SCOTT, Frank Lee, 1943-
A NEW HYPER-UNIFORMITY WITH
APPLICATIONS TO MULTI-VALUED MAPPINGS.

The Ohio State University, Ph.D., 1969
Mathematics

University Microfilms, Inc., Ann Arbor, Michigan
A NEW HYPER-UNIFORMITY WITH
APPLICATIONS TO MULTI-VALUED
MAPPINGS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

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The Ohio State University
1969

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ACKNOWLEDGMENT

I would like to thank Professor Norman Levine for generously giving of his time during the preparation of this dissertation.
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INTRODUCTION

Uniform spaces were introduced in 1937 by Andre Weil. In 1941 Tukey approached the same problem from another direction and showed that his system giving a uniformity in terms of coverings was consistent with the axioms of Weil. The first attempt at placing a uniform-like structure on a hyperspace, meaning some collection of subsets of a given set, belongs to Hausdorff [6] ([ ] refers to the number of the item as listed in the bibliography) who in 1927 defined a metric on the collection of all non-empty, closed, bounded subsets of a given metric space. If \((X, d)\) denotes a metric space and \(\mathcal{H}\) the collection of non-empty, closed, bounded subsets of \(X\), then the Hausdorff metric on \(\mathcal{H}\) is given by \(p(A, B) = \max \{\sup \{d(a, B) : a \in A\}, \sup \{d(b, A) : b \in B\}\}\) where as usual \(d(a, B) = \inf \{d(a, b) : b \in B\}\).

Bourbaki [1] and Kelley [10] define uniform structure in terms of entourages containing the diagonal of \(X \times X\). This has been called the orthodox approach to uniform spaces and will be employed exclusively during the course of this thesis. The heretical method is that of uniform covers which is used extensively in the works of Isbell [5], [7], [8] and [9]. This presents us with a dilemma regarding references to results about the Hausdorff uniformity on the hyperspace. Caulfield [2] and Isbell contain many of the same results that we shall quote without proof. Caulfield employs entourages and
in general his proofs are more complete and self-contained than those of Isbell. Unfortunately, the Isbell book is the only one accessible to the general reader. We shall in general reference Caulfield noting here that the same result in some form may be found in Isbell.

Bourbaki gives a method for uniformizing the hyperspace of an arbitrary uniform space, which in the case of metric uniform spaces reduces to the Hausdorff metric on the non-empty closed bounded subsets. Many authors take the collection of non-empty closed subsets of the given uniform space as the hyperspace. Caulfield considers the collection of all subsets, including the empty set, while Michael [12] deletes only the empty set. We shall follow Caulfield in this since undue complications never arise.

Since the uniformizable spaces are precisely the collection of completely regular topological spaces, if we begin with a completely regular topological space, then there is usually a choice as to which uniformity is to be employed as any one compatible with the topology would suffice. Caulfield has several nice results when the fine or the Cech uniformity is used while Isbell uses things he calls locally fine. Isbell begins with an arbitrary uniformity and through a transfinite process obtains the locally fine one compatible with the topology. Unfortunately, it is not known if the intermediate structures are even uniformities.

Isbell coined the term supercomplete to apply to a space when
the hyperspace was complete in the Hausdorff uniformity. Isbell also obtained several conditions for supercompleteness using his concept of locally fine. Caulfield extended the work of Isbell by giving several conditions sufficient for supercompleteness without introducing the difficulties apparently inherent in local fineness.

Completeness is actually our point of departure from the previous results. Isbell shows that if one has a fine uniformity, then the hyperspace is complete if and only if the original uniform space is paracompact. In particular, we shall see that completeness is not sufficient for supercompleteness as was pointed out by Michael in 1951. Even worse, supercompleteness is not product hereditary as $X$ supercomplete does not force $X \times X$ to be supercomplete.

Thus the question naturally arises: Is it possible to place a different uniformity on the hyperspace which is in some sense natural and which is more stable with regard to completion. As far as we know there has been no attempt to uniformize the hyperspace otherwise than by the Hausdorff uniformity. This is precisely what we propose to do. We shall define a new hyper-uniformity and compare the results obtained to those in the Hausdorff uniformity case.

Once we have this new hyper-uniformity in hand we shall follow Michael and apply the results to multi-valued mappings between uniform spaces.

We shall now discuss briefly some of the merits and defects of the new hyper-uniformity, hereafter denoted by $\mathcal{U}^*$ if $\mathcal{U}$ is the
uniformity for the original space. We shall completely characterize $U^*$ completeness and $U^*$ total boundedness in terms of easily verifiable properties of $U$. The topologies induced by $U$ and $U^*$ will be jointly Hausdorff, whereas in the case of the Hausdorff uniformity, the induced topology is Hausdorff if and only if the $U$ induced topology is discrete. Basic uniform and topological considerations of this nature will involve the first four chapters of the work. In Chapter V we shall investigate certain functions induced on the hyperspace (the precedent here being Michael). Chapter VI will be devoted to multi-valued mappings when the $U^*$ uniformity is used on the hyperspace. In Chapter VII we shall pose problems of further interest; for some of these problems partial results are contained herein while others are new.
BACKGROUND

All concepts not defined in this chapter may be found in Kelley [10] unless noted otherwise. Proofs of the theorems about $\mathcal{U}$ will be omitted but may be found in Caulfield [2] or Isbell [7].

Notation and Definitions.

0.1 $X, Y, X_\alpha, Y_\alpha$ will always denote non-empty sets.

0.2 $\mathcal{T}, \mathcal{T}', \mathcal{T}_\alpha$ will denote topologies, and a topological space will be represented by $(X, \mathcal{T})$.

0.3 $\mathcal{U}(X)$ will denote the family of all uniformities on the set $X$.

0.4 $\mathcal{U}$ and $\mathcal{V}$ will denote uniformities, and entourages in $\mathcal{U}$ and $\mathcal{V}$ will be denoted by $U$ and $V$.

0.5 $\mathcal{T}(\mathcal{U})$ will be the topology generated by the uniformity $\mathcal{U}$.

0.6 $X^* = \{E : E \subseteq X\}$.

0.7 Elements of $X$ will be denoted by small Roman letters $x, y, z, w, a, b$; elements of $X^*$ will be denoted by capital Roman letters $A, B, C, D, E$.

0.8 $\Delta(X) = \{(x, x) : x \in X\}$.

0.9 $\Delta(X^*) = \{(A, A) : A \in X^*\}$.
0.10 If $U \in \mathcal{U} \in \mathcal{U}(X)$ and $A \subseteq X$, then $U[A] = \{x : (a,x) \in U \text{ for some } a \in A\}$ and $U[x] = U[\{x\}]$ for all $x \in X$.

0.11 If $I \neq \emptyset$ and $\{W_\alpha : \alpha \in I, \} \alpha \in I$ is a subbase for some uniformity on $X$, then the uniform structure generated by $\{W_\alpha : \alpha \in I\}$ will be denoted by $U[\{W_\alpha : \alpha \in I\}]$.

0.12 Let $\Delta(X) \subseteq R \subseteq X \times X$. Then $H(R) = \{(A,B) : A \subseteq R[B] \text{ and } B \subseteq R[A]\}$ and $S(R) = \{(A,B) : \emptyset \neq A \times B \subseteq R\} \cup \Delta(X^*)$.

0.13 If $U \in \mathcal{U}(X)$, then $\hat{U} = \mathcal{U}\{U(U) : U \in \mathcal{U}\}$ and $U^* = \mathcal{U}\{U(S(U) : U \in \mathcal{U}\}$. $\hat{U}$ is the Hausdorff uniformity on $X^*$.

0.14 Let $\Delta(X) \subseteq R \subseteq X \times X$ and $Y \subseteq X$. Then $S(R \cap Y \times Y) = \{(A,B) : \emptyset \neq A \times B \subseteq R \cap Y \times Y\} \cup \Delta(Y^*)$.

0.15 Let $U \in \mathcal{U}(X)$ and $Y \subseteq X$. Then $Y \times Y \cap U = \{Y \times Y \cap U : U \in \mathcal{U}\}$. That is, $Y \times Y \cap U$ is the subspace uniformity on $Y$ induced by $U$. Similarly, given $(X,\mathcal{F})$ and $Y \subseteq X$, then $Y \cap \mathcal{F} = \{Y \cap O : O \in \mathcal{F}\}$.

0.16 $(X',\mathcal{U}') \subseteq (X,\mathcal{U})$ will mean $X' \subseteq X$ and $\mathcal{U}' = X' \times X' \cap \mathcal{U}$.

0.17 $f : X \rightarrow Y$ will mean $f$ is a single-valued mapping from $X$ into $Y$. 
0.18 Multi-valued mappings from $X$ to $Y$ will be denoted by Greek letters $\mathcal{F}: X \rightarrow Y^*$. 

0.19 $f : (X,\mathcal{F}) \rightarrow (Y,\mathcal{F}')$ will mean $f : X \rightarrow Y$ and $f$ is continuous with respect to $\mathcal{F}$ and $\mathcal{F}'$. 

0.20 $f : (X,\mathcal{U}) \rightarrow (Y,\mathcal{V})$ will mean $f : X \rightarrow Y$ and $f$ is uniformly continuous with respect to $\mathcal{U}$ and $\mathcal{V}$. If $f$ is bijective and if both $f$ and $f^{-1}$ are uniformly continuous, then $f$ is called a unimorphism. 

0.21 If $f : X \rightarrow Y$ and $V \in \mathcal{V} \subseteq \mathcal{U}(Y)$, then 
\[(f \times f)^{-1} V = \{(x,x') : (f(x),f(x')) \in V\} \subseteq X \times X.\]

0.22 If $f : X \rightarrow Y$, then $f^* : X^* \rightarrow Y^*$ is defined by $f^*(A) = f[A] = \{f(a) : a \in A\}$ for all $A \in X^*$. 

0.23 $i : X \rightarrow X^*$ is defined by $i(x) = \{x\}$. 

0.24 Given $(X,\mathcal{F})$ and $A \subseteq X$. Then $c_{\mathcal{F}}(A)$ will denote the closure of $A$ in $X$ with respect to the topology $\mathcal{F}$. When no confusion can arise, we shall write $c(A)$. $\mathcal{F}$ will denote the complement operator, and $\text{Int}_{\mathcal{F}}$ or $\text{Int}$ will denote the interior operator. 

0.25 Given $(X,\mathcal{U})$ and $\mathcal{A} \subseteq X^*$, then $c^{*}(\mathcal{A})$ will denote the closure of $\mathcal{A}$ in $X^*$ with respect to $\mathcal{F}(\mathcal{U}^*)$, and $\mathcal{C}(\mathcal{A})$ will denote the closure of $\mathcal{A}$ in $X^*$ with respect to $\mathcal{F}(\hat{\mathcal{U}})$. 
0.26 If \( d \) is a pseudo-metric on \( X \), then \( \mathcal{U}(d) \) will denote the uniformity generated by \( d \). Define
\[
U_{d,e} = \{(x,y) \in X \times X \mid d(x,y) < e^2\} \quad \text{and we shall write } U_e \text{ whenever there is no confusion as to which } d \text{ is involved.}
\]

0.27 \( \mathcal{G}(\mathcal{U}) = \{d : d \text{ is a pseudo-metric for } X \text{ and } \mathcal{U}(d) \leq \mathcal{U}_e\} \).

\( \mathcal{G}(\mathcal{U}) \) is called the gage of \( \mathcal{U} \).

0.28 Let \((X,\mathcal{I})\) be given and \( A \subseteq X \). Then \( A \) is called generalized closed iff \( c(A) \subseteq 0 \) whenever \( A \subseteq 0 \in \mathcal{I} \).

0.29 \( T : D \to X \) will denote a net in \( X \), where \((D,\geq)\) is a directed set.

0.30 Given \((X,\mathcal{I})\), then \( 2^X \) will denote the collection of closed subsets of \( X \).

0.31 Let \( \mathcal{U} \in \mathcal{U}(X) \). Then \( \mathcal{U} \) is called fine iff \( \forall \subseteq \mathcal{U} \) whenever \( \mathcal{I}(\mathcal{U}) = \mathcal{I}(\forall) \).

Theorems

0.32 If \( \mathcal{U} \in \mathcal{U}(X) \), then \( \hat{\mathcal{U}} \in \mathcal{U}(X^*) \).

0.33 \( f : (X,\mathcal{U}) \to (Y,\mathcal{I}) \) if \( f^* : (X^*,\hat{\mathcal{U}}) \to (Y^*,\hat{\mathcal{I}}) \).

0.34 If \( \mathcal{U} \in \mathcal{U}(X) \), then \( \hat{\mathcal{U}} \) is totally bounded iff \( \mathcal{U} \) is totally bounded.

0.35 If \( \mathcal{U} \in \mathcal{U}(X) \), then \( \mathcal{I}(\mathcal{U}) \) is compact iff \( \mathcal{I}(\hat{\mathcal{U}}) \) is compact.
0.36 If $U \in \mathcal{U}(X)$ and $U$ is fine, then $\hat{U}$ is complete iff $\mathcal{I}(U)$ is paracompact.

0.37 Given $(X, \mathcal{I})$ and $A$ a generalized closed subset of $X$. If $X$ is compact, complete, normal, paracompact or Lindelöf, then so is $A$ in the relative topology (Levine [11]).
CHAPTER I

The Uniformity \( U^* \)

In this chapter we propose to introduce properties indigenous to the uniformity \( U^* \) and to explore some of the stable relations among \( U, \hat{U} \) and \( U^* \). For the purpose of what follows we shall let \( (X, U) \) denote an arbitrary but fixed uniform space.

1.1 Lemma. If \( U \in \mathcal{U} \), then \( \Delta(X) \subseteq S(U) \).

Proof. See definition 0.12.

1.2 Lemma. If \( I \neq \emptyset \) and \( U_\alpha \in \mathcal{U} \) for each \( \alpha \in I \), then \( S(\bigwedge \{U_\alpha : \alpha \in I\}) = \bigwedge \{S(U_\alpha) : \alpha \in I\} \).

Proof. \((A, B) \in S(\bigwedge \{U_\alpha : \alpha \in I\}) \iff A = B \) or \( \emptyset \neq A \times B \subseteq \bigwedge \{U_\alpha : \alpha \in I\} \iff A = B \) or \( \emptyset \neq A \times B \subseteq U_\alpha \) for each \( \alpha \in I \) iff \((A, B) \in S(U_\alpha) \) for each \( \alpha \in I \) iff \((A, B) \in \bigwedge \{S(U_\alpha) : \alpha \in I\} \).

1.3 Lemma. If \( U \in \mathcal{U} \), then \( S(U^{-1}) = (S(U))^{-1} \).

Proof. \((A, B) \in S(U^{-1}) \iff A = B \) or \( \emptyset \neq A \times B \subseteq U^{-1} \iff A = B \) or \((a, b) \in U^{-1} \) for each \( a \in A \) and each \( b \in B \) iff \( A = B \) or \((b, a) \in U \) for each \( a \in A \) and each \( b \in B \) iff \( B = A \) or \( \emptyset \neq B \times A \subseteq U \) iff \((B, A) \in S(U) \) iff \((A, B) \in (S(U))^{-1} \).
Lemma. If $U \cap V \in \mathcal{U}$, then $U \subseteq V$ iff $S(U) \subseteq S(V)$.

Proof. Assume $U \subseteq V$. If $(A,B) \in S(U)$, then $A = B$ or $\emptyset \neq A \times B \subseteq U$. Therefore, $A = B$ or $\emptyset \neq A \times B \subseteq V$, and hence, $(A,B) \in S(V)$. Now assume $S(U) \subseteq S(V)$, and let $(x,y) \in U$. Then $\emptyset \neq \{x\} \times \{y\} \subseteq U$, and hence, $(\{x\},\{y\}) \in S(U) \subseteq S(V)$. But this implies $\{x\} = \{y\}$ or $\emptyset \neq \{x\} \times \{y\} \subseteq V$. In either case we have $(x,y) \in V$.

Lemma. If $U \cap V \in \mathcal{U}$, then $S(V) \circ S(U) \subseteq S(V \circ U)$.

Proof. Let $(A,B) \in S(V) \circ S(U)$. Then for some $C \in X^*$ $(A,C) \in S(U)$ and $(C,B) \in S(V)$. Thus, $A = C$ or $\emptyset \neq A \times C \subseteq U$; and $B = C$ or $\emptyset \neq C \times B \subseteq V$. If $A = B = C$, then $(A,B) \in \Delta(X^*) \subseteq S(V \circ U)$. If $\emptyset \neq A \times C \subseteq U$ and $B = C$, then $\emptyset \neq A \times C = A \times B \subseteq U \subseteq V \circ U$; and hence $(A,B) \in S(V \circ U)$. Similarly, if $A = C$ and $\emptyset \neq C \times B \subseteq V$, then $\emptyset \neq C \times B = A \times B \subseteq V \subseteq V \times U$; and again $(A,B) \in S(V \circ U)$. Finally, suppose $\emptyset \neq A \times C \subseteq U$ and $\emptyset \neq C \times B \subseteq V$. Then, in particular, $A$, $B$ and $C$ are all non-empty. Fix $x \in C$. Now let $a \in A$ and $b \in B$. Then $(a,x) \in U$ and $(x,b) \in V$ implies $(a,b) \in V \circ U$ for every $a \in A$ and every $b \in B$. Therefore, $\emptyset \neq A \times B \subseteq V \circ U$.

Theorem. If $U \in \mathcal{U}(X)$, then $U^* \in \mathcal{U}(X^*)$. 

Proof. This follows from the preceding lemmas.

**I.7 Lemma.** If $U \in \mathcal{U}$, then $(A,\emptyset) \in S(U)$ or $(\emptyset,A) \in S(U)$ iff $A = \emptyset$.

**Proof.** This follows from definition 0.12.

**I.8 Lemma.** $\{\emptyset\}$ is both open and closed in the $\mathcal{J}(\mathcal{U}^\ast)$ topology.

**Proof.** By Lemma I.7, $S(U)[\{\emptyset\}] = \{\emptyset\}$ for each $U \in \mathcal{U}$. The result now follows.

**I.9 Lemma.** $S(\Delta(X)) = \Delta(X^\ast)$.

**Proof.** $S(\Delta(X)) = \Delta(X^\ast) \cup \{(A,B) : \emptyset \neq A \times B \subseteq \Delta(X)\}$, and clearly, $\{(A,B) : \emptyset \neq A \times B \subseteq \Delta(X)\} \subseteq \Delta(X^\ast)$.

**I.10 Lemma.** $S(U) \subseteq H(U)$ for each symmetric $U \in \mathcal{U}$.

**Proof.** Let $(A,B) \in S(U)$. Then $A = B$ or $\emptyset \neq A \times B \subseteq U$.

If $A = B$, then $A \subseteq U[B]$ and $B \subseteq U[A]$; so $(A,B) \in H(U)$.

If $\emptyset \neq A \times B \subseteq U$, then $\emptyset \neq B \times A \subseteq U$ since $U$ is symmetric. Therefore, $A \subseteq U[B]$ and $B \subseteq U[A]$; it follows then that $(A,B) \in H(U)$.

**I.11 Lemma.** If $\mathcal{U} \in \mathcal{U}(X)$, then $\hat{\mathcal{U}} \subseteq \mathcal{U}^\ast$. 
Proof. Let \( \hat{U} \in \hat{U} \). Then \( H(U) \leq \hat{U} \) for some \( U \in U \). There exists a symmetric \( V \in U \) such that \( V \subseteq U \); this implies that \( H(V) \subseteq H(U) \). Therefore, by Lemma I.10 \( S(V) \subseteq H(V) \subseteq H(U) \subseteq \hat{U} \), and hence, \( \hat{U} \in U^* \).

I.12 Lemma. \( S(X \times X) = \{ (\phi, \phi) \} \cup [X^* \setminus \{ \phi \}] \times [X^* \setminus \{ \phi \}] \).

Proof. This is clear.

I.13 Lemma. If \( U, \gamma \in U(X) \), then \( U \subseteq \gamma \) iff \( U^* \subseteq \gamma^* \).

Proof. Suppose \( U \subseteq \gamma \) and \( U^* \subseteq U^* \). Then \( S(U) \subseteq U^* \) for some \( U \in U \). Therefore, \( U^* \subseteq \gamma^* \) since \( S(U) \in \gamma^* \). Now assume \( U^* \subseteq \gamma^* \) and \( U \in U \). Then \( S(U) \in \gamma^* \), and therefore, \( S(V) \subseteq S(U) \) for some \( V \in \gamma \). Thus, by Lemma I.4 \( V \subseteq U \), which implies \( U \in \gamma \).

I.14 Theorem. Let \( U \in U(X) \), and \( B \subseteq U \). If \( B \) is a base (respectively, subbase) for \( U \), then \( \{ S(B) : B \in B \} \) is a base (respectively, subbase) for \( U^* \).

Proof. We shall prove this only in the case that \( B \) is a base since the argument for subbase is similar. First we claim that \( \{ S(B) : B \in B \} \) is a base for some uniformity on \( X^* \). We apply Kelley page 177 Theorem 2. By definition \( \Delta(X^*) \subseteq S(B) \) for every \( B \in B \). If \( B \in B \), then \( B' \subseteq B^{-1} \) for some \( B' \in B \). Then by Lemmas I.4 and I.3 \( S(B') \subseteq S(B^{-1}) = (S(B))^{-1} \). If \( B \in B \),
then \( B_1 \circ B_1 \subseteq B \) for some \( B_1 \in \mathcal{B} \). It follows then from Lemmas 1.4 and 1.5 that \( S(B_1) \circ S(B_1) \subseteq S(B_1 \circ B_1) \subseteq S(B) \).

Finally, if \( B_1, B_2 \in \mathcal{B} \), then \( B_3 \subseteq B_1 \cap B_2 \) for some \( B_3 \in \mathcal{B} \). Applying Lemmas 1.4 and 1.2 we obtain \( S(B_3) \subseteq S(B_1) \cap S(B_2) = S(B_1 \cap B_2) \). Thus by Kelley's Theorem \( \{ S(B) : B \in \mathcal{B} \} \) is a base for some uniformity on \( X^* \). It is clear that \( \mathcal{U} \{ S(B) : B \in B \} \subseteq \mathcal{U} \), since \( S(B) \in \mathcal{U} \) for every \( B \in \mathcal{B} \). Therefore, it suffices to show that \( \mathcal{U} \subseteq \mathcal{U} \{ S(B) : B \in \mathcal{B} \} \). Let \( U \in \mathcal{U} \). Then \( S(U) \subseteq \mathcal{U} \) for some \( U \in \mathcal{U} \). Since \( \mathcal{B} \) is a base for \( \mathcal{U} \), \( \mathcal{B} \subseteq U \) for some \( B \in \mathcal{B} \). Applying Lemma 1.4 again we obtain \( S(B) \subseteq S(U) \subseteq \mathcal{U} \), and hence, \( \mathcal{U} \subseteq \mathcal{U} \{ S(B) : B \in \mathcal{B} \} \). This implies \( \mathcal{U} \subseteq \mathcal{U} \{ S(B) : B \in \mathcal{B} \} \), as was to be shown.

**1.15 Theorem.** If \( \mathcal{U} \in \mathcal{U}(X) \) and \( \mathcal{V} \in \mathcal{U}(Y) \), then \( (X,\mathcal{U}) \subseteq (Y,\mathcal{V}) \) if and only if \( (X^*,\mathcal{U}^*) \subseteq (Y^*,\mathcal{V}^*) \).

**Proof.** It is clear that \( X \subseteq Y \) if and only if \( X^* \subseteq Y^* \). Thus, it suffices to show that \( \mathcal{U} = X \times X \cap \mathcal{V} \) if and only if \( \mathcal{U}^* = X^* \times X^* \cap \mathcal{V}^* \). Assume \( \mathcal{U} = X \times X \cap \mathcal{V} \). Let \( U^* \in \mathcal{U}^* \). Then \( S(U) \subseteq U^* \) for some \( U \in \mathcal{U} \). But \( U = X \times X \cap V \) for some \( V \in \mathcal{V} \); hence, \( S(X \times X \cap V) \subseteq U^* \). By 0.11 \( S(X \times X \cap V) = X^* \times X^* \cap S(V) \subseteq X^* \times X^* \cap V^* \). It follows that \( U^* \in X^* \times X^* \cap \mathcal{V}^* \), which implies \( \mathcal{U}^* \subseteq X^* \times X^* \cap \mathcal{V}^* \). Now let \( W \in X^* \times X^* \cap \mathcal{V}^* \). Then \( W = X^* \times X^* \cap V^* \) for some \( V^* \in \mathcal{V}^* \). Since \( S(V) \subseteq V^* \) for some \( V \in \mathcal{V} \), it follows that \( X^* \times X^* \cap S(V) \subseteq W \).
But \( X^* \times X^* \land S(V) = S(X \times X \land V) \in X \times X \land \mathcal{V} = \mathcal{U} \). This implies \( W \in \mathcal{U}^* \), and hence, \( X^* \times X^* \land \mathcal{V}^* \in \mathcal{U}^* \). Conversely, suppose \( \mathcal{U}^* = X^* \times X^* \land \mathcal{V}^* \). Let \( U \in \mathcal{U} \). Then \( S(U) \in \mathcal{U}^* \) implies \( S(U) = X^* \times X^* \land \mathcal{V}^* \) for some \( \mathcal{V}^* \in \mathcal{V}^* \). Since \( S(V) \subseteq V^* \) for some \( V \in \mathcal{V} \), we obtain \( X^* \times X^* \land S(V) \subseteq S(U) \). Again, \( X^* \times X^* \land S(V) = S(X \times X \land V) \). Then by Lemma I.4 \( X \times X \land V \subseteq U \). Therefore, \( U \subseteq X \times X \land \mathcal{V} \). Now let \( V \in \mathcal{V} \). Then \( S(X \times X \land V) = X^* \times X^* \land S(V) \in X^* \times X^* \land \mathcal{V}^* = \mathcal{U}^* \). Therefore, \( S(U) \subseteq S(X \times X \land V) \) for some \( U \in \mathcal{U} \). Applying Lemma I.4 yet again we have that \( U \subseteq X \times X \land V \). Then \( X \times X \land V \in \mathcal{U} \), and hence, \( X \times X \land \mathcal{V} \subseteq \mathcal{U} \). This completes the proof.
CHAPTER II

The Topology $\mathcal{J}(\mathfrak{U}^*)$.

In this chapter $(X, \mathcal{U})$ will denote an arbitrary but fixed uniform space. We shall investigate the properties of $\mathcal{J}(\mathfrak{U})$, $\mathcal{J}(\mathfrak{U})$, and $\mathcal{J}(\mathfrak{U}^*)$.

II.1 Theorem. $\mathcal{J}(\mathfrak{U})$ is Hausdorff iff $\mathcal{J}(\mathfrak{U})$ is discrete.

Proof. See Caulfield [2].

II.2 Lemma. $\bigcap \{U: U \in \mathfrak{U} \} = \Delta(X)$ iff $\bigcap \{S(U): U \in \mathfrak{U} \} = \Delta(X^*)$.

Proof. Suppose $\bigcap \{U: U \in \mathfrak{U} \} = \Delta(X)$. Let $(A,B) \in \bigcap \{S(U): U \in \mathfrak{U} \}$. If $A = B$, then $(A,B) \in \Delta(X^*)$. If $A \neq B$ for every $U \in \mathfrak{U}$, then $\emptyset \neq A \times B \subseteq \Delta(X)$; and hence, $A = B = \{x\}$ for some $x \in X$. Thus, in either case, $(A,B) \in \Delta(X^*)$. Now assume

$\bigcap \{S(U): U \in \mathfrak{U} \} = \Delta(X^*)$, and let $(x,y) \in \bigcap \{U: U \in \mathfrak{U} \}$. Then $\emptyset \neq \{x\} \times \{y\} \subseteq U$ for all $U \in \mathfrak{U}$. But this implies $(\{x\},\{y\}) \in S(U)$ for all $U \in \mathfrak{U}$, and thus, $(\{x\},(y)) \in \bigcap \{S(U): U \in \mathfrak{U} \} = \Delta(X^*)$.

Therefore, $\{x\} = \{y\}$, and it follows that $\bigcap \{U: U \in \mathfrak{U} \} = \Delta(X)$.

II.3 Corollary. $\mathcal{J}(\mathfrak{U})$ is Hausdorff iff $\mathcal{J}(\mathfrak{U}^*)$ is Hausdorff.

Proof. This follows from Lemma II.2 since $\mathcal{J}(\mathfrak{U})$ is Hausdorff iff $\bigcap \{U: U \in \mathfrak{U} \} = \Delta(X)$. 

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II.4 Remark. Since \( \hat{U} \subseteq U^* \) for every \( U \in U(X) \), it follows that \( \hat{\tau}(\hat{U}) \subseteq \tau(U^*) \). From the preceding lemmas we note that in general \( \hat{\tau}(\hat{U}) \neq \tau(U^*) \). In particular, if \( X \) is the unit interval and \( U \) is the usual metric induced uniformity, then \( \hat{\tau}(\hat{U}) \) is Hausdorff but not discrete. Therefore, \( \tau(U^*) \) is Hausdorff while \( \hat{\tau}(\hat{U}) \) is not. It follows that, in general \( \tau(U^*) \neq \hat{\tau}(\hat{U}) \) and \( U^* \neq \hat{U} \).

II.5 Lemma. For each \( A \in X^* \) and each \( U \in \mathcal{U} \),
\[
S(U)[A] = \{ A_1 \cup \{ B : \emptyset \neq B \subseteq U[a] \} \text{ for every } a \in A \}
\]

Proof. If \( A = \emptyset \), then \( S(U)[\emptyset] = \{ \emptyset \} \) by Lemma II.7.
Suppose \( A \neq \emptyset \). Then \( B \in S(U)[A] \) iff \( A = B \) or \( \emptyset \neq A \times B \subseteq U \)
iff \( A = B \) or \( \emptyset \neq \{ a \} \times B \subseteq U \) for every \( a \in A \) iff \( A = B \) or \( \emptyset \neq B \subseteq U[a] \) for every \( a \in A \).

II.6 Remark. We should note here a particular case of
Lemma II.5, namely, \( S(U)[\{ x \}] = \{ A : \emptyset \neq A \subseteq U[\{ x \}] \} \). Let \( \emptyset \neq A \subseteq X \) and suppose \( d \) is a pseudo-metric on \( X \). We define
\( d\text{-diam}(A) = \sup \{ d(a,a') : a, a' \in A \} \). If \( d\text{-diam}(A) > 0 \) for any \( d \in \tau(U) \), then it follows easily that \( \{ A \} \) is both open and closed with respect to \( \tau(U^*) \). In particular, if \( \tau(U) \) is Hausdorff and if \( \text{card } A \geq 2 \), then \( \{ A \} \in \tau(U^*) \).

II.7 Theorem. \( \tau(U) \) is discrete iff \( \tau(U^*) \) is discrete.
Proof. Suppose $\mathcal{J}(\mathcal{U}^*)$ is discrete, then $\{\{x\}\} \in \mathcal{J}(\mathcal{U}^*)$ for every $x \in X$. Corresponding to each $x \in X$, there exists a $U \in \mathcal{U}$ such that $S(U)[\{x\}] = \{\{x\}\}$. Then by Lemma II.5, $y \in U[x]$ implies $\{y\} \in S(U)[\{x\}]$, and therefore, $y = x$.

Thus $U[x] = \{x\} \in \mathcal{J}(\mathcal{U})$. It follows then that $\mathcal{J}(\mathcal{U})$ is discrete. Now suppose $\mathcal{J}(\mathcal{U})$ is discrete, and let $\emptyset \neq A \in X^*$. Choose $a \in A$. Then $U[a] = \{a\}$ for some symmetric $U \in \mathcal{U}$. Let $B \in S(U)[A]$, and suppose $\emptyset \neq A \times B \subseteq U$. Then $B \subseteq U[a]$ implies $B = \{a\}$. Thus we have $\{a\} \times A \subseteq U$ since $U$ is symmetric, and therefore, $A \subseteq U[a] = \{a\}$. It now follows that $S(U)[A] = \{A\}$.

II.8 Lemma. In the $\mathcal{J}(\mathcal{U})$ relative topology $2^X$ is Hausdorff and dense in $X^*$.


II.9 Lemma. In the $\mathcal{J}(\mathcal{U}^*)$ relative topology, $2^X$ is Hausdorff.

Proof. $\mathcal{J}(\mathcal{U}) \subseteq \mathcal{J}(\mathcal{U}^*)$. The result now follows from Lemma II.7.

II.10 Remark. In general $2^X$ is not dense in $X^*$ with respect to the $\mathcal{J}(\mathcal{U}^*)$ topology. This is almost clear by Remark II.6. In particular, if $(X, \mathcal{U})$ is the real line with the usual metric induced uniformity and if $A$ is the open unit interval, then $A \notin 2^X$ but $\{A\} \in \mathcal{J}(\mathcal{U}^*)$ by Remark II.6.

Recall that $i : X \to X^*$ is defined by $i(x) = \{x\}$. 
**II.11 Lemma.** $i: (X, U) \rightarrow (X^*, \mathcal{U}^*)$.

**Proof.** Since \( \{S(U) : U \in \mathcal{U} \} \) is a base for \( \mathcal{U}^* \), it suffices to show that \( (i \times i)^{-1} S(U) \in \mathcal{U} \) for all \( U \in \mathcal{U} \). Choose \( U \in \mathcal{U} \) and let \((x, y) \in U\). Then \( \emptyset \neq \{x, y\} \subseteq U \) implies \( \{(x, y) : (x, y) \in S(U)\} \). But then \( (i(x), i(y)) = \{(x, y) : (x, y) \in S(U)\} \), and hence, \( (x, y) \in (i \times i)^{-1} S(U) \). Therefore, \( U \subseteq (i \times i)^{-1} S(U) \), which implies \( (i \times i)^{-1} S(U) \in \mathcal{U} \).

**II.12 Lemma.** \( i[X] \times i[X] \cap \mathcal{U}^* = i[X] \times i[X] \cap \mathcal{U}^* \).

**Proof.** Let \( U \in \mathcal{U} \), \( U \) symmetric. Then by Lemma I.10 \( S(U) \subseteq H(U) \), which implies \( i[X] \times i[X] \cap S(U) \subseteq i[X] \times i[X] \cap H(U) \).

Now let \( (A, B) \in i[X] \times i[X] \cap H(U) \). Then there exists \( a, b \in X \) such that \( A = \{a\} \) and \( B = \{b\} \). But since \( \{(a), (b)\} \in H(U) \), it follows that \( \{b\} \subseteq U[a] \); and therefore, \( (a, b) \in U \). This implies \( \{(a), (b)\} \in S(U) \), and hence, \( i[X] \times i[X] \cap S(U) \subseteq i[X] \times i[X] \cap H(U) \). Since the symmetric entourages form a base for \( \mathcal{U} \), the result then follows.

**II.13 Lemma.** \((X, U)\) is unimorphic to \((i[X], i[X] \times i[X] \cap \mathcal{U}^*)\).

**Proof.** \( i \) is clearly bijective and by Lemma II.10 it is uniformly continuous. Thus, it suffices to show that \( i^{-1} \) is uniformly continuous or that \((i \times i) U \in i[X] \times i[X] \cap \mathcal{U}^* \) whenever \( U \in \mathcal{U} \).

But \((i \times i) U = \{(x, y) : (x, y) \in U\} = i[X] \times i[X] \cap S(U) \) where
the last equality comes from the fact that \((x,y) \in U\) iff 
\([x], [y] \in S(U)\).

\textbf{II.14 Corollary.} \(U\) is \(\mathcal{F}(U) \times \mathcal{F}(U)\) closed iff \(S(U)\) is 
\(\mathcal{F}(U^*) \times \mathcal{F}(U^*)\) closed.

\textbf{Proof.} Assume \(U\) is \(\mathcal{F}(U) \times \mathcal{F}(U)\) closed. Then 
\(U = \cap \{ V \circ U \circ V : V \in U \} \). Therefore, \(S(U) = S(\cap \{ V \circ U \circ V : V \in U \}) = \cap \{ S(V) \circ S(U) \circ S(V) : V \in U \} = c(S(U)) \) in the \(\mathcal{F}(U^*) \times \mathcal{F}(U^*)\) topology. The above follows from 
Lemmas 1.2 and 1.5. The fact that \(c(U) = \cap \{ V \circ U \circ V : V \in U \}\) 
may be found in Kelley page 179. Now assume \(S(U)\) is \(\mathcal{F}(U^*) \times \mathcal{F}(U^*)\) 
closed. Then \(S(U) \cap i[X] \times i[X]\) is closed in the relative topology 
on \(i[X] \times i[X]\). Since \(U = (i \times i)^{-1} [S(U) \cap i[X] \times i[X]]\), it 
follows that \(U\) is \(\mathcal{F}(U) \times \mathcal{F}(U)\) closed.

\textbf{II.15 Lemma.} Let \(\emptyset \neq A \subseteq X\), and \(x \in X\). Then \(A \subseteq c([x])\) 
if and only if \(A \in c([x])\).

\textbf{Proof.} \(A \in c([x])\) iff \(A \in S(V)[[x]]\) for all \(V \in U\) iff 
\([x], A) \in S(V)\) for all \(V \in U\) iff \(A = [x]\) or \([x] \times A \subseteq V\) for 
all \(V \in U\) iff \(A \subseteq V[[x]]\) for all \(V \in U\) iff \(A \subseteq c([x])\).

\textbf{II.16 Theorem.} Let \(\emptyset \neq A \subseteq X\). Then \(c(A) = \{ B : \emptyset \neq B \subseteq c(x) \text{ for some } x \in c(A) \}\).
Proof. Part I. We shall show \( c^*(i[A]) \subseteq \{ B : \emptyset \neq B \subseteq c(\{x\}) \} \) for some \( x \in c(A) \). Let \( B \in c^*(i[A]) \). Choose \( U \in \mathcal{U} \). Then \( V \circ V \subseteq U \) for some symmetric \( V \in \mathcal{U} \). Therefore, \( S(V[B]) \cap i[A] \neq \emptyset \).

We note that \( i[A] \neq \emptyset \) since \( A \neq \emptyset \) and also that \( B \neq \emptyset \) since \( B \in c^*(i[A]) \). Thus, there exists an \( a \in A \) such that \( \{a\} \in S(V[B]) \). This implies \( (B,\{a\}) \in S(V) \) and \( (\{a\},B) \in S(V) \) since \( S(V) \) is symmetric. Then—by Lemma II.5 \( B \subseteq V[a] \). Now choose \( x \in B \). We assert that \( B \subseteq c(\{x\}) \). Since \( x \in V[a] \), it follows that \( (a,x) \in V \) and \( (x,a) \in V \) as \( V \) is symmetric. Therefore, \( B \subseteq V[\{a\}] \subseteq V \circ V[\{x\}] \subseteq U[\{x\}] \). Since \( U \) was arbitrary, it follows that \( B \subseteq \cap \{ U[x] : U \in \mathcal{U} \} = c(\{x\}) \). Also since \( x \in V[\{a\}] \subseteq V[A] \subseteq U[A] \), we have \( x \in c(A) \).

Part II. We shall now show that \( \{ B : \emptyset \neq B \subseteq c(\{x\}) \} \) for some \( x \in c(A) \) \( \subseteq c^*(i[A]) \). Let \( \emptyset \neq B \subseteq c(\{x\}) \) for some \( x \in c(A) \). It suffices to show that for all symmetric \( U \in \mathcal{U} \), \( S(U)[B] \cap i[A] \neq \emptyset \). Choose a symmetric \( U \in \mathcal{U} \). Then \( V \circ V \subseteq U \) for some symmetric \( V \in \mathcal{U} \). Since \( x \in c(A) \) it follows that \( V[x] \cap A \neq \emptyset \). Let \( a \in V[x] \cap A \). Then \( (x,a) \in V \), and hence, \( (a,x) \in V \). This implies \( B \subseteq c(\{x\}) \subseteq V[x] \subseteq V \circ V[a] \subseteq U[a] \).

Therefore, by Lemma II.5 \( \{a\},B \in S(U) \) and \( (B,\{a\}) \in S(U) \) since \( S(U) \) is symmetric. Thus, \( \{a\} \in S(U)[B] \cap i[A] \), and it follows then that \( S(U)[B] \cap i[A] \neq \emptyset \) for all symmetric \( U \in \mathcal{U} \).
II.17 Theorem. Let $A \subseteq X$. Then $i[A]$ is $\mathcal{F}(\mathcal{U}^*)$ generalized closed whenever $A$ is $\mathcal{F}(\mathcal{U})$ closed.

Proof. Let $i[A] \subseteq O^* \in \mathcal{F}(\mathcal{U}^*)$. It suffices to show that $c^*(i[A]) \subseteq O^*$. If $A = \emptyset$, then $i[A] = \emptyset$ and $i[A]$ is $\mathcal{F}(\mathcal{U}^*)$ closed, hence, generalized closed. Therefore, without loss of generality $A \neq \emptyset$. By Lemma II.15, if $B \in c^*(i[A])$, then there exists an $x \in c(A) = A$ such that $B \subseteq c(\{x\})$. But $\{x\} \in i[A] \subseteq O^*$ implies $S(V)[\{x\}] \subseteq O^*$ for some $V \in \mathcal{U}$. Therefore, $B \subseteq c(\{x\}) \subseteq V[\{x\}]$. Then by Lemma II.5 $B \in S(V)[\{x\}]$, and hence, $B \subseteq O^*$. It now follows that $c^*(i[A]) \subseteq O^*$.

II.18 Corollary. $i[X]$ is $\mathcal{F}(\mathcal{U}^*)$ generalized closed.

II.19 Lemma. Let $A \subseteq X$. Then $c^*(i[A]) = i[A]$ iff $A$ is a closed Hausdorff subset of $X$.

Proof. Assume $c^*(i[A]) = i[A]$. Then $A = i^{-1}[i[A]]$ is closed since $i$ is continuous with respect to $\mathcal{F}(\mathcal{U})$ and $\mathcal{F}(\mathcal{U}^*)$. Equality holds since $i$ is injective. To prove Hausdorff it suffices to show that if $x \in A$, then $\{x\}$ is $A \wedge \mathcal{F}(\mathcal{U})$ closed. We shall in fact prove that $c(\{x\}) = \{x\}$ for every $x \in A$. Suppose $x \in A$, and $y \in c(\{x\})$. Then by Theorem II.15 $\{x, y\} \in c^*(i[A])$ since $\{x\} \subseteq c(\{x\})$ and $\{y\} \subseteq c(\{x\})$. But $c^*(i[A]) = i[A]$ implies $\{x, y\} = i(a) = \{a\}$ for some $a \in A$. It follows that $x = y = a$; and therefore, $c(\{x\}) = \{x\}$. Now suppose $c(A) = A$ and $A$ is
is Hausdorff. Let $B \in c^*(i[A])$. If $A = \emptyset$, then $B = \emptyset$; therefore, we may assume $A \neq \emptyset$. Then by Theorem II.12

$$\emptyset \neq B \subseteq c(\{x_0\})$$

for some $x \in c(A) = A$. But $c(\{x_0\}) \subseteq c(A) = A$ implies $c(\{x_0\}) = \{x_0\}$. Thus, $\{x_0\} = B$ and $B \in i[A]$. It now follows that $c^*(i[A]) = i[A]$.

**II.20 Corollary.** If $(X, \mathcal{U})$ is a uniform space, then the following are equivalent: (1) $i[X]$ is $\mathcal{T}(\mathcal{U}^*)$ closed, (2) $\mathcal{T}(\mathcal{U})$ is Hausdorff, and (3) $\mathcal{T}(\mathcal{U}^*)$ is Hausdorff.

The following theorem is of a type that shall be recurrent during the course of this thesis. That is, in an appreciable number of situations, if $(X^*, \mathcal{U}^*)$ has a given uniform or topological property, then this will be a sufficient condition for $(X, \mathcal{U})$ to have the property also.

**II.21 Theorem.** If $(X^*, \mathcal{U}^*)$ is compact, paracompact, Lindelöf, normal, or complete, then $(X, \mathcal{U})$ is also.

**Proof.** This follows from Theorem 0.37 and the fact that $X$ is unimorphic to a generalized closed subset of $X^*$. 
CHAPTER III

Further Relations Among $\mathcal{U}$, $\hat{\mathcal{U}}$ and $\mathcal{U}^*$

In this chapter we shall delve further into uniform theoretic properties involving $\mathcal{U}$, $\hat{\mathcal{U}}$ and $\mathcal{U}^*$. This yields another striking difference between $\hat{\mathcal{U}}$ and $\mathcal{U}^*$; for it will be shown that the operator $*$ and the operator $\sup$ will commute for any family of uniformities on a set $X$, but this is not true if one replaces $*$ with $\wedge$.

III.1 Theorem. $(X, \mathcal{U})$ is pseudo-metrizable iff $(X^*, \mathcal{U}^*)$ is pseudo-metrizable.

Proof. If $\mathcal{U}$ is pseudo-metrizable, then there exists a countable base $\mathcal{B}$ for $\mathcal{U}$. Then by Lemma I.14 $\{ S(B) : B \in \mathcal{B} \}$ is a base for $\mathcal{U}^*$; and since this collection is also countable, it follows that $\mathcal{U}^*$ is pseudo-metrizable. Conversely, if $\mathcal{U}^*$ is pseudo-metrizable, then $i[X] \times i[X] \cap \mathcal{U}^*$ is pseudo-metrizable. And since $X$ is unimorphic to $i[X]$, it follows then that $\mathcal{U}$ is pseudo-metrizable.

III.2 Corollary. If $\mathcal{U} \in \mathcal{U}(X)$, then $\mathcal{U}$ is metrizable iff $\mathcal{U}^*$ is metrizable.

Proof. This follows from III.1 and the fact that $\mathcal{L}(\mathcal{U})$ is Hausdorff iff $\mathcal{L}(\mathcal{U}^*)$ is Hausdorff.
III.3 Definition. A uniformity $\mathcal{U} \in \mathcal{U}(X)$ is said to have a linearly ordered base (l.o.b.) if there exists a base $\mathcal{B}$ for $\mathcal{U}$ such that for each $B, B' \in \mathcal{U}$, then either $B \leq B'$ or $B' \leq B$.

III.4 Lemma. If $\mathcal{U} \in \mathcal{U}(X)$ has a l.o.b., then $\mathcal{U}^*$ has a l.o.b.

Proof. Let $\mathcal{B}$ be a l.o.b. for $\mathcal{U}$. Then by Lemma I.14, $\{S(B) : B \in \mathcal{B}\}$ is a base for $\mathcal{U}^*$. Also, $\{S(B) : B \in \mathcal{B}\}$ is linearly ordered. For if $S(B), S(B') \in \{S(B) : B \in \mathcal{B}\}$, then either $B \leq B'$, which implies $S(B) \leq S(B')$; or $B' \leq B$, which implies $S(B') \leq S(B)$.

III.5 Lemma. If $A \subseteq X$, $\mathcal{U} \in \mathcal{U}(X)$ and $\mathcal{U}$ has a l.o.b., then $A \times A \cap \mathcal{U}$ has a l.o.b.

Proof. If $\mathcal{B}$ is a l.o.b. for $\mathcal{U}$, then $\{A \times A \cap B : B \in \mathcal{B}\}$ is clearly a l.o.b. for $A \times A \cap \mathcal{U}$.

III.6 Lemma. If $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is a unimorphism, then $\mathcal{U}$ has a l.o.b. iff $\mathcal{V}$ has a l.o.b.

Proof. If $\mathcal{B}$ is a l.o.b. for $\mathcal{U}$, then $\{(f \times f)B : B \in \mathcal{B}\}$ is clearly a linearly ordered subset of $\mathcal{V}$. Now let $V \in \mathcal{V}$. Then $B \subseteq (f \times f)^{-1}V$ for some $B \in \mathcal{B}$, since $\mathcal{B}$ is a base for $\mathcal{U}$ and $f$ is uniformly continuous. Thus, $(f \times f)B \subseteq (f \times f)[(f \times f)^{-1}V] = V$, since $f$ is bijective;
hence \( \{ (f \times f) B : B \in \mathcal{B} \} \) is a base for \( \mathcal{Y} \).

From the preceding lemmas we obtain the following:

**III.7 Theorem.** \((X, \mathcal{U})\) has l.o.b. iff \((X^*, \mathcal{U}^*)\) has a l.o.b.

**III.8 Theorem.** Let \( I \neq \emptyset \) and let \( \mathcal{U}_\alpha \in \mathcal{U}(X) \) for each \( \alpha \in I \). Then \( \sup \{ \mathcal{U}_\alpha^*: \alpha \in I \} = (\sup \{ \mathcal{U}_\alpha : \alpha \in I \})^* \).

**Proof.** Since \( \{ S(U) : U \in \mathcal{U}_\alpha \} \) is a base for \( \mathcal{U}_\alpha^* \), it follows that \( \{ S(U) : U \in \mathcal{U}_\alpha, \alpha \in I \} \) is a subbase for \( \sup \{ \mathcal{U}_\alpha^* : \alpha \in I \} \). Therefore, \( \{ S(\mathcal{U}_\alpha \cap \ldots \cap S(\mathcal{U}_\alpha_k) : \mathcal{U}_\alpha_j \in \mathcal{U}_\alpha_j, \alpha_j \in I \} \) is a base for \( \sup \{ \mathcal{U}_\alpha^*: \alpha \in I \} \). Also \( \mathcal{U}_\alpha \cap \ldots \cap \mathcal{U}_\alpha_k : \mathcal{U}_\alpha_k \in \mathcal{U}_\alpha_k, \alpha_k \in I \} \) is a base for \( \sup \{ \mathcal{U}_\alpha : \alpha \in I \} \). Therefore, by Lemma I.14, \( \{ S(\mathcal{U}_\alpha \cap \ldots \cap S(\mathcal{U}_\alpha_k) : \mathcal{U}_\alpha_k \in \mathcal{U}_\alpha_k, \alpha_k \in I \} \) is a base for \( (\sup \{ \mathcal{U}_\alpha : \alpha \in I \})^* \). By Lemma I.2, \( S(\mathcal{U}_\alpha \cap \ldots \cap S(\mathcal{U}_\alpha_k) = S(\mathcal{U}_\alpha \cap \ldots \cap S(\mathcal{U}_\alpha_k) \), and the result follows.

**III.9 Corollary.** If \( I \neq \emptyset \) and \( \mathcal{U}_\alpha \in \mathcal{U}(X) \) for all \( \alpha \in I \), then \( \mathcal{J}(\sup \{ \mathcal{U}_\alpha : \alpha \in I \})^* = \sup \{ \mathcal{J}(\mathcal{U}_\alpha^*) : \alpha \in I \} \).

**Proof.** Since it is known that \( \sup \{ \mathcal{J}(\mathcal{U}_\alpha^*) : \alpha \in I \} \)
\( = \mathcal{J}(\sup \{ \mathcal{U}_\alpha^* : \alpha \in I \}) \), the result now follows from III.8.

**III.10 Lemma.** If \( I \neq \emptyset \) and \( \mathcal{U}_\alpha \in \mathcal{U}(X) \) for all \( \alpha \in I \) and if \( \mathcal{U} = \{ \mathcal{U}_\alpha : \alpha \in I \} \in \mathcal{U}(X) \), then \( \mathcal{U}^* = \{ \mathcal{U}_\alpha^* : \alpha \in I \} \).
Proof. We know that \( \mathcal{U} = \bigcup_{\alpha \in I} \mathcal{U}_\alpha : \alpha \in I \bigcup = \sup \{ \mathcal{U}_\alpha : \alpha \in I \} \).

Therefore, \( \mathcal{U}^* = (\sup \{ \mathcal{U}_\alpha : \alpha \in I \bigcup \})^* = \sup \{ \mathcal{U}_\alpha^* : \alpha \in I \} \). For each \( \alpha \in I \), \( \mathcal{U}_\alpha \leq \mathcal{U} \) implies \( \mathcal{U}_\alpha^* \leq \mathcal{U}^* \), and hence, \( \mathcal{U}_\alpha \subset \subset \mathcal{U} \). Now let \( \mathcal{U}^* \in \mathcal{U} \). Then \( S(U) \subseteq U^* \) for some \( U \in \mathcal{U} \). But \( U \in \mathcal{U} \) implies \( U \in \mathcal{U}_\alpha \) for some \( \alpha \in I \).

Therefore, \( S(U) \in \mathcal{U}_\alpha^* \), and hence, \( \mathcal{U}^* \in \mathcal{U}_\alpha^* \). Thus \( \mathcal{U}^* \in \bigcup_{\alpha \in I} \mathcal{U}_\alpha^* : \alpha \in I \bigcup \).

III.11 Corollary. If \( \mathcal{U} \in \mathcal{U}(X) \), then \( \mathcal{U}^* = \bigcup \{ \mathcal{U}(d)^* : d \in \mathcal{S}(\mathcal{U}) \} \).

Proof. This follows from the preceding lemma and the fact that \( \mathcal{U} = \bigcup_{\alpha \in I} \mathcal{U}(d) : d \in \mathcal{S}(\mathcal{U}) \).

III.12 Remark. Let \( \mathcal{U} \in \mathcal{U}(X) \), and fix \( d \in \mathcal{S}(\mathcal{U}) \), and let \( \epsilon > 0 \). Define \( V_\epsilon = V_{d,\epsilon} = \{ (x,y) : d(x,y) < \epsilon \} \). For \( \phi \neq K \subseteq X \), recall that \( d - \text{diam}(K) = \sup \{ d(x,y) : x \in K, y \in K \} \).

\( S(V_\epsilon) = \{ (A,B) : A = B \text{ or } \phi \neq A \times B \subseteq V_\epsilon \} \). Note that \( \phi \neq A \times B \subseteq V_\epsilon \) iff \( d(a,b) < \epsilon \) for all \( a \in A \) and all \( b \in B \).

Let \( x, y \in A \cup B \), and assume \( \phi \neq A \times B \subseteq V_\epsilon \). If \( x \in A \) and \( y \in B \), then \( d(x,y) < \epsilon \). If \( x \in A \) and \( y \in A \), then \( d(x,y) \leq d(x,b) + d(b,x) < 2 \epsilon \) for any \( b \in B \neq \phi \). Similarly, \( x, y \in B \) implies \( d(x,y) < 2 \epsilon \). Therefore, if \( \phi \neq A \times B \subseteq V_\epsilon \), then \( d - \text{diam}(A \cup B) \leq 2 \epsilon \). Define \( C(d,\epsilon) = \{ (A,B) : A = B \text{ or } A \neq \phi \neq B \text{ and } d - \text{diam}(A \cup B) < \epsilon \} \). It follows then that \( S(V_\epsilon) \subseteq C(d,2\epsilon) \). Now let \( (A,B) \in C(d,\epsilon/2) \). Then \( A = B \) or \( d - \text{diam}(A \cup B) < \epsilon/2 \), and \( A \neq \phi \neq B \). Suppose \( A \neq B \); then
for each \( a \in A \) and each \( b \in B \), \( d(a,b) \leq \varepsilon/2 < \varepsilon \); and hence, \( \emptyset \neq A \times B \subseteq V_{\varepsilon} \). Therefore, \( C(d,\varepsilon/2) \subseteq S(V_{\varepsilon}) \); and it follows that \( \{ C(d,\varepsilon) : d \in \mathcal{D}(U) \text{ and } \varepsilon > 0 \} \) is a base for \( \mathcal{U}^* \).

**III.13 Lemma.** Let \( I \neq \emptyset \) and \( \mathcal{U}_\alpha \in \mathcal{U}(X) \) for each \( \alpha \in I \); then \( \sup \{ \hat{\mathcal{U}}_\alpha : \alpha \in I \} \subseteq \left( \sup \{ \mathcal{U}_\alpha : \alpha \in I \} \right)^\wedge \).

**Proof.** Let \( \hat{U} \in \left( \sup \{ \hat{\mathcal{U}}_\alpha : \alpha \in I \} \right) \). Then there exists \( \alpha_1, \ldots, \alpha_k \in I \) and \( \mathcal{U}_{\alpha_1} \in \mathcal{U}_{\alpha_1}, \ldots, \mathcal{U}_{\alpha_k} \in \mathcal{U}_{\alpha_k} \) such that \( H(\mathcal{U}_{\alpha_1}) \wedge \ldots \wedge H(\mathcal{U}_{\alpha_k}) \subseteq U \). But \( \mathcal{U}_{\alpha_1} \wedge \ldots \wedge \mathcal{U}_{\alpha_k} \in \sup \{ \mathcal{U}_\alpha : \alpha \in I \} \)

implies \( H(\mathcal{U}_{\alpha_1} \wedge \ldots \wedge \mathcal{U}_{\alpha_k}) \in \left( \sup \{ \mathcal{U}_\alpha : \alpha \in I \} \right)^\wedge \). Since

\( H(\mathcal{U}_{\alpha_1} \wedge \ldots \wedge \mathcal{U}_{\alpha_k}) \subseteq H(\mathcal{U}_{\alpha_1}) \wedge \ldots \wedge H(\mathcal{U}_{\alpha_k}) \), it follows that \( \hat{U} \in \left( \sup \{ \mathcal{U}_\alpha : \alpha \in I \} \right)^\wedge \).

**III.14 Example.** The following is an example of a space \( X \) with two uniformities \( \mathcal{U} \) and \( \mathcal{V} \) for which \( \sup \{ \hat{\mathcal{U}}, \hat{\mathcal{V}} \} \neq (\sup \{ \mathcal{U}, \mathcal{V} \})^\wedge \).

Let \( X = \{1,2,3\} \), \( \mathcal{U} = \Delta(X) \cup \{1,2\} \times \{1,2\} \), and \( \mathcal{V} = \Delta(X) \cup \{2,3\} \times \{2,3\} \). Let \( \mathcal{U} = \mathcal{U}\{1,2\} \) and \( \mathcal{V} = \mathcal{V}\{1,2\} \). Then \( \Delta(X) = \mathcal{U} \wedge \mathcal{V} \in \sup \{ \mathcal{U}, \mathcal{V} \} \), and \( H(\Delta(X)) = \Delta(X^*) \). Therefore,

\( (\sup \{ \mathcal{U}, \mathcal{V} \})^\wedge = \mathcal{U}\{ \Delta(X^*) \} \). Since \( \hat{\mathcal{U}} = \mathcal{U}\{H(\mathcal{U})\} \) and \( \hat{\mathcal{V}} = \mathcal{V}\{H(\mathcal{V})\} \), it follows easily that \( \sup \{ \hat{\mathcal{U}}, \hat{\mathcal{V}} \} = \mathcal{U}\{H(\mathcal{U}) \wedge H(\mathcal{V})\} \).

Now \( \{1,3\} \subseteq \mathcal{U}\{1,2,3\} \) and \( \{1,2,3\} \subseteq \mathcal{U}\{1,3,2\} \), since \( (1,2) \in \mathcal{U} \).

Also \( \{1,3\} \subseteq \mathcal{V}\{1,2,3\} \) and \( \{1,2,3\} \subseteq \mathcal{V}\{1,3,2\} \), since \( (3,2) \in \mathcal{V} \).
Thus, \( (\{1,3\}, \{1,2,3\}) \in H(U) \land H(V) \land \gamma (\Delta(X*)) \). It now follows that \( \sup \{ \hat{U}, \hat{V}^\uparrow \} \neq (\sup \{ U, V \})^\uparrow \).

**III.15 Lemma.** Let \( I \neq \emptyset \), and let \( U_\alpha \in U(X) \) for each \( \alpha \in I \); then \( (\inf \{ U_\alpha : \alpha \in I \}^*) \subseteq \inf \{ U_\alpha^* : \alpha \in I \} \).

**Proof.** For each \( \beta \in I \), \( \inf \{ U_\alpha : \alpha \in I \} \subseteq U_\beta \). Therefore, \( (\inf \{ U_\alpha : \alpha \in I \}^*) \subseteq U_\beta^* \) for each \( \beta \in I \); and hence, \( (\inf \{ U_\alpha : \alpha \in I \}^*) \subseteq \inf \{ U_\alpha^* : \alpha \in I \} \).

**III.16 Example.** We shall now construct an example to show that, in general \( (\inf \{ U, V \})^* \neq \inf \{ U^*, V^* \} \). Let \( X, U \) and \( V \) be as in Example II.14. We assert \( \inf \{ U, V \} = \{ X \times X \} \). Let \( W \in \inf \{ U, V \} \); then \( W \in U \), and \( W \in V \). Therefore, \( U \cup V \subseteq W \). Then \( \Delta(X) \cup \{1,2\} \times \{1,2\} \cup \{2,3\} \times \{2,3\} \subseteq U \cup V \); hence \( \{ (1,3), (3,1) \} \subseteq (U \cup V) \circ (U \cup V) \). It follows then that \( X \times X \subseteq (U \cup V) \circ (U \cup V) \), and therefore, \( W \circ W = X \times X \). Thus, \( \inf \{ U, V \} = \{ X \times X \} \), and hence, \( (\inf \{ U, V \})^* = U \{ S(X \times X) \} \).

Let \( \mathcal{A} = X \times \mathcal{E} \). Let \( W^* = \Delta(X^*) \cup \mathcal{A} \times \mathcal{A} \). Since \( W^* \circ W^* = W^* = (W^*)^{-1} \), we have \( \mathcal{U} \{ W^* \} \subseteq \mathcal{U}(X^*) \). We note that \( (\{1,2\}, \{1,2,3\}) \notin W^* \), and therefore, \( W^* \notin S(X \times X) \). It follows then that \( \mathcal{U} \{ W^* \} \notin \mathcal{U} \{ S(X \times X) \} = (\inf \{ U, V \})^* \). Since \( S(U) = \Delta(X^*) \cup \{ \{2\}, \{3\}, \{2,3\} \times \{2\}, \{3\}, \{2,3\} \} \), it follows that \( S(U) \subseteq W^* \). Similarly, \( S(V) \subseteq W^* \). Therefore, \( \mathcal{U} \{ W^* \} \subseteq \mathcal{U} \{ S(U) \} \subseteq \mathcal{U} \{ W^* \} \subseteq \mathcal{U} \{ \mathcal{E} \times \mathcal{E} \} \), which implies \( \inf \{ U^*, V^* \} \neq (\inf \{ U, V \})^* \).
CHAPTER IV

Completeness and Total Boundedness

In this chapter we shall give necessary and sufficient conditions for \( U^* \) to be complete and for \( U^* \) to be totally bounded. The condition for completeness for \( U^* \) is much more natural than that for \( \hat{U} \); while the condition for \( U^* \) total boundedness is rather unfortunate and strikingly dissimilar to that for \( \hat{U} \). From this condition for \( U^* \) total boundedness it will follow that \( \mathcal{I}(U^*) \) is compact if and only if \( X \) is finite or \( U \) is indiscrete. However, we shall conclude the chapter with a positive result when \( \mathcal{I}(U) \) is compact.

IV.1 Lemma. If \( (X^*, U^*) \) is complete, then \( (X, U) \) is complete.

Proof. Since \( i[X] \) is generalized closed in \( X^* \), this result follows immediately from 0.37. However, for the sake of completeness, we shall give a direct proof here. Let \( T : D \to X \) be a \( U \)-cauchy net. Then \( i \circ T : D \to X^* \) is \( U^* \)-cauchy, since \( i \) is uniformly continuous. Thus, there exists an \( A \in X^* \) such that \( \lim i \circ T = A \). Note that \( A \neq \emptyset \). For if \( \lim i \circ T = \emptyset \), then \( i \circ T \) would have to equal \( \emptyset \) eventually, since \( \{ \emptyset \} \in \mathcal{I}(U^*) \); however, \( i \circ T(d) \neq \emptyset \) for all \( d \in D \). Now let \( a \in A \). We assert that \( T \) converges to \( a \). Let \( a \in 0 \in \mathcal{I}(U) \). Then \( U[a] \subseteq 0 \) for some \( U \in \mathcal{U} \). There exists a \( d' \in D \) such that
i \circ T(d) \in S(U)[A] \text{ whenever } d \geq d'. Therefore, (A, i \circ T(d)) = (A, \{ T(d) \}) \in S(U) \text{ for all } d \geq d'. This implies \{ T(d) \} = A \text{ or } \emptyset \neq A \times \{ T(d) \} \subseteq U \text{ whenever } d \geq d'. In either case, (a, T(d)) \in U \text{ for all } d \geq d', \text{ and hence, } T(d) \in U[a] \text{ whenever } d \geq d'. Thus, } \lim T = a.

**IV.2 Lemma.** Let \( U \in \mathcal{U}(X) \). If \( T : D \to X^* \) is \( \mathcal{U}^* \)-cauchy, then for each \( U \in \mathcal{U} \) there exists a \( d(U) \in D \) such that either

1. \( T(d) = T(d') \) for all \( d, d' \geq d(U) \) or
2. \( \emptyset \neq T(d) \times T(d') \subseteq U \) for all \( d, d' \geq d(U) \).

**Proof.** Let \( U \in \mathcal{U} \). Then \( V \circ V \subseteq U \) for some symmetric \( V \in \mathcal{U} \). Since \( T \) is \( \mathcal{U}^* \)-cauchy, there exists a \( d(V) \in D \) such that \( (T(d), T(d')) \in S(V) \) whenever \( d, d' \geq d(V) \). Thus, if \( d \geq d(V) \), then either \( T(d) = T(d(V)) \) or \( \emptyset \neq T(d) \times T(d(V)) \subseteq V \).

If for all \( d \geq d(V) \), we have \( T(d) = T(d(V)) \), then (1) holds. Suppose not. Then for some \( d_1 \geq d(V) \) we have \( T(d_1) \neq T(d(V)) \).

Therefore, \( \emptyset \neq T(d_1) \times T(d(V)) \subseteq V \) since \( (T(d_1), T(d(V))) \in S(V) \).

Also \( T(d(V)) \times T(d_1) \subseteq V^{-1} = V \). Thus, \( T(d(V)) \times T(d_1) \subseteq V \circ V \subseteq U \).

We assert \( \emptyset \neq T(d) \times T(d') \subseteq U \) whenever \( d, d' \geq d(V) \).

**Case 1.** If \( T(d) = T(d') = T(d(V)) \), then \( \emptyset \neq T(d) \times T(d') = T(d(V)) \times T(d(V)) \subseteq U \).

**Case 2.** If \( d, d' \geq d(V) \) and \( T(d) = T(d') \), then \( \emptyset \neq T(d) \times T(d') \subseteq V \subseteq U \), since \( (T(d), T(d')) \in S(V) \subseteq S(U) \).
Case 3. If \( T(d) = T(d') \neq T(d(v)) \) for \( d, d' \geq d(v) \), then 
\[(T(d), T(d(v))) \in S(V) \text{ and } (T(d(v)), T(d')) \in S(V), \]
which implies that \( \phi \neq T(d) \times T(d(v)) \subseteq V \) and \( \phi \neq T(d(v)) \times T(d') \subseteq V \). Therefore, \( T(d) \times T(d') \subseteq V \cup V \subseteq U \). If we let \( d(U) = d(V) \), the result then follows.

**IV.3 Lemma.** If \( I \neq \emptyset \) and \( X_\alpha \neq \emptyset \) for each \( \alpha \in I \), then there exists an \( f : I \to \bigcup \{ X_\alpha : \alpha \in I \} \) with the following properties: (1) \( f(\alpha) \in X_\alpha \) for each \( \alpha \in I \) and (2) \( f(\alpha) = f(\beta) \) whenever \( X_\alpha = X_\beta \) for \( \alpha, \beta \in I \).

**Proof.** Define \( \alpha \sim \beta \) if and only if \( X_\alpha = X_\beta \) for \( \alpha, \beta \in I \). Then \( \sim \) is an equivalence relation on \( I \). Define \( g : I \to I/\sim \) by \( g(\alpha) = [\alpha] = \{ \beta : \alpha \sim \beta \} \). Define \( k : I/\sim \to I \) by \( k([\alpha]) = \beta \) where \( \beta \in [\alpha] \). The Axiom of Choice implies such a well defined \( k \) exists. Define \( h : I/\sim \to \bigcup \{ X_\alpha : \alpha \in I \} \) by \( h([\alpha]) \in X_k([\alpha]) \).

Let \( f = h \circ g \). Then \( f \) clearly satisfies (1) and (2).

**IV.4 Theorem.** If \( (X, \mathcal{U}) \) is complete, then \( (X^*, \mathcal{U}^*) \) is complete.

**Proof.** Note that it is sufficient to show that if \( T : D \to X^* \) is \( \mathcal{U}^* \)-cauchy, then \( T \) converges with respect to \( \mathcal{J}(\mathcal{U}^*) \).

**Case 1.** Suppose there exists \( A \in X^* \) such that \( T(d) = A \) frequently, then \( T \) converges to \( A \).
Case 2. Assume that for each $A \in X^* T(d)$ does not equal $A$ frequently. In particular, there exist a $d(\phi) \in D$ such that $T(d) \neq \phi$ whenever $d \geq d(\phi)$. Define $D(\phi) = \{ d : d \in D$ and $d \geq d(\phi) \}$; and define $P : D(\phi) \to X$ by $P(d) \in T(d)$ for every $d \in D(\phi)$ and further that $P(d) = P(d')$, whenever $T(d) = T(d')$ with $d, d' \in D(\phi)$. Then by Lemma IV.3 $P$ is well-defined. We assert that $P$ is $\mathcal{U}$-cauchy. By Lemma IV.2 for each $U \in \mathcal{U}$ there exists $d(U) \in D$, $d(U) \geq d(\phi)$, such that either

1. $T(d) = T(d')$ for all $d, d' \geq d(U)$ or
2. $\phi \neq T(d) \times T(d') \subseteq U$

for all $d, d' \geq d(U)$. Thus either $P(d) = P(d')$ or $\{P(d')\} \times \{P(d')\} \subseteq U$ whenever $d, d' \geq d(U)$. Therefore, in either case, we have $(P(d), P(d')) \in U$ whenever $d, d' \geq d(U)$. This implies $P$ is $\mathcal{U}$-cauchy as was claimed. Now let $V \in \mathcal{U}$. There exists a symmetric $U \in \mathcal{U}$ such that $U \circ U \subseteq V$. Since $(X, U)$ is complete and $P$ is $\mathcal{U}$-cauchy, there exists an $x \in X$ such that $P$ converges to $x$. By the preceding remarks there exists $d(U) \in D$, $d(U) \geq d(\phi)$, such that $(T(d), T(d')) \in S(U)$, whenever $d, d' \geq d(U)$. Since $P$ converges to $x$, there is a $d_1 \in D(\phi)$, $d_1 \geq d(U)$, such that $(x, P(d)) \in U$ whenever $d \geq d_1$, $d \in D(\phi)$. Since $T(d) = A$ does not hold frequently for any $A$, it cannot hold eventually; therefore, by Lemma IV.2 there exists $d_2 \geq d_1$, $d_2 \in D(\phi)$, such that $T(d) \times T(d') \subseteq U$, whenever $d, d' \geq d_2$. Now let $d \geq d_2$; then $\phi \neq T(d) \times T(d_2) \subseteq U$, which implies $T(d) \times \{P(d_2)\} \subseteq U$. Also since $d_2 \geq d_1$, we have
3k \geq d_2$. But this implies $T(d) \in S(V)[\{x\}]$ whenever $d \geq d_2$; and since $S(V)[\{x\}]$ is a basic neighborhood for $\{x\}$ in $X^*$, it follows then that $T$ converges to $\{x\}$.

From the preceding results we obtain the following:

**IV.5 Theorem.** $(X^*,\mathcal{U}^*)$ is complete iff $(X,\mathcal{U})$ is complete.

**IV.6 Example.** Isbell has shown that if $\mathcal{U}$ is fine, then $\mathcal{U}$ is complete if and only if $\mathcal{F}(\mathcal{U})$ is paracompact. Let $(X,\mathcal{F})$ denote the half-open interval space, and let $\mathcal{U}$ be the fine uniformity that generates $\mathcal{F}$ (i.e. $\mathcal{F}(\mathcal{U}) = \mathcal{F}$). Then it is well-known that $\mathcal{F}$ is paracompact, and hence, $\mathcal{U}$ is complete. However, $(X \times X, \mathcal{F} \times \mathcal{F})$ is the half-open rectangle space which is not paracompact. Let $\mathcal{V}$ be the fine uniformity for $\mathcal{F} \times \mathcal{F}$. Since $\mathcal{F} \times \mathcal{U} \subseteq \mathcal{V}$ and $\mathcal{F}(\mathcal{U} \times \mathcal{U}) = \mathcal{F}(\mathcal{V})$ and $\mathcal{U} \times \mathcal{U}$ is complete, it then follows easily that $\mathcal{V}$ is complete. This then gives an example of a complete uniformity $\mathcal{V}$ for which $\mathcal{V}$ is not complete. Furthermore, it follows that $(X^*,\mathcal{U})$ complete is not sufficient for $((X \times X)^*, \mathcal{U} \times \mathcal{U})$ to be complete.

We now characterize $\mathcal{U}$ total boundedness.

**IV.7 Theorem.** If $\mathcal{U} \in \mathcal{U}(X)$, then $\mathcal{U}^*$ is totally bounded iff either (1) $X$ is finite or (2) $\mathcal{U} = \{x \times x\}$.

**Proof.** If (1) holds, then $X^*$ is finite, and if (2) holds, then $X^* = S(U)[\{X,\emptyset\}]$ for every $U \in \mathcal{U}$. Therefore, in either
case, \(\mathcal{U}^*\) is totally bounded. Conversely, if \(\mathcal{U}^*\) is totally bounded, then it suffices to show not (1) implies (2). Let \(\mathcal{U}^*\) be totally bounded, and assume \(X\) is infinite. Let \(W \in \mathcal{U}\). Then there exists a symmetric \(U \in \mathcal{U}\) such that \(U \circ U \circ U \circ U \subseteq W\).

Since \(\mathcal{U}^*\) is totally bounded, there exists non-empty \(A_1, A_2, \ldots, A_n\) in \(X^*\) so that \(S(U)[A_1] \cup S(U)[A_2] \cup \ldots \cup S(U)[A_n] = X^* \setminus \{\emptyset\}\).

Now let \(x, y \in X\). Since \(X\) is infinite, there exists \(z \in X\) such that \(\{x, z\} \notin A_k\) and \(\{y, z\} \notin A_j\) for \(1 \leq j < k \leq n\).

For some \(k_0\) and some \(j_0\), we have \(\{x, z\} \in S(U[A_{k_0}]\) and \(\{y, z\} \in S(U[A_{j_0}]\). Thus \(A_{k_0} \times \{x, z\} \subseteq U\) and \(A_{j_0} \times \{y, z\} \subseteq U\).

But this implies \((x, z) \in \{x, z\} \times \{x, z\} \subseteq U \circ U\), and \((z, y) \in \{y, z\} \times \{y, z\} \subseteq U \circ U\). Therefore \((x, y) \in U \circ U \circ U \circ U \subseteq W\), which implies \(X \times X \subseteq U \circ U \circ U \circ U \subseteq W\). This means \(\mathcal{U} = \{X \times X\}\) as was claimed.

It is known that \(\hat{\mathcal{U}}\) is totally bounded if and only if \(\mathcal{U}\) is totally bounded; and \(\mathcal{I}(\hat{\mathcal{U}})\) is compact if and only if \(\mathcal{I}(\mathcal{U})\) is compact. Theorem IV.7 implies \(\mathcal{I}(\mathcal{U}^*)\) is compact only when \((X, \mathcal{U})\) is trivial. However, \(\mathcal{I}(\mathcal{U})\) compact does give some positive information about \(\mathcal{U}^*\). Namely,

**IV.8 Theorem.** If \(\mathcal{I}(\mathcal{U})\) is compact, then \(\mathcal{U}^*\) is fine.

**Proof.** We shall employ the following characterization of fineness for a uniformity: \(\mathcal{U}\) is a fine uniformity for \(X\) iff
$f : (X,\mathcal{U}) \to (Y,\mathcal{V})$, whenever $f : (X,F(\mathcal{U})) \to (Y,F(\mathcal{V}))$. Let $F : (X^*,F(\mathcal{U}^*)) \to (Y,F(\mathcal{V}))$. Define $f : X \to Y$ by $f = F \circ i$.

Then $f : (X,F(\mathcal{U})) \to (Y,F(\mathcal{V}))$, since both $F$ and $i$ are continuous; and since $F(\mathcal{U})$ is compact, we have $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$.

Now let $V' \in \mathcal{V}$. There is a symmetric $V \in \mathcal{V}$ such that $V \circ V \circ V \circ V = V'$. Since $f$ is uniformly continuous, $(f \times f)^{-1} V \in \mathcal{U}$. $F$ is continuous at each $\{x\}$ for every $x \in X$; hence there exists $U'(x,V) \in \mathcal{U}$ such that $F[S(U'(x,V))\{x\}] \subseteq V[F(\{x\})]$. Also $U(x,V) \circ U(x,V) \subseteq U'(x,V)$ for some symmetric $U(x,V) \in \mathcal{U}$. By compactness there exist a finite number of points $x_1, x_2, \ldots, x_k$ in $X$ such that $X = \bigcup\{U(x_n,V)\{x_n\} : 1 \leq n \leq k\}$. Let $W = \cap\{U(x_n,V) : 1 \leq n \leq k\} \cap (f \times f)^{-1} V$. Then $W \in \mathcal{U}$ and $W = W^{-1}$. We now assert that for all $x \in X$, $F[S(W)\{x\}] \subseteq V \circ V[F(\{x\})]$. Let $A \in S(W)\{x\}$. Then by Lemma II.5 $\emptyset \neq A \subseteq W(x)$. Pick $a \in A$. Then $(x,a) \in W \subseteq U(x_n,V)$, $1 \leq n \leq k$ and $(x_j,x) \in U(x_j,V)$ for some $j$, $1 \leq j \leq k$. Thus $(x_j,a) \in U(x_j,V) \subseteq U'(x_j,V)$ for all $a \in A$. Therefore, $\{x_j\} \times A \subseteq U'(x_j,V)$, which implies $A \in S(U'(x_j,V))[\{x_j\}]$. Then $F(A) \in V[F(\{x_j\})]$. Also since $\{x_j\} \times \{x\} \subseteq U(x_j,V)$, we have $\{x\} \in S(U'(x_j,V))[\{x_j\}]$, which implies $F(\{x\}) \in V[F(\{x_j\})]$. Finally, we have $(F(\{x\}), F(A)) \in V \circ V$, since $(F(\{x_j\}), F(A)) \in V$ and $(F(\{x\}), F(\{x_j\})) \in V^{-1} = V$. We now assert that $S(W) \subseteq (F \times F)^{-1} V'$. Let $(A,B) \in S(W)$. Then either $A = B$ or
\( \emptyset \neq A \times B \subseteq W \). If \( A = B \), then \( (F(A), F(B)) \in V' \). If
\( \emptyset \neq A \times B \subseteq W \), then choose \( a \in A \) and \( b \in B \). We have
\[ \{b\} \times A \subseteq W^{-1} = W, \{a\} \times B \subseteq W \text{ and } \{a\} \times \{b\} = \{(a, b)\} \subseteq W. \]
Then \( A \in S(W)[\{b\}] \) and \( B \in S(W)[\{a\}] \). But by the previous argument this implies \( F(A) \in V \circ V[F(\{b\})] = V \circ V[f(b)] \), and similarly, \( F(B) \in V \circ V[F(\{a\})] = V \circ V[f(a)] \). Also,
\((a, b) \in W \subseteq (f \times f)^{-1} V \) implies \( (f(a), f(b)) \in V \). Finally, we have \( (F(A), F(B)) \in V \circ V \circ V \circ V \subseteq V' \), hence
\( (F \times F)[S(W)] \subseteq V' \) and \( F : (X^\star, U^\star) \to (Y, V) \).
CHAPTER V

Point Valued Functions

In this chapter we shall study single-valued mappings from one uniform space to another uniform space, together with certain induced mappings on the hyperspaces. The basic result is that a necessary and sufficient condition that $f^*$ be continuous (respectively, uniformly continuous) is that $f$ be continuous (respectively, uniformly continuous).

For Lemmas V.1 through V.11 $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ will denote fixed uniform spaces and $f : X \to Y$. Recall that $f^* : X^* \to Y^*$ is defined by $f^*(A) = f[A] = \{ f(a) : a \in A \}$ for every $A \in X^*$.

V.1 Lemma. $S((f \times f)^{-1} V) \subseteq (f^* \times f^*)^{-1} S(V)$, whenever $V \in \mathcal{V}^0$.

Proof. $(A,B) \in S(f \times f)^{-1} V$ implies $A = B$ or $\emptyset \neq A \times B \subseteq (f \times f)^{-1} V$. This implies $A = B$ or $(f(a), f(b)) \in V$ for all $a \in A$ and all $b \in B$. Therefore, $f[A] \times f[B] \subseteq V$ or $f[A] = f[B]$; and hence, $\emptyset \neq f^*(A) \times f^*(B) \subseteq V$ or $f^*(A) = f^*(B)$. Therefore, $(f^*(A), f^*(B)) \in S(V)$, and so $(A, B) \in (f^* \times f^*)^{-1} S(V)$.

V.2 Lemma. If $f$ is injective, then $S((f \times f)^{-1} V) = (f^* \times f^*)^{-1} S(V)$ whenever $V \in \mathcal{V}^0$.
Proof. Let \((A, B) \in (f^* \times f^*)^{-1} S(V)\). Then either \(f^*(A) = f^*(B)\) or \(\emptyset \neq f^*(A) \times f^*(B) \subseteq V\). This implies either \(f[A] = f[B]\) or \(f[A] \times f[B] \subseteq V\). Since \(f\) is injective, if \(f[A] = f[B]\), then it follows that \(A = B\). If \(\emptyset \neq f[A] \times f[B] \subseteq V\), then \(\emptyset \neq A \times B = (f \times f)^{-1}[f[A] \times f[B]] \subseteq (f \times f)^{-1} V\). Equality holds in this case because \(f\) is injective. Therefore, either \(A = B\) or \(\emptyset \neq A \times B \subseteq (f \times f)^{-1} V\). This implies \((A, B) \in S((f \times f)^{-1} V)\). The result now follows from Lemma V.1.

V.3 Theorem. \(f : (X, U) \rightarrow (Y, \gamma)\) if and only if \(f^* : (X^*, U^*) \rightarrow (Y^*, \gamma^*)\).

Proof. Assume \(f : (X, U) \rightarrow (Y, \gamma)\). To prove \(f^*\) is uniformly continuous it suffices to show that \((f^* \times f^*)^{-1} S(V) \in U^*\), whenever \(V \in \gamma\). Let \(V \in \gamma\). Then \((f \times f)^{-1} V \in U\), and therefore, \(U \subseteq (f \times f)^{-1} V\) for some \(U \in U\). By Lemmas I.4 and V.1 \(S(U) \subseteq S((f \times f)^{-1} V) \subseteq (f^* \times f^*)^{-1} S(V)\), and hence \(S(U) \subseteq (f^* \times f^*)^{-1} S(V)\). This implies \((f^* \times f^*)^{-1} S(V) \in U^*\).

Conversely, assume \(f^* : (X^*, U^*) \rightarrow (Y^*, \gamma^*)\). We need only show that \((f \times f)^{-1} V \in U\), whenever \(V \in \gamma\). Let \(V \in \gamma\). Then \(S(U) \subseteq (f^* \times f^*)^{-1} S(V)\) for some \(U \in U\). We assert \(U \subseteq (f \times f)^{-1} V\). Let \((x, y) \in U\). Then \(\emptyset \neq \{x\} \times \{y\} \subseteq U\) implies \(\{x, y\} \subseteq S(U)\). Therefore, \((f^*([x]), f^*([y])) \in S(V)\). But \(f^*([x]) = \{f(x)\}\) and \(f^*([y]) = \{f(y)\}\), and it follows that \((\{f(x)\}, \{f(y)\}) \in S(V)\). This implies \(\{f(x)\} \subseteq \{f(y)\}\) or
\[ \emptyset \not\subset \{f(x)\} \times \{f(y)\} \subseteq V. \] Then, in either case, we have
\[(f(x), f(y)) \in V, \] which implies \((x, y) \in (f \times f)^{-1} V. \] Thus
\[ U \subseteq (f \times f)^{-1} V. \]

**V.4 Theorem.** \( f : (X, \mathcal{U}(U)) \rightarrow (Y, \mathcal{V}(\gamma')) \) iff
\[ f^* : (X^*, \mathcal{U}(U^*)) \rightarrow (Y^*, \mathcal{V}(\gamma^*)) \] .

**Proof.** Assume \( f^* : (X^*, \mathcal{U}(U^*)) \rightarrow (Y^*, \mathcal{V}(\gamma^*)) \) . It suffices to show that for each \( x \in X \) and \( V \in \mathcal{V}' \), that \( f^{-1}[V[f(x)]] \)
is a \( \mathcal{U}(U) \) neighborhood for \( x \) . Let \( x \in X \) and \( V \in \mathcal{V}' \) be fixed.

Then \( S(V)[f^*({x})] \) is a neighborhood of \( f^*({x}) \) , which implies
\[ f^*[S(U)[{x}] \subseteq S(V)[f^*({x})] \] for some \( U \in \mathcal{U} \) . Let \( y \in U[x] \) .
Then \( \{y\} \in S(U)[{x}] \) , which implies
\[ f^*({y}) = \{f(y)\} \subseteq S(V)[f^*({x})] = S(V)[{f(x)}] \]. Therefore, either
\[ \{f(y)\} = \{f(x)\} \] or \[ \emptyset \not\subset \{f(x)\} \times \{f(y)\} \subseteq V. \] In either case
\( (f(x), f(y)) \in V \) , hence, \( f(y) \in V[f(x)] \). This implies
\[ U[x] \subseteq f^{-1}[V[f(x)]] ; \] thus, \( f : (X, \mathcal{U}(U)) \rightarrow (Y, \mathcal{V}(\gamma')) \) .

Conversely, assume \( f : (X, \mathcal{U}(U)) \rightarrow (Y, \mathcal{V}(\gamma')) \) . Since \( f^*(\emptyset) = \emptyset \)it is clear that \( f^* \) is continuous at \( \emptyset \) . Now let \( \emptyset \not\subset A \in X^* \) .
We assert that \( f^* \) is continuous at \( A \) . Fix \( x \in A \) and \( V_1 \subseteq \mathcal{V}' \).
Then \( V \cup V \subseteq V_1 \) for some symmetric \( V \in \mathcal{V}' \) . Since \( f \) is continuous
at \( x \) , \( f[U_1[x]] \subseteq V[f(x)] \) for some \( U_1 \in \mathcal{U} \) . Furthermore, there
exists a symmetric \( U \in \mathcal{U} \) such that \( U \cup U \subseteq U_1 \) . We assert that
\[ f^*[S(U)[A]] \subseteq S(V_1)[f^*(A)] \] . Let \( B \in S(V)[A] \) . Then \( A = B \) or
\( \emptyset \not\subset A \times B \subseteq U \) . If \( A = B \) , then \( f^*(A) = f^*(B) \) and certainly
\( f^*(B) \in S(V_1)[f^*(A)] \). Suppose \( A \neq B \). It will suffice to show that if \( a \in A \) and \( b \in B \), then \( (f(a), f(b)) \in V_1 \); for it would then follow that \( \emptyset \neq f[A] \times f[B] \subseteq V_1 \), and hence

\( \emptyset \neq f^*(A) \times f^*(B) \subseteq V_1 \). Now \( A \neq B \) implies \( \emptyset \neq A \times B \subseteq U \), and \( \emptyset \neq B \times A \subseteq U^{-1} = U \). Therefore, \( \emptyset \neq A \times A \subseteq U \circ U \). Fix \( a \in A \) and \( b \in B \). Then \( (x, a) \in U \circ U \subseteq U_1 \) and \( (x, b) \in U \subseteq U_1 \). Therefore, \( a \in U_1[x] \) and \( b \in U_1[x] \), which implies \( f(a) \in V[f(x)] \) and \( f(b) \in V[f(x)] \). It follows then that \( (f(a), f(x)) \in V^{-1} = V \) and \( (f(x), f(b)) \in V \); thus \( (f(a), f(b)) \in V \circ V \subseteq V_1 \). This implies \( 0 \neq f[A] \times f[B] = f^*(A) \times f^*(B) \subseteq V_1 \). Therefore,

\[ f^*(B) \in S(V_1)[f^*(A)] \] and so \( f^* : (X^*, J(U^*)) \to (Y^*, J(Y^*)) \).

**V.5 Corollary.** If \( U \in \mathcal{U}(X) \) and \( U^* \) is fine, then \( U \) is fine.

**Proof.** We shall again employ the characterization for fine uniformities used in Theorem IV.8. Let \( \mathcal{U} \in \mathcal{U}(Y) \) and suppose \( f : (X, J(U)) \to (Y, J(\mathcal{U})) \). Then by Theorem V.4

\[ f^* : (X^*, J(U^*)) \to (Y^*, J(\mathcal{U}^*)) \]. But \( U^* \) fine implies \( f^* : (X^*, U^*) \to (Y^*, f^*(\mathcal{U})) \). Thus by Theorem V.3 \( f : (X, U) \to (Y, f^* \mathcal{U}) \). The result now follows.

**V.6 Remark.** Theorem IV.8 is a partial converse to V.5 for \( J(U) \) compact (in this case, \( U \) is very fine because it is unique) implies \( U^* \) is fine. It is a current conjecture that \( U \) fine is sufficient for \( U^* \) to be fine.
V.7 Theorem. \( f : (X, U) \to (Y, \nu) \) if and only if \( f^* : (X^*, \hat{U}) \to (Y^*, \hat{\nu}) \).


V.8 Lemma. If \( f^* : (X^*, \mathcal{J}(\hat{U})) \to (Y^*, \mathcal{J}(\hat{\nu})) \), then \( f : (X, \mathcal{J}(U)) \to (Y, \mathcal{J}(\nu)) \).

Proof. Let \( x \in X \) and \( V \in \nu \). It suffices to show that \( f^{-1}[V[f(x)]] \) is a \( \mathcal{J}(U) \) neighborhood of \( x \). Since \( f^* \) is continuous at \( \{x^2\} \), there exists a symmetric \( U \in \mathcal{U} \) such that \( f^*[H(U)[\{x^2\}] \subseteq H(V)[f^*(\{x^2\})] \). Let \( y \in U[x] \). Then \( \{y^2\} \in H(U)[\{x^2\}] \) which implies \( f^*(\{y^2\}) = \{f(y)^2 \in H(V)[\{f(x)^2\}] \). Thus \( f(y) \in V[f(x)] \), and hence \( f[U[x]] \subseteq V[f(x)] \).

V.9 Example. The following is an example in which \( f \) is continuous with respect to \( \mathcal{T}(U) \) and \( \mathcal{T}(\nu) \) but \( f^* \) is not continuous with respect to \( \mathcal{T}(\hat{U}) \) and \( \mathcal{T}(\hat{\nu}) \). Let \( X \) be the real line with the usual uniformity. Define \( f : X \to X \) by \( f(x) = x^2 \). Then \( f \) is continuous but not uniformly continuous. Let \( V_1 = \{(x, y) : |x - y| < 1\} \). Let \( A \) be the set of natural numbers. We assert that for no \( \varepsilon > 0 \) does \( f^*[H(U_\varepsilon)[A]] \subseteq H(V_1)[f^*(A)] \). Recall \( U_\varepsilon = \{(x, y) : |x - y| < \varepsilon\} \). Fix \( \varepsilon > 0 \). Let \( B = \{y : y = n \text{ for } n \text{ a natural number and } n < 1/\varepsilon \text{ and } y = n + 1/n \text{ for } n > 1/\varepsilon\} \). Then \( B \in H(U_\varepsilon)[A] \). To see this let \( y \in B \), then \( y = n \) or \( y = n + 1/n \). If \( y = n \), then
If \( y = n + 1/n \), then since \( n \in A \), we have
\[ |n - y| = 1/n < \varepsilon. \]
Thus, \( B \in H(U_\varepsilon)[A] \). However, for large \( y \in B \) and any \( x \in A \),
\[ |f(x) - f(y)| = |x^2 - y^2 - 2 - 1/y^2| \geq 2 > 1 \]
so \( f(B) \notin H(V_1)[f(A)] \). It follows then that \( f^* \)
is not continuous with respect to \( \mathcal{F}(U) \) and \( \mathcal{F}(\gamma) \).

\textbf{V.10 Lemma.} If \( f^* : X^* \to Y^* \) is open with respect to \( \mathcal{F}(U^*) \)
and \( \mathcal{F}(\gamma^*) \), then \( f \) is open with respect to \( \mathcal{F}(U) \) and \( \mathcal{F}(\gamma) \).

\textbf{Proof.} Let \( 0 \in \mathcal{F}(U) \). It suffices to show that \( f[0] \in \mathcal{F}(\gamma) \).
If \( 0 = \emptyset \), then \( f[0] = \emptyset \). Therefore, we may assume \( 0 \neq \emptyset \).
Let \( x \in 0 \). Then \( U[x] \subseteq 0 \) for some \( U \in U \). \( S(U)[\{x_3\}] \) is a
\( \mathcal{F}(U^*) \) neighborhood of \( \{x_3\} \), which implies \( f^*[S(U)[\{x_3\}] \) is a
\( \mathcal{F}(\gamma^*) \) neighborhood of \( f^*(\{x_3\}) \), since \( f^* \) is open. Thus,
\[ S(V)[f^*(\{x_3\}] \subseteq f^*[S(U)[\{x_3\}] \] for some \( V \in \gamma \). Let \( y \in V[f(x)] \).
Then \( \{y_3\} \in S(V)[\{f(x)_3\}] \), which implies \( \{y_3\} \in f^*[S(U)[\{x_3\}] \).
Thus, there exists \( \emptyset \neq B \in X^* \) such that \( B \in S(U)[\{x_3\}] \) and
\( f^*(B) = \{y\} \). Let \( x' \in B \). Then \( f(x') = y \) and clearly
\( \{x'_3\} \in S(U)[\{x_3\}] \). But this implies \( x' \in U[x] \). Therefore,
\( y \in f[U[x]] \). It follows then that \( V[f(x)] \subseteq f[U[x]] \subseteq f[0] \) and
hence \( f[0] \in \mathcal{F}(\gamma) \).

\textbf{V.11 Lemma.} If \( f^* : X^* \to Y^* \) is uniformly open with respect
to \( U^* \) and \( \gamma^* \), then \( f \) is uniformly open with respect to \( U \)
and \( \gamma \).
Proof. Let \( U \in \mathcal{U} \). It suffices to show that there exists a \( V \in \mathcal{V} \) such that \( V[f(x)] \subseteq f[U[x]] \) for all \( x \in X \). Since \( f^* \) is uniformly open, there exists a \( V \in \mathcal{V} \) such that \( S(V)[f^*(A)] \subseteq f^*[S(U)[A]] \) for every \( A \in X^* \). Fix \( x \in X \). We assert that \( V[f(x)] \subseteq U[f(x)] \). If \( y \in V[f(x)] \), then \( \{y\} \in S(V)[\{f(x)\}] = S(V)[f^*(\{x\})] \subseteq f^*[S(U)[\{x\}]] \). Thus, there exists \( A \in S(U)[\{x\}] \) such that \( f^*(A) = \{y\} \). Since \( A \neq \emptyset \), pick any \( a \in A \). Then \( f^*({a}) = \{f(a)\} = \{y\} \) implies \( f(a) = y \). Also \( \{a\} \in S(U)[\{x\}] \) implies \( a \in U[x] \), and hence, \( y = f(a) \in f[U[x]] \). The conclusion now follows.

V.12 Example. \( f \) uniformly continuous, uniformly open, and onto is not sufficient for \( f^* \) to be open with respect to \( J(\mathcal{U}^*) \) and \( J(\mathcal{V}^*) \). Let \( X = [-1,1] \), \( Y = [0,1] \), and let \( \mathcal{U} \) and \( \mathcal{V} \) be the usual metric induced uniformities for \( X \) and \( Y \), respectively. Define \( f : X \to Y \) by \( f(x) = |x| \). Then \( f \) is onto, uniformly continuous and uniformly open. The first two assertions are obvious and the last follows since \( f[U_\varepsilon[x]] = V_\varepsilon[f(x)] \). To see this let \( |x - x'| < \varepsilon \). Then \( ||x| - |x'|| \leq |x - x'| < \varepsilon \), and hence, \( |x'| \in V_\varepsilon[|x|] \). It follows that \( f[U_\varepsilon[x]] \subseteq V_\varepsilon[f(x)] \). Let \( y \in V_\varepsilon[|x|] \). Then \( |x - y| < \varepsilon \). But \( f \) is surjective so \( |x'| = y \) for some \( x' \in X \). Suppose \( x \leq 0 \), then either \( ||x| - |x'|| = |x - x'| \) or \( ||x| - |x'|| = |x - (-x')| \). Thus it follows that either \( x' \in U_\varepsilon[x] \) or \( -x' \in U_\varepsilon[x] \). A similar argument holds if \( x < 0 \), and hence \( V_\varepsilon[f(x)] \subseteq f[U_\varepsilon[x]] \). Let
A = \{-1, +1\} then \( f^*(A) = \{1\} \). We claim that \( S(U_1)[A] = \{A\} \).

Suppose not. Then there exists \( B \neq \emptyset \) such that \( \emptyset \neq A \times B \subseteq U_1 \).

Let \( b \in B \). Then \( A \times \{b\} \subseteq U_1 \) which implies \(|1 - b| < 1 \) and \(|-1 - b| < 1\). This would imply \( 2 = |1 - (-1)| < 1 + 1 = 2 \), a contradiction. We assert now that for every \( \varepsilon > 0 \), \( S(V_\varepsilon)[\{1\}] \neq \{1\} \).

This follows from the fact that if \(|1 - y| < \varepsilon\), then \( \{y\} \in S(V_\varepsilon)[\{1\}] \). Therefore \( f^* \) is not open.

**V.13 Lemma.** Let \( U \in \mathcal{U}(X) \), \( V \in \mathcal{U}(Y) \), and \( f : X \to Y \). If \( V^* \) is the quotient uniformity for \( X^* \), then \( V \) is the quotient uniformity for \( Y \).

**Proof.** Let \( W \in \mathcal{U}(Y) \), and assume \( f : (X, U) \to (Y, W) \). Then by Theorem V.3 \( f^* : (X^*, U^*) \to (Y^*, W^*) \). Since \( V^* \) is the quotient uniformity, it follows that \( W^* \subseteq V^* \).

Therefore, by Lemma I.13 \( W \subseteq V \), and hence, \( V \) is the quotient uniformity.

**V.14 Theorem.** Let \( U \in \mathcal{U}(X) \), \( V \in \mathcal{U}(Y) \), and \( f : X \to Y \). If \( U^* \) is the weak uniformity on \( X^* \) induced by \( f^* \) and \( V^* \), then \( U \) is the weak uniformity on \( X \) induced by \( f \) and \( V \).

**Proof.** Let \( W \in \mathcal{U}(X) \) and suppose \( f : (X, W) \to (Y, V) \). Then by Theorem V.3 \( f^* : (X^*, W^*) \to (Y^*, V^*) \), which implies \( U^* \subseteq W^* \), since \( U^* \) is minimal for \( f^* \) uniform continuity. Thus, by Lemma I.13 \( U \subseteq W \), and the result follows.
V.15 Lemma. Let \( V \in \mathcal{U}(Y) \) and \( f : X \to Y \) injective. If \( U \) is the weak uniformity on \( X \) induced by \( f \) and \( V \), then \( U^* \) is the weak uniformity on \( X^* \) induced by \( f^* \) and \( V^* \).

Proof. Let \( U^* \in U^* \) and suppose \( f^* : (X^*,U^*) \to (Y^*,V^*) \).

\( S(U) \subseteq U^* \) for some \( U \in U \). Then \( U \supseteq (f \times f)^{-1} V \) for some \( V \in V \), since \( U \) is the weak uniformity. Then

\( U^* \supseteq S(U) \supseteq S((f \times f)^{-1} V) = (f^* \times f^*)^{-1} S(V) \in W^* \). Therefore, \( U^* \subseteq W^* \).

V.16 Theorem. If \( U \in \mathcal{U}(X) \), and \( g : X^* \to 2^X \) as defined by \( g(A) = \mathcal{C}(A) \), then (1) \( g \) is onto, (2) \( g \) restricted to \( 2^X \) is the identity, (3) \( g : (X^*,U^*) \to (2^X,2^X \times 2^X \wedge U^*) \), and (4) \( (g \times g) S(U) \in 2^X \times 2^X \wedge U^* \), whenever \( U \in U \).

Proof. (1) and (2) are trivial. To prove (3) let \( U \in U \).

Then \( V \circ V \circ V \subseteq U \) for some symmetric \( V \in U \). We assert

\((g \times g)(S(V)) \subseteq 2^X \times 2^X \wedge S(U) \). Let \((A,B) \in S(V)\). Then \( A = B \) or \( \emptyset \neq A \times B \subseteq V \). If \( A = B \), then \( \mathcal{C}(A) = \mathcal{C}(B) \), which implies

\((\mathcal{C}(A),\mathcal{C}(B)) \in 2^X \times 2^X \wedge S(U) \). Assume \( \emptyset \neq A \times B \subseteq V \). Let

\( x \in \mathcal{C}(A) \) and \( y \in \mathcal{C}(B) \). Then \( a \in V[x] \) and \( b \in V[y] \) for some \( a \in A \) and some \( b \in B \). Therefore, \((x,a) \in V \), \((a,b) \in V \) and \((b,y) \in V^{-1} = V \). This implies \((x,y) \in V \circ V \circ V \subseteq U \). It follows then that \( \emptyset \neq \mathcal{C}(A) \times \mathcal{C}(B) \subseteq U \), hence,

\((\mathcal{C}(A),\mathcal{C}(B)) \in 2^X \times 2^X \wedge S(U) \). Therefore,

\( g : (X^*,U^*) \to (2^X,2^X \times 2^X \wedge U^*) \). To prove (4) we note that
$2^X \times 2^X \cap S(U) \subseteq (g \times g)(S(U))$, since $g \times g$ is the identity on $2^X \times 2^X$. The result follows.

**V.17 Lemma.** Let $U \in \mathcal{U}(X)$ and $h : X \to X^*$ be defined by $h(x) = c([x])$ for each $x \in X$. Then (1) $h : (X, \mathcal{U}) \to (X^*, \mathcal{U}^*)$, and (2) $(h \times h) U \in h[x] \times h[x] \cap \mathcal{U}^*$ for every $U \in \mathcal{U}$.

**Proof.** $h = g \circ i$ and the result follows from Theorem V.16 and Lemma III.12.
CHAPTER VI

Multi-Valued Functions

In this chapter we propose to investigate multi-valued mappings from one uniform space to another uniform space. The problem has extensive literature. Caulfield considered the problem in depth when the $\hat{U}$-uniformity is employed on the hyperspace. We shall use the $\ast$-uniformity developed previously and compare this to results obtained in the $\wedge$ case.

If one has a multi-valued mapping from one uniform space $(X, \mathcal{U})$ to another $(Y, \mathcal{V})$, then one obtains a natural single-valued function from $(X, \mathcal{U})$ to $(Y, \mathcal{V})$. Thus it indeed is sensible to speak of a multi-valued mapping as being continuous or uniformity continuous. We shall show that a multi-valued mapping $\psi : X \to Y^*$ is continuous (respectively, uniformly continuous) if and only if a certain family of functions from $X$ to $Y$ is equi-continuous (respectively, uniformly equi-continuous). That this does not hold if one replaces $\ast$ with $\wedge$ will be shown by example. We shall show further that in the case that $\hat{\mathcal{U}}(\mathcal{U})$ is connected and $\hat{\mathcal{V}}(\mathcal{V})$ is Hausdorff the only $\mathcal{U}^*$ continuous multi-valued mappings are the trivial ones. Finally, employing the fact that $\hat{\mathcal{U}} \subseteq \mathcal{U}^*$ and certain results obtained by Caulfield, we shall obtain a few further results about $\psi$.

Let $\psi : X \to Y^*$ with the qualification that $\psi(x) \neq \emptyset$ for
every \( x \in X \). Then we make the following,

\section*{VI.1 Definition.} \( F(\varphi) = \{ f : (1) f : X \to Y , (2) f(x) \in \varphi(x) \}
\text{for all} \ x \in X , \text{and (3)} f(x) = f(x') \ \text{whenever} \ \varphi(x) = \varphi(x') \} .

For example, suppose both \( X \) and \( Y \) are the real line, and suppose \( \varphi : X \to Y^* \) is defined by \( \varphi(x) = y \), for every \( x \in X \). Then, in this case \( F(\varphi) \) consists of precisely the collection of real-valued constant functions.

Now let \( U e U(X) \) and \( V e U(Y) \). Let \( F \) be a family of functions from \( X \) to \( Y \).

\section*{VI.2 Definition.} \( F \) is called equicontinuous at \( x \in X \) if for every \( V \in V \) there exists a \( U \in U \), \( U \) depending on both \( x \) and \( V \), such that \( f[U[x]] \subseteq V[f(x)] \) for every \( f \in F \).

\section*{VI.3 Definition.} \( F \) is called uniformly equicontinuous on \( X \) if for every \( V \in V \) there exists a \( U \in U \) such that \( U \subseteq (f \times f)^{-1} V \) for every \( f \in F \).

We note here that \( F \) is uniformly equicontinuous on \( X \) if and only if for every \( V \in V \), there exists a \( U \in U \) such that \( f[U[x]] \subseteq V[f(x)] \) for every \( f \in F \) and for every \( x \in X \).

\section*{VI.4 Lemma.} Let \( \varphi : X \to Y^* \) and suppose \( \varphi(x) \neq \emptyset \) for every \( x \in X \). Then for every \( x \in X \), \( \varphi(x) = \{ f(x) : f \in F(\varphi) \} \).
Proof. Fix \( z \in X \). Then \( f(z) \in \mathcal{V}(z) \) for every \( f \in \mathcal{F}(\mathcal{V}) \). Therefore, \( \{ f(z) : f \in \mathcal{F}(\mathcal{V}) \} \subseteq \mathcal{V}(z) \). Now define \( x \sim x' \) if and only if \( \mathcal{V}(x) = \mathcal{V}(x') \). Then \( \sim \) is clearly an equivalence relation on \( X \). Let \( g : X \to X/\sim \) be defined by \( g(x) = [x] = \{ x' : x' \sim x \} \). Pick \( y \in \mathcal{V}(z) \). Define \( k : X/\sim \to X \) by \( k([x]) = x' \) for some \( x' \in [x] \). Define \( F : X/\sim \to Y \) by \( F([z]) = y \in \mathcal{V}(z) \) and \( F([x]) \in \mathcal{V}(k([x])) \) arbitrary for \( [z] \neq [x] \). Let \( f = F \circ g \). Then \( f : X \to Y \); and \( f(x) = F \circ g(x) = F([x]) = \mathcal{V}(F([x])) = \mathcal{V}(x) \), since \( k([x]) \in [x] \).

Also, \( f(z) = F \circ g(z) = F([z]) = y \in \mathcal{V}(k[z]) = \mathcal{V}(z) \). If \( \mathcal{V}(x) = \mathcal{V}(x') \), then \( f(x) = F \circ g(x) = F([x]) = F([x']) = F \circ g(x') = f(x') \), since \( [x] = [x'] \). Therefore \( f \in \mathcal{F}(\mathcal{V}) \); and hence, \( y \in \{ f(z) : f \in \mathcal{F}(\mathcal{V}) \} \). This implies \( \mathcal{V}(z) \subseteq \{ f(z) : f \in \mathcal{F}(\mathcal{V}) \} \).

VI.5 Theorem. Let \( \mathcal{U} \in \mathcal{U}(X) \) and \( \mathcal{V} \in \mathcal{U}(Y) \). Suppose \( \mathcal{V} : X \to Y^* \) such that \( \mathcal{V}(x) \neq \emptyset \) for every \( x \in X \). Let \( z \in X \).

Then \( \mathcal{V} \) is continuous at \( z \) with respect to \( \mathcal{F}(\mathcal{U}) \) and \( \mathcal{F}(\mathcal{V}^*) \) if and only if \( \mathcal{F}(\mathcal{V}) \) is equicontinuous at \( z \) with respect to \( \mathcal{F}(\mathcal{U}) \) and \( \mathcal{V} \).

Proof. Assume \( \mathcal{V} \) is continuous at \( z \) with respect to \( \mathcal{F}(\mathcal{U}) \) and \( \mathcal{F}(\mathcal{V}^*) \). Fix \( V \in \mathcal{V} \). Then \( \mathcal{V}[U[z]] \subseteq S(V)[\mathcal{V}(z)] \) for some \( U \in \mathcal{U} \). Let \( x \in U[z] \). Then \( \mathcal{V}(x) \in S(V)[\mathcal{V}(z)] \), which implies \( \mathcal{V}(x) = \mathcal{V}(z) \) or \( \mathcal{V}(z) \cap \mathcal{V}(x) \subseteq V \). If \( \mathcal{V}(x) = \mathcal{V}(z) \), then \( f(x) = f(z) \) for every \( f \in \mathcal{F}(\mathcal{V}) \). This implies \( f(x) \in V[f(z)] \) for every \( f \in \mathcal{F}(\mathcal{V}) \). If \( \mathcal{V}(z) \cap \mathcal{V}(x) \subseteq V \), then...
\((f(z), f(x)) \in \psi(z) \times \psi(x) \subseteq V\) for every \(f \in \mathcal{F}(\psi)\). And again we have \(f(x) \in V[f(z)]\) for every \(f \in \mathcal{F}(\psi)\). It follows then that \(f[U[z]] \subseteq V[f(z)]\) for every \(f \in \mathcal{F}(\psi)\); and thus, \(\mathcal{F}(\psi)\) is equicontinuous at \(z\) with respect to \(\mathcal{J}(U)\) and \(V\).

Conversely, suppose \(\mathcal{F}(\psi)\) is equicontinuous at \(z\) with respect to \(\mathcal{J}(U)\) and \(V\). Fix \(V \in \mathcal{V}\). To prove \(\psi\) is continuous at \(z\) it suffices to show that \(\psi^{-1}[S(V)[\psi(z)]]\) is a \(\mathcal{J}(U)\) neighborhood of \(z\). There exists a \(U \in \mathcal{U}\) such that \(f[U[z]] \subseteq V[f(z)]\) for every \(f \in \mathcal{F}(\psi)\). Let \(x \in U[z]\). We assert that \(\psi(x) \in S(V)[\psi(z)]\). If \(\psi(x) = \psi(z)\), then certainly \(\psi(x) \in S(V)[\psi(z)]\). Suppose now that \(\psi(z) \neq \psi(x)\). Pick \(y \in \psi(z)\) and \(y' \in \psi(x)\). Define \(\sim\), \(g\), and \(k\) as in Lemma VI.4. Let \(H : X/\sim \rightarrow Y\) be defined by \(H([z]) = y \in \psi(z) = \psi(k([z]))\), \(H([x]) = y' \in \psi(x) = \psi(k([x]))\), and \(H([x']) = \psi(k([x']))\) arbitrary for \([x] \neq [x'] \neq [z]\). Define \(f : X \rightarrow Y\) by \(f = H \circ g\). Then \(f(w) = H \circ g(w) = H([w]) \in \psi(k([w]) = \psi(w))\) for every \(w \in X\). Also if \(\psi(x) = \psi(x')\), then \([x] = [x']\) which implies \(f(x) = f(x')\). Thus \(f \in \mathcal{F}(\psi)\). Now \(x \in U[z]\) implies \(f(x) \in V[f(z)]\). Therefore, \((y, y') = (f(z), f(x)) \in V\). This implies \(\psi(z) \times \psi(x) \subseteq V\). It follows then that \(\psi(x) \in S(V)[\psi(z)]\); and hence, \(\psi[U[z]] \subseteq S(V)[\psi(z)]\). Thus, \(\psi\) is continuous at \(z\) with respect to \(\mathcal{J}(U)\) and \(\mathcal{J}(Y^*)\) as was to be shown.

VI.6 Theorem. Let \((X, U)\), \((Y, V)\) and \(\psi\) be as in the preceding theorem. Then \(\psi : (X, U) \rightarrow (Y^*, V^*)\) if and only if \(\mathcal{F}(\psi)\) is a uniformly equicontinuous family.
Proof. Assume \( \varphi : (X, \mathcal{U}) \rightarrow (Y^*, \mathcal{F}^*) \). Fix \( V \in \mathcal{V} \). There exists a \( U \in \mathcal{U} \) such that \( U \subseteq (\varphi \times \varphi)^{-1} \cdot S(V) \). Let \((x, x') \in U\). Then \((\varphi(x), \varphi(x')) \in S(V)\). This implies \( \varphi(x) = \varphi(x') \) or \( \varphi \neq \varphi(x) \times \varphi(x') \subseteq V \). If \( \varphi(x) = \varphi(x') \), then \( f(x) = f(x') \) for every \( f \in \mathcal{F}(\varphi) \); and hence \((f(x), f(x')) \in V \) for every \( f \in \mathcal{F}(\varphi) \). If \( \varphi(x) \times \varphi(x') \subseteq V \), then \((f(x), f(x')) \in \varphi(x) \times \varphi(x') \) for every \( f \in \mathcal{F}(\varphi) \). This implies \((f(x), f(x')) \in V \) for every \( f \in \mathcal{F}(\varphi) \). Therefore, \( U \subseteq (f \times f)^{-1} \cdot V \) for every \( f \in \mathcal{F}(\varphi) \); hence, \( \mathcal{F}(\varphi) \) is uniformly equicontinuous with respect to \( U \) and \( V \).

Conversely, assume \( \mathcal{F}(\varphi) \) is uniformly equicontinuous with respect to \( U \) and \( V \). Fix \( V \in \mathcal{V} \). Then there exists a \( U \in \mathcal{U} \) such that \( (f \times f) U \subseteq V \), for every \( f \in \mathcal{F}(\varphi) \). We assert that \( U \subseteq (\varphi \times \varphi)^{-1} \cdot S(V) \). Let \((x, x') \in U \). If \( \varphi(x) = \varphi(x') \), then \((\varphi(x), \varphi(x')) \in S(V) \). Suppose \( \varphi(x) \neq \varphi(x') \). Let \((y, y') \in \varphi(x) \times \varphi(x') \). We now employ the technique of the previous proof to obtain an \( f \in \mathcal{F}(\varphi) \) with the property that \( f(x) = y \) and \( f(x') = y \). Then \((x, x') \in U \) implies \((y, y') = (f(x), f(x')) \in V \). It follows then that \( \varphi(x) \times \varphi(x') \subseteq V \), and hence, \((\varphi(x), \varphi(x')) \in S(V) \). Thus, \( U \subseteq (\varphi \times \varphi)^{-1} \cdot S(V) \) as was claimed. Therefore, \( \varphi : (X, \mathcal{U}) \rightarrow (Y^*, \mathcal{F}^*) \).

VI.7 Example. We now construct an example to show that \( \varphi : (X, \mathcal{U}) \rightarrow (Y^*, \mathcal{F}^*) \) is not sufficient for \( \mathcal{F}(\varphi) \) to be even a family of continuous functions. Let \( I = [0,1] \). Suppose \( \varphi : I \rightarrow I^* \) is defined by \( \varphi(x) = \{x, 1-x\} \). Let \( \mathcal{U} \) be the usual metric induced
uniformity on $I$. We assert $\psi: (I, U) \to (I^*, \hat{U})$. Note that $|x-x'| < \varepsilon$ if and only if $|1-x-(1-x')| < \varepsilon$. Let $V_\varepsilon = \{(x, x') : |x-x'| < \varepsilon\}$ for any $\varepsilon > 0$. Thus, $(x, x') \in V_\varepsilon$ if and only if $(1-x, 1-x') \in V_\varepsilon$. It follows then that $\psi[V_\varepsilon[x]] \subseteq H(V_\varepsilon)[\psi(x)]$ for every $x \in I$ and every $\varepsilon > 0$. Thus, $\psi$ is uniformly continuous. Note that $\psi(x) = \psi(x')$ if and only if $x = x'$ or $x = 1-x'$. Let $f: I \to I$ be defined by the following:

$$f(x) = \begin{cases} 
  x, & x \text{ rational and } 0 \leq x \leq 1/2 \\
  1-x, & x \text{ irrational and } 0 \leq x \leq 1/2 \\
  1-x, & x \text{ rational and } 1/2 \leq x \leq 1 \\
  x, & x \text{ irrational and } 1/2 \leq x \leq 1 
\end{cases}$$

Then, $f(1-x) = \begin{cases} 
  1-(1-x) = x, & x \text{ rational and } 0 \leq x \leq 1/2 \\
  1-x, & x \text{ irrational and } 0 \leq x \leq 1/2 \\
  1-x, & x \text{ rational and } 1/2 \leq x \leq 1 \\
  1-(1-x) = x, & x \text{ irrational and } 1/2 \leq x \leq 1 
\end{cases}$

Thus for every $x \in I$, $f(x) = f(1-x)$. Also, $f(x) \in \psi(x)$ for every $x \in I$. It follows then that $f \in \mathcal{F}(\psi)$. However, $f$ is continuous only at $x = 1/2$; hence $\mathcal{F}(\psi)$ is not a family of continuous functions.
The converse does obtain as the following theorem shows.

**VI.8 Theorem.** Let \( U \in \mathcal{U}(X) \) and \( V \in \mathcal{U}(Y) \). Suppose \( \varphi : X \rightarrow Y^* \) such that \( \varphi(x) \neq \emptyset \) for every \( x \in X \). If \( J(\varphi) \) is equicontinuous at \( z \in X \) (respectively, uniformly equicontinuous on \( X \)), then \( \varphi \) is continuous at \( z \) with respect to \( J(U) \) and \( J(V) \) (respectively, \( \varphi : (X,U) \rightarrow (Y^*,V) \)).

**Proof.** This follows immediately from Theorems VI.5 and VI.6, and the fact that \( V \subseteq Y^* \) and \( J(V) \subseteq J(V^*) \).

The preceding results give a satisfactory characterization for \( \varphi : X \rightarrow Y^* \) to be continuous or uniformly continuous. However, if the domain is locally connected and if the range is Hausdorff, then an even stronger characterization obtains as the following theorems show.

**VI.9 Theorem.** Let \( U \in \mathcal{U}(X) \) and \( V \in \mathcal{U}(Y) \). Assume further that \( J(U) \) is connected and \( J(V) \) is Hausdorff. If 
\[ \varphi : (X,J(U)) \rightarrow (Y^*,J(V^*)) \]
then either (1) \( \varphi \) is constant, or (2) \( \text{card } \varphi(x) = 1 \) for every \( x \in X \).

**Proof.** Case 1. Suppose there exists an \( x \in X \) such that 
\[ \varphi(x) = \emptyset \]. By Lemma I.8 \( \{ \emptyset \} \) is both open and closed with respect to \( J(V^*) \). But \( \varphi \) is continuous, which implies \( \varphi^{-1}[[\emptyset]] \) is both \( J(U) \)-open and \( J(U) \)-closed. Also, \( x \in \varphi^{-1}[[\emptyset]] \) implies 
\[ \varphi^{-1}[[\emptyset]] \neq \emptyset \). Therefore, \( \varphi^{-1}[[\emptyset]] = X \) as \( J(U) \) is connected.
It follows then that \( \varphi \) is constant.
Case 2. Suppose \( \varphi(x) \neq \emptyset \) for every \( x \in X \). It suffices to show that not (2) implies (1). Therefore, we assume that there exists an \( a \in X \) such that \( \text{card } \varphi(a) \geq 2 \). Pick \( y \neq y' \) such that \( y, y' \in \varphi(a) \). Since \( T(\varphi) \) is Hausdorff, there exists a \( W \in \mathcal{V} \) such that \( W[y] \cap W[y'] = \emptyset \). Also, \( V \circ V \subseteq W \) for some symmetric \( V \in \mathcal{V} \). Note that \( \varphi^{-1}[\varphi(a)] \neq \emptyset \). Also, by Corollary II.3 \( T(\varphi) \) Hausdorff implies \( T(\varphi^*) \) is Hausdorff; thus \( \{ \varphi(a) \} \) is \( T(\varphi^*) \)-closed. This implies \( \varphi^{-1}[\{ \varphi(a) \}] \) is \( T(\mathcal{U}) \) closed since \( \varphi \) is continuous. We assert now that \( \varphi^{-1}[\{ \varphi(a) \}] \subseteq T(\mathcal{U}) \). Let \( b \in \varphi^{-1}[\{ \varphi(a) \}] \). Continuity implies that there exists a \( U \in \mathcal{U} \), \( U \) depending on both \( b \) and \( V \), such that \( \varphi(U[b]) \subseteq S(V)[\varphi(b)] \). We claim that \( \varphi(x) = \varphi(b) \) for every \( x \in U[b] \). Suppose not. Then for some \( x \in U[b] \), we have \( \varphi(x) \neq \varphi(b) \). But \( \varphi(x) \in S(V)[\varphi(b)] \). It follows then that \( \emptyset \neq \varphi(b) \times \varphi(x) \subseteq V \). Let \( z \in \varphi(x) \). Then \( (y, z) \in \varphi(b) \times \varphi(x) \subseteq V \) and \( (y', a) \in \varphi(b) \times \varphi(x) \subseteq V \), since \( y, y' \in \varphi(a) \) and \( \varphi(a) = \varphi(b) \). Therefore, \( (y, y') \in V \circ V^{-1} = V \circ V \subseteq W \). This would imply that \( y' \in W[y] \cap W[y'] \), a contradiction. Thus, if \( x \in U[b] \), then \( \varphi(x) = \varphi(b) \). It now follows that if \( b \in \varphi^{-1}[\{ \varphi(a) \}] \), then \( U[b] \subseteq \varphi^{-1}[\{ \varphi(a) \}] \) for some \( U \in \mathcal{U} \). Thus \( \varphi^{-1}[\{ \varphi(a) \}] \subseteq T(\mathcal{U}) \). Since \( T(\mathcal{U}) \) was assumed to be connected, we have \( \varphi^{-1}[\{ \varphi(a) \}] = X \), and hence \( \varphi \) is constant on \( X \).

We can actually strengthen the previous theorem as follows:
VI.10 Theorem. Let $U \in \mathcal{U}(X)$ and $\mathcal{V} \in \mathcal{U}(Y)$. If $\mathcal{F}(U)$ is connected, $\mathcal{F}(\mathcal{V})$ is Hausdorff, $\varphi : (X, \mathcal{F}(U)) \to (Y, \mathcal{F}(\mathcal{V}))$ is not constant, then there exists an $f : (X, \mathcal{F}(U)) \to (Y, \mathcal{F}(\mathcal{V}))$ such that $\varphi = i \circ f$ where $i : Y \to Y^*$ by $i(y) = \{ y \}$.

Proof. By the previous theorem $\operatorname{card} \varphi(x) = 1$ for every $x \in X$. Therefore, applying Theorem VI.5, $\mathcal{F}(\varphi)$ is an equicontinuous family of functions from $X$ to $Y$. Let $f \in \mathcal{F}(\varphi)$. If $g \in \mathcal{F}(\varphi)$, then $g(x) \in \varphi(x)$ for every $x \in X$, which implies $g(x) = f(x)$, since $\operatorname{card} \varphi(x) = 1$ for every $x \in X$. Therefore, $f = g$; and hence, $\mathcal{F}(\varphi) = \{ f \}$. We now apply Lemma VI.4 and obtain $\varphi(x) = \{ f(x) : f \in \mathcal{F}(\varphi) \} = \{ f(x) \} = i \circ f(x)$ for every $x \in X$.

VI.11 Theorem. Let $U \in \mathcal{U}(X)$ and $\mathcal{V} \in \mathcal{U}(Y)$. If $\mathcal{F}(U)$ is connected, $\mathcal{F}(\mathcal{V})$ is Hausdorff, $\varphi : (X, \mathcal{F}(U)) \to (Y, \mathcal{F}(\mathcal{V}))$ and if $\varphi$ is not constant, then there exists an $f : (X, \mathcal{F}(U)) \to (Y, \mathcal{F}(\mathcal{V}))$ such that $\varphi = i \circ f$, where $i : Y \to Y^*$ by $i(y) = \{ y \}$.

Proof. Card $\varphi(x) = 1$ for every $x \in X$ implies $\mathcal{F}(\varphi) = \{ f \}$. Also $\mathcal{F}(\varphi)$ is uniformly equicontinuous so $f : (X, \mathcal{F}(U)) \to (Y, \mathcal{F}(\mathcal{V}))$. Thus $\varphi(x) = \{ f(x) : f \in \mathcal{F}(\varphi) \} = \{ f(x) \} = i \circ f(x)$ for every $x \in X$.

VI.12 Corollary. Let $U \in \mathcal{U}(X)$ and $\mathcal{V} \in \mathcal{U}(Y)$ such that $\mathcal{F}(\mathcal{V})$ is Hausdorff. Suppose $\varphi : (X, \mathcal{F}(U)) \to (Y, \mathcal{F}(\mathcal{V}))$ (respectively, $\varphi : (X, \mathcal{F}(U)) \to (Y, \mathcal{F}(\mathcal{V}))$). If $D$ is any component of $X$, 

then either (1) \( \psi \) restricted to \( D \) is constant or (2) there exists an \( f : (D, D \cap \mathcal{U}) \to (Y, \mathcal{Y}(\gamma)) \) (respectively, \( f : (D, D \times D \cap \mathcal{U}) \to (Y, \gamma) \)) such that \( \psi \) restricted to \( D \) equals \( i \circ f \).

**Proof.** This follows immediately from the preceding results.

A partial converse to this result also holds as we shall now show.

**VI.13 Theorem.** Let \( U \in \mathcal{U}(X) \) and \( \gamma \in \mathcal{U}(Y) \) such that \( \mathcal{J}(U) \) is locally connected and \( \mathcal{J}(\gamma) \) is Hausdorff. If for every component \( D \) of \( X \) either \( \psi \) restricted to \( D \) is constant or \( \psi \) restricted to \( D \) equals \( i \circ f \) for some \( f : (D, D \cap \mathcal{J}(U)) \to (Y, \mathcal{J}(\gamma)) \), then \( \psi : (X, \mathcal{J}(U)) \to (Y^*, \mathcal{J}(\gamma^*)) \).

**Proof.** Fix \( x \in X \). Let \( D(x) \) denote the component of \( X \) containing \( x \). Then \( D(x) \) is both open and closed since \( X \) is locally connected. Now fix \( \gamma \in \mathcal{V} \). If \( \psi \) restricted to \( D(x) \) is constant, then \( \psi[D(x)] \subseteq S(\gamma)[\psi(x)] \). If \( \psi \) equals \( i \circ f \) for some \( f : (D, D \cap \mathcal{J}(U)) \to (Y, \mathcal{J}(\gamma)) \), then for some \( \gamma \in \mathcal{J}(U) \) such that \( x \in \gamma \) we have that \( f[0 \cap D(x)] \subseteq \gamma[f(x)] \). This implies that \( \psi[0 \cap D(x)] = i[f[0 \cap D(x)]] \subseteq i[\gamma[f(x)]] \subseteq S(\gamma)[f(x)] = S(\gamma)[\psi(x)] \). It follows then that \( \psi : (X, \mathcal{J}(U)) \to (Y^*, \mathcal{J}(\gamma^*)) \).

**VI.14 Example.** We shall now show that the previous theorem need not hold if the condition on local connectedness is relaxed.
Let $I = [0,1]$ and let $X = \{0, \frac{1}{k}, \frac{1}{2}, \frac{1}{3}, \ldots \}$ with the usual Euclidean metric. Let $\varphi : X \to I^*$ be defined by $\varphi\left(\frac{1}{n}\right) \times I = \left\{\frac{1}{n}\right\}$ and $\varphi\{0\} \times I = \{1\}$. Let $f : X \to I$ by $f(\frac{1}{k}, x) = \frac{1}{k}$ and $f(0, x) = 1$ for every $x \in I$. Then $\mathcal{F}(\varphi) = \{f\}$. But $f$ is not continuous which implies $\varphi$ is not continuous even though $\varphi$ restricted to all components is a single-valued constant.

We shall now develop some algebraic properties of multi-valued mappings. Suppose $\varphi : X \to Y^*$ and $\psi : Y \to Z^*$. Then we make the following

**VI.15 Definition.** $\bar{\varphi} : X^* \to Y^*$ by $\bar{\varphi}(A) = \bigcup\{\varphi(a) : a \in A\}$ and $\bar{\psi} \bar{\varphi} = \bar{\psi} \circ \bar{\varphi}$.

**VI.16 Lemma.** $\bar{\psi} \circ \bar{\varphi} = \bar{\psi} \circ \bar{\varphi} = \bar{\psi} \circ \bar{\varphi}$.

**Proof.** Let $A \in X^*$. Then $\bar{\psi} \circ \bar{\varphi}(A) = \bar{\psi}(\bar{\varphi}(A)) = \bigcup\{\psi(y) : y \in \bar{\varphi}(A)\} = \bigcup\{\psi(y) : y \in \bigcup\{\varphi(a) : a \in A\}\}$ = $\bigcup\{\psi(y) : y \in \varphi(a)\}$ for some $a \in A$ = $\bigcup\{\psi(y) : y \in \varphi(a)\} : a \in A\} = \bigcup\{\psi(\varphi(a)) : a \in A\} = \bigcup\{\psi \circ \varphi(a) : a \in A\} = \bar{\psi} \circ \bar{\varphi}(A)$.

**VI.17 Lemma.** If $\varphi_1 : X \to Y^*$, $\varphi_2 : Y \to Z^*$, $\varphi_3 : Z \to W^*$, then $\varphi_3 \circ (\varphi_2 \circ \varphi_1) = (\varphi_3 \circ \varphi_2) \circ \varphi_1$.

**Proof.** $\varphi_3 \circ (\varphi_2 \circ \varphi_1) = \varphi_3 \circ (\bar{\varphi}_2 \circ \varphi_1) = \bar{\varphi}_3 \circ (\bar{\varphi}_2 \circ \varphi_1) = \bar{\varphi}_3 \circ (\bar{\varphi}_2 \circ \varphi_1) = (\varphi_3 \circ \varphi_2) \circ \varphi_1 = (\varphi_3 \circ \varphi_2) \circ \varphi_1$.
**VI.18 Lemma.** If \( i : X \to X^* \) as usual and \( \varphi : X \to Y^* \), then \( \varphi \circ i = \varphi \).

**Proof.** \( \varphi \circ i(x) = \overline{\varphi}(i(x)) = \overline{\varphi}(\{x\}) = \bigcup \{ \varphi(x') : x' \in \{x\} \} = \varphi(x) \).

**VI.19 Lemma.** If \( i : X \to X^* \) as usual and \( \psi : Y \to X^* \), then \( i \circ \psi = \psi \).

**Proof.** Let \( y \in Y \). Then \( i \circ \psi(y) = \overline{i}(\psi(y)) = \bigcup \{ i(x) : x \in \psi(y) \} = \bigcup \{ \{x\} : x \in \psi(y) \} = \psi(y) \).

**VI.20 Definition.** \( Y^X = \{ f : f \in X^Y \} \).

For \( \varphi, \psi \in (Y^*)^X \) we make the following

**VI.21 Definition.** \( \varphi \leq \psi \) if and only if \( \varphi(x) \leq \psi(x) \) for every \( x \in X \). Then \( \leq \) is a reflexive proper partial order on \( (Y^*)^X \).

**VI.22 Theorem.** \( ((Y^*)^X, \leq) \) is a complete distributive lattice.

**Proof.** If we define \( \varphi_1 \land \varphi_2(x) = \varphi_1(x) \land \varphi_2(x) \) and \( \varphi_1 \lor \varphi_2(x) = \varphi_1(x) \lor \varphi_2(x) \), then it follows easily that \( \varphi_1 \land (\varphi_2 \lor \varphi_3) = (\varphi_1 \land \varphi_2) \lor (\varphi_1 \land \varphi_3) \). Let \( \varphi_m : X \to Y^* \) by \( \varphi_m(x) = \varnothing \) and \( \varphi_M : X \to Y^* \) by \( \varphi_M = Y \). Then \( \varphi_m \leq \varphi \leq \varphi_M \) for every \( \varphi \in (Y^*)^X \). See Caulfield [2] for the remainder.

Let \( \varphi : X \to Y^* \) such that \( \varphi(x) \neq \varnothing \) for all \( x \in X \). We note
that \( \mathcal{F}(\psi) = \{ f : f \in Y^X \text{ and } i \circ f \leq \psi \} \). Also \( \psi = \bigvee \{ i \circ f : f \in \mathcal{F}(\psi) \} \) as was shown previously. Indeed, a stronger result actually holds as the following theorem shows.

**VI.23 Theorem.** If \( \psi \in (Y^*)^X \) and \( \psi(x) \neq \emptyset \) for all \( x \in X \), then there exists \( \mathcal{F} \subseteq Y^X \) such that \( \psi = \bigvee \{ i \circ f : f \in \mathcal{F} \} \) and \( \{ i \circ f : f \in \mathcal{F} \} \) is a disjoint family (A family \( \mathfrak{A} \subseteq (Y^*)^X \) is called disjoint if \( \psi \land \psi' = \psi \land \psi' \) whenever \( \psi, \psi' \in \mathfrak{A} \)).

**Proof.** See Caulfield [2].

**VI.24 Theorem.** Let \( \nu : X^{**} \to X^* \) be defined by
\[
\nu(\mathcal{A}) = \bigcup \{ A : A \in \mathcal{A} \} \quad \text{for every } \mathcal{A} \in X^{**}.
\]
Then
\[
\nu : (X^{**}, \mathcal{U}^{**}) \to (X^*, \mathcal{U}^*) \quad \text{for any } \mathcal{U} \in \mathcal{U}(X).
\]

**Proof.** Since \( \{ S(U) : U \in \mathcal{U} \} \) is a base for \( \mathcal{U}^* \), it follows from Theorem I.14 that \( \{ S \circ S(U) : U \in \mathcal{U} \} \) is a base for \( \mathcal{U}^{**} \). Let \( \mathcal{V} \in \mathcal{U} \). Then \( U \circ U \subseteq \mathcal{V} \) for some symmetric \( U \in \mathcal{U} \). It suffices to show that \( (\nu \times \nu)^{-1} S(\mathcal{V}) \in \mathcal{U}^{**} \). We claim that \( S \circ S(U) \subseteq (\nu \times \nu)^{-1} S(\mathcal{V}) \). Suppose \( (\mathcal{A}, \mathcal{B}) \in S \circ S(U) \). Then \( \mathcal{A} = \mathcal{B} \) or \( \mathcal{A} \times \mathcal{B} \subseteq S(U) \). If \( \emptyset \notin \mathcal{A} \), and \( \mathcal{A} \times \mathcal{B} \subseteq S(U) \), then \( \mathcal{B} = \{ \emptyset \} \) since \( B \in \mathcal{B} \) implies \( (\emptyset, B) \in S(U) \), and hence, \( B = \emptyset \). It follows then that if \( \emptyset \notin \mathcal{A} \cup \mathcal{B} \), then \( \mathcal{A} = \mathcal{B} = \{ \emptyset \} \) provided \( \mathcal{A} \times \mathcal{B} \subseteq S(U) \). If \( \mathcal{A} = \mathcal{B} \), then \( \nu(\mathcal{A}) = \nu(\mathcal{B}) \) which implies \( (\nu(\mathcal{A}), \nu(\mathcal{B})) \in S(\mathcal{V}) \). Therefore, with no loss of generality, we may assume \( \mathcal{A} \neq \mathcal{B} \) and \( \emptyset \notin \mathcal{A} \cup \mathcal{B} \). Suppose there exists an \( A \neq \emptyset \), such that \( A \in \mathcal{A} \setminus \mathcal{B} \). Then \( \mathcal{A} \times \mathcal{B} \subseteq S(U) \), since
(A, B) ∈ S ◦ S(U). This implies (A, B) ∈ S(U) for every B ∈ B. But A ∉ B for every B ∈ B and therefore A × B ⊆ U for every B ∈ B. It now follows that A × (U { B : B ∈ B}) ⊆ U. Now pick any A' ∈ A. Suppose A' ∉ A. If A' ∉ B, then by the preceding argument we have A' × (U { B : B ∈ B}) ⊆ U. If A' ∈ B, then A × A' ⊆ U. This implies A' × A ⊆ U⁻¹ = U and since A × (U { B : B ∈ B}) ⊆ U, it follows then that A' × (U { B : B ∈ B}) ⊆ U o U ⊆ V. Thus, in either case, we have that (U { A : A ∈ A}) × (U { B : B ∈ B}) ⊆ U o U ⊆ V. Therefore, (ν(A), ν(B)) ∈ S(V). Finally, if there exists a B ∈ B, then similarly we obtain (U { A : A ∈ A}) × (U { B : B ∈ B}) ⊆ U o U ⊆ V. Thus, S ◦ S(U) ⊆ (ν × ν)'⁻¹ S(V) as was to be shown.

VI.25 Lemma. If ψ : X → Y*, then ν o ψ* = ρ.

Proof. Note that ψ* : X* → Y**. Let A ∈ X*. Then

ν o ψ*(A) = ν(ψ*(A)) = ν(∪{ψ(a) : a ∈ A}) = ∪{ψ(a) : a ∈ A} = ρ(A).

VI.26 Theorem. If ψ : (X, U) → (Y, V*) and ψ : (Y, V) → (Z, W*), then ψ ◦ ψ : (X, U) → (Z, W*).

Proof. ψ ◦ ψ = ψ o ψ = ν o ψ o ψ. ν is uniformly continuous by Theorem VI.23 and ψ* is uniformly continuous since ψ is. The result follows.

VI.27 Theorem. Let U ∈ U(X), V ∈ U(Y) and Σ ⊆ (Y*)X. If Σ is equicontinuous (respectively, uniformly equicontinuous) with
respect to \( \mathcal{U} \) and \( \hat{\mathcal{V}} \), then \( \forall \xi : (X, \mathcal{J}(\mathcal{U})) \rightarrow (Y^*, \mathcal{J}(\hat{\mathcal{V}})) \) (respectively, \( \forall \zeta : (X, \mathcal{H}) \rightarrow (Y^*, \hat{\mathcal{H}}) \)).

**Proof.** See Caulfield [2].

VI.28 Example. We now construct an example to show that the preceding theorem need not hold for \( * \) in place of \( ^\wedge \). Let \( X = Y = [0,1] \) with the usual metric induced uniformities. Define: \( f : X \rightarrow Y \) by \( f(x) = x \) and \( g : X \rightarrow Y \) by \( g(x) = 1-x \) for all \( x \in X \). Note that for any \( \epsilon > 0 \) and any \( x \in X \) we have that \( f[U_\epsilon[x]] = U_\epsilon[x] \) and \( g[U_\epsilon[x]] = U_\epsilon[1-x] \). It follows then that \( i \circ f[U_\epsilon[x]] = i[U_\epsilon[x]] \subseteq S(U_\epsilon)[\{x\}] \) and \( i \circ g[U_\epsilon[x]] = i[U_\epsilon[1-x]] \subseteq S(U_\epsilon)[\{1-x\}] \). Therefore, \( \{ i \circ f, i \circ g \} \) is uniformly equicontinuous with respect to \( \mathcal{U} \) and \( \mathcal{V}^* \). Let \( F = (i \circ f) \lor (i \circ g) : X \rightarrow Y^* \) by \( F(x) = \{x, 1-x\} \). We claim that \( F \) is not even continuous with respect to \( \mathcal{J}(\mathcal{U}) \) and \( \mathcal{J}(\mathcal{V}^*) \). Note that \( F(0) = \{0,1\} \). Let \( 1 > \epsilon > 0 \). We assert that \( U_{\epsilon}[0] \subseteq F^{-1}[S(V_{1/4})[F(0)]] \). Choose \( x \in U_{\epsilon}[0] \) such that \( 0 < x < \epsilon \). Suppose \( F(x) \in S(V_{1/4})[F(0)] \). Then \( \{x, 1-x\} \subseteq S(V_{1/4})[\{0,1\}] \) and \( x \neq 0, x \neq 1 \) implies that \( \{0,1\} \times \{x, 1-x\} \subseteq V_{1/4} \). Therefore, \( |1-x| < 1/4 \) and \( |0-x| < 1/4 \) which implies that \( |1-0| < 1/2 \), a contradiction. Thus, \( F \) is not continuous at \( 0 \).

For the next (and final) theorem let \( X^* = \{ E : \emptyset \neq E \subseteq X \} \), i.e., we are not considering the empty set as an element of the hyperspace. Let \( I \neq \emptyset \) and for each \( \alpha \in I \), \( (X_\alpha, \mathcal{U}_\alpha) \) is a uniform space. Let \( X = \bigcup_{\alpha \in I} X_\alpha \). Note that \( F \in X^* \) iff \( F \subseteq X \).
Define $h : X^* \to \bigcup \{ X^*_\alpha : \alpha \in I \}$ by $h(F)(\alpha) = \{ f(\alpha) : f \in F \} \subseteq X^*_\alpha$ for each $F \subseteq X$. Let $(X, \mathcal{U}) = \bigcup \{ (X^*_\alpha, \mathcal{U}_\alpha) : \alpha \in I \}$.

**VI.29 Theorem.** $h : (X^*, \mathcal{U}^*) \to (\bigcup \{ (X^*_\alpha, \mathcal{U}^*_\alpha) : \alpha \in I \})$ and $h$ is surjective.

**Proof.** Let $F \subseteq \bigcup \{ X^*_\alpha : \alpha \in I \}$ such that $F(\alpha) \subseteq X^*_\alpha$. That is, $F(\alpha) \subseteq X^*_\alpha$ and $F(\alpha) \neq \emptyset$, for each $\alpha \in I$. Let $F = \{ f : f : I \to \bigcup \{ X^*_\alpha : \alpha \in I \} \}$ such that $f(\alpha) \subseteq F(\alpha) \subseteq X^*_\alpha$ for each $\alpha \in I$. Then for every $\alpha \in I$, $h(F)(\alpha) = \{ f(\alpha) : f \in F \} = F(\alpha)$, thus $h(F) = F$. Therefore, $h$ is surjective. Let $P_\alpha : X \to X^*_\alpha$ be the usual projection and let $T_\alpha : X \to X^*_\alpha$ be the usual projection. Then

$$\left\{ (P_\alpha \times P_\alpha)^{-1} V_\alpha : V_\alpha \in \mathcal{U}_\alpha, \alpha \in I \right\}$$

is a subbase for $\mathcal{U}$; therefore,

$$\{ (T_\alpha \times T_\alpha)^{-1} V_\alpha : V_\alpha \in \mathcal{U}_\alpha, \alpha \in I \}$$

is a subbase for $\mathcal{U}^*$. Also note that

$$\left\{ (T_\alpha \times T_\alpha)^{-1} S(V_\alpha) : V_\alpha \in \mathcal{U}_\alpha, \alpha \in I \right\}$$

is a subbase for $X\{ V^*_\alpha : \alpha \in I \}$. Fix $\alpha \in I$ and $V_\alpha \in \mathcal{U}_\alpha$. We assert that

$$S((P_\alpha \times P_\alpha)^{-1} V_\alpha) \subseteq (h \times h)^{-1} [(T_\alpha \times T_\alpha)^{-1} S(V_\alpha)] .$$

Let

$$(F, G) \in S((P_\alpha \times P_\alpha)^{-1} V_\alpha) .$$

Then $F = G$ or $F \times G \subseteq (P_\alpha \times P_\alpha)^{-1} V_\alpha$.

If $F = G$, then $h(F) = h(G)$; and hence,

$$(h(F), h(G)) \in (T_\alpha \times T_\alpha)^{-1} S(V_\alpha) .$$

If $F \times G \subseteq (P_\alpha \times P_\alpha)^{-1} V_\alpha$, then $F(\alpha) \times G(\alpha) \subseteq V_\alpha$. By definition of $h$, $h(F)(\alpha) = F(\alpha)$ and $h(G)(\alpha) = G(\alpha)$, thus, $h(F)(\alpha) \times h(G)(\alpha) \subseteq V_\alpha$. This implies

$$(h(F)(\alpha), h(G)(\alpha)) \in S(V_\alpha) .$$

It follows then that
(h(F), h(G)) \in (T_{\alpha} \times T_{\alpha})^{-1} S(V_{\alpha}). Therefore, h is uniformly continuous as was claimed.

VI.30 Remark. The previous theorem states that 
\( X \{ (x_{\alpha}, y_{\alpha}) : \alpha \in I \} \) is a uniformly continuous image of 
\( (X \{ (x_{\alpha}, y_{\alpha}) : \alpha \in I \})^* \). That this result is the best possible will be shown now using a simple cardinal arithmetic argument. Let N denote the natural numbers. Then card N = \( \aleph_0 \) and card 
N* = c. Also card \( (X N)^* = \aleph_0 \) = c. Therefore, card 
X(N*) = c = c while card \( (X N)^* = 2^c \). Thus, it is impossible to map \( X N^* \) onto \( (X N)^* \). Note that in general the function h defined in VI.29 cannot be injective.
CHAPTER VII

Areas for Further Investigation

We shall now pose a few questions related to the present effort. The unifying thread here is that all of these problems arose during the course of this work and their solution would extend (and perhaps simplify) the preceding.

Problem 1. The first question would inquire into the existence of other natural hyper-uniformities and their relation to the two under discussion. This is, of course, an open ended problem which has no literature. We would remark further that the hyper-uniformity is much more workable in terms of entourages in lieu of covers, as was acknowledged by Isbell [7] page 28.

Problem 2. Isbell conjectured that if $\mathcal{U}$ and $\mathcal{V}$ are uniformities for $X$, then $\mathcal{U} \neq \mathcal{V}$ if and only if $\mathcal{I}(\mathcal{U}) \neq \mathcal{I}(\mathcal{V})$. This plausibility was buried by an example due to Ward [14] and another by Isbell [9]. Caulfield did show that Isbell was essentially correct for $\mathcal{U} \neq \mathcal{V}$ if and only if $\mathcal{I}((\mathcal{U})^\wedge) \neq \mathcal{I}((\mathcal{V})^\wedge)$ on $X^{**}$. Thus the question arises, is the $\mathcal{U}$-uniformity a finer screen in the sense of the Isbell conjecture, i.e. is $\mathcal{U} \neq \mathcal{V}$ if and only if $\mathcal{I}(\mathcal{U}^\wedge) \neq \mathcal{I}(\mathcal{V}^\wedge)$? We note that Lemma II.5 and Remark II.6 will probably provide a negative answer to this in general.
Problem 3. We have shown that if $U^*$ is fine then so is $U$; and $T(U)$ compact forces $U^*$ to be fine. Is it true that $U$ fine is necessary and sufficient for $U^*$ to be fine? Or does possibly another condition hold, namely, does $T(U)$ compact imply $U^*$ is unique. The only general results about unique uniformities for completely regular topological spaces are those of Newns [13] and Doss [4]. Unfortunately, the conditions required by these authors appear difficult in application.

Problem 4. Lemma IV.2 was derived to prove that $U$ and $U^*$ are jointly complete. The full power of this Lemma was never realized. What further insight into the structure of $U^*$ does this lemma give?

Problem 5. The theorems about $T(\mathcal{U})$ put severe restrictions on the multi-valued uniformly continuous functions. Suppose $F : (X^*;U^*) \to (Y^*;Y^*)$. Is it possible to characterize $F$ in terms of some $\mathcal{F} \subseteq Y^X$?
BIBLIOGRAPHY


