking[6] is a reference for these and other topics involving covering classes.

In Chapter Three we introduce a relation \( \delta(P) \) defined directly from a class \( P \) of covers of a space \( X \). We say for \( A, B \subseteq X \) that \( A \delta(P) B \) iff \( \{X-A, X-B\} \) is not refined by any member of \( P \). Conditions on \( P \) are given which imply that \( \delta(P) \) is a semi-proximity or proximity relation. We also show that \( \delta(d) = \delta(P(d)) \) and \( \delta(u) = \delta(P(u)) \). Thus, the proximity relations \( \delta(d) \) and \( \delta(u) \) can be considered as arising from those covering classes used in the construction of \( m \) and \( u \). Similarly, the Rogers-Sion covers give rise to just the semi-proximity relation we need. Namely; given \( C \) and \( \varphi \) as before, and a class \( P \) of covers of \( X \), if we let \( p(E) = \sup \{ p(P)(E) : P \in P \} \) where \( p(P)(E) = \inf \{ \varphi(C_i) : \{ C_i \} \subseteq C \text{ is countable}, E \subseteq \bigcup C_i, \{ C_i \} \text{ refines } P \} \) for \( E \subseteq X \), it follows that \( p \) is a \( \delta(P) \)-measure on \( X \). By letting \( P \) be \( P(d), P(u) \), or Rogers-Sion covers, \( p \) is just \( m, u \), or \( r \). Thus our construction of \( p \) is the general outer measure construction we seek.
Now the second problem posed at the beginning of this introduction, concerning the existence of a non-trivial outer measure invariant on "congruent" sets, has been studied extensively for locally compact topological groups and locally compact Hausdorff spaces. The former case is the well-known theory of Haar measure, for which Berberian [2] is a good reference. Banach[3] utilizes an abstract theory of congruence in locally compact Hausdorff spaces, and Steinlage[14] states a condition, Condition A, which is sufficient for the existence of a non-trivial, invariant measure for these spaces.

Now both topological groups and locally compact Hausdorff spaces admit canonical proximity relations, and the invariant measures discussed in the preceding paragraph are $\delta$-measures with respect to these proximity relations. Thus a proximity space seems to be a reasonable space in which to consider invariant outer measure. Since the situation for locally compact spaces is well-known, local compactness was not assumed in this paper. Along this line, Alexandroff[1] has studied invariant measures in topological groups which are not locally compact. He has shown that without local compactness, countable subadditivity of such a measure cannot be assured, if we require that the measure be non-trivial. For example, let $Q$ be the rational numbers and let $A,B \subset Q$ be congruent just when $A$ is a translate of $B$. Then if $v$ is an invariant
\(\delta(d)\)-measure on \(Q\), where \(d\) is the usual metric for the rationals, the points of \(Q\) all have the same measure \(q \geq 0\), and the measure of a set \(A \subset Q\) is just the number of points in \(A\) multiplied by \(q\). Thus if \(q = 0\), \(\nu\) is the zero measure and so is trivial. If \(q > 0\), then all open subsets of \(Q\) have infinite measure, since an open set contains an infinite number of points. This is also an unwanted situation. However, Alexandroff has shown that given a locally [totally] bounded topological group \(X\) there exists a non-trivial invariant measure on \(X\) having all the properties of a \(\delta\)-measure except that countable subadditivity is replaced by finite subadditivity. We call such a measure an Alexandroff measure. It is this type of measure which we consider in Chapter Six.

The context for our study of Alexandroff measure is that of a translation space, a concept introduced in this paper. Basically, this means we have a proximity space \((X, \delta)\) together with a group \(G\) of one-to-one mappings of \(X\) onto \(X\) which satisfy \(g[A] \delta g[B]\) iff \(A \delta B\). We also assume the group \(G\) is non-trivial by requiring that it be weakly transitive, and we require that it satisfy a modification of Steinlage's Condition A. Then \(A, B \subset X\) are congruent iff there is \(g \in G\) such that \(A = g[B]\). We also define the concept of a locally totally bounded translation space. Our major result is then completely analogous to
the existence theorem of Alexandroff: There exists an Alexandroff measure on a given translation space iff the space is locally totally bounded.

In our work on Alexandroff measure we have considered only the question of existence. Other topics of interest, discussed by Alexandroff in his paper, would include: uniqueness of Alexandroff measure, integration in a space having an Alexandroff measure, and measurability of sets with respect to an Alexandroff measure.
CHAPTER I
PROXIMITY RELATIONS

Definition 1.1: By a semi-proximity space \((X, \delta)\) we mean a non-empty set \(X\) together with a binary relation \(\delta\) between the subsets of \(X\) which satisfies:

\(\text{[P0]}\) For all \(A, B \subseteq X\), either \(A \delta B\) or \(A \not\delta B\), but not both. If \(A \delta B\) we say "\(A\) is near \(B\)" and if \(A \not\delta B\) we say "\(A\) is far from \(B\)".

\(\text{[P1]}\) For all \(A, B \subseteq X\), \(A \delta B\) iff \(B \delta A\).

\(\text{[P2]}\) For all \(A, B, C \subseteq X\), \(C \delta (A \cup B)\) iff \(C \delta A\) or \(C \delta B\).

\(\text{[P3]}\) For all \(A, B \subseteq X\), \(A \cap B \neq \emptyset\) implies \(A \delta B\).

\(\text{[P4]}\) For all \(A \subseteq X\), \(A \not\delta \emptyset\).

Theorem 1.2: Let \((X, \delta)\) be a semi-proximity space, and let \(A, B \subseteq X\) such that \(A \not\delta B\). Then for all \(A^* \subseteq A\), \(B^* \subseteq B\) there holds \(A^* \not\delta B^*\).

Proof: Now \(A \not\delta (B \cup B^*)\) since \(B \cup B^* = B\). Thus by \([P2]\) we have \(A \not\delta B^*\). Then \((A \cup A^*) \not\delta B^*\), so by \([P1]\) we have \(B^* \not\delta (A \cup A^*)\). The desired result then follows from \([P2]\).
Corollary 1.3: Let \((X, \delta)\) be a semi-proximity space, and let \(A, B \subseteq X\) such that \(A \delta B\). Then for all \(A \subseteq A^*\), \(B \subseteq B^*\) there holds \(A^* \delta B^*\).

Definition 1.4: Let \((X, \delta)\) be a semi-proximity space. Then we define a relation \(\subseteq\) between the subsets of \(X\) by:

For \(A, B \subseteq X\), \(A \subseteq B\) iff \(A \delta (X-B)\).

The relation \(\subseteq\) is called the semi-proximal containing relation induced by \(\delta\).

Theorem 1.5: Let \((X, \delta)\) be a semi-proximity space. Then the semi-proximal containing relation \(\subseteq\) induced by \(\delta\) satisfies the following properties:

\([\text{PC0}]\) For all \(A, B \subseteq X\) either \(A \subseteq B\) or \(A \not\subseteq B\), but not both. Thus, \(\subseteq\) is a binary relation between the subsets of \(X\).

\([\text{PC1}]\) \(X \subseteq X\).

\([\text{PC2}]\) For all \(A, B \subseteq X\), \(A \subseteq B\) implies \(A \subseteq B\).

\([\text{PC3}]\) For all \(A, B, C, D \subseteq X\), \(A \subseteq B \subseteq C \subseteq D\) implies \(A \subseteq D\).

\([\text{PC4}]\) For all \(A \subseteq X\) and all \(B_1, \ldots, B_n \subseteq X\) such that \(A \subseteq B_i\) for \(1 \leq i \leq n\), there holds \(A \subseteq \cap \{B_i : 1 \leq i \leq n\}\).

\([\text{PC5}]\) For all \(A, B \subseteq X\), \(A \subseteq B\) implies \((X-A) \subseteq (X-B)\).

Proof: \([\text{PC0}]\) follows immediately from \([\text{PO}]\) and the definition of \(\subseteq\).
Now $X \not\in \emptyset$ by [P4], so since $\emptyset = (X-X)$ we have $X \subseteq X$ by the definition of $\subseteq$; thus [PC1] holds.

To show [PC2], let $A \subseteq B$. Then $A \not\in (X-B)$ so by [P3] $A \cap (X-B) = \emptyset$. But this means $A \subseteq B$.

Now let $A \subseteq B \subseteq C \subseteq D$. Then it follows that $A \subseteq B$, $B \not\in (X-C)$, and $(X-D) \subseteq (X-C)$. Using Theorem 1.2 we obtain $A \not\in (X-D)$. But this just means that $A \subseteq D$, proving [PC3].

To show [PC4], first let $A \subseteq B_1$ and $A \subseteq B_2$. Then $A \not\in (X-B_1)$ and $A \not\in (X-B_2)$. By [P2] it follows that $A \not\in [(X-B_1) \cup (X-B_2)]$ and therefore $A \not\in (X-B_1 \cap B_2)$. Thus $A \subseteq (B_1 \cap B_2)$ by definition of $\subseteq$. The general result follows immediately by induction.

Finally, to show [PC5], let $A \subseteq B$. Then $A \not\in (X-B)$ so $(X-B) \not\in A$ by [P1]. Hence $(X-B) \not\in (X-A)$. But this just means that $(X-B) \subseteq (X-A)$.

Theorem 1.6: Let $(X, \delta)$ be a semi-proximity space, and let $A_i \subseteq B_i$ for $1 \leq i \leq n$. Then

$$\cap \{A_i: 1 \leq i \leq n\} \subseteq \cap \{B_i: 1 \leq i \leq n\}$$
$$\cup \{A_i: 1 \leq i \leq n\} \subseteq \cup \{B_i: 1 \leq i \leq n\}.$$

Proof: For convenience let

$$A = \bigcup_{i=1}^{n} A_i, \quad A^* = \bigcap_{i=1}^{n} A_i, \quad B = \bigcup_{i=1}^{n} B_i, \quad B^* = \bigcap_{i=1}^{n} B_i$$

Now by [PC3], $A^* \subseteq B_i$ for $1 \leq i \leq n$, so by [PC4] $A^* \subseteq B^*$. Again using [PC3] we obtain $A_i \subseteq B$ for $1 \leq i \leq n$. Then by [PC5] $(X-B) \subseteq (X-A_i)$ for $1 \leq i \leq n$ and so
(X-B) \subseteq \cap \{(X-A_i): 1 \leq i \leq n\} by [PC4]. Again taking complements, we obtain A \subseteq B.

The semi-proximal containing relation can be taken as the defining relation for a semi-proximity space, as indicated in the next three theorems.

**Theorem 1.7:** Let X be a non-empty set and let \(\subseteq\) be a relation between the subsets of X which satisfies [PC0]-[PC5]. Then define a relation \(\delta\) between the subsets of as follows:

For A,B \subseteq X, A \delta B iff A \not\subseteq (X-B).

Then \(\delta\) is a semi-proximity relation on X, and is called the semi-proximity relation induced by \(\subseteq\).

**Proof:** [PO] follows immediately from [PC0].

To show [P1], let A \delta B. Then A \not\subseteq (X-B), so by [PC5] B \not\subseteq (X-A). Then by definition of \(\delta\) we have B \delta A.

Now let A,B,C \subseteq X and first suppose C \delta A. Then C \not\subseteq (X-A) and it follows from [PC3] that also C \not\subseteq [(X-A) \cap (X-B)]. But then C \not\subseteq X-(A \cup B), so by the definition of \(\delta\) we have C \delta (A \cup B). Now suppose C \delta A and C \delta B. Then using [PC0] and the definition of \(\delta\) we have C \subseteq (X-A) and C \subseteq (X-B). Applying [PC4] we obtain C \subseteq [(X-A) \cap (X-B)] and hence C \subseteq X-(A \cup B). Then it follows that C \delta (A \cup B). This proves [P2].

Now if A \delta B we have that A \subseteq (X-B), so A \subseteq (X-B) by [PC2]. Hence A \cap B = \emptyset. This proves [P3].
Finally, to show \([P4]\), let \(A \subseteq X\). It follows from \([PC1]\) and \([PC3]\) that \(A \subseteq X\). But this means that \(A \subseteq \emptyset\).

**Theorem 1.8:** Let \((X, \delta)\) be a semi-proximity space, and let \(\subseteq\) be the semi-proximal containing relation induced by \(\delta\), and let \(\delta^*\) be the semi-proximity relation induced by \(\subseteq\). Then \(\delta = \delta^*\).

**Proof:** For \(A \in (\subseteq B)\) iff \(A \subseteq (X-B)\) iff \(A \delta (X-B)\) iff \(A \subseteq B\).

**Theorem 1.9:** Let \(X\) be a non-empty set and let \(\subseteq\) be a relation between the subsets of \(X\) which satisfies \([PC0]-[PC5]\). Let \(\delta\) be the semi-proximity relation induced by \(\subseteq\), and let \(\subseteq^*\) be the semi-proximal containing relation induced by \(\delta\). Then \(\subseteq = \subseteq^*\).

**Proof:** For \(A \in (\subseteq B)\) iff \(A \subseteq (X-B)\) iff \(A \delta (X-B)\) iff \(A \subseteq^* B\).

Hence the concepts of a semi-proximity relation and a semi-proximal containing relation are equivalent. Hereafter it will be understood that the symbol \(\subseteq\) will mean the semi-proximal containing relation induced by \(\delta\), whenever we are speaking in the context of a given semi-proximity space \((X, \delta)\).

**Theorem 1.10:** Let \((X, \delta)\) be a semi-proximity space. For \(A \subseteq X\) let \(cA = \{x: [x] \delta A\}\). Then \(c\) satisfies the axioms of a Čech closure operator; namely:
[C1] \( c\emptyset = \emptyset \).

[C2] For all \( A \subseteq X \), \( A \subseteq cA \).

[C3] For all \( A, B \subseteq X \), \( c(A \cup B) = cA \cup cB \).

Proof: Now if \( x \in c\emptyset \) we have \( \{x\} \cap \emptyset \), contradicting [P4]. Hence \( c\emptyset = \emptyset \), which proves [C1]. Now let \( A \subseteq X \) and let \( x \in A \). Then \( \{x\} \subseteq A \), by [P3], so \( x \in cA \). This proves [C2]. Using [P2] and the definition of \( c \) we have:

\[ x \in c(A \cup B) \iff \{x\} \subseteq (A \cup B) \iff \{x\} \subseteq A \text{ or } \{x\} \subseteq B \iff x \in cA \text{ or } x \in cB. \]

This proves [C3]. For more on Čech closure operators see Čech[5].

Definition 1.11: Let \( (X, \delta) \) be a semi-proximity space.

Then a set \( F \subseteq X \) is said to be \( \delta \)-strongly-closed iff there is a sequence \( \{F_n\}_{n=1}^\infty \) of subsets of \( X \) such that 

\[ F = \bigcap_{n \geq 1} F_n \text{ and } F_{n+1} \subseteq F_n \text{ for all } n \geq 1. \]

Definition 1.12: Let \( (X, \delta) \) be a semi-proximity space.

Then a set \( G \subseteq X \) is said to be \( \delta \)-strongly-open iff there is a sequence \( \{G_n\}_{n=1}^\infty \) of subsets of \( X \) such that 

\[ G = \bigcup_{n \geq 1} G_n \text{ and } G_{n+1} \subseteq G_n \text{ for all } n \geq 1. \]

Theorem 1.13: Let \( (X, \delta) \) be a semi-proximity space.

Then the complement of a \( \delta \)-strongly-closed set is \( \delta \)-strongly-open, and conversely.

Proof: The proof is straightforward using [PC5] and the definitions of \( \delta \)-strongly-closed and \( \delta \)-strongly-open.
Definition 1.14: A semi-proximity space \((X, \delta)\) is said to be a proximity space iff \(\delta\) satisfies:

\[\text{[P5]}\] For all \(A, B \subseteq X\) such that \(A \not\subseteq B\) there exist sets \(C, D \subseteq X\) such that:

1. \(A \subseteq C, B \subseteq D,\)
2. \(C \cap D = \emptyset,\) and
3. \(A \not\subseteq (X-C)\) and \(B \not\subseteq (X-D).\)

Theorem 1.15: A semi-proximity space \((X, \delta)\) is a proximity space iff \(\subseteq\) satisfies:

\[\text{[PC6]}\] For all \(A, B \subseteq X\) such that \(A \subseteq B\) there is a set \(C \subseteq X\) such that \(A \subseteq C \subseteq B.\)

Proof: Suppose \(\delta\) satisfies \([P5]\) and let \(A \subseteq B.\) Then \(A \not\subseteq (X-B)\) so there are sets \(C, D \subseteq X\) such that \(A \subseteq C,\)

\((X-B) \subseteq D,\) \(C \cap D = \emptyset,\) \(A \not\subseteq (X-C),\) and \((X-B) \not\subseteq (X-D).\) Now \(A \not\subseteq (X-C)\) implies \(A \subseteq C\) and \((X-B) \not\subseteq (X-D)\) implies \((X-D) \not\subseteq (X-B)\) which implies \((X-D) \subseteq B.\) Since \(C \cap D = \emptyset\) we have \(C \subseteq (X-D).\) Thus we have \(A \subseteq C \subseteq (X-D) \subseteq B\) which by \([PC3]\) yields \(A \subseteq C \subseteq B\) as desired.

Now suppose \(\subseteq\) satisfies \([PC6]\) and let \(A \not\subseteq B.\) Then \(A \not\subseteq (X-B),\) so there is \(C \subseteq X\) such that \(A \subseteq C \subseteq (X-B).\) Then we have:

1. \(A \subseteq C, B \subseteq (X-C),\)
2. \(C \cap (X-C) = \emptyset\)
3. \(A \not\subseteq (X-C)\) and \(B \not\subseteq X-(X-C).\)

Thus we see that \(\delta\) satisfies \([P5].\)
Lemma 1.16: Let $(X, \delta)$ be a proximity space and let the operator $c$ be defined as in Theorem 1.10; that is, for $A \subseteq X$, $cA = \{x : \{x\} \delta A\}$. Then for $A, B \subseteq X$, $A \subseteq B$ implies $cA \subseteq cB$.

Proof: Since $(X, \delta)$ is a proximity space, given $A \subseteq B$ there is $C \subseteq X$ such that $A \subseteq C \subseteq B$. Now if $x \in (X-C)$ then $\{x\} \delta A$ and hence $x \notin cA$. Hence $x \in cA$ implies $x \in C$; thus $cA \subseteq C \subseteq B$, so $cA \subseteq cB$.

Theorem 1.17: Let $(X, \delta)$ be a proximity space. Then the operator $c$ satisfies [C1]-[C3] and also satisfies:

[C4] For all $A \subseteq X$, $ccA = cA$.

Thus $c$ is a Kuratowski closure operator.

Proof: That $c$ satisfies [C1]-[C3] follows immediately from Theorem 1.10. To prove [C4] we first note that [C2] already gives $cA \subseteq ccA$. Conversely, if $x \notin cA$ we have $\{x\} \delta A$. This yields $A \subseteq (X-\{x\})$, which by Lemma 1.16 gives $cA \subseteq (X-\{x\})$. It follows that $\{x\} \delta cA$ which means $x \notin ccA$.

Remark 1.18: Let $(X, \delta)$ be a proximity space. Then since $c$ is a Kuratowski closure operator we may introduce a topology on $X$, which is called the proximal topology, and which is denoted by $\mathcal{J}(\delta)$, and for which $c$ is the closure operator in the topology. We use the notation $A^\circ$ for the interior of a set $A \subseteq X$ in the topology $\mathcal{J}(\delta)$. 
**Theorem 1.19:** Let \((X, \delta)\) be a proximity space. Then \(A \subseteq B\) implies \(cA \subseteq B^0\), for all \(A, B \subseteq X\).

**Proof:** Using Lemma 1.16 we have \(cA \subseteq B\). Thus \((X-B) \subseteq (X-cA)\) so using Lemma 1.16 again we obtain \(c(X-B) \subseteq (X-cA)\). But this yields \(X-(X-cA) \subseteq X-c(X-B)\) which is the same as \(cA \subseteq B^0\).

**Theorem 1.20:** Let \((X, \delta)\) be a proximity space. Then for all \(A \subseteq X\) we have:

\[
cA = \bigcap \{B : A \subseteq B\} \quad \text{and} \quad A^0 = \bigcup \{C : C \subseteq A\}.
\]

**Proof:** Now \(A \subseteq B\) implies \(cA \subseteq B\) so also \(cA \subseteq B\).

Thus it follows that \(cA \subseteq \bigcap \{B : A \subseteq B\}\). Now let \(x \notin cA\). Then \(\{x\} \notin A\) so \(\{x\} \subseteq (X-A)\). Then there is \(C \subseteq X\) such that \(\{x\} \subseteq C \subseteq (X-A)\). But then \(A \subseteq (X-C)\) and \(x \notin (X-C)\).

Thus \(x \notin \bigcap \{B : A \subseteq B\}\). This proves the first part of the theorem. For the second part we have \(A^0 = X-c(X-A) = X - \bigcap \{B : (X-A) \subseteq B\} = \bigcup \{X-B : (X-B) \subseteq A\} = \bigcup \{C : C \subseteq A\} = X - \bigcap \{B : A \subseteq B\} = \bigcup \{C : C \subseteq A\}.

**Theorem 1.21:** Let \((X, \delta)\) be a proximity space. Then a \(\delta\)-strongly-closed set is a closed \(G_\delta\), and a \(\delta\)-strongly-open set is an open \(F_\delta\), with respect to the topology \(J(\delta)\).

**Proof:** Let \(F\) be a \(\delta\)-strongly-closed subset of \(X\).

Then \(F = \bigcap_{n=1}^{\infty} F_n\) where \(F_{n+1} \subseteq F_n\) for \(n \geq 1\). Now since \(F \subseteq F_{n+1} \subseteq F_n\) for \(n \geq 1\) we have \(F \subseteq F_n\) for \(n \geq 1\). Then \(cF \subseteq F_n^0\) for \(n \geq 1\). Thus we have:
\[ \bigcap_{n=1}^{\infty} F_n = F = cF \subseteq \bigcap_{n=1}^{\infty} F_n^0 \subseteq \bigcap_{n=1}^{\infty} F_n. \]

Thus \( F = cF \), and so \( F \) is closed. Clearly, \( F \) is a \( G_6 \).

The second assertion of the theorem follows by taking complements and using Theorem 1.13.

We shall now use the concept of a distance function to give examples of semi-proximity and proximity spaces.

**Definition 1.22:** Let \( X \neq \emptyset \) be a set and let \( d \) be a mapping from \( X \times X \) into the real numbers. Consider the following axioms:

- [M0] For all \( (x,y) \in X \times X \), \( d(x,y) \geq 0 \).
- [M1] For all \( x \in X \), \( d(x,x) = 0 \).
- [M2] For all \( (x,y) \in X \times X \), \( d(x,y) = d(y,x) \).
- [M3] For all \( x,y,z \in X \), \( d(x,y) \leq d(x,z) + d(z,y) \).
- [M4] For all \( (x,y) \in X \times X \), \( d(x,y) = 0 \) implies \( x = y \).

Then (1) \( (X,d) \) is called a semi-pseudo-metric space iff \( d \) satisfies [M0]-[M2], (2) \( (X,d) \) is called a pseudo-metric space iff \( d \) satisfies [M0]-[M3], (3) \( (X,d) \) is called a semi-metric space iff \( d \) satisfies [M0]-[M2] and [M4], and (4) \( (X,d) \) is called a metric space iff \( d \) satisfies [M0]-[M4].

**Definition 1.23:** Let \( (X,d) \) be a semi-pseudo-metric space. We then define:
For \( A, B \subseteq X \), \( d(A, B) = \inf \{d(a, b) : a \in A, b \in B \} \) for
\[
A \neq \emptyset, B \neq \emptyset, \\
= \infty \text{ otherwise.}
\]

For \( x \in X \), \( \varepsilon > 0 \), \( S^0(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon \} \).

For \( A \subseteq X \), \( \varepsilon > 0 \), \( S^0(A, \varepsilon) = \{x \in X : d(A, \{x\}) < \varepsilon \} \).

For \( \emptyset \neq A \subseteq X \), \( \text{diam}(A) = \sup \{d(a, b) : a, b \in A \} \).

**Theorem 1.24:** Let \((X, d)\) be a semi-pseudo-metric space and define a relation \( \delta(d) \) between the subsets of \( X \) by:

For \( A, B \subseteq X \), \( A \delta(d) B \) iff \( d(A, B) = 0 \).

Then \( \delta(d) \) is a semi-proximity relation on \( X \), and if \((X, d)\) is a pseudo-metric space, \( \delta(d) \) is a proximity relation on \( X \).

**Proof:** First suppose \((X, d)\) is a semi-pseudo-metric space. Then \( \delta(d) \) clearly satisfies \([P0]\), \([P1]\), and \([P3]\).

Now let \( A, B, C \subseteq X \). First suppose \( C \delta(d) (A \cup B) \).

Then there is \( \alpha > 0 \) such that \( d(C, A \cup B) \geq \alpha \). Then it is easy to verify that \( d(C, A) \geq \alpha \) and \( d(C, B) \geq \alpha \). Hence \( C \delta(d) A \) and \( C \delta(d) B \). Conversely, suppose that \( C \delta(d) A \) and \( C \delta(d) B \). Then there are \( \alpha_1, \alpha_2 > 0 \) such that \( d(C, A) \geq \alpha_1 \) and \( d(C, B) \geq \alpha_2 \). It is easy to verify that \( d(C, A \cup B) \geq \min[\alpha_1, \alpha_2] > 0 \), so that \( C \delta(d) (A \cup B) \). Thus \( \delta(d) \) satisfies \([P2]\).

Now suppose there is an \( A \subseteq X \) such that \( A \delta(d) \emptyset \).

Then \( d(A, \emptyset) = 0 \), but by definition \( d(A, \emptyset) = \infty \). Thus we see that \([P4]\) holds.
Now suppose that \((X,d)\) is a pseudo-metric space, and let \(A, B \subseteq X\) such that \(A \not\subseteq B\). Thus there is \(\alpha > 0\) such that \(d(A,B) \geq 2\alpha\). Let \(C = S^\alpha(A,\alpha)\) and let \(D = S^\alpha(B,\alpha)\). It is easy to verify that \(C\) and \(D\) meet all the requirements of \([P5]\).

**Theorem 1.25:** Let \((X,d)\) be a pseudo-metric space. Then define \(\mathcal{J}(d)\) by:

\[
A \in \mathcal{J}(d) \iff \text{ for all } a \in A, \text{ there is } \varepsilon > 0 \text{ such that } S^\varepsilon(a,\varepsilon) \subseteq A.
\]

Then \(\mathcal{J}(d) = \mathcal{J}(\delta(d))\) and the \(\delta(d)\)-strongly-closed sets are just the sets closed in \(\mathcal{J}(d)\).

**Proof:** Let \(\mathcal{c}\) denote the closure operator induced by \(\delta(d)\). Then we have that \(A \in \mathcal{J}(d)\) iff for all \(a \in A\) there is \(\varepsilon > 0\) such that \(S^\varepsilon(a,\varepsilon) \subseteq A\) iff for all \(a \in A\) there is \(\varepsilon > 0\) such that \(S^\varepsilon(a,\varepsilon) \cap (X-A) = \emptyset\) iff for all \(a \in A\), \([a] \delta(d) (X-A)\) iff for all \(a \in A\), \(a \not\in c(X-A)\) iff \(c(X-A) \subseteq (X-A)\) iff \(A \in \mathcal{J}(\delta(d))\). Thus \(\mathcal{J}(d) = \mathcal{J}(\delta(d))\).

Now suppose \(F\) is closed in \(\mathcal{J}(d)\). For each \(n \geq 1\) let \(F_n = S^{\delta(d)}(F,\frac{1}{n})\). It is easy to verify that \(F_{n+1} \subseteq \delta(d) (X-F_n)\) for all \(n \geq 1\) and that \(F = \bigcap \{F_n : n \geq 1\}\). Thus \(F\) is \(\delta(d)\)-strongly-closed. Since \(\delta(d)\) is a proximity relation, we have by Theorem 1.21 that a \(\delta(d)\)-strongly-closed set is closed in \(\mathcal{J}(\delta(d)) = \mathcal{J}(d)\). This completes the proof.

We shall now discuss continuity in semi-proximity spaces. All mappings are single-valued.
Definition 1.26: Let \((X, \delta)\) and \((X^*, \delta^*)\) be semi-proximity spaces. Then a mapping \(f\) from \(X\) to \(X^*\) is said to be proximally continuous (or \(\delta-\delta^*\) continuous) iff for all \(A, B \subseteq X\), \(A \quad \delta \quad B\) implies \(f[A] \quad \delta^* \quad f[B]\).

The following two characterizations of proximal continuity are immediate consequences of the definitions involved.

Theorem 1.27: Let \((X, \delta)\) and \((X^*, \delta^*)\) be semi-proximity spaces. Then a mapping \(f\) from \(X\) to \(X^*\) is proximally continuous iff for all \(A^*, B^* \subseteq X^*\), \(A^* \quad \subset^* \quad B^*\) implies \(f^{-1}[A^*] \quad \subset^* \quad f^{-1}[B^*]\).

Theorem 1.28: Let \((X, \delta)\) and \((X^*, \delta^*)\) be semi-proximity spaces, with \(\subset\) and \(\subset^*\), respectively, the induced semi-proximal containing relations. Then a mapping \(f\) from \(X\) to \(X^*\) is proximally continuous iff for all \(A^*, B^* \subseteq X^*\), \(A^* \quad \subset^* \quad B^*\) implies \(f^{-1}[A^*] \quad \subset \quad f^{-1}[B^*]\).

The next theorem gives one relationship between the concepts of proximal continuity and topological continuity.

Theorem 1.29: Let \((X, \delta)\) and \((X^*, \delta^*)\) be proximity spaces, with \(c\) and \(c^*\) the respective induced closure operators. Let \(f\) be a mapping from \(X\) to \(X^*\) which is proximally continuous. Then \(f\) is topologically continuous with respect to \(\mathcal{J}(\delta)\) and \(\mathcal{J}(\delta^*)\).
Proof: We need only show that $f[cA] \subset c*f[A]$ for all $A \subset X$. Therefore let $A \subset X$ and $x \in cA$ be given. Then $\{x\} \delta A$ so $f[\{x\}] \delta^* f[A]$. But then $\{f(x)\} \delta^* f[A]$ so $f(x) \in c*f[A]$.

Definition 1.30: Let $(X, \delta)$ and $(X^*, \delta^*)$ be semi-proximity spaces. Then a one-to-one mapping $f$ from $X$ onto $X^*$ is said to be a proximal isomorphism iff both $f$ and $f^{-1}$ are proximally continuous.

We now define an order relation on the set of semi-proximity relations on a space $X$, and state a theorem which is immediate from the definition.

Definition 1.31: Let $X$ be a non-empty set and let $\delta$ and $\delta^*$ be two semi-proximity relations on $X$. Then we say that $\delta \leq \delta^*$ iff $A \delta B$ implies $A \delta^* B$ for all $A, B \subset X$.

Theorem 1.32: Let $X$ be a non-empty set and let $\delta$ and $\delta^*$ be two semi-proximity relations on $X$ with the induced semi-proximal containing relations $\subseteq$ and $\subseteq^*$, respectively. Then the following statements are equivalent:

1. $\delta \leq \delta^*$
2. For all $A, B \subset X$, $A \delta B$ implies $A \delta^* B$.
3. For all $A, B \subset X$, $A \subseteq B$ implies $A \subseteq^* B$.

We now state and prove a theorem for use in Chapter Six. Following this are a few miscellaneous examples.
Theorem 1.33: Let \((X, \delta)\) be a proximity space. Let \(A, B \subseteq X\) such that \(A\) is closed in \(\mathfrak{S}(\delta)\) and \(B\) is compact in \(\mathfrak{S}(\delta)\). Then \(A \mathfrak{F} B\) iff \(A \cap B = \emptyset\).

Proof: Let \(A \cap B = \emptyset\). Then \(\{b\} \mathfrak{F} A\) for all \(b \in B\). Thus for each \(b \in B\) there are disjoint sets \(C_b, D_b \subseteq X\) such that \(A \subseteq C_b\) and \(\{b\} \subseteq D_b\). Then \(b \in D_b^0\) for each \(b \in B\) so since \(B\) is compact there are \(b_1, \ldots, b_n \in B\) such that \(B \subseteq \bigcup \{D_{b_i}^0 : 1 \leq i \leq n\} = D\). Now \(A \subseteq \bigcap \{C_{b_i} : 1 \leq i \leq n\} = C\), and clearly \(C \cap D = \emptyset\). It follows that \(A \mathfrak{F} B\). The converse is immediate from \([P3]\).

Remark 1.34: Let \(X\) be any set and let \(A, B \subseteq X\). Then let \(A \mathfrak{F}_m B\) iff \(A \neq \emptyset\) and \(B \neq \emptyset\); let \(A \mathfrak{F}_M B\) iff \(A \cap B \neq \emptyset\); and let \(A \mathfrak{F}_\infty B\) iff \(A \cap B \neq \emptyset\) or both \(A\) and \(B\) are infinite sets. Then \(\mathfrak{F}_m, \mathfrak{F}_M, \) and \(\mathfrak{F}_\infty\) are proximity relations on \(X\). If \(\delta\) is any proximity relation on \(X\) there holds \(\mathfrak{F}_m \leq \delta \leq \mathfrak{F}_M\). Now \(\mathfrak{S}(\mathfrak{F}_m) = \{\emptyset, X\}\), and \(\mathfrak{S}(\mathfrak{F}_M) = \mathfrak{S}(\mathfrak{F}_\infty) = \) all subsets of \(X\).

This last relationship shows that different proximity relations can induce the same topology on a space.

Now consider \((X, \mathfrak{F}_\infty)\) for an uncountable set \(X\), and let \(F\) be a countably infinite subset of \(X\). Then \(F\) is both open and closed in \(\mathfrak{S}(\mathfrak{F}_\infty)\) and hence is a closed \(G_\delta\). But \(F\) is not \(\mathfrak{F}_\infty\)-strongly-closed, for if \(F \subseteq F^*\) then \((X-F^*)\) must be finite, so that the intersection of a countable number of such sets must be an uncountable subset of \(X\), and thus can never equal \(F\).
CHAPTER II

UNIFORMITY AND PROXIMITY

Remark 2.1: We here define some of the notations to be used in this chapter:

For a set $X$, $i_X$ will denote the diagonal of $X \times X$.
For $U \subseteq X \times X$, $U^{-1} = \{(x,y): (y,x) \in U\}$.
For $U, V \subseteq X \times X$, $U \circ V = \{(x,y): \text{there is } z \in X \text{ such that } (x,z) \in V \text{ and } (z,y) \in U\}$.
For $U \subseteq X \times X$ and $x \in X$, $U[x] = \{y: (x,y) \in U\}$.
For $U \subseteq X \times X$ and $A \subseteq X$, $U[A] = \{y: \text{there is } a \in A \text{ such that } (a,y) \in U\}$.
For a mapping $f$ from $X$ to $Y$ and $U \subseteq X \times X$, $(f \times f)[U] = \{(f(x), f(y)) : (x,y) \in U\}$.

Remark 2.2: The following are immediate consequences of the definitions in Remark 2.1.

For all $x \in X$, $U \subseteq X \times X$, $U[x] = U[\{x\}]$.
For all $A \subseteq X$, $U \subseteq X \times X$, $U[A] = U \cup \{U[a] : a \in A\}$.
For all $A \subseteq X$, $U, V \subseteq X \times X$, $U \circ V[A] = U[V[A]]$.
For all $A \subseteq X$, $U, V \subseteq X \times X$, $(U \cap V)[A] \subseteq U[A] \cap V[A]$.
For all $U \subseteq X \times X$, $U^{-1} = (U \cap U^{-1})^{-1}$.
Definition 2.3: Let $X \neq \emptyset$ be a set and let $U \neq \emptyset$ be a family of subsets of $X \times X$ which satisfies:

[U1] For all $U \in U$, $i_X \subseteq U$.

[U2] $U \in U$ implies $U^{-1} \in U$.

[U3] $U, V \in U$ implies $U \cap V \in U$.

[U4] $V \in U$, $V \subseteq U \subseteq X \times X$ implies $U \in U$.

Then $(X, U)$ is said to be a semi-uniform space. If in addition $U$ satisfies:

[U5] For all $U \in U$ there is $V \in U$ such that $V \circ V \subseteq U$,

then $(X, U)$ is said to be a uniform space.

Definition 2.4: Let $X \neq \emptyset$ be a set and let $U \neq \emptyset$ be a family of subsets of $X \times X$ which satisfies:

[B1] For all $U \in U$, $i_X \subseteq U$.

[B2] For all $U, V \in U$ there is $W = W^{-1} \in U$ such that $W \subseteq U \cap V$.

Then $(X, U)$ is said to be a semi-pre-uniform space. If in addition $U$ satisfies:

[B3] For all $U \in U$ there is $V \in U$ such that $V \circ V \subseteq U$,

then $(X, U)$ is said to be a pre-uniform space.

Theorem 2.5: A semi-uniform space is a semi-pre-uniform space, and a uniform space is a pre-uniform space.

Proof: The proof is straightforward using the definitions involved and the remarks at the beginning of this chapter.
Definition 2.6: For $X \neq \emptyset$ and $\mathcal{U} \neq \emptyset$, a family of subsets of $X \times X$, define $\mathcal{U}$, a family of subsets of $X \times X$, as follows:

$\mathcal{U} = \{ U \subset X \times X : \text{there is } V \in \mathcal{U} \text{ such that } V \subset U \}.$

Theorem 2.7: If $(X, \mathcal{U})$ is a semi-pre-uniform space, then $(X, \mathcal{U})$ is a semi-uniform space; if $(X, \mathcal{U})$ is a pre-uniform space, then $(X, \mathcal{U})$ is a uniform space.

Proof: First suppose that $(X, \mathcal{U})$ is a semi-pre-uniform space. That $\mathcal{U}$ satisfies $[U4]$ follows from the definition, and $[B1]$ and the definition imply that $\mathcal{U}$ satisfies $[U1]$. To show $[U2]$, let $U \in \mathcal{U}$ be given. Then there is $V \in \mathcal{U}$ such that $V \subset U$. By $[B2]$ there is $W = W^{-1} \in \mathcal{U}$ such that $W \subset V \cap V \subset U$. But then $W = W^{-1} \subset U^{-1}$ so that $U^{-1} \in \mathcal{U}$.

To show $[U3]$, let $U_1, U_2 \in \mathcal{U}$ be given and find $V_1, V_2 \in \mathcal{U}$ such that $V_1 \subset U_1$, $V_2 \subset U_2$. By $[B2]$ there is $W \in \mathcal{U}$ such that $W \subset V_1 \cap V_2$. Then also $W \subset U_1 \cap U_2$, so that $U_1 \cap U_2 \in \mathcal{U}$.

Now suppose $\mathcal{U}$ also satisfies $[B3]$ and let $U \in \mathcal{U}$ and $V \in \mathcal{U}$ with $V \subset U$ be given. Then using $[B3]$ find $W \in \mathcal{U}$ such that $W \circ W \subset V$. Then $W \circ W \subset U$ and since $U \subset \mathcal{U}$, $W \in \mathcal{U}$. Thus $(X, \mathcal{U})$ is a uniform space.

We shall now prove some general results concerning proximity relations induced by uniform structures. The main result, Theorem 2.14, is preceded by several lemmas.
Lemma 2.8: Let $X \neq \emptyset$ be a set and let $U, V, W \subset X \times X$. Then $U \circ (V \circ W) = (U \circ V) \circ W$. Thus we may write $U \circ V \circ W$ for either of these expressions.

Lemma 2.9: Let $X \neq \emptyset$ be a set and let $U, V \subset X \times X$. Then $(U \circ V)^{-1} = V^{-1} \circ U^{-1}$.

Definition 2.10: Let $X \neq \emptyset$ be a set. For $U \subset X \times X$, define $U^n$ for $n \geq 1$ by:

$U^1 = U$ and $U^n = U \circ U^{n-1}$ for $n \geq 2$.

Thus $U^n = U \circ \ldots \circ U$ $n$ times which in view of Lemma 2.8 may be written without parentheses or with parentheses in any manner. In particular, $U^n = U^{n-1} \circ U$.

Lemma 2.11: Let $X \neq \emptyset$ be a set and let $U \subset X \times X$ such that $U = U^{-1}$. Then for $n \geq 1$ we have $(U^n)^{-1} = U^n$.

Proof: That the theorem is true for $n = 1$ follows immediately from the definition of $U^n$. Now assume the truth of the assertion for $n-1$. Then using Lemma 2.9, for $n \geq 2$:

$(U^n)^{-1} = (U \circ U^{n-1})^{-1} = (U^{n-1})^{-1} \circ U^{-1} = U^{n-1} \circ U = U^n$, as desired.

Lemma 2.12: Let $X \neq \emptyset$ be a set and let $V \subset X \times X$ such that $V = V^{-1}$. Then for all $A, B \subset X$ we have $V[A] \cap B = \emptyset$ iff $V[B] \cap A = \emptyset$.

Lemma 2.13: Let $(X, \mu)$ be a pre-uniform space. Then for $n \geq 1$ and $U \in \mu$ there is $V = V^{-1} \in \mu$ such that $V^n \subset U$. 

Proof: The case $n = 1$ follows directly from [B2].

For $n = 2$ first find $W \in U$ such that $WoW \subset U$ by using [B3]. Then using [B2] find $V = V^{-1} \in U$ such that $V \subset W$. Then $V^2 = VoV \subset U$. Finally, assume the truth of the assertion for $n-1$. Then there is $W = W^{-1} \in U$ such that $W^{n-1} \subset U$.

Using [B2] and [B3] find $V = V^{-1} \in U$ such that $VoV \subset W$. Then $V \subset W$ by [B1]. Since we may assume that $n \geq 3$ we then obtain: $V^n = VoVoV^{n-2} \subset WoW^{n-2} = W^{n-1} \subset U$.

Theorem 2.14: Let $(X,U)$ be a semi-pre-uniform space. For each $n \geq 1$ define a relation $\mathcal{C}_n(U)$ between the subsets of $X$ by:

For $A, B \subset X$, $A \in \not C_n(U) B$ iff there is $U \in U$ such that $U^n[A] \subset B$. We may choose $U = U^{-1}$.

Then we have:

(1) For each $n \geq 1$, $\mathcal{C}_n(U)$ is a semi-proximal containing relation on $X$. We denote by $\delta_n(U)$ the corresponding induced semi-proximity relation on $X$.

(2) $\delta_m(U) \leq \delta_n(U)$ whenever $n \leq m$.

(3) If $(X,U)$ is a pre-uniform space, then $\mathcal{C}_n(U) = \mathcal{C}_m(U)$ and hence $\delta_n(U) = \delta_m(U)$, for all $n,m \geq 1$.

Let $\mathcal{C}(U)$ and $\delta(U)$ denote these common relations. Then $\delta(U)$ is a proximity relation on $X$.

Proof: To prove (1), fix $n \geq 1$. Then $\mathcal{C}_n(U)$ clearly satisfies [PC0] and [PC1]. Since $U$ satisfies [B1], we have
for all $U \in \mathcal{U}$, $A \subseteq X$ that $A \subseteq U^n(A)$. From this it is easy
to verify that $\mathcal{C}_n(U)$ satisfies [PC2] and [PC3]. Now if
$U^n(A) \subseteq B_k$ for $1 \leq k \leq r$ then $U^n(A) \subseteq \bigcap \{B_k: 1 \leq k \leq r\}$,
from which it follows that $\mathcal{C}_n(U)$ satisfies [PC4]. To show
[PC5], suppose $U^n(A) \subseteq B$ for some $U = U^{-1} \in \mathcal{U}$. Then we
have $U^n(A) \cap (X-B) = \emptyset$ so also $U^n(X-B) \cap A = \emptyset$ by Lemma
2.12 since $(U^n)^{-1} = U^n$ by Lemma 2.11. Thus $U^n(X-B) \subseteq (X-A)$
and it is easy to see that $\mathcal{C}_n(U)$ satisfies [PC5].

That (2) holds follows from Theorem 1.32 and the fact
that if $n \leq m$ then $U^m(A) \subseteq B$ implies $U^n(A) \subseteq B$.

To show (3), let $n \geq 1$ be fixed and let $A \in \mathcal{C}_1(U) B$.
Then there is $U \in \mathcal{U}$ such that $U[A] \subseteq B$. Since $U$ satisfies
[B3] it follows from Lemma 2.13 that there is $V = V^{-1} \in \mathcal{U}$
such that $V^n \subseteq U$. Then $V^n(A) \subseteq B$, so $A \in \mathcal{C}_n(U) B$. From (2)
we have $A \in \mathcal{C}_n(U) B$ implies $A \in \mathcal{C}_1(U) B$. Therefore we see
that $\mathcal{C}_n(U) = \mathcal{C}_1(U)$. The assertion of equality in (3)
then follows. To prove that $\delta(U)$ is a proximity relation
suppose $A \subseteq (U) B$. Then there is $U \in \mathcal{U}$ such that $U^2[A] \subseteq B$
and we see that $A \subseteq U[A] \subseteq (U) B$. Thus $\mathcal{C}(U)$ satisfies
[PC6]. This completes the proof of (3).

To show that [B1] and [B2] are not sufficient for the
equality of the $\delta_n(U)$ relations consider the following
example:

$X = \{a,b,c\}$ and $U = \{U\}$ where $U = i_X \cup \{(a,b), (b,a),
(b,c), (c,b)\}$
Clearly, \(u\) satisfies \([B1]\) and \([B2]\) but does not satisfy \([B3]\), since \(U \cup U = X \times X\). But we have:

\[U(a) \cap \{c\} = \emptyset \text{ so } \{a\} \subseteq (U) \cap \{c\},\]

\[c \in U^2[a], \text{ so } \{a\} \subseteq (U) \cap \{c\}.

**Theorem 2.15**: Let \((X,\mathcal{U})\) be a semi-pre-uniform space. Then \(\delta_n(u) = \delta_n(\bar{u})\) for \(n \geq 1\).

**Proof**: Fix \(n \geq 1\). Now \(A \subseteq (\mathcal{U})\) \(B\) means there is \(U \in \mathcal{U}\) such that \(U^n[A] \subseteq B\). But there is \(V \in \mathcal{U}\) such that \(V \subseteq U\). Then \(V^n[A] \subseteq B\) so that \(A \subseteq (\mathcal{U})\) \(B\). Conversely, \(A \subseteq (\mathcal{U})\) \(B\) implies \(A \subseteq (\mathcal{U})\) \(B\) since \(\mathcal{U} \subseteq \bar{\mathcal{U}}\).

**Corollary 2.16**: Let \((X,\mathcal{U}_1)\) and \((X,\mathcal{U}_2)\) be semi-pre-uniform spaces such that \(\mathcal{U}_1 = \mathcal{U}_2\). Then \(\delta_n(u_1) = \delta_n(u_2)\) for all \(n \geq 1\).

**Proof**: For \(\delta_n(u_1) = \delta_n(\bar{u}_1) = \delta_n(\bar{u}_2) = \delta_n(u_2)\).

**Definition 2.17**: Let \((X,\mathcal{U})\) and \((X^*,\mathcal{U}^*)\) be semi-pre-uniform spaces. Then a mapping \(f\) from \(X\) to \(X^*\) is said to be uniformly continuous (\(\mathcal{U}-\mathcal{U}^*\) continuous) iff for all \(U^* \in \mathcal{U}^*\) there is \(U \in \mathcal{U}\) such that \(f[U[x]] \subseteq U^*[f(x)]\) for all \(x \in X\).

**Theorem 2.18**: Let \((X,\mathcal{U})\) and \((X^*,\mathcal{U}^*)\) be semi-pre-uniform spaces and let \(f\) be a mapping from \(X\) to \(X^*\). Then the following statements are equivalent:
(1) The mapping \( f \) is uniformly continuous.

(2) For all \( U \in \mathcal{U}^* \) there is \( U \in \mathcal{U} \) such that 
\[ (f \times f)[U] \subseteq U^*. \]

(3) For all \( U^* \in \mathcal{U}^* \) there is \( U \in \mathcal{U} \) such that 
\[ U \subseteq (f \times f)^{-1}[U^*]. \]

(4) For all \( U^* \in \mathcal{U}^* \) there holds \( (f \times f)^{-1}[U^*] \in \mathcal{U}. \)

**Proof:** The proof is completely analogous to the well-known proof in the case of uniform spaces.

**Lemma 2.19:** Let \( X \) and \( X^* \) be non-empty sets, let \( f \) be a mapping from \( X \) to \( X^* \), and let \( U, V \subseteq X \times X \). Then:

1. \( (f \times f)[U \cup V] \subseteq (f \times f)[U] \cup (f \times f)[V] \), and
2. \( (f \times f)[U^n] \subseteq [(f \times f)[U]]^n \) for all \( n \geq 1 \).

**Proof:** Let \((x, y) \in U \cup V\). Then there is \( z \in X \) such that \((x, z) \in V\) and \((z, y) \in U\). Thus \((f(x), f(z)) \in (f \times f)[V]\) and \((f(z), f(y)) \in (f \times f)[U]\). From this we see that \((f(x), f(y)) \in (f \times f)[U] \cup (f \times f)[V]\) which proves (1).

Clearly (2) holds for \( n = 1 \), so suppose it holds for \( n-1 \) and let \( n \geq 2 \) be given. Then using (1) we obtain:
\[
(f \times f)[U^n] = (f \times f)[U \cup U^{n-1}] \subseteq (f \times f)[U] \cup (f \times f)[U^{n-1}] \subseteq [(f \times f)[U]]^n \cap [(f \times f)[U]]^{n-1} = [(f \times f)[U]]^n.
\]

**Theorem 2.20:** Let \((X, \mathcal{U})\) and \((X^*, \mathcal{U}^*)\) be semi-pre-uniform spaces and let \( f \) be a uniformly continuous mapping from \( X \) to \( X^* \). Then \( f \) is \( \delta_n(\mathcal{U}) - \delta_n(\mathcal{U}^*) \) continuous for \( n \geq 1 \).
Proof: Let $A, B \subseteq X$ such that $A \delta_n(u) B$ and let $U^* \in U$. Then there is $U \in U$ such that $(f \times f)[U] \subseteq U^*$. Now there is $b \in U^n[A] \cap B$ since $A \delta_n(u) B$ and so there is $a \in A$ such that $(a, b) \in U^n$. Now since $(f \times f)[U^n] \subseteq \{(f \times f)[U]\}^n \subseteq (U^*)^n$ we have $(f(a), f(b)) \in (U^*)^n$. Thus $f(b) \in \{(U^*)^n[f[A]] \cap f[B]\}$. Then $f[A] \delta_n(U^*) f[B]$ since $U^* \in U^*$ was arbitrary.

We now consider some results which give an alternate way of looking at the relations $\delta_n(u)$.

**Theorem 2.21:** Let $X \neq \emptyset$ be a set and let $U \neq \emptyset$ be a family of subsets of $X \times X$. For $n \geq 1$ let $U^n = \{U^n : U \in U\}$. Then for $n \geq 1$, if $U$ satisfies $[B1]$, $[B2]$, or $[B3]$, then $U^n$ also has the same property.

**Proof:** Let $n \geq 1$ be fixed. Then it is clear that if $U$ satisfies $[B1]$, then $U^n$ does also. Now suppose $U$ satisfies $[B2]$ and let $U, V \in U$ be given. Then there is $W = W^{-1} \in U$ such that $W \subseteq U \cap V$. Now $(W^n)^{-1} = W^n$ by Lemma 2.11, and also $W^n \subseteq U^n \cap V^n$. Since $U^n$ and $V^n$ are arbitrary members of $U^n$ we obtain the result that $U^n$ satisfies $[B2]$. Finally, suppose that $U$ satisfies $[B3]$ and let $U \in U$ be given. Then there is $V \in U$ such that $V \subseteq U$. By Definition 2.10 we obtain $V^n \subseteq (V^n)^n \subseteq U^n$. This completes the proof of the theorem.
**Remark 2.22:** From Theorem 2.21 we see for each $n \geq 1$ that if $(X, u)$ is a semi-pre-uniform space, then so is $(X, u^n)$, and if $(X, u)$ is a pre-uniform space, then so is $(X, u^n)$.

**Theorem 2.23:** Let $(X, u)$ be a semi-pre-uniform space, and let $n \geq 1$, $m \geq 1$ be given. Then $\delta_{nm}(u) = \delta_n(u^m)$.

**Proof:** Let $A, B \subset X$. Then $A \mathcal{F}_{nm}(u) B$ iff there is $U \in u$ such that $U^{nm}[A] \cap B = \emptyset$ iff there is $V \in u^m$ such that $V^n[A] \cap B = \emptyset$ iff $A \mathcal{F}_n(u^m) B$.

**Remark 2.24:** Thus we may consider the relation $\delta_n(u)$ to be defined directly on $u$, or to be the relation $\delta_1(u^n)$ defined on $u^n$.

We now prove one relationship of uniformity, proximity, and topology.

**Theorem 2.25:** Let $(X, u)$ be a pre-uniform space and let $\mathcal{J}(u) = \{A \subset X: \text{for all } x \in A \text{ there is } U \in u \text{ such that } U[x] \subset A\}$. Then $\mathcal{J}(u)$ is a topology on $X$ and $\mathcal{J}(u) = \mathcal{J}(\delta(u))$.

**Proof:** Since $(X, u)$ is a pre-uniform space, $\delta(u)$ is a proximity relation on $X$ by Theorem 2.14. Thus by Remark 1.18, $\mathcal{J}(\delta(u))$ is a topology on $X$. Hence the present theorem will be proved if we show $\mathcal{J}(u) = \mathcal{J}(\delta(u))$. To facilitate this, let $c$ denote the closure operator induced by $\delta(u)$. 
Now let $A \in \mathcal{F}(U)$ and let $x \in A$. Then there is $U \in \mathcal{U}$ such that $U[x] \subseteq A$. Thus $\{x\} \in \mathcal{F}(U)$ $(X-A)$. But this means that $x \not\in c(X-A)$. Hence it follows that $c(X-A) \subseteq (X-A)$ so $c(X-A) = (X-A)$. But this means that $A \in \mathcal{F}(\delta(U))$.

Conversely, suppose $A \in \mathcal{F}(\delta(U))$. Then $c(X-A) = (X-A)$ so for all $x \in A$ there holds $\{x\} \in \mathcal{F}(U)$ $(X-A)$. But this means that for all $x \in A$ there is $U \in \mathcal{U}$ such that $U[x] \subseteq A$. Thus $A \in \mathcal{F}(U)$. 
CHAPTER III

COVERING CLASSES

**Definition 3.1:** Let $X \neq \emptyset$ be a set and let $C$ and $P$ be families of subsets of $X$. Then we say that $C$ refines $P$ iff for every $C \in C$ there is $P \in P$ such that $C \subseteq P$.

**Definition 3.2:** Let $X \neq \emptyset$ be a set and let $P \neq \emptyset$ be a family of subsets of $X$. Then for $x \in X$ define:

$$\text{st}(x,P) = \bigcup \{P \in P : x \in P\} = \text{the star of } P \text{ at } x.$$ 

For $A \subseteq X$ define:

$$\text{st}(A,P) = \bigcup \{P \in P : P \cap A \neq \emptyset\} = \text{the star of } P \text{ at } A.$$ 

Then define:

$$\text{st}P = \{\text{st}(x,P) : x \in X\} = \text{the star of } P.$$ 

The following results are immediate from these definitions:

1. For all $x \in X$, $\text{st}(x,P) = \text{st}(\{x\},P)$.
2. For all $A \subseteq X$, $\text{st}(A,P) = \bigcup \{\text{st}(a,P) : a \in A\}$.
3. If $A \subseteq \bigcup \{P : P \in P\}$ then $A \subseteq \text{st}(A,P)$.

**Definition 3.3:** Let $X \neq \emptyset$ be a set and let $C$ and $P$ be non-empty families of subsets of $X$. Then we say that $C$ star-refines $P$ iff $\text{st}C$ refines $P$. Thus $C$ star-refines $P$ iff for all $x \in X$ there is $P \in P$ such that $\text{st}(x,C) \subseteq P$. 

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Definition 3.4: Let \( X \neq \emptyset \) be a set and let \( \mathcal{P} \neq \emptyset \) be a class of families of subsets of \( X \). Then \( \mathcal{P} \) is said to be a semi-uniformizing class iff \( \mathcal{P} \) satisfies:

[UC1] Every member of \( \mathcal{P} \) is a cover of \( X \). (That is, for all \( P \in \mathcal{P} \), \( X = \bigcup \{ P: P \in \mathcal{P} \} \).

[UC2] Every two members of \( \mathcal{P} \) have a common refinement in \( \mathcal{P} \). (Note that this axiom extends by induction to: for all \( \{ P_1, \ldots, P_n \} \subset \mathcal{P} \) there is \( P \in \mathcal{P} \) such that \( P \) refines \( P_i \), \( 1 \leq i \leq n \).

If in addition \( \mathcal{P} \) satisfies:

[UC3] Every member of \( \mathcal{P} \) has a star-refinement in \( \mathcal{P} \), then \( \mathcal{P} \) is said to be a uniformizing class.

Theorem 3.5: Let \( X \neq \emptyset \) be a set and let \( \mathcal{U} \neq \emptyset \) be a family of subsets of \( X \times X \). For \( U \in \mathcal{U} \) let \( P(U) = \{ U(x): x \in X \} \). Then define \( \mathcal{P}(\mathcal{U}) = \{ P(U): U \in \mathcal{U} \} \). Then:

1. If \( \mathcal{U} \) satisfies [Bl], then \( \mathcal{P}(\mathcal{U}) \) satisfies [UC1].
2. If \( \mathcal{U} \) satisfies [Bl] and [B2], then \( \mathcal{P}(\mathcal{U}) \) satisfies [UC1] and [UC2].
3. If \( \mathcal{U} \) satisfies [Bl]-[B3], then \( \mathcal{P}(\mathcal{U}) \) satisfies [UC1]-[UC3].

Proof: First suppose that \( \mathcal{U} \) satisfies [Bl] and let \( U \in \mathcal{U} \) be given. Then \( i_X \subset U \), so \( x \in U(x) \) for all \( x \in X \). Thus we have \( X \subset \bigcup \{ U(x): x \in X \} \). This proves (1).

Now suppose that \( \mathcal{U} \) satisfies [Bl] and [B2] and let \( U, V \in \mathcal{U} \) be given. Then there is \( W = W^{-1} \in \mathcal{U} \) such that
W ⊂ U ∩ V. Then for all x ∈ X we have W[x] ⊂ U[x] and
W[x] ⊂ V[x]. Thus P(W) refines both P(U) and P(V).

Finally, suppose that U satisfies [B1]-[B3] and let
U ∈ U be given. Then using [B2] and [B3] we can find
W = W' ∈ U such that WoW ⊂ U. Then for x ∈ X we have
st(x, P(W)) = U [W[y]: x ∈ W[y]] = U [W[y]: (y, x) ∈ W].
Thus z ∈ st(x, P(W)) iff z ∈ W[y] and (y, x) ∈ W for some
y ∈ X iff (y, z) ∈ W and (x, y) ∈ W' = W for some y ∈ X
iff (x, z) ∈ WoW iff z ∈ WoW[x]. Hence st(x, P(W)) ⊂
⊂ WoW[x] ⊂ U[x] for all x ∈ X, from which it follows
that stP(W) refines P(U).

Remark 3.6: We see from the proof of Theorem 3.5-(3)
that for W = W' ⊂ X × X there holds st(x, P(W)) = WoW[x]
for all x ∈ X.

Theorem 3.7: Let X ≠ ∅ be a set and let P ≠ ∅ be a
class of families of subsets of X. For P ∈ P let U(P) =
= {P × P: P ∈ P}. Then define U(∅) = {∅}. Then:

(1) If P satisfies [UC1], then U(P) satisfies [B1].
(2) If P satisfies [UC1] and [UC2], then U(P)
satisfies [B1] and [B2].
(3) If P satisfies [UC1]-[UC3], then U(P) satisfies
[B1]-[B3].

Proof: First, suppose P satisfies [UC1] and let x ∈ X
and P ∈ P be given. Then there is P ∈ P such that x ∈ P,
since \( P \) is a cover of \( X \). Thus \( (x, x) \in P \times P \subseteq U(P) \). Hence \( i_X \subseteq U(P) \) for all \( P \in \mathcal{P} \).

Now let \( U(P_1), U(P_2) \in \mathcal{U}(P) \) and let \( P \in \mathcal{P} \) be a common refinement of \( P_1 \) and \( P_2 \). Now \( (x, y) \in U(P) \) implies \( (x, y) \in P \times P \) for some \( P \in \mathcal{P} \). But there are sets \( P_1 \in \mathcal{P}_1 \) and \( P_2 \in \mathcal{P}_2 \) such that \( P \subseteq P_1 \) and \( P \subseteq P_2 \). Hence we have \( (x, y) \in P_1 \times P_1 \subseteq U(P_1) \) and \( (x, y) \in P_2 \times P_2 \subseteq U(P_2) \). Thus \( U(P) \subseteq U(P_1) \cap U(P_2) \). Now \( U(P) = [U(P)]^{-1} \) and so \( U(P) \) meets all the requirements of \([B2]\).

Finally, let \( U(P) \in \mathcal{U}(P) \) and let \( P* \in \mathcal{P} \) be a star-refinement of \( P \). Now \( (x, y) \in U(P*)oU(P*) \) means there is \( z \in X \) such that \( (x, z) \in U(P*) \) and \( (z, y) \in U(P*) \). Then \( (x, z) \in P*_1 \times P*_1 \) for some \( P*_1 \in \mathcal{P}_* \) and \( (z, y) \in P*_2 \times P*_2 \) for some \( P*_2 \in \mathcal{P}_* \). Thus \( [x, y] \in st(z, P*) \). But there is \( P \in \mathcal{P} \) such that \( st(z, P*) \subseteq P \). Hence, \( (x, y) \in P \times P \subseteq U(P) \). Thus \( U(P*)oU(P*) \subseteq U(P) \).

**Definition 3.8:** Let \( X \neq \emptyset \) be a set and let \( \mathcal{P} \neq \emptyset \) be a class of families of subsets of \( X \). Define \( \mathcal{P} = \{P': P' \text{ is a family of subsets of } X \text{ and there is a } P \in \mathcal{P} \text{ such that } P \text{ refines } P' \} \). It is a straightforward exercise to show that the class \( \mathcal{P} \) satisfies \([UC1]\), \([UC2]\), or \([UC3]\) iff \( \mathcal{P} \) satisfies the same property.

**Theorem 3.9:** Let \( X \neq \emptyset \) be a set and let \( \mathcal{P} \) be a semi-uniformizing class on \( X \). Then \( \mathcal{U}(P) = \mathcal{U}(\mathcal{P}) \).
Proof: Now $U \in U(\mathcal{P})$ iff there is $V \in u(\mathcal{P})$ such that $V \subseteq U$ iff there is $\mathcal{P}' \in \mathcal{P}$ such that $U(\mathcal{P}') \subseteq U$ iff there is $\mathcal{P}' \in \mathcal{P}$ such that $\bigcup \{ \mathcal{P}' \times \mathcal{P}' : \mathcal{P}' \in \mathcal{P}' \} \subseteq U$ iff there is $\mathcal{P} \in \mathcal{P}$ such that $\bigcup \{ \mathcal{P} \times \mathcal{P} : \mathcal{P} \in \mathcal{P} \} \subseteq U$ iff there is $\mathcal{P} \in \mathcal{P}$ such that $U(\mathcal{P}) \subseteq U$ iff there is $W \in U(\mathcal{P})$ such that $W \subseteq U$ iff $U \in U(\mathcal{P})$.

Lemma 3.10: Let $X \neq {}^0$ be a set and let $\mathcal{P} \neq {}^0$ be a class of families of subsets of $X$. Then:

1. For all $x \in X$ and all $\mathcal{P} \in \mathcal{P}$, $U(\mathcal{P})[x] = st(x, \mathcal{P})$.
2. For all $A \subseteq X$ and all $\mathcal{P} \in \mathcal{P}$, $U(\mathcal{P})[A] = st(A, \mathcal{P})$.
3. For all $\mathcal{P} \in \mathcal{P}$, $\mathcal{P}(U(\mathcal{P})) = st \mathcal{P}$.

Proof: First, given $x \in X$ and $\mathcal{P} \in \mathcal{P}$ we have $y \in U(\mathcal{P})[x]$ iff $(x, y) \in U(\mathcal{P})$ iff $(x, y) \in \bigcup \{ \mathcal{P} \times \mathcal{P} : \mathcal{P} \in \mathcal{P} \}$ iff there is $\mathcal{P} \in \mathcal{P}$ such that $x \in \mathcal{P}$ and $y \in \mathcal{P}$ iff $y \in \bigcup \{ \mathcal{P} \in \mathcal{P} : x \in \mathcal{P} \}$ iff $y \in st(x, \mathcal{P})$. This proves (1).

Now given $A \subseteq X$ and $\mathcal{P} \in \mathcal{P}$ we have $U(\mathcal{P})[A] = \bigcup \{ U(\mathcal{P})[a] : a \in A \} = \bigcup \{ st(a, \mathcal{P}) : a \in A \} = st(A, \mathcal{P})$. This proves (2).

Finally, for $\mathcal{P} \in \mathcal{P}$ we have $\mathcal{P}(U(\mathcal{P})) = \{ U(\mathcal{P})[x] : x \in X \} = \{ st(x, \mathcal{P}) : x \in X \} = st \mathcal{P}$. This proves (3).

Lemma 3.11: Let $X \neq {}^0$ be a set and let $\emptyset \neq U \subseteq X \times X$. Then $U(\mathcal{P}(U)) = U^{-1} \circ U$.

Proof: We have that $(x, y) \in U(\mathcal{P}(U))$ iff $(x, y) \in \bigcup \{ \mathcal{P} \times \mathcal{P} : \mathcal{P} \in \mathcal{P}(U) \}$ iff $(x, y) \in \bigcup \{ U[z] \times U[z] : z \in X \}$.
iff there is \( z \in X \) such that \( x \in U[z] \) and \( y \in U[z] \) iff there is \( z \in X \) such that \((x,z) \in U\) and \((y,z) \in U\) iff there is \( z \in X \) such that \((x,z) \in U\) and \((z,y) \in U^{-1}\) iff \((x,y) \in U^{-1} \circ U\).

**Theorem 3.12:** Let \( X \neq \emptyset \) be a set and let \( \mathcal{P} \) be a uniformizing class on \( X \). Then \( \mathcal{P}(\mathcal{U}(\mathcal{P})) = \mathcal{P} \).

**Proof:** Since by Theorem 3.9 we have \( \mathcal{U}(\mathcal{P}) = \mathcal{U}(\mathcal{P}) \) we need only show that \( \mathcal{P}(\mathcal{U}(\mathcal{P})) = \mathcal{P} \). Now \( \mathcal{P}' \in \mathcal{P}(\mathcal{U}(\mathcal{P})) \) iff there is \( \mathcal{P}'' \in \mathcal{P}(\mathcal{U}(\mathcal{P})) \) such that \( \mathcal{P}'' \) refines \( \mathcal{P}' \) iff there is \( \mathcal{U} \in \mathcal{U}(\mathcal{P}) \) such that \( \mathcal{P}(\mathcal{U}) \) refines \( \mathcal{P}' \) iff there is \( \mathcal{V} \in \mathcal{U}(\mathcal{P}) \) such that \( \mathcal{P}'' \) refines \( \mathcal{P}' \) iff there is \( \mathcal{P}^* \in \mathcal{P} \) such that \( \mathcal{P}(\mathcal{P}^*) \) refines \( \mathcal{P}' \) iff there is \( \mathcal{P} \in \mathcal{P} \) such that \( \mathcal{P} \) refines \( \mathcal{P}' \) iff \( \mathcal{P}' \in \mathcal{P} \).

**Theorem 3.13:** Let \((X, \mathcal{U})\) be a pre-uniform space. Then \( \mathcal{P}(\mathcal{U}) \) is a uniformizing class and \( \mathcal{U}(\mathcal{P}(\mathcal{U})) = \mathcal{U} \).

**Proof:** That \( \mathcal{P}(\mathcal{U}) \) is a uniformizing class follows from 3.5 and 3.8. Now let \( W \subset X \times X \). Then \( W \in \mathcal{U}(\mathcal{P}(\mathcal{U})) \) iff there is \( V \in \mathcal{U}(\mathcal{P}(\mathcal{U})) \) such that \( V \subset W \) iff there is \( \mathcal{P}' \in \mathcal{P}(\mathcal{U}) \) such that \( \mathcal{U}(\mathcal{P}') \subset W \) iff there is \( \mathcal{P} \in \mathcal{P}(\mathcal{U}) \) such that \( \mathcal{U}(\mathcal{P}) \subset W \) iff there is \( \mathcal{U} \in \mathcal{U} \) such that \( \mathcal{U}(\mathcal{P}(\mathcal{U})) \subset W \) iff there is \( \mathcal{U} \in \mathcal{U} \) such that \( \mathcal{U}^{-1} \circ \mathcal{U} \subset W \) iff \( W \in \mathcal{U} \).

We shall now consider covering classes in proximity and semi-proximity spaces.
Definition 3.14: Let \((X, \delta)\) be a semi-proximity space. Then a finite family \(P = \{P_i: 1 \leq i \leq n\}\) of subsets of \(X\) is said to be a \(\delta\)-cover, or proximal cover, of \(X\) iff there are sets \(Q_i \subseteq P_i\) for \(1 \leq i \leq n\) such that \(X = \bigcup \{Q_i: 1 \leq i \leq n\}\). The symbol \(P(\delta)\) will be used to denote the class of all \(\delta\)-covers for a given semi-proximity space \((X, \delta)\).

Theorem 3.15: Let \((X, \delta)\) be a semi-proximity space. Then for all \(A, B \subseteq X\), \(\{X-A, X-B\} \in P(\delta)\) implies \(A \subseteq B\).

Proof: Now if \(\{X-A, X-B\} \in P(\delta)\) there are sets \(Q_1 \subseteq (X-A)\) and \(Q_2 \subseteq (X-B)\) such that \(X = Q_1 \cup Q_2\). Then \(A \subseteq (X-Q_1) \subseteq Q_2 \subseteq (X-B)\) which implies \(A \subseteq (X-B)\). But this just means \(A \subseteq B\).

Theorem 3.16: Let \((X, \delta)\) be a proximity space. Then for all \(A, B \subseteq X\), \(\{X-A, X-B\} \in P(\delta)\) iff \(A \subseteq B\).

Proof: Now \(A \subseteq B\) implies \(A \subseteq (X-B)\) so since \(\delta\) is a proximity relation on \(X\) there is \(C \subseteq X\) such that \(A \subseteq C \subseteq (X-B)\). But then \(X-C \subseteq (X-A)\), \(C \subseteq (X-B)\), and \(X = C \cup (X-C)\), so we see that \(\{X-A, X-B\}\) is a \(\delta\)-cover of \(X\). In view of Theorem 3.15, this completes the proof.

Definition 3.17: Let \(X \neq \emptyset\) be a set and let \(P \neq \emptyset\) be a class of families of subsets of \(X\). Then we define a relation \(\delta(P)\) between the subsets of \(X\) as follows:

For \(A, B \subseteq X\), \(A \delta(P) B\) iff \(\{X-A, X-B\}\) is not refined by any member of \(P\).
It is easy to see that $A \delta(P) B$ iff $\{X-A, X-B\} \not\in \mathcal{P}$ and thus $\delta(P) = \delta(\mathcal{P})$. Also, $\mathcal{P}_1 \subset \mathcal{P}_2$ implies $\delta(\mathcal{P}_1) \leq \delta(\mathcal{P}_2)$.

**Theorem 3.18**: Let $X \not= \emptyset$ be a set and let $\mathcal{P} \not= \emptyset$ be a class of families of subsets of $X$. Then for $A, B \subset X$, $A \delta(P) B$ iff $\text{st}(A, P) \cap B \not= \emptyset$ for all $P \in \mathcal{P}$.

**Proof**: Let $A, B \subset X$. Now if $P \in \mathcal{P}$ and $\text{st}(A, P) \cap B = \emptyset$ we have that for all $P \in \mathcal{P}$, $P \cap A \not= \emptyset$ implies $P \cap B = \emptyset$. Thus, $P \not\in (X-A)$ implies $P \subset (X-B)$, for all $P \in \mathcal{P}$. It then follows that $P$ refines $\{X-A, X-B\}$, and so $A \delta(P) B$.

Conversely, let $A, B \subset X$ and suppose that for all $P \in \mathcal{P}$ there holds $\text{st}(A, P) \cap B \not= \emptyset$. Then for fixed $P \in \mathcal{P}$ there is $P \in \mathcal{P}$ such that $P \cap A \not= \emptyset$ and $P \cap B \not= \emptyset$. Thus $P \not\in (X-A)$ and $P \not\in (X-B)$ so that $P$ does not refine $\{X-A, X-B\}$. Since $P$ was an arbitrary member of $\mathcal{P}$ it follows that $A \delta(P) B$.

**Theorem 3.19**: Let $X \not= \emptyset$ be a set and let $\mathcal{P}$ be a semi-uniformizing class on $X$. Then $\delta(\mathcal{P})$ is a semi-proximity relation on $X$. If $\mathcal{P}$ is a uniformizing class, then $\delta(\mathcal{P})$ is a proximity relation on $X$.

**Proof**: First suppose that $\mathcal{P}$ is a semi-uniformizing class. Clearly, $\delta(\mathcal{P})$ satisfies [P0] and [P1].

To show [P2], let $A, B, C \subset X$ and first suppose that $C \delta(\mathcal{P}) (A \cup B)$. Then there is $P \in \mathcal{P}$ such that $P$ refines $\{X-C, X-(A \cup B)\} = \{X-C, (X-A) \cap (X-B)\}$. Then if $P \in \mathcal{P}$ and $P \not\in (X-C)$ we must have that $P \subset (X-A) \cap (X-B)$. From this it follows that $P$ refines both $\{X-C, X-A\}$ and $\{X-C, X-B\}$ and
so $C \notin(P) A$ and $C \notin(P) B$. Conversely, suppose now that $C \notin(P) A$ and $C \notin(P) B$. Then $[X-C,X-A]$ and $[X-C,X-B]$ have refinements in $P$, and so by $[UC2]$ have a common refinement $P \in P$. Then if $P \in P$ and $P \notin (X-C)$ we must have that $P \subset (X-A)$ and $P \subset (X-B)$. From this it follows that $P$ refines $[X-C,(X-A) \cap (X-B)] = [X-C,X-(A \cup B)]$ and hence $C \notin(P) (A \cup B)$.

To show $[P3]$, suppose $A,B \subset X$ are such that $A \cap B \neq \emptyset$. Then $(X-A) \cup (X-B) \neq X$ so that $[X-A,X-B]$ is not a cover of $X$, and hence cannot be a member of $P$. Thus $A \notin(P) B$.

To show $[P4]$, note that for all $A \subset X$ the family $[X-A,X]$ is refined by every member of $P$. Since $P \neq \emptyset$, this implies that $A \notin(P) \emptyset$.

Now suppose that $P$ also satisfies $[UC3]$ and let $A,B \subset X$ such that $A \notin(P) B$. Then there is $P \in P$ such that $P$ refines $[X-A,X-B]$. Now by $[UC3]$ there is $P^* \in P$ such that $P^*$ star-refines $P$; clearly, $P^*$ also star-refines $[X-A,X-B]$. Let $C = \text{st}(A,P^*)$ and $D = \text{st}(B,P^*)$. Now since $P^*$ covers $A$ and $B$ we have that $A \subset C$ and $B \subset D$. It is also easy to see that $P^*$ refines both $[X-A,C]$ and $[X-B,D]$ so that $A \notin(P) (X-C)$ and $B \notin(P) (X-D)$. Now if $C \cap D \neq \emptyset$ there are sets $P^*_1,P^*_2 \in P^*$ and points $a,b,c \in X$ such that

$a \in A \cap P^*_1$, $b \in B \cap P^*_2$, and $c \in P^*_1 \cap P^*_2$. Then $\{a,b\} \subset P^*_1 \cup P^*_2 \subset \text{st}(c,P^*)$. Now since $P^*$ star-refines $[X-A,X-B]$ we must have either $\text{st}(c,P^*) \subset (X-A)$ or $\text{st}(c,P^*) \subset (X-B)$. But this implies that either $a \in (X-A)$ or $b \in (X-B)$, a
contradiction. Hence \( C \cap D = \emptyset \), so \( C \) and \( D \) meet all the requirements of [P5]. This completes the proof of the theorem.

**Theorem 3.20:** Let \((X, \delta)\) be a semi-proximity space.
Then \( \mathcal{P}(\delta) \) is a semi-uniformizing class on \( X \). If \((X, \delta)\) is a proximity space, then \( \mathcal{P}(\delta) \) is a uniformizing class.

**Proof:** First suppose that \((X, \delta)\) is a semi-proximity space. Then \( \mathcal{P}(\delta) \) satisfies \([UCl]\) since all members of \( \mathcal{P}(\delta) \) are covers of \( X \). Now let \( P_1, P_2 \in \mathcal{P}(\delta) \). Then we have that
\[
P_1 = \{P_i : 1 \leq i \leq n\}
\]
where there are sets \( Q_i \subseteq P_i \) for \( 1 \leq i \leq n \) such that \( X = \bigcup \{Q_i : 1 \leq i \leq n\} \), and that
\[
P_2 = \{R_j : 1 \leq j \leq m\}
\]
where there are sets \( S_j \subseteq R_j \) for \( 1 \leq j \leq m \) such that \( X = \bigcup \{S_j : 1 \leq j \leq m\} \). Then define
\[
P = \{P_i \cap R_j : 1 \leq i \leq n, 1 \leq j \leq m\}.
\]
By Theorem 1.6 we have \( (Q_i \cap S_j) \subseteq (P_i \cap R_j) \) for all \( i, j \) where \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \). Since \( X = \bigcup \{Q_i \cap S_j : 1 \leq i \leq n, 1 \leq j \leq m\} \) we see that \( P \) is a \( \delta \)-cover of \( X \). Also, \( P \) is a common refinement of \( P_1 \) and \( P_2 \). Thus \( \mathcal{P}(\delta) \) satisfies \([UC2]\).

Now suppose that \((X, \delta)\) is a proximity space and let \( P = \{P_1, P_2\} \) be a \( \delta \)-cover of \( X \) consisting of two sets. Then there are sets \( Q_1 \subseteq P_1 \), \( Q_2 \subseteq P_2 \) such that \( X = Q_1 \cup Q_2 \).

Then we have \( (X-P_2) \subseteq (X-Q_2) \subseteq Q_1 \subseteq P_1 \) so that \( (X-P_2) \subseteq P_1 \). Thus we may find sets \( C_1, C_2, C_3, C_4, C', C'' \subseteq X \) such that:
\[
(X-P_2) \subseteq C_1 \subseteq C' \subseteq C_2 \subseteq C_3 \subseteq C'' \subseteq C_4 \subseteq P_1.
\]
Now let $P_1^* = C_2$, $P_2^* = (C_4-C_1)$, and $P_3^* = (X-C_3)$. We assert that $P^* = \{P_1^*, P_2^*, P_3^*\}$ is a δ-cover of $X$ which star-refines $\mathcal{P} = \{P_1, P_2\}$. Now we have:

$$C' \subseteq C_2 = P_1^*,$$

$$(C''-C') = C'' \cap (X-C') \subseteq C_4 \cap (X-C_1) = (C_4-C_1) = P_2^*,$$

$$(X-C'') \subseteq (X-C_3) = P_3^*,$$

$$X = C' \cup (C''-C') \cup (X-C'').$$

Hence we see that $P^*$ is a δ-cover of $X$. To show that $P^*$ star-refines $\mathcal{P}$, note that $\text{st}P^* = \{\text{st}(x, P^*): x \in X\} = \{P_1^*, P_2^*, P_3^*, P_1^* \cup P_2^*, P_2^* \cup P_3^*, P_1^* \cup P_2^* \cup P_3^*\}$. But $P_1^* \cup P_2^* \subseteq P_1$ and $P_2^* \cup P_3^* \subseteq P_2$, so we see that $P^*$ star-refines $\mathcal{P}$.

Now let $\mathcal{P} = \{P_i: 1 \leq i \leq n\}$ be an arbitrary δ-cover of $X$. Then there are sets $Q_i \subseteq P_i$ for $1 \leq i \leq n$ such that $X = \bigcup \{Q_i: 1 \leq i \leq n\}$. Now $Q_i \subseteq (X-P_i)$ for $1 \leq i \leq n$, so by Theorem 3.16 we see that $P_i = \{X-Q_i, P_i\}$ is a δ-cover of $X$ for all $1 \leq i \leq n$. Hence by the preceding paragraph there is a δ-cover $P_i^*$ of $X$ for each $1 \leq i \leq n$ such that $P_i^*$ star-refines $P_i$. We have also shown that $P(\delta)$ satisfies [UC2], so we may find a δ-cover of $X$, say $P^*$, which is a common refinement of all the $P_i^*$, $1 \leq i \leq n$. Clearly, $\text{st}P^*$ refines $\text{st}P_i^*$ for $1 \leq i \leq n$ so that $P^*$ star-refines all the $P_i$, $1 \leq i \leq n$. Now let $x \in X$. Then $x \in Q_k$ for some $k$, $1 \leq k \leq n$, so $x \not\in (X-Q_k)$. Thus $\text{st}(x, P^*) \not\subseteq (X-Q_k)$. Thus since $P^*$ star-refines $P_k$ we must have $\text{st}(x, P^*) \subseteq P_k$. Since $x$ was an arbitrary point of $X$, it follows that $P^*$ star-refines $\mathcal{P}$. This completes the proof.
Remark 3.21: By the preceding theorem and Theorem 3.7, we see that given a semi-proximity space \((X, \delta)\) we may consider the induced semi-pre-uniform space \((X, \nu(\mathcal{P}(\delta)))\) which, of course, is a pre-uniform space whenever \(\delta\) is a proximity relation. Since we do not use this idea of defining a uniform space from a proximity space in our work on measure theory, we include only one more result on the topic, Theorem 3.26. See Thron[15] for further results.

Theorem 3.22: Let \((X, \delta)\) be a semi-proximity space. Then \(\delta(\mathcal{P}(\delta)) \leq \delta\). If \((X, \delta)\) is a proximity space, then \(\delta(\mathcal{P}(\delta)) = \delta\).

Proof: First suppose that \((X, \delta)\) is a semi-proximity space and let \(A, B \subseteq X\) such that \(A \not\subseteq (\mathcal{P}(\delta)) B\). Then there is \(P \in \mathcal{P}(\delta)\) such that \(st(A, P) \cap B = \emptyset\). Then we have that \(P = \{P_i: 1 \leq i \leq n\}\) where there are sets \(Q_i \subseteq P_i\) for \(1 \leq i \leq n\) such that \(X = \bigcup \{Q_i: 1 \leq i \leq n\}\). Now

\[
A \subseteq \bigcup \{Q_i: 1 \leq i \leq n, \quad Q_i \cap A \neq \emptyset\} 
\subseteq 
\bigcup \{P_i: 1 \leq i \leq n, \quad Q_i \cap A \neq \emptyset\} \subseteq st(A, P) \subseteq (X-B).
\]

Hence, \(A \subseteq (X-B)\), so \(A \not\subseteq B\). Thus, \(\delta(\mathcal{P}(\delta)) \leq \delta\).

Now suppose that \((X, \delta)\) is a proximity space and let \(A, B \subseteq X\) such that \(A \not\subseteq B\). Then by Theorem 3.16 we have that \([X-A, X-B] \in \mathcal{P}(\delta)\). But this means \(A \not\subseteq (\mathcal{P}(\delta)) B\). In view of the preceding paragraph we obtain \(\delta(\mathcal{P}(\delta)) = \delta\) in this case.
**Theorem 3.23**: Let \( X \neq \emptyset \) be a set and let \( \mathcal{P} \) be a semi-uniformizing class on \( X \). Then \( \mathcal{P}(\delta(\mathcal{P})) \subset \mathcal{P} \).

**Proof**: Since \( \delta(\mathcal{P}) = \delta(\mathcal{P}) \) the assertion is the same as \( \mathcal{P}(\delta(\mathcal{P})) \subset \mathcal{P} \). But this will follow immediately if we show that \( \mathcal{P}(\delta(\mathcal{P})) \subset \mathcal{P} \). To this end let \( \mathcal{P}' \in \mathcal{P}(\delta(\mathcal{P})) \).

Then \( \mathcal{P}' = \{ P_i : 1 \leq i \leq n \} \) where there are sets \( Q_i \subset X \) for \( 1 \leq i \leq n \) such that \( Q_i \in \delta(\mathcal{P}) \) \( (X - P_i) \) for \( 1 \leq i \leq n \) and such that \( X = \bigcup \{ Q_i : 1 \leq i \leq n \} \). Thus \( \{ P_i, X - Q_i \} \in \mathcal{P} \) for \( 1 \leq i \leq n \), so there are families \( \mathcal{P}_i \in \mathcal{P} \) for \( 1 \leq i \leq n \) such that \( \mathcal{P}_i \) refines \( \{ P_i, X - Q_i \} \). Since \( \mathcal{P} \) satisfies [UC2] there is a common refinement \( \mathcal{P} \in \mathcal{P} \) of the \( \mathcal{P}_i \), \( 1 \leq i \leq n \). Thus, \( \mathcal{P} \) also refines all the families \( \{ P_i, X - Q_i \} \). Now for \( \emptyset \neq P \in \mathcal{P} \) there is some \( Q_k \), \( 1 \leq k \leq n \), such that \( P \cap Q_k \neq \emptyset \); that is, \( P \notin (X - Q_k) \). Thus we must have \( P \subseteq P_k \). It follows that \( \mathcal{P} \) refines \( \mathcal{P}' \). Hence \( \mathcal{P}' \in \mathcal{P} \).

In general, even assuming [UC3], we cannot replace the subset sign of the assertion by an equal sign, since every member of \( \mathcal{P}(\delta(\mathcal{P})) \) is refined by a family (some \( \delta(\mathcal{P}) \)-cover of \( X \)) which contains only finitely many sets, while all members of \( \mathcal{P} \) may not have this property. For example, consider \( \mathcal{P} = \{ \{ x \} : x \in X \} \) and \( X \) is an infinite set.

**Remark 3.24**: Given a non-empty set \( X \) and a non-empty covering class \( \mathcal{P} \) of \( X \) which satisfies [UC1] and [UC2], it is possible to define other semi-proximity relations on \( X \).
in addition to the relation \( \delta(P) \). One approach, analogous to the approach in semi-pre-uniform spaces of Theorems 2.21-2.23, is to define \( \text{st}^nP \) inductively by \( \text{st}^nP = \text{st}(\text{st}^{n-1}P) \) for \( n \geq 2 \), and then define \( A \delta_n(P) B \) iff \( \{X-A,X-B\} \) is not refined by any family of the form \( \text{st}^nP \) for \( P \in P \). Our relation \( \delta(P) \) could then be thought of as \( \delta_0(P) \). The key point for our purposes is the idea of defining such relations by a refinement condition on the family \( \{X-A,X-B\} \), for it is this condition which plays a central role in Chapter Five, in our generalized outer measure construction. Thus the study of the relation \( \delta(P) \) is sufficient for our purposes.

We remark that if \( P \) also satisfies [UC3], all the relations \( \delta_n(P) \) defined above are equal.

In the next theorem we show that for a semi-pre-uniform space \((X,U)\) we have \( \delta(P(U)) = \delta_2(u) \). More general relations of this type involving \( \delta_n(P(U)) \) and \( \delta_n(u) \) exist.

**Theorem 3.25:** Let \((X,U)\) be a semi-pre-uniform space. Then \( \delta(P(U)) = \delta_2(u) \).

**Proof:** Let \( A,B \subset X \). Then \( A \delta(P(U)) B \) iff there is \( P \in P(U) \) such that \( \text{st}(A,P) \cap B = \emptyset \) iff there is \( P \in P(U) \) such that \( U(P)[A] \cap B = \emptyset \) iff there is \( U \in U \) such that \( U(P(U))[A] \cap B = \emptyset \) iff there is \( U \in U \) such that \\

\[ V^*= V \] and such that \( \text{st}(A,P) \cap B = \emptyset \) iff \( \emptyset \). See Lemmas 3.10 and 3.11.
Theorem 3.26: Let \((X, \delta)\) be a semi-proximity space. Let \(U(\delta)\) denote the semi-pre-uniformity \(U(P(\delta))\). Then \(P(U(\delta)) \subseteq P(\delta)\). If \((X, \delta)\) is a proximity space, then \(P(U(\delta)) = P(\delta)\) and \(\delta(U(\delta)) = \delta\).

Proof: Since \(U(\delta) = U(P(\delta))\) we see that the members of \(P(U(\delta))\) are just those covers of \(X\) of the form \(P(U(\mathcal{P}))\) for some \(P \in P(\delta)\). But by Lemma 3.10, \(P(U(\mathcal{P})) = stP\). Now \(P\) refines \(stP\), so every member of \(P(U(\delta))\) is refined by a member of \(P(\delta)\). Hence, \(P(U(\delta)) \subseteq P(\delta)\).

If \((X, \delta)\) is a proximity space we have that given \(P \in P(\delta)\) there is \(P^* \in P(\delta)\) such that \(P^*\) star-refines \(P\). Then \(stP^* = P(U(P^*)) \in P(U(\delta))\) and \(stP^*\) refines \(P\), so that \(P \in P(U(\delta))\). Thus \(P(U(\delta)) = P(\delta)\) in this case. Hence in this case, \(\delta = \delta(P(\delta)) = \delta(P(\delta)) = \delta(P(U(\delta))) = \delta(U(\delta)) = \delta(U(\delta)).\) The last equality holds since \(U(\delta)\) is a uniformity whenever \(\delta\) is a proximity relation.

We now prove a theorem for covering classes in semi-pseudo-metric spaces, to use in Chapter Five. Following this, we consider covering classes and continuity.

Theorem 3.27: Let \((X, d)\) be a semi-pseudo-metric space. For each \(\epsilon > 0\) let \(P_\epsilon = \{S^0(x, \epsilon) : x \in X\}\). Then define \(P(d) = \{P_\epsilon : \epsilon > 0\}\). Then \(\delta(P(d)) \leq \delta(d)\), and if \((X, d)\) is a pseudo-metric space, then \(\delta(P(d)) = \delta(d)\).

Proof: Let \(A, B \subseteq X\). Now if \(A \in P(d)\) then \(P_\epsilon\) refines \(\{X-A, X-B\}\) for some \(\epsilon > 0\). We assert that then
d(A, B) ≥ ε. For if not there are a ∈ A, b ∈ B such that
d(a, b) < ε. But then \( S^0(a, ε) \not\subseteq (X-B) \) and \( S^0(b, ε) \not\subseteq (X-A) \)
so that \( P_ε \) does not refine \( \{X-A, X-B\} \), a contradiction.
Thus, d(A, B) > 0, and so A \( \notin \delta(d) \) B. Since A and B were
arbitrary subsets of X, this proves that \( \delta(P(d)) \leq \delta(d) \).

Now suppose that (X, d) is a pseudo-metric space, and
let A, B ⊆ X such that A \( \notin \delta(d) \) B. Then there is ε > 0 such
that d(A, B) ≥ 2ε. We assert that \( P_ε \) refines \( \{X-A, X-B\} \).
To this end let x ∈ X be given. Now if \( S^0(x, ε) \subseteq (X-A) \)
we have nothing more to show, so suppose that
\( S^0(x, ε) \not\subseteq (X-A) \). Then there is a ∈ A such that d(x, a) < ε.
Now if there is b ∈ B such that d(b, x) < ε we would have
d(b, a) < 2ε, which is impossible. Thus we must have
\( S^0(x, ε) \subseteq (X-B) \). This completes the proof.

**Definition 3.28:** Let X and X* be non-empty sets and
let \( C \) and \( C^* \) be families of subsets of X and X*, respec­
tively. Then for a mapping \( f \) from X to X* define:
\[
\begin{align*}
f[C] &= \{f[C]: C ∈ C\} \text{ and} \\
f^{-1}[C^*] &= \{f^{-1}[C^*]: C^* ∈ C^*\}.
\end{align*}
\]

**Theorem 3.29:** Let (X, U) and (X*, U*) be semi-pre­uniform spaces, and let \( f \) be a mapping from X to X*. Then
if \( f \) is uniformly continuous, given \( P^* ∈ P(U^*) \) there is
\( P ∈ P(U) \) such that \( P \) refines \( f^{-1}[P^*] \). Moreover, if (X*, U*)
is a pre-uniform space, \( f \) is uniformly continuous iff the
above condition holds.
Proof: To prove the first part of the theorem, let $f$ be uniformly continuous and let $P^* \in \mathcal{P}(U^*)$ be given. Then there is $U^* \in U^*$ such that $P^* = \{U^*[x^*]: x^* \in X^*\}$. Now by the definition of uniform continuity, there is $U \in U$ such that for all $x \in X$ there holds $f[U[x]] \subseteq U^*[f(x)]$. Then for $x \in X$ we have $U[x] \subseteq f^{-1}[f[U[x]]] \subseteq f^{-1}[U^*[f(x)]]$, and so $\{U[x]: x \in X\} \in \mathcal{P}(U)$ refines $f^{-1}[P^*]$. This proves the first assertion of the theorem.

To prove the second assertion, suppose that $U^*$ also satisfies [B3], and let $U^* \in U^*$ be given. Then there is $W^* = (W^*)^{-1} \in U^*$ such that $W^*oW^* \subseteq U^*$. Now by the assumed condition there is $U \in U$ such that $P(U)$ refines $f^{-1}[P(W^*)]$. Thus, given $x \in X$ we have that $U[x] \subseteq f^{-1}[W^*[x^*]]$ for some $x^* \in X^*$, so for $z \in U[x]$ we have both $f(z), f(x) \in W^*[x^*]$. From this it follows that $(f(x), f(z)) \in W^*oW^* \subseteq U^*$. Thus, $f[U[x]] \subseteq U^*[f(x)]$ for all $x \in X$, and so $f$ is uniformly continuous.

Theorem 3.20: Let $(X, \delta)$ and $(X^*, \delta^*)$ be semi-proximity spaces, and let $f$ be a mapping from $X$ to $X^*$. Then if $f$ is proximally continuous, $f^{-1}[P^*] \in \mathcal{P}(\delta)$ for all $P^* \in \mathcal{P}(\delta^*)$. Moreover, if $(X^*, \delta^*)$ is a proximity space, $f$ is proximally continuous iff the above condition holds.

Proof: To prove the first part of the theorem, let $f$ be proximally continuous and let $P^* \in \mathcal{P}(\delta^*)$ be given. Then $P^* = \{P^*_i: 1 \leq i \leq n\}$ where there are sets $Q^*_i \subseteq P^*_i$ for
\[ 1 \leq i \leq n \text{ such that } X^* = \bigcup \{ Q_i^*: 1 \leq i \leq n \}. \] Since \( f \) is proximally continuous, we have by Theorem 1.28 that \( f^{-1}[Q_i^*] \subseteq f^{-1}[P_i^*] \) for \( 1 \leq i \leq n \). Then since \( X = \bigcup \{ f^{-1}[Q_i^*]: 1 \leq i \leq n \} \) we see that \( \{ f^{-1}[P_i^*]: 1 \leq i \leq n \} = f^{-1}[P^*] \subseteq P(\delta). \) This proves the first assertion of the theorem.

To prove the second assertion suppose \( \delta^* \) also satisfies \([P5]\), and let \( A^*, B^* \subseteq X^* \) such that \( A^* \notin \delta^* B^* \) be given. Now by Theorem 3.16 we have that \( [X^*-A^*,X^*-B^*] \subseteq P(\delta^*), \) so by the assumed condition \( f^{-1}[[X^*-A^*,X^*-B^*]] = \{ X-f^{-1}[A^*], X-f^{-1}[B^*] \} \subseteq P(\delta). \) Hence, by Theorem 3.15 we have \( f^{-1}[A^*] \subseteq f^{-1}[B^*]. \) But this means that \( f \) is proximally continuous.

**Theorem 3.31:** Let \( X \) and \( X^* \) be non-empty sets and let \( P \) and \( P^* \) be semi-uniformizing classes on \( X \) and \( X^* \), respectively. Then for a mapping \( f \) from \( X \) to \( X^* \), if \( f^{-1}[P^*] \subseteq P \) for all \( P^* \subseteq P^* \), \( f \) is \( \delta(P)-\delta(P^*) \) continuous.

**Proof:** Let \( A^*, B^* \subseteq X^* \) such that \( A^* \notin \delta(P^*) B^* \). Then there is \( P^* \subseteq P^* \) such that \( P^* \) refines \([X^*-A^*,X^*-B^*]\). Thus, given \( P^* \subseteq P^* \), either \( P^* \subseteq (X^*-A^*) \) or \( P^* \subseteq (X^*-B^*) \), so either \( f^{-1}[P^*] \subseteq X-f^{-1}[A^*] \) or \( f^{-1}[P^*] \subseteq X-f^{-1}[B^*] \). Thus, \( f^{-1}[P^*] \) refines \([X-f^{-1}[A^*],X-f^{-1}[B^*]] \) and so \( f^{-1}[A^*] \subseteq(P) f^{-1}[B^*] \) since by assumption \( f^{-1}[P^*] \subseteq P \). It follows that \( f \) is \( \delta(P)-\delta(P^*) \) continuous.
CHAPTER IV

MEASURE AND PROXIMITY

Definition 4.1: By an outer measure \( v \) on a set \( X \) we mean a mapping \( v \) from the subsets of \( X \) to the extended real numbers which satisfies:

1. \( v(\emptyset) = 0. \)
2. For all \( E \subseteq X \), \( v(E) \geq 0. \)
3. For all \( E \subseteq F \subseteq X \), \( v(E) \leq v(F). \) (That is, \( v \) is monotone).
4. For any countable family \( \{E_k\} \) of subsets of \( X \) there holds \( v(\bigcup E_k) \leq \sum v(E_k). \) (The family \( \{E_k\} \) may be finite or countably infinite. We say that \( v \) is countably subadditive).

Definition 4.2: Let \( X \) be a set and let \( v \) be an outer measure on \( X \). Then a set \( E \subseteq X \) is said to be \( v \)-measurable iff for all \( A \subseteq X \) there holds:

\[
v(A) \geq v(A \cap E) + v(A - E).
\]

Note that this condition holds iff it holds for all \( A \subseteq X \) such that \( v(A) < \infty \). Also note that in view of [OM4] the condition is equivalent to:

\[
v(A) = v(A \cap E) + v(A - E)
\]

for all \( A \subseteq X \) such that \( v(A) < \infty \).
Theorem 4.3: Let X be a set and let \( \nu \) be an outer measure on X. Then the \( \nu \)-measurable subsets of X form a \( \sigma \)-algebra; that is, complements and countable unions of \( \nu \)-measurable sets are \( \nu \)-measurable. The sets X and \( \emptyset \) are \( \nu \)-measurable. For a countable family \( \{E_k\} \) of pairwise disjoint \( \nu \)-measurable subsets of X there holds \( \nu\left(\bigcup E_k\right) = \sum \nu(E_k) \). Countable intersections of \( \nu \)-measurable sets are \( \nu \)-measurable. These are all standard results. See Berberian[2] for proofs.

Definition 4.4: Let \((X, \delta)\) be a semi-proximity space and let \( \nu \) be an outer measure on X. Then \( \nu \) is said to be a \( \delta \)-measure on X iff for all \( A, B \subseteq X \) such that \( A \subseteq B \) there holds \( \nu(A \cup B) = \nu(A) + \nu(B) \).

Lemma 4.5: Let \((X, \delta)\) be a semi-proximity space and let \( \nu \) be a \( \delta \)-measure on X. Then if \( \{A_i: 1 \leq i \leq n\} \) is a finite family of subsets of X such that \( A_i \not\subseteq A_j \) for \( i \neq j \) there holds \( \nu\left(\bigcup \{A_i: 1 \leq i \leq n\}\right) = \sum \nu(A_i): 1 \leq i \leq n \).

Proof: By the definition of \( \delta \)-measure the lemma holds for \( n = 2 \). Now assume the truth of the lemma for \( n-1 \).

Since by [P2] we have \( \bigcup \{A_i: 1 \leq i \leq n-1\} \subseteq A_n \) we obtain:

\[
\begin{align*}
\nu\left(\bigcup \{A_i: 1 \leq i \leq n\}\right) &= \nu\left(\bigcup \{A_i: 1 \leq i \leq n-1\} \cup A_n\right) \\
&= \nu\left(\bigcup \{A_i: 1 \leq i \leq n-1\}\right) + \nu(A_n) \\
&= \Sigma \{\nu(A_i): 1 \leq i \leq n-1\} + \nu(A_n) = \Sigma \{\nu(A_i): 1 \leq i \leq n\}.
\end{align*}
\]
Lemma 4.6: (Caratheodory Lemma). Let \((X, \delta)\) be a semi-proximity space and let \(v\) be a \(\delta\)-measure on \(X\). Let \(F\) be a \(\delta\)-strongly-closed subset of \(X\) and let \(B \subseteq X\) such that \(B \cap F = \emptyset\). Now there is a sequence \(\{F_n\}_{n=1}^{\infty}\) of subsets of \(X\) such that \(F = \bigcap \{F_n : n \geq 1\}\) where \(F_{n+1} \delta (X-F_n)\) for all \(n \geq 1\). Let \(B_n = (B-F_n)\) for each \(n \geq 1\). We assert that \(\{v(B_n)\}_{n=1}^{\infty}\) is a non-decreasing sequence and that 
\[
\lim v(B_n) = v(B).
\]

Proof: Since \(B_n \subseteq B_{n+1} \subseteq B\) for \(n \geq 1\), we see that 
\(\{v(B_n)\}_{n=1}^{\infty}\) is a non-decreasing sequence and that 
\(v(B_n) \leq v(B)\) for \(n \geq 1\). Hence, \(\lim v(B_n) \leq v(B)\).

Now let \(D_n = (B_{n+1} - B_n)\) for \(n \geq 1\). Then for each \(n \geq 1\) we have \(D_n = B \cap (F_n-F_{n+1})\) and 
\(B = B_{2n} \cup ( \bigcup \{D_k : k \geq 2n\}) = B_{2n} \cup ( \bigcup_{k=n}^{\infty} D_{2k}) \cup ( \bigcup_{k=n}^{\infty} D_{2k+1})\). Then we have:

\[
v(B) \leq v(B_{2n}) + \sum_{k=n}^{\infty} v(D_{2k}) + \sum_{k=n}^{\infty} v(D_{2k+1}) \text{ for all } n \geq 1.
\]

Case I: Both series on the right converge. Then given \(\varepsilon > 0\) there is \(m \geq 1\) such that for \(n \geq m\) the sum of the two series is less than \(\varepsilon\). Thus we have 
\(\lim v(B_n) \leq v(B) \leq v(B_{2m}) + \varepsilon \leq \lim v(B_{2n}) + \varepsilon = \lim v(B_n) + \varepsilon\). Since \(\varepsilon > 0\) was arbitrary, we have that \(\lim v(B_n) = v(B)\) in this case.

Case II: One or both of the series on the right diverges. Since the proof is the same whichever series diverge, suppose \(\sum_{k=n}^{\infty} v(D_{2k})\) diverges.
Now for $n,k \geq 1$ we have that $F_{n+2k} \in (X-F_{n+1})$ which implies that $[B \cap (F_{n+2k}-F_{n+1})] \subseteq [B \cap (F_n-F_{n+1})]$ for all $n,k \geq 1$, since these sets are subsets of the former sets. But the latter relation is just that $D_{n+2k} \subseteq D_n$ for $n,k \geq 1$. In particular, the even-numbered $D_n$ (and also the odd-numbered $D_n$), are mutually far apart. Thus using Lemma 4.5 we obtain for all $n > 1$ that

$$\sum_{k=1}^{n-1} v(D_{2k}) = \sum_{k=1}^{n-1} v(\bigcup_{k=1}^{n-1} D_{2k}) \leq v(B_{2n}),$$

the last inequality holding since $\bigcup \{D_{2k}: 1 \leq k \leq n-1\} \subseteq B_{2n}$ for all $n > 1$. From this we see that $\lim v(B_n) =$

$$\lim v(B_{2n}) = \infty \text{ since } \sum_{k=1}^{\infty} v(D_{2k}) \text{ diverges.}$$

Now since $v$ is monotone, it follows that $v(B) = \infty$ since $B_n \subseteq B$ for $n \geq 1$.

Thus $\lim v(B_n) = v(B)$ in this case also. This completes the proof of the lemma.

**Theorem 4.7:** Let $(X,\delta)$ be a semi-proximity space and let $v$ be a $\delta$-measure on $X$. Then the $\delta$-strongly-closed subsets of $X$ are $v$-measurable. Hence, the smallest $\sigma$-algebra in $X$ containing all $\delta$-strongly-closed subsets of $X$ ( = the smallest $\sigma$-algebra in $X$ containing all $\delta$-strongly-open subsets of $X) contains only $v$-measurable sets.

**Proof:** Let $F$ be a $\delta$-strongly-closed subset of $X$ and let $A \subseteq X$ such that $v(A) < \infty$. Now let $\{F_n\}_{n=1}^{\infty}$ be a sequence
of subsets of X such that \( F = \cap \{ F_n : n \geq 1 \} \) and \( F_{n+1} \subseteq F_n \) for \( n \geq 1 \). Now for each \( n \geq 1 \) let \( A_n = (A-F)^{-1} F_n = (A-F_n) \).

Then \( \cup \{ A_n : n \geq 1 \} = (A-F) \). Now \( F \not\subseteq (X-F_n) \) implies \( (A \cap F) \not\subseteq (A-F_n) \), so we see that \( (A \cap F) \not\subseteq A_n \) for \( n \geq 1 \).

Thus since \( \nu \) is a \( \delta \)-measure on \( X \) we have:

\[
\nu(A) \geq \nu((A \cap F) \cup A_n) = \nu(A \cap F) + \nu(A_n) \quad \text{for} \quad n \geq 1.
\]

But by Lemma 4.6, with \( B = (A-F) \) and \( B_n = A_n \) we have:

\[
\lim \nu(A_n) = \nu(A-F),
\]

so we obtain:

\[
\nu(A) \geq \nu(A \cap F) + \nu(A-F).
\]

Hence, \( F \) is \( \nu \)-measurable.

**Theorem 4.8:** Let \((X, \delta)\) be a proximity space and let \( \nu \) be an outer measure on \( X \). Then \( \nu \) is a \( \delta \)-measure iff the \( \delta \)-strongly-closed subsets of \( X \) are \( \nu \)-measurable.

**Proof:** The only if part follows from Theorem 4.7.

Therefore, suppose all \( \delta \)-strongly-closed subsets of \( X \) are \( \nu \)-measurable and let \( A, B \subset X \) such that \( A \not\subseteq B \). Then using [PC6] repeatedly we can inductively choose a sequence \( \{ F_n \}_{n=1}^{\infty} \) of subsets of \( X \) such that for each \( n \geq 1 \) there holds \( A \subseteq F_{n+1} \subseteq F_n \subseteq (X-B) \). Now \( F = \cap \{ F_n : n \geq 1 \} \) is \( \delta \)-strongly-closed and hence \( \nu \)-measurable. Then since \( (A \cup B) \cap F = A \) and \( (A \cup B) - F = B \) we obtain \( \nu(A \cup B) = \nu((A \cup B) \cap F) + \nu((A \cap B) - F) = \nu(A) + \nu(B) \). This shows that \( \nu \) is a \( \delta \)-measure.
CHAPTER V

MEASURE CONSTRUCTION

**Definition 5.1:** Let \( X \) be a set. Then \( (C, \varphi) \) is said to be a pre-measure system on \( X \) iff \( C \) and \( \varphi \) satisfy:

- \([\text{PM1}]\) \( C \) is a family of subsets of \( X \).
- \([\text{PM2}]\) \( \varphi \) is a real valued function on \( C \).
- \([\text{PM3}]\) \( \emptyset \in C \).
- \([\text{PM4}]\) \( \varphi(\emptyset) = 0 \).
- \([\text{PM5}]\) For all \( C \in C \), \( 0 \leq \varphi(C) < \infty \).

**Lemma 5.2:** Let \( X \neq \emptyset \) be a set and let \( (C, \varphi) \) be a pre-measure system on \( X \). Then given a property \( R \) of subsets of \( X \), define a mapping \( v[C, \varphi, R] \) from the subsets of \( X \) to the extended real numbers by:

For \( E \subset X \), \( v[C, \varphi, R](E) = \inf \{ \Sigma \varphi(C_i) : \{C_i\} \subset C \text{ is countable, } E \subset \bigcup C_i \text{, and each } C_i \text{ satisfies } R \} \).

(Note the infimum of the empty set has value \(+\infty\)).

Then \( v[C, \varphi, R] \) is an outer measure on \( X \).

**Proof:** For simplicity let \( v \) denote \( v[C, \varphi, R] \). Now it is easy to see that \( v \) satisfies \([\text{OM1}]-[\text{OM3}]\). To show that \( v \) satisfies \([\text{OM4}]\), let \( \{E_k\} \) be a countable family of subsets of \( X \), and let \( \varepsilon > 0 \) be given. Clearly, we may
assume that $\Sigma v(E_k) < \infty$. Hence, $v(E_k) < \infty$ for all $k$. Thus for each $k$ there is a countable family $\{C^k_i\} \subset \mathcal{C}$ such that $E_k \subset \cup_i C^k_i$, $\Sigma \varphi(C^k_i) \leq v(E_k) + \varepsilon/2^k$, and $C^k_i$ satisfies $R$, for all $i$. Now $\cup E_k \subset \cup_k \cup_i C^k_i$, $\{C^k_i\}, k \subset \mathcal{C}$ is countable, and $C^k_i$ satisfies $R$ for all $i,k$. Hence we have $v(\cup E_k) \leq \Sigma i,k \varphi(C^k_i) = \Sigma_k \Sigma_i \varphi(C^k_i) \leq \Sigma_k \{v(E_k) + \varepsilon/2^k\} \leq \varepsilon + \Sigma v(E_k)$.

The desired result then follows since $\varepsilon > 0$ was arbitrary.

**Lemma 5.3:** Let $X$ be a set and let $\{v_\alpha : \alpha \in \mathcal{J}\}$ be a non-empty family of outer measures on $X$. ($\mathcal{J}$ is a non-empty index set). For $E \subset X$ define $v(E) = \sup \{v_\alpha(E) : \alpha \in \mathcal{J}\}$. Then $v$ is an outer measure on $X$.

**Proof:** It is easy to see that $v$ satisfies [OM1]-[OM3]. To show that $v$ satisfies [OM4], let $\{E_k\}$ be a countable family of subsets of $X$. Then given $\alpha \in \mathcal{J}$ we have that $v_\alpha(\cup E_k) \leq \Sigma_\alpha (E_k) \leq \Sigma v(E_k)$. Since $\alpha \in \mathcal{J}$ was arbitrary, the desired result follows.

**Definition 5.4:** Let $X \neq \emptyset$ be a set and let $(\mathcal{C}, \varphi)$ be a pre-measure system on $X$. Then given a family $\mathcal{P}$ of subsets of $X$ define a mapping $p[\mathcal{C}, \varphi, \mathcal{P}]$ from the subsets of $X$ to the extended real numbers by:

For $E \subset X$, $p[\mathcal{C}, \varphi, \mathcal{P}](E) = \inf \{\Sigma \varphi(C^i) : \{C^i\} \subset \mathcal{C}$ is countable, $E \subset \cup C^i$, and $\{C^i\}$ refines $\mathcal{P}\}.

By taking the property $R$ to be "for all $i$ there is $P \in \mathcal{P}$ such that $C^i \subset P,"$ we see that $p[\mathcal{C}, \varphi, \mathcal{P}]$ is an outer measure.
on $X$ by Lemma 5.2, since then property $R$ is just that $\{C_i\}$ refines $\mathcal{P}$.

**Definition 5.5**: Let $X \neq \emptyset$ be a set and let $(C, \varphi)$ be a pre-measure system on $X$. Then for a non-empty class $\mathcal{P}$ of families of subsets of $X$ define a mapping $p[C, \varphi, \mathcal{P}]$ from the subsets of $X$ to the extended real numbers by:

For $E \subseteq X$, $p[C, \varphi, \mathcal{P}](E) = \sup \{p[C, \varphi, \mathcal{P}](E) : \mathcal{P} \in \mathcal{P}\}$.

That $p[C, \varphi, \mathcal{P}]$ is an outer measure on $X$ follows from Lemma 5.3 and Definition 5.4.

**Theorem 5.6**: Let $X \neq \emptyset$ be a set and let $(C, \varphi)$ be a pre-measure system on $X$. Then for a non-empty class $\mathcal{P}$ of families of subsets of $X$ there holds $p[C, \varphi, \mathcal{P}] = p[C, \varphi, \mathcal{P}]$.

**Proof**: Since $\mathcal{P} \subseteq \mathcal{P}$ it follows for all $E \subseteq X$ that $\sup \{p[C, \varphi, \mathcal{P}](E) : \mathcal{P} \in \mathcal{P}\} \leq \sup \{p[C, \varphi, \mathcal{P}](E) : \mathcal{P} \in \mathcal{P}\}$, so that $p[C, \varphi, \mathcal{P}] \leq p[C, \varphi, \mathcal{P}]$. To show the reverse inequality let $\mathcal{P}' \in \mathcal{P}$ be given and find $\mathcal{P} \in \mathcal{P}$ such that $\mathcal{P}$ refines $\mathcal{P}'$. Then for $E \subseteq X$ we have $p[C, \varphi, \mathcal{P}'](E) \leq p[C, \varphi, \mathcal{P}](E) \leq p[C, \varphi, \mathcal{P}](E)$. Since $\mathcal{P}' \in \mathcal{P}$ and $E \subseteq X$ were arbitrary, the desired result follows.

**Theorem 5.7**: (Main Theorem). Let $X \neq \emptyset$ be a set, let $(C, \varphi)$ be a pre-measure system on $X$, and let $\mathcal{P}$ be a semi-uniformizing class on $X$. Then $p = p[C, \varphi, \mathcal{P}]$ is a $\delta(\mathcal{P})$-measure on $X$.

**Proof**: We showed in Theorem 5.5 that $p$ is an outer measure on $X$. Thus we must show that for $A, B \subseteq X$ such that
A \( P \) \( B \) there holds \( p(A \cup B) = p(A) + p(B) \). In showing this we may assume that \( p(A \cup B) < \infty \). Thus also \( p(A) < \infty \) and \( p(B) < \infty \). Hence, letting \( p[P] \) denote \( p[C,\varphi,P] \), given \( \varepsilon > 0 \) there are \( P_1, P_2 \in P \) such that \( p[P_1](A) \geq p(A) - \frac{\varepsilon}{3} \) and \( p[P_2](B) \geq p(B) - \frac{\varepsilon}{3} \). Now since \( A \ \overline{P} \ \ B \) we have that \( \{X - A, X - B\} \in P \). Let \( P \in P \) be a common refinement of \( P_1, P_2, \) and \( \{X - A, X - B\} \). Now \( p[P](A \cup B) \leq p(A \cup B) < \infty \) so there is a countable family \( \{C_i\} \subset C \) such that \( (A \cup B) \subset \bigcup C_i \), \( \{C_i\} \) refines \( P \), and \( p[P](A \cup B) + \frac{\varepsilon}{3} \geq \Sigma \varphi(C_i) \). Now since \( \{C_i\} \) refines \( \{X - A, X - B\} \), no \( C_i \) can meet both \( A \) and \( B \). Hence we have \( p(A \cup B) + \frac{\varepsilon}{3} \geq p[P](A \cup B) + \frac{\varepsilon}{3} \geq \Sigma \varphi(C_i) \geq \Sigma \{\varphi(C_i): C_i \cap A \neq \emptyset\} + \Sigma \{\varphi(C_i): C_i \cap B \neq \emptyset\} \geq p[P](A) + p[P](B) = p(A) - \frac{\varepsilon}{3} + p(B) - \frac{\varepsilon}{3} \). Since \( \varepsilon > 0 \) was arbitrary and all the quantities involved are finite, we obtain \( p(A \cup B) \geq p(A) + p(B) \). Since \( p \) is an outer measure on \( X \), the reverse inequality follows from \([OM4]\). This proves the theorem.

**Remark 5.8:** (Measure construction in proximity spaces). Let \((X, \delta)\) be a semi-proximity space and let \((C, \varphi)\) be a pre-measure system on \( X \). Then by Theorem 5.7 it follows that \( p = p[C,\varphi,P(\delta)] \) is a \( \delta(P(\delta)) \)-measure on \( X \). In general \( p \) will not be a \( \delta \)-measure on \( X \), but if \((X, \delta)\) is a proximity space we have \( \delta(P(\delta)) = \delta \) so that \( p \) is a \( \delta \)-measure in this case, and the \( \delta \)-strongly-closed subsets of \( X \) are \( p \)-measurable.
We shall now consider some well-known constructions of outer measures in metric, uniform, and topological spaces, and show how they can be obtained as special cases of our construction introduced in the Main Theorem. For brevity, we shall from now on denote $p[C,\varphi,P]$ by $p$, and $p[C,\varphi,F]$ by $p[F]$, where $P$ and $F$ are appropriate for the example or theorem under discussion.

**Remark 5.9:** Let $(X,d)$ be a metric space and let $(C,\varphi)$ be a pre-measure system on $X$. Then the standard outer measure $m$ (see Munroe[10]) defined on $X$ is as follows:

For $E \subseteq X$ and $\varepsilon > 0$ let $m_\varepsilon(E) = \inf \{\Sigma \varphi(C_i) : \{C_i\} \subseteq C$

is countable,

$E \subseteq \bigcup C_i$, and

$diam(C_i) < \varepsilon$ for all $i$].

Then for $E \subseteq X$ let $m(E) = \sup \{m_\varepsilon(E) : \varepsilon > 0\}$.

Then for $A,B \subseteq X$ such that $d(A,B) > 0$ there holds $m(A \cup B) = m(A) + m(B)$, and the sets closed in $J(d)$ are $m$-measurable.

We assert that these results follow from the theory of the outer measure $p$.

**Proof:** We assert that $m = p[C,\varphi,P(d)] = p$. For it is easy to verify that for $\varepsilon > 0$ a family $\{C_i\} \subseteq C$ refines $P_\varepsilon$ iff $diam(C_i) < 2\varepsilon$ for all $i$. Hence $p[P_\varepsilon] = m_{2\varepsilon}$. Taking supremums over $\varepsilon > 0$ gives $p = m$. This implies that $m$ is a $\delta(P(d))$-measure on $X$, and hence also a $\delta(d)$-measure,
since by Theorem 3.27 we have \( \delta(\mathcal{P}(d)) = \delta(d) \) in this case. Hence we obtain that \( d(A, B) > 0 \) implies \( m(A \cup B) = m(A) + m(B) \). Moreover, since \( m \) is a \( \delta(d) \)-measure on \( X \) it follows that the \( \delta(d) \)-strongly-closed subsets of \( X \) are \( m \)-measurable. But by Theorem 1.25, these are just the sets closed in \( \mathcal{S}(d) \). Thus the standard results of the outer measure \( m \) follow from the theory of the outer measure \( p \).

**Remark 5.10:** Let \((X, \mathcal{U})\) be a semi-pre-uniform space. Then given a pre-measure system \((\mathcal{C}, \varphi)\) on \( X \), it follows that \( p = p[\mathcal{C}, \varphi, \mathcal{P}(u)] \) is a \( \delta(\mathcal{P}(u)) \)-measure on \( X \). But by Theorem 3.25 we have that \( \delta(\mathcal{P}(u)) = \delta_2(u) \), so \( p \) is a \( \delta_2(u) \)-measure on \( X \). Thus, for \( A, B \subseteq X \), if there is \( U \in \mathcal{U} \) such that \( U^2[A] \subseteq B \) then \( p(A \cup B) = p(A) + p(B) \).

**Theorem 5.11:** Let \((X, \mathcal{U})\) be a uniform space, and let \((\mathcal{C}, \varphi)\) be a pre-measure system on \( X \). Then the standard outer measure \( u \) (see Willmott[17]) defined on \( X \) is as follows:

For \( E \subseteq X \) and \( U \in \mathcal{U} \) let \( u_U(E) = \inf \{ \varphi(C_i) : \{C_i\} \subseteq \mathcal{C} \) is countable, \( E \subseteq \bigcup C_i \), and \( C_i \times C_i \subseteq U \) for all \( i \} \).

Then for \( E \subseteq X \) let \( u(E) = \sup \{ u_U(E) : U \in \mathcal{U} \} \).

We assert that \( u = p[\mathcal{C}, \varphi, \mathcal{P}(u)] = p \).
Proof: Let \( U \in \mathcal{U} \) be given. Then find \( W = W^{-1} \in \mathcal{U} \) such that \( W_0 W \subseteq U \). Let \( \{C_i\} \subseteq \mathcal{C} \) refine \( \mathcal{P}(W) \). Thus, for each \( i \) there is \( x_i \in X \) such that \( C_i \subseteq W[x_i] \). Then \((a,b) \in C_i \times C_i \) implies \( a \in W[x_i] \) and \( b \in W[x_i] \). Hence, \((a,b) \in W_0 W \subseteq U \), and so \( C_i \times C_i \subseteq U \). Thus, if \( \{C_i\} \subseteq \mathcal{C} \) refines \( \mathcal{P}(W) \), then \( C_i \times C_i \subseteq U \) for all \( i \). From this it follows that \( u_U \leq p[\mathcal{P}(W)] \leq p \). Hence, \( u \leq p \), since \( U \in \mathcal{U} \) was arbitrary.

Now if \( U \in \mathcal{U} \) and \( C \subseteq X \) are such that \( C \times C \subseteq U \), then \( c,x \in C \) implies \( (c,x) \in U \) so that \( x \in U[c] \). Thus if \( \{C_i\} \subseteq \mathcal{C} \) is such that \( C_i \times C_i \subseteq U \in \mathcal{U} \) for all \( i \), we see that \( \{C_i\} \) refines \( \mathcal{P}(U) \) since given \( i \), \( C_i \subseteq U[c_i] \) for any \( c_i \in C_i \), by the preceding remark. From this it follows that \( p[\mathcal{P}(U)] \leq u_U \leq u \), and so \( p \leq u \) since \( U \in \mathcal{U} \) was arbitrary.

Thus we see that \( u \) is a \( \delta(U) \)-measure on \( X \) by Remark 5.10, since \( \delta_2(U) = \delta(U) \) in this case. Hence the \( \delta(U) \)-strongly-closed subsets of \( X \) are \( u \)-measurable. It is a straightforward exercise to show that a set \( F \subseteq X \) is \( \delta(U) \)-strongly-closed iff there is a sequence \( \{U_n\}_{n=1}^{\infty} \subseteq \mathcal{U} \) such that \( F = \bigcap \{U_n[F]: n \geq 1\} \) and \( U_{n+1} \cup U_{n+1} \subseteq U_n \) for all \( n \geq 1 \).

Remark 5.12: Let \((X,\mathcal{U})\) be a topological space, and let \((\mathcal{C},\varphi)\) be a pre-measure system on \( X \). Then Rogers and Sion[12] have defined an outer measure \( r \) on \( X \) as follows:
Let $\mathcal{P}(\mathcal{J}) = \{\mathcal{P}: \mathcal{P} \text{ is a cover of } X \text{ consisting of a finite number of sets, each of which is the difference of two open sets}\}$. 

For $E \subset X$ and $\mathcal{P} \in \mathcal{P}(\mathcal{J})$ let $r_\mathcal{P}(E) = \inf \{r_\mathcal{P}(C_1): \{C_1\} \subset C \text{ is countable, } E \subset \bigcup C_1, \text{ and } \{C_1\} \text{ refines } \mathcal{P}\}$. 

For $E \subset X$ let $r(E) = \sup \{r_\mathcal{P}(E): \mathcal{P} \in \mathcal{P}(\mathcal{J})\}$. 

Rogers and Sion show directly that the closed subsets of $X$ are $r$-measurable. Since evidently $r = p[\mathcal{P}, \mathcal{P}(\mathcal{J})]$, we shall show that this result follows from our theory. 

**Proof:** Clearly, $\mathcal{P}(\mathcal{J})$ satisfies [UC1]. To show that $\mathcal{P}(\mathcal{J})$ satisfies [UC2], let $\mathcal{P}_1, \mathcal{P}_2 \in \mathcal{P}(\mathcal{J})$. Then 

$\mathcal{P}_1 = \{G^1_i - G^2_i: 1 \leq i \leq n\}$ where $X = \bigcup \{G^1_i - G^2_i: 1 \leq i \leq n\}$ and $G^1_i, G^2_i \in \mathcal{P}(\mathcal{J})$ for $1 \leq i \leq n$, and 

$\mathcal{P}_2 = \{O^1_j - O^2_j: 1 \leq j \leq m\}$ where $X = \bigcup \{O^1_j - O^2_j: 1 \leq j \leq m\}$ and $O^1_j, O^2_j \in \mathcal{P}(\mathcal{J})$ for $1 \leq j \leq m$. 

Now let $\mathcal{P} = \{(G^1_i - G^2_i) \cap (O^1_j - O^2_j): 1 \leq i \leq n, 1 \leq j \leq m\}$. Then clearly $\mathcal{P}$ is a cover of $X$ consisting of a finite number of sets, and $\mathcal{P}$ refines both $\mathcal{P}_1$ and $\mathcal{P}_2$. Now for fixed $i, j$ where $1 \leq i \leq n$ and $1 \leq j \leq m$ we have:
\[(G^1_1 - G^2_1) \cap (O^1_2 - O^2_2) = G^1_1 \cap (X-G^2_1) \cap O^1_2 \cap (X-O^2_2) =
\]
\[(G^1_1 \cap O^1_2) \cap (X-(G^2_1 \cup O^2_2)) = (G^1_1 \cap O^1_2)-(G^2_1 \cup O^2_2)
\]

which is the difference of two open sets. Hence \(P \in \mathcal{P}(\mathbb{S})\), and we see that \(\mathcal{P}(\mathbb{S})\) satisfies [UC2].

Now suppose that \(F\) is a closed subset of \(X\). Then \([F,X-F] \in \mathcal{P}(\mathbb{S})\) since \(X = F \cup (X-F)\) and \(F = X-(X-F)\), \((X-F) = (X-F)-\emptyset\) are the differences of two open sets. Thus we have \((X-F) \in \mathcal{P}(\mathbb{S})\) \(F\). Now if we let \(F_n = F\) for all \(n \geq 1\) we see that \(F = \bigcap \{F_n : n \geq 1\}\) and \(F_{n+1} \in \mathcal{P}(\mathbb{S})\) \((X-F_n)\) for all \(n \geq 1\), so that \(F\) is \(\mathcal{P}(\mathbb{S})\)-strongly-closed, and hence is \(p[C,\phi,\mathcal{P}(\mathbb{S})]\)-measurable. Since \(p = r\), and \(F\) was an arbitrary closed subset of \(X\), this completes the proof.

We now prove some theorems concerning the construction process of the outer measure \(p\); Theorems 5.13 and 5.14 are essentially the approximation theorems of Knowles[7], and Theorems 5.15-5.17 give necessary and sufficient conditions that \(p\) extend \(\phi\).

**Theorem 5.13:** Let \(X \neq \emptyset\) be a set, let \((C,\phi)\) be a pre-measure system on \(X\), and let \(P\) be a semi-uniformizing class on \(X\). Then given \(A \subset X\) such that \(p(A) < \infty\) there is a sequence \(\{\mathcal{C}_k : k \geq 1\} \subset \mathcal{P}\) such that:

1. \(\mathcal{C}_{k+1}\) refines \(\mathcal{C}_k\) for \(k \geq 1\), and
2. For all \(p\)-measurable \(E \subset A\), \(p(E) \leq p[\mathcal{C}_k](E) + \frac{1}{k}\) for each \(k \geq 1\).
Proof: Now since \( p(A) < \infty \), for each \( k \geq 1 \) there is \( \mathcal{P}_k \in \mathcal{P} \) such that \( p[\mathcal{P}_k^*(A)](A) \geq p(A) - \frac{1}{k} \). Let \( \mathcal{P}_1 = \mathcal{P}_1^* \) and for \( k \geq 2 \) inductively choose \( \mathcal{P}_k \in \mathcal{P} \) so that \( \mathcal{P}_k \) refines all the families \( \mathcal{P}_1, \ldots, \mathcal{P}_{k-1}, \mathcal{P}_k^* \). Then since \( \mathcal{P}_k \) refines \( \mathcal{P}_k^* \) we have that \( p[\mathcal{P}_k^*] \geq p[\mathcal{P}_k^*] \) so that \( p[\mathcal{P}_k^*](A) \geq p(A) - \frac{1}{k} \). Thus for any \( p \)-measurable \( E \subseteq A \) and \( k \geq 1 \) we have:

\[
p(E) + p(A - E) = p(A) \leq p[\mathcal{P}_k^*](A) + \frac{1}{k} \leq p[\mathcal{P}_k](E) + p(A - E) + \frac{1}{k}.
\]

Since all the quantities involved are finite we obtain (2). This completes the proof.

Theorem 5.14: Let \( X \neq \emptyset \) be a set, let \((\mathcal{C}, \varphi)\) be a pre-measure system on \( X \), and let \( \mathcal{P} \) be a semi-uniformizing class on \( X \). Suppose \( \mathcal{C} \) contains only \( p \)-measurable sets, and let \( A \subseteq X \) be \( p \)-measurable such that \( p(A) < \infty \). Then for any \( E \subseteq A \) there is a \( C_0^\infty \) set \( I \) such that \( E \subseteq I \) and \( p(E) = p(I \cap A) \).

Proof: Let \( \{ \mathcal{P}_k : k \geq 1 \} \) be as given for the set \( A \) by Theorem 5.13. Then given \( E \subseteq A \) we have for each \( k \geq 1 \) that \( p[\mathcal{P}_k](E) \leq p[\mathcal{P}_k](A) \leq p(A) < \infty \). Thus given \( k \geq 1 \) there is a countable family \( \{ C_n^k : n \in \mathbb{N} \} \subseteq \mathcal{C} \) such that \( E \subseteq \bigcup_n C_n^k \), \( \{ C_n^k \} \) refines \( \mathcal{P}_k \), and \( \Sigma_n \varphi(C_n^k) \leq p[\mathcal{P}_k](E) + \frac{1}{k} \). Then for each \( k \geq 1 \) let \( I_k = \bigcup_n C_n^k \) and then let \( I = \cap I_k \). Clearly, \( E \subseteq I \) and \( I \) is a \( C_0^\infty \) set. Now for each \( k \geq 1 \), \( (I_k \cap A) \) is a \( p \)-measurable subset of \( A \), and \( p[\mathcal{P}_k](I_k \cap A) \leq \Sigma_n \varphi(C_n^k) \). Thus for each \( k \geq 1 \) we have:
\[ p(E) \leq p(I \cap A) \leq p(I_k \cap A) \leq p[p_k](I_k \cap A) + \frac{1}{k} \leq \sum \varphi(C^k_n) + \frac{1}{k} \leq p[p_k](E) + \frac{1}{k} \leq p(E) + \frac{2}{k}. \]

Since \( p(E) < \infty \) and \( k \geq 1 \) was arbitrary, the desired result follows.

The following two lemmas are immediate from the definition of \( p[C, \varphi, P] \).

**Lemma 5.15:** Let \( X \neq \emptyset \) be a set, let \((C, \varphi)\) be a pre-measure system on \( X \), let \( \mathcal{P} \neq \emptyset \) be a class of families of subsets of \( X \), and let \( q \geq 0 \), \( E \subset X \), and \( P \in \mathcal{P} \) be given. Then \( q \leq p[C, \varphi, P](E) \) iff for all countable families \( \{C^k_n\} \subset C \) which refine \( P \) and such that \( E \subset \bigcup C^k_n \) there holds \( q \leq \sum \varphi(C^k_n) \).

**Lemma 5.16:** Let \( X \neq \emptyset \) be a set, let \((C, \varphi)\) be a pre-measure system on \( X \), let \( \mathcal{P} \neq \emptyset \) be a class of families of subsets of \( X \), and let \( Q \geq 0 \), \( E \subset X \), and \( P \in \mathcal{P} \) be given. Then \( p[C, \varphi, P](E) \leq Q \) iff for all \( \varepsilon > 0 \) there is a countable family \( \{C^*_k\} \subset C \) which refines \( P \) and such that \( E \subset \bigcup C^*_k \) and \( \sum \varphi(C^*_k) \leq Q + \varepsilon \).

**Theorem 5.17:** Let \( X \neq \emptyset \) be a set, let \((C, \varphi)\) be a pre-measure system on \( X \), and let \( \mathcal{P} \neq \emptyset \) be a class of families of subsets of \( X \). Then the following are equivalent:

1. \( p(C) = \varphi(C) \) for all \( C \in C \).
2. \( p[\mathcal{P}](C) = \varphi(C) \) for all \( C \in C \) and all \( \mathcal{P} \in \mathcal{P} \).
3. Given \( C \in C \) and \( \mathcal{P} \in \mathcal{P} \) there hold:
(a) For all countable families \(\{C_k\} \subseteq C\) which refine \(\mathcal{P}\) and such that \(C \subseteq \bigcup C_k\) we have 
\[\varphi(C) \leq \Sigma \varphi(C_k).\]

(b) For all \(\epsilon > 0\) there is a countable family 
\(\{C^*_k\} \subseteq C\) which refines \(\mathcal{P}\) and such that 
\(C \subseteq \bigcup C^*_k\) and \(\Sigma \varphi(C^*_k) \leq \varphi(C) + \epsilon.\)

**Proof:** Letting \(E = C\) and \(q = Q = \varphi(C)\), we see that (2) is equivalent to (3) by Lemmas 5.15 and 5.16. Clearly, (2) implies (1). To show that (1) implies (2), suppose (1) holds and let \(C \in C, \mathcal{P} \in \mathcal{P}\), and \(\epsilon > 0\) be given. Then since 
\[p[\mathcal{P}](C) \leq p(C) = \varphi(C) < \infty,\]
there is a countable family 
\(\{C_k\} \subseteq C\) which refines \(\mathcal{P}\) and such that \(C \subseteq \bigcup C_k\) and
\[p[\mathcal{P}](C) + \epsilon \geq \Sigma \varphi(C^*_k).\]

But \(\Sigma \varphi(C^*_k) = \Sigma p(C) \geq p(C) = \varphi(C),\) so therefore 
\[\varphi(C) \geq p[\mathcal{P}](C) \geq \varphi(C) - \epsilon.\]
The desired result then follows.

**Theorem 5.18:** Let \(X \neq \emptyset\) be a set, let \((C, \varphi)\) be a pre-measure system on \(X\), and let \(\mathcal{P} \neq \emptyset\) be a class of families of subsets of \(X\). Let \(G\) be a group of one-to-one mappings of \(X\) onto \(X\) which satisfies:

1. For all \(g \in G\) and all \(\mathcal{P} \in \mathcal{P}, g[\mathcal{P}] \in \mathcal{P}.\)
2. For all \(g \in G\) and all \(C \in C, g[C] \in C.\)
3. For all \(g \in G\) and all \(C \in C, \varphi(g[C]) = \varphi(C).\)

Then for all \(E \subseteq X\) and all \(g \in G\) there holds 
\[p(g[E]) = p(E).\]

**Proof:** Let \(E \subseteq X, g \in G,\) and \(\mathcal{P} \in \mathcal{P}\) be given. Then if 
\(\{C_k\} \subseteq C\) is a countable family which refines \(\mathcal{P}\) and such that
E ⊆ \bigcup C_k, we have \{g[C_k] \subseteq C\} is a countable family which refines g[P] and such that g[E] ⊆ \bigcup g[C_k]. Hence we have p[g[P]](g[E]) ≤ \Sigma \varphi(g[C_k]) = \Sigma \varphi(C_k), so that p[g[P]](g[E]) ≤ p[\varphi](E). But this also implies that p[\varphi](E) = p[g^{-1}[g[P]](g^{-1}[g[E]]) ≤ p[g][g(E)] by replacing g by g^{-1}. Thus we see that p[\varphi](E) = p[g][g(E)] for all E ⊆ X, g ∈ G, and P ∈ P. Now for fixed g ∈ G, g[P] runs over P as P runs over P, since G is a group. Hence, by taking the supremum over P ∈ P of both sides of the last equality, we obtain the desired result.

Remark 5.19: Let X ≠ ∅ be a set, let (C, \varphi) be a pre-measure system on X, and let P be a semi-uniformizing class on X. Then for each n ≥ 0 we may define outer measures p_n on X, all of which are equal if P is a uniformizing class, as follows:

\begin{align*}
p_0 &= p[C, \varphi, P], \text{ and for } n \geq 1, 
p_n &= p[C, \varphi, \{\text{st}^nP : P ∈ P}\].
\end{align*}

Then p_n is a δ_{k}(P) measure on X for k ≥ n. Also p_n ≤ p_m for n ≥ m ≥ 0.
CHAPTER VI

ALEXANDROFF MEASURE

Definition 6.1: Let \((X, \delta)\) be a proximity space and let \(G\) be a group of proximal isomorphisms from \(X\) onto \(X\). Then \(G\) is said to satisfy Condition \(A^*\) iff for all \(E, F \subseteq X\) such that \(E \# F\) there is \(\emptyset \neq 0 \in \mathcal{J}(\delta)\) such that for all \(g \in G\) either \(g[0] \cap E = \emptyset\) or \(g[0] \cap F = \emptyset\).

Condition \(A^*\) was suggested by Condition \(A\) which was introduced by Steinlage[14], and is defined as follows: Let \((X, \mathcal{J})\) be a locally compact Hausdorff space and let \(G\) be a group of homeomorphisms of \(X\) onto \(X\). Then \(G\) is said to satisfy Condition \(A\) iff for each pair of disjoint compact sets \(B, C \subseteq X\) there is \(\emptyset \neq 0 \in \mathcal{J}\) such that for all \(g \in G\) either \(g[0] \cap B = \emptyset\) or \(g[0] \cap C = \emptyset\). Now a locally compact Hausdorff space \((X, \mathcal{J})\) is completely regular; for such a space there is a proximity relation \(\delta\) on \(X\) such that \(\mathcal{J}(\delta) = \mathcal{J}\) (see Thron[15] for details). Since compact subsets of a Hausdorff space are closed, we see by Theorem 1.33 that \(B \# C\) in the definition of Condition \(A\). Thus Condition \(A^*\) is essentially a re-definition of Condition \(A\) in the case where local compactness is not assumed. This is done in a proximity space, where the group \(G\) is "nice" relative to the proximity relation involved.
Remark 6.2: It is easy to see that, in Definition 6.1, $G$ satisfies Condition $A^*$ iff for all $E, F \subseteq X$ such that $E \notin F$ there is $\emptyset \neq 0 \in \mathcal{J}(\delta)$ such that for all $g \in G$ either $g[0] \subseteq E$ or $g[0] \subseteq F$. For if $\emptyset \neq 0 \in \mathcal{J}(\delta)$ has the property of Condition $A^*$ for the sets $E, F$, then noting that $x \in 0^*$ implies $\{x\} \subseteq 0^*$, we can find $\emptyset \neq 0 \subseteq O^*$ and since all the mappings $g \in G$ are proximal isomorphisms, it follows that $0$ has the desired property. From this it is clear that $G$ satisfies Condition $A^*$ iff for all $E \subseteq F \subseteq X$ there is $\emptyset \neq 0 \in \mathcal{J}(\delta)$ such that for all $g \in G$, $g[0] \cap E \neq \emptyset$ implies $g[0] \subseteq F$. Since all members of $G$ are proximal isomorphisms, this is also equivalent to: $G$ satisfies Condition $A^*$ iff for all $E \subseteq F \subseteq X$ there is $\emptyset \neq 0 \in \mathcal{J}(\delta)$ such that for all $g_1, g_2 \in G$, $g_1[0] \cap g_2[F] \neq \emptyset$ implies $g_1[0] \subseteq g_2[F]$.

Definition 6.3: Let $(X, \delta)$ be a proximity space and let $G$ be a group of proximal isomorphisms from $X$ onto $X$. Then $G$ is said to be weakly transitive iff for every $\emptyset \neq 0 \in \mathcal{J}(\delta)$ there holds $X = \bigcup \{g[0]: g \in G\}$.

Definition 6.4: By a translation space we mean a proximity space $(X, \delta)$ together with a weakly transitive group $G$ of proximal isomorphisms from $X$ onto $X$ which satisfies Condition $A^*$. The symbol $(X, \delta, G)$ will be used to denote a translation space. For brevity, we will sometimes use the notation $\mathcal{J}^*$ for the non-empty members of $\mathcal{J}(\delta)$.
As an example of a translation space, consider the real numbers \( X \) with \( \delta \) the proximity relation induced by the usual metric on \( X \). Let \( G \) be all rational translations; that is, \( g \in G \) iff \( g(x) = x + r \) for some rational number \( r \).

**Definition 6.5:** Let \((X, \delta, G)\) be a translation space. Then a set \( E \subset X \) is said to be totally bounded iff given \( \emptyset \neq 0 \in J(\delta) \) there are \( g_1, \ldots, g_n \in G \) such that \( E \subset \bigcup \{ g_i[0] : 1 \leq i \leq n \} \).

**Definition 6.6:** A translation space \((X, \delta, G)\) is said to be locally totally bounded iff each \( x \in X \) is in some totally bounded open set. Since \( G \) is weakly transitive, this is equivalent to the existence of a non-empty totally bounded open subset of \( X \). For brevity, we will sometimes use the notation \( J^{**} \) for the non-empty totally bounded members of \( J(\delta) \).

**Remark 6.7:** Let \((X, \delta, G)\) be a translation space. The following are immediate consequences of the definition of totally bounded:

1. For all \( x \in X \), \( \{ x \} \) is totally bounded.
2. Subsets of totally bounded sets are totally bounded.
3. Finite unions of totally bounded sets are totally bounded.
4. If \( E \subset X \) is totally bounded, then \( g[E] \) is totally bounded for all \( g \in G \).
Theorem 6.8: Let \((X, \delta, G)\) be a locally totally bounded translation space. Then every totally bounded subset of \(X\) is proximally contained in a totally bounded open subset of \(X\).

Proof: Let \(E \subset X\) be totally bounded. If \(E = \emptyset\), then \(E \subset E\) and \(E\) is a totally bounded open set, so suppose \(E \neq \emptyset\). Now since \((X, \delta, G)\) is locally totally bounded there are \(0, 0^* \in \mathfrak{J}^*\) such that \(0 \subset O^*\). Then since \(E\) is totally bounded there are \(g_1, \ldots, g_n \in G\) such that 
\[
E \subset \bigcup \{g_i[0]: 1 \leq i \leq n\}. \]
Since \(G\) consists of proximal isomorphisms we have that \(g_i[0] \subset g_i[0^*]\) for each \(i, 1 \leq i \leq n\), and hence 
\[
\bigcup \{g_i[0]: 1 \leq i \leq n\} \subset \bigcup \{g_i[0^*]: 1 \leq i \leq n\}. \]
Thus \(E \subset \bigcup \{g_i[0^*]: 1 \leq i \leq n\}\), which by Remark 6.7 is a totally bounded open set.

Definition 6.9: Let \((X, \delta, G)\) be a translation space. Then a mapping \(\phi\) from the subsets of \(X\) to the extended real numbers is said to be an Alexandroff measure on \(X\) iff:

[A1] \(0 \leq \phi(A)\) for all \(A \subset X\).

[A2] \(A^* \neq \emptyset\) implies \(\phi(A) > 0\).

[A3] There is \(\emptyset \neq E^0 \subset X\) such that \(\phi(E) < \infty\).

[A4] \(A \subset B \subset X\) implies \(\phi(A) \leq \phi(B)\).

[A5] For all \(A, B \subset X\), \(\phi(A \cup B) \leq \phi(A) + \phi(B)\).

[A6] For all \(A, B \subset X\) such that \(A \nsubseteq B\) there holds 
\[
\phi(A \cup B) = \phi(A) + \phi(B).\]

[A7] For all \(E \subset X\) and all \(g \in G\), \(\phi(g[E]) = \phi[E]\).
Since $\emptyset \subseteq \emptyset$, it follows from [A3],[A4], and [A6] that $\varphi(\emptyset) = 0$.

**Theorem 6.10:** Let $(X,\delta,G)$ be a translation space, and suppose that $(X,\delta,G)$ is not locally totally bounded. Then there cannot exist an Alexandroff measure on $X$.

**Proof:** On the contrary, suppose that $\varphi$ is an Alexandroff measure on $X$. Then by [A4] and [A3] there is $0' \in J^*$ such that $\varphi(0') < \infty$. Now there is $0^* \in J^*$ such that $0^* \subseteq 0'$. Then using Condition A* there is $0_1 \in J^*$ such that for all $g \in G$, $g[0_1] \cap 0^* \neq \emptyset$ implies $g[0_1] \subseteq 0'$.

Now since $(X,\delta,G)$ is not locally totally bounded, $0^*$ cannot be totally bounded. Thus there is $0_2 \in J^*$ such that the family $\{g[0_2] : g \in G\}$ contains no finite subfamily which covers $0^*$. Moreover, by the weak transitivity of $G$ we may assume that $0_2 \subseteq 0_1$. Now find $0_3 \in J^*$ such that $0_3 \subseteq 0_2$.

Again using Condition A* there is $0 \in J^*$ such that for all $g, g' \in G$, $g[0] \cap g'[0_3] \neq \emptyset$ implies $g[0] \subseteq g'[0_3]$. We may assume that $0 \subseteq 0_3$. Hence, we have in particular that for all $g, g' \in G$, $g[0] \cap g'[0_3] \neq \emptyset$ implies $g[0] \subseteq g'[0_2]$.

Now by the weak transitivity of $G$ there is $g_1 \in G$ such that $g_1[0] \cap 0^* \neq \emptyset$. We assert there is $g_2 \in G$ such that $g_2[0] \cap 0^* \neq \emptyset$ and $g_2[0] \cap g_1[0] = \emptyset$. For if not we have $0^* \subseteq \bigcup \{g[0] : g[0] \cap 0^* \neq \emptyset\} \subseteq \bigcup \{g[0] : g[0] \cap g_1[0] \neq \emptyset\} = g_1[0_2]$, which is impossible by the choice of $0_2$.

Now suppose that $g_1, \ldots, g_n \in G$ have been found such that $g_i[0] \cap 0^* \neq \emptyset$ for $1 \leq i \leq n$ and $g_i[0] \cap g_j[0] = \emptyset$ for
We assert there is $g_{n+1} \in G$ such that $g_{n+1}[0] \cap O^* \neq \emptyset$ and $g_{n+1}[0] \cap g_j[0] = \emptyset$ for $1 \leq j \leq n$.

For if not we have:

$$\emptyset \subset \bigcup \{g[0]: g[0] \cap 0^* \neq \emptyset\} \subset \bigcup \{g[0]: g[0] \cap g_j[0] \neq \emptyset\} \subset \bigcup \{g_j[0]: 1 \leq j \leq n\},$$

which is impossible. Hence, we may inductively choose a sequence $\{g_n\}_{n=1}^\infty \subset G$ such that $g_n[0] \cap 0^* \neq \emptyset$ for all $n \geq 1$ and $g_i[0] \cap g_j[0] = \emptyset$ for $i \neq j$.

Since $0 \subset O_1$, we see by the choice of $O_1$ that $g_n[0] \subset O'$ for all $n \geq 1$. Now find $O'' \in 3^*$ such that $O'' \subset 0$. Then $g_n[O''] \subset O'$ for all $n \geq 1$ and $g_i[O''] \cap g_j[O'']$ for $i \neq j$, since the $g_n$ are proximal isomorphisms. Thus given $k \geq 1$:

$$\varphi(O') \geq \varphi(\bigcup \{g_i[O'']: 1 \leq i \leq k\} = \sum \varphi(g_i[O'']): 1 \leq i \leq k\}
= \sum \varphi(O''): 1 \leq i \leq k\} = k\varphi(O'').$$

Since $k$ was arbitrary and $\varphi(O') < \infty$ we must have $\varphi(O'') = 0$. But this contradicts [A2] since $O'' \neq \emptyset$. Hence there cannot exist an Alexandroff measure on $X$.

**Definition 6.11:** Let $(X, \delta, G)$ be a translation space. Then a mapping $\sigma$ from the non-empty open sets $3^*$ to the extended real numbers is said to be an Alexandroff pre-measure on $X$ iff:

- $[\text{AP1}]$ \quad $\sigma(O) > 0$ for all $O \in 3^*$.
- $[\text{AP2}]$ \quad $O_1, O_2 \in 3^*$ and $O_1 \subset O_2$ imply $\sigma(O_1) \leq \sigma(O_2)$. 

[AP3] There is $0^* \in \mathcal{G}^*$ such that $\sigma(0^*) < \infty$.

[AP4] For all $0_1, 0_2 \in \mathcal{G}^*$ there holds
$$\sigma(0_1 \cup 0_2) \leq \sigma(0_1) + \sigma(0_2).$$

[AP5] For all $0_1, 0_2 \in \mathcal{G}^*$ such that $0_1 \not\subseteq 0_2$ there
holds $\sigma(0_1 \cup 0_2) = \sigma(0_1) + \sigma(0_2)$.

[AP6] For all $0 \in \mathcal{G}^*$ and all $g \in G$, $\sigma(g[0]) = \sigma(0)$.

The same proof as in Theorem 6.10 show that there
cannot exist an Alexandroff pre-measure on $X$ unless $(X, \delta, G)$
is locally totally bounded.

**Theorem 6.12:** Let $(X, \delta, G)$ be a locally totally bounded
translation space, and let $\sigma$ be an Alexandroff pre-measure
on $X$. Then for $E \subseteq X$ define:
$$\varphi(E) = \inf \{ \sigma(0) : E \subseteq 0 \}.$$  

Then $\varphi$ is an Alexandroff measure on $X$.

**Proof:** First note that $\sigma(0) \leq \varphi(0)$ for all $0 \in \mathcal{G}^*$.

It is clear that $\varphi$ satisfies [A1], [A2], and [A4].

Now if $A \subseteq 0_1$ and $B \subseteq 0_2$, then $(A \cup B) \subseteq (0_1 \cup 0_2)$.

[[A5]] follows from this.

To show that $\varphi$ satisfies [A6], let $A, B \subseteq X$ such that
$A \not\subseteq B$. We may assume that both $\varphi(A)$ and $\varphi(B)$ are finite,
for if not, $\varphi(A \cup B) = \infty$ by [A4], so that [A6] holds. Then
by [[A5]], $\varphi(A \cup B) < \infty$, so given $\varepsilon > 0$ there is $0 \in \mathcal{G}^*$ such
that $A \cup B \subseteq 0$ and $\sigma(0) \leq \varphi(A \cup B) + \varepsilon$. Now since $A \not\subseteq B$
there are $0_1, 0_2 \in \mathcal{G}^*$ such that $A \subseteq 0_1$, $B \subseteq 0_2$ and $0_1 \not\subseteq 0_2$.
Then $A \subseteq 0 \cap 0_1$, $B \subseteq 0 \cap 0_2$, and $(0 \cap 0_1) \not\subseteq (0 \cap 0_2)$. 


Then we have $\varphi(A) + \varphi(B) \leq \sigma(0 \cap O_1) + \sigma(0 \cap O_2) = \\
\sigma((0 \cap O_1) \cup (0 \cap O_2)) \leq \sigma(0) \leq \sigma(A \cup B) + \varepsilon$. The result then follows since $\varepsilon > 0$ was arbitrary.

To show that $\varphi$ satisfies [A3], let $E$ be any totally bounded subset of $X$ with non-empty interior, and let $O^* \in \mathcal{F}^*$ be such that $\sigma(O^*) < \omega$. By Theorem 6.8 there is a totally bounded open set $O$ such that $E \subseteq O$. Now there are $g_1, \ldots, g_n \in G$ such that $0 \subseteq \bigcup \{g_i[O^*]: 1 \leq i \leq n\}$. Thus $\varphi(E) \leq \sigma(0) \leq \sum \{\sigma(g_i[O^*]): 1 \leq i \leq n\} = n\sigma(O^*) < \omega$.

Now for $E \subseteq X$ and $g \in G$ we have $\varphi(g[E]) = \inf \{\sigma(O): g[E] \subseteq O\} = \inf \{\sigma(g^{-1}[O]): g[E] \subseteq O\} = \inf \{\sigma(O): g[E] \subseteq g[0]\} = \inf \{\sigma(O): E \subseteq O\} = \varphi(E)$. Thus, $\varphi$ also satisfies [A7].

We now prove that there exists an Alexandroff measure on an arbitrary locally totally bounded translation space. Our procedure is to construct an Alexandroff pre-measure on $X$, using the standard Weil proof (see Weil[16] and Steinlage [14]), and then apply the preceding theorem. This procedure is broken up into a series of lemmas.

Lemma 6.13: Let $(X, \mathcal{F}, G)$ be a locally totally bounded translation space. Then for $0, A \in \mathcal{F}^{**}$ define:

$$t(0, A) = \min \{n: \text{there is a family } \{g_i: 1 \leq i \leq n\} \subseteq G \text{ such that } 0 \subseteq \bigcup \{g_i[A]: 1 \leq i \leq n\}\}.$$

Then $t$ satisfies:

(1) $0 < t < \omega$. 
(2) \( 0_1 \subset 0_2 \) implies \( t(0_1, A) \leq t(0_2, A) \) for all \( A \).

(3) \( t(0_1 \cup 0_2, A) \leq t(0_1, A) + t(0_2, A) \) for all \( 0_1, 0_2, A \).

(4) \( t(g[0], A) = t(0, A) \) for all \( g \in G \), all \( 0, A \).

(5) \( t(0, A) \leq t(0, A^*) + t(A^*, A) \) for all \( 0, A, A^* \).

(6) Given \( 0_1, 0_2 \in \mathcal{J}^{**} \) such that \( 0_1 \not\subset 0_2 \) there is \( A^* \in \mathcal{J}^{**} \) such that if \( A \in \mathcal{J}^{**} \) and \( A \subset A^* \) then

\[
t(0_1 \cup 0_2, A) = t(0_1, A) + t(0_2, A).
\]

**Proof:** (1)-(4) are obvious.

If \( 0 \subset \bigcup \{ g_i[A^*] : 1 \leq i \leq n \} \) and \( A^* \subset \bigcup \{ g_j^*[A] : 1 \leq i \leq m \} \)

then \( 0 \subset \bigcup \{ g_i^*[A] : 1 \leq i \leq n, 1 \leq j \leq m \} \). (5) follows from this.

In (6), let \( 0' \) be as given by Condition \( A^* \) for the sets \( 0_1 \) and \( 0_2 \). Then any non-empty totally bounded open set \( A^* \subset 0' \) satisfies the desired property.

**Lemma 6.14:** Let \( (X, \delta, G) \) be a locally totally bounded translation space. Fix a non-empty totally bounded open set \( A_0 \). Then for \( A \in \mathcal{J}^{**} \) define \( t_A \) on \( \mathcal{J}^{**} \) by:

\[
t_A(0) = t(0, A) / t(A_0, A).
\]

Then \( t_A \) satisfies:

(1) For all \( 0_1 \subset 0_2 \), \( t_A(0_1) \leq t_A(0_2) \).

(2) For all \( 0_1, 0_2 \in \mathcal{J}^{**} \), \( t_A(0_1 \cup 0_2) \leq t_A(0_1) + t_A(0_2) \).

(3) For all \( g \in G \), all \( 0 \in \mathcal{J}^{**} \), \( t_A(g[0]) = t_A(0) \).

(4) \( 1 / t_A(0) \leq t_A(0) \leq t(0, A_0) \) for all \( 0 \in \mathcal{J}^{**} \).

(5) For all \( 0_1, 0_2 \in \mathcal{J}^{**} \) such that \( 0_1 \not\subset 0_2 \) there is \( A^* \in \mathcal{J}^{**} \) such that if \( A \in \mathcal{J}^{**} \) and \( A \subset A^* \) then

\[
t_A(0_1 \cup 0_2) = t_A(0_1) + t_A(0_2).
\]
Proof: Now we have that \( t(A_o, A) \leq t(A_o, 0)t(0, A) \leq t(A_o, 0)t(0, A_0)t(A_0, A) \), so by dividing by \( t(A_o, 0)t(A_0, A) \) we obtain (4). The other assertions are obtained by dividing the corresponding assertion of Lemma 6.13 by \( t(A_o, A) \).

Lemma 6.15: Let \((X, \delta, G)\) be a locally totally bounded translation space. Then there is a mapping \( L \) from the non-empty totally bounded open sets \( \mathcal{J}^* \) to the real numbers which satisfies:

1. \( 0 < L < \infty \).
2. \( L(0_1 \cup 0_2) \leq L(0_1) + L(0_2) \), for all \( 0_1, 0_2 \).
3. \( 0_1 \not\subseteq 0_2 \) implies \( L(0_1 \cup 0_2) = L(0_1) + L(0_2) \).
4. \( 0_1 \subseteq 0_2 \) implies \( L(0_1) \leq L(0_2) \).
5. For all \( g \), all \( 0 \), \( L(g[0]) = L(0) \).

Proof: For each \( 0 \in \mathcal{J}^* \) let \( X_0 = [1/t(A_o, 0), t(0, A_0)] \), a closed interval, and let \( \mathcal{J}_0 \) be the usual topology on \( X_0 \). Then let \( X^* = \prod \{X_0: 0 \in \mathcal{J}^*\} \). By the Tychonoff Theorem, \( X^* \) with the product topology is a compact Hausdorff space. Clearly, \( t_A \in X^* \) for all \( A \in \mathcal{J}^* \). Then for each \( A^* \in \mathcal{J}^* \) let \( K(A^*) = \{t_A: A \subseteq A^*, A \in \mathcal{J}^*\} \). Letting \( c \) denote the closure operator in \( X^* \), we obtain for \( A_1, \ldots, A_n \in \mathcal{J}^* \) that:

\[
\emptyset \neq cK(\bigcap_{i=1}^n A_i) \subseteq c(\bigcap_{i=1}^n K(A_i)) \subseteq \bigcap_{i=1}^n cK(A_i).
\]

Thus the family \( \{cK(A): A \in \mathcal{J}^*\} \) satisfies the finite intersection property, so since \( X^* \) is compact, there is an \( L \in \cap \{cK(A): A \in \mathcal{J}^*\} \). Clearly, \( L \) satisfies (1).
Now for each $0 \in \mathcal{F}^*$, the projection map $p_0$, defined on $X^*$ by $p_0(s) = s(0)$ for $s \in X^*$, is continuous. ($s(0)$ is the $0^{th}$ co-ordinate of the point $s$).

Now for all $A_1, A_2 \in \mathcal{F}^*$ the set
$$S(A_1, A_2) = \{ s \in X^* : s(A_1) = p_{A_1}(s) \leq p_{A_2}(s) = s(A_2) \}$$
in closed in $X^*$, since the projection maps are continuous. If $A_1 \subseteq A_2$, then $t_A \in S(A_1, A_2)$ for all $A \in \mathcal{F}^*$. Thus for $A \in \mathcal{F}^*$ we have $cK(A) \subseteq S(A_1, A_2)$, in this case. Hence $L \in S(A_1, A_2)$; that is, $L(A_1) \leq L(A_2)$. Since $A_1, A_2 \in \mathcal{F}^*$ were arbitrary, we have that $L$ satisfies (4).

Similarly, the sets
$$R(A') = \{ s \in X^* : s(A') = s(g[A']) \text{ for all } g \in G \}$$
and
$$Q(A_1, A_2) = \{ s \in X^* : s(A_1 \cup A_2) \leq s(A_1) + s(A_2) \}$$
defined for $A', A_1, A_2 \in \mathcal{F}^*$, are closed in $X^*$ and contain $t_A$ for every $A \in \mathcal{F}^*$. It follows as before that $L \in R(A')$ for all $A' \in \mathcal{F}^*$ and $L \in Q(A_1, A_2)$ for all $A_1, A_2 \in \mathcal{F}^*$, and so $L$ satisfies (2) and (5).

Now if $A_1 \not\subseteq A_2$ there is $A^* \in \mathcal{F}^*$ such that if $A \in \mathcal{F}^*$ and $A \subseteq A^*$, then $t_A(A_1 \cup A_2) = t_A(A_1) + t_A(A_2)$. Thus
$$W(A_1, A_2) = \{ s \in X^* : s(A_1 \cup A_2) = s(A_1) + s(A_2) \}$$
is a closed set in $X^*$ and contains $t_A$ for all $A \subseteq A^*$. Hence, $L \in cK(A^*) \subseteq W(A_1, A_2)$, and so $L$ satisfies (3).

Remark 6.16: Let $(X, 0, G)$ be a locally totally bounded translation space, and let $L$ be as in Lemma 6.15. Then define a mapping $\sigma$ from $\mathcal{F}^*$ to the extended real numbers by:
\[ \sigma(0) = L(0) \text{ if } 0 \text{ is totally bounded,} \]
\[ = \infty \text{ if } 0 \text{ is not totally bounded.} \]

It is easy to verify that \( \sigma \) is an Alexandroff pre-measure on \( X \). Then using Theorems 6.10 and 6.12 we obtain:

**Theorem 6.17:** Let \((X, \delta, G)\) be a translation space. Then there exists an Alexandroff measure on \( X \) iff \((X, \delta, G)\) is locally totally bounded.
BIBLIOGRAPHY


