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INTEGRAL BASES IN BANACH SPACES

DISSERTATION

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the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

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INTRODUCTION

In [6], R. E. Edwards introduced the concept of integral bases for topological vector spaces. This is an extension of the concept of series bases for such spaces. One major difference between integral bases and other extensions of this notion (Markushevich bases, generalized bases, etc.) is the lack of discreteness of the index set. As in some other extensions (extended bases, for instance), the index set is also not assumed to be discrete. Because of the topological structure which is put on them, one is able to consider these larger index sets and still retain an idea of converging partial sums (partial integrals).

In place of a sequence of scalars corresponding to an element $x \ [x = \sum a_i x_i \leftrightarrow (a_i)]$ we use a measure, $\mu_x$, on the index set. The general setting of Edwards' work is as follows: Let $F$ be a locally convex topological vector space and $E$ a subspace whose topology is at least as strong as its relative $F$-topology. The index set is a locally compact, usually Hausdorff, space $T$, on which is defined an $E$-valued index function $u$. Our object is to represent each $x \in E$ as an integral, $\int_T u \, d\mu$, for some measure $\mu$ on $T$. Some

(1) Throughout this paper, we shall omit the indices from infinite sequences, series, products, etc., unless some confusion could arise.
interpretation must be chosen for these integrals. If $K$ is a
$\mu$-measurable subset of $T$, then the WF integral $\int_K u \, d\mu$ exists whenever
the map $t \rightarrow \langle u(t), f \rangle$ is $\mu$-integrable over $K$ for each $f \in F^*$. Here
$\langle \cdot, \cdot \rangle$ is the pairing of $F$ and its dual, $F^*$.

The value of such an integral is an element of $(F^*)^\#$ which is given by

$$(\int_K u \, d\mu) \cdot f = \int_K \langle u, f \rangle \, d\mu$$

for all $f \in F^*$. Under suitable conditions, this integral will be an
element of $F$. We interpret $\int_T u \, d\mu$ as the limit of a sequence

$$x_n = \int_{K_n} u \, d\mu$$

where $(K_n)$ is a fixed increasing sequence of compact
subsets of $T$. The measures to be used are chosen from some locally
convex space of measures on $T$.

**Definition 0.1.** Suppose $\{ u, T, (K_n), M \}$ are given as
above. Further suppose

(I) $\int_{K_n} u \, d\mu \in F$ for each $n, \mu \in M$.

(II) There is a total subset $S$ of $F^*$ such that the linear map

$$\mu \rightarrow \int_{K_n} \langle u, f \rangle \, d\mu$$

is continuous on $M$ for each $n, f \in S$.

---

(2) We shall denote by $F^*$ the topological (continuous) dual of $F$
and by $F^\#$ the algebraic dual.
(III) For each \( x \in E \) there exists a unique \( \mu_x \in M \) such that
\[
S_n(x) = \int_{K_n} u d\mu_x \quad \text{for each } n
\]
and, in the topology of \( E \),
\[
x = \lim_{n \to \infty} S_n(x).
\]
Then we say \( \{ u, T, (K_n), M \} \) is an integral basis for \( E \).

This integral basis differs from the L basis of Dyer [5] in several ways. Dyer points out that his bases, which consist of linearly ordered subsets of the topological space \( E \), are distinct from the usual series (Schauder) bases in that neither is a special case of the other. A series basis is, however, a special case of the integral basis (let \( T = N \) and \( K_n = \{ 1, \ldots, n \} \) with \( u(j) = x_j \) if \( (x_j) \) is a series basis). Further, the set over which the integration occurs in Dyer's bases is a subset of \( E \) and its topology, which comes from its order structure, need not be locally compact.

In his paper, Edwards proves several results in the setting of inductive limit spaces. In particular, he gives analogues to several standard series basis theorems. Included in the work are the Banach-Neuwns theorem, the Arsove similar bases theorem and the theorem relating Schauder bases for the weak topology to Schauder bases for the original topology on \( E \).

**Theorem 0.2. (Banach-Neuwns)** Suppose \( E \) is the strict inductive limit of Fréchet spaces \( (E_{\alpha}) \), that conditions I-III of definition 0.1 hold and that \( M \) is a Fréchet space. Then
(i) The mapping \( x \rightarrow \mu_x \) is continuous from \( E \) into \( M \).

(ii) The mappings \( (S_n) \), are equicontinuous from \( E \) into itself.

Theorem 0.3. Suppose \( E \) is the external inductive limit of a sequence of Fréchet spaces \( (E_n) \) and maps \( (\lambda_n) \) such that

\[
E = \bigcup_n \lambda_n(E_n),
\]

that \( M \) is any topological vector space of measures on \( T \) such that \( \{ u, T, (K_n), M \} \) is a weak integral basis for \( E \), that the map \( x \rightarrow \mu_x \) is continuous from \( E \) into \( M \) and that for each \( n, \xi_{K_n} \cdot \mu \in M \)

for \( \mu \in M \). Then \( \{ u, T, (K_n), M \} \) is an integral basis for \( E \) in its original topology.

Remark 0.4. We list here three properties of strict inductive limits of Fréchet spaces which are essential in the proofs of Edwards' results. Following his notation, we list them as A1-A3.

(A1) \( E \) is the internal inductive limit of a countable increasing directed family \( (E_n) \) of Fréchet spaces, each \( E_n \) being sequentially closed in \( E \).

(A2) Each bounded subset of \( E \) is bounded in \( E_n \), for some \( n \) depending on the set involved.

(A3) If \( (x_n) \) is a bounded sequence in \( E_j \) and Cauchy in \( E \) then it converges in \( E \) to some \( x \in E_j \) and \( p_j(x) \leq \sup_n p_j(x_n) \) for each \( p_j \) in a defining family of seminorms for the topology of \( E_j \).
Edwards notes that these conditions are enough to insure the validity of theorem 0.2. Since there are spaces E which satisfy A1-A3 but are not strict inductive limits of Fréchet spaces, theorem 0.2 can be slightly generalized. We now also get the following results.

**Theorem 0.5.** Let E satisfy A1-A3 and have integral basis \{ u, T, (K_n), M \} with M a Fréchet space. Then \( x \to \mu_x \) is an isomorphism of E onto M.

**Corollary 0.6.** If E, F satisfy A1-A3 and have integral bases \{ u, T, (K_n), M \}, \{ v, T, (C_n), M \}, respectively, with M a Fréchet space, then E is isomorphic to F.

In this work we shall confine our attention to integral bases in Banach spaces. In this setting some other familiar theorems of basis theory are also obtained. The first chapter deals with the existence of a basis constant as in the classic characterization theorem for series bases.

**Theorem 0.7.** Let X be a Banach space and \( (x_n) \) a sequence of elements of X. Then \( (x_n) \) is basic (is a basis for \( \text{sp} \{ x_n \} \)) if and only if there exists a constant \( K \geq 1 \) such that for any positive integers n and p and any scalar sequence \( (a_i) \) we have

\[
\| \sum_{i=1}^{n} a_i x_i \| \leq K \| \sum_{i=1}^{n+p} a_i x_i \|.
\]

**Definition 0.8.** \( \{ X_k ; E_k \} \) is a Schauder decomposition of a Banach space X if \( (E_k) \) is an equicontinuous family of projections.
on $X$ with $E^i_k(X) = X_k$ and $X_k \cap X_j = \emptyset$ if $k \neq j$ and such that for each $x \in X$,

$$x = \lim_{n \to \infty} \sum_{j=1}^{n} E_j(x).$$

We shall write $X = \bigoplus \sum X_i$.

In the theorem above, we get a natural method for constructing a Schauder decomposition for $X$ which corresponds to the given series basis $\{ E_j \mid \sum a_j x_j \}$. From the analogue to theorem 0.7, we get the same type of decomposition.

The second chapter is made up of three parts. The first part concerns Schauder decompositions $\{ X_k ; E_k \}$ with each $X_k$ finite dimensional. We show that any Banach space with such a decomposition has an integral basis which corresponds to the decomposition.

Another familiar theorem from the theory of series bases is the following.

**Theorem 0.9.** Let $X$ be a Banach space with Schauder decomposition $\{ X_k ; E_k \}$. Suppose each $X_k$ has a basis $(x^j_k)^{j \geq 1}$ whose basis constant is $N_k$ (see theorem 0.7). Finally, assume there is an $N$ such that $N_k \leq N$ for every $k$. Then $X$ has a basis.

We give the integral basis analogue to this theorem in part two of the second chapter: If a Banach space $X$ has Schauder decomposition $\{ X_k ; E_k \}$ with each $X_k$ having an integral basis,
then, under the assumption that the corresponding basis constants are bounded, we show X has an integral basis.

The final part of chapter two is devoted to an investigation of the structure of the space of measures associated with an integral basis. If a Banach space X has an integral basis \( \{ u, T, (K_n), M \} \) then M is the completion with respect to some norm \( \rho \) of the space \( U \cup \text{rca}(K_n) \). Based on this and using a result of D. W. Dean [3], we are able to give an example of a Banach space with a Schauder decomposition which does not come from an integral basis.

We have seen that an integral basis gives rise to a Schauder decomposition but that the converse does not hold. We might ask, therefore, do certain special Schauder decompositions always give us integral bases? The first two parts of chapter two give such special cases. Using the results of the last part of chapter two, we look at the space \( U \cup \text{rca}(K_n) \), and some completion of it, for an increasing sequence of compact sets \( (K_n) \). We show that if X is a Banach space with Schauder decomposition \( \{ X_k; E_k \} \) and \( \bigoplus_{i=1}^{n} X_i \) is isomorphic to \( \text{rca}(K_n) \) for all \( n \) with \( K_n \subset \text{Int} K_{n+1} \) (\( \text{Int} A \) is the interior of A), then X has an integral basis. The assumption that \( K_n \subset \text{Int} K_{n+1} \) is used to get the correct representation and may hopefully be dropped if a better method of proof can be found. This is the material comprising the first part of chapter three.
In the second part of chapter three, we drop one of the assumptions we will make on the index function, u. Without assuming that u is bounded on each $K_n$, we show that any Banach space has an integral basis.

**Definition 0.9.** Let $X$ be a Banach space with Schauder decomposition $\{X_k; E_k\}$.

(i) $\{X_k; E_k\}$ is said to be shrinking if, for every $f \in X^*$, 
$$
||f|| [X_{n+1}, X_{n+2}, \ldots] || = ||f||_n \to 0 \text{ as } n \to \infty,
$$
where $[X_{n+1}, X_{n+2}, \ldots]$ denotes the closed linear span of $\{X_{n+1}, X_{n+2}, \ldots\}$.

(ii) $\{X_k; E_k\}$ is said to be boundedly complete if, whenever $(x_j) \in \prod X_j$ such that 
$$
sup_n \left|\sum_{j=1}^{n} x_j \right| < \infty
$$
then $\sum x_j$ converges.

We say an integral basis is shrinking (boundedly complete) if the associated Schauder decomposition is shrinking (respectively, boundedly complete). Chapter four contains a discussion of integral bases with these properties, especially with regard to reflexivity.

B. L. Sanders [11] has shown that if a Banach space $X$ has Schauder decomposition $\{X_k; E_k\}$ with each $X_k$ reflexive, then $X$ is reflexive if and only if $\{X_k; E_k\}$ is both boundedly complete and
shrinking. This result is a natural extension of R. C. James' result for a space with a series basis [7]. M. Zippin [12] has strengthened some earlier results of I. Singer and A. Pelczynski [10]. Zippin shows that if X has a basis \( \langle x_j \rangle \) and if all bases of X are shrinking (or all bases are boundedly complete) then X is reflexive. In the fourth chapter, we use and try to copy these results for spaces with integral bases. The result of Sanders applies directly and by the use of bases with parentheses, we can extend Zippin's result to some spaces with integral bases.

The paper concludes in the second part of chapter four with the posing of several questions which arise from this work. Some are analogues of familiar theorems, the proofs of which are so similar to the proofs of the model theorems that they are left out or indicated only briefly. The Paley-Wiener stability theorem for series bases and a further characterization of reflexivity are included in the discussion. Other questions are posed for which the author has no answer, but which ought to be solved.
THE BASIC THEOREM

In this chapter we prove the analogue of theorem (0.7) for integral bases in Banach spaces. We will show that a system \{u, T, (K_n), M\} is an integral basis for X if and only if there is a basis constant for the system. We take for our basic spaces a Banach space X and a locally compact Hausdorff space T. No distinction will be made between X and its canonical embedding in its second dual X**.

We suppose that T is the union of an increasing sequence of compact subsets, \( (K_n) \).

We build a space of "coefficient sequences", actually a space of measures, in the following manner. For each \( n \geq 1 \), let \( M(K_n) \) be a Banach space of regular Borel measures on T whose supports lie in \( K_n \). The norm, \( \| \cdot \|_n \), on \( M(K_n) \) is assumed to generate a topology which is at least as strong as that generated by the restriction of the total variation norm, \( |\cdot| \), to \( M(K_n) \). We define a metric, \( \rho \), on the space \( \bigcup M(K_n) \) and let \( M_1 \) be the completion of this space with respect to \( \rho \).

**Definition 1.1.** For \( \mu \in \bigcup M(K_n) \) let

\[
\rho(\mu) = \rho(\mu, 0) = \sum 2^{-n} \frac{\| \xi_{K_n} \mu \|_n}{1 + \| \xi_{K_n} \mu \|_n}
\]

\[
\rho(\mu, \nu) = \rho(\mu - \nu, 0) = \rho(\mu - \nu).
\]

Here, for all measurable subsets, \( A \), of T, we take, for each \( j \geq 1 \)

\[
\xi_{K_j} \mu(A) = \mu(A \cap K_j).
\]
Finally, we suppose we have any $X$-valued function $u$, defined on $T$, with one special property. We suppose that, for each $n \geq 1$, the number

$$||| u |||_n = \sup_{t \in K_n} || u(t) ||$$

is finite. Here, $||\cdot||$ designates the norm on $X$. We may note here that this condition is always satisfied in the series case since we use for $K_n$ the set $\{1, 2, \ldots, n\}$. The stronger assumption of continuity or even weak continuity of $u$, also satisfied in the series case because the index set is discrete, seems to be too strong. We are able to obtain the results of this chapter without this condition being imposed and the condition is hard to recover when we try to construct integral bases. On the other hand, if we make no assumption on $u$, the system is too weak. We will show, in fact, that with no restrictions on $u$, we are able to obtain far too many Banach spaces with bases and that the results of this chapter cannot be proved.

The assumptions made above on the system $(u, T, (K_n), M)$ with $M \subseteq M_1$, are referred to as the "general hypotheses".

All integrals $\int_A u \, d\mu$ are to be taken as WF integrals as in Edwards [6]. That is, we say this integral exists if $t \to \langle u(t), f \rangle$ is $\mu$-integrable for all $f \in X^*$ where $\langle \cdot, \cdot \rangle$ is the pairing of $X$ and its dual $X^*$. The integral is an element of $X^{**}$ given by

$$\left( \int_A u \, d\mu \right) f = \int_A \langle u, f \rangle \, d\mu.$$ 

When we say $\int_A u \, d\mu \in X$, we mean that the integral is an element of the canonical image of $X$ in $X^{**}$, which we identify with $X$. 
**Definition 1.2.** Let $X$ be a Banach space and $(u, T, (K_n), M)$ be a system satisfying the general hypotheses as above. If

(I) $\int_{K_n} u \, d\mu \in X$ for every $n$ and $\mu \in M$; 

(II) there exists a total subset $S \subset X^*$ such that, for each $n$ and $f \in S$, the linear map

$$\mu \mapsto \int_{K_n} \langle u, f \rangle \, d\mu$$

is continuous on $M$; 

(III) for each $x \in X$ there exists a unique $\mu_x \in M$ such that

$$S_n(x) = \int_{K_n} u \, d\mu_x \in X$$

for all $n$ and such that

$$\lim_{n \to \infty} \| x - S_n(x) \| = 0;$$

then we say $(u, T, (K_n), M)$ is an integral basis for $X$.

**Proposition 1.3.** Under the general hypotheses above, if:

(1) $L_n = \{ \int_{K_n} u \, d\mu \mid p \leq n, \mu \in M \}$ is a closed subspace of $X$ for each $n$,

(2) whenever

$$\int_{K_n} u \, d\mu = \int_{K_n} u \, d\nu$$

then we must have

$$\xi_{K_n} \cdot \mu = \xi_{K_n} \cdot \nu,$$
(3) there exists a constant $C \geq 1$ such that, for all positive integers $n$ and $p$ and any $\mu \in M$, we have

$$|| \int_{K_n} u d\mu || \leq C || \int_{K_{n+p}} u d\mu ||,$$

then $\{ u, T, (K_n), M \}$ is an integral basis for $\Lambda$, the completion in $X$ of $u L_n$.

**Proof of proposition.** For each $n$ and any $\mu \in M$ we have by definition

$$\int_{K_n} u d\mu \in L_n \subseteq \Lambda.$$ 

So condition (I) of the definition of integral basis is satisfied.

We shall take for a separating family the whole conjugate space, $X^*$. Fixing $f \in X^*$ and a positive integer $n_0$, we suppose the sequence $(\mu_j)$ converges to 0 in the $\rho$-topology of $M$. By the definition of $\rho$, we must have

$$\lim_{j \to \infty} || \varepsilon_{K_n} \mu_j ||_n = 0$$

for each $n$. Further, we have assumed that the $|| \cdot ||_n$-topology on $M(K_n)$ is stronger than the $|| \cdot ||$-topology ($|| \cdot ||$ is total variation norm). So we have, for each $n$,

$$\lim_{j \to \infty} || \varepsilon_{K_n} \mu_j || = 0.$$

Hence, for any $\eta > 0$, we may choose $N_\eta$ such that if $k \geq N_\eta$ we have

$$||\mu_k||_n(\varepsilon_{K_n}) = ||\varepsilon_{K_n} \mu_k||_n < \eta ||f||^{-1} ||u||^{-1}_n.$$

Then we have, for $k \geq N_\eta$
That is, the map
\[ \mu \rightarrow \int \langle u, f \rangle \, d\mu \]
is continuous for each \( f \in X^* \) and each \( n \geq 1 \). We have therefore satisfied condition (II) of the definition of integral basis.

We must finally show that, for each \( x \in X \), there exists a unique \( \mu_x \in M \) such that
\[ x = \lim_{j \to \infty} (\int_{K_j} u \, d\mu_{x_j}) = \int_{T} u \, d\mu_x. \]

To see that this condition [(III) of our definition] is fulfilled, we need the following several lemmas.

**Lemma 1.4.** For every \( x \in L = \bigcup L_n \), there exists a \( \mu_x \in M \) such that

1. \( x = \lim_{j \to \infty} (\int_{K_j} u \, d\mu_{x_j}) \)
2. the map \( \varphi \), given by \( \varphi(x) = \mu_x \), is linear from \( L \) into \( M \).

**Proof of lemma.** We first note that, for any integers \( n \) and \( p \), we have \( L_n \subseteq L_{n+p} \). For each \( x \in L \) define
\[ I(x) = \{ n \geq 1 \mid x \in L_n \}. \]
Since \( x \in L = \bigcup L_n \) we have \( I(x) \neq \emptyset \) and so we may define
\[ n(x) = \min \{ j \mid j \in I(x) \}. \]
We see that \( x \in L_{n(x)} \) so there is a \( \mu \in M \) such that
\[ x = \int_{K_{n(x)}} u \, d\mu. \]
Define:
\[ M(x) = \{ \mu \in M \mid x = \int_{K_n(x)} u \, d\mu \} \]

Then \( M(x) \neq \emptyset \) and if \( \mu \in M(x) \) and \( \nu \in M(x) \), we must have by assumption (2)
\[ \mathbb{E}_{K_n(x)} \mu = \mathbb{E}_{K_n(x)} \nu \tag{1.4.1} \]

We now define \( \varphi \) taking \( L \) into \( M \) by
\[ \varphi(x) = \mathbb{E}_{K_n(x)} \mu \text{ for some } \mu \in M(x). \]

By (1.4.1), \( \varphi \) is well-defined and we note that the support of \( \varphi(x) \) lies in \( K_n(x) \). To see that \( \varphi(x) = \mu_{\cdot y} \) is the required measure, fix \( x_0 \in L \) and let \( n \geq n(x_0) \). Since the support of \( \mu_{\cdot x_0} \) is contained in \( K_n(x_0) \), we have
\[ \int_{K_n(x)} u \, d\mu_{\cdot x_0} = \int_{K_n(x_0)} u \, d\mu_{\cdot x_0} = x_0. \]

So that

\[ \begin{aligned}
(1) \quad x_0 &= \lim_{j \to \infty} \int_{K_j} n \, d\mu_{\cdot x_0} = \int_{T} u \, d\mu_{\cdot x_0}.
\end{aligned} \]

Let \( x \) and \( y \) be any elements of \( L \). Then we have
\[ \varphi(x + y) = \mu_{\cdot x+y}. \]

Without loss of generality, we suppose \( n(x) \geq n(y) \).

(a) Suppose \( n(x) = n(y) \).

Since \( n(y) \leq n(x) \), we have \( y \in L_n(x) \) and so \( x + y \in L_n(x) \) and \( n(x + y) \leq n(x) \). The support of \( \mu_{\cdot x+y} \) is in \( K_n(x+y) \subset K_n(x) \) so we
have by assumption (2)

\[ \mu_{x+y} = \xi_{K_{n(x)}} \quad \mu_{x+y} = \xi_{K_{n(x)}} \quad (\mu_x + \mu_y) = \mu_x + \mu_y. \]

Thus, if \( n(x) = n(y) \), \( \varphi(x + y) = \varphi(x) + \varphi(y) \).

(b) Suppose \( n(x) > n(y) \).

If \( y \not\in L_{n(x+y)} \), then \( n(y) > n(x + y) \) so we see that

\[ x + y \in L_{n(y)} \]

and so \( x = (x + y) - y \in L_{n(y)} \) which gives us

\[ n(x) \leq n(y) < n(x). \]

This contradiction forces \( y \in L_{n(x+y)} \). But then

\[ x = (x + y) - y \in L_{n(x+y)} \]

so that \( n(x) \leq n(x + y) \). As we noted in

(a), we always have \( n(x) \geq n(x + y) \). Since this yields

\[ n(x) = n(x + y), \]

and since the support of \( \mu_y \) lies in \( K_{n(y)} \subset K_{n(x)} \),

we have

\[ \int_{K_{n(x+y)}} u \, d\mu_{x+y} = \int_{K_{n(x)}} u \, d\mu_{x+y} = x + y = \]

\[ = \int_{K_{n(x+y)}} u \, d(\mu_x + \mu_y). \]

Again using assumption (2), we obtain

\[ \mu_{x+y} = \xi_{K_{n(x)}} \quad \mu_{x+y} = \xi_{K_{n(x)}} \quad (\mu_x + \mu_y) = \mu_x + \mu_y \]

and so

\[ \varphi(x + y) = \varphi(x) + \varphi(y). \]  \hspace{1cm} (1.4.2)

Now let \( \alpha \neq 0 \) be any scalar and \( x \in L \). One sees

immediately that \( I(\alpha x) = I(x) \) and hence that \( n(\alpha x) = n(x) \). Thus

\[ \int_{K_{n(\alpha x)}} u \, d\mu_{\alpha x} = \int_{K_{n(x)}} u \, d\mu_{\alpha x} = \alpha x = \alpha \int_{K_{n(x)}} u \, d\mu = \]
By assumption (2), \( \mu \alpha_x = \varepsilon \cdot \mu \alpha_x = \varepsilon \cdot \mu \alpha_x = \mu \alpha_x \). This, together with (1.4.2) shows that \( \varphi \) is a linear map.

**Lemma 1.5.** The sequence of linear maps \( (S_n) \), given by

\[
S_n(x) = \int_{\mathbb{K}} u \, d\mu_x \quad \text{for } x \in L
\]

is an equicontinuous family. In fact, if \( C \) is as in (3) then for any \( n \geq 1 \), we have \( \| S_n \| \leq C \).

**Proof of lemma.** Let \( x = \int_{\mathbb{K}} u \, d\mu_x \in L \) and fix an \( m \geq 1 \).

If \( m < n(x) \), let \( p = n(x) - m \) and then

\[
\| S_m(x) \| = \| \int_{\mathbb{K}} u \, d\mu_x \| \leq C \| \int_{\mathbb{K}} u \, d\mu_x \| = C \| x \|.
\]

If \( m \geq n(x) \), then \( S_m(x) = x \) and so \( \| S_m(x) \| = \| x \| \). Since we have assumed that \( C \geq 1 \), \( \| S_m(x) \| \leq C \| x \| \). So we have, for any \( m \geq 1 \) and any \( x \in L \), \( \| S_m(x) \| \leq C \| x \| \). Thus, for any \( m \geq 1 \),\( \| S_m \| \leq C \).

**Corollary 1.5.** For every \( n \geq 1 \) there exists a unique linear map, \( \hat{S}_n \) from \( L \) into \( L \) with the following properties

(i) \( \hat{S}_n(x) = S_n(x) \) for all \( x \in L \)

(ii) \( x = \lim_{n \to \infty} \hat{S}_n(x) \) for all \( x \in L \)

(iii) \( \| \hat{S}_n \| = \| S_n \| \).
**Proof of corollary.** The existence of a sequence \((\hat{S}_n)\) of linear maps satisfying (i) and (iii) follows from lemma (1.4) and the principle of uniform boundedness. To see that (ii) holds, we note that, for \(x \in L\), we have \(S_n(x) = x\) for all \(n \geq n(x)\). Thus, for \(x \in L\), \(x = \lim_{n \to \infty} \hat{S}_n(x)\). Since \(L\) is dense in \(\Lambda\) and \(\hat{S}_n\) is the extension of \(\hat{S}_n\mid L\), (ii) holds.

**Lemma 1.6.** For each \(x \in L\), the corresponding \(\mu_\chi \in M\) of lemma (1.3) is unique.

**Proof of lemma.** We first note that for any integers \(n \geq p \geq 1\), we have

\[
\xi^K_n \cdot \mu = \xi^K_p \cdot \mu = \xi^K_p \cdot \mu.
\]

With this in mind, we note for any \(\mu \in M\)

\[
\rho(\mu, \xi^K_n \cdot \mu) = \sum_{j \geq 1} 2^{-j} \frac{|| \xi^K_j \cdot \mu - \xi^K_n \cdot \mu ||_j}{1 + || \xi^K_j \cdot \mu - \xi^K_n \cdot \mu ||_j} = \sum_{j \geq n + 1} 2^{-j} \frac{|| \xi^K_j \cdot \mu - \xi^K_n \cdot \mu ||_j}{1 + || \xi^K_j \cdot \mu - \xi^K_n \cdot \mu ||_j} \leq \sum_{j \geq n + 1} 2^{-j}
\]

for any positive integer \(n\). Thus we see that

\[
\lim_{n \to \infty} \rho(\mu, \xi^K_n \cdot \mu) = 0. \quad (1.6.1)
\]
Now, suppose there is an \( x_0 \in L \) and \( \mu_{x_0} \in M \) such that

\[
\int_T u \, d\mu_{x_0} = x_0 = \int_T u \, d\mu_{x_0}.
\]

Then we have

\[
x_0 = \hat{S}_n(x_0) \left( \int_T u \, d\mu_{x_0} \right) = \hat{S}_n(x_0) \left( \int_T u \, d\mu_{x_0} \right) = \hat{S}_n(x_0) \left( \lim_{k \to \infty} \int_T u \, d\mu_{x_0} \right) = \lim_{k \to \infty} \hat{S}_n(x_0) \left( \int_T u \, d\mu_{x_0} \right) = \lim_{k \to \infty} \int_{K_n} u \, d\mu_{x_0} = \int_{K_n} u \, d\mu_{x_0}.
\]

But \( x_0 = \int_{K_n} u \, d\mu_{x_0} \), so for any \( n \geq n(x_0) \),

\[
\int_{K_n} u \, d\mu_{x_0} = \hat{S}_n \left( \int_T u \, d\mu_{x_0} \right) = \hat{S}_n(x_0) = x_0 = \int_{K_n} u \, d\mu_{x_0}.
\]

Thus, by (2), we have for all \( n \geq n(x_0) \)

\[
\xi_{K_n} \cdot u_{x_0} = \xi_{K_n} \cdot \mu_{x_0}.
\]  

(1.6.2)

Now let \( \eta > 0 \) be given. By (1.6.1) we choose \( N_{\eta, x_0} \) such that if \( n \geq N_{\eta, x_0} \) we have \( \rho(\mu_{x_0}, \xi_{K_n} \cdot \mu_{x_0}) < \frac{\eta}{2} \) and \( \rho(\nu_{x_0}, \xi_{K_n} \cdot \nu_{x_0}) < \frac{\eta}{2} \).

Then by (1.6.2), if \( n \geq \max \{ N_{\mu_{x_0}, x_0}, n(x_0) \} \)

\[
\rho(\mu_{x_0}, \nu_{x_0}) \leq \rho(\mu_{x_0}, \xi_{K_n} \cdot \mu_{x_0}) + \rho(\xi_{K_n} \cdot \mu_{x_0}, \xi_{K_n} \cdot \nu_{x_0}) + \rho(\xi_{K_n} \cdot \nu_{x_0}, \nu_{x_0}) < \frac{\eta}{2} + \frac{\eta}{2} + \frac{\eta}{2} = \eta.
\]
But \( \eta \) was arbitrary so that \( \rho(x_\eta, \mu(x_\eta)) = 0 \). Hence \( \mu(x_\eta) = v(x_\eta) \) and the measure is unique.

**Lemma 1.8.** For every \( n \geq 1 \), the mapping \( \varphi \) from \( L_n \) into \( M \) is continuous with respect to the \( \rho \)-topology of \( M \). \( \varphi(x) = \mu(x) \)

**Proof of lemma.** Since \( L_n \) is a Banach space [it is closed in \( X \) by assumption (1)] and \( M \) is a Frechet space, we use the closed graph theorem. Suppose the sequence \( \{(x_j, \mu(x_j))\} \) converges in \( L_n \times M \) to \( (x_\eta, v) \). Then \( v \in M \) and we must show that \( \mu(x_\eta) = \varphi(x_\eta) = v \). Now \( v \in M \) so there exists:

\[
\int_{K_n} u \, dv = x_\eta \in L_n
\]

and

\[
\varphi(x_\eta) = \xi_{K_n} \cdot v.
\]

We are assuming that, in the \( \rho \)-topology of \( M \), \( \lim_{k \to \infty} \mu(x_k) = v \); so we also have that, in the \( \| \cdot \|_n \)-topology of \( M(K_n) \),

\[
\lim_{k \to \infty} \xi_{K_n} \cdot \mu(x_k) = \xi_{K_n} \cdot v.
\]

This implies, by our general hypotheses, that \( \lim_{k \to \infty} \| \xi_{K_n} \cdot \mu(x_k) - \xi_{K_n} \cdot v \| = 0 \).

Let \( f \in X^* \) and let \( \eta > 0 \) be given. Then, as we have noted, we may choose \( N_{\eta,f} \) such that, if \( k \geq N_{\eta,f} \), we have:

\[
|v - \mu(x_k)| (K_n) < \|f\|^{-1} \|u\|_{K_n}^{-1} \eta.
\]

Then we have, for \( k \geq N_{\eta} \),

\[
|f(x_\eta - x_k)| = \left| \int_{K_n} <u,f> \, d(x_\eta - x_k) \right|
\leq \|f\| \|u\|_n \|v - \mu(x_k)| (K_n) < \eta.
\]
Thus \((x_j)\) converges weakly to \(x_v\). But \((x_j)\) converges in norm and hence weakly to \(x_0\). Since the weak topology of a Banach space is Hausdorff, we must have \(x_0 = x_v\). From this we see that:

\[
\varphi(x_0) = \varphi(x_v) = \ell K_n \cdot v.
\]

If we could now show that the support of \(v\) is contained in \(K_n\), we would be done. For then we would have \(\ell x_n \cdot v = v\) and so \(\varphi(x_0) = v\). Suppose this is not the case. Then there is a compact set \(A_0\) contained in the complement of \(K_n\) and an \(a_0 > 0\) such that \(|v(A_0)| = a_0\). Now \((\mu_{x_j})\) converges to \(v\) in the \(p\)-topology and so in the pointwise topology of \(M\). That is, given \(\eta > 0\) and any Borel set \(B\), there exists an \(N_\eta,B\) such that if \(j \geq N_\eta,B\)

\[
|\langle v - \mu_{x_j}, B \rangle| < \eta.
\]

In particular, for \(j \geq N_{a_0/2, A_0}\), we have:

\[
a_0/2 > |\langle v - \mu_{x_j}, A_0 \rangle| = |v(A_0) - \mu_{x_j}(A_0)| = |v(A_0)| = a_0,
\]

which is a contradiction. Thus the support of \(v\) is contained in \(K_n\), as we wished to show.

**Corollary 1.9.** The mapping \(\varphi \circ S_m\) from \(L\) into \(M\) is continuous for each positive integer \(m\).

**Proof of corollary.** Fix \(m_0\) and let \((x_j)\) converge to \(x_0\) in \(L\). Then \((S_{m_0}(x_j))\) converges to \(S_{m_0}(x_0)\) in \(L_{m_0}\) and so, by lemma 1.7:
\[
\lim_{j \to \infty} \varphi \circ S_m(x_j) = \lim_{j \to \infty} \varphi(S_m(x_j)) = \varphi(\lim_{j \to \infty} S_m(x_j)) = \varphi(S_m(x_o)) = \varphi \circ S_m(x_o).
\]

**Lemma 1.10.** The mapping \( \varphi \) from \( L \) into \( M \) is continuous.

**Proof of lemma.** Suppose \( (x_j) \) converges to \( x_o \) in \( L \). Then, by lemma 1.9, there exists for all positive integers \( m \),

\[
\lim_{j \to \infty} \varphi \circ S_m(x_j) = \varphi \circ S_m(x_o). \tag{1.10.1}
\]

Let \( \eta > 0 \) be given and then:

(a) We may choose \( N_{\eta} \) so that

\[
\sum_{j \geq N_{\eta}} 2^{-j} \eta^2 < \eta/2
\]

and

(b) By (1.10.1), choose, for each \( k = 1, 2, \ldots, N_{\eta} \), an \( N_{k, \eta} \) such that for \( j \geq N_{k, \eta} \)

\[
\left\| \xi_k \mu_{x_o} - \xi_k \mu_{x_j} \right\|_k < \frac{\eta}{2} N_{\eta}.
\]

Then, for any \( m \geq \max \{N_1, \eta; N_2, \eta; \ldots; N_{N_{\eta}}, \eta\} \)

\[
\rho(\mu_{x_o}, \mu_{x_m}) = \sum_{j = 1}^{N_{\eta}} 2^{-j} \left( \frac{\left\| \xi_k \mu_{x_o} - \xi_k \mu_{x_j} \right\|_k}{1 + \left\| \xi_k \mu_{x_o} - \xi_k \mu_{x_j} \right\|_k} \right)
\]

\[
\leq \sum_{j = 1}^{N_{\eta}} 2^{-j} \left\| \xi_k \mu_{x_o} - \xi_k \mu_{x_j} \right\|_j + \sum_{j \geq N_{\eta}} 1 \cdot 2^{-j}
\]

\[
< \frac{\eta}{2} + \frac{\eta}{2} = \eta.
\]
That is, \((\varphi(x_m))\) converges to \(\varphi(x_0)\) in the \(\rho\)-topology of \(M\), so that \(\varphi\) is continuous.

**Lemma 1.11.** For each \(x \in \Lambda\), there is a \(\mu_x \in M\) such that

\[
\lim_{n \to \infty} \int T_n u \, d\mu_x = \int T u \, d\mu_x = x.
\]

**Proof of lemma.** By lemma 1.4 there is such a \(\mu_x\) for any \(x \in \Lambda\). The linear function \(\varphi_x\), given by \(\varphi(x) = \mu_x\), is continuous from \(L\) into \(M\). Thus we may extend \(\varphi\) to a mapping \(\hat{\varphi}\) taking all of \(\Lambda\) into \(M\). We shall, of course, use this extension to get the measures we seek. We define, for each \(x \in \Lambda\), \(\mu_x = \hat{\varphi}(x)\) and note that, as \(\hat{\varphi}\) extends \(\varphi\), we have not changed the measures corresponding to the elements of \(L\).

We have already noted that each \(S_n\) has an extension, \(\hat{S}_n\), taking \(\Lambda\) into \(\Lambda\). Also, by corollary 1.5, we know that for each \(x \in \Lambda\),

\[
x = \lim_{n \to \infty} S_n(x).
\]

Now define for each \(x \in \Lambda\),

\[
\sigma_n(x) = \int K_n u \, d\mu_x.
\]

If we could show that \(\sigma_n(x) = S_n(x)\) then we would have that \(\sigma_n(x)\) converges to \(x\) and we would be done. To see that this is true, fix \(x_0 \in \Lambda\) and let \((x_j)\) be a sequence in \(L\) converging to \(x_0\). Fix a positive integer \(n_0\) and let \(f \in X^*\). Let \(\eta > 0\) be given. Since \(\hat{\varphi}\) is continuous, we must have \((\mu_{x_j})\) converging in the \(\rho\)-topology of \(M\) to \(\mu_{x_0}\) and so in the \(\|\cdot\|_n\)-topology, as above. Since this gives
convergence in the total variation norm, we can choose $N_{\eta, n_0}$ such that if $j \geq N_{\eta, n_0}$ we have:

$$|\mu_{x_0} - \mu_{x_j}| (K_{n_0}) < ||f||^{-1} ||u|| n_0^{-1} \eta .$$

Then, if $j \geq N_{\eta, n_0}$ we have:

$$| \int_{K_{n_0}} \langle u, f \rangle d (\mu_{x_0} - \mu_{x_j}) | \leq ||f|| \cdot ||u|| n_0 \mu_{x_0} - \mu_{x_j} (K_{n_0}) < \eta .$$

So we see that, in the weak topology of $X$,

$$\lim_{j \to \infty} \hat{S}_{n_0} (x_j) = \lim_{j \to \infty} \int_{K_{n_0}} u d\mu_{x_j} = \sigma_{n_0} (x_0) .$$

But, we already know that $(\hat{S}_{n_0} (x_j))$ converges in norm and hence, weakly, to $\hat{S}_{n_0} (x_0)$. Since the weak topology of $X$ is Hausdorff, we have $\sigma_{n_0} (x_0) = \hat{S}_{n_0} (x_0)$. But, $n_0$ and $x_0$ were arbitrary so that for any positive integer $n$ and any $x \in X$, $\hat{S}_n (x) = \sigma_n (x)$.

**Lemma 1.12.** For each $x \in \Lambda$, the corresponding measure $\mu_x \in M$ of lemma 1.11 is unique.

**Proof of lemma.** Suppose, for some $x \in \Lambda$, there is $\mu_x \in M$ $\nu_x \in M$ such that

$$\int_T u d\mu_x = x = \int_T u d\nu_x .$$

Then for any positive integer $n$, we have,
Then by assumption 2 we have for any positive integer $n$:

$$\int_{K_n} ud\mu = S_n(x) = \int_{K_n} ud\nu_x.$$  

Then by assumption 2 we have for any positive integer $n$:

$$\xi_{K_n} \cdot \mu_x = \xi_{K_n} \cdot \nu_x. \quad (1.12.1)$$

We have seen above that for any $\mu \in M$ the sequence $(\xi_{K_n} \cdot \mu)$ converges in the $\rho$-topology of $M$ to $\mu$. Then, given $\eta > 0$, choose $N_{\eta, x}$ so that, if $j \geq N_{\eta, x}$ we have

$$\rho(\mu_x, \xi_{K_j} \cdot \mu_x) < \eta/2 \text{ and } \rho(\nu_x, \xi_{K_j} \cdot \nu_x) < \eta/2.$$  

Then, using this and (1.12.1) above, for any $j \geq N_{\eta, x}$

$$0 \leq \rho(\mu_x, \nu_x) \leq$$

$$\leq \rho(\mu_x, \xi_{K_j} \cdot \mu_x) + \rho(\xi_{K_j} \cdot \mu_x, \xi_{K_j} \cdot \nu_x) + \rho(\xi_{K_j} \cdot \nu_x, \nu_x)$$

$$< \eta/2 + o + \eta/2 = \eta.$$  

Since $\eta$ was arbitrary we have $\mu_x = \nu_x$. This concludes the proof of condition (III) and, hence, the proof of proposition 1.3.

We now turn to the second half of the basic theorem. That is, we show that under the general hypotheses of the first section, the converse of proposition 1.3 holds. To obtain condition 3 of proposition 1.3, we shall need a very familiar looking lemma. Because the lemma is not unexpected and because it is useful by
itself, we separate it from the proof of the proposition.

**Lemma 1.13.** Let $X$ be a Banach space with norm $\| \cdot \|$. Suppose $X$ has an integral basis $\{u, T, (K_n), M\}$. As before, we take $S_n(x) = \int_{K_n} u \, d\mu$. Then we have:

(a) $\sup_{n \geq 1} \| S_n(x) \| < \infty$ for all $x \in X$.

(b) $\lim_{n \to \infty} S_n(x) = x$ for all $x \in X$.

(c) There exists a constant $C$ such that, for any $n \geq 1$, we have $\| S_n \| \leq C$.

**Proof of lemma.** We note that (b) is part of the definition of integral basis. To see that (a) holds, fix $x_0 \in X$ and choose $N_{x_0}$ such that, for any $j \geq N_{x_0}$ we have

$$\| x_0 - S_j(x_0) \| \leq 1.$$ 

Then we have, for any $j \geq N_{x_0}$,

$$\| S_n(x_0) \| \leq 1 + \| x_0 \|.$$ 

Now let $K_{x_0} = \max \{ \| S_1(x_0) \|, \| S_2(x_0) \|, \ldots, \| S_{N_{x_0}}(x_0) \| \}$, $1 + \| x_0 \|$ and then we have:

$$\sup_{n \geq 1} \| S_n(x_0) \| \leq K_{x_0} < \infty.$$
To obtain (c) we use (a) and the principle of uniform boundedness to find an \( \eta > 0 \) such that, if \( ||x|| \leq \eta \) then we have

\[
||S_n(x)|| < 1 \quad \text{for all } n.
\]

Then for any \( x \neq 0 \) we have

\[
||S_n(x)|| = || \frac{x}{\eta} \sum_{k=1}^{n} e_k x || \\
\leq \frac{1}{\eta} ||x|| || \sum_{k=1}^{n} e_k x || < \frac{1}{\eta} ||x||.
\]

This holds for any positive integer \( n \), so if we let \( C = 1/\eta \), we have

\[
||S_n|| \leq C \quad \text{for all } n.
\]

We shall now prove the converse of proposition 1.3 and give several straightforward but important corollaries to it and to lemma 1.13.

**Proposition 1.14.** Let \( X \) be a Banach space with an integral basis \( \{u, T, (K_n), M\} \). Then we have:

1) \( L_n = \left\{ \int_{K_n} ud\mu \mid p \leq n, \mu \in M \right\} \) is a closed subspace of \( X \) for each \( n \geq 1 \).

2) Whenever \( \int_{K_n} ud\mu = \int_{K_n} ud\nu \), then we have \( \xi_{K_n} \cdot \mu = \xi_{K_n} \cdot \nu \).

3) There exists a constant \( C \geq 1 \) such that, for all positive integers \( n \) and \( \eta \) and for all \( \mu \in M \), we have
Proof of proposition. Suppose for some $\mu \in M$, $\nu \in M$ we have

$$\int_{K_n} u d\mu = \int_{K_n} u d\nu \quad \text{for some } n.$$

Then we have:

$$\int_{T K_n} u d\mu \cdot \nu = \int_{T K_n} u d\mu = \int_{T K_n} u d\nu = \int_{T K_n} u d\nu \cdot \nu.$$

But representation is unique so that we must have $dK_n \cdot \mu = dK_n \cdot \nu$ and we see that (2) holds.

By lemma 1.13, we have a $C$ such that $||S_n|| \leq C$ for any positive integer $n$. Then for any $n$, $p$ and any $\mu \in M$ we have

$$||\int_{K_n} u d\mu|| = ||S_n \int_{K_n} u d\mu|| \leq C ||\int_{K_n} u d\mu||.$$

Thus we see that (3) holds.

We must now show (1) that each $L_n$ is closed in $X$. To see this, fix $m_0$ and let $x_0$ be in the norm closure of $L_{m_0}$ in $X$. Choose a sequence $(x_j)$ of elements of $L_{m_0}$ which converges to $x_0$ in the norm topology. By lemma 1.13, the projections $(S_n)$ (as no confusion arises, we drop the $^\wedge$) are all continuous. Further, since we have an integral basis, the map $\varphi$ (given by $\varphi(x) = \mu_x$) is also continuous so that the sequence $(\mu_{x_j})$ converges in $M$ to $\mu_{x_0}$. This puts $\mu_{x_0} \in M$ and hence
\[
S_{m_0}(x) = \int_{K_{m_0}} u d\mu_{x_0} \in L_{m_0}.
\]

But \( S_{m_0} \) is continuous and \( x_j \in L_{m_0} \) for each \( j \) so that \( x_j = S_{m_0}(x_j) \) and
\[
\lim_{j \to \infty} x_j = \lim_{j \to \infty} S_{m_0}(x_j) = S_{m_0}(x_0).
\]

But, \( \lim_{j \to \infty} x_j = x_0 \). Thus \( x_0 = S_{m_0}(x_0) \in L_{m_0} \).

Thus \( L_{m_0} \) is closed and (1) holds. This completes the proof of the proposition.

**Corollary 1.15.** Let \( X \) and \( Y \) be Banach spaces. Suppose \( X \) has an integral basis \( \{u, T, (K_n), M\} \) and that there is an isomorphism \( \psi \) of \( X \) onto \( Y \). Then \( Y \) has an integral basis \( \{v, S, (C_n), N\} \).

**Proof of corollary.** We take \( S = T \), \( N = M \), and \( C_n = K_n \) for each \( n \). Finally, define \( v \) from \( S \) to \( Y \) by
\[
v(t) = \psi(u(t)).
\]

This gives the required basis and the associated basis constant is \( C ||\psi|| ||\psi^{-1}|| \) where \( C \) is the constant for the basis in \( X \).

In particular, the space of measures \( M = \varphi(X) \) which is associated with \( X \), has an integral basis.

**Corollary 1.16.** Let \( X \) be a Banach space with an integral basis \( \{u, T, (K_n), M\} \). Then \( X \) has a Schauder decomposition \( \{X_k; E_k\} \). Further, this decomposition is naturally associated with the given basis.
Proof of corollary. Let $S_n(x) = 0$ for each $x \in X$ and define $E_n(x) = (S_n - S_{n-1}) (x)$ for each $x \in X$. Then, for each $n \geq 1$, $E_n$ is a linear operator in $X$ and we have for any $x \in X$:

$$E_n \circ E_n (x) = S_n \circ (S_n - S_{n-1})x - S_{n-1} \circ (S_n - S_{n-1}) x$$

$$= S_n (x) - S_n \circ S_{n-1} (x) - S_{n-1} \circ S_n (x) + S_{n-1} (x)$$

$$= S_n (x) - S_{n-1} (x) - S_{n-1} (x) + S_{n-1} (x)$$

$$= S_n (x) - S_{n-1} (x) = E_n (x).$$

So we see that the $E_n$ are all projections and if $C$ is the basis constant we have

$$||E_n (x)|| = ||(S_n - S_{n-1}) x|| \leq ||S_n (x) + S_{n-1} (x)|| \leq 2C ||x||$$

so that $||E_n|| \leq 2C$ for every $n$. Finally:

$$x = \lim_{n \to \infty} S_n (x) = \lim_{n \to \infty} \sum_{j=1}^{n} E_j (x).$$

Thus, if we set $X_j = E_j (X)$, we have that $\{X_k; E_k\}$ is a basis of subspaces for $X$. But McArthur and Retherford [9] have shown that since $X$ is tonnelé, the decomposition must be a Schauder decomposition.

Because of corollary 1.16 we are provided with examples of Banach spaces with no integral bases. Dean [3] has shown that a large class of Banach spaces; including (m) and those spaces $C(H)$ for which $H$ is compact, Hausdorff and extremely disconnected; have
no Schauder decomposition. Since an integral basis always yields such a decomposition, these spaces cannot have one.

A natural question at this point would be: since integral bases give Schauder decompositions, will such decompositions yield integral bases? In Chapter 2 we consider this question and show that certain types of decompositions do give rise to such bases. On the other hand, we shall show that there are spaces with Schauder decompositions which are not derived from any integral basis. Thus the concept of integral bases is strictly stronger than that of Schauder decompositions.
SOME EXAMPLES

We shall give here some examples of integral bases constructed from some particular Schauder decompositions. We shall also show that there are some Schauder decompositions of certain spaces which have no integral basis subordinate to them in the sense that the basis gives back the original decomposition. The question of whether an integral basis must already be a series basis if the index set is countable will also be investigated.

Our first construction involves a space $X$ with a Schauder decomposition $\{X_j; E_j\}$ where, for each $j$, we have $\dim X_j = m_j < \infty$. We remark here that the existence of such a decomposition in $X$ is equivalent to the statement that $X$ has a basis with parentheses [8]. $((x_n))$ is a basis with parentheses for $X$ if there is a sequence $(n_k)$ of integers and $(f_n)$ in $X^*$ biorthogonal to $(x_n)$ such that for each $x \in X$ the series $\sum_{m,n+1}^{m} f_i(x)x_i$ converges to $x$. The fact that these are equivalent conditions on $X$ is easy to see. If $X$ has a Schauder decomposition as above, then for each $j$, $X_j = [x_1^{(j)}, \ldots, x_{m_j}^{(j)}]$. Thus, since for all $x \in X$ we have $x = \sum E_j(x)$, we may take the $(x_k^{(j)})$ in the natural order $(x_1^{(1)}, x_2^{(1)}, \ldots, x_{m_1}^{(1)}, x_1^{(2)}, \ldots, x_{m_2}^{(2)}, \ldots)$ to get a basis with parentheses. Given a basis with parentheses $(x_j)$ we let $X_k = [x_{n_k+1}, \ldots, x_{n_k+1}]$ and we obtain a
Schauder decomposition for $X$.

We shall show that a space with a basis with parentheses (or, equivalently, with a Schauder decomposition into finite dimensional subspaces) has an integral basis. To do this, we shall show that the basis with parentheses is an integral basis.

We take the natural numbers [with the discrete topology] as our index set $T$. We are assuming that $X$ has a basis with parentheses $(e_j)$, whose corresponding Schauder decomposition is $[X_k; E_k]$. We shall denote by $m_k$ the dimension of the subspace $X_k = E_k(X)$.

Taking $t_n = \sum_{j=1}^{n} m_j$, we let $K_n = \{1, 2, \ldots, t_n\}$. We see that the sets $K_n$ are compact and increasing in $T$, and that $T$ is locally compact and Hausdorff. For our index function, we simply take the map $u(k) = e_k$. Then since each $K_n$ is a finite set, we have $||u||_n = \max \{||u(j)|| \mid j \in K_n\}$ is finite.

We must now construct a space of measures, $\mathcal{M}$. To do this, we shall build a copy of $X$ in (s) the space of all sequences of scalars.

**Definition 2.1.** Denote by $\delta_i$ the point mass centered at $i$.

(a) For each $i \in T$, let $\mu_{e_i} = \delta_i$.

(b) For each $y_j \in X_j$ we have $y_j = \sum_{i=1}^{m_j} a_i e_{i+m_j-1}$ so

$$\mu_{y_j} = \sum_{i=1}^{m_j} a_i \delta_{i+m_j-1}$$

(c) For each positive integer $n$, let
Lemma 2.2.

(1) The set $\mathcal{U} \mathcal{M}_n$ together with $\| \cdot \|$ is a normed linear space, and

(2) For each positive integer $n$, $M_n$ is a linear subspace of $\mathcal{U} \mathcal{M}_n$ which is isometric, as a normed space, to $\bigoplus_{j=1}^{n} X_j$.

Proof of lemma. The proof of statement (1) is clear. To see that (2) holds, define a map $\varphi$ by

$$\varphi(\sum_{j=1}^{n} y_j) = \sum_{j=1}^{n} \mu_j y_j.$$

Then $\varphi$ is a map from $\bigoplus_{j=1}^{n} X_j$ into $M_n$, and by the definition of the norm in $M_n$, is an isometry.

We now place a copy of $X$ into $(s)$ by mapping an element $x = \sum_{n \geq 0} (t_{n+1} a_i e_i)$ to the sequence $(a_i)$ (here we take $t_0 = 0$). We norm this subspace by letting $||a_i|| = \sum_{n \geq 0} (t_{n+1} a_i e_i)$ to get an isometric copy of $X$ in $(s)$. We now return to our space of measures, $M$.

Definition 2.3. Denote by $\psi$ the mapping given above from $X$ into $(s)$. We now define a map $\sigma$ from $\mathcal{U} \mathcal{M}_n$ into $\psi(X)$ by
\[ \sigma \left( \sum_{j=1}^{n} \mu y_j \right) = \psi \left( \sum_{j=1}^{n} y_j \right). \]

**Lemma 2.4.** Take \( M \) to be the completion of \( uM_n \) with respect to its norm \( || \cdot || \). Then there is an extension, \( \hat{\sigma} \), of \( \sigma \) to all of \( M \) which is an isometry onto \( \psi(X) \) in (s).

**Proof of lemma.** Let \( \mu = \sum_{j=1}^{n} \mu y_j \in uM_n \). Then we have

\[ ||\sigma(\mu)|| = ||\psi \sum_{j=1}^{n} y_j|| = ||\sum_{j=1}^{n} y_j|| = ||\mu||. \]

Thus \( \sigma \) is an isometry from \( uM_n \) into \( \psi(X) \). We may extend \( \sigma \) to \( \hat{\sigma} \), an isometry of \( M \) into \( \psi(X) \). To see that \( \hat{\sigma} \) is onto, let \( (b_j) = \psi(x_0) \in \psi(X) \). Then the sequence \( (y_k) \) given by \( y_k = \sum_{i=0}^{k} \Sigma_{j=t_i+1}^{t_{i+1}} b_j e_j \), is in \( X \) and converges to \( x_0 \). Further, the sequence \( (\mu_n) \) is in \( M \). Since \( \varphi \) was an isometry from \( u \bigoplus_{j=1}^{\infty} X_j \) into \( M \), it has an extension \( \hat{\varphi} \) from \( X \) into \( M \). In fact \( \hat{\varphi}(y_k) = \mu y_k \) so that \( \hat{\varphi}(y_k) \) converges to \( \varphi(x_0) \in M \). Then we must have \( \hat{\sigma} \circ \hat{\varphi}(y_k) = \sigma \circ \varphi(y_k) = \psi(y_k) \) which converges to \( \psi(x_o) \) since \( \psi \) is continuous. Thus \( \hat{\sigma}(\varphi(x_0)) = \psi(x_o) \) and we see that \( \hat{\sigma} \) is onto. In fact \( \hat{\sigma} \circ \hat{\varphi} = \hat{\psi} \).

**Definition 2.5.** For each \( \mu \in M \) we have \( \sigma(\mu) = (a_1(\mu), a_2(\mu), ...) \) in \( \psi(X) \). For each positive integer \( n \), let

\[ \xi_n^\mu = \sigma^{-1} (a_1(\mu), ..., a_t(\mu), \omega, \omega, ...) \].

**Lemma 2.6.** For each \( \mu \in M \), \( \xi_n^\mu \in M_n \) for all \( n \) and so \( \mu \) is a measure on \( K_n \).
Proof of lemma. \[ \sigma^{-1}(a_1(\mu), \ldots, a_{\tau_n}(\mu), \circ, \circ, \ldots) = \]
\[ = \sigma^{-1}(\sum_{i=0}^{n-1} \sum_{j=t_i+1}^{t_{i+1}} a_j(\mu) e_j) \]
\[ = \varphi\left(\sum_{i=0}^{n-1} \sum_{j=t_i+1}^{t_{i+1}} a_j(\mu) e_j\right) \in \mathbb{M}_n. \]

Lemma 2.7. For each \( \mu \in \mathcal{M} \) and for any integer \( n \), there exists \( \int_{\mathcal{K}_n} u d\mu \in \mathcal{X} \).

Proof of lemma. Let \( \mu \in \mathcal{M} \) and fix \( n_0 \). Then \( \xi_{\mathcal{K}_{n_0}} \cdot \mu_{n_0} \in \mathcal{M}_{n_0} \)
thus \( \xi_{\mathcal{K}_{n_0}} \cdot \mu = \Sigma_{i=1}^{n_0} y_i \) for some \( y_i \in \mathcal{X}_i \). Now, for any \( f \in \mathcal{X}^* \) we have
\[ \int_{\mathcal{K}_{n_0}} foud\mu_{n_0} = \int_{\mathcal{K}_{n_0}} foud \left( \sum_{i=1}^{n_0} y_i \right) \]
\[ = \sum_{i=1}^{n_0} \int_{\mathcal{K}_{n_0}} foud y_i \]
\[ = \sum_{i=1}^{n_0} \int_{\mathcal{K}_{n_0}} foud \left( \sum_{j=1}^{m_i} a_j(\delta_j) \right) \]
\[ = \sum_{i=1}^{n_0} \sum_{j=1}^{m_i} a_j(\delta_j) \]
\[ = f\left( \sum_{i=1}^{n_0} \sum_{j=1}^{m_i} a_j(\delta_j) \right) = f\left( \sum_{i=1}^{n_0} y_i \right). \]

Then we have
\[ \int_{\mathcal{K}_{n_0}} u d\mu = \sum_{i=1}^{n_0} y_i = \psi^{-1}(\xi_{\mathcal{K}_{n_0}} \cdot \mu). \]
Lemma 2.8. \( L_n = \left\{ \int_{K_p} u \, d \mu \mid p \leq n, \mu \in \mathcal{M} \right\} = \bigoplus_{j=1}^{n} X_j \) and so is a closed subspace of \( X \), for any integer \( n \).

**Proof of the lemma.** First let \( \mu \in \mathcal{M} \). Then \( \sum_{p} \mu \in \mathcal{M} \) for any integer \( p \) so that \( \sum_{j=1}^{n} \mu_y \) for some \( y \in X_j \). Then we have, by lemma 2.7,

\[
\int_{K_p} u \, d \mu = \int_{K_p} u \, d \sum_{p} \mu = \sum_{j=1}^{n} y_j \in \bigoplus_{j=1}^{n} X_j.
\]

Thus we see that \( L_n \) is contained in \( \bigoplus_{j=1}^{n} X_j \) for any positive integer \( n \).

On the other hand, let \( x = \sum_{j=1}^{n} x_j \) be in \( \bigoplus_{j=1}^{n} X_j \).

Then we have, again by lemma 2.7:

\[
x = \sum_{j=1}^{n} x_j = \int_{K_n} u \, d \left( \sum_{j=1}^{n} \mu x_j \right) \in L_n.
\]

Thus the reverse inclusion holds and we have

\[
L_n = \bigoplus_{j=1}^{n} X_j.
\]

Lemma 2.9. Whenever \( \int_{K_n} u \, d (\mu - \nu) = 0 \) we have \( \sum_{j=1}^{n} \mu x_j = \sum_{j=1}^{n} \nu y_j \).

**Proof of the lemma.** Suppose there are \( \mu, \nu \) in \( \mathcal{M} \) such that \( \int_{K_n} u \, d (\mu - \nu) = 0 \) for some \( n \). Let

\[
\sum_{j=1}^{n} \mu x_j = \sum_{j=1}^{n} \nu y_j.
\]
Then we have:
\[
\sum_{j=1}^{n} x_j = \int_{K_n} u \hat{\mu} = \int_{K_n} u d \nu = \sum_{j=1}^{n} y_j.
\]

But then
\[
\xi_{K_n} \cdot \mu = \sum_{j=1}^{n} \mu x_j = \sum_{j=1}^{n} \nu y_j = \xi_{K_n} \cdot \nu.
\]

**Lemma 2.10.** \{u, T, (K_n), M\} is an integral basis for its closed linear span.

**Proof of lemma.** We shall employ the basic theorem of Chapter 1. By lemmas 2.8 and 2.9 we have satisfied two of the conditions needed. What remains to be shown is that there is a constant \( C \geq 1 \) such that for any \( \mu \in M \) and any positive integers \( p, n \) we have
\[
||\int_{K_n} u \hat{\mu}|| \leq C \left( ||\int_{K_n} u d \mu|| \right).
\]

Because we have a Schauder decomposition \((X_k, E_k)\) we know there exists a \( C \geq 1 \) such that if \( S_n = \sum_{j=1}^{n} E_j \), then
\[
||S_n|| \leq C \text{ for any } n. \quad [9]
\]
We shall use this \( C \). Fix \( \mu \in M \) and integers \( n, p \). Let \( (x_i) = \varphi^{-1}(\mu) \) (i.e., \( \xi_{X_n} \cdot \mu = \sum_{j=1}^{n} \mu x_j \)). Then
\[
||\int_{K_n} u \hat{\mu}|| = ||\int_{K_n} u \xi_{X_n} \cdot \mu||
\]
\[
= ||\int_{K_n} u \sum_{j=1}^{n} \mu x_j||
\]
Thus \( \{u, T, (K^), M\} \) is a basis for the closure of \( UL_n \) in \( X \).

**Proposition 2.11.** \( X \) has an integral basis whenever it has a basis with parentheses, or, equivalently, a Schauder decomposition into finite dimensional subspaces \( (X_j) \).

**Proof of proposition.** By the preceding lemmas, we have constructed \( \{u, T, (K^), M\} \) which is integral basic (i.e., is an integral basis for its closed span). On the other hand, we have seen that, for any \( n \):

\[
L_n = \bigoplus_{j=1}^{n} X_j.
\]

Since we have a Schauder decomposition, we must have the closed span of \( U(\bigoplus_{j=1}^{n} X_j) \) equal to the space \( X \). But this is the same as the closed span of \( UL_n \). Thus our \( \{u, T, (K^), M\} \) is an integral basis for \( X \).

We have seen that a given basis with parentheses must be an integral basis. A natural question arises. Can we force a basis with parentheses to be a series basis? Or, more generally, does an integral basis with countable index set \( T \) have to be a
series basis? We shall answer the first question (and hence the second) negatively with the following construction.

**Proposition 2.12.** There exists a Banach space $X$ which has a basis with parentheses $(x_j)$ which cannot be a series basis for $X$.

**Proof of proposition.** We shall construct a space $X$ with a Schauder decomposition into two-dimensional subspaces and give a basis with parentheses which is not a series basis.

For each positive integer $j$, let

$$x_0^{(j)} = (1, 0) \quad \text{and} \quad x_1^{(j)} = (1, 1/j) .$$

We take $X_j$ to be $[x_0^{(j)}, x_1^{(j)}]$ with the $\ell_1$ norm. That is

$$||a_0 x_0^{(j)} + a_1 x_1^{(j)}|| = |a_0 + a_1| + \frac{|a_1|}{j} .$$

For the corresponding coordinate projections

$$E_i^{(j)} (a_0 x_0^{(j)} + a_1 x_1^{(j)}) = a_i x_i^{(j)} , \quad i = 0, 1$$

we have

$$||E_0^{(j)}|| \cdot ||a_0 x_0^{(j)} + a_1 x_1^{(j)}|| = ||E_0^{(j)}|| \cdot (|a_0 + a_1| + 1/j|a_1|) \geq$$

$$\geq ||a_0 x_0^{(j)}|| = |a_0| .$$

In particular, if $a_1 = -a_0 = -1$ then

$$||E_0^{(j)}|| \geq j .$$
We also have

\[ ||E_1^{(j)}|| \geq |a_1 x_1^{(j)}| = |a_1| + 1/j|a_1| \]

In particular, if \( a_1 = -a_o = -1 \) then

\[ ||E_1^{(j)}|| (1/j) \geq 1/j + 1 \quad \text{so} \quad ||E_1^{(j)}|| \geq j + 1 \]

So we have, for each \( j \), that if there is a \( K \) so that

\[ ||a_1 x_1^{(j)}|| \leq K||a_x^{(j)} + a_1 x_1^{(j)}||, \quad i = 0, 1 \]

then we must have \( K \geq j \).

Now let \( X = \sum x_{j} \) \( \ell_1 = \{ (x_j) \in \sum x_{j} : ||x_j|| = ||(x_j)|| < \infty \} \).

Then we have a Schauder decomposition \( (X_j; E_j) \) for \( X \) and the collection \( \{ x_o^{(1)}, x_1^{(1)}, x_o^{(2)}, \ldots, x_o^{(n)}, x_1^{(n)}, \ldots \} \) is a basis with parentheses for \( X \).

If this collection is to be a series basis \( (x_j) \) then we must have:

- \( a(1) \): \( x_1 = x_o^{(1)} \) and \( x_2 = x_1^{(1)} \) or \( b(1) \): \( x_1 = x_1^{(1)} \) and \( x_2 = x_o^{(1)} \)

- \( a(n) \): \( x_{2n-1} = x_o^{(n)} \) and \( x_{2n} = x_1^{(n)} \) or \( b(n) \): \( x_{2n-1} = x_1^{(n)} \) and \( x_{2n} = x_o^{(n)} \)

Now let \( S_n \) denote the usual \( n \)th partial sum operator \( \sum_{n}^{n} \sum_{j=1}^{n} a_j x_j \) and suppose \( K \) is any constant such that \( ||S_n|| \leq K \)
for all \( n \).

We now choose \( j > K \). Let \( a_i = 0 \) for \( 1 \leq i \leq 2j - 2 \) and \( a_{2j-1} = -a_{2j} = -1 \). Then we have:

\[
||s_{2j-1}|| \geq ||s_{2j-1}|| - \sum_{i=1}^{2j-1} a_i x_i
\]

\[
\geq \sum_{i=1}^{2j-1} a_i x_i
\]

\[
= ||a_{2j-1}x_{2j-1}||.
\]

Then we have either

(1) \( ||s_{2j-1}|| \geq ||x^{(j)}_o - x^{(j)}_1|| \geq ||x^{(j)}_o|| \)

and so \( ||s_{2j-1}|| \geq j > K \) or

(2) \( ||s_{2j-1}|| \geq ||x^{(j)}_1 - x^{(j)}_o|| \geq ||x^{(j)}_1|| \)

and so \( ||s_{2j-1}|| \geq j+1 > K \)

both of which conclusions are contradictions. Then the sequence \( (x_j) \) cannot be a series basis.

In this section we shall give another example of a space with an integral basis. The main result will be the analogue of a result for series bases. Let \( X \) be a Banach space with a Schauder decomposition \( (X_k; E_k) \). We suppose that each \( X_k \) has an integral basis \( \{u_k, T_k, (k_n^{(k)}), k_k\} \) whose associated projections \( (S_n^{(k)}) \) from
onto $L_n^{(k)} = \{ \int_{\mathbb{R}^k} u_\mu \, d\mu \mid p \leq n, \mu \in \mathcal{M}_k \}$ have norms bounded by $C_k$ for all $n$. Finally, suppose there exists a constant $C$ such that $C$ is greater than any of the $C_j$. Under these assumptions, we shall show that $X$ has an integral basis.

**Definition 2.13.** For a collection of non-empty sets $(A_\alpha)_{\alpha \in \Delta}$, we shall denote by $\bigcup_{\alpha \in \Delta} A_\alpha$ the disjoint union of the collection. That is, we take the sets to be pairwise disjoint and form their union.

Let $T = \bigcup_{\alpha \in \Delta} T_\alpha$ and put a topology on $T$ as follows. We say that $\theta$ is an open neighborhood of $t$ in $T$ if $t \in \theta$ and $\theta = \bigcup_{\alpha \in \Delta} \theta_\alpha$ where, for each $\alpha \in \Delta$, $\theta_\alpha$ is the open subset of some $T_\alpha$. The collection of all such neighborhood systems forms a subbase for a topology $\tau$ on $T$.

**Lemma 2.14.** $T$ with the topology $\tau$ given above is a locally compact and Hausdorff space. Further, for any positive integers $n, j$, the set $K_n^{(j)}$ is a compact subset of $T$.

**Proof of lemma.** First, let $t_1, t_2$ be in $T$. If $t_1, t_2$ are in the same $T_j$, then they are separated by open subsets of $T_j$ since it is Hausdorff. If $t_1, t_2$ are in different $T_j$, then these two sets are open, disjoint and separate $t_1, t_2$. Thus $T$ is seen to be Hausdorff. To see that $T$ is locally compact, we shall first show that any subset $A$ of $T_j$ which was compact in $T_j$ is compact in $T$. This also shows that $K_n^{(j)}$ is compact for any $n, j$. Let $A$ be a compact subset of $T_j$, and let $A \subseteq \bigcup_{\alpha \in \Delta} \theta_\alpha$ some open cover
of A. For each $t \in A$, choose $\theta_{\alpha_t} \in \{\theta_{\alpha} \}_{\alpha \in \Delta}$ such that $t \in \theta_{\alpha_t}$. Then $\theta_{\alpha_t} \cap T_j$ is open in $T_j$ and $A \subseteq U \{\theta_{\alpha_t} \cap T_j\}$. But $A$ is compact in $T_j$ so there exists $t_1, \ldots, t_n$ such that $A \subseteq U_{k=1}^n \theta_{\alpha_{t_k}} \cap T_j$. Then $A \subseteq U_{k=1}^n \theta_{\alpha_{t_k}}$ so that $A$ is compact. Finally, since each $t \in T$ has a compact neighborhood in $T_j$, it has the same compact neighborhood in $T$. Thus $T$ is locally compact.

We now have our index set and we must find a space of measures $M$. We shall use the given spaces $M_j$ to form $M$. Denote by $\varphi_j$ the map from $X_j$ into $M_j$ given by $\varphi_j(x) = \mu_x$.

**Definition 2.15.** We shall let $M$ be defined by

$$M = \{(\mu_j)_{j \in \mathbb{N}} \mid \| (\mu_j) \| = \| \sum_{j=1}^\infty \varphi_j^{-1}(\mu_j) \| < \infty \}.$$ 

**Lemma 2.16.** The space $M$ given above is a normed linear space and for each $(\mu_j) \in M$ there is a regular Borel measure on $T$ associated with $(\mu_j)$.

**Proof of lemma.** $M$ is clearly a normed linear space. To get the measures we seek, we shall give an inner measure and form a measure from it. We note that for any compact subset $A$ of $T$, there exists $i_1, \ldots, i_k$ such that $A \subseteq U_{j=1}^k T_{i_j}$ and that $k$ is the least such (that is, $A \cap T_{i_j} \neq \emptyset$ for $1 \leq j \leq k$). Now fix $(\mu_j) \in M$. We define, for $A$ compact

$$(\mu_j)(A) = \sum_{j=1}^k \mu_{i_j}(A \cap T_{i_j}) = \sum_{j=1}^k \mu_{i_j}(A \cap T_{i_j}) < \infty.$$
This is well-defined since the \((T_j)\) are pairwise disjoint. Now for any subset \(B\) of \(T\) let

\[
(\mu_j)(B) = \sup \{ (\mu_j)(A) | A \text{ compact}, A \subseteq B \}.
\]

Now if \(A_1 \cap A_2 = \emptyset\) with \(A_1\) compact then

\[
(\mu_j)(A_1 \cup A_2) = (\mu_j)(A_1) + (\mu_j)(A_2).
\]

Further, let \(B_1 \cap B_2 = \emptyset\) with \(B_i\) any subset of \(T\). Now, \((\mu_j)(B_i)\) \(\leq (\mu_j)(B_1 \cup B_2)\) for \(i = 1, 2\) so if either \((\mu_j)(B_i)\) or \((\mu_j)(B_2)\) is infinite, so is \((\mu_j)(B_1 \cup B_2)\). Then, in this case, \((\mu_j)(B_1 \cup B_2) = (\mu_j)(B_2) + (\mu_j)(B_1)\). On the other hand, if both \((\mu_j)(B_1)\), \((\mu_j)(B_2)\) are finite, we can find for any \(\eta > 0\) sets \(A_1 \subseteq B_1\), and \(A_2 \subseteq B_2\) such that \(A_1\) and \(A_2\) are compact and

\[
(\mu_j)(A_i) \geq (\mu_j)(B_i) - \eta/2 \text{ for } i = 1, 2.
\]

then

\[
(\mu_j)(B_1 \cup B_2) \geq (\mu_j)(A_1 \cup A_2)
\]

\[
= (\mu_j)(A_1) + (\mu_j)(A_2)
\]

\[
\geq (\mu_j)(B_1) + (\mu_j)(B_2) - \eta.
\]

Thus

\[
(\mu_j)(B_1 \cup B_2) = (\mu_j)(B_1) + (\mu_j)(B_2).
\]
So we have that $(\mu_j)$ is a finitely additive set function on the semi-algebra generated by the compact sets. Then there is a unique measure $(\mu_j)$ on the $\sigma$-algebra generated by them (namely, the Borel sets), such that this measure agrees with the inner measure defined above. The measure will be regular because of the way it was defined.

We now turn to the construction of an index function $u$ from $T$ into $X$. First define $\hat{u}_j$ from $T$ into $X$ by $\hat{u}_j(t) = u_j(t)$ if $t \in T_j$ and $\hat{u}_j(t) = 0$ if $t \in T_j$. Then, since for each $t \in T$ there is a unique $j_t$ such that $t \in T_{j_t}$ we define

$$u(t) = \sum_{j_t} \hat{u}_j(t) = u_{j_t}(t).$$

Then $u$ takes $T$ into $X$. Finally, we need a collection of compact, nested subsets $(K_n)$ of $T$. For each $n$, let

$$K_n = \bigcup_{i=1}^n K(i).$$

Then each $K_n$ is compact and $K_n \subseteq K_{n+1}$ for all $n$. Now

$$||u||_n = \sup \{ ||u(t)|| : t \in K_n \}$$

$$= \sup \{ ||u(t)|| : t \in \bigcup_{i=1}^n K(i) \}$$

$$\leq \max \{ ||u_1||_1, \ldots, ||u_n||_n \} < \infty$$

so our condition on $u$ is fulfilled.

**Lemma 2.17.** There is an isometry of $X$ onto $M$. 

Proof of lemma. We define \( \varphi \) from \( X \) into \( M \) by

\[
\varphi(x) = \varphi(\sum_j x_j) = \sum_j \varphi(x_j).
\]

Then \( \varphi \) is an isometry of \( X \) onto \( M \) and we let \( \mu_X = \varphi(x) \).

Lemma 2.18. For any \( \mu \in M \) and any positive integer \( n \), there exists \( \int ud\mu \in X \).

Proof of lemma. Fix \( \mu \in M \) and an integer \( n \). Let \( f \in X^* \), then

\[
\int_{K_n} <f, uzmu> = \sum_{i=1}^n \int_{K_n(i)} <f, uzd\varphi(x)>
\]

\[
= \sum_{i=1}^n \int_{K_n(i)} <f, uzd\varphi(x)>
\]

\[
= \sum_{i=1}^n \int_{K_n(i)} <f, uzd\varphi(x)>
\]

So we have:

\[
\int_{K_n} ud\mu = \sum_{i=1}^n S_n^{(i)}(x_i) eX
\]

where \( \mu = \varphi(x) = \varphi(\sum x_i) \).  

Lemma 2.19. For any positive integer $n$ and any $f \in X^*$ the map $\mu \to \int_{K_n} \langle f, u \rangle d\mu$ is continuous.

Proof of lemma. Suppose $(\mu_n')$ converges to $0$ in $M$, say $\mu_n' = \varphi(x(\alpha))$. Then we have:

$$\left| \int_{K_n} \langle f, u \rangle d\varphi(x(\alpha)) \right| = \left| \int_{-n}^{n} S_n^{(i)}(x(\alpha)) \right|$$

$$\leq \| f \| \left| \int_{-n}^{n} S_n^{(i)}(x(\alpha)) \right|$$

$$\leq \max_{1 \leq i \leq n} \left| S_n^{(i)}(x(\alpha)) \right|$$

$$\leq C \| f \| \left| \int_{-n}^{n} x_i(\alpha) \right| .$$

Now, we have assumed a Schauder decomposition for $X$ and that $x(\alpha) = \sum_{i=1}^{\infty} x_{i}(\alpha)$ is the representation of $x(\alpha)$. Then there is a constant, $B$, such that for any integers $n$, $p$ we have:

$$\| \sum_{i=1}^{n} E_i(x) \| \leq B \| \sum_{i=1}^{n+p} E_i(x) \| .$$

Thus we see:

$$\left| \int_{K_n} \langle f, u \rangle d\mu_n' \right| \leq C \| f \| \left| \sum_{i=1}^{n} x_i(\alpha) \right|$$

$$\leq CB \| f \| \left| \sum_{i=1}^{n} x_i(\alpha) \right|$$

$$= CB \| f \| \| \varphi(x(\alpha)) \| = CB \| f \| \| \mu_n' \| .$$
Then, for any $\eta > 0$, choose $\alpha_\eta$ such that if $\alpha \geq \alpha_\eta$ we have

$$||\mu_\alpha|| < (CB||f||)^{-1}\eta$$

and we have for $\alpha \geq \alpha_\eta$:

$$\int \langle f, u \rangle d\mu_\alpha < \eta.$$  

Thus,

$$\lim_{\alpha} \int_{K_n} \langle f, u \rangle d\mu_\alpha = 0$$  

and so the map is continuous.

We now turn back to the basic theorem of Chapter 1 to show that $\{u, T, (K_n), M\}$ is an integral basis for $X$.

Lemma 2.20. If $\int_{K_n} u d\mu = \int_{K_n} u d\nu$ then we have $\xi_{K_n} \mu = \xi_{K_n} \nu$.

Proof of lemma. If $\int_{K_n} u d\mu = \int_{K_n} u d\nu$ and $\mu = \phi(\sum \lambda_i x_i)$ then we have:

$$\sum_{i=1}^n \sum_{j=1}^n \int_{K_n} (i) u_i d\varphi_i (x_i) = \sum_{i=1}^n \sum_{j=1}^n \int_{K_n} (i) u_i d\varphi_i (y_j)$$.

And so we have:

$$\sum_{i=1}^n S_n^{(i)} (x_i) = \sum_{i=1}^n S_n^{(i)} (y_i).$$

But $\sum_{i=1}^n S_n^{(i)} (x_i)$ and $\sum_{i=1}^n S_n^{(i)} (y_i)$ are in $\bigoplus_{i=1}^n X_i$ and in particular, $S_n^{(i)} (x_i) \in X_i, S_n^{(i)} (y_i) \in X_i$. Since we have a Schauder decomposition, we must have, for each $i$, $S_n^{(i)} (x_i) = S_n^{(i)} (y_i)$. But then

$$\int_{K_n} (i) u_i d\varphi_i (x_i) = \int_{K_n} (i) u_i d\varphi_i (y_i).$$
Now each $X_i$ had an integral basis so:

$$
\xi_{K_n}(i), \varphi_{i}(x_i) = \xi_{K_n}(i), \varphi_{i}(y_i) \quad \text{for each } i.
$$

Finally, since the sets $(K_n(i))$ are pairwise disjoint and the support of $\varphi_i(x_i)$ lies in $T_i$ for any $i$, we have:

$$
\xi_{K_n} \cdot \varphi \left( \sum x_i \right) = \sum_{j=1}^{n} \xi_{K_n(j)} \cdot \varphi \left( \sum x_i \right) = \sum_{j=1}^{n} \sum_{i=1}^{n} \xi_{K_n(j)} \cdot \varphi_i(x_i) = \sum_{j=1}^{n} \xi_{K_n(j)} \cdot \varphi_j(x_j) = \sum_{j=1}^{n} \xi_{K_n(j)} \cdot \varphi_j(y_j) = \xi_{K_n} \cdot \varphi \left( \sum y_j \right).
$$

So we have $\xi_{K_n} \cdot \mu = \xi_{K_n} \cdot \nu$.

**Lemma 2.21.** $L_n = \left\{ \int p \, d\mu : p \leq n, \mu \in M \right\}$ is a closed subspace of $X$.

**Proof of lemma.** First, let $x \in L_n$. Then there is a

$$
\sum y_i \in X
$$

such that

$$
x = \int p \, d\varphi \left( \sum_{K_n} y_i \right).
$$
In fact, we have seen that
\[ x = \sum_{i=1}^{n} s_{n}^{(i)}(y_{i}) \, . \]

But then \( x \in \bigoplus_{i=1}^{n} X_{i} \) so that \( L_{n} \) is contained in \( \bigoplus_{i=1}^{n} X_{i} \). Since \( \bigoplus_{i=1}^{n} X_{i} \) is a closed subspace of \( X \), the closure of \( L_{n} \) will also be contained in \( \bigoplus_{i=1}^{n} X_{i} \). Next, consider \( x = \sum_{i=1}^{n} x_{i} \in \bigoplus_{i=1}^{n} X_{i} \).

Then \( \varphi(x) = \sum_{i=1}^{n} \varphi_{i}(x_{i}) \) and so we have, for \( f \in X^{\ast} \)
\[ \int_{T} <f, u> d \varphi(x) = \sum_{i=1}^{n} \int_{T_{i}} <f, u> d \varphi_{i}(x_{i}) \]
\[ = f \left( \sum_{i=1}^{n} x_{i} \right) . \]

So we see that \( x = \int_{T} u d \varphi(x) \). Using this, we now can finish the lemma. Suppose \( x_{0} \) in the closure of \( L_{n} \) and that \( (x_{j}) \) is a sequence in \( L_{n} \) converging to \( x_{0} \). We must show \( x_{0} \in L_{n} \). To do this, since \( x_{0} \in \bigoplus_{i=1}^{n} X_{i} \) and so \( x_{0} = \int_{T} u d \varphi(x_{0}) \), it suffices to show that \( \varphi(x_{0}) \) has support in \( K_{n} \). Now, since \( (x_{j}) \) converges to \( x_{0} \) and \( \varphi \) is continuous by lemma 2.17, we have
\[ \lim_{j \to \infty} \varphi(x_{j}) = \varphi(x_{0}) \, . \]

But the support of \( \varphi(x_{j}) \) is contained in \( K_{n} \) for each \( j \) so that support \( \varphi(x_{0}) \) is also.

**Lemma 2.22.** \( \{u, T, (K_{n}), M\} \) is an integral basis for the closed span of \( \mathcal{U} L_{n} \).
Proof of lemma. The previous two lemmas give the first two conditions of the basic theorem. We have only to show the existence of a basis constant to apply the theorem.

Fix integers $n$, $p$ and a measure $\mu = \varphi(x_\mu) e^M$. Then we have:

$$
|| \int_{K_n}^p u d\varphi(x_\mu) || = || \sum_{i=1}^n \int_{K_n}^p (i) u d\varphi(x_\mu) ||
\leq C || \sum_{i=1}^n \int_{K_n}^{n+p} (i) u d\varphi(x_\mu) ||
\leq CB || \int_{K_n}^{n+p} u d\varphi(x_\mu) ||
$$

Then we may use CB as our basis constant.

Theorem 2.23. Under the assumptions made in this section, $X$ has an integral basis $\{u, T, (K_n), M\}$.

Proof of theorem. We have constructed $\{u, T, (K_n), M\}$ to be an integral basis for the closed span of $\cup_{i=1}^n X_i$. Now let $x \in \bigoplus_{i=1}^n X_i$, say $x = \sum_{i=1}^n x_i$. Then we have:

$$
x = \sum_{i=1}^n x_i = \sum_{i=1}^n \int_{T_i} u d\varphi(x_i)
= \sum_{i=1}^n \int_{T_i} u d\varphi(x_i)
= \sum_{i=1}^n (\lim_{k \to \infty} \int_{K_k} (i) u d\varphi(x_i))
$$
Thus \( x \) is in the closed span of \( U_{\mathbb{L}} \), and so \( \bigcup \bigoplus_{i=1}^{n} X_i \) is contained in the closed span of \( U_{\mathbb{L}} \). Then so is its closed span, which is just \( X \). Thus \( \{u, T, (K_n), M\} \) is an integral basis for \( X \).

In the first two parts of this chapter, we have seen examples of when certain Schauder decompositions give rise to integral bases. In this section, we shall discuss the structure of the space of measures, \( M \), corresponding to a space, \( X \), with an integral basis. [We show that \( M \) must be the completion with respect to some norm, \( \rho \), of the space \( \mathfrak{M}(K_n) \).] Because of this structure and the structure it imposes on \( X \), we shall find a space with a Schauder decomposition which cannot be the decomposition arising from an integral basis.

We suppose \( X \) has integral basis \( \{u, T, (K_n), M\} \). Denote by \( S_n \) the projection map

\[
S_n(x) = \int_{K_n} u \, d\mu_x .
\]

As before, we take \( L_n = \left\{ \int_{K_p} u \, d\mu \mid p \leq n, \mu \in M \right\} \), and note that \( L_n = S_n(X) \).

As in Chapter 1 our spaces \( \mathfrak{M}(K_n) \) of measures on \( T \) with support in \( K_n \) are assumed to have a norm topology at least as strong as that induced by the total variation norm on \( K_n \). Finally, let the function \( \varphi \) be the map selecting measures for the elements of \( X \) (i.e., \( \varphi(x) = \mu_x \)).
Lemma 2.24. For each integer \( n \), we have

\[
\varphi(L_n) = M(K_n) = \{ \mu \in M \mid \text{supp } \mu \subseteq K_n \}.
\]

**Proof of lemma.** Let \( \{ \mu \in M \mid \text{supp } \mu \subseteq K_n \} = M_n \). By definition, \( M(K_n) \) is contained in \( M_n \). If \( \mu \in M_n \), then \( \mu \in M \) and there is a sequence \( (\mu_k) \) in \( \cup M(K_n) \) with \( \lim_{k \to \infty} \mu_k = \mu \) in the metric \( \rho \) of \( M \). By the definition of \( \rho \), we must have \( \lim_{k \to \infty} \xi_{K_n} \cdot \mu_k = \xi_{K_n} \cdot \mu \) in the topology of \( M(K_n) \).

But \( \mu \in M_n \) so \( \xi_{K_n} \cdot \mu = \mu \). Since \( M(K_n) \) is complete, \( \xi_{K_n} \cdot \mu = \mu \in M(K_n) \).

Thus, we see \( M_n = M(K_n) \).

Now let \( y \in S_n(x) = L_n \). Then for all \( m \geq n \), \( S_m(y) = y \), so that \( \xi_{K_m} \cdot y = \xi_{K_n} \cdot y \). But \( \mu_y = \lim_{m \to \infty} \xi_{K_m} \cdot \mu_y = \xi_{K_n} \cdot \mu \). Then the support of \( \mu_y \) is contained in \( K_n \) so that \( \varphi(L_n) \) is contained in \( M_n \).

On the other hand, if \( \mu \in M_n \), \( \int_{K_n} ud\mu = \int_{K_n} u d\mu L_n \). Then \( \mu = \varphi(\int_{K_n} ud\mu) \in \varphi(L_n) \) so that \( \varphi(L_n) \) contains \( M_n \) and we have \( \varphi(L_n) = M_n \).

Lemma 2.25. For any positive integer \( n \), we have that \( M(K_n) = \text{rca}(K_n) \), as point sets.

**Proof of lemma.** First, let \( \mu \in M(K_n) \). Then \( \mu \) is a regular Borel measure on \( T \). Since \( K_n \) is a compact subset of \( T \), we must have \( |\mu|(K_n) \) finite. But then \( \mu \in \text{rca}(K_n) \) and so \( M(K_n) \) is contained in \( \text{rca}(K_n) \). On the other hand, let \( \mu \in \text{rca}(K_n) \). For \( f \in X^* \) we have:

\[
\left| \int_{K_n} foud\mu \right| \leq ||f|| \cdot ||u|| \cdot \xi_{K_n} \cdot |\mu| < \infty.
\]
Then there exists \( \int u \mu e \). So there exists \( \int u \mu e \). Then \( \mu e \mathcal{M} \) and since the support of \( \mu \) is contained in \( \mathcal{K}^n \), we must have \( \mu e \mathcal{M}(\mathcal{K}^n) \) by lemma 2.24.

**Corollary 2.26.** As topological spaces, \( \mathcal{M}(\mathcal{K}^n) \) in the \( \rho \)-topology and \( \text{rca}(\mathcal{K}^n) \) in the total variation topology, are the same.

**Proof of corollary.** By lemma 2.25 the spaces consist of the same point sets. Further, under the given topologies, both spaces are Fréchet spaces. Finally, both topologies contain the topology of convergence on Borel sets which is Hausdorff. Hence the topologies are the same.

**Theorem 2.27.** There is a Banach space \( X \) with a Schauder decomposition \( (X^*_n; E^*_n) \) but which has no integral basis which yields the same decomposition (i.e., has \( L_n = \bigoplus_{i=1}^{\mathcal{N}} X_i \)).

**Proof of theorem.** We let

\[
X = \left( \sum_{(m)} \right)_{C_0} = \left\{ (a_1) \in \mathcal{M} \left| \left\{ (a_1) \right\} = \sup \left\{ \left\| a_1 \right\| < \infty, \right. \right. \right. \\
\left. \left. \left. \left. \text{and } \left\| a_1 \right\| \to 0 \right\} \right. \right. \right. \\
\text{Then } X \text{ has Schauder decomposition in which, for each } j, X_j = (m).
\]

Suppose \( X \) has integral basis \( \{u, T, (K^*_n), \mathcal{M}\} \) giving the same decomposition. In particular, we would have \( \varphi(L_1) = \mathcal{M}(\mathcal{K}_1) \). Now \( \varphi \) is an isomorphism and we have seen in corollary 2.26 that \( \mathcal{M}(\mathcal{K}_1) \) is isomorphic to \( \text{rca}(\mathcal{K}_1) \). On the other hand, we are assuming that \( L_1 \) is just \( (m) \). Then we have \( (m) \) isomorphic to \( \text{rca}(\mathcal{K}_1) \). Since \( \mathcal{K}_1 \) is a compact, Hausdorff space, we know that \( \text{rca}(\mathcal{K}_1) \) is weakly complete. However, \( (m) \) is not weakly complete, and we have a contradiction.
A CHARACTERIZATION THEOREM

In the third section of Chapter 2, we saw that a space with integral basis must be isomorphic to a completion of $Urca(K_n^r)$ with respect to some norm $\rho$. The natural question arises as to whether the converse is true. A complete answer is not given here, but under certain assumptions on the sets $(K_n^r)$, we can show that the converse does hold.

Let $T$ be a locally compact, Hausdorff space and let $(K_n^r)$ be an increasing sequence of compact subsets of $T$ such that $\bigcup K_n = T$. We shall take $Y$ to be the completion of $Urca(K_n^r)$ with respect to some norm, $\rho$. We make two further assumptions (which have been shown to obtain if $Y$ has an integral basis):

1°) If $\sigma_n : Y \rightarrow rca(K_n^r)$ by

\[ \sigma_n(\mu) = \xi_{K_n^r} \mu \]

then, for each $n$

\[ |\sigma_n| = \sup \left\{ \rho(\xi_{K_n^r} \mu) | \rho(\mu) \leq 1 \right\} \leq K. \]

This is the basis constant.

2°) For each $n$ there exist positive numbers $a_n$, $b_n$ such that for all $\mu \in Y$ we have

\[ a_n \rho(\xi_{K_n^r} \mu) \leq |\xi_{K_n^r} \mu| \leq b_n \rho(\xi_{K_n^r} \mu). \]

(Here $|\cdot|$ denotes total variation - this says $M(K_n) \text{ is isomorphic to } rca(K_n^r).$)
Lemma 3.1. There exists an $\varepsilon_\rho > 0$ such that for any $\mu$ 
$\mu \in \mathcal{U rca}(K_n)$ we have $\varepsilon_\rho \rho(\mu) \leq |\mu|$.

Proof of Lemma. By our assumption $(2^*)$ there is a sequence 
$(a_n)$ of positive numbers such that for any $\mu \in \mathcal{Y}$ we have $a_n \rho(\xi_{K_n} \cdot \mu) \leq |\xi_{K_n} \cdot \mu|$. If $\mu \in \mathcal{U rca}(K_n)$, then $|\mu| < \infty$ and so $a_n \rho(\xi_{K_n} \cdot \mu) \leq |\xi_{K_n} \cdot \mu| \leq |\mu|$. Now, suppose $(a_n)$ is unbounded. Then for each $k$ we find $n_k > n_{k-1}$ such that $a_{n_k} > k$. Fix $\mu \in \mathcal{Rca}(K_n)$ such that 
$|\mu| = 1$. Then $\xi_{K_{n_k}} \cdot \mu = \mu$ for any $m \geq n_o$ and we have: $1 = |\mu| \geq a_{n_k} \rho(\xi_{K_{n_k}} \cdot \mu) = a_{n_k} \rho(\mu)$ for all $k$ such that $n_k \geq n_o$. Since 
$|\mu| = 1$ and $\rho$ is a norm, we have $\rho(\mu) > 0$ and so $1/\rho(\mu) \geq a_{n_k} > k$ for all $n_k > n_o$. This last statement is a contradiction, so $(a_n)$ must be bounded. Then let: 
$$
\varepsilon_\rho = \sup \{ a_n \mid n \geq 1 \}.
$$

Let $\mu \in \mathcal{U rca}(K_n)$, say $\mu \in \mathcal{Rca}(K_{n_o})$. Now $|\mu| \geq a_n \rho(\xi_{K_n} \cdot \mu)$ for all $n$ so that $|\mu| \geq \varepsilon_\rho \rho(\xi_{K_n} \cdot \mu)$ for all $n$. Then, for $n \geq n_o$ we have 
$|\mu| \geq \varepsilon_\rho \rho(\xi_{K_n} \cdot \mu) = \varepsilon_\rho \rho(\mu)$.

We shall now need an index function $u$ from $T$ into $Y$. To get this let $u(t) = \delta_t$, the point mass at $t$.

Lemma 3.2. $|||u|||_n = \sup \{ \rho(u(t)) \mid t \in K_n \}$ is finite.

Proof of Lemma. By lemma 3.1 there is an $\varepsilon_\rho$ such that 
$\rho(u(t)) = \rho(\delta_t) \leq 1/\varepsilon_\rho |\delta_t| = 1/\varepsilon_\rho$. Thus we have $|||u|||_n \leq 1/\varepsilon_\rho$, for any $n$.

We also need to construct a space of measures. Keeping the conditions of the basis theorem of Chapter 1 in mind, we define a
metric for $\cup \text{rca}(K_n)$ and then complete this space with respect to that metric.

**Definition 3.3.** Let $|\cdot|$ denote total variation and define, for $\mu \in \cup \text{rca}(K_n)$:

$$\delta(\mu,0) = \delta(\mu) = \sum 2^{-j} \frac{|\xi_{K_j} \cdot \mu|}{1 + |\xi_{K_j} \cdot \mu|}.$$  

Let $\delta(\mu,\nu) = \delta(\mu-\nu)$ and we see that $\delta$ is a metric on $\cup \text{rca}(K_n)$. Denote by $M_1$ the completion of the space with respect to $\delta$.

**Lemma 3.4.** There is a continuous linear map $\varphi$ from $Y$ into $M_1$.

**Proof of lemma.** Let $\varphi$ be defined on $\cup \text{rca}(K_n)$ by $\varphi(\mu) = \mu$. Suppose we have a sequence $(\mu_n)$ and a point $\mu$ in $\cup \text{rca}(K_n)$ such that, in the $\rho$-topology, $\lim_{n \to \infty} \mu_n = \mu$. Let $\eta > 0$ and choose $N_{\eta}$ such that $\sum_{i \geq N_{\eta}} 2^{-i} < \eta/2$. Now, for $1 \leq j \leq N_{\eta}$, we have, by assumptions $(1^0)$ and $(2^0)$

$$|\xi_{K_j} \cdot (\mu - \mu_k)| \leq b_j \rho(\xi_{K_j} \cdot (\mu - \mu_k)) \leq b_j K \rho(\mu - \mu_k) \leq \max \{b_j \mid 1 \leq j \leq N_{\eta}\} K \rho(\mu - \mu_k) = b_{\eta} K \rho(\mu - \mu_k).$$

We may choose $p_{\eta}$ such that for $k > p_{\eta}$ we have:

$$\rho(\mu - \mu_k) < (b_{\eta} K N_{\eta})^{-1} \eta/2.$$  

Then if $k \geq p_{\eta}$, we have:

$$\delta(\mu - \mu_k) = \sum_{j=1}^{N_{\eta}} 2^{-j} \frac{|\xi_{K_j} \cdot (\mu - \mu_k)|}{1 + |\xi_{K_j} \cdot (\mu - \mu_k)|} + \sum_{j > N_{\eta}+1} 2^{-j} \frac{|\xi_{K_j} \cdot (\mu - \mu_k)|}{1 + |\xi_{K_j} \cdot (\mu - \mu_k)|}.$$
Thus, in the $\delta$-topology, we have $\lim_{n \to \infty} \mu_n = \mu$. Since $\varphi$ is continuous on $\bigcup \text{rca}(K_n)$ and $M_1$ is complete, we may extend $\varphi$ to all of $Y$. We denote by $M$ the image $\varphi(Y)$ in $M_1$.

We have now set up the situation as it appears above the basic theorem. If we know that the representation $f(\mu) = \int f \mu$ for each $\mu \in Y$ and $f \in Y^*$ holds, then we need only to show that the three conditions of the theorem hold to get that $\{\mu, \cdot, (K_n), M\}$ is an integral basis for $Y$. If the representation is valid for $\bigcup \text{rca}(K_n)$, then we shall be done for then we get an integral basis for the closure of this union. For the moment, we shall assume the representation and finish the proof. We shall then consider the question of when the representation is valid.

**Lemma 3.5.** $L_n = \left\{ \int_{K_n} \mu \, | \, \mu \leq n, \mu \in M \right\}$ is a closed subspace of $Y$ for every $n$.

**Proof of lemma.** We note that, if $\mu \in \text{rca}(K_n)$, then $\xi_{K_n} \cdot \mu = \sigma_n(\mu) = \mu$ so that $\text{rca}(K_n) \subseteq \sigma_n(Y)$. On the other hand, if $\mu \in Y$, then $\sigma_n(\mu) = \xi_{K_n} \cdot \mu \in \text{rca}(K_n)$ so we have $\text{rca}(K_n) = \sigma_n(Y)$, which is closed in $Y$. If we now show that $L_n = \text{rca}(K_n)$ for each $n$, we shall be done.
Fix $n$ and let $\nu \in L_n$, say $\nu = \int_{K_n} \mu$. Then, for $f \in Y^*$, we have

$$f(\nu) = \int_{K_n} f \circ \mu = \int_{T} f(\xi_{K_n} \cdot \mu) = f(\xi_{K_n} \cdot \mu).$$

Thus, $\nu = \xi_{K_n} \cdot \mu \in \text{rca}(K_n)$ and so $L_n \subseteq \text{rca}(K_n)$. Now let $\mu \in \text{rca}(K_n)$. Then there exists $\int_{K_n} \mu \in L_n$. If $f \in Y^*$ we have

$$\int_{K_n} f \circ \mu = \int_{T} f(\int_{K_n} \mu) = f(\mu).$$

Thus $\mu = \int_{K_n} \mu \in L_n$ so that $\text{rca}(K_n) \subseteq L_n$.

**Lemma 3.6.** If $\int_{K_n} \mu = \int_{K_n} \nu$ then $\xi_{K_n} \cdot \mu = \xi_{K_n} \cdot \nu$.

**Proof of lemma.** Suppose that for some such $\mu, \nu$ and $n$ we have $\xi_{K_n} \cdot \mu \neq \xi_{K_n} \cdot \nu$. Then there is an $f \in Y^*$ such that $f(\xi_{K_n} \cdot \mu) \neq f(\xi_{K_n} \cdot \nu)$.

But:

$$0 = f(\int_{K_n} (\mu - \nu)) = f(\int_{T} (\xi_{K_n} \cdot (\mu - \xi_{K_n} \cdot \nu))) = f(\xi_{K_n} \cdot (\mu - \xi_{K_n} \cdot \nu)) \neq 0$$

Since we have reached a contradiction, we must have $\xi_{K_n} \cdot \mu = \xi_{K_n} \cdot \nu$.

**Lemma 3.7.** There exists a $C > 1$ such that for any $\mu \in M$ and integers $n, p$ we have

$$\rho(\int_{K_n} \mu) \leq C \rho(\int_{K_{n+p}} \mu).$$
Proof of lemma. By our assumption (1°), $||\sigma_n|| \leq K$ for all $n$. Let $\mu \in M$ and fix integers $n, p$. Then we have:

$$\rho \left( \int_{K_n} ud\mu \right) = \rho \left( \int_{K_n} ud\xi_{K_n} \cdot \mu \right)$$

$$= \rho \left( \sigma_n \left[ \int_{K_n} ud\xi_{K_n} \cdot \mu \right] \right)$$

$$= \rho \left( \sigma_n \left[ \int_{K_n+p} ud\mu \right] \right)$$

$$\leq K \rho \left( \int_{K_n+p} ud\mu \right).$$

Then we are done if we let $C = K$.

Corollary 3.8. $[u, T, (K_n), M]$ is an integral basis for $Y$.

Proof of corollary. By the lemmas above, we have fulfilled the conditions of the basic theorem. $\{u, T, (K_n), M\}$ is, therefore, an integral basis for the closed span of $\cup L_n$. But this is the same, as we saw in the proof of lemma 3.5, as the closed span of $\cup rca(K_n)$, which is $Y$.

We now turn back to the question of the validity of our representation. In order to get this, we make an additional assumption (which may or may not be necessary). We shall assume that, for each $n$, there exists an $m$ such that $K_n \subseteq \text{Int } K_m$. (Int $K_m$ is the interior of $K_m$.) We show that this assumption allows us to say that for all $\mu \in \cup rca(K_n)$ we have $f(\mu) = \int_T Fou \, d\mu$ for any $f \in Y^*$. 
Definition 3.9. Let $Y^\#$ denote the algebraic dual of $Y$.

Define a map, $\varphi$, from $C(T)$ into $Y^\#$ by $\varphi f(\mu) = \int_T f d\mu$.

Our assumption about the topology in $T$ will allow us to put enough functions from $C(T)$ into $Y^*$ (by $\varphi$) to establish the representation.

Lemma 3.10. For each $f \in C(K)$ there is an $\tilde{f} \in C(T)$ such that

a) $\tilde{f}$ is an extension of $f$,

b) $\varphi \tilde{f} \in Y^*$.

Proof of lemma. Fix $n_0$ and choose $m_0$ such that $K_n \subseteq \text{Int } K_m$.

For $f \in C(K_n)$ we have, by the Tietze extension theorem, that there is an $\tilde{f} \in C(T)$ such that $\tilde{f}|_{K_n} = f$ and with the support of $\tilde{f}$ lying in $K_m$.

We know $\varphi \tilde{f} \in Y^\#$ and we have:

$$|\varphi \tilde{f}(\mu)| = \left| \int_T \tilde{f} d\mu \right| = \left| \int_{K_m} \tilde{f} d\mu \right|$$

$$\leq \left| \int_{K_m} d\mu \right| \mu(K_m)$$

$$= \left| \int_{K_m} |\xi_{K_m} \cdot \mu|(K_m) \right|$$

$$\leq \left| \int_{K_m} b_m \rho(K_n) \cdot \mu \right|$$

Where $b_m$ and $K$ are as in the work above. Then we have

$$\left| \varphi \tilde{f} \right| = \sup \left| \tilde{f}(\mu) : \rho(\mu) \leq 1 \right| \leq \left| \tilde{f} \right| b_m K.$$
Lemma 3.11. Let $\Omega = \varphi [C(T)] \cap Y^*$. Then $\Omega$ is dense in the $\sigma(Y^*, U \text{rca}(K_n))$ topology of $Y^*$. That is, if $f \in Y^*$, there exists $(f^{(n)})_{\alpha \in A}$ in $\Omega$ such that $f(\mu) = \lim_{\alpha} f^{(n)}(\mu)$ for all $\mu \in U \text{rca}(K_n)$.

Proof of lemma. Let $g \in Y^*$. Then $g|_{\text{rca}(K_n)} = g_n \in [\text{rca}(K_n)]^*$.

But $C(K_n)$ is $\omega^*$-dense in $[\text{rca}(K_n)]^*$ so there is a net $(f^{(n)})_{\alpha \in A_n}$ in $C(K_n)$; such that $\lim_{\alpha} f^{(n)}(\mu) = g_n(\mu)$ for all $\mu \in \text{rca}(K_n)$. We now define a new net in $Y^*$. Let

$$D = \bigcup(A_j \times P) \quad (P \text{ the natural numbers}).$$

We order $D$ as follows:

$$(\alpha_1, n_1) \geq (\alpha_2, n_2) \text{ if and only if either}$$

$$n_1 \geq n_2 \text{ or } n_1 = n_2 \text{ and } \alpha_1 \geq \alpha_2.$$

Let $g_{(\alpha, n)} = \varphi f^{(n)}_{\alpha}$ for each $(\alpha, n) \in D$. Then $(g_{(\alpha, n)})$ is a net in $\Omega$ and if $\mu \in \bigcup \text{rca}(K_n)$ we have $\mu \in \text{rca}(K_m)$ for some $m$. For $m \geq m_o$, we have $\mu(g_{m_o}) = \mu(g_{m_o}) = \lim_{n \to \infty} g_n(\mu) = g(\mu)$. Thus, for $m \geq m_o$

$$g(\mu) = \mu(g_{m_o}) = \lim_{\alpha \in A_m} g_{(\alpha, m)}(\mu).$$

That is, $(g_{(\alpha, m)})$ converges to $g$ in the $\sigma(Y^*, U \text{rca}(K_n))$ topology.

Definition 3.12. For each $\mu \in \text{rca}(T)$, let $\hat{\varphi}$ denote the scalar field and
\( \mathcal{G}(\mu) = \{ f: T \to \mathbb{F} \mid f \text{ is } \mu\text{-measurable} \} \).

Let

\[ \mathcal{G} = \bigcap \{ \mathcal{G}(\mu) \mid \mu \in \text{rca}(T) \} \].

Define a map, \( \psi \), from \( \mathcal{G} \) into \( Y \) by \( \psi f(\mu) = \int \limits_T f d\mu \).

**Lemma 3.13.** Let \( \mathcal{G} \) and \( \psi \) be as above. Then if \( \hat{\Omega} = \psi(\mathcal{G}) \cap Y^* \), we have \( \hat{\Omega} = Y^* \).

**Proof of lemma.** If \( f \in C(T) \) then \( f \) is \( \mu\text{-measurable} \) for all \( \mu \in \text{rca}(T) \) and so \( C(T) \subseteq \mathcal{G} \). Further, if \( \phi \) is the map of definition 3.9, we have \( \psi| C(T) = \phi \). Then

\[ \Omega \subseteq \hat{\Omega} \subseteq Y^* \].

Let \( f \in Y^* \), then by lemma 3.11 there is a net \( (f_\alpha) \) \( \alpha \in A \) in \( \Omega \) such that, for each \( \mu \in \text{rca}(K_n) \),

\[ f(\mu) = \lim \alpha \in A f_\alpha(\mu) \].

Let us say \( f_\alpha = \phi_\alpha h_\alpha \) for \( (h_\alpha) \) in \( C(T) \). In particular, for any \( t \in T \) we have

\[ f(\delta_t) = \lim \alpha \in A h_\alpha(t) = \lim \alpha \in A f_\alpha(\delta_t) \].

Define \( h \) from \( T \) into the scalar field \( \mathbb{F} \) by

\[ h(t) = \lim \alpha \in A h_\alpha(t) = f(\delta_t) \].

Then,

\[ \sup \{ |h(t)| \mid t \in T \} = \sup \{ |f(\delta_t)| \mid t \in T \} \leq ||f|| \sup \{ \rho(\delta_t) \mid t \in T \} \leq ||f|| (1/\rho) \].
Thus, h is bounded and is the pointwise limit of \( \mu \)-measurable functions. Then h is a \( \mu \)-measurable function for any \( \mu \in \mathcal{U} \) \( \text{rca}(K_n) \). That is, \( h \in \mathcal{G} \). But then \( \psi h \in Y^* \). On the other hand, for any \( \mu \in \mathcal{U} \) \( \text{rca}(K_n) \), we have \( \psi h(\mu) = f(\mu) \). This means that \( \psi h \) must be in \( (\mathcal{U} \text{rca}(K_n))^* = Y^* \) and so \( f = \psi h \). Then \( f \in \hat{\Omega} \) and we see \( Y^* \subseteq \hat{\Omega} \).

**Corollary 3.14.** For every \( \mu \in \mathcal{U} \) \( \text{rca}(K_n) \) we have

\[
\int f(\mu) = \int <f,u> \, d\mu \quad \text{for all } f \in Y^* .
\]

**Proof of corollary.** Fix \( \mu \in \text{rca}(K_n) \) and let \( f \in Y^* \), say \( f = \psi h \) for \( h \in \mathcal{G} \). Then \( f(\delta_t) = \psi h(\delta_t) = h(t) \) so we have

\[
\int_{\mathcal{T}} <f,u> \, d\mu = \int_{K_n} <f,u> \, d\mu
\]

\[
= \int_{K_n} h \, d\mu
\]

\[
= \int_{\mathcal{T}} h \, d\mu = \psi h(\mu) = f(\mu) .
\]

In this section we digress a bit to get a rather alarming result. We shall show the power of our assumption about the boundedness of the index function on the subsets \( (K_n) \). If we drop this assumption, we are able to show that every Banach space has an integral basis type system and further that every Banach space with a Schauder decomposition has such a system with the same decomposition.

**Lemma 3.15.** Let \( X \) be any non-empty set. Then there is a topology for which \( X \) is compact and Hausdorff.
Proof of lemma. Fix \( x_0 \in X \) and define open sets as follows: \( \emptyset \) is open if either \( x_0 \notin \emptyset \) or \( x_0 \notin \emptyset \) and \( X - \emptyset \) is a finite set. This defines, for each \( x_0 \in X \), a topology on \( X \) which makes \( X \) both compact and Hausdorff.

Lemma 3.16. Let \( X \) be a Banach space. Then there exists a compact Hausdorff space \( T \) and a map \( u \) from \( T \) into \( X \) such that for every \( x \in X \) there is a \( \mu_x \), a regular Borel measure on \( T \), such that \( x = \int_T \mu_x \). The integral is taken in the sense that for all \( f \in X^* \) we have \( f(x) = \int_T <f, u> \, d\mu_x \).

Proof of lemma. \( U(X^*) = \{ f \in X^* \mid ||f|| \leq 1 \} \) is compact. Define \( \varphi \) from \( X \) into the space \( C(U(X^*)) \) by

\[ \varphi(x)(f) = f(x) \]

Then \( \varphi \) is an isometry into. We take for \( T \) the image \( \varphi(X) \). By lemma 3.15 there is a topology which makes \( T \) a compact Hausdorff space. Define \( u \) from \( T \) into \( X \) by \( u(\varphi(x)) = x \). To get measures to represent elements of \( X \) we let \( \mu_x = \delta_{\varphi(x)} \) the point mass of \( \varphi(x) \). Then if \( x \in X \), \( f \in X^* \) we have:

\[ \int_T <f, u> \, d\mu_x = \varphi(u)(\varphi(x)) = f(x) \]

Theorem 3.17. Let \( X \) be a Banach space with Schauder decomposition \( (X_k; E_k) \). Then \( X \) has an integral basis type system \( \{u, T, (K_n), M\} \) whose associated decomposition is just \( (X_k; E_k) \).
Proof of theorem. Each $X_k$ is a Banach space so construct $q_k, u_k, T_k, \mu_k(x)$ as in lemma 3.16. Let $T = \bigcup T_j$ with its natural topology. That is, $\theta$ is an open neighborhood of $x \in T$ if $x \in \theta$ and $\theta = \bigcup \theta_j$ with $\theta_j$ open in $T_j$. This gives a locally compact Hausdorff topology for $T$. If we let $K_n = \bigcup_{k=1}^{n} T_k$ then $K_n$ is compact and $K_n \subseteq K_{n+1}$ for each $n$. To get an index function, let $u$ take $T$ into $X$ by $u(t) = \sum u_j(t)$ where $u_j(t) = u_j(t)$ for $t \in T_j$ and $u_j(t) = 0$ if $t \notin T_j$. We must finally construct a space of measures $M$. For $x \in X$ we have $x = \sum x_j$. For compact subsets, $C$, of $T$ let

$$p(C) = \sum_{x \in C} p(x) \quad (C \subseteq T_j).$$

Since $C$ is compact and the support of $\mu_j(\cdot)$ lies in $T_j$, this makes sense. (The $T_j$ are disjoint open subsets of $T$ so a finite number, at most, intersect $C$.) As we did in lemma 2.16, we construct a regular Borel measure, $\mu_x$, on $T$ by letting

$$\mu_x(A) = \sup \{ \mu_x(C) \mid C \text{compact, } C \subseteq A \}.$$  

This gives a regular inner measure which yields a measure on the $\sigma$-algebra generated by the compact sets, namely, the Borel sets. We shall let $M = \{ \mu_x \mid x \in X \}$ and norm $M$ by $||\mu_x|| = ||x||$. This completes the construction of $\{u, T, (K_n), M\}$. We shall use the next three lemmas to show that we have an integral basis in the sense of Edwards. Since we do not have the complete set of conditions to use the basic theorem, we shall work from the definition.
Lemma 3.18. For every \( x \in X \) and every integer \( n \), there exists \( \sum_{j=1}^{n} \langle f, u \rangle d\mu_x \in X \).

Proof of lemma. Let \( f \in X^* \) and we have:

\[
\sum_{j=1}^{n} \langle f, u \rangle d\mu_x = \sum_{j=1}^{n} \langle f, u \rangle d\mu_j(E_j(x))
\]

So that, for each \( n \) and \( x = \sum_{j=1}^{n} x_j \), \( \sum_{j=1}^{n} \langle f, u \rangle d\mu_x = \sum_{j=1}^{n} x_j \).

Lemma 3.19. For every positive integer \( n \) and any \( f \in X^* \), the map \( \mu \to \int_X \langle f, u \rangle \, d\mu \) is continuous.

Proof of lemma. Since we have a Schauder decomposition, there exists a \( K \geq 1 \) such that for any \( n \) we have \( \| \sum_{j=1}^{n} E_j \| \leq K \).

Now, fix \( n \), \( f \in X^* \) and suppose there is a net \( (u_A, \alpha) \) in \( M \) such that

\[
\lim_{\alpha \to \infty} \mu_{x_{\alpha}} = 0.
\]

Then for any \( \eta > 0 \) choose \( \alpha_\eta \) such that if \( \alpha \geq \alpha_\eta \) then \( \| \mu_{x_{\alpha}} \| < (2K\| f \|)^{\eta} \). Thus, if \( \alpha \geq \alpha_\eta \) we have

\[
\| \int_X \langle f, u \rangle \, d\mu \| = \| f \| \int_X \| \sum_{j=1}^{n} E_j(x_{\alpha}) \| d\mu_x
\]

= \( \| f \| \| \sum_{j=1}^{n} E_j(x_{\alpha}) \| \)
Lemma 3.20. For each $x \in X$ we have

$$\lim_{n \to \infty} S_n(x) = x$$

Proof of lemma. Fix $x \in X$, $x = \sum E_j(x)$ and we have

$$S_n(x) = \sum_{k=1}^n E_j(x)$$

But $\lim_{n \to \infty} S_n(x) = x$. Thus $\{u, T, (K_n), M\}$ is an integral basis for $X$.

Lemma 3.21. The associated decomposition is $L_n = \bigoplus \sum_{j=1}^n X_j$.

Proof of lemma. Let $S_n(x) = \int K_n^X$ as usual. Then:

$$(S_n - S_{n-1}) X = \{ \int_{K_n} u d\mu_x - \int_{K_{n-1}} u d\mu_x \mid x \in X\}$$

$$= \{ \sum_{j=1}^n E_j(x) - \sum_{j=1}^{n-1} E_j(x) \mid x \in X\}$$

$$= \{ E_n(x) \mid x \in X\} = E_n(X)$$

So $L_n = \bigoplus \sum_{j=1}^n (S_j - S_{j-1}) X = \bigoplus \sum_{j=1}^n E_j(X) = \bigoplus \sum_{j=1}^n X_j$.
As a final remark in this chapter, let us again consider the question of validity of representation involved in proving our characterization theorem (corollary 3.8). In some special cases, we need not make out assumption $K_n \subseteq \text{Int } K_m$ for some $m$. The assumption is used only in the proof of lemma 3.10 where we showed that there are a lot of functions from $C(T)$ in $Y^*$. The rest of our argument depends only on having these functions in the dual and not on the assumption. As the argument stands, we can establish the results following lemma 3.10, if we know that our function $\varphi$ carries all of $C(T)$ into $Y^*$. If we have $T$ a compact Hausdorff space and $Y = \text{rca}(T)$, then $Y$ has an integral basis, taking $K_n = T$ for each $n$ and using the argument following lemma 3.10 to get the representation we need.
SERIES BASIS ANALOGUES

In the first part of this chapter, we consider the question of integral basis structures in the dual of a space with an integral basis. If $X$ has an integral basis whose associated Schauder decomposition is $(X^*_k; E^*_k)$, we give necessary and sufficient conditions for $[X^*_j]$, the closed span of $\{X^*_1, X^*_2, \ldots\}$ to have an integral basis.

**Theorem 4.1.** Let $X$ be a Banach space with integral basis \{u, t, (K_n), M\} whose corresponding Schauder decomposition is $(X^*_k; E^*_k)$. Then each $X^*_k$ is finite dimensional if and only if $[X^*_j]$ has an integral basis $\{u', T', (K'_n), M'\}$ such that

$$\bigoplus_{j=1}^{n} X^*_j = L' = \left\{ \int_{K'_p} u'd\mu' \mid p \leq n, \mu' \in M' \right\}.$$

**Proof of theorem.** First suppose each $X^*_k$ is finite dimensional. Then for each $k$ there is an isomorphism, $\varphi_k$, of $X^*_k$ onto $X^*_k$. From these maps, we define maps $\hat{\varphi}_k$ from $\bigcup \bigoplus_{j=1}^{n} X^*_j$ into $X^*_k$ by $\varphi_k(x) = \hat{\varphi}_k(x) = \varphi_k(E_k(x))$. Finally, define $\varphi$ from $\bigcup (\bigoplus_{j=1}^{n} X^*_j)$ to $\bigcup (\bigoplus_{j=1}^{n} X^*_j)$ by

$$\varphi(x) = \sum \hat{\varphi}_j(x).$$

This sum makes sense because $x$ is in the finite span of $\{X^*_k\}$ so that all but finitely many $\hat{\varphi}_j(x)$ are $0$. We claim further that $\varphi$ is one
to one, onto and linear from $U(\bigoplus_{j=1}^{n} X_j)$ to $U(\bigoplus_{j=1}^{n} X_j^*)$. If $x_1 + x_2 \in U(\bigoplus_{j=1}^{n} X_j)$ then, for some $k$, $E_k(x_1) \neq E_k(x_2)$. But $\phi_k$ is one to one so $\phi_k(E_k(x_1)) \neq \phi_k(E_k(x_2))$ and so $\phi(x_1) \neq \phi(x_2)$. Thus, $\phi$ is one to one. Let $y \in U(\bigoplus_{j=1}^{n} X_j^*)$, say $y = \sum_{j=1}^{n} y_j$ with $y_j \in X_j^*$. Again, $\phi_k$ was an isomorphism so there exists $\phi_k^{-1}(y_k) \in X_k$ for each $k$. Then

$$
\phi\left(\sum_{j=1}^{n} \phi_j^{-1}(y_j)\right) = \sum_{k=1}^{n} \phi_k\left(\sum_{j=1}^{n} \phi_j^{-1}(y_j)\right) = \sum_{j=1}^{n} y_j = y.
$$

So we have $\phi$ is onto. Let $x = \sum_{j=1}^{n} x_j$ with $x_j \in X_j$. Then $\alpha x = \sum_{j=1}^{n} \alpha x_j$ and we have

$$
\phi(\alpha x) = \phi\left(\sum_{j=1}^{n} \alpha x_j\right) = \sum_{k=1}^{n} \phi_k\left(\sum_{j=1}^{n} \alpha x_j\right) = \sum_{k=1}^{n} (\phi_k(\alpha E_k(x_k))) = \sum_{j=1}^{n} \phi_j(\alpha x_j) = \alpha \sum_{j=1}^{n} \phi_j(x_j) = \alpha \phi(x).
$$

Finally, let $x + y \in U(\bigoplus_{j=1}^{n} X_j)$, say $x = \sum_{j=1}^{n} E_j(x)$ and $y = \sum_{j=1}^{m} E_j(y)$. We may assume $n \geq m$ and write $y = \sum_{j=1}^{m} E_j(y)$ where $E_k(y) = 0$ for $m + 1 \leq k \leq n$. Then we have

$$
\phi(x + y) = \phi\left(\sum_{j=1}^{n} [E_j(x) + E_j(y)]\right) = \sum_{k=1}^{n} \phi_k\left(\sum_{j=1}^{n} [E_j(x) + E_j(y)]\right) = \sum_{k=1}^{n} \phi_k(E_k(x) + E_k(y)).
$$
\[ \sum_{k \geq 1} \hat{\phi}_k(E_k(x)) + \sum_{k \geq 1} \hat{\phi}_k(E_k(y)) = \phi(x) + \phi(y). \]

Hence, \( \phi \) is linear and so is an algebraic isomorphism of \( \bigoplus \bigoplus_{j=1}^n X_j \) and \( \bigoplus \bigoplus_{j=1}^n X_j^* \). We now proceed to the construction of an integral basis. We use the basis \( \{u, T, (K), M\} \) of \( X \). Let \( T' = \bigcup K_n \) and \( K'_n = K_n \) for each \( n \). We use the same space of measures, \( M \), and define \( u' \) on \( T' \) by \( u'(t) = \phi(u(t)) \). Then \( u' \) is \( [X_j^*] \)-valued and we have, for each \( n \),

\[ \|u'\|_n = \sup_{t \in K_n} \|u'(t)\| \]

\[ = \sup_{t \in K_n} \|\phi u(t)\| \]

\[ \leq \|\phi\|[X_1, \ldots, X_n]\| \|u\|_n < \infty. \]

We select measures in \( M' = M \) by letting \( \nu_y = \mu_{\phi^{-1}}(y) \) for each \( y \in \bigoplus \bigoplus_{j=1}^n X_j^* \). Our system \( \{u', T', (K'_n), M\} \) satisfies the general hypotheses of Chapter one and we must show it satisfies the conditions of proposition 1.3. First, \( y \in L'_n \to y = \int_{K_n} u'd\nu \)

\[ \to y = \int_{K_n} \phi u d\mu_{\phi^{-1}}(y) \]

\[ \to \phi^{-1}(y) = \int_{K_n} u d\mu_{\phi^{-1}}(y) \]
so that \( L_n' = \varphi(L_n) = \bigoplus_{j=1}^{n} X_j^* \) and so \( L_n' \) is a closed subspace of \([X_j^*]\). Suppose \( \mu, \nu \) are in \( M' \) and that

\[
\int_{K_n'} u'd(\mu - \nu) = 0
\]

Then \( \int_{K_n'} ud(\mu - \nu) = 0 \) so \( \xi_{K_n'}, \mu = \xi_{K_n'}, \nu \).

Now, let \( v \in M' \) and fix integers \( n \) and \( p \). Let \( C^* \) be the constant of the Schauder decomposition \( \bigoplus_{j=1}^{n} X_j^* \). Then for

\[
\int_{K_p'} u'dv = \sum_{j=1}^{q} x_j^*
\]

we have

\[
|\int_{K_p'} u'dv| = \left| \sum_{j=1}^{q} x_j^* \right|
\]

\[
\leq C^* \left| \sum_{j=1}^{n+p} x_j^* \right| = C^* \left| \int_{K_n'} u'dv \right|
\]

Then we apply proposition 1.3 to get that \( \{u', T', (K_n'), M'\} \) is an integral basis for the completion in \([X_j^*]\) of \( \bigcup L_n' \). But, for each \( n \), \( L_n' = \bigoplus_{j=1}^{n} X_j^* \) so that \( \{u', T', (K_n'), M'\} \) is an integral basis for \([X_j^*]\). To get the converse statement, suppose \([X_j^*]\) has integral basis \( \{v, S, (C_n), N\} \) such that \( L_n' = \bigoplus_{j=1}^{n} X_j^* \). As we have seen in
corollary 2.26, we have $L_n$ isomorphic to $\text{rca}(C_n)$. On the other hand, we have $L_n$ isomorphic to $\text{rca}(K_n)$. Thus, we have $(\text{rca}(K_n))^* = L_n^* = L_n' = \text{rca}(C_n)$. But then $\text{rca}(K_n)$, and hence $\bigoplus_{j=1}^n X_j'$, is finite dimensional. Then we must have $X_k$ finite dimensional for all $k \geq 1$.

We now consider reflexivity of spaces with integral bases. We use the works of Zippin [12] and Sanders [11] as models to apply the concepts of shrinking and boundedly complete Schauder decompositions to find necessary and sufficient conditions for reflexivity.

Definition 4.2. Let $X$ be a Banach space with Schauder decomposition $\{X_k; E_k\}$.

(i) $\{X_k; E_k\}$ is shrinking if, for each $f \in X^*$,

$$\|f\|_n = \|f[X_{n+1}, X_{n+2}, \ldots]\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$  

(ii) $\{X_k; E_k\}$ is boundedly complete if, for any $(x_j) \in X_j$ such that

$$\sup_n \left| \sum_{j=1}^n x_j \right| < \infty,$$

then $\sum x_j$ converges.

Definition 4.3. Let $X$ be a Banach space with an integral basis $\{u, T, (K_n), M\}$. The integral basis is said to be shrinking (respectively, boundedly complete) if its associated Schauder decomposition is shrinking (respectively, boundedly complete).

Lemma 4.4. Let $X$ be a Banach space with Schauder decomposition $\{X_k; E_k\}$. If, for any $(x_j) \in X_j$, we have

$$\|f[X_{n+1}, X_{n+2}, \ldots]\| \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for each } f \in [X_j]^*$$
then \( X^* = \bigoplus \Sigma X_j^* \).

**Proof of lemma.** We know \( \bigoplus \Sigma X_j^* \subseteq X^* \) so choose \( f \in X^* \) and show \( f \in \bigoplus \Sigma X_j^* \). For each \( k \), let \( f_k = f_{\text{e}X_k} \). Then \( f_k \in X_k^* \) and, for \( x \in X \), \( f(x) = f(\sum E_k(x)) = \sum f_k(x) \). Thus \( f = \lim_{n \to \infty} \sum_{j=1}^n f_j \) in the \( \sigma(X^*, X) \)-topology. Suppose \( \sum_{j=1}^n f_j \) does not converge to \( f \) in norm. Then \( (\sum_{j=1}^n f_j) \) is not a norm Cauchy sequence, for if it converges at all, it must be to \( f \). Then there exists \( \eta > 0 \) and an increasing set of integers \( (p_k) \) such that
\[
| \sum_{j=1}^{p_{k+1}} - \sum_{j=1}^{p_k} f_j | > \eta .
\]

Then for each \( k \), there is an \( x^{(k)} \in X \) with \( ||x^{(k)}|| \leq 1 \) and such that
\[
| \sum_{j=p_k+1}^{p_{k+1}} f_j(x^{(k)}) | = | \left( \sum_{j=1}^{p_{k+1}} f_j - \sum_{j=1}^{p_k} f_j \right)x^{(k)} | \geq \eta/2
\]

We let \( x_j = E_j(x^{(k)}) \) for \( p_k < j \leq p_{k+1} \) and we have \( (x_j) \in X_j \). But, for any \( k \),
\[
| f \left( \sum_{j=p_k+1}^{p_{k+1}} x_j \right) | \geq | f \left( \sum_{j=p_k+1}^{p_{k+1}} x_j \right) |
\]
\[
= | \left( \sum_{j=p_k+1}^{p_{k+1}} f_j \right) \left( \sum_{j=p_k+1}^{p_{k+1}} x_j \right) | \geq \eta/2 > 0.
\]

This contradicts our assumption on sequences \( (x_j) \in X_j \). Thus \( f = \sum f_j \) so that \( f \in \bigoplus \Sigma X_j^* \).

**Theorem 4.5.** Let \( X \) be a Banach space with a shrinking integral basis \( \{ u, T, (K_n), M \} \). If \( \{ X_k; E_k \} \) is the associated
Schauder decomposition then \( X^* = \bigoplus_j X_j^* \).

**Proof of theorem.** Let \( (x_j)_{j\in\mathbb{N}} \) and \( f \in \mathcal{H}(X_0)^* \). Then there exists \( f_{\infty} \in X^* \) such that \( ||\hat{f}|| = ||f|| \) and \( \hat{f} | [x_j] = f \). Then,

\[
||f| [x_{n+1}, x_{n+2}, \ldots]|| = ||\hat{f}| [x_{n+1}, x_{n+2}, \ldots]||
\]

\[
\leq ||\hat{f}| [x_{n+1}, x_{n+2}, \ldots]|| \rightarrow 0 \text{ as } n \rightarrow \infty.
\]

Thus \( X \) satisfies the hypotheses of lemma 4.4, so \( X^* = \bigoplus \sum X_j^* \).

**Lemma 4.6.** Let \( X \) be a reflexive Banach space with an integral basis \( \{u, T, (K_n), M\} \). Then \( X \) has a basis with parentheses. In particular, if \( X \) is reflexive and has an integral basis, \( X \) must be separable.

**Proof of lemma.** If \( X \) is reflexive then the closed subspace \( L_n = \{ \sum_{K_p} u \mu | p \leq n, \mu \in M \} \) is also reflexive. Since \( L_n = \text{rca}(K_n) \) by corollary 2.26 of Chapter two, we must have \( L_n \) finite dimensional. But then the associated Schauder decomposition \( (X_k; E_k) \) must have \( X_k \) finite dimensional for each \( k \). As we have seen in the first part of Chapter two, this is equivalent to the existence of a basis with parentheses for \( X \).

Because of the way in which we define shrinking and boundedly complete integral bases, we may apply a result of Sanders [11] to get a necessary and sufficient condition for reflexivity in Banach spaces with integral bases. If \( X \) has an integral basis whose associated Schauder decomposition \( (X_k; E_k) \) has \( X_k \) finite dimensional for all \( k \) (that is, \( X \) has a basis with parentheses) then \( X \) is reflexive if and
only if the integral basis is both shrinking and boundedly complete.

We can also give conditions for reflexivity in terms of shrinking or boundedly complete integral bases in the same way that Zippen [11] has done for series bases. Again, we assume $X$ has a basis with parentheses and that all integral bases are shrinking or that all are boundedly complete. Under these assumptions, $X$ is reflexive. A straightforward modification of Zippen's proof for series bases gives the result for bases with parentheses.

Another familiar theorem from the theory of series bases has an analogue in integral bases. The Paley-Wiener theorem on equivalence of series basis says that if $(x_j)$ is a basis and $(y_j)$ is close to $(x_j)$, then $(y_j)$ is also a basis.

**Theorem 4.7.** Let $\{u, T, (K_n), M\}$ be an integral basis for a Banach space $X$. Let $v: T \to X$ and suppose there is a $\lambda (0 < \lambda < 1)$ such that

$$\left| \int_{K_n} (u-v) d\mu \right| \leq \lambda \left| \int_{K_n} u d\mu \right|$$

for all $n$, $\mu \in M$. Then $\{v, T, (K_n), M\}$ is an integral basis for $X$.

We close the paper with some problems.

**Problem 4.8.** In our characterization theorem, how can the restriction $K_n \subseteq \text{Int} K_{n+1}$ be removed?

**Problem 4.9.** We have noted that a reflexive space with an integral basis is separable. Is separability, along with boundedly complete and shrinking a sufficient condition for reflexivity of a space with an integral basis?
BIBLIOGRAPHY


