TURPIN, Richard Harold, 1939-

BASIS TRANSFORMATION, LEAST SQUARE, AND CHARACTERISTIC MODE TECHNIQUES FOR THIN-WIRE SCATTERING ANALYSIS.

The Ohio State University, Ph.D., 1969
Engineering, electrical

University Microfilms, Inc., Ann Arbor, Michigan
BASIS TRANSFORMATION, LEAST SQUARE, AND CHARACTERISTIC MODE
TECHNIQUES FOR THIN-WIRE SCATTERING ANALYSIS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

Richard Harold Turpin, B.S.E.E., B.S. Math., M.S.E.E.

*****

The Ohio State University
1969

Approved by

[Signature]
Adviser
Department of
Electrical Engineering
ACKNOWLEDGMENT

The author is pleased to acknowledge the guidance and encouragement given so unselfishly by his adviser, Professor J.H. Richmond, and by colleague Dr. R.J. Garbacz. Also deserving of recognition are Professors W.H. Peake, D.B. Hodge, and L. Peters, Jr., and Dr. G.A. Thiele for their critical reading of the manuscript as well as for earlier suggestions and assistance. The assistance and cooperation of all his associates at the ElectroScience Laboratory are greatly appreciated.

The support of The Ohio State University Computer Center, which provided computer time for portions of this work, is also gratefully acknowledged.
VITA

July 30, 1939
Born - Manning, Iowa

May, 1962....
B.S.E.E., B.S. Math.
Iowa State University, Ames, Iowa

1962-1964....
Hughes Master's Work-Study Fellow
Hughes Aircraft Company,
Ground Systems Group,
Fullerton, California

January, 1964
M.S.E.E., The University of Southern
California, Los Angeles, California

1964-present.
Research Associate, ElectroScience
Laboratory, The Ohio State
University, Columbus, Ohio

FIELDS OF STUDY

Major Field: Electrical Engineering

Studies in Electromagnetism.
  Professor R.G. Kouyoumjian

Studies in Antennas and Propagation.
  Professors C.H. Walter and J.H. Richmond

Studies in Communication Theory.
  Professors C.E. Warren and W.C. Davis

Studies in Applied Mathematics.
  Professors W.J. Davis and R.J. Bumcrot

Studies in Quantum Chemistry.
  Professor W.J. Taylor
# CONTENTS

<table>
<thead>
<tr>
<th>CONTENTS</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACKNOWLEDGMENTS</td>
<td>ii</td>
</tr>
<tr>
<td>VITA</td>
<td>iii</td>
</tr>
<tr>
<td>TABLES</td>
<td>vi</td>
</tr>
<tr>
<td>ILLUSTRATIONS</td>
<td>vii</td>
</tr>
<tr>
<td>Chapter</td>
<td></td>
</tr>
<tr>
<td>I  INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>II BASIS TRANSFORMATION AND LEAST SQUARE TECHNIQUES WITH APPLICATION TO SCATTERING BY THIN WIRES...</td>
<td>6</td>
</tr>
<tr>
<td>A. A Numerical Basis Transformation</td>
<td>6</td>
</tr>
<tr>
<td>B. Least Square Solutions of Scattering by Thin Wires</td>
<td>8</td>
</tr>
<tr>
<td>C. Application to Scattering by Thin Wires</td>
<td>17</td>
</tr>
<tr>
<td>(i) Closed Loops</td>
<td>18</td>
</tr>
<tr>
<td>(ii) Wires of Open Configuration</td>
<td>20</td>
</tr>
<tr>
<td>(iii) Computer Time and Storage</td>
<td>28</td>
</tr>
<tr>
<td>III THE DETERMINATION OF CHARACTERISTIC MODES AND THEIR USE IN THE SOLUTION OF SCATTERING BY THIN WIRES...</td>
<td>33</td>
</tr>
<tr>
<td>A. Summary of Characteristic Mode Properties</td>
<td>34</td>
</tr>
<tr>
<td>B. A Technique for Determining Characteristic Modes of Wires</td>
<td>37</td>
</tr>
<tr>
<td>C. Scattering Solutions Using Characteristic Modes</td>
<td>46</td>
</tr>
<tr>
<td>(i) Backscattering Cross Section</td>
<td>46</td>
</tr>
<tr>
<td>(ii) Tumble Average Cross Section</td>
<td>51</td>
</tr>
<tr>
<td>(iii) Input Admittance</td>
<td>57</td>
</tr>
<tr>
<td>D. Examples of Characteristic Modes and Scattering Solutions</td>
<td>59</td>
</tr>
<tr>
<td>(i) Circular Loops</td>
<td>61</td>
</tr>
<tr>
<td>(ii) Elliptical Loops</td>
<td>62</td>
</tr>
<tr>
<td>(iii) Straight Wires</td>
<td>67</td>
</tr>
<tr>
<td>Chapter</td>
<td>Page</td>
</tr>
<tr>
<td>------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>(iv) Circular Arcs</td>
<td>73</td>
</tr>
<tr>
<td>(v) Helices</td>
<td>79</td>
</tr>
<tr>
<td>IV SUMMARY AND CONCLUSIONS</td>
<td>90</td>
</tr>
<tr>
<td>Appendixes</td>
<td></td>
</tr>
<tr>
<td>A THE LEAST SQUARE SOLUTION</td>
<td>93</td>
</tr>
<tr>
<td>OF A SET OF N COMPLEX</td>
<td></td>
</tr>
<tr>
<td>EQUATIONS IN M UNKNOWNS</td>
<td></td>
</tr>
<tr>
<td>WITH M &lt; N</td>
<td></td>
</tr>
<tr>
<td>B PIECEWISE UNIFORM IMPEDANCE MATRIX ELEMENTS</td>
<td>95</td>
</tr>
<tr>
<td>C COMPUTER PROGRAMS</td>
<td>97</td>
</tr>
<tr>
<td>D CHARACTERISTIC MODE DATA</td>
<td>138</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>148</td>
</tr>
</tbody>
</table>
### TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Figures of Merit for Least Square Solutions of Scattering by a Straight Wire of Length $L = 0.465\lambda$ and Radius $a = 0.005\lambda$</td>
<td>15</td>
</tr>
<tr>
<td>2</td>
<td>A Comparison of Computation Times</td>
<td>30</td>
</tr>
<tr>
<td>3</td>
<td>Phase Angles $\alpha_m$ for a Circular Loop of Radius $0.25\lambda$ with Wire Radius $0.0025\lambda$</td>
<td>46</td>
</tr>
<tr>
<td>4</td>
<td>Computer Programs and Subprograms</td>
<td>98</td>
</tr>
<tr>
<td>5</td>
<td>Characteristic Modes of Circular Loops with a Loop Radius-to-Wire Radius Ratio $R/a = 100$</td>
<td>141</td>
</tr>
<tr>
<td>6</td>
<td>Characteristic Modes of Elliptical Loops with $1.1\lambda$ Circumference and Wire Radius $a = 0.00175\lambda$ as a Function of Axial Ratio</td>
<td>142</td>
</tr>
<tr>
<td>7</td>
<td>Characteristic Modes of Straight Wires with a Length-to-Diameter Ratio $L/2a = 100$</td>
<td>144</td>
</tr>
<tr>
<td>8</td>
<td>Characteristic Modes of a One-Turn Helix with a Wire Length-to-Diameter Ratio $L/2a = 100$ and a Pitch $p = 10a$</td>
<td>146</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>1</td>
<td>Incident and scattered fields on axis for a plane wave incident on a straight wire</td>
<td>13</td>
</tr>
<tr>
<td>2</td>
<td>Least square solutions for the total tangential electric field on a straight wire of near resonant length</td>
<td>16</td>
</tr>
<tr>
<td>3</td>
<td>Geometry for the circular loop</td>
<td>19</td>
</tr>
<tr>
<td>4</td>
<td>Echo area of circular loops at the broadside aspect</td>
<td>20</td>
</tr>
<tr>
<td>5</td>
<td>The ratio of the echo area at broadside aspect to that at edge aspect for circular loops. (The ratio in dB = ( \log_{10} ) (ratio).)</td>
<td>22</td>
</tr>
<tr>
<td>6</td>
<td>Backscattering cross section as a function of incidence angle ( \theta ) for a circular loop of radius 0.2( \lambda )</td>
<td>23</td>
</tr>
<tr>
<td>7</td>
<td>Broadside- and edge-aspect backscattering cross section for an ellipse as a function of the axial ratio</td>
<td>24</td>
</tr>
<tr>
<td>8</td>
<td>Broadside backscattering cross section for a straight wire of length 0.45( \lambda ) and radius 0.005( \lambda ). A comparison of convergence as a function of ( M ), the number of modes, using cosine functions and even Chebyshev polynomials of the second kind</td>
<td>25</td>
</tr>
<tr>
<td>9</td>
<td>Broadside backscattering cross section for straight wires of near first resonant lengths and radius 0.005( \lambda )</td>
<td>26</td>
</tr>
<tr>
<td>10</td>
<td>Broadside backscattering cross section of straight wires with length-to-diameter ratio ( L/2\alpha = 100 )</td>
<td>27</td>
</tr>
<tr>
<td>11</td>
<td>Broadside backscattering cross section of an open circular arc of fixed wire length (0.46( \lambda )) as a function of arc radius, with a wire of radius 0.00388( \lambda )</td>
<td>28</td>
</tr>
<tr>
<td>12</td>
<td>Computation times for a piecewise uniform method and for the basis transformation method</td>
<td>31</td>
</tr>
</tbody>
</table>
Figure | Page
--- | ---
13 | Computer time as a function of $M$, the number of modes, with $N = 100$.
14 | Eigenvalues $\varepsilon_m(\alpha)$ for a circular loop of radius $0.25\lambda$ with a wire radius of $0.0025\lambda$.
15 | Phase angles $\alpha_m$, $m = 0, 1, 2, 3$, for circular loops as a function of loop radius.
16 | Backscattering and tumble average cross sections for circular loops with a loop radius-to-wire radius ratio $R/a = 100$.
17 | Ratio of the backscattering cross section at broadside aspect to that at edge aspect for circular loops with $R/a = 100$. (Ratio = $10 \log_{10} \sigma_{bs}/\sigma_{edge}$).
18 | Input admittance for circular loops with loop radius-to-wire radius ratio $R/a = 100$.
19 | Phase angles $\alpha_m$, $m = 0, 1, 2$, for elliptical loops with circumference $C = 1.1\lambda$ and wire radius $a = 0.00175\lambda$.
20 | Backscattering and tumble average cross sections for elliptical loops with circumference $C = 1.1\lambda$ and wire radius $a = 0.00175\lambda$.
21 | Phase angles $\alpha_m$, $m = 1, 2, 3, 4$, for straight wires with a length-to-diameter ratio $L/2a = 100$.
22 | Normalized characteristic current distributions $I_m$, $m = 1, 2, 3, 4$, for a straight wire of length $L = 1.5\lambda$ with $L/2a = 100$.
23 | Broadside and tumble average backscattering cross sections of straight wires with $L/2a = 100$.
24 | Backscattering cross section as a function of incidence angle $\theta$ for a straight wire of length $L = 1.5\lambda$ with $L/2a = 100$.
25 | Phase angles $\alpha_m$, $m = 1, 2, 3, 4$, for straight wires of constant radius $a = 0.005\lambda$.
26 | Broadside and tumble average backscattering cross section of straight wires with $a = 0.005\lambda$. 
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>27</td>
<td>Phase angle $\alpha$ as a function of arc radius for the dominant mode of a circular wire arc with wire length $L = 0.475\lambda$ and wire radius $a = 0.0008467\lambda$...</td>
</tr>
<tr>
<td>28</td>
<td>Backscattering and tumble average cross section data for a circular arc with $L = 0.475\lambda$ and $a = 0.0008467\lambda$...</td>
</tr>
<tr>
<td>29</td>
<td>Characteristic mode current coefficients for the dominant mode of a circular arc with wire length $L = 0.475\lambda$ and wire radius $a = 0.0008467\lambda$. (Each coefficient is normalized to its value for a radius $R = 0.15\lambda$)...</td>
</tr>
<tr>
<td>30</td>
<td>Phase angle $\alpha$ for a helix with wire length $L = 0.475\lambda$, wire radius $a = 0.0008467\lambda$, and pitch $p = 30a$...</td>
</tr>
<tr>
<td>31</td>
<td>Backscattering and tumble average cross sections of a helix with $L = 0.475\lambda$, $a = 0.0008467\lambda$, and $p = 30a$ for linear polarization...</td>
</tr>
<tr>
<td>32</td>
<td>Backscattering and tumble average cross sections of a helix with $L = 0.475\lambda$, $a = 0.0008467\lambda$, and $p = 30a$ for circular polarization...</td>
</tr>
<tr>
<td>33</td>
<td>Phase angles $\alpha_m$, $m=1,2,3$, for a helix with $L = 0.6\lambda$, $a = 0.0008467\lambda$, and $p = 30a$...</td>
</tr>
<tr>
<td>34</td>
<td>Backscattering and tumble average cross sections of a helix with $L = 0.6\lambda$, $a = 0.0008467\lambda$, and $p = 30a$ for linear polarization...</td>
</tr>
<tr>
<td>35</td>
<td>Backscattering and tumble average cross sections of a helix with $L = 0.6\lambda$, $a = 0.0008467\lambda$, and $p = 30a$ for circular polarization...</td>
</tr>
<tr>
<td>36</td>
<td>Phase angles $\alpha_m$, $m=1,2,3,4$, for a one-turn helix with wire length-to-diameter ratio $L/2a = 100$ and pitch $p = 10a$...</td>
</tr>
<tr>
<td>37</td>
<td>Backscattering and tumble average cross sections of a one-turn helix with $L/2a = 100$ and $p = 10a$ for linear polarization...</td>
</tr>
<tr>
<td>38</td>
<td>Backscattering and tumble average cross sections of a one-turn helix with $L/2a = 100$ and $p = 10a$ for circular polarization...</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>39</td>
<td>Comparison of tumble average cross sections of a straight wire and a one-turn helix with wire length-to-diameter ratio L/2a = 100 and a helix pitch p = 10a.</td>
</tr>
<tr>
<td>40</td>
<td>Computer method for locating α_m corresponding to relative minimum ε_{min}(α) = ε_{min}(α_m).</td>
</tr>
<tr>
<td>41</td>
<td>Example input data for determining characteristic modes of a straight wire of length 1.0a and radius 0.005a.</td>
</tr>
<tr>
<td>42</td>
<td>Example data set for computing backscattering and tumble average cross section data for a straight wire with length SL = 1.0a and radius AL = 0.005a using two even and one odd characteristic modes.</td>
</tr>
<tr>
<td>43</td>
<td>Main program for the basis transformation method.</td>
</tr>
<tr>
<td>44</td>
<td>Main program for the piecewise uniform method.</td>
</tr>
<tr>
<td>45</td>
<td>Main program for the determination of characteristic modes of wires.</td>
</tr>
<tr>
<td>46</td>
<td>Main program for computing backscattering and tumble average cross sections of wires using characteristic modes.</td>
</tr>
<tr>
<td>47</td>
<td>Wire geometry subprogram for circular and elliptical loops.</td>
</tr>
<tr>
<td>48</td>
<td>Wire geometry subprogram for helical and straight wires and circular arcs.</td>
</tr>
<tr>
<td>49</td>
<td>Subprogram for the piecewise uniform impedance matrix of symmetric wire geometries.</td>
</tr>
<tr>
<td>50</td>
<td>Subprogram for the modal impedance matrix derived as a transformation of a piecewise uniform impedance matrix for symmetric wire geometries.</td>
</tr>
<tr>
<td>51</td>
<td>Subprogram for the modal impedance matrix derived as a transformation of a piecewise uniform impedance matrix for elliptical wire geometries.</td>
</tr>
<tr>
<td>52</td>
<td>Sine and cosine basis functions for wire loops.</td>
</tr>
<tr>
<td>53</td>
<td>Sine and cosine basis functions for wires of open configuration.</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>54</td>
<td>Chebyshev polynomial basis functions</td>
</tr>
<tr>
<td>55</td>
<td>Subroutine for locating $e_{\text{min}}(\alpha_m)$</td>
</tr>
<tr>
<td>56</td>
<td>Subroutine for computing $[B(\alpha)]$</td>
</tr>
<tr>
<td>57</td>
<td>Subroutine for computing the eigenvalues and eigenvectors of $[B(\alpha)]$</td>
</tr>
<tr>
<td>58</td>
<td>Subroutine for averaging $\sigma(\theta,\phi)$ over $\theta$ and $\phi$</td>
</tr>
</tbody>
</table>
CHAPTER I
INTRODUCTION

Computer solutions to scattering by straight or continuously curved, perfectly conducting wires (without generators or lumped loads) often are obtained by point-matching techniques whereby the boundary condition of zero total tangential electric field everywhere along the wire surface is approximated by enforcing this condition at, say, $N$ points along the wire, where $N$ is finite. The point-matching problem is represented by a set of $N$ linear equations expressed in matrix notation as

$$\begin{align*}
\mathbf{Z} \mathbf{I} + \mathbf{E} &= \mathbf{0},
\end{align*}$$

where the $N$-dimensional column vector $\mathbf{E}$ is the tangential component of the incident electric field and the components $I_n$ of the current vector $\mathbf{I}$ are coefficients of an expansion of the current induced on the wire in terms of a set of orthonormal basis functions $i_1, i_2, \ldots, i_N$. The $mn^{th}$ element of the $N \times N$ impedance matrix $\mathbf{Z}$ gives the tangential component of the electric field at the $m^{th}$ point on the wire due to the $n^{th}$ basis function of current distributed over the wire length.

Selection of the basis functions is an important part of the point-matching methods. For the familiar piecewise uniform methods, \cite{1-8}
the basis functions are unit pulses of width $L/N$, so that a step approximation is obtained for the induced current. Other segmental techniques include the piecewise linear[5,6] and the piecewise sinusoidal[8-11] methods.

A somewhat different class of point-matching methods, which will be called "modal methods", assumes a set of continuously differentiable, "modal" basis functions.[12-14] Unlike the segmental basis functions which are nonzero over only a small portion of the wire, the modal basis functions generally extend over the wire length. The cosine and sine functions $\cos \left(2\pi n \frac{\ell}{L}\right)$ and $\sin \left(2\pi n \frac{\ell}{L}\right)$, for example, where $\ell$, $-L/2 \leq \ell \leq L/2$, is measured along the wire center-line, form a set of continuous basis functions well suited to certain shapes, such as closed wire loops.

These two classes of point-matching methods, i.e., the segmental and the modal methods, are further distinguished by certain characteristics encountered upon application. The segmental approaches usually afford efficient derivation of the set of linear equations for wires of arbitrary shape. The number of equations needed for good solution accuracy can become great, however, and since the time required to solve a set of $N$ linear equations is proportional to $N^3$, computer time can become restrictive. Likewise, computer storage soon limits the size of the scatterer to which these methods can be applied.

On the other hand, sufficient solution accuracy often can be achieved with a relatively small number of linear equations when modal functions form the basis. The derivation of the linear
equations, is, except in the case of straight wires,[14] generally not efficient, however, usually requiring considerable numerical integration resulting in substantial computer time.[15]

Having derived a set of linear equations relative to a certain basis, it is possible to make a linear transformation to a new basis. This basis transformation is discussed in Chapter II. It is shown that a least square solution may be extracted from the new set of equations. Specific application of the basis transformation and least square techniques is made to a set of equations derived assuming the piecewise uniform basis where the transformation matrix has a very simple form. It is also shown that as a result of this transformation the number of linear equations required for good solution accuracy can be reduced considerably with consequent reductions in computer time and storage requirements. As examples of the basis transformation method, backscattering cross sections are computed for circular and elliptical loops, straight wires and open circular arcs.

To the author's knowledge, the basis transformation, although a familiar mathematical tool, has not previously been applied to the numerical solution of scattering problems in the manner discussed here. The least square method, however, has been applied to various problems including the following. Kenneah[16] obtained a least square solution by a field-matching technique in which multipole expansions were used to represent electromagnetic fields. Successful calculations were carried out for prolate and oblate
spheroids illuminated by a plane wave incident along the axis of symmetry. Fenlon[17] obtained least square solutions for the acoustic radiation field at the surface of a finite cylinder. Hildebrand and Crout[18] have treated the least square solution of integral equations and Crout[19] has described a least square procedure for solving homogeneous integral equations.

The determination of "characteristic mode" currents,[20,21] particular modal currents which are characteristic of a given scatterer and which satisfy certain requirements, is another application of point-matching methods which is greatly facilitated by the basis transformation and least square techniques. The concept of characteristic modes was formulated by Garbacz,[20] who gave examples of such modes associated with several geometries including the infinite circular cylinder, the sphere, infinitely long, thin parallel wires arrayed arbitrarily in space, two coaxial circular loops of thin wire, and a single circular loop scatterer. Consideration of thin finite wires of arbitrary shape awaited the development of a mode-determining technique such as the one described in Chapter III of the present work. Included is the derivation, in terms of the characteristic mode pattern functions, of expressions for the tumble average back-scattering cross section of a scatterer for arbitrary transmitter and receiver polarizations. Results are given in graphical form for the backscattering and tumble average cross sections of circular and elliptical loops, straight wires, circular arcs, and helices.
Considerable scattering data have been calculated for objects such as straight wires, circular loops and open circular arcs. Relatively little effort, however, has been directed toward the determination of the tumble average cross section as discussed in Chapter III. Curtis[22] has computed average cross section data for straight wires and circular wire arcs, and dipoles have been treated by Borison,[23] Palermo and Bauer,[24] Hessemer,[25] and Morrow.[26] This tumble average would be the "best estimate" or the expected value of the cross section of an object placed with arbitrary orientation in the radar beam or, from another viewpoint, the average cross section of a large number of the particular object, assuming sufficient separation that the elements may be considered decoupled. The mode approach discussed in Chapter III provides an efficient method for obtaining tumble average cross section data as well as other subsidiary information, such as the input impedance of an arbitrarily placed gap on the scattering element, and bistatic and monostatic cross sections.

An \( e^{i\omega t} \) time convention is assumed throughout this work and is suppressed for convenience. Matrices and column vectors are denoted by [ ] and ( ), respectively.
CHAPTER II
BASIS TRANSFORMATION AND LEAST SQUARE TECHNIQUES
WITH APPLICATION TO SCATTERING BY THIN WIRES

A. A Numerical Basis Transformation

In the following discussion a method of transforming the basis of a set of linear equations is described. The resulting transformation technique may be applied, in conjunction with existing methods, to the solution of scattering by thin wires. In succeeding discussions some uses and advantages of the technique are demonstrated by means of specific applications.

Assume that the set of linear equations,

\[ [Z] (I) + (E) = (0) \]

has as a basis the set of orthonormal functions \( i_1, i_2, \ldots, i_N \). We can, if we choose, refer Eq. (2) to a new basis \( i'_1, i'_2, \ldots, i'_N \), by means of a linear transformation\[27\]

\[ (I) = [T] (I') \],

where \([T]\) is the transformation matrix and \((I')\) is a column vector representing the current \( I \) on the wire relative to the new basis functions, i.e.,

\[ I = I'_1 i'_1 + I'_2 i'_2 + \cdots + I'_N i'_N \].
where the $I'_n$, $n=1,2,\ldots,N$, are the components of the vector $(I')$. Substituting Eq. (3) into Eq. (2),

$$\begin{align*}
(5) \quad [Z] [T] (I') + (E) &= (0).
\end{align*}$$

Letting

$$\begin{align*}
(6) \quad [Z'] = [Z] [T],
\end{align*}$$

we write

$$\begin{align*}
(7) \quad [Z'] (I') + (E) &= (0).
\end{align*}$$

Equation (7) is equivalent to Eq. (2), referred to the new basis.

Consider, for example, the transformation from the basis of the piecewise uniform method to a basis of modal orthonormal functions, such as the cosine and sine functions. In this application we find that the matrix $[T]$ of Eq. (3) is very simply defined. In fact, inspection of Eq. (3) reveals that each member of the new set of basis functions forms a column of the matrix $[T]$. (Note: Moving from top to bottom in a column of $[T]$ in this case corresponds to moving segment-by-segment from one end of the wire to the other.) If, for example, the cosine functions $\cos(n\pi x)/L$, $n=1,2,\ldots,N$, form the new basis, then the $n^{th}$ column of $[T]$ represents an $n^{th}$ order cosine function distributed over the wire, i.e., the $m^{th}$ element of $[T]$ is $\cos(m\pi x_m)/L$, where $x_m$ is the location of the $m^{th}$ match point on the center-line of the wire. Equation (3) would be,
The simple definition of the transformation matrix afforded by this example is a consequence of the fact that each component $I_n$ of the current vector ($I$) in the piecewise uniform basis system equals the current at $\xi_n$ ($-L/2 \leq \xi_n \leq L/2$), on the wire. This is true since only the $n^{th}$ basis function is nonzero at $\xi_n$. It is not true in general, however. For example, it is not true of the new current vector ($I'$).

Having determined the desired basis and having thus specified the transformation matrix $[T]$, the impedance matrix $[Z]$ is transformed as in Eq. (6), and the new set of equations, Eq. (7), is obtained with reference to the new basis.

B. Least Square Solutions of Scattering by Thin Wires

Often one finds that the components $I'_m$ of the current solution vector ($I'$) of Eq. (7) are negligible for $m > M$ ($M < N$), i.e., the solution
is quite well approximated by retaining only the first \( M \) terms (assuming proper ordering of the \( I_m' \)). In fact, it is not unusual to obtain sufficient accuracy with \( M < N/10 \). Recalling that the computer solution time for \( N \) linear equations is proportional to \( N^3 \), one of the advantages afforded by such a basis transformation is evident.

When this approximation can be made we need only to solve Eq. (7) for \( M \) of the \( N \) unknowns. The \( N \times N \) impedance matrix \([Z']\) may be replaced by an \( N \times M \) rectangular matrix \([Z']_{\text{a}}\) formed by taking the first \( M \) columns of \([Z']\), and the column vector \((I')\) is replaced by an \( M \)-dimensional column vector \((I'_{\text{a}})\) whose components are the first \( M \) components of \((I')\). The field vector \((E)\) remains unchanged. The result is the overdetermined set of \( N \) equations in \( M \) unknowns,

\[
\begin{pmatrix}
Z'_{a11} & Z'_{a12} & \cdots & Z'_{a1M} \\
Z'_{a21} & Z'_{a22} & \cdots & Z'_{a2M} \\
\vdots & \vdots & \ddots & \vdots \\
Z'_{aN1} & Z'_{aN2} & \cdots & Z'_{aNM}
\end{pmatrix}
\begin{pmatrix}
I'_{a1} \\
I'_{a2} \\
\vdots \\
I'_{aN}
\end{pmatrix}
= 
\begin{pmatrix}
E_1 \\
E_2 \\
\vdots \\
E_N
\end{pmatrix}
\]
for which several solutions exist, depending upon which subset of
M equations is chosen. To avoid this non-uniqueness in an un-
ambiguous fashion, we multiply (10) by the complex conjugate transpose
of $[Z_a']$ to get

\[
\begin{align*}
\begin{bmatrix}
Z_a^{c_{11}} & Z_a^{c_{12}} & \cdots & Z_a^{c_{N1}} \\
Z_a^{c_{12}} & Z_a^{c_{22}} & \cdots & Z_a^{c_{N2}} \\
\vdots & \vdots & \ddots & \vdots \\
Z_a^{c_{1M}} & Z_a^{c_{2M}} & \cdots & Z_a^{c_{NM}}
\end{bmatrix}
\begin{bmatrix}
Z_i^{a_{11}} \\
Z_i^{a_{12}} \\
\vdots \\
Z_i^{a_{1M}}
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
I_i^{a_1} \\
I_i^{a_2} \\
\vdots \\
I_i^{a_M}
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
Z_i^{a_{11}} \\
Z_i^{a_{12}} \\
\vdots \\
Z_i^{a_{1M}}
\end{bmatrix}
\begin{bmatrix}
Z_i^{a_{11}} & Z_i^{a_{12}} & \cdots & Z_i^{a_{1M}} \\
Z_i^{a_{21}} & Z_i^{a_{22}} & \cdots & Z_i^{a_{2M}} \\
\vdots & \vdots & \ddots & \vdots \\
Z_i^{a_{N1}} & Z_i^{a_{N2}} & \cdots & Z_i^{a_{NM}}
\end{bmatrix}
\begin{bmatrix}
I_i^{a_1} \\
I_i^{a_2} \\
\vdots \\
I_i^{a_M}
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
Z_i^{a_{11}} & Z_i^{a_{12}} & \cdots & Z_i^{a_{1M}} \\
Z_i^{a_{21}} & Z_i^{a_{22}} & \cdots & Z_i^{a_{2M}} \\
\vdots & \vdots & \ddots & \vdots \\
Z_i^{a_{N1}} & Z_i^{a_{N2}} & \cdots & Z_i^{a_{NM}}
\end{bmatrix}
\begin{bmatrix}
E_1 \\
E_2 \\
\vdots \\
E_N
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
I_i^{a_1} \\
I_i^{a_2} \\
\vdots \\
I_i^{a_M}
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
Z_i^{a_{11}} & Z_i^{a_{12}} & \cdots & Z_i^{a_{1M}} \\
Z_i^{a_{21}} & Z_i^{a_{22}} & \cdots & Z_i^{a_{2M}} \\
\vdots & \vdots & \ddots & \vdots \\
Z_i^{a_{N1}} & Z_i^{a_{N2}} & \cdots & Z_i^{a_{NM}}
\end{bmatrix}
\begin{bmatrix}
E_1 \\
E_2 \\
\vdots \\
E_N
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
I_i^{a_1} \\
I_i^{a_2} \\
\vdots \\
I_i^{a_M}
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
Z_i^{a_{11}} & Z_i^{a_{12}} & \cdots & Z_i^{a_{1M}} \\
Z_i^{a_{21}} & Z_i^{a_{22}} & \cdots & Z_i^{a_{2M}} \\
\vdots & \vdots & \ddots & \vdots \\
Z_i^{a_{N1}} & Z_i^{a_{N2}} & \cdots & Z_i^{a_{NM}}
\end{bmatrix}
\begin{bmatrix}
E_1 \\
E_2 \\
\vdots \\
E_N
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
I_i^{a_1} \\
I_i^{a_2} \\
\vdots \\
I_i^{a_M}
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
Z_i^{a_{11}} & Z_i^{a_{12}} & \cdots & Z_i^{a_{1M}} \\
Z_i^{a_{21}} & Z_i^{a_{22}} & \cdots & Z_i^{a_{2M}} \\
\vdots & \vdots & \ddots & \vdots \\
Z_i^{a_{N1}} & Z_i^{a_{N2}} & \cdots & Z_i^{a_{NM}}
\end{bmatrix}
\begin{bmatrix}
E_1 \\
E_2 \\
\vdots \\
E_N
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
I_i^{a_1} \\
I_i^{a_2} \\
\vdots \\
I_i^{a_M}
\end{bmatrix}
\end{align*}
\]
where the superscript \( c \) denotes the complex conjugate. The product 
\( [Z']^* [Z'] \), where \( * \) denotes the matrix complex conjugate transpose, 
yields an \( M \times M \) complex symmetric matrix. The solution[28-31] of 
Eq. (11) provides a least mean square "match" by \( M \) modes of the 
boundary conditions at \( N \) points on the wire.* Only \( M \) simultaneous 
equations need be solved, where \( M \), the number of modes, may be less 
than 10, while the number of match points, \( N \), may be greater than 
100, for example. There can be substantial savings in computer time 
and storage requirements when this technique is used, as is demon­
strated by examples given in later discussions (Sec. C-iii).

The above mentioned least square solution of problems associated 
with wires of closed configuration, such as circular and elliptical 
loops, causes no problems, but the application of least square methods 
to wires of open configuration, such as straight wires and open cir­
cular arcs, requires special consideration.

The field in the vicinity of the ends of wires of open configu­
ration exhibits certain discontinuities depending on the geometry of 
the wire termination. In the case where flat ends are assumed, an 
edge-type singularity results. The approximations inherent in 
numerical calculations cause this singularity to introduce erroneous 
values for the scattered field on the wire in the vicinity of the 
ends. Any solution method which satisfies the boundary conditions on 
the wire in a least square sense must account for this approximation 
effect.

*See Appendix A.
A simple example will serve to clarify the effect which the field near the wire ends will have on a least square solution. Consider a straight wire scatterer with a plane wave incident from the broadside aspect. The incident electric field is assumed to have unit magnitude and zero phase on the wire axis. Ideally the induced current would produce the scattered field shown in Fig. 1-a, so that the total tangential field \((E^i + E^s)\) on the wire is zero. Numerical solutions, however, provide only an approximation (Fig. 1-b) to that field. In deriving a least square solution it is seen that, roughly speaking, the scaling coefficient \(A\) in Fig. 1-b will be adjusted to produce a least mean squared error in the total field. The large error in the approximate fields near the wire ends forces \(A\) to be less than unity. The associated induced current is thus reduced in magnitude and is in error. If, however, in obtaining a least square solution, the match points within a certain distance, \(d\) (Fig. 1-b), from the wire ends are ignored (i.e., the equations corresponding to these points are deleted from the set of equations) a better solution results. This procedure may be viewed in the context of least square solutions as the inclusion of a weighting function in the integral to be minimized which is unity for all values of the differential (arc length, in this case) except for incremental regions near the limits of integration (wire ends), where it is zero.
Fig. 1. -- Incidence and scattered fields on axis for a plane wave incident on a straight wire.

The need for these special considerations is not unusual. Other numerical solution methods[32-34] have shown a critical dependence on the choice of match points.

There is a need, then, for some measure of the error in the field matching achieved by a given solution both for the purpose of optimizing the solution conditions just discussed and to serve as an estimate of the accuracy of the final solution. A useful figure of merit[32,33,35] is

\[ P = -\frac{1}{2} \text{Re} \left\{ \int_{L} J^C \cdot E^T \, dx \right\} , \]

where \( J^C \) is the complex conjugate of the induced current density and \( E^T \) is the total electric field (i.e., the error field) on the wire. \( P \) is the average power radiated, in the absence of the scatterer, by
an equivalent source current $J$ which generates an electric field intensity $E^T$. A negative value for $P$ corresponds to the absorption of power while a positive value corresponds to radiation. Because the wire is a perfectly conducting scatterer it can neither absorb nor radiate power. In this case the condition $P = 0$ corresponds to satisfying the condition of conservation of power. It is desirable, then, to minimize the magnitude of $P$.

It must be pointed out, however, that to obtain $|P| = 0$ in a numerical solution such as that being discussed here does not necessarily imply that the solution is exact, for the figure of merit is itself computed by approximate methods. Equation (12) is, in this work, approximated by

$$P \sim -\frac{1}{2} \text{Re}\left\{ \sum_{n=1}^{N} I_n^c \cdot E_n^T \right\},$$

where $I_n$ and $E_n^T$ are the current and the total electric field intensity at the $N$ match points.

In Table 1 are given figures of merit for a straight wire of length $L = 0.465\lambda$ with radius $a = 0.005\lambda$ as a function of the distance from the wire ends within which corresponding equations were ignored. The number of equations ignored at each end is also given for each case. A very significant improvement is often obtained by simply deleting the point nearest each end, though this is not the optimum in this case. The optimum distance depends on the wire radius and on the accuracy of the numerical impedance matrix. The impedance matrix $[Z_a']$ used to obtain these solutions was derived by a transformation from a piecewise
uniform basis[1-3] of dimension \( N = 100 \) to a cosine basis. Only the ten most significant cosine terms were retained, i.e., \( M = 10 \).

TABLE 1
FIGURES OF MERIT FOR LEAST SQUARE SOLUTIONS OF SCATTERING FROM A STRAIGHT WIRE OF LENGTH 0.465\( \lambda \) AND RADIUS 0.005\( \lambda \)

<table>
<thead>
<tr>
<th>NUMBER OF EQUATIONS DELETED AT EACH END</th>
<th>DISTANCE FROM WIRE ENDS IN WIRE RADIUS</th>
<th>FIGURE OF MERIT</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.00</td>
<td>0.75 \times 10^{-2}</td>
</tr>
<tr>
<td>1</td>
<td>0.93</td>
<td>0.33 \times 10^{-5}</td>
</tr>
<tr>
<td>2</td>
<td>1.86</td>
<td>0.27 \times 10^{-5}</td>
</tr>
<tr>
<td>3</td>
<td>2.79</td>
<td>0.37 \times 10^{-6}</td>
</tr>
<tr>
<td>4</td>
<td>3.72</td>
<td>0.60 \times 10^{-5}</td>
</tr>
<tr>
<td>5</td>
<td>4.65</td>
<td>0.11 \times 10^{-4}</td>
</tr>
</tbody>
</table>

Two curves of least square solutions for the total tangential electric field for the same straight wire are given in Fig. 2. One (the upper, dashed line) was derived using all \( N \) equations. The other (the lower, solid line), which shows a significant improvement in field matching, was derived from the same \( N \) equations minus the six equations corresponding to the three points nearest each end. The figures of merit for these two cases are the first and fourth values given in Table 1. The values for the broadside echo area are, respectively, 0.712\( \lambda^2 \) and 0.813\( \lambda^2 \), the latter value corresponding to the minimum
Fig. 2.--Least square solutions for the total tangential electric field on a straight wire of near resonant length.
figure of merit. (Reference 12 gives a value of approximately $0.82\lambda^2$ for the same wire by means of a linear equation method.)

C. Application to Scattering by Thin Wires

Application of the basis transformation and least square techniques just discussed follows the general procedure outlined below. In each case the wire is considered perfectly conducting and its radius is assumed to be much smaller than its length $L$ and the wavelength $\lambda$. The assumption is made that the surface current density on the thin wire has only an axial component and is distributed uniformly around the circumference of the wire.

A segmental approximation to the wire is computed, giving the $x$, $y$, and $z$ coordinates of the center of each linear wire segment along with orientation angles $\beta$, the angle of rotation from the positive $x$-axis, and $\alpha$, the angle out of the $xy$ plane ($\alpha$ is considered positive if the segment is tilted toward the negative $z$-axis). The matrix $[T]$ of Eq. (3) is then derived with each column equal to one of the set of new basis functions. Having computed and stored $[T]$, the transformation of Eq. (6) is performed. The piecewise uniform impedance matrix coefficients* are computed[1-3] one at a time while forming the new impedance matrix $[Z'_a]$ of Eq. (10) by means of running sums. The tangential component of the incident electric field (plane wave excitation is assumed) is then computed for each of the $N$ match points, completing the

*See Appendix B.
formation of Eq. (10). Following pre-multiplication of the matrix equation by the complex conjugate transpose of \([Z']\), the resultant complex symmetric matrix equation (Eq. (11)) is solved.\[31\]

Substituting the solution \((I'_a)\) into Eq. (3), a piecewise uniform representation of the current is obtained from which the far field is computed. Solution for the cross section follows by application of the definition

\[
\sigma = \lim_{r_0 \to \infty} 4\pi r_0^2 \left( \frac{|E^s|^2}{|E_0|^2} \right),
\]

where \(E^s\) is the scattered electric field intensity at a far distance \(r_0\) from the origin in the scatterer and \(E_0\) is the electric field intensity incident on the scatterer.

(i) Closed Loops

The backscattering cross section of circular loops (Fig. 3) was computed by transforming from the piecewise uniform basis to a basis of cosine functions, \(\cos(2(m-1)\pi \lambda)/L, m = 1, 2, \ldots, M\), where \(L\) is the wire length (the loop circumference) and \(\lambda\) is measured along the wire center-line. The loop was approximated by \(N = 120\) straight-wire segments and \(M = 5\).

In Fig. 4 the broadside echo area is compared with measured values\[36\] and with values derived\[36\] by a variational method. A similar comparison is made in Fig. 5 of the ratio of the echo area at broadside aspect to that at edge aspect. In both cases the loop radius varies from 0.06\(\lambda\) to 0.40\(\lambda\). Figure 6 gives the backscattering cross
section as a function of incidence angle for both theta- and phi-polarization for a 0.2λ radius loop. The wire radius for these examples is 0.005λ.

A suitable basis for elliptical loops is the set of cosine and sine functions, \( \cos(2(m-1)\pi\ell)/L \) and \( \sin(2m\pi\ell)/L \), \( m = 1, 2, \ldots, M \), where \( L \) is the circumference of the ellipse. Just as for the circular loops, \( N = 120 \) straight wire segments approximate the elliptical configuration. Six cosine functions and five sine functions were retained in the solutions. Backscattering cross sections are shown in Fig. 7 for both theta- and phi-polarizations at broadside- and edge-aspects as a function of the axial ratio of the ellipse. The wire radius is \( a = 0.00175\lambda \).

![Fig. 3.--Geometry for the circular loop.](image-url)
Fig. 4.—Echo area of circular loops at the broadside aspect.

(ii) Wires of Open Configuration

It has been shown[37] that the asymptotic form of the current distribution along any portion of a perfectly conducting wire, whether the wire is straight or curved, is

\[ I(\xi) = A \cos \frac{2\pi \xi}{\lambda} + B \sin \frac{2\pi \xi}{\lambda} \]

as the wire radius approaches zero. This sinusoidal approximation was
first discovered by Pocklington.\[38\] For wires of finite radius, however, the current distribution deviates from this sinusoidal behavior, especially near the wire ends. Knowing this, it might be expected that an accurate solution may be obtained for smaller M if some functions other than the sine and cosine functions are chosen as basis functions, in particular if the basis functions selected more closely approximate the true current distribution or perhaps better approximate the behavior of the current near the wire ends. A comparison is given in Fig. 8 of the broadside echo area of a wire of first resonant length as a function of M, the number of basis functions retained, for cosine functions* \(\cos (n\pi \xi) / L\) and Chebyshev polynomials of the second kind.\[39\] The Chebyshev polynomials, which better approximate the current near the wire ends, are seen to give a slight improvement in this case.

The broadside echo area was computed for straight wires of lengths from 0.3\(\lambda\) to 0.6\(\lambda\) and of 0.005\(\lambda\) radius (Fig. 9), with \(N = 80\) match points and \(M = 6\) even Chebyshev polynomials as basis functions. Also shown are data derived[12] by a linear equation method based on the expansion of the current distribution in a series of 15 cosine modes. Figure 10 gives the backscattering cross section at the broadside aspect of straight wires with a length-to-diameter ratio \(L/2a = 100\). Six even

---

*It was found that for least square solutions a complete, though truncated, set of basis functions is needed. In other words, when using the sine and cosine functions as basis functions, for example, it is necessary to retain all orders including the constant term and those which are non-zero at the wire ends. This differs from the usual practice of retaining only those functions which vanish at the wire ends, where the current must be zero.
Chebyshev modes were used with from $N = 100$ to 160 match points evenly spaced along the wires. Comparison is made with similar results[40] obtained by an independent method.

![Graph of loop radius vs. ratio (dB)]

Fig. 5.—The ratio of the echo area at broadside aspect to the echo area at edge aspect for circular loops. (The ratio in dB = $\log_{10}$ (ratio).)

The broadside backscattering cross section of an open circular arc is shown in Fig. 11 as a function of arc radius. The wire length
and radius are $0.46\lambda$ and $0.00388\lambda$, respectively, and the arc was approximated by $N = 150$ straight-wire segments. The currents were derived with $M = 10$ even Chebyshev polynomials as basis functions. Included in the figure are piecewise uniform results using $N = 80$ match points. [41,42]

![Diagram](image)

Fig. 6.—Backscattering cross section as a function of incidence angle $\theta$ for a circular loop of radius $0.2\lambda$. 
Fig. 7.—Broadside- and edge-aspect backscattering cross sections for an ellipse as a function of the axial ratio.
Fig. 8.—Broadside echo area for a straight wire of length $0.45\lambda$ and radius $0.005\lambda$. A comparison of convergence as a function of $M$, the number of modes, using cosine functions and even Chebyshev polynomials of the second kind.
Fig. 9.—Broadside backscattering cross section for straight wires of near first resonant lengths and radius $0.005\lambda$. 
Fig. 10.—Broadside backscattering cross section of straight wires with length-to-diameter ratio, L/2a = 100.
Basis Transformation Method
Thiele's Piecewise Uniform Solution

Fig. 11.--Broadside backscattering cross section of an open circular arc of fixed wire length (0.46λ) as a function of arc radius, with a wire of radius 0.00388λ.

(iii) Computer Time and Storage

A comparison was made between the computer time required to derive and solve a set of N linear equations for the unknown current using a piecewise uniform method and the time required if the same N equations were transformed to a new, continuous basis and reduced to a set of M equations from which a least square solution for the current was extracted. The wires for this comparison were two straight wires of lengths 0.45λ and 1.5λ with 0.005λ radii. The computer times and the
corresponding values for the backscattering cross sections (broadside aspect) are given in Table 2. A graphical time comparison is given in Fig. 12.

The piecewise uniform solution time is seen to have approximately a cubic dependence on \( N \), being determined primarily by the time needed to invert an \( N \)-dimensional system of equations. By applying the basis transformation the time was reduced to a square law dependence on \( N \), depending dominantly on the time required to compute the impedance matrix and to perform the transformation. The time consumed inverting the resultant \( M \)-dimensional system is almost negligible. Figure 13 shows the nearly linear dependence of computer time on \( M \). Furthermore, the storage capacity of the computer (IBM 7094) limits the piecewise uniform program to \( N \leq 100 \), while application of a basis transformation allows an increase to \( N \leq 500 \) with \( M \leq 20 \).
TABLE 2

A COMPARISON OF COMPUTATION TIMES

<table>
<thead>
<tr>
<th>NUMBER OF MATCH POINTS</th>
<th>PIECEWISE UNIFORM METHOD</th>
<th>PIECEWISE UNIFORM WITH BASIS TRANSFORMATION</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>T (SEC)</td>
<td>T (SEC)</td>
</tr>
<tr>
<td></td>
<td>L = 0.45λ L = 1.5λ</td>
<td>L = 0.45λ L = 1.5λ</td>
</tr>
<tr>
<td></td>
<td>σ/λ²</td>
<td>σ/λ²</td>
</tr>
<tr>
<td></td>
<td>M = 6</td>
<td>M = 10</td>
</tr>
<tr>
<td>N</td>
<td>0.65</td>
<td>0.97</td>
</tr>
<tr>
<td></td>
<td>0.7918 0.7323</td>
<td>0.7918 0.7323</td>
</tr>
<tr>
<td></td>
<td>3.33</td>
<td>2.73</td>
</tr>
<tr>
<td></td>
<td>0.7686 1.097</td>
<td>0.7801 1.097</td>
</tr>
<tr>
<td></td>
<td>9.72</td>
<td>5.32</td>
</tr>
<tr>
<td></td>
<td>0.8391 1.080</td>
<td>0.8427 1.074</td>
</tr>
<tr>
<td></td>
<td>21.50</td>
<td>8.78</td>
</tr>
<tr>
<td></td>
<td>0.8453 0.9354</td>
<td>0.8463 0.9911</td>
</tr>
<tr>
<td></td>
<td>40.35</td>
<td>12.12</td>
</tr>
<tr>
<td></td>
<td>0.8460 0.9202</td>
<td>0.8454 0.9652</td>
</tr>
<tr>
<td></td>
<td>—</td>
<td>18.35</td>
</tr>
<tr>
<td></td>
<td>—</td>
<td>1.027</td>
</tr>
<tr>
<td></td>
<td>—</td>
<td>30.05</td>
</tr>
<tr>
<td></td>
<td>—</td>
<td>1.016</td>
</tr>
<tr>
<td>120</td>
<td>—</td>
<td>27.88</td>
</tr>
<tr>
<td></td>
<td>—</td>
<td>1.115</td>
</tr>
<tr>
<td></td>
<td>—</td>
<td>45.46</td>
</tr>
<tr>
<td></td>
<td>—</td>
<td>1.124</td>
</tr>
<tr>
<td>150</td>
<td>—</td>
<td>47.76</td>
</tr>
<tr>
<td></td>
<td>—</td>
<td>1.136</td>
</tr>
<tr>
<td></td>
<td>—</td>
<td>78.24</td>
</tr>
<tr>
<td></td>
<td>—</td>
<td>1.145</td>
</tr>
<tr>
<td>200</td>
<td>—</td>
<td>73.17</td>
</tr>
<tr>
<td></td>
<td>—</td>
<td>1.124</td>
</tr>
<tr>
<td></td>
<td>—</td>
<td>119.88</td>
</tr>
<tr>
<td></td>
<td>—</td>
<td>1.139</td>
</tr>
</tbody>
</table>
Fig. 12.--Computation times for a piecewise uniform method and for the basis transformation method.
Fig. 13.—Computer time as a function of $M$, the number of modes, with $N = 100$. 
CHAPTER III
THE DETERMINATION OF CHARACTERISTIC MODES AND THEIR USE
IN THE SOLUTION OF SCATTERING BY THIN WIRES

Basis transformation and least square techniques have been discussed in the preceding chapter. Their use in conjunction with a piecewise uniform method has been demonstrated by application to the direct solution of scattering by thin wires. In the following discussion these same techniques are applied to the determination of characteristic modes of thin wires, which are in turn used to describe the scattering by these wires. The usefulness of this less direct approach to scattering problems will become evident later.

It has been shown[20,21] that every perfectly conducting obstacle has associated with it a particular set of surface currents and corresponding radiated fields which are characteristic of the obstacle shape and independent of any specific excitation. This set of so called "characteristic modes" forms a useful basis set in which to expand fields radiated or scattered at a great distance from the obstacle. Once these modes are known for a given obstacle, the scattering of plane waves incident from arbitrary source directions into arbitrary receiver directions may be evaluated concisely. In general, however, the determination of these characteristic modes is not a simple task.
Following a brief discussion of characteristic modes, a method for determining characteristic mode current distributions on wires of general shape is given. As examples of the method, the characteristic modes associated with a variety of perfectly conducting wires are derived and are then used to compute their backscattering properties.

A. Summary of Characteristic Mode Properties

The brief discussion of characteristic modes given in this section does not reflect the generality of the theory. For a more comprehensive treatment the reader is directed to References 20 and 21.

Consider a loss-free obstacle with a surface $S$. Assume a specified angular frequency $\omega$. It has been shown[20] that a special set of characteristic vector fields, which are determined uniquely by the scatterer shape and composition and by the frequency, may be associated with this scatterer. Once they are found, these fields may be used in the solution of a variety of problems involving the obstacle; of particular interest here is their use in evaluating the field scattered in arbitrary directions for a plane wave incident on the obstacle from any direction.

If the obstacle is perfectly conducting there corresponds to this set of characteristic vector fields a set of characteristic current density distributions, $J_m$, on $S$ which satisfies the following conditions:
(i) Each member, $J_m$, of the set is real on $S$.

(ii) Each $J_m$, radiating in free space, generates an electric field intensity whose tangential component on $S$ is equiphase and lags $J_m$ by an angle $\alpha_m$, where $\pi/2 < \alpha_m < 3\pi/2$. (The ordering of the $J_m$ is chosen such that the magnitudes $|\cos \alpha_m|$ form a non-increasing sequence of numbers with increasing integers $m$.)

(iii) For each mode of finite order $m$, $J_m$ is normalized so that it radiates unit power.

(iv) On the sphere $\Sigma$ at $r=\infty$, $J_m$ radiates the electric field

\[ E_m^{\text{ext}}(r,\theta,\phi) = -\frac{i\omega}{4\pi} \frac{e^{-ikr}}{r} F_m(\theta,\phi), \]

where $F_m(\theta,\phi)$ is the $m$th characteristic mode pattern function.

The normalization to unit radiated power (iii) above) implies that

\[ \langle F_m^*, F_m \rangle = \frac{1}{Z_0} \int_{\Sigma} |F_m|^2 \sin \theta \, \text{d}\theta \, \text{d}\phi = \left(\frac{4\pi}{kZ_0}\right)^2, \]

where $\langle,\rangle$ denotes a scalar product. (RMS values are assumed here to agree with the notation of Reference 20.) Furthermore

\[ \langle F_m^*, F_n \rangle = \frac{1}{Z_0} \int_{\Sigma} F_m^* \cdot F_n^c \sin \theta \, \text{d}\theta \, \text{d}\phi = 0, \, n \neq m, \]

where the superscript $c$ denotes the complex conjugate. Equations (17) and (18) imply that the characteristic pattern functions, $F_m(\theta,\phi)$, form an orthonormal set (with respect to radiated power) on the sphere $\Sigma$ at infinity.
Knowledge of the $F_m(\theta, \phi)$ and the associated characteristic values $a_m = |\cos \alpha_m| e^{i\alpha_m}$ permits a compact bilinear expansion for the far field scattered in any direction $(r, \theta^S, \phi^S)$ due to a unit plane wave illuminating the obstacle from any direction $(\theta^i, \phi^i)$. If both the source and the observer are at extreme distances from the scatterer,

\begin{equation}
E(r, \theta^S, \phi^S; \theta^i, \phi^i) = \frac{i\omega}{4\pi} \sum_{m=1}^{M} a_m F_m(\theta^S, \phi^S) F_m(\theta^i, \phi^i) \cdot \hat{h}^i
\end{equation}

\begin{equation}
= -\frac{i\omega}{4\pi} \frac{e^{-ikr}}{r} F(\theta^S, \phi^S; \theta^i, \phi^i) \cdot \hat{h}^i
\end{equation}

where $\hat{h}^i$ is the polarization state vector of the plane wave and

\begin{equation}
F(\theta^S, \phi^S; \theta^i, \phi^i) = -\sum_{m=1}^{M} a_m F_m(\theta^S, \phi^S) F_m(\theta^i, \phi^i)
\end{equation}

is the dyadic pattern function represented as a bilinear expansion in the characteristic pattern functions, $F_m(\theta, \phi)$. The bilinear expansion has been truncated after M terms, all terms beyond these being negligible contributors to the scattered field. The bistatic cross section is then given by

\begin{equation}
\sigma(\theta^S, \phi^S; \theta^i, \phi^i) = \frac{(\omega\mu)^2}{4\pi} |F(\theta^S, \phi^S; \theta^i, \phi^i)|^2.
\end{equation}

Those characteristic modes for which the phase angle $\alpha_m$ is near $\pi$ are the most effective contributors to radiated power. For this reason, if $\alpha_m = \pi$, the mode is said to be in scattering resonance. Modes for which $\alpha_m = \pi/2$ or $3\pi/2$ do not contribute at all to radiated power and are said to be in scattering null.
The determination of these characteristic fields and their characteristic values, $\alpha_m$, is not a simple task. In the following section a computational technique is described and used to derive characteristic mode current distributions and the associated angles, $\alpha_m$, for perfectly conducting thin wire scatterers of general configuration. Use is made of the defining conditions (i) and (ii), given in the above discussion, in developing the digital computer method.

B. A Technique for Determining Characteristic Modes of Wires

A modal impedance matrix, $[Z]$, can be derived for a wire such that

$$[Z] \ (I) = (E),$$

where $(E)$ is a vector representing the tangential component of the electric field intensity at $N$ points on the wire due to a current $I$ existing on the wire surface. $(I)$ is a vector whose components are the coefficients of a Fourier expansion of the current $I$ in terms of $M$ orthonormal basis functions ($M \leq N$). The $ij^{th}$ element of the impedance matrix, then, gives the tangential electric field at the $i^{th}$ point on the wire due to the $j^{th}$ basis function of current distributed over the wire length. Thus, $[Z]$ is an $N \times M$ matrix and $(I)$ and $(E)$ are column vectors of dimensions $M$ and $N$, respectively.

In a common application of Eq. (22) to scattering problems, the $M$ dimensional current vector $(I)$ is sought which generates a scattered
field equal to the negative of the incident field (i.e., \(E) = -(E^i)\)) at \(M\) points along the wire, thereby approximating the exact boundary condition of zero total tangential electric field everywhere along the wire in a "point-matching" sense. A less common application[16], (Chapter II) of Eq. (22) is the approximation of the exact boundary condition in a "least mean square" sense. In this method, an \(M\) dimensional current vector \(I\) is sought which generates a scattered field that differs from the negative of the incident field such that the norm, \(|(E) + (E^i)|^2\), of the difference vector, when averaged over \(N > M\) points on the wire, is a minimum. Application of Eq. (22) to the determination of characteristic modes will make rather novel use of the least mean square criterion by minimizing the phase variation along the wire rather than the field differences. This is accomplished by imposing, as constraints on Eq. (22), conditions (i) and (ii) of the preceding section, which specify that the \(m^{th}\) characteristic current \(I_m\), represented by the vector \((I_m)\) in the chosen basis system, be real and that the associated characteristic field \(E_m\), represented by the vector \((E_m)\) in the chosen basis system, be equiphase over the wire surface and lag \(I_m\) in phase by an angle \(\alpha_m\). The problem is to determine the phase angles \(\alpha_m\) and the associated characteristic current distributions \(I_m\) for the significant modes. The characteristic pattern functions \(F_m(\theta, \phi)\) derived from the \(I_m\), together with the characteristic values \(a_m\) derived from the \(\alpha_m\), then provide a concise evaluation of the scattered electric field according to Eq. (19).
To this end, Eq. (22) is rewritten as

\[(23) \quad [Z] (I) = (E')e^{-i\alpha},\]

where a constant phase term, \(\alpha\), is removed from the field vector and is shown explicitly. This phase factor is then transferred to the left side of Eq. (23) and finally is absorbed into the impedance matrix, i.e.,

\[(24) \quad e^{i\alpha} [Z] (I) = [Z'(\alpha)] (I) = (E'),\]

where \([Z'(\alpha)] = e^{i\alpha}[Z].\)

If \(\alpha\) and \((I)\) of Eqs. (23) and (24) were the phase and corresponding real current vector associated with a characteristic mode, i.e., if \(\alpha = \alpha_m\) and \((I) = (I_m)\), \((E')\) would be a real vector. At this point in the problem, however, both \(\alpha_m\) and \((I_m)\) are unknown. Thus it is expected that, initially, \((E')\) has some residual phase factor which is a function of the distance \(l\) along the wire center-line. To reflect this fact, Eq. (24) is rewritten as

\[(25) \quad [Z'(\alpha)] (I) = (\text{RE}(E')) + i(\text{IM}(E')).\]

Letting \((\phi) = (\text{IM}(E'))\), and assuming that \((I)\) is a real vector,

\[(26) \quad [\text{IM}(Z'(\alpha))] (I) = (\phi),\]

where \([\text{IM}(Z'(\alpha))]\) is a real matrix. The real vector \((\phi)\) is thereby related to the deviation of the phase of \((E')\) from an equiphase
condition. Ideally, $(\delta)$ would equal the null vector when $(I) = (I_m)$.

We shall approximate this null vector with one which is minimum in a least mean square sense, i.e., one whose norm,

\begin{equation}
\varepsilon = (\delta)^T (\delta),
\end{equation}

is minimized. (The superscript $T$ denotes the transpose operation.)

Following the suggestion of Eq. (27), Eq. (26) is multiplied by its transpose, with the result,

\begin{equation}
(I)^T [\text{Im}(Z'(\alpha))]^T [\text{Im}(Z'(\alpha))] (I) = (\delta)^T (\delta) = \varepsilon(\alpha),
\end{equation}

where the mean squared error, $\varepsilon(\alpha)$, is recognized explicitly as a function of $\alpha$. Let

\begin{equation}
[B(\alpha)] = [\text{Im}(Z'(\alpha))]^T [\text{Im}(Z'(\alpha))],
\end{equation}

where $[B(\alpha)]$ is a real symmetric matrix of dimensions $M \times M$. Assume for convenience that the normalization $(I)^T (I) = 1$ applies. Then Eq. (28) becomes

\begin{equation}
[B(\alpha)] (I) = \varepsilon(\alpha) (I)
\end{equation}

or

\begin{equation}
[B(\alpha) - \varepsilon(\alpha) I] (I) = (0),
\end{equation}

where $[I]$ is the identity matrix and $(0)$ is the null vector. The equation
\[(32) \quad \det [B(\alpha) - \varepsilon(\alpha)I] = 0 \]

is recognized as the characteristic equation of the matrix \([B(\alpha)]\). The \(M\) roots \(\varepsilon(\alpha) = \varepsilon_m(\alpha), m=1,2,\ldots,M\), of this equation are the eigenvalues of \([B(\alpha)]\) and the associated \(M\) vectors \((I) = (I_m(\alpha))\) are its eigenvectors. Because \([B(\alpha)]\) is real and symmetric the \(\varepsilon_m(\alpha)\) must be real; [43] furthermore, because of the mean square condition enforced by Eq. (13) they must be non-negative. To assist in the following discussion, the eigenvalues \(\varepsilon_m(\alpha)\) have been plotted in Fig. 14 as a function of the phase angle \(\alpha\) for a specific example to be developed later.

Equation (32) determines the functional dependence of the eigenvalues \(\varepsilon_m\) on \(\alpha\). At this point it may be well to remember that \(\varepsilon_m(\alpha)\) is a measure of the deviation of the phase of \((E)\) from an equiphase condition. The smaller the \(\varepsilon_m(\alpha)\) the closer this ideal condition is approached. In this light it is clear that we are interested in those values of \(\alpha\) which cause relative minima in \(\varepsilon_{\text{min}}(\alpha)\), where \(\varepsilon_{\text{min}}(\alpha)\) denotes the smallest of the \(M\) eigenvalues \(\varepsilon_m(\alpha)\) for a specified \(\alpha\). These relative minima, denoted by \(\varepsilon_{\text{min}}(\alpha_m)\), will occur for \(\alpha = \alpha_m\), \(m=1,2,\ldots,M\), corresponding to each of the eigenvectors \((I_{\text{min}}(\alpha_m))\) of Eq. (31). The angle \(\alpha_m\) is the phase of the \(m^{th}\) equiphase surface field \((E_m)\) and the associated eigenvector \((I_{\text{min}}(\alpha_m))\), hereafter denoted simply by \((I_m)\), describes the \(m^{th}\) characteristic current distribution. For each pair, \(\alpha_m\) and \((I_m)\), the equiphase constraint on the tangential electric field is satisfied in a least mean square sense at the \(N\) selected points along the wire. The \(m^{th}\) characteristic
Fig. 14.—Eigenvalues $\epsilon_m(\alpha)$ for a circular loop of radius 0.25$\lambda$ with a wire radius of 0.0025$\lambda$. 
mode current distribution may now be expressed explicitly as a Fourier expansion in terms of the aforementioned set of orthonormal basis functions (e.g., cosine and sine functions), the coefficients of which are the components of the real vector \( (I_m) \).

In applying this procedure it is convenient to break up the matrix \([B(\alpha)]\) of Eq. (29), separating out those parts which are independent of \( \alpha \). Recalling that \( [Z'(\alpha)] = e^{i\alpha}[Z] \), where \([Z]\) is a complex matrix, let

\[
[A] = [\text{Im}(Z'(\alpha))] = [\text{Im}(e^{i\alpha}[Z])]
= [\text{Im}(Z) + \tan \alpha \text{Re}(Z)] \cos \alpha.
\]

Then, according to Eq. (29),

\[
[B(\alpha)] = [A]^T [A]
= [\text{Im}(Z)^T \text{Im}(Z) + \tan \alpha (\text{Re}(Z)^T \text{Im}(Z))
+ \text{Im}(Z)^T \text{Re}(Z)] + \tan^2 \alpha \text{Re}(Z)^T \text{Re}(Z)] \cdot \cos^2 \alpha
\]

Letting

\[
[P] = [\text{Im}(Z)^T \text{Im}(Z)],
\]
\[
[Q] = [\text{Re}(Z)^T \text{Im}(Z) + \text{Im}(Z)^T \text{Re}(Z)],
\]
and

\[
[R] = [\text{Re}(Z)^T \text{Re}(Z)],
\]

Eq. (34) may be written as

\[
\frac{1}{\cos^2 \alpha} [B(\alpha)] = [P] + \tan \alpha [Q] + \tan^2 \alpha [R].
\]
It is efficient to derive the matrices \([P]\), \([Q]\), and \([R]\) from which the matrix \([B(\alpha)]\) can be formed for each \(\alpha\) as \(\alpha\) is varied in seeking relative minima of \(\varepsilon_{\text{min}}(\alpha)\). The term \(1/\cos^2\alpha\) is ignored from this point on since it has no effect on the location of these minima.

When both even and odd basis functions are used for the current expansion and these functions are ordered so that the first \(M_e\) components of the current vector \((I)\) correspond to the even and the last \(M_o\) components correspond to the odd basis functions (where \(M = M_e + M_o\) is the total number of basis functions), the matrix \([B(\alpha)]\) has a block diagonal form, i.e.,

\[
[B(\alpha)] = \begin{bmatrix}
[B_e(\alpha)] & 0 \\
0 & [B_o(\alpha)]
\end{bmatrix},
\]

where the matrices \([B_e(\alpha)]\) and \([B_o(\alpha)]\) are associated with the even and the odd basis functions, respectively. The even and odd characteristic mode current distributions can then be derived separately from the respective even and odd characteristic equations

\[
\det [B_e(\alpha) - \varepsilon_e(\alpha)I] = 0
\]

and

\[
\det [B_o(\alpha) - \varepsilon_o(\alpha)I] = 0.
\]

Solution of the two above equations of dimensions \(M_e\) and \(M_o\), respectively, consumes less computer time than the solution of one characteristic equation of dimension \(M = M_e + M_o\). As an added
benefit, the separation into even and odd problems aids the location of minima in $\epsilon_{\text{min}}(\alpha)$ when an even and an odd characteristic mode both have equal or very nearly equal phase angles.

An example of the behavior of the eigenvalues, $\epsilon_m(\alpha)$, as functions of $\alpha$ has been given in Fig. 14. The scatterer in this example is a circular loop of radius $0.25\lambda$, with a wire radius of $0.0025\lambda$. The loop is approximated by 120 straight-wire segments and five cosine functions form the basis of the Fourier expansion of the current distributions. Due to the rotational symmetry of the loop and the resulting second order degeneracy of the modes, the eigenvalues associated with the sine functions are the same as those associated with the cosine functions. The characteristic phase angles, $\alpha_m$, $m=0,1,2,3,4$, corresponding to the relative minima $\epsilon_{\text{min}}(\alpha_m)$, are given in Table 3 where they are compared with those angles calculated[20] analytically by independent methods.

The circular loop is a very special example in that the characteristic mode current distributions and the basis functions used for their expansion are one and the same, i.e., cosine and sine functions. For this reason each eigenvalue $\epsilon_m(\alpha)$ of $[B(\alpha)]$ assumes a minimum $\epsilon_{\text{min}}(\alpha_m)$ for its respective angle $\alpha_m$ as seen in Fig. 14. For more general scatterers the eigenvalue behavior is not this simple but

*Because the characteristic currents turn out to be the familiar $\cos m\phi$ functions in this example, we have chosen to order the modes on $m$ in the conventional manner, beginning with $m = 0$. In more general problems one would order the modes so as to make $|\cos \alpha_m|$ a non-increasing sequence with increasing $m$. 
no extra complications arise. In any case, the smallest eigenvalue, $\varepsilon_{\text{min}}(\alpha)$, is the only eigenvalue of interest and regardless of which $\varepsilon_m(\alpha)$ happens to be the smallest eigenvalue for the choice of $\alpha$ under study, it is the relative minima of this $\varepsilon_{\text{min}}(\alpha)$ which are sought. To emphasize this point $\varepsilon_{\text{min}}(\alpha)$ is shown for the preceding example as a heavy line in Fig. 14.

### TABLE 3

<table>
<thead>
<tr>
<th>PHASE ANGLES $\alpha_m$ FOR A CIRCULAR LOOP OF RADIUS $0.25\lambda$ WITH WIRE RADIUS $0.0025\lambda$</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>MODE $m$</th>
<th>$\alpha_m$ (DEGREES)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ANALYTICAL METHOD</td>
</tr>
<tr>
<td>0</td>
<td>103.4</td>
</tr>
<tr>
<td>1</td>
<td>107.5</td>
</tr>
<tr>
<td>2</td>
<td>263.1</td>
</tr>
<tr>
<td>3</td>
<td>269.7</td>
</tr>
<tr>
<td>4</td>
<td>$\sim 270.0$</td>
</tr>
</tbody>
</table>

### C. Scattering Solutions Using Characteristic Modes

(i) Backscattering Cross Section

Assuming a unit plane wave incident field, the scattered electric field intensity for an object is given by Eq. (19). The equation is repeated here for convenience.
The matrix equivalent of $F$ is

$$[F] = \begin{bmatrix}
\sum_{m=1}^{M} a_m F_m^{(\theta^i, \phi^i)} F_m^{(\theta^S, \phi^S)} & \sum_{m=1}^{M} a_m F_m^{(\theta^i, \phi^i)} F_m^{(\theta^S, \phi^S)} \\
\sum_{m=1}^{M} a_m F_m^{(\theta^i, \phi^i)} F_m^{(\theta^S, \phi^S)} & \sum_{m=1}^{M} a_m F_m^{(\theta^i, \phi^i)} F_m^{(\theta^S, \phi^S)}
\end{bmatrix}$$

where $F_m^{\theta}$ and $F_m^{\phi}$ are the $\theta$- and $\phi$-components of the vector $F_m$. The quantities $F_{\theta \theta}$, $F_{\theta \phi}$, $F_{\phi \theta}$ and $F_{\phi \phi}$ are functions (complex in general) of the incidence direction $(\theta_i, \phi_i)$ and the scattering direction $(\theta^S, \phi^S)$ although the dependence is not explicitly shown for simplicity in later discussions.

Both bistatic and monostatic scattering solutions may be derived from these expressions. Only the backscattering (monostatic) case, for which $\theta^i = \theta^S = \theta$, $\phi^i = \phi^S = \phi$ and $F_{\theta \phi} = F_{\phi \theta}$, is considered here. The matrix representation of Eq. (42) is

$$\mathbf{E}^S = -\frac{i\omega}{4\pi} e^{-ikr} \mathbf{r} \cdot \mathbf{F} \left[ (\mathbf{h}^t) \right]$$

where $(\mathbf{h}^t)$ is the transmitting antenna vector height[44] expressed in the $\theta$-\phi basis system.
The radar cross section is proportional to $|V|^2$, with

$$V(\theta, \phi) = (h^r)^T (E^s),$$

where $(h^r)$ is the vector height of the receiving antenna and the superscript $T$ denotes the vector transpose. Substituting Eq. (44) into Eq. (45),

$$V(\theta, \phi) = -\frac{i\omega}{4\pi} \frac{e^{-ikr}}{r} (h^r)^T [F] (h^t)$$

The backscattering cross section is then

$$\sigma_{bs}(\theta, \phi) = \lim_{r \to \infty} 4\pi r^2 |V(\theta, \phi)|^2 = \frac{\omega^2}{4\pi} |(h^r)^T [F] (h^t)|^2.$$  

It may be helpful at this point to note the similarity between the matrix $[F]$ and the familiar scattering or polarization matrix which will be denoted by $[S]$. In fact, $[S] = [F]$, so that

$$s_{11} = -\sum_{m=1}^{M} a_m F_m(\theta, \phi^i) F_m(\theta^S, \phi^S), \text{ etc.}$$

Thus, the matrix $[F]$ is simply an explicit expression of the polarization matrix in terms of the characteristic mode pattern functions $F_m(\theta, \phi)$.

Equation (47) compactly expresses the plane wave backscattering cross section of an obstacle in terms of its characteristic pattern functions and the transmitter and receiver vector antenna heights. Since it will be useful later on to have explicit expressions for
certain vector antenna height combinations, we now devote some effort to their derivation. Of general interest are the transmitter-receiver polarization combinations of (1) vertical-to-vertical, (2) horizontal-to-horizontal, (3) vertical-to-horizontal (equal to horizontal-to-vertical in the monostatic case), (4) right circular-to-right circular, (5) left circular-to-left circular, and (6) right circular-to-left circular (equal to left circular-to-right circular in the monostatic case).

Letting the theta- and phi-components of the transmitter and receiver vector antenna heights be expressed explicitly, i.e.,

\[
(h^t) = \begin{pmatrix} h^t_\theta \\ h^t_\phi \end{pmatrix} \quad \text{and} \quad (h^r) = \begin{pmatrix} h^r_\theta \\ h^r_\phi \end{pmatrix},
\]

and substituting into Eq. (47),

\[
\sigma_{bs} = \frac{(\omega u)^2}{4\pi} \left[ |F_{\theta \theta}|^2 |h^t_\theta|^2 |h^r_\theta|^2 + |F_{\phi \phi}|^2 |h^t_\phi|^2 |h^r_\phi|^2 
+ |F_{\theta \phi}|^2 (|h^t_\phi|^2 |h^r_\theta|^2 + |h^t_\theta|^2 |h^r_\phi|^2)
+ h^t_\phi h^c_\phi h^r_\theta h^r_\phi + h^t_\theta h^c_\theta h^r_\phi h^r_\phi
+ 2 \text{Re}\{F_{\theta \phi}^c F_{\phi \theta}^c (h^t_\phi h^c_\phi h^r_\theta h^r_\phi + h^t_\theta h^c_\theta h^r_\phi h^r_\phi)}
+ F_{\theta \phi}^c F_{\phi \theta}^c (|h^t_\theta|^2 h^c_\theta h^r_\phi + h^t_\phi h^c_\phi h^r_\theta) \right].
\]

The backscattering cross section for specific transmitter and receiver polarizations are obtained from Eq. (49) by assigning the
proper values to the vector antenna height components. Expressions for the polarizations mentioned earlier are given here:

(1) vertical-to-vertical \((h^t_\theta = h^r_\theta = 1, h^t_\phi = h^r_\phi = 0)\),

\[
\sigma_{VV} = \frac{(\omega \mu)^2}{4\pi} |F_{\theta\theta}|^2
\]

(2) horizontal-to-horizontal \((h^t_\theta = h^r_\theta = 0, h^t_\phi = h^r_\phi = 1)\),

\[
\sigma_{HH} = \frac{(\omega \mu)^2}{4\pi} |F_{\phi\phi}|^2
\]

(3) vertical-to-horizontal \((h^t_\theta = h^r_\theta = 1, h^t_\phi = h^r_\phi = 0)\),

\[
\sigma_{VH} = \frac{(\omega \mu)^2}{4\pi} |F_{\theta\phi}|^2
\]

(4) right circular-to-right circular \((h^t_\theta = h^r_\theta = \frac{1}{\sqrt{2}}, h^t_\phi = h^r_\phi = -\frac{1}{\sqrt{2}})\),

\[
\sigma_{RR} = \frac{(\omega \mu)^2}{4\pi} \left[ \frac{1}{4} \left( |F_{\theta\theta}|^2 + |F_{\phi\phi}|^2 \right) + |F_{\theta\phi}|^2 \right.
- \frac{1}{2} \text{Re} \{F_{\theta\theta} F_{\phi\phi}^c\} + \text{Im} \{F_{\theta\theta} F_{\phi\phi}^c\} \left. \right]
\]

(5) left circular-to-left circular \((h^t_\theta = h^r_\theta = \frac{1}{\sqrt{2}}, h^t_\phi = h^r_\phi = \frac{1}{\sqrt{2}})\),

\[
\sigma_{LL} = \frac{(\omega \mu)^2}{4\pi} \left[ \frac{1}{4} \left( |F_{\theta\theta}|^2 + |F_{\phi\phi}|^2 \right) + |F_{\theta\phi}|^2 \right.
- \frac{1}{2} \text{Re} \{F_{\theta\theta} F_{\phi\phi}^c\} - \text{Im} \{F_{\theta\theta} F_{\phi\phi}^c\} \left. \right]
\]

and (6) right circular-to-left circular \((h^t_\theta = h^r_\theta = \frac{1}{\sqrt{2}}, h^t_\phi = -h^r_\phi = -\frac{1}{\sqrt{2}})\),

\[
\sigma_{RL} = \frac{(\omega \mu)^2}{4\pi} \left[ \frac{1}{4} \left( |F_{\theta\theta}|^2 + |F_{\phi\phi}|^2 \right) + \frac{1}{2} \text{Re} \{F_{\theta\theta} F_{\phi\phi}^c\} \right]
\]
The \( \theta \)- and \( \phi \)-dependence of the elements of the matrix \([F]\) and of the cross section \(\sigma\) is assumed and is suppressed for ease in notation. The \( \theta \) and \( \phi \) subscripts signify the particular elements of the matrix \([F]\) (see Eq. (43)).

(ii) Tumble Average Backscattering Cross Section

Having determined a suitable number of characteristic values and characteristic pattern functions for a given obstacle, it is a simple matter to apply the bilinear expansion of Eq. (42) and obtain plane wave scattering data, bistatic or monostatic, in an economical fashion. The ensuing compact expression (Eq. (47)) for backscattering cross section is particularly useful when one must consider an obstacle which is tumbling in space, and the quantity of interest is not backscattering cross section for some fixed orientation, but backscattering cross section averaged over all tumble aspects of the obstacle. Thus, if we denote the backscattering cross section with transmitter and receiver located in a direction \((\theta, \phi)\) by \(\sigma_{bs}(\theta, \phi)\), and if we assume a spherical probability density function for aspect likelihood, then the tumble average cross section can be defined by

\[
\langle \sigma_{bs} \rangle = \frac{1}{\pi} \int_0^{\pi} \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \sigma_{bs}(\theta, \phi) \sin \theta \, d\theta \, d\phi \, d\alpha.
\]

The angle \(\alpha\) is a measure of the rotation of the obstacle about the line-of-sight. Since \(\sigma_{bs}(\theta, \phi)\) is expressible so concisely in terms of characteristic pattern functions via Eq. (21) or Eq. (47), the calculation of \(\langle \sigma_{bs} \rangle\) is greatly expedited.
In the following discussion the average over $\alpha$ will be performed analytically for arbitrary transmitter and receiver polarizations. The average over $\theta$ and $\phi$, however, must be done numerically in general.

In the far field, the most general polarization, elliptical polarization, may be represented[45-47] by a combination of right circular and left circular polarization vectors, i.e.,

$$ (h_e) = (h_L) + (h_R)e^{j2\alpha} $$

where the subscripts $L$ and $R$ denote left and right circular polarization, respectively. The angle $\alpha$ (not to be confused with the characteristic phase angles $\alpha_m$) is the angle of rotation, measured in a C.C.W. direction (looking in the direction of propagation), of the major axis of the polarization ellipse from the vertical axis. From the general expression of Eq. (57) can be derived left circular ($(h_R) = 0$), right circular ($(h_L) = 0$), vertical ($(h_L) = (h_R)$, $\alpha = 0$), and horizontal ($(h_L) = (h_R)$, $\alpha = \pi/2$) polarization vectors.

The norm of $(h_e)$ will be set equal to unity, i.e.,

$$ |h_e| = \sqrt{|h_L|^2 + |h_R|^2} = 1. $$

The axial ratio, $r$, of the ellipse is

$$ r = \frac{|h_R| + |h_L|}{|h_R| - |h_L|}. $$

It is desired to obtain an expression for the average backscattering cross section for general transmitter and receiver polarization states. To this end assume a transmitter polarization
(60) \( (h^t) = (h^t_L) + (h^t_R) e^{j2\alpha t} \)

and a receiver polarization

(61) \( (h^r) = (h^r_L) + (h^r_R) e^{j2\alpha r} \).

Substituting Eqs. (60) and (61) into Eq. (46) (suppressing the constant terms for convenience),

(62) \( V(\theta, \phi) = (h^r_L + h^r_R e^{j2\alpha r})^T [F] (h^t_L + h^t_R e^{j2\alpha t}) \).

Then

(63) \[ |V(\theta, \phi)|^2 = |(h^r_L + h^r_R e^{j2\alpha r})^T [F] (h^t_L + h^t_R e^{j2\alpha t})|^2 \]

\[ = |(h^r_L)^T [F] (h^t_L)|^2 + |(h^r_R)^T [F] (h^t_L)|^2 \]

\[ + |(h^r_R)^T [F] (h^t_R)|^2 + |(h^r_L)^T [F] (h^t_R)|^2 \]

\[ + 2 \text{Re} \left\{ [(h^r_L)^T [F] (h^t_L)] [(h^r_R)^T [F] (h^t_L)] e^{-j2(\alpha_t - \alpha_r)} \right\} \]

\[ + [(h^r_L)^T [F] (h^t_L)] [(h^r_L)^T [F] (h^t_R)] e^{-j2\alpha_r} \]

\[ + [(h^r_R)^T [F] (h^t_L)] [(h^r_L)^T [F] (h^t_R)] e^{-j2\alpha_r} \]

\[ + [(h^r_L)^T [F] (h^t_R)] [(h^r_R)^T [F] (h^t_L)] e^{-j2(\alpha_t + \alpha_r)} \]

\[ + [(h^r_R)^T [F] (h^t_R)] [(h^r_R)^T [F] (h^t_R)] e^{-j2\alpha_t} \]

\[ + [(h^r_L)^T [F] (h^t_L)] [(h^r_R)^T [F] (h^t_R)] e^{-j2\alpha_t} \].
where $c$ denotes the complex conjugate. Let $\alpha_r = \alpha_t + \delta$. The last five terms of Eq. (63), then, contain terms $e^{-j2\alpha_t}$ and $e^{-j4\alpha_t}$. Integrating over $\alpha_t$ corresponds to an averaging of the backscattering cross section as the object rotates on an axis parallel to the direction of propagation. Since

$$\frac{1}{\pi} \int_{0}^{\pi} e^{-j2\alpha} d\alpha = \frac{1}{\pi} \int_{0}^{\pi} e^{-j4\alpha} d\alpha = 0,$$

these five terms will not contribute to the average.

The first five terms of Eq. (63) are seen to be independent of $\alpha_t$. The fifth term is dependent on $\delta$, the angle of separation between the transmitter and the receiver major axes of polarization.

Denoting the average over $\alpha_t$ by $\bar{\cdot}$, Eq. (63) reduces to

$$\hat{\sigma}_{bs}(\theta, \phi) = |(h_L^r)^T [F] (h_L^t)|^2 + |(h_R^r)^T [F] (h_L^t)|^2 + |(h_L^r)^T [F] (h_R^t)|^2 + |(h_R^r)^T [F] (h_R^t)|^2 + 2 \text{Re}\left\{ [(h_L^r)^T [F] (h_L^t)] [(h_L^r)^T [F] (h_L^t)]^c e^{j2\delta} \right\}.$$

The vector antenna heights expressed in the $\theta$-$\phi$ basis system are now specified as noted by their subscripts, i.e.,

$$(64) \quad a) \quad (h_L^r) = L_r \left( \begin{array}{c} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{array} \right), \quad \text{left circular receive;}$$

$$(65) \quad b) \quad (h_R^r) = R_r \left( \begin{array}{c} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{array} \right), \quad \text{right circular receive;}$$
c) \( (h^t_L) = L_t \left( \frac{1}{\sqrt{2}} \right) \), left circular transmit; and

\[
d) \quad (h^t_R) = R_t \left( \frac{1}{\sqrt{2}} \right), \quad \text{right circular transmit;}
\]

where \( R \) and \( L \) represent the right circular and left circular magnitudes, respectively. Substituting into Eq. (64), normalizing according to Eq. (58), averaging over \( \theta \) and \( \phi \) to obtain the average over all tumble angles (denoted by \( <> \)), and expressing the proportionality constant explicitly, we get

\[
<\sigma_{bs}> = \frac{(\omega \mu)^2}{4\pi} \frac{1}{4(R^2_r + L^2_r)(R^2_t + L^2_t)} \left\{ (R^2_r R^2_t + L^2_r L^2_t) \right\}
\]

\[
\times \left[ <|F_{\theta\theta}|^2> + <|F_{\phi\phi}|^2> + 4<|F_{\theta\phi}|^2> - 2<Re\{F_{\theta\theta} F_{\phi\phi}\}> \right]
\]

\[
+ 4[(R^2_r R^2_t - L^2_r L^2_t)] \left<Im\{(F_{\theta\theta} - F_{\phi\phi})F^{C}\}\right>
\]

\[
+ [(L^2_r L^2_t)^2 + (R^2_r R^2_t)^2 + 2(R^2_r L^2_t)(L^2_r R^2_t)\cos 2\delta]
\]

\[
\times \left[ <|F_{\theta\theta}|^2> + <|F_{\phi\phi}|^2> + 2<Re\{F_{\theta\theta} F^{C}_{\phi\phi}\}> \right].
\]

From Eq. (66) we obtain tumble average expressions for the following specific transmitter-receiver polarizations:

(i) right circular-to-right circular \((R_r = R_t = 1, L_r = L_t = 0)\)

\[
<\sigma_{RR}> = \frac{(\omega \mu)^2}{4\pi} \frac{1}{4} \left( <|F_{\theta\theta}|^2> + <|F_{\phi\phi}|^2> + 4<|F_{\theta\phi}|^2>
\]

\[
-2<Re\{F_{\theta\theta} F^{C}_{\phi\phi}\}> + 4<Im\{(F_{\theta\theta} - F_{\phi\phi})F^{C}_{\theta\phi}\}> \right)
\]
(ii) left circular-to-left circular \( (R_r=R_t=0, L_r=L_t=1) \)

\[
\langle \sigma_{LL} \rangle = \frac{(\omega u)^2}{4\pi} \frac{1}{4} \left( |F_{\theta\theta}|^2 + |F_{\phi\phi}|^2 + 4|F_{\theta\phi}|^2 \right. \\
-2\text{Re}(F_{\theta\theta} F_{\phi\phi}^C) - 4\text{Im}((F_{\theta\theta} - F_{\phi\phi}) F_{\theta\phi}^C) \left. \right),
\]

(iii) right circular-to-left circular \( (R_r=L_t=0, R_t=L_r=1) \)

\[
\langle \sigma_{RL} \rangle = \frac{(\omega u)^2}{4\pi} \frac{1}{4} \left( |F_{\theta\theta}|^2 + |F_{\phi\phi}|^2 + 2\text{Re}(F_{\theta\theta} F_{\phi\phi}^C) \right),
\]

(iv) left circular-to-right circular \( (R_r=L_t=1, R_t=L_r=0) \)

\[
\langle \sigma_{LR} \rangle = \frac{(\omega u)^2}{4\pi} \frac{1}{4} \left( |F_{\theta\theta}|^2 + |F_{\phi\phi}|^2 + 2\text{Re}(F_{\theta\theta} F_{\phi\phi}^C) \right),
\]

(v) linear-to-linear \( (R_r=L_r=R_t=L_t=1, \delta = 0) \)

\[
\langle \sigma_{ll} \rangle = \frac{(\omega u)^2}{4\pi} \left( \frac{3}{8} (|F_{\theta\theta}|^2 + |F_{\phi\phi}|^2) + \frac{1}{2} |F_{\theta\phi}|^2 \right. \\
+ \frac{1}{4} \text{Re}(F_{\theta\theta} F_{\phi\phi}^C) \left. \right),
\]

(vi) linear-to-cross linear \( (R_r=L_r=R_t=L_t=1, \delta = \pi/2) \)

\[
\langle \sigma_{l\perp} \rangle = \frac{(\omega u)^2}{4\pi} \left( \frac{1}{8} (|F_{\theta\theta}|^2 + |F_{\phi\phi}|^2) + \frac{1}{2} |F_{\theta\phi}|^2 \right. \\
- \frac{1}{4} \text{Re}(F_{\theta\theta} F_{\phi\phi}^C) \left. \right).
\]

Having expressions (67) through (72), Eq. (66) can be written as
\begin{equation}
\langle \sigma_{bs} \rangle = \frac{1}{(R_r^2 + L_r^2)(R_t^2 + L_t^2)} \cdot \left( (R_r R_t)^2 \langle \sigma_{RR} \rangle + (L_r L_t)^2 \langle \sigma_{LL} \rangle 
\right.
+ [(R_r L_t)^2 + (L_r R_t)^2 + 2(L_r L_t)(L_r R_t)\cos 2\delta] \langle \sigma_{RL} \rangle)
\end{equation}

or as
\begin{equation}
\langle \sigma_{bs} \rangle = \frac{1}{(R_r^2 + L_r^2)(R_t^2 + L_t^2)} \cdot \left( (R_r R_t)^2 - (L_r L_t)^2 \langle \sigma_{RR} \rangle 
\right.
+ [4(L_r L_t)^2 - (R_r L_t)^2 - (L_r R_t)^2 - 2(L_r L_t)(L_r R_t)\cos 2\delta] \langle \sigma_{UL} \rangle 
+ [(R_r L_t)^2 + (L_r R_t)^2 + 2(L_r L_t)(L_r R_t)\cos 2\delta] \langle \sigma_{\parallel} \rangle),
\end{equation}

where Eq. (73) is in terms of circular polarization averages and Eq. (74) is in terms of circular and linear polarization averages.

(iii) Input Admittance

The design of loaded scatterers requires a knowledge of the driving point impedance (or admittance) of the element when it is excited as an antenna. The scatterer is usually resonated by placing a passive load at the set of terminals such that the reactive component is cancelled. The characteristic mode theory offers a new approach toward determining the desired antenna impedance for wire-type scatterers.

An expression for the input admittance can be derived\cite{48} in terms of the characteristic currents and the characteristic values of a given wire. The current evaluated at point \( \xi \) on the wire due to a unit plane wave incident from a direction \((\phi^i_1, \phi^i_2)\) is\cite{20,48}
\begin{equation}
I(\varepsilon) = \sum_{m=1}^{M} A_m I_m(\varepsilon),
\end{equation}

where $I_m(\varepsilon)$ is real and represents the $m^{th}$ normalized characteristic current distribution evaluated at $\varepsilon$, and $A_m$, the weight\cite{20} of the $m^{th}$ term in the expansion, is complex and is given by the reaction\cite{49-51} of the $m^{th}$ radiated field, $E_{m}^{\text{ext}}(r,\theta,\phi)$, with the source current, i.e.,

\begin{equation}
A_m = \iiint_{V} E_{m}^{\text{ext}}(r,\theta,\phi) \cdot J(r,\theta,\phi) \, dV,
\end{equation}

where the integration is over the volume of the source. If the source is an electric dipole located at $(r^i,\theta^i,\phi^i)$ which causes an essentially plane wave in the vicinity of the wire, Eq. (76) becomes

\begin{equation}
A_m = a_m F_m(\theta^i,\phi^i) \cdot \hat{h}^i,
\end{equation}

where $\hat{h}^i$ is the polarization state vector of the plane wave, $a_m$ is the characteristic value associated with the $m^{th}$ mode, and $F_m(\theta^i,\phi^i)$ is the $m^{th}$ characteristic mode pattern function evaluated at $(\theta^i,\phi^i)$. Substituting Eq. (77) into Eq. (75),

\begin{equation}
I(\varepsilon) = \sum_{m=1}^{M} a_m F_m(\theta^i,\phi^i) \cdot \hat{h}^i I_m(\varepsilon).
\end{equation}

From reciprocity,

\begin{equation}
I(\varepsilon) = -\hat{h}^i \cdot F^r(\theta^i,\phi^i),
\end{equation}

Where $F^r(\theta^i,\phi^i)$ is the radiation pattern in the $(\theta^i,\phi^i)$ direction due to unit voltage, $V = 1$, across the gap at point $\varepsilon$ along the wire. Then
Recalling the orthonormality of the characteristic pattern functions, the complex power radiated is seen to be \([20, 48]\)

\[
(80) \quad F^r(\theta^i, \phi^i) = - \sum_{m=1}^{M} a_m F_m(\theta^i, \phi^i) I_m(\varepsilon).
\]

where \(a_m = -\frac{1}{1 + i \lambda_m}\) and \(\lambda_m\) is the ratio of net reactive to net radiated power. The admittance is, then,

\[
(82) \quad Y(\varepsilon) = \frac{P}{|V|^2} = - \sum_{m=1}^{M} a_m I_m(\varepsilon)^2.
\]

D. Examples of Characteristic Modes and Scattering Solutions

In this section the technique described in Section B above is used to determine characteristic mode current distributions and characteristic values of circular and elliptical loops, straight wires, circular arcs, and helices. The modes are used to compute backscattering and tumble average cross sections and input admittance data as outlined in Section C. Where possible, the results are compared with solutions derived by independent methods.

In each case, in deriving the impedance matrix \([Z]\) of Eq. (22), the wire is considered perfectly conducting and its radius is assumed to be much smaller than its length \(L\) and the wavelength \(\lambda\). The assumption is made that the surface current density on the thin wire has only an axial component and is distributed uniformly around the circumference of the wire.
The modal impedance matrix ([Z] of Eq. (22)) for each wire configuration is computed by means of the numerical basis transformation method used in conjunction with a piecewise uniform method.[1-3] The P, Q, and R matrices (Eqs. (35), (36), and (37)) are formed from [Z] and from them the matrix [B(a)] is derived explicitly in terms of a. The characteristic equation (Eq. (32)) is then solved for the smallest eigenvalue $e_{\text{min}}(a)$. By stepping through a range of $a$'s, we can locate the relative minima of $e_{\text{min}}(a) = e_{\text{min}}(a_m)$, thereby specifying the phase angles $a_m$ and allowing the determination of the corresponding eigenvectors ($I_m$) of Eq. (31). The components of these eigenvectors are the Fourier expansion coefficients of the characteristic mode current distributions in terms of the chosen set of basis functions.

More than one characteristic mode generally is needed to describe a scatterer. Often, however, one mode is sufficient, as is the case, for example, with short wires, i.e., wires of first resonant length or less. This single mode character holds true for quite general wire configurations. For short wires which are effective scatterers, the dominant mode has associated with it a phase angle $a_m$ within about $\pm \pi/4$ of the $a = \pi$ resonant angle; other modes characteristic of such a scatterer are found to have phase angles very near $\pi/2$ or $3\pi/2$, implying that their contributions to scattering are relatively small and usually negligible. As wires increase in length more modes are required to describe their scattering properties accurately. The number of modes needed is also dependent, though to a lesser degree, on the wire geometry.
Certain wire geometries support multiply degenerate modes. As mentioned previously, the circular loop is doubly degenerate since each phase angle $\alpha_m$ (i.e., each characteristic value, $\alpha_m$) has associated with it two mode current distributions, one of the form $\cos m\phi$, the other of the form $\sin m\phi$. This fact is due, of course, to the rotational symmetry of the loop.

(i) Circular Loops

Characteristic modes were determined for circular loops with a loop radius-to-wire radius ratio $R/a = 100$. The first four phase angles $\alpha_m$ are shown in Fig. 15 for a range of frequencies. The curves

Fig. 15.—Phase angles $\alpha_m$, $m = 0,1,2,3$, for circular loops as a function of loop radius.
compare favorably with independently derived results presented in Reference 20. The backscattering cross section data derived using these modes are shown in Figs. 16 and 17. The edge aspect results are compared with solutions obtained by Harrington.\[52\] The ratio of broadside-to-edge aspect cross section given in Fig. 17 is compared with similar data,\[36\] both measured and analytical, for loops with a wire radius \(a = 0.005\lambda\).

The real and imaginary parts of the input admittance of circular loops with a loop radius-to-wire radius ratio \(R/a = 100\) are plotted in Fig. 18 and are compared with Harrington's solutions.\[52\] The admittance, especially its imaginary part, is more difficult to compute accurately than the far zone scattering parameters. This example is given, together with part (iii) of Section C, above, to show the manner in which the admittance can be obtained from the characteristic modes. More demanding application to admittance calculations would require more accurate solutions for the characteristic modes, which would in turn require more sophisticated (than piecewise uniform) methods for computing the impedance matrix \([Z]\) of Eq. (22).

(ii) Elliptical Loops

Having noted in the preceding discussion that the characteristic modes of a circular loop are doubly degenerate, it is of interest to observe the effect which distortions of the circular geometry have on the modes. For example, one may study the modes of a planar elliptical loop whose axial ratio diverges from unity. Figure 19 shows
Fig. 16.—Backscattering and tumble average cross sections for circular loops with a loop radius-to-wire radius ratio $R/a = 100$. 
Fig. 17.—Ratio of the backscattering cross section at broadside aspect to that at edge aspect for circular loops with $R/a = 100$. (Ratio = $10 \log_{10} \sigma_{bs}/\sigma_{edge}$.)
Fig. 18.—Input admittance for circular loops with loop radius-to-wire radius ratio $R/a = 100$. 
several phase angles associated with the modes of an ellipse as a function of the axial ratio. The circumference is held constant at 1.1λ. The wire radius is a = 0.00175λ and the circumference-to-wire radius ratio is 200π, i.e., the wire is of the same dimensions as
that used in the preceding circular loop example for a loop radius $R = 0.175\lambda$. For an axial ratio of unity the angles $\alpha_m$ are, of course, those shown in Fig. 15 for the corresponding circular loop. As the axial ratio is varied from unity the rotational symmetry of the loop is lost and the modes are no longer doubly degenerate. The back-scattering cross section data for these examples are shown in Fig. 20.

In the loop examples just given 120 match points were used in solving for the modes, with six cosine and five sine terms in the Fourier expansions of the characteristic currents. The loops are in the $xy$-plane and the origin $\lambda = 0$ is on the positive $x$-axis.

(iii) Straight Wires

The characteristic phase angles for straight wires with a length-to-diameter ratio $L/2a = 100$ are shown in Fig. 21 as a function of frequency. The characteristic current distributions $I_{\pm m}$ for a wire of length $L = 1.5\lambda$ are shown in Fig. 22, where each current has been normalized to radiate unit power. Figures 23 and 24 give backscattering and tumble average cross section data obtained using the characteristic modes associated with the phase angles of Fig. 21. The broadside data given in Fig. 23 are compared with results obtained by Harrington and Mautz.[40] The backscattering cross section as a function of incidence angle $\theta$ is shown in Fig. 24 for a wire length $L = 1.5\lambda$.

Similar phase angle and scattering data are shown in Figs. 25 and 26 for straight wires of radius $a = 0.005\lambda$. The results in Fig. 26 for the broadside backscattering cross section of wires of lengths near the first resonance are compared with those obtained by Richmond.[12]
Fig. 20.--Backscattering and tumble average cross sections for elliptical loops with circumference $C = 1.1\lambda$ and wire radius $a = 0.00175\lambda$. 
Fig. 21.--Phase angles $\alpha_m$, $m=1,2,3,4$, for straight wires with a length-to-diameter ratio $L/2a = 100$.

A comparison of the phase angle curves with the scattering behavior as a function of wire length for the straight wire examples just given demonstrates the dependence of the scattering or radiating efficiency of a mode upon its characteristic value or phase. In each case considered (where one mode is the dominant contributor) a peak in the average cross section corresponds to a wire length for which the phase associated with the dominant mode is equal to or approximately equal to $\pi$. 

Fig. 22.—Normalized characteristic current distributions $I_m$, $m=1,2,3,4$, for a straight wire of length $L = 1.5\lambda$ with $L/2a = 100$. 
Fig. 23.—Broadside and tumble average backscattering cross sections of straight wires with $L/2a = 100$. 
Fig. 24.—Backscattering cross section as a function of incidence angle $\theta$ for a straight wire of length $L = 1.5\lambda$ with $L/2\alpha = 100$.

In determining the characteristic modes for the straight wires considered above, up to 160 match points were used with six even and five odd Chebyshev polynomials of the second kind in the Fourier expansions of the characteristic currents.
(iv) Circular Arcs

The examples given so far have been scatterers for which more than one mode were needed to describe the scattering properties accurately. In particular instances, e.g., the case of straight wires with $L \leq 0.6\lambda$, only one mode was needed to obtain reasonably accurate plane wave scattering data. In this section we show that such short
Fig. 26.—Broadside and tumble average backscattering cross sections of straight wires with $a = 0.005\lambda$. 
wires, even when formed into circular arcs, retain this "single-mode" nature.

The phase angle associated with the dominant characteristic mode for a lossless open circular arc of wire of length $L = 0.475\lambda$ and radius $a = 0.0008467\lambda$ is shown in Fig. 27 as a function of the arc radius, $R$. Using only this dominant mode, the backscattering data of Fig. 28 were derived. The broadside- and edge-aspect cross section results are compared with solutions obtained[53] by a piecewise uniform, point-matching method for tungsten wires.

It is interesting to note the dependence of the dominant mode current distribution on the arc radius. The distribution appears to be nearly independent of the arc radius to a radius of approximately $0.085\lambda$. This can be seen by noting the nearly constant values obtained for the Fourier coefficients of the series approximations to be current. Figure 29 shows the first five coefficients as a function of arc radius. Each coefficient has been normalized to its value for a radius $R = 0.15\lambda$ for convenience in plotting. This suggests that the mode current distribution for a short wire of a given length may be, to a degree, independent of the wire configuration. The characteristic value, or phase angle, associated with the current, however, may show considerable variation and can give evidence of significant variations in scattering cross section. For example, Fig. 28 shows a variation of more than 3 dB in backscattering cross sections for circular arcs ranging in arc radii from $0.085\lambda$ to $0.15\lambda$, in spite of
the constancy of current distribution implied by Fig. 29. Only a substantial change in phase angle can be the explanation, and Fig. 27 bears this out.

Fig. 27.—Phase angle $\alpha_1$ as a function of arc radius for the dominant mode of a circular wire arc with wire length $L = 0.475\lambda$ and wire radius $a = 0.0008467\lambda$. 
Fig. 28.--Backscattering and tumble average cross section data for a circular arc with $L = 0.475\lambda$ and $a = 0.0008467\lambda$. 
Fig. 29.—Characteristic mode current coefficients for the dominant mode of a circular arc with wire length $L = 0.475\lambda$ and wire radius $a = 0.0008467\lambda$. (Each coefficient is normalized to its value for a radius $R = 0.15\lambda$.)
(v) Helices

Helices were formed with a wire of length $L = 0.475\lambda$ and radius $a = 0.0008467\lambda$. The helix pitch, $p$, the distance between turns, is equal to thirty times the wire radius. Just as for the circular arcs formed by wires of the same dimensions, a single mode describes the scattering with sufficient accuracy.

The phase angle associated with the dominant mode as a function of the helix radius, $R$, is shown in Fig. 30. Backscattering and tumble average cross section data for linear polarization are given in Fig. 31 and that for circular polarization in Fig. 32. The broadside aspect results in Fig. 31 are compared with Thiele's[54] piecewise uniform, point-matching solutions for tungsten wires.

A helix formed of a wire whose length, $L$, is slightly greater than the straight-wire resonant length, viz., $L = 0.6\lambda$, is considered next. In this case it is necessary to include the contributions of up to three modes, all of which are relatively small. The phase angles are given in Fig. 33 for the first two even modes and the first odd mode. With these modes the linear and circular polarization backscattering and tumble average cross section curves of Figs. 34 and 35 were calculated.

Finally, a one-turn helix with a wire length-to-diameter ratio $L/2a = 100$ and a pitch $p = 10a$ is investigated as a function of frequency. The phase angles corresponding to the first four characteristic modes are shown in Fig. 36. Figures 37 and 38 give the backscattering cross sections at various fixed aspects, as well as the
tumble average backscattering cross sections, for linear and circular polarizations, respectively. It is interesting to compare these results with those (Figs. 21 and 23) obtained for a straight wire of the same dimensions. A comparison is made in Fig. 39 of the tumble average cross sections of these configurations for linear-to-linear transmitter-receiver polarizations. Of particular interest, for
Fig. 31.—Backscattering and tumble average cross sections of a helix with $L = 0.475\lambda$, $a = 0.0008467\lambda$, and $p = 30a$ for linear polarization.
Fig. 32.—Backscattering and tumble average cross sections of a helix with $L = 0.475\lambda$, $a = 0.0008467\lambda$, and $p = 30a$ for circular polarization.
example, is the relation of the bandwidth of the first resonance to the slope of the corresponding phase angle curve for $\alpha_1$ in the vicinity of $180^\circ$. The slope of the $\alpha_1$-curve for the helix (Fig. 36) is noticeably greater than that for the straight wire (Fig. 21). The reduced bandwidth of the first resonance for the helix is evident in Fig. 39.
Fig. 34.—Backscattering and tumble average cross sections of a helix with $L = 0.6\lambda$, $\alpha = 0.0008467\lambda$, and $p = 30\alpha$ for linear polarization.
Fig. 35.—Backscattering and tumble average cross sections of a helix with $L = 0.6\lambda$, $a = 0.0008467\lambda$, and $p = 30a$ for circular polarization.
Fig. 36.—Phase angles $\alpha_m$, $m = 1, 2, 3, 4$, for a one-turn helix with wire length-to-diameter ratio $L/2a = 100$ and pitch $p = 10a$. 
Fig. 37.—Backscattering and tumble average cross sections of a one-turn helix with \( L/2a = 100 \) and \( p = 10a \) for linear polarization.
Fig. 38.--Backscattering and tumble average cross sections of a one-turn helix with $L/2a = 100$ and $p = 10a$ for circular polarization.
Fig. 39.—Comparison of tumble average cross sections of a straight wire and a one-turn helix, with wire length-to-diameter ratio $L/2a = 100$ and a helix pitch $p = 10a$. 
CHAPTER IV
SUMMARY AND CONCLUSIONS

A numerical basis transformation is described which transforms the set of point-matching linear equations from one basis to another. It is shown that it is possible to reduce the number of linear equations needed for good accuracy and, as a result, to reduce computer solution time and storage requirements. For example, a transformation from the piecewise uniform basis to a basis of modal functions, such as cosines and sines, can reduce the number of equations, N, by a factor of 10 or more with a consequent reduction from a cubic to a square law dependence of solution time on N and at least a five-fold reduction in storage requirements. It is also shown that, following the basis transformation and the subsequent reduction from N to M equations, a least square solution is obtained. The transformation and least square techniques are applied to the solution of scattering by several thin wire objects. In particular, perfectly conducting elements in the shape of circular and elliptical loops, straight wires, and circular arcs are considered.

In the examples given here the basis transformation technique is applied only in transforming from the piecewise uniform basis to a modal basis. The technique can be employed equally well in con-
junction with other point-matching methods. It may also find application with numerical methods other than those based on point-matching. For example, the multi-segment induced emf technique, a technique based on the reciprocity theorem and reaction concepts, could be combined with the basis transformation technique in the same manner as is discussed here. It should also be noted that the transformation and least square techniques are not restricted to perfectly conducting objects, although only loss-free scatterers are considered here.

Having shown the advantages afforded by a transformation of basis as applied to the solution of scattering by wires, other applications can be suggested. The point-matching technique has been applied to bodies of revolution and to planar scatterers such as circular and rectangular plates. A basis transformation could significantly reduce the number of equations in such problems, a factor of particular importance for three-dimensional bodies. Also of interest are the application of the basis transformation and least square techniques to point-matching solutions of loaded wire scatterers and antennas and to multiple-wire scatterers such as coaxial loops.

Further application is made of the basis transformation and least square techniques to the determination of characteristic modes of loss-free, thin wires of general configuration. A method of determining characteristic mode current distributions and associated phase angles of thin wires is developed and is
applied to circular and elliptical loops, straight wires, circular arcs, and helices.

Explicit expressions are derived, in terms of the characteristic mode pattern functions, for backscattering and tumble average backscattering cross sections for arbitrary transmitter and receiver polarizations. The characteristic modes derived for the aforementioned objects are then used to compute backscattering and tumble average cross section data for the respective shapes. Also computed for heuristic purposes are input admittance data for circular loops. Where possible, the results are compared with solutions derived by independent methods.

The examples given in the preceding discussion serve to demonstrate the determination and the application of characteristic modes for the solution of scattering by wire objects. Of particular interest is the usefulness of the characteristic mode approach in calculating the tumble average cross section. Application of the method to other single-wire configurations would be of interest. The method could also be extended to multiple-wire scatterers, such as coaxial wire loops, and to wires with sharp bends and/or lumped loads. At the present time, however, the characteristic mode method is limited in a practical sense to objects for which a small number of modes, say, less than four or five, accurately describe the scattering properties.
APPENDIX A

THE LEAST SQUARE SOLUTION OF A SET OF N LINEAR COMPLEX EQUATIONS IN M Unknowns, WITH M ≤ N

The least square solution of a set of N equations,

\[ \sum_{j=1}^{M} Z_{ij} I_j = E_i, \quad i = 1, 2, \ldots, N; \quad M \leq N, \]

is one which minimizes the quantity

\[ S = \sum_{i=1}^{N} \left( \sum_{j=1}^{M} Z_{ij} I_j \right) \left( \sum_{j=1}^{M} Z_{ij} I_j \right)^C, \]

where \( c \) denotes the complex conjugate. To find a minimum in \( S \) consider its first derivative with respect to each \( I_k, k = 1, 2, \ldots, M \), i.e.,

\[ \frac{\partial S}{\partial I_k} = -\sum_{i=1}^{N} \left( Z_{ik} [E_i - \sum_{j=1}^{M} Z_{ij} I_j]^C + Z_{ik}^C [E_i - \sum_{j=1}^{M} Z_{ij} I_j] \right), \]

\[ k = 1, 2, \ldots, M. \]

Setting \( \partial S/\partial I_k = 0 \) for a minimum in \( S \), Eq. (85) yields

\[ \Re \left( \sum_{i=1}^{N} Z_{ik}^C [E_i - \sum_{j=1}^{M} Z_{ij} I_j] \right) = 0, \quad k = 1, 2, \ldots, M. \]

The matrix equivalent to Eq. (86) is

\[ \Re [Z]^* [Z] (I) = \Re [Z]^* (E), \]
where the superscript $^*$ denotes the complex conjugate transpose and the brackets $[\ ]$ and parenthesis $(\ )$ denote matrices and column vectors, respectively.

Returning to Eq. (85), we may also satisfy the condition $\partial S/\partial I_k = 0$ by setting each of the right hand terms equal to zero, i.e.,

\begin{equation}
\sum_{i=1}^{N} Z_{ik} [E_i - \sum_{j=1}^{M} Z_{ij} I_{ij}]^C = 0, \quad k = 1, 2, \ldots, M,
\end{equation}

and

\begin{equation}
\sum_{i=1}^{N} Z_{ik}^C [E_i - \sum_{j=1}^{M} Z_{ij}^C I_{ij}] = 0, \quad k = 1, 2, \ldots, M.
\end{equation}

Equation (88) is merely the complex conjugate of Eq. (89). The matrix equivalent to these equations is seen to be

\begin{equation}
\end{equation}

The solution of either Eq. (87) or Eq. (90) is a least square solution of the set of equations given in Eq. (83).
The thin-wire scatterers considered in this treatise are approximated in each case by a number (N) of short, straight-wire segments. The elementary scatterer to be considered under the piecewise uniform approximation is then a short slender wire with a uniform current density on its surface.

Consider a perfectly conducting wire segment of length s and radius a to have a harmonic electric current induced uniformly over its surface. Let the axis of the wire coincide with the z axis and let the center of the wire be the coordinate origin. If the current density is \( \mathbf{J} = \hat{z} J \), where \( J \) is a constant and \( \hat{z} \) denotes a unit vector parallel with the z-axis, the current in amperes is given by

\[
I = 2\pi a J.
\]

In general, the free-space field of this source is expressed as a surface integral over the surface of the wire. In our application, however, the general expressions simplify as follows

\[
E_\rho(\rho, \phi, z) = \frac{\rho i \sqrt{\mu / \varepsilon}}{4 \pi j k} (1 + j kr) \frac{e^{-j kr}}{r} \left| \begin{array}{c} r_2^2 \\ r_1 \end{array} \right|,
\]

(92) 

\[
E_\phi(\rho, \phi, z) = 0,
\]

(93)
and

\[(94) \quad E_z(\rho, \phi, z) = \frac{I}{4\pi j k} \int_{-s/2}^{s/2} \left[ 2r^2(1+jkr) - (\rho^2 + a^2) \right. \]
\[\left. - (3 + 3jkr - k^2 r^2) \right] \frac{e^{-jkr}}{r^5} \, dt, \]

where \( \rho, \phi, \) and \( z \) are the cylindrical coordinates of the observation point,

\[(95) \quad k = \omega \sqrt{\mu \varepsilon}, \]

\[(96) \quad r = \sqrt{\rho^2 + a^2 + (z - t)^2}, \]

\[(97) \quad r_1 = \sqrt{\rho^2 + a^2 + (z + s/2)^2}, \]

and

\[(98) \quad r_2 = \sqrt{\rho^2 + a^2 + (z - s/2)^2}. \]

These expressions for the field are accurate if \( \rho = 0 \) or if \( r_1^2 \) and \( r_2^2 \) are large in comparison with the quantity \( a\rho \). Equations (92) through (98) may be used to calculate the impedance matrix elements

\[(99) \quad Z_{ij} = \frac{E_{ij}^T}{I_j}, \]

where \( E_{ij}^T \) is the component of the field at the center of and tangential to wire segment \( i \) due to a unit current \( I_j = 1 \) on segment \( j \).
The data given in the body of this treatise and in Appendix D were computed by means of the main programs and subprograms listed in Table 4. Statement listings are given in Figs. 43 through 58. All programming was done in the Fortran IV language, Version 13, for the Ohio State University IBM 7094 IBSYS Operating System. A brief discussion of each program and subprogram is given here.

A. Main Programs

(1) MAIN1

The main program MAIN1 derives the backscattering cross section by application of the basis transformation technique. The program receives the transformed impedance matrix \([Z'_a]\) from the subroutine SUB2(b) and performs the multiplication \([Z'_a]^* [Z'_a]\) of Eq. (11). The Crout reduction procedure which follows is interrupted following statement 118 in order to specify the desired angles of incidence and the polarization state of the incident plane wave. The tangential components of the incident electric field are calculated and are multiplied by \([Z'_a]^*\). The Crout routine is then completed yielding a least square solution for the current on the scatterer in the form of coefficients of a series expansion in terms of the new basis functions. Finally,
<table>
<thead>
<tr>
<th>Program or Subprogram</th>
<th>Description</th>
<th>Figure</th>
</tr>
</thead>
<tbody>
<tr>
<td>MAIN1</td>
<td>Backscattering cross section by the basis transformation technique</td>
<td>43</td>
</tr>
<tr>
<td>MAIN2</td>
<td>Backscattering cross section by the piecewise uniform method</td>
<td>44</td>
</tr>
<tr>
<td>MAIN3</td>
<td>Determination of characteristic modes of wires</td>
<td>45</td>
</tr>
<tr>
<td>MAIN4</td>
<td>Backscattering and tumble average cross sections of wires using characteristic modes</td>
<td>46</td>
</tr>
<tr>
<td>SUB1</td>
<td>Wire geometry subroutines</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(a) circular and elliptical loops</td>
<td>47</td>
</tr>
<tr>
<td></td>
<td>(b) straight wires, circular arcs and helices</td>
<td>48</td>
</tr>
<tr>
<td>SUB2</td>
<td>Impedance matrix subroutines</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(a) piecewise uniform; circular loops, circular arcs, straight wires, and helices</td>
<td>49</td>
</tr>
<tr>
<td></td>
<td>(b) piecewise uniform with basis transformations; circular loops, circular arcs, straight wires, and helices</td>
<td>50</td>
</tr>
<tr>
<td></td>
<td>(c) piecewise uniform with basis transformation; elliptical loops</td>
<td>51</td>
</tr>
<tr>
<td>SUB3</td>
<td>Basis function subroutines</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(a) sine and cosine functions for closed loops</td>
<td>52</td>
</tr>
<tr>
<td></td>
<td>(b) sine and cosine functions for wires of open configuration</td>
<td>53</td>
</tr>
<tr>
<td></td>
<td>(c) Chebyshev polynomials of the second kind</td>
<td>54</td>
</tr>
<tr>
<td>SUB4</td>
<td>Subroutine for locating $\varepsilon_{\text{min}}(\alpha_m)$</td>
<td>55</td>
</tr>
<tr>
<td>SUB5</td>
<td>Subroutine for computing $[B(\alpha)]$</td>
<td>56</td>
</tr>
<tr>
<td>SUB6</td>
<td>Subroutine for computing eigenvalues and eigenvectors of $[B(\alpha)]$</td>
<td>57</td>
</tr>
<tr>
<td>SUB7</td>
<td>Subroutine for averaging $\sigma(\theta,\phi)$ over $\theta$ and $\phi$</td>
<td>58</td>
</tr>
</tbody>
</table>
the theta- and phi-polarized and total backscattering cross sections, EAT, EAP, and EA, respectively, are computed.

(2) MAIN2

The second main program, MAIN2, is similar to MAIN1 with the exception that a piecewise uniform impedance matrix is returned by SUB2(a), multiplication by the complex transpose of the impedance matrix is deleted, and the Crout reduction yields a piecewise uniform approximation to the induced current.

(3) MAIN3

The main program, MAIN3, for determining the characteristic modes of wires follows the steps outlined here. The number of separate cases is specified by the parameter NRUNS. The wire geometry for each case is calculated by the appropriate subroutine SUB1(a) or (b). Having established the wire geometry, the impedance matrix \([Z]\) of Eq. (22) is obtained by a call to SUB2(b) or (c). The main program next proceeds to determine the phase angles \(\alpha_m\) and the associated characteristic current distributions, finding first the even and then the odd modes. The \(P\), \(Q\), and \(R\) matrices of Eqs. (35)-(37) are derived from \([Z]\). Data cards then specify the number (NMODES) of modes to be found, the maximum number (NTRY) of "tries" to be made in seeking \(\alpha_m\), an estimate (ALP) of \(\alpha_m\) (ALP = \(\alpha_m - 180^\circ\)), the initial step size (ASTEP) to be used in searching for \(\alpha_m\) and a lower limit (ERROR) in the step size. The parameters ALP, ASTEP, and ERROR are in degrees. A new estimate for \(\alpha_m\) is computed by the subroutines SUB4 (ANULL),
Fig. 40.—Computer method for locating $\alpha_m$ corresponding to the relative minimum $\varepsilon_{\text{min}}(\alpha) = \varepsilon_{\text{min}}(\alpha_m)$.

SUB5 (EGNVLU), and SUB6 (MATEIG). To do this the minimum eigenvalue $\varepsilon_{\text{min}}(\alpha)$ is computed for each of three values of $\alpha$, i.e., for $A1=A2-\text{ASTEP}$, $A2$, and $A3=A2+\text{ASTEP}$. SUB4 calculates the angle $\alpha_{\text{min}}$ corresponding to the minimum of the quadratic curve defined by the three points (see Fig. 40). If no minimum is found for $A1 < \alpha < A3$, ...
i.e., if the first derivative is non-zero or if the second derivative is negative in this region, either \( A_1 \) or \( A_3 \), whichever corresponds to the smaller minimum eigenvalue \( \varepsilon_{\text{min}}(\alpha) \) (denoted by \( \text{EV}_1 \) and \( \text{EV}_3 \), respectively), becomes the new estimate of \( \alpha_m \) and is assigned to "A2". If a minimum in \( \varepsilon_{\text{min}}(\alpha) \) is found within the limits \( A_1 < \alpha < A_3 \), the step size \( \text{ASTEP} \) is reduced and the process is repeated (up to \( \text{NTRY} \) times or until \( \text{ASTEP} \leq \text{ERROR} \)), taking as the new "A2" the angle \( \alpha_{\text{min}} \). In this manner the relative minima \( \varepsilon_{\text{min}}(\alpha_m) \) are located, thereby specifying the phase angles \( \alpha_m \) and allowing the determination of the corresponding eigenvalues \( (I_m) \).

When either \( \text{NTRY} \) attempts have been made to locate \( \alpha_m \) or the specified error limit (\( \text{ERROR} \)) has been achieved, the eigenvector \( (I_m) \) associated with the last estimate of \( \alpha_m \) is obtained by a final call to \( \text{SUB5} \) and thence to \( \text{SUB6} \). The current distribution \( \text{CUR} (I_m) \), represented by the eigenvector coefficients \( \text{EF} ((I_m)) \), is then normalized to radiate unit power and the resultant normalized vector \( (I_m) \), along with the angle \( \alpha_m \), specify the \( m^{\text{th}} \) characteristic mode for the given obstacle.

An example set of input data is shown in Fig. 41 for a straight wire of length \( 1.0\lambda \) and radius \( 0.005\lambda \).

(4) \text{MAIN4}

The main program \text{MAIN4}, which calculates the backscattering cross sections of an object using its characteristic modes, assumes that a significant number of characteristic mode current distributions are known for the given obstacle; that each mode is specified by the
coefficients of a Fourier series expansion in terms of predetermined basis functions and has been normalized to radiate unit power; and that the characteristic value \( a_m = |\cos \alpha_m|e^{j\alpha_m} \) corresponding to each current distribution is known and is specified by the phase angle \( \alpha_m \), \( \pi/2 < \alpha_m < 3\pi/2 \).

\begin{verbatim}
$DATA
1
150
  1.0  1.0  0.0,
  0.005
  6  5
  1
  3.0
  0  1
  2
  10
   -65.0  2.0  0.01
  10
   89.0  0.2  0.01
  1
  10
   -30.0  5.0  0.01
\end{verbatim}

Fig. 41.--Example data for determining characteristic modes (two even and one odd) of a straight wire with length \( SL = 1.0\lambda \) and radius \( AL = 0.005\lambda \).

After establishing the wire geometry by a call to SUB1(a) or (b) and the appropriate basis functions by a call to SUB3(a), (b), or (c), the main program reads the parameters PH and TH, which determine the limits on the integrations over phi and theta, and IPH and ITH, which specify the number of points in phi and theta in the integrations as well as the angles of incidence for which
backscattering cross section data are calculated. For wires such as straight wires and circular loops, whose scattering is independent of phi and is symmetric in theta, PH=0° and TH=90° are sufficient. Ellipses require PH=90° and TH=90°, circular arcs, PH=180° and TH=90°, and helices PH=180° and TH=180° or PH=360° and TH=90°.

The matrix \([F]\), corresponding to the dyadic pattern function \(F\) of Eq. (20), is derived in the steps through statement 500. The backscattering cross sections for the various linear and circular polarizations are then computed. In calculating the average backscattering cross section, the expressions \(|F_{\theta\theta}|^2, |F_{\phi\phi}|^2, |F_{\theta\phi}|^2, \text{Re}(F_{\theta\theta}F_{\phi\phi}^C),\text{Im}(F_{\theta\theta}F_{\phi\phi}F_{\phi\phi}^C)\) are averaged separately. The tumble average cross sections are then formed by taking weighted combinations of these terms. The averaging over theta and phi is accomplished through a call to SUB7 (DBLINT).

An example set of input data is shown in Fig. 42 for a straight wire of length 1.0\(\lambda\) and radius 0.005\(\lambda\).

B. Subprograms

(1) SUB1

The geometries of the various wire configurations were computed by one of two subroutines (WGRID) given in Figs. 47 and 48. The wires were represented by \(N\) straight-wire segments. The subroutines derive the \(x\), \(y\), and \(z\) coordinates of the center of each wire segment along with orientation angles \(\beta\), the angle of rotation from the positive \(x\)-axis, and \(\alpha\), the angle out of the \(xy\) plane (\(\alpha\) is
Fig. 42.—Example data set for computing backscattering and tumble average cross section data for a straight wire with length $SL = 1.0\lambda$ and radius $AL = 0.005\lambda$ using two even and one odd characteristic modes.

Considered positive if the segment is tilted toward the negative $z$-axis.

The subroutine of Fig. 47 generates circular and elliptical loops according to the expression

$$ (100) \quad R(\phi) = \frac{A_1^2 \cdot A_2^2}{\sqrt{A_1^2 \sin^2 \phi + A_2^2 \cos^2 \phi}}. $$

In the subroutine, $\alpha = CAB$ and $\beta = SAB$.

Helices, straight wires, and circular arcs are generated by the subroutines of Fig. 48. The wire length is $SL$, the helix pitch (distance between turns) is $S$, and $R$ is the helix radius. The helix is symmetric in $\phi$ about the positive $x$-axis and symmetric in $z$ about $z = 0$. For a straight wire $SL = S$ and $R = 0$. For
circular arcs $S = 0$ and $R$ is the arc radius with $SL$ being the arc length.

(3) SUB2

Each of the impedance matrix subroutines (ZMAT), Figs. 49, 50, and 51, are based on the piecewise uniform method formulated and programmed by Richmond.[1-3]

The program given in Fig. 49 computes the elements $C(I,J)$ of an $N \times N$ piecewise uniform impedance matrix. Due to the symmetric geometry of the wires considered it was necessary to calculate only the first row of the matrix. The portion of the matrix on and above the main diagonal is completed by noting that $C_{i+1,j+1} = C_{ij}$, i.e., all elements on a given diagonal are equal. Finally, the elements below the main diagonal are set equal to their transpose since the matrix is symmetric.

The program shown in Fig. 50 is identical to that of Fig. 49 except that the impedance matrix is transformed from the piecewise uniform basis to the basis derived by a subroutine call to SUB3 (a) or (b). The piecewise uniform impedance matrix elements $C$ are computed one at a time while forming the new impedance matrix elements $CC(I,J)$. The resulting matrix is $N \times M$ in dimension $(M\leq N)$, where $N$ is the number of match points and $M$ is the number of modes retained in the new basis.

For the elliptical geometries the symmetries assumed in SUB2 (a) and (b) are not valid. The impedance matrix subroutine SUB2 (c) written for elliptical wires takes into account the more limited
symmetries of the ellipse. In all other respects SUB2 (c) is identical to SUB2 (b).

(3) SUB3

The programs given in Figs. 52, 53, and 54 generate basis functions and form the transformation matrix [T] of Eq. (3) when called by SUB2 (b) or (c). The first, Fig. 52, computes sine and cosine functions suitable as a basis for the Fourier expansion of the current induced on closed wire loops. The modes are defined by

\begin{align}
(101) \quad A_J(J,K) &= \cos \frac{2\pi (K-1)SLJ}{SL}, \quad K = 1,2,\ldots,MC \\
(102) \quad A_J(J,MC+K) &= \sin \frac{2\pi KSLJ}{SL}, \quad K = 1,2,\ldots,MS
\end{align}

where $A_J(J,K)$ is an element of the transformation matrix, $SL$ is the wire length in wavelengths, $MC$ is the number of cosine modes, $MS$ is the number of sine modes, and $SLJ$ is the location of the $J$\textsuperscript{th} match point along the center-line of the wire.

The subroutine shown in Fig. 53 derives a transformation matrix similar to that just described but for wires of open configuration. The matrix elements are, in this case,

\begin{align}
(103) \quad A_J(J,K) &= \cos \frac{\pi (K-1)SLJ}{SL}, \quad K = 1,2,\ldots,MC \\
(104) \quad A_J(J,MC+K) &= \sin \frac{\pi KSLJ}{SL}, \quad K = 1,2,\ldots,MS
\end{align}
The Chebyshev polynomial basis functions are computed by the subroutine given in Fig. 54. The elements of the transformation matrix are

\[
\begin{align*}
AJ(J,K) &= \frac{\sin((2K-1)\cos^{-1}(2SLJ/SL))}{\sin(\cos^{-1}(2SLJ/SL))}, \quad K=1,2,\ldots,MC \\
AJ(J,MC+K) &= \frac{\sin((2K \cos^{-1}(2SLJ/SL))}{\sin(\cos^{-1}(2SLJ/SL))}, \quad K=1,2,\ldots,MS
\end{align*}
\]

(4) SUB4

The subroutine SUB4 (ANULL) receives three values for \(\alpha\) (A1, A2, and A3) from MAIN3, obtains the corresponding minimum eigenvalues \(\epsilon_{\text{min}}(\alpha)\) (EV1, EV2, and EV3) from SUB5, then fits a quadratic equation to these three points (see Fig. 40) to estimate \(\alpha_m\), the phase angle associated with a relative minimum \(\epsilon_{\text{min}}(\alpha) = \epsilon_{\text{min}}(\alpha_m)\).

(5) SUB 5

Given an angle \(\alpha\) (ALP) and the matrices \([P]\), \([Q]\), and \([R]\), the subroutine SUB5 derives the matrix \([B(\alpha)]\) according to Eq. (38).

(6) SUB6

The subroutine SUB6 computes the eigenvalues and eigenvectors of \([B(\alpha)]\).

(7) SUB7

The subroutine SUB7 uses Simpson's rule integration in performing a double integration of the function \(S\) over theta and phi. The limits of integration are 0 to PH on phi and 0 to TH on theta using NP points in phi and NT points in theta. If PH = 0, a single integration on theta is performed.
Fig. 43.—Main program for the basis transformation method.
START CROUT REDUCTION
CROUT METHOD FOR COMPRESSION OF SYMMETRIC MATRIX.
DO 118 L=1,N
LLL=L-1
DO 118 I=L,M
IF(LLL.EQ.0) GO TO 1171
DO 117 K=1,LLL
R1=AR(L,K)
R2=AR(K,L)
AI1=AI(L,K)
AI2=AI(K,L)
AR(L,L)=AR(L,L)-R1*R2+AI1*AI2
1171 CONTINUE
1 M I = L - 1
IF(LMI) 105,118,105
105 AR(I,L)=AR(L,I)
AI(I,L)=-AI(L,I)
R2=AR(L,L)
AR(L,I)=AR(L,I)/R2
AI(L,I)=AI(L,I)/R2
118 CONTINUE
C COMPUTE INCIDENT E-FIELD AND MULTIPLY BY (CC)COMPLEX TRANSPOSE.
C TO OBTAIN EQUALS COLUMN.
READ (5,7) THETI,THTF,THETA,PHII,PHIF,DPHI,ETH,EPH
ITH=ETHETA-TETTHETA/ETHETA+1.1
IPH=PHIF-PHII/DPHI+1.1
THETA=THETI
DO 250 IT=1,N
TTA=TA*THETA
CTHT=COS(TTA)
STHT=SIN(TTA)
STCP=STHET*CPHI
CTCP=CTHET*CPHI
CTSP=CTHET*SPHI
DO 850 EI(I)=(ETH*Q(I)+EPH*P(I))*EXP(A(I))
DO 220 K=1,M
AR(K,MM)=0.
220 AI(K,MM)=0.
DO 1001 I=N1,N2
R2=REAL(EI(I))
AI2=-AIMAG(EI(I))
DO 1001 K=1,M
R1=REAL(CC(I,K))
AI1=AIMAG(CC(I,K))
AR(K,MM)=AR(K,MM)+R1*R2+AI1*AI2
1001 AI(K,MM)=AI(K,MM)+R1*AI2=R2*AI1
C RESUME CROUT REDUCTION
DO 120 L=1,N
LLL=L-1
Fig. 43.
IF(LLL.EQ.0) GO TO 1191
DO 119 K=1,LLL
R1=AR(L,K)
A11=AI(L,K)
R2=AR(K,MM)
A12=AI(K,MM)
AR(L,MM)=AR(L,MM)-R1*R2+A11*A12
119 A(L,MM)=AI(L,MM)-R1*A12-R2*A11
CONTINUE
R2=AR(L,L)
AR(L,MM)=AR(L,MM)/R2
DO 120 L=2,MM
I=MM-L
I1=I+1
120 A1(L,MM)=AI(L,MM)/R2
CONTINUE
DO 122 L=2,MM
I=MM-L
I1=I+1
END CROUT REDUCTION
SOLUTION IS IN MM COLUMN
CHECK ON FIELD MATCHING
WRITE (6,10)
DO 231 J=1,MM
CR=AR(J,MM)
CI=AI(J,MM)
231 A(J)=CMPLX(CR+CI)
PNM=(0.0,0.0)
PNM=(0.0,0.0)
DO 240 I=1,N
EP=(0.0,0.0)
DO 230 J=1,MM
ET=EI(I)+EP
ETM=CABS(ET)
WRITE (6,12) EI(I),EP,ET,ETM
EI(I)=(0.0,0.0)
DO 150 J=1,MM
150 EI(I)=EI(I)+AJ(I,J)*A(J)
240 PNM=PN+CONJG(EI(I))*ET
PNMAG=CABS(PNM)
WRITE (6,11) PNM,PNMAG
WRITE (6,3) (A(I),I=1,MM)
WRITE (6,4)
DO 160 I=1,N
CMAG=CABS(EI(I))
CR=REAL(EI(I))
CI=AIMAG(EI(I))
PH=TD*ATAN2(CI,CR)
160 WRITE (6,6) I,CR,CI,CMAG,PH
COMPUTE FAR FIELD AND ECHO AREA
ET=(0.0,0.0)
EP=(0.0,0.0)
DO 400 I=1,N
S=SI(I)*ZZ/2.
F=EI(I)*EXP(I)*CMPLX(0.0,-S)
ET=F*Q(I)+ET
400 EP=EP+F*P(I)
ETM=CABS(ET)
EPM=CABS(EP)

Fig. 43.
\begin{verbatim}
TB = ETM ** 2
TC = EPM ** 2
EAT = CNSNT * TB
EAP = CNSNT * TC
EA = EAT + EAP
WRITE (6*70) THEA, PHI, ETH, EPH
WRITE (6*2 ) EAT, EAP, EA
ETR = REAL (ET)
ETI = AIMAG (ET)
EPR = REAL (EP)
EPI = AIMAG (EP)
PHT = TD * ATAN2 (ETI, ETR)
PHP = TD * ATAN2 (EPI, EPR)
WRITE (6*5)
WRITE (6*1) ETR, ETI, ETM, PHT
WRITE (6*9)
WRITE (6*1) EPR, EPI, EPM, PHP
250 THEA = THEA + DTHETA
300 PHI = PHI + DPHI
1000 CONTINUE
STOP
END
\end{verbatim}

Fig. 43.
Fig. 44.—Main program for the piecewise uniform method.
EXPA(I)=CEXP(CMPLX(0.,TP*ARG))

P(I)=CPHI*SAB(I)-SPHI*CAB(I)

Q(I)=CT*P*SAB(I)+CTSP*SAB(I)+STHE*TALP(I)

C(I+NN)=-(ETH*Q(I)+EPH*P(I))*EXPA(I)

DO 120 L=1,N
LLL=L-1
IF(LLL.EQ.0) GO TO 1191
DO 119 K=1,LLL
119 C(L+NN)=C(L+NN)-C(K+NN)*C(K,NN)
1191 CONTINUE

DO 122 K=1,N
122 C(L+NN)=C(L+NN)/C(L,L)

C=CEXP(CMPLX(0.,TP*ARG))

THETA = THETA + DTHETA
PHI = PHI + DPHI
1000 CONTINUE
STOP
END

Fig. 44.
Fig. 45.—Main program for the determination of characteristic modes of wires.
MN=MC
MORDER=1
30 DO 50 I=1,MD
   IC=I+MN
DO 50 J=1,MD
   JC=J+MN
   PIJ=0.
   QIJ=0.
   RIJ=0.
DO 40 K=N1*N2
   RKI=REAL(CC(K+IC))
   RKJ=REAL(CC(K+JC))
   AIKI=AIMAG(CC(K+IC))
   AIKJ=AIMAG(CC(K+JC))
   PIJ=PIJ+AIKI*AIKJ
   QIJ=QIJ+RKI*AIKJ+RKJ*AIKI
40 RIJ=RIJ+RKI*RKJ
   P(I,J)=PIJ
   Q(I,J)=QIJ
   R(I,J)=RIJ
   P(J,I)=PIJ
   Q(J,I)=QIJ
50 R(J,I)=RIJ
WRITE (6,120)
DO 60 I=1,MD
60 WRITE (6,150) (P(I,J),J=1,MD)
WRITE (6,130)
DO 70 I=1,MD
70 WRITE (6,150) (Q(I,J),J=1,MD)
WRITE (6,140)
DO 80 I=1,MD
80 WRITE (6,150) (R(I,J),J=1,MD)
WRITE (6,9)
IF(NPUNH+EQ,0) GO TO 90
   PUNCH 160* ((P(I,J),J=1,MD),I=1,MD)
   PUNCH 160* ((Q(I,J),J=1,MD),I=1,MD)
   PUNCH 160* ((R(I,J),J=1,MD),I=1,MD)
90 READ (5,8) NMODES
DO 425 N=M1,NMODES
WRITE (6,15)
C FIND ALPHA CORRESPONDING TO MINIMUM EIGENVALUE
   AAA=0.
READ (5,8) NTRY
READ (5,7) ALP,ASTEP,ERROR
   A2=TA*ALP
   ERROR=TA*ERROR
   ASTEP=TA*ASTEP
   DO 100 NT=1,NTRY
92 A1=A2-ASTEP
   A3=A2+ASTEP
   IF(A1+1.5707963)*96,96,94
94 IF(A3-1.5707963)*98,96,96
96 ASTEP=ASTEP/2.*
   GO TO 92
98 CALL ANULL(MD,A1,A2,A3,ASTEP,ERROR)
   IF(AAA+EQ,1.) GO TO 200
   ALPHA=TA*ALP
   WRITE (6,2) ALPHA
   ASTEP=ASTEP/5.*
   IF(ASTEP*LT,ERROR) ASTEP=ERROR
   IF(ASTEP*EQ,ERROR) AAA=1.
100 CONTINUE

Fig. 45.
200  ALPHA=TD*A2
WRITE (6,10) AAA
TANALP=SIN(A2)/COS(A2)
WRITE (6,3) ALPHA*TANALP
YTIME=CLOCK(XTIME)
WRITE (6,7) YTIME
C COMPUTE EIGENVECTOR CORRESPONDING TO ALPHA
MM=0.
CALL EGNVLU(MD,A2,EV,EF,MM)
C COMPUTE CURRENT DISTRIBUTION AND E-FIELD PHASE
CMAK=0.
ERROR=0.
PT=(0.,0.)
EALP=CEXP(CMPLX(0.,A2))
DO 300 I=1+N
EI=(0.,0.)
CUR(I)=0.
DO 250 J=1,MD
JC=J+M N
EI=EI+C(EXP(CMPLX(0.,J))*EF(J))
CUR(I)=CUR(I)+MJ(JC)*EF(J)
PT=PT+EI*C(ALP(I))*SI(I)
IF(ABS(CUR(I))>CMAK) CMAX=ABS(CUR(I))
EI=EALP*EI
ER=-REAL(EI)
EAI=AIMAG(EI)
PH=ATAN2(EAI,ER)
300  TANPH(I)=SIN(PH)/COS(PH)
PTOTAL=REAL(PT)
PNORM=SQRT(PTOTAL)
DO 350 I=1,MD
350  EF(I)=EF(I)/PNORM
IF(MPUN*H*EQ.0) GO TO 375
PUNCH 14+ (EF(I),I=1,MD)
375 IF(MORDER) 390,380,390
380  WRITE (6,4) (EF(I),I=1,MD)
GO TO 395
390  WRITE (6,12) (EF(I),I=1,MD)
395  WRITE (6,5)
DO 400 I=1+N
CU=CUR(I)/PNORM
CUR(I)=CUR(I)/CMAK
ERROR=ERROR+ABS(TANPH(I)*CUR(I))
400  WRITE (6,6) I+CUR(I)*CU*TANPH(I)
FN=N
ERROR=ERROR/FN
WRITE (6,11) ERROR
YTIME=CLOCK(XTIME)
WRITE (6,7) YTIME
425 CONTINUE
IF(MS=EQ.0) GO TO 500
IF (MORDER) 28,28,500
500  CONTINUE
STOP
END

Fig. 45.
Fig. 46.—Main program for computing backscattering and tumble average cross sections of wires using characteristic modes.
DO 100 IP=1,IPH
  FTT(IT,IP)=(0.,0.,0.)
  FPP(IT,IP)=(0.,0.,0.)
100 FTP(IT,IP)=(0.,0.,0.)
  WRITE (6,1)
  READ (5,5) ME,EVEN,MO,ODD
  NMODES=ME+MO
  MCMS=MC
  FMODE=EVEN
  DO 500 NM=1,NMODES
    IF(NM.LE.ME) GO TO 110
    MCMS=MS
    FMODE=ODD
110 READ(5,14) (A(I),I=1,MCMS)
  READ (5,7) ALPHA
  WRITE (6,2) NM,FMODE,ALPHA,(A(I),I=1,MCMS)
  ARG=TA#ALPHA
  AMAG=ABS(COS(ARG))
  CEXALP=EXP(CMPLX(0.,ARG))
  ALPX=AMAG*CEXALP
  DO 160 IT=1,N
  CI=0.
  DO 150 J=1,MCMS
    CI=CI+A(J)#ALPX
 150 CI=CI+A(J)#ALPX
  J=J+1
  IF(NM.GT.ME) J=J+MC
160 C(NM+I)=CI
  PHI=0.
  DO 450 IP=1,IPH
  CPHI=COS(PHI)
  SPHI=SIN(PHI)
  THEETA=0.
  DO 400 IT=1,ITH
    ETTT=(0.,0.,0.)
    EPTT=(0.,0.,0.)
    CTHETA=COS(THEETA)
    STHETA=SIN(THEETA)
    STCP=STHETA*CPHI
    STSP=STHETA*SPHI
    CTCP=CTHETA*CPHI
    CTSP=CTHETA*SPHI
    DO 1001 I=1,N
      ARG=X(I)*STCP+Y(I)*STSP+Z(I)*CTHET
      EXPA=EXP(CMPLX(0.,TP*ARG))
      P=CPHI#SAB(I)-SPHI#CAB(I)
      Q=CTCP#AB(I)+CTSP#SAB(I)+STHETA#SALP(I)
      FF=C(NM,I)*EXP#SI(I)
      ETTT=ETTT+FF
1001 EPTT=EPTT+FF
  ET(IT,IP)=ETTT
  EP(IT,IP)=EPTT
  F(IT,IP)=CABS(ETTT)**2+CABS(EPTT)**2
400 THEETA=THEETA+DTH
450 PHI=PHI+DPM
  PTOTAL=ZZ*DBLINT(F,PHI,TH,IPH,ITH)/4.
  WRITE (6,12) PTOTAL
  DO 500 IT=1,ITH
  DO 500 IP=1,IPH
    FTT(IT,IP)=FTT(IT,IP)+ET(IT,IP)**2*ALPX
    FPP(IT,IP)=FPP(IT,IP)+EP(IT,IP)**2*ALPX
500 FTP(IT,IP)=FTP(IT,IP)+ET(IT,IP)*EP(IT,IP)*ALPX
  WRITE (6,3)
  DO 501 I=1,N
501 WRITE (6,4) I,C(NM+I),NM=1,NMODES)

Fig. 46.
Fig. 46.
**Fig. 47.**—Wire geometry subprogram for circular and elliptical loops.
Fig. 48.—Wire geometry subprogram for helical and straight wires and circular arcs.
Fig. 49.—Subprogram for the piecewise uniform impedance matrix of symmetric wire geometries.
123

DO 280 J=1,N
   IF (I-J) 103,280,103
   S=S(I)
   ST=S/2+
   SS=S*S
   CABJ=CAB(J)
   SABJ=SAB(J)
   XIJ=XI-X(J)
   YIJ=YI-Y(J)
   ZIJ=ZI-Z(J)
   ZP=XIJ*YIJ*SABJ-ZIJ*SALP(J)
   RS=XIJ*YIJ+YIJ*ZIJ+ZIJ*ZI
   RH2=RS-ZP*ZP
   RB=RH2+B2
   RK=FP&P*RB
   KK=B
   EZR=0.
   EZI=0.
   ERR=0.
   ERI=0.
   W=-1.
   T=ST
   SSS=2*SS
   IF (RS*GT*SSS) KK=1
   KKK=KK+1
   D(S)=1/(7*KK)
   FKK=KK
   DEL=S/FKK
   DO 50 K=1,KKK
      R2=RB+(ZP-T)*(ZP-T)
      RR=SQRT(R2)
      RK=TP*RR
      COSB=COS(RK)/RR/R2
      SINB=SIN(RK)/RR/R2
      RKCS=RK*COSB*SINB
      CRKS=COSB+RK*SINB
      TRBR=2-*J*RB/R2
      IF (D(K)-7.) 104,104,49
   104 ERR=ERR+w*CRY*CRK
      ERI=ERI-w*CRY*CRK
      W=1.
   49 EZR=ERR+D(K)*(TRBR*CRK+RB*K)*SINB
      EZI=EZI-D(K)*(RB*COSB+TRBR*CRK)
   50 T=T+DEL
      EZR=ERR*CCST*DEL
      EZI=EZI*CCST*DEL
      Q1=CABJ*CABJ+SABI*SABJ+SALP(I)*SALP(J)
      Q2=XIJ*YIJ+YIJ*ZIJ+ZIJ*ZI
      EIC=EZI-ZP*ERI
      ERC=ERR-ZP*ERR
      C(I,J)=Q1*CMPLX(ERR+ERI)+Q2*CMPLX(ERR,ERI)
280 CONTINUE
300 CONTINUE
   DO 400 I=2,N
      C(I,J)=C(I,J)
   DO 400 J=1,N
      J=J+1
   CC=C(I,J)
   C(I,J)=C
   400 C(J,J)=C
RETURN
END

Fig. 49.
Fig. 50.—Subprogram for the modal impedance matrix derived as a transformation of a piecewise uniform impedance matrix for symmetric wire geometries.
COMPUTE POINT-TO-POINT IMPEDANCE MATRIX COMPONENTS
AND CONVERT TO MODE-TO-POINT IMPEDANCE MATRIX

\[ S = S_I(1) \]

42

\[ \begin{align*}
S_T &= S / 2. \\
\text{RR} &= \sqrt{B_x^2 + S_T^2} \\
\text{SINB} &= S_T / \text{RR} \\
\text{COSB} &= A / \text{RR} \\
\text{TANB} &= (1 + \text{SINB} + \text{COSB})^2 \\
\text{RK} &= T_P * \text{RR} \\
\text{RK2} &= \text{RK}^2 \\
\text{ER} &= -P_I / 3 * S^2 * (2 - B_2^2 / 3 + B_4 / 60 + \text{RK2} * (B_2 - B_4) / 120) \\
\text{EI} &= \text{SINB}^2 / (R_2 + 5 + \text{RK2} * (1 - 7 / 60 * B_2 / 8 + \text{RK2} * R_2 / 240)) \\
2 - (1 - B_2 / 4 + 7 / 480 * B_4) & \times \text{ALOG (TANB)} \\
\text{CII} &= \text{ZZ} * \text{CMPLX (ER, EI)} \\
\end{align*} \]

45

\[ \begin{align*}
\text{XI} &= X(1) \\
\text{YI} &= Y(1) \\
\text{ZI} &= Z(1) \\
\text{CABI} &= \text{CAB}(1) \\
\text{SABI} &= \text{SAB}(1) \\
\text{DO 280} & \text{ JN=1} \text{+N} \text{J=N+1} \\
\text{IF (J=EQ+1)} & \text{GO TO 240} \\
\end{align*} \]

103

\[ \begin{align*}
S &= S_I(J) \\
\text{ST} &= S / 2. \\
\text{SS} &= S * S \\
\text{CABJ} &= \text{CAB}(J) \\
\text{SABJ} &= \text{SAB}(J) \\
\text{XIJ} &= X(J) \\
\text{YIJ} &= Y(J) \\
\text{ZIJ} &= Z(J) \\
\text{ZP} &= \text{XIJ} * \text{CABJ} + \text{YIJ} * \text{SABJ} - \text{ZIJ} * \text{SALP}(J) \\
\text{RS} &= \text{XIJ} * \text{XIJ} + \text{YIJ} * \text{YIJ} + \text{ZIJ} * \text{ZIJ} \\
\text{RH2} &= \text{RS} - \text{ZP} * \text{ZP} \\
\text{RB} &= \text{RH2} + 2 \text{B2} \\
\text{RBK} &= \text{FPP} * \text{RB} \\
\text{KK} &= 8 \\
\text{EZ} &= 0. \\
\text{EZI} &= 0. \\
\text{ERR} &= 0. \\
\text{ERI} &= 0. \\
\text{W} &= -1. \\
\text{T} &= \text{ST} \\
\text{SSS} &= 2 * S * S \\
\text{IF (RS GT SSS)} & \text{KK=4} \\
\text{KKK} &= \text{KK} + 1 \\
\text{D(5)} &= (7 * \text{KK}) / 4 \\
\text{FKK} &= \text{KK} \\
\text{DEL} &= S / \text{FKK} \\
\text{DO 50} & \text{ K=1} \text{+KK} \\
\text{R2} &= \text{RB} + (ZP - T) * (ZP - T) \\
\text{RR} &= \sqrt{\text{R2}} \\
\text{RK} &= \text{TP} * \text{RR} \\
\text{COSB} &= \text{COS} * \text{RK} / \text{RR} / \text{R2} \\
\text{SINB} &= \text{SIN} * \text{RK} / \text{RR} / \text{R2} \\
\text{RKCS} &= \text{RK} * \text{COSB} - \text{SINB} \\
\text{CRKS} &= \text{COSB} + \text{RK} * \text{SINB} \\
\text{TRBR} &= 2 * -3 * \text{RB} / \text{R2} \\
\text{IF (D(K)=7s)} & 104 \text{104} \text{+49} \\
\end{align*} \]
104 ERR=ERR+W*CRY*RKCS
ERI=ERI-W*CRY*CRKS
W=1,
49 EZR=EZR+D(K)*(TRBR*RKCS-RBK*SINB)
EZI=EZI-D(K)*(RBK*COSB+TRBR*CRKS)
50 T=T+DEL
ERI=ERI*CCST*DEL
EZI=EZI*CCST*DEL
Q1=CABI*CABJ+SABI*SALP(1)*SALP(J)
Q2=XIJ*CABI+YIJ*SABI-ZIJ*SALP(1)
EIC=EZI-ZP*ERI
ERC=ERZ-ZP*ERR
C =Q1*CMPLX(ERC+EIC)+Q2*CMPLX(ERR+ERI)
240 IF(J.EQ.1) C=C1/2.
DO 250 L=1*M
CC(1+L)=CC(1+L)+C*AJ(J+L)
250 CC(J+L)=CC(J+L)+C*AJ(1+L)
IF(J.EQ.N) GO TO 280
K=1
JP=J+1
DO 270 I=JP+N
K=K+1
DO 260 L=1+M
CC(K+L)=CC(K+L)+C*AJ(I+L)
CC(I+L)=CC(I+L)+C*AJ(K+L)
260 CONTINUE
270 CONTINUE
280 CONTINUE
RETURN
END

Fig. 50.
SUBROUTINE ZMAT(AL, MC, MS, N)
COMMON X(250), Y(250), Z(250), CAB(250), SAB(250), SIZE/S1(250)
COMMON /COEF/ CC(250), SALP(250)
COMPLEX C, CC, Cl
DIMENSION D(11)
7 FORMAT (7F10.5)
13 FORMAT (13H1WIRE RADIUS=F7.5, 14H (WAVELENGTHS))
TA=0.1745329
ZZ=376.72727
PI=3.1415926
P2=PI/2.*
TP=2.*PI
P5=PI*PI
CRY=ZZ/8.*PI
CCST=ZZ/180.*PI
READ(5,7) AL
WRITE(6,13) AL
B2=AL**2
BK=TP*AL
BK2=BK*BK
BK4=BK2*BK2
D(1)=7.
D(2)=32.
D(3)=12.
D(4)=32.
D(5)=14.
D(6)=32.
D(7)=12.
D(8)=32.
D(9)=7.
CALL CBASIS(MC, MS, N)
M=MC+MS
DO 12 J=1, M
DO 12 I=1, N
12 CC(I, J)=(0.0, 0.0)
DO 40 K=1, N
ALP=TA*CAB(K)
BET=TA*SAB(K)
CALP=COS(ALP)
SALP(K)=SIN(ALP)
CBET=COS(BET)
SBET=SIN(BET)
CAB(K)=CALP*CBET
40 SAB(K)=CALP*SBET

Fig. 51.—Subprogram for the modal impedance matrix derived as
a transformation of a piecewise uniform impedance
matrix for elliptical wire geometries.
N2 = N/2
N4 = N/4
DO 300 I=1+N4
S = S(I)
ST = S/2
RR = SQRT(ST*ST)
SINB = ST/RR
COSB = AL/RR
TANB = (1+SIGNB+COSB)/(1-SIGNB+COSB)
RK = TP*RR
RK2 = RK*KR
ER = PI/J+S*(2+6K2/3+8K4/60+RK2*(8K2-8))/120*
EI = SIGNB*(1/RK2+B+RK2*(1-7/60*8K2)/8+RK2*8K2/240)*/
2*(1+6K2/4+7/480+8K4)A.LOG(TANB)
C1 = ZZ*CMPLX(ER*EI)
CONTINUE
X I = X(I)
Y I = Y(I)
Z I = Z(I)
CABI = CAB(I)
SABI = SAB(I)
II = N2-1+1
I2 = N2+1
I3 = N-1+1
DO 280 J=1+N
IF(I=J) 47+46+47
46 C = C1
GO TO 52
47 S = S(I)
ST = S/2
SS = S*S
CABJ = CAB(J)
SABJ = SAB(J)
XIJ = X(I)
YIJ = Y(I)
ZIJ = Z(I)
ZP = XIJ*CABJ+YIJ*SABJ-ZIJ*SALP(J)
RS = XIJ*XIJ+YIJ*YIJ+ZIJ*ZIJ
RH2 = RS*ZP*ZP
RB = RH2+B2
RK = FP*PRB
KK = 8
ERZ = 0
EIZ = 0
ERR = 0
ERI = 0
W = 1
J = ST
SSS = 25*SS
IF(RS+GT*SSS) KK = 4
KKK = KK+1
D(S) = (7*KK)/4
FKK = KK
DEL = S/FKK
Fig. 51
DO 50 K=1,99
R2=R8+(ZP-T)*(ZP-T)
RR=SQR(R2)
RK=TP*RR
COSB=COS(RK)/RR/R2
SINB=SIN(RK)/RR/R2
RKCS=RK*COSB-SINB
CRKS=COSB+RK*SINB
TRBR=2*3*R8/R2
IF(D(K)>7)104,104,49
104 ERR=ERR+W*CRY*RKCS
ERI=ERI+W*CRY*CRKS
W=1.
49 EZR=EZR+D(K)*(TRBR*RKCS-RBK*SINB)
EZI=EZI-D(K)*(RBK*COSB+TRBR*CRKS)
50 T=T+DEL
EZP=EZP+CCST*DEL
EZI=EZI+CCST*DEL
Q1=CABI*CAJ+SABI*SABJ+SALP(I)*SALP(J)
Q2=XIJ*CABI+YIJ*SABI-ZIJ*SALP(I)
EIC=EZI-ZP*ERI
ERC=EZP-ZP*ERR
C=Q1*CMPLX(ERC,EIC)+Q2*CMPLX(ERR,ERI)
IF(J*GT*N2) GO TO 55
52 J1=N2-J+1
J2=N2+J
J3=N-J+1
GO TO 60
55 J1=N-J+N2+1
J2=J-N2
J3=N-J+1
60 DO 280 L=1*M
CC(I,L)=CC(I,L)+C*AJ(J*L)
CC(I1,L)=CC(I1,L)+C*AJ(J1*L)
CC(I2,L)=CC(I2,L)+C*AJ(J2*L)
280 CC(I3,L)=CC(I3,L)+C*AJ(J3*L)
300 CONTINUE
RETURN
END

Fig. 51.
$ IBFTC SUB3   DECK
SUBROUTINE CBASIS(MC, MS, N)
COMMON /CURR/AJ(250, 20)/SIZE/SI(250)
8 FORMAT (14I5)
9 FORMAT (15H1BASIS CURRENTS, //H12F10.6))
PI=3.1415926
SL=0.
DO 40 I=1,N
40 SL=SL+SI(I)
N21=N/2+1
READ (5*8) MC, MS
M=MC+MS
PS=2.*PI/SL
SLJ=.5*SI(N21)
DO 500 J=N21,N
JJ=N-J+1
IF(MC.EQ.0) GO TO 350
DO 300 K=1,M
FK=K
FKC=(FK-1.)*PS
AJ(J,K)=COS(FKC*SLJ)
300 AJ(JJ,K)=AJ(J,K)
IF(MS.EQ.0) GO TO 450
350 DO 400 K=1,MS
KK=K+MC
FK=K
FKS=FK*PS
AJ(J,KK)=SIN(FKS*SLJ)
400 AJ(JJ,KK)=-AJ(J,KK)
450 J1=J+1
500 SLJ=SLJ+.5*(SI(J)+SI(J1))
WRITE (6*9) ((AJ(J,K), J=1,N), K=1,M)
RETURN
END

Fig. 52.— Sine and cosine basis functions for wire loops.
$IBFTC SUB3 DECK
SUBROUTINE CBASIS(MC, MS, N)
COMMON /CURR/AJ(250, 20)/SIZE/SI(250)
8 FORMAT (14(15))
9 FORMAT (15H1BASIS CURRENTS, /)
PI=3.1415926
SL=0.
DO 40 I=1, N
40 SL=SL+SI(I)
N21=N/2+1
READ (5,8) MC, MS
M=MC+MS
PS=PI/SL
SLJ=5*SI(N21)
DO 500 J=N21, N
JJ=N-J+1
IF(MC.EQ.0) GO TO 350
DO 300 K=1, MC
FK=K
FKC=(FK-1.)*PS
AJ(J, K)=COS(FKC*SLJ)
300 AJ(JJ, K)=AJ(J, K)
IF(MS.EQ.0) GO TO 450
350 DO 400 K=1, MS
KK=K+MC
FK=K
FKS=FK*PS
AJ(J, KK)=SIN(FKS*SLJ)
400 AJ(JJ, KK)=AJ(J, KK)
450 J1=J+1
500 SLJ=SLJ+5*(SI(J)+SI(J1))
WRITE (6,9) ((AJ(J, K), J=1, N, K=1, M)
RETURN
END

Fig. 53.—Sine and cosine basis functions for wires of open configuration.
SUBROUTINE CBASIS(MC, MS, N)
COMMON /CURR/AJ, SIZE/SI
2 FORMAT (15H1BASIS CURRENTS //)
8 FORMAT (14I5)
9 FORMAT (1H *12FlO<6)
ARCCOS(X)=SQRT(1.-X)*(1.5707288-0.2121144*X+0.074261*X*X
2 -0.0187293*X*X*X)
SL=0.
DO 40 I = 1, N
40 SL=SL+SI(I)
N21=N/2+1
READ (5*8) MC, MS
M=MC+MS
SL2=SL/2.
SLJ=5*SI(N21)
DO 500 J=N21, N
JJ=N-J+1
EL=SLJ/SL2
ACOS=ARCCOS(EL)
IF(MC) 200 » 350 » 200
200 DO 300 K=1,MC
FK=2*(K-1)
AJ(J,K)=SIN((FK+1.)*ACOS)/SIN(ACOS)
300 AJ(JJ,K)=AJ(J,K)
IF(MS) 350 » 450 » 350
350 DO 400 K=1,MS
KK=K+MC
FK=2*K-1
AJ(J,KK)=SIN((FK+1.)*ACOS)/SIN(ACOS)
400 AJ(JJ,KK)=-AJ(J,KK)
450 J1=J+1
500 SLJ=SLJ+5*(SI(J)+SI(J1))
WRITE (6*2)
DO 600 K=1,M
600 WRITE (6*9) (AJ(J,K), J=1,N)
RETURN
END

Fig. 54.—Chebyshev polynomial basis functions.
SUBROUTINE ANULL(M, A1, A2, A3, ASTEP, ERROR)

      FORMAT (1H .3<FI5.6*E10.0>)
      COMMON /POR/P<30,20),O <20,20),R(2C,20)
      DIMENSION EF{20)
      DOUBLE PRECISION P,Q,R
      MM=1
      CALL EGNVLU(M*A1,EV1,EF,MM)
      CALL EGNVLU(M*A2,EV2,EF,MM)
      CALL EGNVLU(M*A3,EV3,EF,MM)
      WRITE (6*5) A1,EV1,A2,EV2,A3,EV3
      A12=A1###2
      A22=A2###2
      A32=A3###2
      D=A22*A1-A32*A2-A12*A3+A32*A1+A12*A2-A22#A1
      IF(D) 0,6,0
      6 S=0.
      GO TO 10
      R=A/D
      IF(S.LE.0.) GO TO 10
      B = A12*EV2-A22*EV1-A12*EV3+A32*EV1+A22*EV3-A32*EV2
      A0=-B/(2.*A)
      WRITE (6,5) A0
      IF(A0.LT.A1,OR.A0.GT.A3) GO TO 10
      A2=A0
      RETURN
      10 IF(EV3-EV1) 20,30,30
      20 A0=A3
      GO TO 40
      30 A0=A1
      40 ASTEP=5.0*ASTEP
      A2=A0
      RETURN
      END

Fig. 55.— Subroutine for locating $e_{\text{min}}(\alpha_m)$. 
SUBROUTINE EGVL (M, ALP, EV, EF, MM)
COMMON /PPQR/ P(20, 20), Q(20, 20), R(20, 20)
DIMENSION EF(20)
DOUBLE PRECISION P, Q, R, B, D, TALP, TALP2
COMMON /BDAK/ B(20, 20), D(20, 20)
TALP = SIN(ALP) / COS(ALP)
TALP2 = TALP**2
DO 20 J = 1, M
   DO 20 I = J, M
      B(I, J) = P(I, J) + TALP*Q(I, J) + TALP2*R(I, J)
   20 CONTINUE
CALL MATEIG(M, MM)
EV = B(1, 1)
DO 30 I = 1, M
   EF(I) = D(I, I)
30 CONTINUE
RETURN
END

Fig. 56.—Subroutine for computing \([B(a)]\).
Fig. 57.—Subroutine for computing the eigenvalues and eigenvectors of $[B(a)]$. 
X = 2 * A(L+M) * SINC
Y = A(L+1) * COSX + A(M+M) * SINC2 - X
X = A(L+1) * SINC2 + A(M+M) * COSX2 + X
A(L+M) = (A(L+L) - A(M+M)) * SINC + A(L+M) * (COSX2 - SINC2)
A(L+L) = Y
A(M+M) = X
DO 100 I = 1, N
A(L+I) = A(I+L)
A(M+I) = A(I+M)
IF(MV = 1) 95, 100, 95
95 X = R(I+L) * COSX - R(I+M) * SINC
R(I+M) = R(I+L) * SINC + R(I+M) * COSX
R(I+L) = X
100 CONTINUE
149 CONTINUE
150 IF (IND = 1) 160, 155, 160
155 IND = 0
GO TO 50
160 IF (THR = ANRMX) 165, 165, 45
165 DO 185 I = 1, N
DO 185 J = 1, N
IF (A(I+1) - A(J+J)) 170, 185, 185
170 X = A(I+1)
A(I+1) = A(J+J)
A(J+J) = X
DO 180 K = 1, N
X = R(K+I)
R(K+I) = R(K+J)
180 R(K+J) = X
185 CONTINUE
RETURN
END

Fig. 57.
Fig. 58.—Subroutine for averaging $\sigma(\theta, \phi)$ over $\theta$ and $\phi$. 
APPENDIX D
CHARACTERISTIC MODE DATA

The coefficients of the series approximations to the characteristic current distributions and the associated phase angles, \( \alpha_m \), are given in Tables 5 through 8 for the circular and elliptical loops, straight wires, and one-turn helices. The modes of all but the circular loops are ordered such that the magnitudes \(|\cos \alpha_m|\) form a non-increasing sequence of numbers with increasing integer \(m\). The modes of the circular loops are ordered on \(m\) in the conventional manner, beginning with \(m = 0\). In each case, the modes are normalized to radiate unit power. The tables are as follows:

Table 5—Circular Loops

The left-most column of Table 5 is the loop radius in wavelengths. Each row to the right of a given radius contains the coefficients of the \(m^{th}\) order cosine or sine function for the \(m^{th}\) even or odd characteristic current distribution, respectively, for \(m = 0,1,2,3\). (Recall that in the case of the circular loop the characteristic current distributions are simply cosine and sine functions.) The corresponding phase angles, \(\alpha_m\), are given in parenthesis following each coefficient.

As an example, the \(I_m^{(\text{even})}\) characteristic current, \(I_m\), of a loop of radius \(R = 0.35\lambda\) for, say, \(m = 2\), is
where $L$ is the loop diameter. The corresponding phase angle is $\alpha_2 = 161.59^\circ$.

Table 6—Elliptical Loops

The first column in Table 6 gives the axial ratios of the elliptical loops in descending order, starting with a ratio of unity (a circular loop). For a given axial ratio, the next column to the right specifies the mode number $m$ and the mode character, i.e., even or odd. The associated phase angles, $\alpha_m$, are found in the next column and to the right of that, the coefficients of the cosine (for even modes) or sine (for odd modes) series approximations to the characteristic currents. The basis functions are the cosine and sine functions $\cos \frac{2\pi n \xi}{L}$, $n=0,1,2,\ldots$, and $\sin \frac{2\pi n \xi}{L}$, $n=1,2,\ldots$, where $L$ is the ellipse circumference ($L=1.1\lambda$) and $\xi$ is measured from $\xi = 0$ on the positive $x$ axis. (See the ellipse geometry in Fig. 19, Section D of Chapter III.)

For example, the characteristic current for the dominant mode ($m = 1$) of an ellipse with axial ratio 0.890 is approximated by the series

\begin{equation}
I_1(\xi) \approx (7.7450 \times 10^{-2}) \sin \frac{2\pi \xi}{L} - (7.3738 \times 10^{-11}) \sin \frac{4\pi \xi}{L} \\
- (6.4297 \times 10^{-5}) \sin \frac{6\pi \xi}{L} + (8.8768 \times 10^{-9}) \sin \frac{8\pi \xi}{L} \\
+ (6.0340 \times 10^{-6}) \sin \frac{10\pi \xi}{L}.
\end{equation}
Table 7—Straight Wires

These data are similar to the data given in Table 6, except that the basis functions are the even and odd Chebyshev polynomials of the second kind, and the first column, as noted, gives the wire length. The polynomials are denoted by $U_n$, even modes having even subscripts and odd modes having odd subscripts. For example, the characteristic current $I_m$ of a straight wire of length $L = 1.5\lambda$, with $m = 2$ (an odd mode, as noted in the second column), is approximated by

\begin{equation}
I_m = 3.7329 \times 10^{-2} U_1 - 2.6274 \times 10^{-2} U_3 \\
+ 6.5850 \times 10^{-3} U_5 - 8.5873 \times 10^{-4} U_7, + 3.6305 \times 10^{-5} U_9.
\end{equation}

Table 8—One-Turn Helix

These data are similar to the data given in Table 7.
### TABLE 5

**CHARACTERISTIC MODES OF CIRCULAR LOOPS WITH A LOOP RADIUS-TO-WIRE RADIUS RATIO R/a = 100.**

<table>
<thead>
<tr>
<th>RADIUS</th>
<th>( I )</th>
<th>( \cos \frac{2\pi L}{L} )</th>
<th>( \sin \frac{2\pi L}{L} )</th>
<th>( \cos \frac{4\pi L}{L} )</th>
<th>( \sin \frac{4\pi L}{L} )</th>
<th>( \cos \frac{6\pi L}{L} )</th>
<th>( \sin \frac{6\pi L}{L} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.100</td>
<td>0.187730 (51.37)</td>
<td>0.122641 (265.92)</td>
<td>0.321039 (269.84)</td>
<td>0.268318 (269.75)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.125</td>
<td>0.122874 (52.51)</td>
<td>0.102554 (259.70)</td>
<td>0.230386 (269.60)</td>
<td>0.210891 (269.52)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.150</td>
<td>0.087114 (54.08)</td>
<td>0.090180 (240.74)</td>
<td>0.200619 (269.41)</td>
<td>0.170737 (269.13)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.162</td>
<td>0.075924 (54.99)</td>
<td>0.085800 (210.00)</td>
<td>0.158000 (268.80)</td>
<td>0.141005 (268.24)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.175</td>
<td>0.066591 (55.98)</td>
<td>0.082291 (170.81)</td>
<td>0.140050 (268.00)</td>
<td>0.130500 (267.80)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.187</td>
<td>0.059085 (57.05)</td>
<td>0.075485 (140.47)</td>
<td>0.121000 (266.80)</td>
<td>0.113541 (266.22)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.200</td>
<td>0.052965 (58.21)</td>
<td>0.072250 (126.05)</td>
<td>0.106000 (264.50)</td>
<td>0.106000 (264.50)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.210</td>
<td>0.048800 (59.20)</td>
<td>0.075800 (119.70)</td>
<td>0.096000 (262.80)</td>
<td>0.106000 (264.50)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.220</td>
<td>0.045500 (60.00)</td>
<td>0.074700 (115.50)</td>
<td>0.089000 (261.60)</td>
<td>0.106000 (264.50)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.230</td>
<td>0.042266 (61.20)</td>
<td>0.073609 (112.30)</td>
<td>0.082000 (259.60)</td>
<td>0.106000 (264.50)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.240</td>
<td>0.039700 (62.10)</td>
<td>0.072800 (110.00)</td>
<td>0.076000 (257.20)</td>
<td>0.106000 (264.50)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.250</td>
<td>0.037191 (63.32)</td>
<td>0.072083 (108.12)</td>
<td>0.071000 (254.80)</td>
<td>0.106000 (264.50)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.275</td>
<td>0.032471 (66.07)</td>
<td>0.070617 (104.84)</td>
<td>0.063000 (250.40)</td>
<td>0.106000 (264.50)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.300</td>
<td>0.028995 (68.84)</td>
<td>0.069040 (102.78)</td>
<td>0.056000 (245.16)</td>
<td>0.106000 (264.50)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.325</td>
<td>0.026415 (71.56)</td>
<td>0.066745 (101.56)</td>
<td>0.050000 (240.00)</td>
<td>0.106000 (264.50)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.340</td>
<td>0.025200 (73.00)</td>
<td>0.064800 (101.50)</td>
<td>0.045000 (234.96)</td>
<td>0.106000 (264.50)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.350</td>
<td>0.024505 (74.11)</td>
<td>0.063361 (101.06)</td>
<td>0.040000 (229.92)</td>
<td>0.106000 (264.50)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.360</td>
<td>0.024000 (75.33)</td>
<td>0.061608 (101.50)</td>
<td>0.036000 (224.98)</td>
<td>0.106000 (264.50)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.375</td>
<td>0.023115 (76.40)</td>
<td>0.059929 (101.18)</td>
<td>0.032000 (219.94)</td>
<td>0.106000 (264.50)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.400</td>
<td>0.022142 (78.12)</td>
<td>0.058369 (101.93)</td>
<td>0.028000 (214.90)</td>
<td>0.106000 (264.50)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
TABLE 6
CHARACTERISTIC MODES OF ELLIPTICAL LOOPS WITH 1.1λ CIRCUMFERENCE AND WIRE RADIUS a = 0.00175λ AS A FUNCTION OF AXIAL RATIO

<table>
<thead>
<tr>
<th>AXIAL RATIO</th>
<th>MODE</th>
<th>ALPHA</th>
<th>(\frac{1}{\sin^{2\pi L}})</th>
<th>(\frac{\cos^{2\pi L}}{L})</th>
<th>(\frac{\cos^{4\pi L}}{L})</th>
<th>(\frac{\cos^{6\pi L}}{L})</th>
<th>(\frac{\cos^{8\pi L}}{L})</th>
<th>(\frac{\cos^{10\pi L}}{L})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.000</td>
<td>1 (EVEN)</td>
<td>170.81</td>
<td>-0.</td>
<td>8.229E-02</td>
<td>-0.</td>
<td>-0.</td>
<td>-0.</td>
<td>-0.</td>
</tr>
<tr>
<td></td>
<td>2 (ODD)</td>
<td>170.81</td>
<td>-0.</td>
<td>8.229E-02</td>
<td>-0.</td>
<td>-0.</td>
<td>-0.</td>
<td>-0.</td>
</tr>
<tr>
<td></td>
<td>3 (EVEN)</td>
<td>95.98</td>
<td>6.6591E-02</td>
<td>-0.</td>
<td>-0.</td>
<td>-0.</td>
<td>-0.</td>
<td>-0.</td>
</tr>
<tr>
<td></td>
<td>4 (EVEN)</td>
<td>269.12</td>
<td>-0.</td>
<td>1.7600E-01</td>
<td>-0.</td>
<td>-0.</td>
<td>-0.</td>
<td>-0.</td>
</tr>
<tr>
<td></td>
<td>5 (ODD)</td>
<td>269.12</td>
<td>-0.</td>
<td>1.7600E-01</td>
<td>-0.</td>
<td>-0.</td>
<td>-0.</td>
<td>-0.</td>
</tr>
<tr>
<td>0.943</td>
<td>1 (ODD)</td>
<td>174.01</td>
<td>7.980E-02</td>
<td>-7.373E-11</td>
<td>-6.429E-07</td>
<td>8.876E-09</td>
<td>6.034E-06</td>
<td>-6.442E-06</td>
</tr>
<tr>
<td></td>
<td>3 (EVEN)</td>
<td>95.98</td>
<td>6.6690E-02</td>
<td>0.</td>
<td>-5.638E-04</td>
<td>0.</td>
<td>-1.933E-05</td>
<td>0.</td>
</tr>
<tr>
<td></td>
<td>5 (EVEN)</td>
<td>269.14</td>
<td>-1.717E-03</td>
<td>-2.982E-08</td>
<td>1.763E-01</td>
<td>-1.840E-08</td>
<td>9.478E-04</td>
<td>1.494E-08</td>
</tr>
<tr>
<td>0.890</td>
<td>1 (ODD)</td>
<td>177.42</td>
<td>7.745E-02</td>
<td>-7.373E-11</td>
<td>-6.429E-07</td>
<td>8.876E-09</td>
<td>6.034E-06</td>
<td>-6.442E-06</td>
</tr>
<tr>
<td></td>
<td>2 (EVEN)</td>
<td>163.09</td>
<td>3.34E-12</td>
<td>8.802E-02</td>
<td>-4.160E-11</td>
<td>-1.962E-04</td>
<td>5.195E-09</td>
<td>-6.442E-06</td>
</tr>
<tr>
<td></td>
<td>3 (EVEN)</td>
<td>95.93</td>
<td>6.703E-02</td>
<td>0.</td>
<td>-1.094E-03</td>
<td>0.</td>
<td>-4.077E-05</td>
<td>0.</td>
</tr>
<tr>
<td></td>
<td>4 (ODD)</td>
<td>269.12</td>
<td>7.3742E-08</td>
<td>1.754E-01</td>
<td>2.934E-08</td>
<td>1.973E-03</td>
<td>3.792E-08</td>
<td>1.958E-03</td>
</tr>
<tr>
<td></td>
<td>5 (EVEN)</td>
<td>269.14</td>
<td>-3.535E-03</td>
<td>1.172E-08</td>
<td>1.770E-01</td>
<td>-5.645E-08</td>
<td>1.958E-03</td>
<td>1.646E-08</td>
</tr>
<tr>
<td>0.838</td>
<td>1 (ODD)</td>
<td>179.93</td>
<td>7.5264E-02</td>
<td>-1.201E-10</td>
<td>-6.391E-05</td>
<td>9.529E-09</td>
<td>1.437E-05</td>
<td>-2.547E-06</td>
</tr>
<tr>
<td></td>
<td>3 (EVEN)</td>
<td>95.98</td>
<td>6.747E-02</td>
<td>0.</td>
<td>-2.314E-16</td>
<td>0.</td>
<td>-2.132E-04</td>
<td>0.</td>
</tr>
<tr>
<td></td>
<td>4 (ODD)</td>
<td>269.10</td>
<td>8.44E-08</td>
<td>1.745E-01</td>
<td>2.916E-08</td>
<td>2.928E-03</td>
<td>4.348E-08</td>
<td>1.621E-15</td>
</tr>
<tr>
<td></td>
<td>5 (EVEN)</td>
<td>269.15</td>
<td>-5.352E-03</td>
<td>1.267E-08</td>
<td>1.781E-01</td>
<td>-6.7459E-08</td>
<td>2.951E-08</td>
<td>1.969E-08</td>
</tr>
<tr>
<td>0.788</td>
<td>1 (ODD)</td>
<td>182.35</td>
<td>7.3192E-02</td>
<td>-2.313E-10</td>
<td>-4.262E-05</td>
<td>1.364E-08</td>
<td>2.735E-05</td>
<td>1.587E-06</td>
</tr>
<tr>
<td></td>
<td>2 (EVEN)</td>
<td>152.38</td>
<td>-3.742E-09</td>
<td>9.491E-02</td>
<td>-2.565E-08</td>
<td>-5.020E-04</td>
<td>7.075E-09</td>
<td>1.587E-06</td>
</tr>
<tr>
<td></td>
<td>3 (EVEN)</td>
<td>95.98</td>
<td>6.8199E-02</td>
<td>0.</td>
<td>-2.314E-16</td>
<td>0.</td>
<td>-1.283E-04</td>
<td>0.</td>
</tr>
<tr>
<td></td>
<td>4 (ODD)</td>
<td>269.10</td>
<td>9.301E-08</td>
<td>1.733E-01</td>
<td>-2.445E-08</td>
<td>3.875E-03</td>
<td>5.737E-08</td>
<td>5.737E-08</td>
</tr>
<tr>
<td></td>
<td>5 (EVEN)</td>
<td>269.17</td>
<td>-7.2752E-03</td>
<td>-9.584E-09</td>
<td>1.798E-01</td>
<td>-8.758E-06</td>
<td>3.9711E-03</td>
<td>3.511E-08</td>
</tr>
<tr>
<td>0.758</td>
<td>1 (ODD)</td>
<td>183.89</td>
<td>7.1993E-02</td>
<td>-2.598E-10</td>
<td>-1.850E-05</td>
<td>1.290E-08</td>
<td>3.793E-05</td>
<td>3.826E-06</td>
</tr>
<tr>
<td></td>
<td>2 (EVEN)</td>
<td>149.00</td>
<td>-3.336E-11</td>
<td>9.738E-02</td>
<td>-2.132E-10</td>
<td>-6.313E-04</td>
<td>1.101E-08</td>
<td>3.826E-06</td>
</tr>
<tr>
<td></td>
<td>3 (EVEN)</td>
<td>95.69</td>
<td>6.8799E-02</td>
<td>-7.4687E-13</td>
<td>-2.504E-03</td>
<td>1.767E-11</td>
<td>-1.6959E-04</td>
<td>-4.639E-10</td>
</tr>
<tr>
<td></td>
<td>4 (ODD)</td>
<td>269.09</td>
<td>1.085E-07</td>
<td>1.725E-01</td>
<td>-3.6827E-08</td>
<td>4.4439E-03</td>
<td>5.754E-08</td>
<td>3.607E-03</td>
</tr>
<tr>
<td></td>
<td>5 (EVEN)</td>
<td>269.18</td>
<td>-8.538E-03</td>
<td>1.164E-07</td>
<td>1.812E-01</td>
<td>-8.5333E-08</td>
<td>4.6074E-03</td>
<td>3.990E-08</td>
</tr>
<tr>
<td>0.739</td>
<td>0.720</td>
<td>0.693</td>
<td>0.646</td>
<td>0.602</td>
<td>0.515</td>
<td>0.306</td>
<td></td>
<td></td>
</tr>
<tr>
<td>--------</td>
<td>--------</td>
<td>--------</td>
<td>--------</td>
<td>--------</td>
<td>--------</td>
<td>--------</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 (ODD)</td>
<td>184.18</td>
<td>185.30</td>
<td>186.03</td>
<td>187.49</td>
<td>188.71</td>
<td>190.43</td>
<td>186.16</td>
<td></td>
</tr>
<tr>
<td>2 (EVEN)</td>
<td>145.85</td>
<td>163.89</td>
<td>139.81</td>
<td>133.32</td>
<td>126.07</td>
<td>115.50</td>
<td>97.17</td>
<td></td>
</tr>
<tr>
<td>3 (EVEN)</td>
<td>95.68</td>
<td>6.9373E-11</td>
<td>7.0373E-12</td>
<td>7.0373E-12</td>
<td>5.8937E-12</td>
<td>6.3073E-12</td>
<td>5.6925E-02</td>
<td></td>
</tr>
<tr>
<td>4 (ODD)</td>
<td>269.08</td>
<td>2.8534E-07</td>
<td>1.7160E-01</td>
<td>1.0078E-01</td>
<td>1.2083E-01</td>
<td>7.1534E-02</td>
<td>5.8925E-02</td>
<td></td>
</tr>
<tr>
<td>5 (EVEN)</td>
<td>269.20</td>
<td>9.2434E-03</td>
<td>1.8194E-01</td>
<td>9.3845E-01</td>
<td>1.5293E-01</td>
<td>1.5293E-01</td>
<td>5.3150E-02</td>
<td></td>
</tr>
</tbody>
</table>

**TABLE 6--Continued**
### TABLE 7
CHARACTERISTIC MODES OF STRAIGHT WIRES WITH A LENGTH-TO-DIAMETER RATIO \( L/2a = 100 \)

| LENGTH | MODE | ALPHA | \( u_0 \) \( u_1 \) | \( u_2 \) \( u_3 \) | \( u_4 \) \( u_5 \) | \( u_6 \) \( u_7 \) | \( u_8 \) \( u_9 \) | \( u_{10} \) \( u_{11} \) |
|--------|------|-------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 0.300  | 1 (EVEN) | 265.26 | 1.3436E-01 | -3.7299E-02 | -3.1200E-04 | -2.6043E-04 | -7.9192E-05 | -6.7434E-05 |
| 0.400  | 1 (EVEN) | 248.07 | 1.0244E-01 | -2.3852E-02 | 2.8354E-04 | -1.1801E-04 | -5.5523E-05 | -4.7831E-05 |
| 0.450  | 1 (EVEN) | 211.29 | 9.1924E-02 | -2.8876E-02 | 5.3394E-04 | -1.6606E-04 | -4.7326E-05 | -4.1133E-05 |
| 0.475  | 1 (EVEN) | 179.10 | 8.7526E-02 | -2.7779E-02 | 6.5187E-04 | -1.5756E-04 | -4.3803E-05 | -3.8285E-05 |
| 0.500  | 1 (EVEN) | 153.43 | 8.3584E-02 | -2.6810E-02 | 7.6614E-04 | -1.5049E-04 | -4.0585E-05 | -3.5605E-05 |
| 0.550  | 1 (EVEN) | 130.34 | 7.6819E-02 | -2.5897E-02 | 9.6678E-04 | -1.4023E-04 | -3.4914E-05 | -3.1139E-05 |
| 0.600  | 1 (EVEN) | 122.06 | 7.1229E-02 | -2.4996E-02 | 1.1976E-03 | -1.3457E-04 | -3.0022E-05 | -2.7314E-05 |
| 0.750  | 1 (EVEN) | 115.14 | 5.9114E-02 | -2.1348E-02 | 1.8003E-03 | -1.4141E-04 | -1.8312E-05 | -1.8590E-05 |
| 0.850  | 1 (EVEN) | 113.77 | 5.3507E-02 | -2.0339E-02 | 2.2567E-03 | -1.6663E-04 | -1.0170E-05 | -9.8731E-06 |
| 0.900  | 1 (EVEN) | 227.26 | 7.3809E-02 | -2.0273E-02 | 3.5423E-03 | -3.2825E-04 | -7.7505E-05 | -8.5274E-06 |
| 0.950  | 1 (EVEN) | 186.52 | 6.8057E-02 | -3.6087E-02 | 3.7782E-03 | -3.3580E-04 | -7.2271E-05 | -7.4722E-06 |
|       | 2 (EVEN) | 113.72 | 4.9133E-02 | -1.9641E-02 | 2.6522E-03 | -2.0640E-04 | -4.3892E-05 | -4.1622E-06 |
|       | 3 (EVEN) | 268.94 | -2.3364E-03 | 1.5682E-01 | -4.9842E-02 | -8.8035E-03 | -8.1101E-04 | -2.1268E-04 |
TABLE 7--Continued

<table>
<thead>
<tr>
<th>1.000</th>
<th>1 (ODD)</th>
<th>147.70</th>
<th>6.3192E-02</th>
<th>-3.4586E-02</th>
<th>4.0850E-03</th>
<th>-3.6570E-04</th>
<th>-7.6321E-05</th>
</tr>
</thead>
<tbody>
<tr>
<td>(EVEN)</td>
<td>246.29</td>
<td>4.7250E-02</td>
<td>-1.9534E-02</td>
<td>2.9721E-03</td>
<td>-2.4204E-04</td>
<td>-7.0623E-07</td>
<td>-8.8829E-06</td>
</tr>
<tr>
<td>1.050</td>
<td>1 (ODD)</td>
<td>132.63</td>
<td>5.8945E-02</td>
<td>-3.3060E-02</td>
<td>4.3218E-03</td>
<td>-3.8861E-04</td>
<td>-6.3385E-05</td>
</tr>
<tr>
<td>(EVEN)</td>
<td>113.87</td>
<td>4.5559E-02</td>
<td>-1.9332E-02</td>
<td>3.1403E-03</td>
<td>-2.7361E-04</td>
<td>3.3172E-06</td>
<td>-7.7714E-06</td>
</tr>
<tr>
<td>(EVEN)</td>
<td>267.68</td>
<td>3.2406E-03</td>
<td>9.5215E-02</td>
<td>-6.5156E-02</td>
<td>7.4656E-03</td>
<td>-7.9924E-04</td>
<td>-1.7846E-04</td>
</tr>
<tr>
<td>1.150</td>
<td>1 (ODD)</td>
<td>121.62</td>
<td>5.2025E-02</td>
<td>-3.0660E-02</td>
<td>4.7952E-03</td>
<td>-4.5169E-04</td>
<td>-4.0039E-05</td>
</tr>
<tr>
<td>(EVEN)</td>
<td>114.27</td>
<td>4.5559E-02</td>
<td>-1.9045E-02</td>
<td>3.5806E-03</td>
<td>-3.5038E-04</td>
<td>1.2494E-05</td>
<td>-5.8790E-06</td>
</tr>
<tr>
<td>(EVEN)</td>
<td>264.93</td>
<td>3.2406E-03</td>
<td>9.5215E-02</td>
<td>-6.5156E-02</td>
<td>7.4656E-03</td>
<td>-7.9924E-04</td>
<td>-1.7846E-04</td>
</tr>
<tr>
<td>1.250</td>
<td>1 (ODD)</td>
<td>117.98</td>
<td>4.6650E-02</td>
<td>-2.8901E-02</td>
<td>5.2757E-03</td>
<td>-5.3708E-04</td>
<td>-7.8322E-06</td>
</tr>
<tr>
<td>(EVEN)</td>
<td>114.74</td>
<td>4.0081E-02</td>
<td>-1.8999E-02</td>
<td>4.0554E-03</td>
<td>-4.4964E-04</td>
<td>2.4243E-05</td>
<td>-4.5705E-06</td>
</tr>
<tr>
<td>(EVEN)</td>
<td>258.39</td>
<td>6.5933E-03</td>
<td>7.0162E-02</td>
<td>-5.2963E-02</td>
<td>7.8790E-03</td>
<td>-8.3851E-04</td>
<td>-9.7918E-05</td>
</tr>
<tr>
<td>1.300</td>
<td>1 (ODD)</td>
<td>117.12</td>
<td>4.4392E-02</td>
<td>-2.8201E-02</td>
<td>5.5215E-03</td>
<td>-5.8832E-04</td>
<td>-8.0966E-06</td>
</tr>
<tr>
<td>(EVEN)</td>
<td>114.97</td>
<td>3.8944E-02</td>
<td>-1.8899E-02</td>
<td>4.3200E-03</td>
<td>-5.1106E-04</td>
<td>3.1680E-05</td>
<td>-4.0966E-06</td>
</tr>
<tr>
<td>(EVEN)</td>
<td>251.50</td>
<td>9.5461E-03</td>
<td>7.0162E-02</td>
<td>-5.2963E-02</td>
<td>7.8790E-03</td>
<td>-8.3851E-04</td>
<td>-9.7918E-05</td>
</tr>
<tr>
<td>1.400</td>
<td>1 (EVEN)</td>
<td>212.62</td>
<td>1.0884E-02</td>
<td>5.8670E-02</td>
<td>-4.4209E-02</td>
<td>8.2039E-03</td>
<td>-9.2380E-04</td>
</tr>
<tr>
<td>(EVEN)</td>
<td>115.96</td>
<td>3.3641E-02</td>
<td>-1.9026E-02</td>
<td>4.9600E-03</td>
<td>-6.7238E-04</td>
<td>5.2085E-05</td>
<td>-3.8266E-06</td>
</tr>
<tr>
<td>(ODD)</td>
<td>268.73</td>
<td>8.9135E-03</td>
<td>1.3637E-01</td>
<td>-1.0086E-01</td>
<td>1.5208E-02</td>
<td>-2.0749E-03</td>
<td></td>
</tr>
<tr>
<td>(ODD)</td>
<td>115.09</td>
<td>3.8557E-02</td>
<td>-2.6176E-02</td>
<td>6.3176E-03</td>
<td>-7.6276E-04</td>
<td>2.3814E-05</td>
<td>-3.8899E-06</td>
</tr>
<tr>
<td>(EVEN)</td>
<td>115.69</td>
<td>3.3641E-02</td>
<td>-1.9161E-02</td>
<td>5.3320E-03</td>
<td>-7.7632E-04</td>
<td>1.4597E-02</td>
<td>-1.9028E-03</td>
</tr>
<tr>
<td>(ODD)</td>
<td>268.23</td>
<td>1.0799E-02</td>
<td>1.1995E-01</td>
<td>-9.0287E-02</td>
<td>1.4856E-02</td>
<td>-2.0264E-03</td>
<td></td>
</tr>
<tr>
<td>1.500</td>
<td>1 (EVEN)</td>
<td>149.95</td>
<td>1.2002E-02</td>
<td>4.9766E-02</td>
<td>-4.0024E-02</td>
<td>8.5188E-03</td>
<td>-1.0315E-03</td>
</tr>
<tr>
<td>(ODD)</td>
<td>116.05</td>
<td>3.7382E-02</td>
<td>-2.6276E-02</td>
<td>6.5850E-03</td>
<td>-8.5873E-04</td>
<td>3.6305E-05</td>
<td>-4.2478E-06</td>
</tr>
<tr>
<td>(EVEN)</td>
<td>115.93</td>
<td>3.4911E-02</td>
<td>-1.9355E-02</td>
<td>5.7556E-03</td>
<td>-8.9596E-04</td>
<td>6.5899E-05</td>
<td>-3.8899E-06</td>
</tr>
<tr>
<td>(ODD)</td>
<td>267.50</td>
<td>1.2274E-02</td>
<td>1.0604E-01</td>
<td>-8.3065E-02</td>
<td>1.4597E-02</td>
<td>-1.9028E-03</td>
<td></td>
</tr>
<tr>
<td>1.600</td>
<td>1 (EVEN)</td>
<td>129.11</td>
<td>1.3245E-02</td>
<td>4.2528E-02</td>
<td>-3.6699E-02</td>
<td>8.8495E-03</td>
<td>-1.1646E-03</td>
</tr>
<tr>
<td>(ODD)</td>
<td>116.40</td>
<td>3.3091E-02</td>
<td>-2.0034E-02</td>
<td>6.8865E-03</td>
<td>-1.2394E-03</td>
<td>1.3325E-04</td>
<td>-6.3205E-06</td>
</tr>
<tr>
<td>(ODD)</td>
<td>116.14</td>
<td>3.4911E-02</td>
<td>-2.5649E-02</td>
<td>7.1595E-03</td>
<td>-1.0537E-03</td>
<td>6.4596E-05</td>
<td>-1.9028E-03</td>
</tr>
<tr>
<td>(ODD)</td>
<td>265.03</td>
<td>1.4311E-02</td>
<td>8.4046E-02</td>
<td>7.0531E-02</td>
<td>-1.4178E-02</td>
<td>1.9028E-03</td>
<td>1.9028E-03</td>
</tr>
</tbody>
</table>

\[
\psi_n(\ell) = \frac{\sin((n+1) \cos^{-1}(\ell/\ell))}{\sin(\cos^{-1}(2\ell/\ell))}
\]
<table>
<thead>
<tr>
<th>LENGTH</th>
<th>MODE</th>
<th>ALPHA</th>
<th>( u_0 )</th>
<th>( u_1 )</th>
<th>( u_2 )</th>
<th>( u_4 )</th>
<th>( u_6 )</th>
<th>( u_8 )</th>
<th>( u_{10} )</th>
<th>( u_{12} )</th>
<th>( u_{14} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.300</td>
<td>EVEN</td>
<td>269.31</td>
<td>3.3951E-01</td>
<td>-1.3968E-01</td>
<td>2.3234E-02</td>
<td>-2.6440E-03</td>
<td>-8.3942E-05</td>
<td>-7.5260E-05</td>
<td>7.2608E-05</td>
<td>-1.10E-05</td>
<td></td>
</tr>
<tr>
<td></td>
<td>ODD</td>
<td>269.52</td>
<td>1.5140E-01</td>
<td>-8.9017E-02</td>
<td>1.3108E-02</td>
<td>-7.5689E-04</td>
<td>5.2783E-06</td>
<td>5.2783E-06</td>
<td>5.2783E-06</td>
<td>5.2783E-06</td>
<td></td>
</tr>
<tr>
<td>0.450</td>
<td>EVEN</td>
<td>250.86</td>
<td>2.5379E-01</td>
<td>-7.6440E-02</td>
<td>1.4992E-03</td>
<td>-1.0065E-03</td>
<td>-2.6000E-04</td>
<td>-1.0352E-04</td>
<td>-3.5038E-05</td>
<td>-3.5038E-05</td>
<td></td>
</tr>
<tr>
<td></td>
<td>ODD</td>
<td>268.25</td>
<td>1.0321E-01</td>
<td>-6.0435E-02</td>
<td>8.7943E-03</td>
<td>-5.1144E-04</td>
<td>3.8338E-06</td>
<td>3.8338E-06</td>
<td>3.8338E-06</td>
<td>3.8338E-06</td>
<td></td>
</tr>
<tr>
<td></td>
<td>ODD</td>
<td>268.12</td>
<td>1.0115E-01</td>
<td>-5.9225E-02</td>
<td>8.6110E-03</td>
<td>-5.0112E-04</td>
<td>2.2165E-06</td>
<td>2.2165E-06</td>
<td>2.2165E-06</td>
<td>2.2165E-06</td>
<td></td>
</tr>
<tr>
<td></td>
<td>ODD</td>
<td>267.96</td>
<td>9.3276E-02</td>
<td>-5.8036E-02</td>
<td>8.4898E-03</td>
<td>-4.6365E-04</td>
<td>2.4245E-05</td>
<td>2.4245E-05</td>
<td>2.4245E-05</td>
<td>2.4245E-05</td>
<td></td>
</tr>
<tr>
<td>0.500</td>
<td>EVEN</td>
<td>102.84</td>
<td>2.2974E-01</td>
<td>-6.2513E-02</td>
<td>2.4712E-03</td>
<td>-6.2809E-04</td>
<td>-2.6722E-04</td>
<td>-1.0538E-04</td>
<td>-3.5238E-05</td>
<td>-3.5238E-05</td>
<td></td>
</tr>
<tr>
<td></td>
<td>ODD</td>
<td>267.51</td>
<td>9.3822E-02</td>
<td>-5.4920E-02</td>
<td>8.0022E-03</td>
<td>-4.6146E-04</td>
<td>7.7087E-06</td>
<td>7.7087E-06</td>
<td>7.7087E-06</td>
<td>7.7087E-06</td>
<td></td>
</tr>
<tr>
<td>0.600</td>
<td>EVEN</td>
<td>95.73</td>
<td>1.8349E-01</td>
<td>-4.2533E-02</td>
<td>6.3013E-03</td>
<td>-1.4233E-03</td>
<td>-2.5261E-04</td>
<td>-9.7776E-05</td>
<td>-1.2269E-05</td>
<td>-1.2269E-05</td>
<td></td>
</tr>
<tr>
<td>0.750</td>
<td>ODD</td>
<td>258.51</td>
<td>6.6646E-02</td>
<td>-3.8772E-02</td>
<td>5.5537E-03</td>
<td>-3.1639E-04</td>
<td>7.0508E-06</td>
<td>7.0508E-06</td>
<td>7.0508E-06</td>
<td>7.0508E-06</td>
<td></td>
</tr>
<tr>
<td>0.900</td>
<td>ODD</td>
<td>239.75</td>
<td>5.8350E-02</td>
<td>-3.3953E-02</td>
<td>4.7991E-03</td>
<td>-2.7286E-04</td>
<td>-3.9212E-06</td>
<td>-3.9212E-06</td>
<td>-3.9212E-06</td>
<td>-3.9212E-06</td>
<td></td>
</tr>
<tr>
<td></td>
<td>EVEN</td>
<td>263.87</td>
<td>1.9211E-02</td>
<td>5.8349E-02</td>
<td>3.9089E-02</td>
<td>4.4157E-03</td>
<td>-7.4218E-04</td>
<td>-7.4218E-04</td>
<td>-7.4218E-04</td>
<td>-7.4218E-04</td>
<td></td>
</tr>
</tbody>
</table>
TABLE 8--Continued

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 (ODD)</td>
<td>211.12</td>
<td>5.4515E-02</td>
<td>-3.1732E-02</td>
<td>4.4340E-03</td>
<td>-2.6344E-04</td>
<td>2.8587E-06</td>
<td>-1.0123E-05</td>
<td>-5.8245E-06</td>
<td></td>
</tr>
<tr>
<td>2 (EVEN)</td>
<td>260.73</td>
<td>1.1786E-02</td>
<td>5.6444E-02</td>
<td>-3.6377E-02</td>
<td>4.5551E-03</td>
<td>-7.2794E-04</td>
<td>-8.3723E-05</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 (EVEN)</td>
<td>98.26</td>
<td>8.0568E-02</td>
<td>-1.5997E-02</td>
<td>-4.8551E-03</td>
<td>2.4766E-04</td>
<td>-8.9670E-05</td>
<td>-6.3058E-05</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.100</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 (ODD)</td>
<td>170.53</td>
<td>5.1659E-02</td>
<td>-2.9980E-02</td>
<td>4.1439E-03</td>
<td>-2.4245E-04</td>
<td>2.5158E-06</td>
<td>-9.7838E-06</td>
<td>-5.7948E-06</td>
<td></td>
</tr>
<tr>
<td>2 (EVEN)</td>
<td>256.12</td>
<td>7.1616E-03</td>
<td>5.1991E-02</td>
<td>-3.4527E-02</td>
<td>4.7691E-03</td>
<td>-7.3339E-04</td>
<td>-6.7101E-05</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 (EVEN)</td>
<td>99.69</td>
<td>6.8786E-02</td>
<td>-1.4203E-02</td>
<td>-3.9678E-03</td>
<td>2.4874E-04</td>
<td>-6.2605E-05</td>
<td>-5.3971E-05</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4 (ODD)</td>
<td>268.67</td>
<td>-3.2408E-02</td>
<td>-7.4102E-02</td>
<td>8.9281E-02</td>
<td>-3.0681E-02</td>
<td>5.2599E-03</td>
<td>-5.8796E-04</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.250</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 (ODD)</td>
<td>135.53</td>
<td>4.6995E-02</td>
<td>-2.7981E-02</td>
<td>3.7008E-03</td>
<td>-2.0530E-04</td>
<td>5.9668E-06</td>
<td>-4.9119E-06</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 (EVEN)</td>
<td>244.20</td>
<td>3.2804E-03</td>
<td>4.8768E-02</td>
<td>-3.2952E-02</td>
<td>5.2346E-03</td>
<td>-7.6512E-04</td>
<td>-4.7294E-05</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 (EVEN)</td>
<td>102.13</td>
<td>5.5853E-02</td>
<td>-1.2763E-02</td>
<td>-2.6774E-03</td>
<td>2.3408E-04</td>
<td>-3.0567E-05</td>
<td>-4.0366E-05</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4 (ODD)</td>
<td>267.40</td>
<td>-2.5815E-02</td>
<td>-5.9835E-02</td>
<td>7.2501E-02</td>
<td>-2.4512E-02</td>
<td>3.6323E-03</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.400</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 (ODD)</td>
<td>121.83</td>
<td>4.6740E-02</td>
<td>-2.6790E-02</td>
<td>3.5470E-03</td>
<td>-1.9950E-04</td>
<td>-1.6491E-05</td>
<td>-2.9551E-05</td>
<td>-5.7493E-06</td>
<td></td>
</tr>
<tr>
<td>3 (EVEN)</td>
<td>104.79</td>
<td>4.6905E-02</td>
<td>-1.2605E-02</td>
<td>-1.3518E-03</td>
<td>1.8514E-04</td>
<td>-1.5646E-05</td>
<td>-2.9551E-05</td>
<td>-5.7493E-06</td>
<td></td>
</tr>
<tr>
<td>4 (ODD)</td>
<td>265.03</td>
<td>2.1345E-02</td>
<td>5.0493E-02</td>
<td>-6.0599E-02</td>
<td>2.0391E-02</td>
<td>-3.1543E-03</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.500</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 (ODD)</td>
<td>113.50</td>
<td>4.5146E-02</td>
<td>-2.5778E-02</td>
<td>3.3807E-03</td>
<td>-2.0955E-04</td>
<td>-1.4243E-05</td>
<td>-2.0836E-06</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 (EVEN)</td>
<td>108.47</td>
<td>3.8751E-02</td>
<td>-1.4627E-02</td>
<td>1.3465E-03</td>
<td>-2.3614E-05</td>
<td>7.6881E-06</td>
<td>-1.6076E-05</td>
<td>-2.0836E-06</td>
<td></td>
</tr>
<tr>
<td>4 (ODD)</td>
<td>258.77</td>
<td>1.7211E-02</td>
<td>4.2026E-02</td>
<td>-4.9800E-02</td>
<td>1.6631E-02</td>
<td>-2.5439E-03</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ u_n(l) = \frac{\sin((n+1)\cos^{-1}(2l/L))}{\sin(\cos^{-1}(2l/L))} \]
REFERENCES


21. Garbacz, R.J., "Introduction to Characteristic Modes for Chaff Applications," Report 2584-6 (to be published), ElectroScience Laboratory, Department of Electrical Engineering, The Ohio State University; prepared under Contract F 33615-68-C-1252 for Air Force Avionics Laboratory, Wright-Patterson Air Force Base, Ohio.


43. Smirnov, loc. cit., pp. 151, 152.

44. Kennaugh, E.M., "Effects of Type of Polarization on Echo Characteristics," Report 389-12, 1 March 1952, ElectroScience Laboratory, Department of Electrical Engineering, The Ohio State University; prepared under Contract AF 28(099)-90 for Rome Air Development Center. (AD 2494)


48. Garbacz, R.J., personal correspondence.


50. Richmond, J.H., Reciprocity Theorems and Plane Surface Waves, Bulletin 176, Engineering Experiment Station, The Ohio State University, 1959.

