SIMILAR AND NON-SIMILAR SOLUTIONS
OF
THE FULLY VISCOUS FLOW IN SLENDER CHANNELS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the degree Doctor of Philosophy in the Graduate
School of the Ohio State University

By

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*** *** ***

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PART I: FLOWS WITH NO SLIP AT THE WALL

(1) Introduction:

After more than sixty years of development of boundary layer theory, many viscous flow problems have been systematically studied and effectively solved by the famous "thin" boundary layer idea. However, there are certain categories of viscous flow problems which do not fit into the basic boundary layer assumption and thus cannot be treated by using the well-known boundary layer theory. One of these particular categories is the fully viscous flow inside the channels; i.e., the channel flow with such a low Reynolds number that no inviscid flow region can be found inside the whole channel.

When the Reynolds number of the flow is sufficiently high, the viscous effect of the fluid tends to be confined inside a very thin boundary layer near the wall, hence the flow problems inside the channel can be solved by using the inviscid flow theory together with a contour correction for the viscous boundary layer displacement thickness. But, as the Reynolds number of the flow becomes lower and lower, the thin boundary layer near the channel wall then grows thicker and thicker, and eventually merges at the center-line to establish a fully viscous
flow field inside the channel. In this case, there is no "external flow" which can be used as a reference or an outer boundary condition, and the flow problem cannot be solved directly by applying the boundary layer theory.

The equations of motion governing the fully viscous flow inside channels are the Navier-Stokes equations,\(^{(1)}\) which are usually very difficult to solve even for a quite simple problem. Thus, only a few special cases such as the Hagan-Poiseuille flow and incompressible convergent and divergent channel flow have been solved and illustrated in the literature. All of these classical exact solutions actually result from the simple geometry of the channel, and belong to the so-called similar solutions which theoretically have the same velocity profile throughout the whole channel. Many investigations and developments about the fully viscous channel flow \(^{(2,3)}\) have been accomplished recently because of their increasing importance and applications. One of the more remarkable contributions is the study of the similar solutions of the fully viscous flow in slender channels by Dr. James Williams III who, for the first time, successfully obtained the similar solutions for several different flow conditions. The similar solution he found for the two-dimensional, incompressible flow is no surprise in the convergent and divergent straight channel
case. However, his finding for the case of axisymmetric, incompressible flow is somewhat unusual since the similar solution does not exist in a conical nozzle, but in a nozzle with an exponential wall contour. Unfortunately, these kinds of similar solutions are extremely limited in number and restricted from practical applications.

The present work is intended to be a more general study of the fully viscous flow in slender channels, which will cover both the similar and non-similar solutions for several different flow conditions. The scheme employed in the analysis is the integral method which, in some way, corresponds to the Von Karman-Pohlhausen's momentum integral method in the boundary layer theory, though the boundary conditions are not the same. For boundary layer flow, both the inner and outer boundary conditions of the velocity profile are given; however, for fully viscous channel flow, the "outer" boundary condition at the centerline of the channel is unknown and has to be determined simultaneously with the solution itself. The great advantage of the present integral method is that it provides not only a simple way for finding the similar solutions but also a powerful means to solve the non-similar solutions. However, the accuracy of the solution is expected to be within the same degree as the Von Karman-Pohlhausen result, since the velocity
Due to the recent advances in aerodynamics and space sciences, the low density, fully viscous flow inside the channel has received much attention for various applications. Some of the important, feasible applications are listed below for further interest and reference;

a) The micro-thrust rocket which is used for high altitude (low density) control and maneuvering of satellites and spacecraft.

b) The low-density wind tunnel nozzle which is used for hypersonic testing.

c) The plasma or ion propulsion system which may be used in the near future for long range space flight.
(2) Equations of Motion:

The equations of motion governing the low density, fully viscous flow inside the channels are the Navier-Stokes equations, even if it is within the slip flow region (please refer to Part II for this argument). By using the nomenclature in figure 1, the incompressible and compressible Navier-Stokes equations for the two-dimensional and axisymmetric flow can be written as below.

For steady, two-dimensional flow

**continuity equation**

\[ \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial z}(\rho v) = 0 \]  \hspace{1cm} (1a)

**x-momentum equation**

\[ \rho \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial z} + \frac{\partial p}{\partial x} = \frac{\partial}{\partial x}\left[ \mu \left( \frac{\partial u}{\partial x} - \frac{2}{3} \frac{\partial v}{\partial z} \right) \right] + \frac{\partial}{\partial z}\left[ \mu \left( \frac{\partial u}{\partial z} + \frac{\partial v}{\partial x} \right) \right] \]  \hspace{1cm} (1b)

**r-momentum equation**

\[ \rho \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial z} + \frac{\partial p}{\partial z} = \frac{\partial}{\partial x}\left[ \mu \left( \frac{\partial v}{\partial x} - \frac{2}{3} \frac{\partial u}{\partial z} \right) \right] + \frac{\partial}{\partial z}\left[ \mu \left( \frac{\partial v}{\partial z} + \frac{\partial u}{\partial x} \right) \right] \]  \hspace{1cm} (1c)

**energy equation**

\[ \rho \frac{\partial e}{\partial x} + \rho v \frac{\partial e}{\partial z} = \frac{\partial}{\partial x}\left[ \frac{\mu}{\gamma - 1} \left( \frac{\partial u}{\partial x} \right)^2 + \frac{\partial v}{\partial z} \frac{\partial v}{\partial x} \right] + \frac{\partial}{\partial z}\left[ \frac{\mu}{\gamma - 1} \left( \frac{\partial v}{\partial z} \right)^2 + \frac{\partial u}{\partial x} \frac{\partial v}{\partial z} \right] \]  \hspace{1cm} (1d)

**equation of state for perfect gas**

\[ \rho = \rho R T \]  \hspace{1cm} (1e)

For steady, axisymmetric flow

**continuity equation**
\[
\frac{\partial}{\partial x} \left( \rho u u \right) + \frac{\partial}{\partial z} \left( \rho u v \right) = 0
\]  \hspace{1cm} (2a)

**x-momentum equation**

\[
\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial z} + \frac{\partial p}{\partial x} = \frac{\partial}{\partial x} \left[ \mu \left( \frac{\partial u}{\partial x} - \frac{v}{3} \frac{\partial v}{\partial z} - \frac{u}{3} \frac{\partial u}{\partial z} \right) \right] \\
+ \frac{\partial}{\partial z} \left[ \mu \left( \frac{\partial u}{\partial z} \right) \right] + \frac{\mu}{\rho} \left( \frac{\partial u}{\partial x} \right)^2
\]  \hspace{1cm} (2b)

**r-momentum equation**

\[
\rho u \frac{\partial u}{\partial r} + \rho v \frac{\partial u}{\partial z} + \frac{\partial p}{\partial r} = \frac{\partial}{\partial r} \left[ \mu \left( \frac{\partial u}{\partial r} - \frac{v}{3} \frac{\partial v}{\partial z} - \frac{u}{3} \frac{\partial u}{\partial z} \right) \right] \\
+ \frac{\partial}{\partial z} \left[ \mu \left( \frac{\partial u}{\partial z} \right) \right] + \frac{\mu}{\rho} \left( \frac{\partial u}{\partial r} \right)^2
\]  \hspace{1cm} (2c)

**energy equation**

\[
\rho u \frac{\partial e}{\partial x} + \rho v \frac{\partial e}{\partial z} = \frac{\partial}{\partial x} \left[ \frac{u}{2} \mu \left( \frac{\partial u}{\partial x} - \frac{v}{3} \frac{\partial v}{\partial z} - \frac{u}{3} \frac{\partial u}{\partial z} \right) \right] \\
+ \frac{\partial}{\partial z} \left[ \frac{u}{2} \mu \left( \frac{\partial u}{\partial z} \right) \right] + \frac{u}{2} \mu \left( \frac{\partial u}{\partial x} \right)^2 \\
+ u \left( \frac{\partial u}{\partial r} \right)^2
\]  \hspace{1cm} (2d)

**equation of state for perfect gas**

\[
\bar{\rho} = \rho \bar{R} T
\]  \hspace{1cm} (2e)

**Boundary conditions:**

- at channel wall \( r=r_w \); \( u=0, \ v=0, \ h=h_w \)
- at centerline \( r=0 \); \( u=u_0, \ v=0, \ h=h_0 \)

Due to the non-linearity and coupling, the above systems of equations (1) and (2) are quite complicated and involved, and can be only solved by numerical schemes with proper initial and boundary conditions.
Figure 1: Two-dimensional and Axisymmetric, Fully Viscous Flow inside the Slender Channels
(3) Dimensional Analysis:

Before trying to solve the systems of equations (1) and (2) completely, it is naturally desirable to investigate and simplify those equations at first by utilizing dimensional analysis. Following the same idea as Prandtl's thin boundary layer concept, the systems of equations (1) and (2) can be approximated and simplified to the boundary layer type equations though their boundary conditions are not the same.

The approximations of equations (1) and (2) for slender channels are actually based on the fact that the length of the slender channel is much greater than its width, and, as a result, the axial velocity, \( u \), is much larger than the transverse velocity, \( v \) (see fig. 1), i.e., \( r \ll 1 \), \( v \ll u \), and \( r, v \) are in the same small order of magnitude \( \delta \), hence,

\[
\begin{align*}
u & \approx O(1), \quad \nu \approx O(\delta), \quad \lambda \approx O(1), \quad L \approx O(\delta) \\
\frac{\partial u}{\partial x} & \approx O(1), \quad \frac{\partial u}{\partial z} \approx O(\delta), \quad \frac{\partial \nu}{\partial x} \approx O(\frac{1}{\delta}), \quad \frac{\partial \nu}{\partial z} \approx O(1).
\end{align*}
\]

After neglecting the small terms in equations (1) and (2), the equations of motion for the fully viscous flow in slender channels have the relatively simply form as follows(4):
For steady, two-dimensional flow

continuity equation
\[ \frac{1}{x} (\rho u_x) + \frac{1}{y} (\rho v) = 0 \]  \hspace{1cm} (3a)

x-momentum equation
\[ \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \frac{\partial}{\partial x} \left( \mu \frac{\partial u}{\partial x} \right) = \frac{1}{2} \frac{\partial}{\partial x} \left( \mu \frac{\partial u}{\partial y} \right) \]  \hspace{1cm} (3b)

r-momentum equation
\[ \frac{\partial}{\partial r} (r \rho u) = 0 \]  \hspace{1cm} (3c)

energy equation
\[ \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial v}{\partial y} = \frac{\partial}{\partial x} \left( \mu \frac{\partial u}{\partial x} \right) + \frac{1}{2} \frac{\partial}{\partial x} \left( \mu \frac{\partial v}{\partial y} \right) \]  \hspace{1cm} (3d)

equation of state
\[ p = \rho R T \]  \hspace{1cm} (3e)

For steady, axisymmetric flow

continuity equation
\[ \frac{1}{r} (\rho u_r) + \frac{1}{r} (\rho v) = 0 \]  \hspace{1cm} (4a)

x-momentum equation
\[ \rho u \frac{\partial u_r}{\partial r} + \rho v \frac{\partial u_r}{\partial z} + \frac{\partial}{\partial r} \left( \mu \frac{\partial u_r}{\partial r} \right) = \frac{1}{2} \frac{\partial}{\partial r} \left( \mu \frac{\partial u_r}{\partial z} \right) \]  \hspace{1cm} (4b)

r-momentum equation
\[ \frac{1}{r} \frac{\partial}{\partial r} (r \rho v) = 0 \]  \hspace{1cm} (4c)
energy equation
\[ \rho u \frac{\partial u}{\partial r} + \rho v \frac{\partial u}{\partial z} = \rho \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \alpha (\frac{\partial u}{\partial z})^2 \] (4d)
equation of state
\[ \rho = \rho R T \] (4e)

Boundary conditions:
at channel wall \((r=r_w)\); \(u=0, \ v=0, \ h=h_w\)
at centerline \((r=0)\); \(u=u_0, \ v=0, \ h=h_0\)

Examining the above systems of equations (3) and (4), it is not hard to find that they are exactly similar to the boundary layer equations, except one of their boundary conditions is different. For boundary layer flow, the outer boundary condition is usually known from the external inviscid flow, however, for this fully viscous flow inside slender channels, the outer boundary conditions along the centerline such as \(u_0, \ P_0, \ \rho_0\) are all indetermined and coupled within the solution itself.
(4) Integral Equations:

The systems of equations (3) and (4) for two-dimensional and axisymmetric flow in the previous section can be written together for simplicity.

continuity equation

\[ \frac{\partial}{\partial x} (\rho u x^2) + \frac{\partial}{\partial z} (\rho v z^2) = 0 \]  

(5)

x-momentum equation

\[ \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial z} + \frac{\partial p}{\partial x} = \frac{1}{2} \frac{\partial}{\partial x} (\rho x \frac{\partial x}{\partial x}) \]  

(6)

r-momentum equation

\[ \frac{1}{r} \frac{\partial}{\partial r} (r \rho x \frac{\partial x}{\partial r}) = 0 \]  

(7)

energy equation

\[ \rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial z} = \frac{1}{2} \frac{\partial}{\partial x} (\rho x \frac{\partial x}{\partial x}) + \rho (\frac{\partial u}{\partial x})^2 \]  

(8)

equation of state

\[ \rho = \rho \rho T \]  

(9)

where \( \{ j = 0 \) for two-dimensional flow \( \{ j = 1 \) for axisymmetric flow

boundary conditions:

at channel wall \( r = r_w \); \( u=0, v=0, h=h_w, \rho = \rho_w \)  

(10)

at the centerline \( r = 0 \); \( u=u_0, v=0, h=h_0, \rho = \rho_0, \frac{\partial}{\partial r} = 0 \) (symmetric profiles)  

(11)

By using the integral technique and the boundary conditions (10) and (11), the following integral equations are obtained.
Momentum Integral Equation. Integrate the momentum equation (6) across the channel from the centerline to the wall, it is obtained,

\[
\int_0^{\infty} \left( \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial z} + \rho \frac{\partial \theta}{\partial x} \right) dx = \left[ \rho \frac{\partial u}{\partial x} \right]_0^\infty \tag{12}
\]

from continuity equation (5)

\[
\rho \frac{\partial u}{\partial x} = - \int_0^z (\rho u \frac{\partial u}{\partial z}) dz \tag{13}
\]

substitute eq. (13) into eq. (12), it becomes

\[
\int_0^{\infty} \left[ \rho u \left( \frac{\partial u}{\partial x} - \frac{\partial \theta}{\partial z} \right) \frac{\partial u}{\partial z} + \rho \frac{\partial \theta}{\partial x} \right] dx = \left[ \rho \frac{\partial u}{\partial x} \right]_0^\infty \tag{14}
\]

by using the boundary conditions (10) and (11), the second term of eq. (14) can be integrated by parts as

\[
\int_0^{\infty} \left[ \frac{\partial}{\partial x} \int_0^z (\rho u \frac{\partial u}{\partial z}) dz \right] dx = \int_0^{\infty} \left\{ \frac{\partial}{\partial x} \left[ \int_0^z (\rho u \frac{\partial u}{\partial z}) dz \right] - \rho \frac{\partial u}{\partial x} \frac{\partial u}{\partial z} \right\} dx \]

\[
= - \int_0^{\infty} \rho \frac{\partial u}{\partial x} (\rho u \frac{\partial u}{\partial z}) dx
\]

and the compatibility condition at the wall gives

\[
\frac{\partial}{\partial x} \left[ \frac{1}{\rho} \frac{\partial (\rho u \frac{\partial u}{\partial z})}{\partial x} \right] = 0
\]

hence, eq. (14) becomes

\[
\int_0^{\infty} \rho u \frac{\partial u}{\partial x} dx + \int_0^{\infty} \rho \frac{\partial u}{\partial x} (\rho u \frac{\partial u}{\partial z}) dx = \left[ \rho \frac{\partial u}{\partial x} \right]_0^\infty dx = \left[ \rho \frac{\partial u}{\partial x} \right]_0^\infty \tag{14}
\]
This is the momentum integral equation though it appears in a different form from that of Von Karman. The left-hand side of eq. (15) is the change of total momentum along the x-direction, while the first term on the right-hand side is the shear stress at the wall, and the second term, variation of the shear stress at the wall. Physically, it represents that the total momentum loss inside the channel is equal to the dissipation of the shear stress and its variation on the wall.

**Energy Integral Equation.** Multiply the x-momentum equation by \( u \), and substitute into the energy equation, it is easy to obtain

\[
\rho u \frac{d}{dx} \left( \frac{h + \frac{u^2}{2}}{2} \right) + \rho u \frac{d}{dx} \left( \frac{h + \frac{u^2}{2}}{2} \right) = \frac{1}{2} \frac{d}{dx} \left[ \rho u \frac{d}{dx} \left( \frac{h + \frac{u^2}{2}}{2} \right) \right] + \frac{1}{2} \frac{d}{dx} \left[ \frac{1}{2} \frac{d}{dx} \left( \frac{h - 1}{\frac{h^2}{2}} \right) \right]
\]
within the slender channel approximation \((v \ll u)\),
\[
\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 = \mathcal{H}_0 + \frac{1}{2} (u^2 + v^2) \approx \mathcal{H}_0 + \frac{1}{2} \frac{v^2}{u^2}
\]
assume the case, \(P_\infty = 1\), then the above energy equation has the following simple form
\[
\rho u \frac{dH}{dx} + \rho v \frac{dH}{dz} = \frac{1}{2} \frac{1}{u^2} \left( u \frac{dH}{dz} \right)
\] (16)
integrate the energy equation (16) across the channel as it did in the momentum equation
\[
\int_0^{\infty} \rho u \frac{dH}{dz} \, dz + \int_0^{\infty} \rho v \frac{dH}{dz} \, dz = \left[ \frac{u \frac{dH}{dz}}{u} \right]_0^{\infty}
\] (17)
by using the continuity equation, the second term of eq. (17) can be performed as,
\[
\int_0^{\infty} \rho v \frac{dH}{dz} \, dz = - \int_0^{\infty} \frac{d}{dz} \left[ \int_0^{\infty} (\rho u \dot{v}) \, dz \right] \, dz
\]
\[
= - \int_0^{\infty} \left\{ \frac{d}{dz} \left[ \mathcal{H}_0 \frac{d}{dz} (\rho u \dot{v}) \right] - \mathcal{H}_0 \frac{d}{dz} (\rho u \dot{v}) \right\} \, dz
\]
\[
= \int_0^{\infty} \mathcal{H}_0 \frac{d}{dz} (\rho u \dot{v}) \, dz - \mathcal{H}_0 \int_0^{\infty} \frac{d}{dz} (\rho u \dot{v}) \, dz
\] (18)
substitute eq. (18) into eq. (17)
\[
\int_0^{\infty} \rho u \frac{dH}{dz} \, dz + \int_0^{\infty} \mathcal{H}_0 \frac{d}{dz} (\rho u \dot{v}) \, dz = \mathcal{H}_0 \int_0^{\infty} \frac{d}{dz} (\rho u \dot{v}) \, dz + \left[ u \frac{dH}{dz} \right]_0^{\infty}
\]
rearrange and use the Leibnitz's rule again, we finally have
\[
\frac{d}{dz} \int_0^{\infty} \rho u \dot{H} \, dz = \mathcal{H}_0 \frac{d}{dz} \int_0^{\infty} \rho u \dot{v} \, dz + u \frac{d}{dz} \left[ u \frac{dH}{dz} \right]
\] (19)
It is not difficult to recognize that the first term at the right-hand side of eq. (19) is the product of wall enthalpy and mass flow variation. For solid wall channel, the mass flow rate is always constant, hence eq. (19) can be reduced to

$$\frac{d}{dx} \int_{-\infty}^{\infty} \rho u H \frac{dh}{dx} \, dx = \rho \frac{d}{dx} \left( \int_{-\infty}^{\infty} \rho u \, dx \right) = \left( \dot{m} \right)_{w}$$  \hspace{1cm} (20)

Eq. (20) is the energy integral equation which is also different in form from the usual one. The left-hand side of the equation represents the change of the total enthalpy flow rate, while the right-hand side is the heat transfer to the wall.

**Mass Flow Rate Integral Equation.** The above momentum and energy integral equations can be theoretically solved for $u$ and $h$ by assuming the proper velocity and enthalpy profiles. However, it must be recalled that one of the boundary conditions $u_0$, on which the velocity profile is based is still unknown, and must be determined by an additional equation further than the momentum and energy. The condition of constant mass flow rate inside a solid wall channel provides this additional relation,

$$\dot{m} = 2 \int_{-\infty}^{\infty} \rho u (\pi \Delta) \frac{dh}{dx} \, dx = \text{Constant}$$  \hspace{1cm} (21)
Equations (15), (20) and (21) form the system of integral equations which will be solved simultaneously for velocities $u$, $u_0$ and enthalpy $h$. The main procedure is to assume a velocity profile and an enthalpy profile with the unknown boundary condition at the centerline, then substitute these profiles into the momentum, energy and mass flow rate integral equations. After performing the integration, a system of relatively simple, first-order, ordinary differential equations results. However, these equations can be combined together to become a single equation which contains only the pressure gradient parameter, $\lambda$, and the channel width, $2\epsilon$, as variables. Thus, it is possible to either fix the value of the pressure gradient parameter to seek for the similar solutions, or to solve the non-similar problems for any given channel wall shape.

Due to the intrinsic characteristics and mathematical difficulties, it is much more convenient to solve the incompressible, compressible, two-dimensional and axisymmetric cases separately.
Incompressible, Two-dimensional Flow Case.

For incompressible, two-dimensional flow, \( j=0 \), \( \rho = \text{const} \), and \( \nu = \text{constant} \). The energy equation has the simple trivial solution, \( h=\text{constant} \), and is thus decoupled from the momentum and mass flow equations.

After introducing the kinematic viscosity, \( \nu = \frac{\mu}{\rho} \), the integral equations (15) and (21) have the following forms:

Momentum integral equation

\[
\frac{d}{dz} \int_0^{2w} u^2 \, d\eta = \nu \left( \frac{\partial^2 h}{\partial \eta^2} \right)_{\eta=2w} - \nu \omega \left( \frac{\partial^2 u}{\partial \eta^2} \right)_{\eta=2w} \tag{22}
\]

Mass flow integral equation

\[m = 2 \int_0^{2w} \rho u \, d\eta = \text{Constant} \tag{23}\]

Now, assume a Pohlhausen type velocity profile

\[\frac{u}{u_c} = a_0 + a_1 \eta + a_2 \eta^2 + a_3 \eta^3 + a_4 \eta^4\]

where \( \eta = r/\rho \) is a non-dimensional parameter.

The coefficients \( a_i \)'s of the velocity profile are actually not a constant, and will be determined by the boundary and the compatibility conditions.

Boundary conditions;

at center line ( \( \eta = 0 \)), \( u/u_0 = 1 = a_0 \) \( \therefore a_0 = 1 \)

\[\frac{\partial (u/u_c)}{\partial \eta} = 0 = a_1 \quad \therefore a_1 = 0 \]

at channel wall ( \( \eta = \pm 1 \)), \( u/u_0 = 1 + a_2 + a_3 + a_4 = 0 \)

\[u/u_0 = 1 + a_2 - a_3 + a_4 = 0 \]

\[\therefore a_3 = 0 \quad \text{and} \quad a_4 = -(1 + a_2) \tag{24}\]
Compatibility condition:

$$\frac{\partial p}{\partial x} = \left( \frac{1}{\mu} \frac{\partial (\mu \frac{\partial u}{\partial x})}{\partial x} \right)_{n=x_{0}} = \frac{\mu \nu_{c}}{2 \nu_{x}} \left( 2a_{2} + 12a_{4} \right)$$  \hfill (25)

Solving eq. (24) and (25) for \(a_{2}\) as well as \(a_{4}\), it is obtained

$$a_{2} = -\frac{1}{10} \left( \frac{\rho_{w}}{\mu \nu_{c}} \frac{\partial p}{\partial x} + 12 \right) = \lambda$$  \hfill (26)

$$a_{4} = -(1 + \lambda)$$

Then the velocity profile becomes,

$$\frac{u}{\nu_{c}} = 1 + \lambda \eta^{2} - (1+\lambda) \eta^{4}$$  \hfill (27)

Here \(\lambda\) is the pressure gradient parameter though it appears in different form from the Pohlhausen's. It is not surprising that the velocity profile depends entirely on the pressure gradient parameter.

Performing the integration by employing the velocity profile in the integral equations, the following is obtained,

$$\int_{0}^{2\omega} u^{2} \, dx = \nu_{c}^{2} \int_{0}^{1} (\frac{u}{\nu_{c}})^{2} \, d\eta = \frac{8 \nu_{w} \nu_{c}^{2}}{315} (\lambda^{2} + 8\lambda + 28)$$

$$\int_{0}^{2\omega} u \, dx = \frac{2}{3} \nu_{w} \nu_{c} (1 + 6)$$

$$\left( \frac{\partial u}{\partial x} \right)_{n=x_{0}} = -\frac{\nu_{c}}{2 \nu_{x}} (2\lambda + 4).$$
\[
\left( \frac{3}{2} \frac{u_c}{\lambda} \right)_{x=2w} = -\frac{U_c}{2w} (10 \lambda + 12)
\]

Then, the momentum integral equation becomes,
\[
\frac{d}{dx} \left[ (\lambda^2 + 8 \lambda + 28) 2w \, u_c^2 \right] = \frac{3U_c}{2w} u_c (\lambda + 1) \quad (28)
\]

and the mass flow integral equation becomes,
\[
\dot{m} = \frac{4}{15} \frac{\dot{m}_w}{u_c} \, u_c (\lambda + 6) = \text{constant} \quad (29)
\]

Eliminating \( u_c \) from eq. (28) by using the relation of eq. (29), one obtains the final single equation
\[
\frac{d}{dx} \left( \frac{\lambda^2 + 8 \lambda + 28}{(\lambda + 6)^2} \frac{1}{2w} \right) = 8 \frac{\dot{m}}{\dot{m}_w} (\frac{\lambda + 1}{\lambda + 6}) \frac{1}{2w} \quad (30)
\]

Eq. (30) contains only \( \lambda \) and \( r_w \) as variables, thus we can either fix the value of \( \lambda \) to find the channel contour, \( r_w \) for similar solutions, or calculate the pressure gradient parameter \( \lambda \) along the x-axis for any given non-similar shape of the channel. After determining the values of \( \lambda \) (the value of \( \lambda \) will be a constant for similar solution, but a function of \( x \) for non-similar problems.), it is then easy to calculate the centerline velocity, \( u_c \) from eq. (29), and determine the velocity profile by eq. (27).
Most of the classical and similar solutions are obtained from the result of a very simple channel geometric shape that reduces the equations of motion to the simplest form, thus, this integral method becomes a useful and important method for solving those non-similar, arbitrary channel shape problems.

**Similar solutions.** As mentioned before, the velocity profile inside the slender channel of a two-dimensional, incompressible flow is governed merely by the pressure gradient parameter, \( \lambda \). If the value of \( \lambda \) is constant throughout the channel, then the shape of the velocity profile will be similar to each other in every cross-section, and the solution becomes the so-called similar solution. Thus, it is reasonable for us to find the channel contour which has the similar solution by assuming the value of \( \lambda \) constant throughout the channel.

With \( \lambda = \text{constant} \) and \( \frac{d\lambda}{dz} = 0 \), eq. (30) reduces to

\[
\frac{d\omega}{dz} = -84 \omega \alpha \frac{(\alpha+1)(\alpha+6)}{(\alpha+2)(\alpha+7)} = C_1
\]

\[
\therefore \quad \rho = C_1 z + C_2
\]  \( \text{(31)} \)

where \( C_1 \) and \( C_2 \) are constant.

The result of eq. (31) clearly shows that the similar solution under the present flow condition exists only when the channel wall is linear. It can be either a convergent
or a divergent straight channel depending on the value of \( c \) negative or positive respectively. For the special case, \( \alpha_1 = 0 \), then \( \eta_w = \alpha_2 \) and the result reduces to the classical parallel flow solution. The above conclusion is not only exactly the same as Dr. Williams finding from his self-similar method, but also verifies that those classical exact solutions which were obtained from the simple geometric contour shape are merely the special cases of the similar solutions.

In order to test the accuracy and demonstrate the simplicity of this integral method, two typical similar solutions will be solved in the following:

A) Parallel Flow

The incompressible flow through a straight channel forms the typical parallel flow since it has only one velocity component \( u \) inside the channel. In this case, 
\[
\eta_w = \text{constant}, \quad \frac{\partial \eta}{\partial z} = \text{constant}, \quad \text{and consequently}
\]
\[
\eta_c = \frac{E}{24} \left( \frac{15}{4} \frac{\partial u}{\partial x} + \frac{15}{2} \frac{\partial^2 u}{\partial x^2} \right) = \text{constant}
\]
\[
\lambda = -\frac{1}{10} \left( \frac{2 \eta_c u}{u_c} \frac{\partial \eta_c}{\partial z} + 12 \right) = \text{constant}
\]

Since both \( \eta_w \) and \( \lambda \) are constant, the left-hand side of the eq. (30) is equal to zero, and the pressure gradient parameter, \( \lambda \) has the following two simple solutions; \( \lambda = -1 \), and \( \lambda = -6 \). However,
\( \lambda = -6 \) is a singular solution corresponding to the case of zero mass flow rate which so far has no physical significance (for this matter, please refer to eq. 29). Thus

\[
\lambda = -1 \quad \Rightarrow \quad \frac{\mu}{uc} = (1 - \eta^2) = (1 - \frac{x^2}{2})
\]

then from eq. (26),

\[
uc = -\frac{2x}{2x} \frac{df}{dx}
\]

\[
u = -\frac{1}{2u} \frac{df}{dx} (x^2 - x^2)
\]

This is exactly the same as the classical parabolic velocity profile solution.

B) Convergent and Divergent Channel Flow

Consider the flow problem inside a slender convergent or divergent straight channel as shown in the figure 2. The included angle of the channel, \( 2\theta \), is 10°, then from geometrical relationship

\[
\alpha = \pm (\tan 5°) x = \pm 0.0875 x
\]

\[
\frac{d\alpha}{dx} = \pm 0.0875 \begin{cases} + \text{for divergent} \\ - \text{for convergent} \end{cases}
\]
Figure 2: Convergent and Divergent Channel with Plane Walls
In order to compare the result to the classical Hamel's and Pohlhausen's solution, a Reynolds number for this case is defined as $Re = \frac{u_0 L}{\nu}$. By using the mass flow rate equation, it results

$$Re = \frac{15}{4 \sin \beta \cos \alpha} \frac{1}{\alpha + \beta}$$

(33)

It is not hard to see that the Reynolds number of this flow is a constant throughout the channel if the mass flow rate is kept constant, since the value of the $\lambda$ will be actually a constant for this linear channel wall shape. Hence, eq. (30) becomes

$$\lambda^2 + \left(8 \pm \frac{4 \pi \beta^2}{Re} \right) \lambda + \left(28 \pm \frac{9 \pi \beta^2}{Re} \right) = 0$$

(34)

where, + for divergent, - for convergent

The above equation is a simple, quadratic equation for $\lambda$ which depends on the value of Reynolds number only. Once the Reynolds number of the flow is given, the value of the $\lambda$, as well as the velocity profile and the center line velocity can all be determined from eq. (34), eq. (29) and eq. (27).

It is interesting to point out here that the velocity at the center line, $u_c$ is inversely proportional to $\sqrt{\lambda}$, or more precisely, to $\pi$, since the Reynolds number of the flow will be constant. In this manner, the solution of
$u_c$ at $x=0$ becomes infinite, which then corresponds to a singular point of either a sink or source with the infinite velocity at that point.

In order to compare the result of this integral method to those available solutions, several calculations of the velocity profile have been carried out and plotted in figure 3 and 4 for the following Reynolds number.

for convergent channel;

(1) $R_e = 684$, \quad $\lambda = -0.625$

(2) $R_e = 1342$, \quad $\lambda = -0.1226$

for divergent channel;

(1) $R_e = 684$, \quad $\lambda = -1.318$

(2) $R_e = 1342$ \quad $\lambda = -1.58$

It is not surprising that the solution for the low Reynolds number cases give a much better comparison than those of higher Reynolds number, since the low Reynolds number flow is close to the previously assumed conditions and requirements. The solution for the convergent channel has also shown superior agreement than those of divergent channel. This is perhaps due to the reason that the Reynolds number for the divergent channel flow is already too high and the flow starts to separate inside the channel.
Figure 3: Similar Velocity Profiles of the Convergent Channel with Plane Walls
Figure 4: Similar Velocity Profiles of the Divergent Channel with Plane Walls
(6) Incompressible, Axisymmetric Flow Case

For incompressible, axisymmetric flow, $j=1$, $\rho = \text{const}$, and $\mu = \text{constant}$. Similar to the two-dimensional case, the energy equation here will also have the simple solution, $h = \text{constant}$. Thus, the momentum integral and the mass flow integral equations can be solved independently from the energy integral equation.

Momentum integral equation

$$\frac{d}{dz} \int_{\infty}^{\infty} u^2 \rho \, dz = \frac{2}{R} \left[ \left( \frac{\partial u}{\partial \rho} \right)_{\rho = \infty} - \left( \frac{\partial \phi}{\partial \rho^2} \right)_{\rho = \infty} \right]$$

Mass flow integral equation

$$\dot{m} = 2\pi \int_{0}^{\infty} \rho u \, dz = \text{constant.}$$

Following the same procedure as in the two-dimensional case, a velocity profile is introduced. The coefficients of the velocity profile is again determined by using the boundary and compatibility conditions. After rearranging, a quite familiar velocity profile is found;

$$\frac{u}{u_c} = 1 + \lambda_1 \frac{\eta^2}{(1+\lambda_1) \eta^4}$$

Where

$$\lambda_1 = -\frac{1}{12} \left( \frac{\rho_{\infty}}{\mu \nu_c} \frac{d^2 \rho}{dz^2} + 16 \right)$$

Comparing eq. (38) to eq. (26), it is found that both the pressure gradient parameter $\lambda_1$ and $\lambda$ have
the same form, but different values on the constant coefficients.

By employing the velocity profile, eq. (37) in the integrations, it is easy to obtain the following:

\[ \int_{0}^{a} u^2 \, dr = \frac{V_c^2 \, \omega}{10} (\lambda_i^2 + 7\lambda_i + 16) \]

\[ \left( \frac{\partial V_c}{\partial x} \right)_{r=a_0} = -2 \, V_c (\lambda_i + 2) \]

\[ \left( \frac{\partial^2 V_c}{\partial r^2} \right)_{r=a_0} = -2 \, V_c (5 \lambda_i + 6) \]

Then, the momentum equation becomes,

\[ \frac{d}{dx} \left[ (\lambda_i^2 + 7\lambda_i + 16) \, V_c^2 \, \omega^2 \right] = 240 \, V_c^2 \, (\lambda_i + 1) \]

(39)

and the mass flow rate equation becomes,

\[ \dot{m} = \frac{\pi}{6} \rho \, \omega^2 \, V_c (\lambda_i + 4) = \text{constant} \]

(40)

Eliminating \( U_0 \) from eq. (39) by using the relation eq. (40) it is thus obtained

\[ \frac{d}{dx} \left[ \frac{\lambda_i^2 + 7\lambda_i + 16}{(\lambda_i + 4)^2} \, \frac{1}{\omega^2} \right] = 40 \, \pi \left( \frac{\omega}{\omega_0} \right) \frac{1}{\omega_0^2} \left( \frac{\lambda_i + 1}{\lambda_i + 4} \right) \]

(41)
This is the final equation for determining either the contour shape \( l_n \) or the pressure gradient parameter \( \lambda_1 \), depending totally on which one of the two variables is given. Eq. (41), which is corresponding to eq. (30) in the previous two-dimensional case, has nearly the same form as eq. (30).

**Similar Solutions.** Since the velocity profile of the incompressible, axisymmetric flow depends also merely on the pressure gradient parameter \( \lambda_1 \), the similar solutions of this flow condition then can be sought in the same manner as in the two-dimensional case by assuming a constant value of \( \lambda_1 \) throughout the channel. Thus, from eq. (41), it is obtained:

\[
\frac{1}{l_n} \frac{dl_n}{dx} = -20 \pi \frac{\xi (\lambda_1 + 4)(\lambda_1 + 1)}{\lambda_1^2 + 7\lambda_1 + 16} = C_3
\]

or

\[
\ln l_n = C_3 x + C_4.
\]

\[
(42)
\]

so

\[
l_n = e^{(C_3 x + C_4)} = C_5 e^{C_3 x}
\]

where \( C_3, C_4 \) and \( C_5 \) are all constants.

According to eq. (42), it is surprising that the similar solution for the incompressible, axisymmetric flow is not a convergent, divergent straight wall nozzle,
but a nozzle with an exponential wall contour shape. This finding is in complete agreement with the result of Dr. Williams who used a direct similar method for the investigation of similar solutions. There is also a special case for this similar solution, i.e., the case when $C_3 = 0$, then $\mathcal{F}_w = C_5$; it becomes the classical Hagan-Poiseuille circular pipe solution.

A) Hagan-Poiseuille Flow.

For circular pipe, both the pipe radius $R_w$ and the pressure gradient $\frac{dp}{dz}$ are constant, thus

$$
\nu_c = \frac{3}{8} \left( \frac{6 \pi \nu_c}{\rho R_w} + \frac{\rho \omega}{12 \pi \mu} \frac{dp}{dz} \right) = \text{constant}
$$

$$
\lambda_1 = -\frac{1}{12} \left( \frac{\rho \omega}{12 \mu u_c} \frac{dp}{dz} + 16 \right) = \text{constant}
$$

The left-hand side of eq. (41) is then identical zero, and the pressure gradient parameter $\lambda_1$ has the simple solutions; $\lambda_1 = -1$ and $\lambda_1 = -4$. However, the solution of the case $\lambda_1 = -4$ is nothing but a singular solution, since it corresponds to the zero mass flow case (please refer to eq. 40). Hence

$$
\lambda_1 = -1 \quad \& \quad \frac{v}{u_c} = (1 - \eta^2) = (1 - \frac{\eta^2}{R_w})
$$
however,

\[ u_c = - \frac{\omega^2}{4 \pi} \frac{dP}{dx} \]

\[ u = - \frac{1}{4 \pi \mu} \frac{dP}{dx} (\omega^2 - r^2) \]  \hspace{1cm} (43)

This result again shows a parabolic velocity profile across the pipe, which is exactly the same as the classical Hagan-Poiseuille solution.

B) Flow in an exponential wall contour nozzle.

It is found in the previous section that the incompressible flow through a nozzle with an exponential wall contour will result in the similar solution. Thus, the present nozzle flow problem shown in the figure 5 is a similar problem. The equation of the exponential wall contour is given as below,

\[ \frac{R_\infty}{R_o} = \xi \epsilon \left( \pm 0.0804 \frac{L}{R_o} \right) \]

\[ \begin{cases} + & \text{for divergent} \\ - & \text{for convergent} \end{cases} \]  \hspace{1cm} (44)

where \( R_o \) is the radius of the nozzle throat.

Define a Reynolds number for this flow case as

\[ Re = \frac{u_c R_o}{\nu} \]
Figure 5: Convergent and Divergent Nozzle with the Exponential Wall Contour
where \( \nu_c = \frac{6 \sin \theta}{\pi \rho A_0^2 (\lambda + 4)} \) is the center-line velocity at the nozzle throat, \( n = n_0 \).

Then,

\[
\frac{d \nu_c}{d \lambda} = \pm 0.0804 \frac{\nu_c}{n_0} \quad \begin{cases} 
+ \text{ for divergent} \\
- \text{ for convergent}
\end{cases}
\]

\[
Re = \frac{6 (\sin \theta \nu_c)}{n \rho A_0^2 (\lambda + 4) n_0} = \text{constant}
\] (45)

\[
\lambda_1 = \left[ \frac{6 (\sin \theta \nu_c)}{n \rho A_0^2 \nu_c / n_0 \nu_c} - 4 \right] = \text{constant}
\]

Which agrees very well with our former observation that \( \lambda_1 \) will be a constant, if a similar solution exists.

With the relation (44) and (45), eq. (41) can be finally reduced to a quadratic equation

\[
\lambda_1^2 + (7 \pm \frac{14 \sqrt{2} \phi}{n \rho A_0^2 \nu_c / n_0 \nu_c}) \lambda_1 + (16 \pm \frac{14 \sqrt{2} \phi}{n \rho A_0^2 \nu_c / n_0 \nu_c}) = 0
\] (46)

where the + sign is for the divergent case, and the - sign is for the convergent case.

In order to compare the result and accuracy of this method, some calculations about the velocity profile have
been carried out for several specific Reynolds numbers; For convergent case,

\[ \lambda_1 = -\left(7 - \frac{1492.5}{Re} \right) \pm \sqrt{\left(7 - \frac{1492.5}{Re} \right)^2 - 4\left(16 - \frac{1492.5}{Re} \right)} \]

when \( Re = 47 \) then \( \lambda_1 = -0.62 \)

\( Re = 107 \) \( \lambda_1 = +0.3 \)

For divergent case,

\[ \lambda_1 = -\left(7 + \frac{1492.5}{Re} \right) \pm \sqrt{\left(7 + \frac{1492.5}{Re} \right)^2 - 4\left(16 + \frac{1492.5}{Re} \right)} \]

when \( Re = 59 \) then \( \lambda_1 = -1.38 \)

\( Re = 98 \) \( \lambda_1 = -1.51 \)

The result both for the convergent and the divergent cases are shown in figures 6 and 7 with the comparison to those similar solutions obtained from the similar method in ref. 2. It is not hard to see that the present results show the same characteristic as in the two-dimensional flow case; i.e., the convergent case has a much better agreement than the divergent one.
Figure 6: Similar Velocity Profiles of the Convergent Nozzle with the Exponential Wall Contour
Figure 7: Similar Velocity Profiles of the Divergent Nozzle with the Exponential Wall Contour
Non-similar Solutions. Thus so far the theory has been used to demonstrate the effectiveness of obtaining and solving similar solutions with the integral method. However, the main interest and emphasis of the present integral approach is to solve the arbitrary, non-similar problems, since the similar solutions are usually very limited in number and lack practical application. Hence, a simple, typical non-similar example of the flow about a conical nozzle is presented in the following.

C) Convergent and Divergent Conical Nozzle Flow

Consider now a convergent or divergent conical nozzle as shown in the figure 8. Its included apex angle, \( 2\theta \) is 10°, and the nozzle section starts from -10 to -20 for convergent case, and 10 to 20 for divergent case, so the total length of the nozzle is 10.

It is known from the previous finding that this simple conical nozzle does not result in a similar solution as expected from the two-dimensional case; instead it is a non-similar problem with \( \lambda_1 \) varying along the \( x \). Thus, the essential part of this solution is to determine the values of \( \lambda_1 \) along the \( x \)-axis by using the given geometric shape and the flow condition of the nozzle.
Figure 8: Convergent and Divergent Conical Nozzle
From geometric relationship,

\[ L = \pm 0.0875 \alpha \]

\[ \therefore \frac{1}{L} \frac{d\alpha}{d\tau} = \frac{1}{\alpha} \]

substitute this eq. (47) into eq. (41), then it becomes

\[ \frac{d\lambda}{d\tau} = \frac{(\lambda_1 + q)^2}{(\lambda_1 - q)} \left[ 40 \pi \frac{\tau}{m} (\lambda_1 + 1) + \frac{2(\lambda_1^2 + \lambda_1 + 1)}{(\lambda_1 + 1)} \frac{1}{\alpha} \right] \]  (48)

Equation (48) is the unique equation to determine the value of \( \lambda \), both for the convergent and divergent cases, however, the sign of the geometric length \( x \) in that equation should not be misused; for the convergent nozzle, \( x \) is negative, for the divergent nozzle, \( x \) is positive.

Due to the non-linear characteristics, it is difficult to obtain the analytical solution of eq. (48) though it is only a first-order ordinary differential equation. Thus a numerical computation of the value of \( \lambda \) by computer is necessary. The results of the computation are tabulated on table 1 for several different mass flow rate similarity parameters, \( \frac{\rho \nu}{\rho_0 \nu_0} \). They are also plotted in figure 9 and 10 such that the variation of \( \lambda \) along the \( x \) can be clearly seen.
### TABLE 1: VALUES OF THE PRESSURE GRADIENT PARAMETER

#### A. Convergent Case

<table>
<thead>
<tr>
<th>x (in.)</th>
<th>-20</th>
<th>-19</th>
<th>-18</th>
<th>-17</th>
<th>-16</th>
<th>-15</th>
<th>-14</th>
<th>-13</th>
<th>-12</th>
<th>-11</th>
<th>-10</th>
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<tbody>
<tr>
<td>$\frac{m}{2x}$</td>
<td>21.46</td>
<td>-1.0</td>
<td>-0.939</td>
<td>-0.936</td>
<td>-0.933</td>
<td>-0.928</td>
<td>-0.923</td>
<td>-0.918</td>
<td>-0.912</td>
<td>-0.904</td>
<td>-0.895</td>
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<tr>
<td>$\frac{m}{2y}$</td>
<td>85.84</td>
<td>-1.0</td>
<td>-0.767</td>
<td>-0.740</td>
<td>-0.724</td>
<td>-0.706</td>
<td>-0.686</td>
<td>-0.662</td>
<td>-0.635</td>
<td>-0.602</td>
<td>-0.564</td>
</tr>
<tr>
<td>$\frac{m}{2z}$</td>
<td>171.67</td>
<td>-1.0</td>
<td>-0.604</td>
<td>-0.479</td>
<td>-0.427</td>
<td>-0.384</td>
<td>-0.338</td>
<td>-0.285</td>
<td>-0.221</td>
<td>-0.145</td>
<td>-0.053</td>
</tr>
<tr>
<td>$\frac{m}{2\ell}$</td>
<td>214.59</td>
<td>-1.0</td>
<td>-0.545</td>
<td>-0.353</td>
<td>-0.269</td>
<td>-0.208</td>
<td>-0.145</td>
<td>-0.072</td>
<td>+0.015</td>
<td>+0.120</td>
<td>+0.251</td>
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</tbody>
</table>

#### B. Divergent Case

<table>
<thead>
<tr>
<th>x (in.)</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
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<tr>
<td>$\frac{m}{2x}$</td>
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<td>-1.087</td>
<td>-1.081</td>
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<td>-1.066</td>
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<tr>
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<tr>
<td>$\frac{m}{2z}$</td>
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<td>-1.505</td>
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<tr>
<td>$\frac{m}{2\ell}$</td>
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<td>-1.756</td>
<td>-1.759</td>
<td>-1.74</td>
<td>-1.71</td>
<td>-1.676</td>
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<td>$\frac{m}{2\sigma}$</td>
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<td>-1.598</td>
<td>-1.798</td>
<td>-1.885</td>
<td>-1.921</td>
<td>-1.927</td>
<td>-1.914</td>
<td>-1.89</td>
<td>-1.859</td>
<td>-1.824</td>
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</table>
Figure 9: Variations of the Pressure Gradient Parameter along the Convergent Conical Nozzle.
Figure 10: Variations of the Pressure Gradient Parameter along the Divergent Conical Nozzle.
It is found that the values of the solution, \( \lambda_i \), is only good for certain special region of the mass flow rate similarity parameter, \( \left( \frac{\dot{m}}{\mu} \right) \). If the value of \( \left( \frac{\dot{m}}{\mu} \right) \) is too high, the solution of \( \lambda_i \) will break down shortly after the entrance. This implies either an over mass flow rate or a weak viscosity of the flow such that fully viscous flow has not yet been attained inside the nozzle. On the other hand, if the value of \( \left( \frac{\dot{m}}{\mu} \right) \) is too small, then the solution of \( \lambda_i \) becomes nearly a constant throughout the nozzle. This of course hints that a similar solution will eventually result; however, for such a small value of \( \left( \frac{\dot{m}}{\mu} \right) \), the effects of the slip and molecular phenomena might be dominant and alter the present tendency. The region of the values of \( \left( \frac{\dot{m}}{\mu} \right) \) which will result in a good solution to \( \lambda_i \) are different for convergent and divergent cases; for the convergent nozzle, the value of \( \left( \frac{\dot{m}}{\mu} \right) \) cannot be greater than 171 \( \mu \), otherwise a velocity overshoot starts, for the divergent nozzle, the limit value of \( \left( \frac{\dot{m}}{\mu} \right) \) is around 300 \( \mu \) (refer fig. 9, fig. 10 and fig. 11).

After determining the values of \( \lambda_i \) along the nozzle, it is only a minor job to complete the solutions of the problem by calculating the center-line velocity, \( \mathcal{U} \), from eq. (40) and the velocity profile from the eq. (37).
Figure 11: Family of Velocity Profiles for the equation,
\[ \frac{u}{u_c} = 1 + \lambda \eta^2 - (1 + \lambda) \eta^4 \]
with various values of \( \lambda \).
along the nozzle for different flow conditions, four particular sections A, B, C, and D along the nozzle are chosen. For the convergent case, section A is at x=-19, B at x=-17, C at x=-14 and D at x=-11. While for divergent case, section A is at x=11, B at x=14, C at x=17 and D at x=19. The velocity profiles of several different values of \( \frac{\pi n}{\mu} \) at these particular sections are plotted consequently on figures 12, 13, 14, 15 and figures 16, 17, 18, 19, for the convergent and divergent cases respectively.

The velocity at the center line, \( \mu_c \) not only depends on the parameter \( \frac{\pi n}{\mu} \), but also on the density of the flow. Thus, any two flows with the same value of \( \frac{\pi n}{\mu} \), but different values of density, will have exactly the same velocity profile, however, different values of center-line velocity. It is also to say that the corresponding velocity across the nozzle section for these two flows will be different though their velocity profiles are similar. The change of the center line velocity, for several combinations of \( \frac{\pi n}{\mu} \), and for an arbitrary flow density are shown in figures 20 and 21 for the convergent and divergent nozzles, respectively.
Figure 12: Velocity Profiles of the Convergent Conical Nozzle at Section A.
Figure 13: Velocity Profiles of the Convergent Conical Nozzle at Section B.
Figure 14: Velocity Profiles of the Convergent Conical Nozzle at Section C.
Figure 15: Velocity Profiles of the Convergent Conical Nozzle at Section D.
Figure 16: Velocity Profiles of the Divergent Conical Nozzle at Section A.
Figure 17: Velocity Profiles of the Divergent Conical Nozzle at Section B.
Figure 18: Velocity Profiles of the Divergent Conical Nozzle at Section C.
Figure 19: Velocity Profiles of the Divergent Conical Nozzle at Section D.
Figure 20: Centerline Velocity Variations along the Convergent Conical Nozzle.
Figure 21: Centerline Velocity Variations along the Divergent Conical Nozzle.
(7) Compressible, Two-dimensional Flow Case.

For compressible, two-dimensional flow, $j = 0$, but, $ho$ and $\mu$ are not constant any more. Thus the energy equation does not have the simple, trivial solution, $T = \text{constant}$ as in the incompressible flow case. Consequently, it is necessary to solve the energy and the momentum integral equations simultaneously together with the mass flow integral equation in order to obtain a complete solution.

Momentum integral equation

$$\frac{d}{dx} \int_{-\infty}^{2\omega} \rho u^2 \, dz = \left( \frac{\partial T}{\partial z} \right)_{z=2\omega} - \left[ \frac{\partial}{\partial z} \left( \rho \frac{d \mu}{dz} \right) \right]_{z=2\omega}$$

Energy integral equation

$$\frac{d}{dx} \int_{-\infty}^{2\omega} \rho u H \, dz = \left( \frac{\partial T}{\partial z} \right)_{z=2\omega} = \tilde{q}_{\omega}$$

Mass flow integral equation

$$\dot{m} = 2 \int_{-\infty}^{2\omega} \rho u \, dz = \text{constant}$$

The boundary conditions of the velocity for this case are identically the same as those of the incompressible flow. However, the boundary conditions of the temperature for this case are much complicated and different from those in the incompressible flow; they could
be a constant wall temperature case, or an adiabatic wall case, or a constant heat transfer case, or a more complicated, arbitrary heat transfer case. However, for the reason of simplicity, only the adiabatic wall case is considered in this section, i.e.,

$$\left( k \frac{dT}{dz} \right)_{z=2w} = \dot{q}_w = 0$$

where $\dot{q}_w$ is the heat transfer at the channel wall.

Furthermore, assume the case of Prandtl number equal unity, then the recovery factor is approximately one for most of the cases, hence

$$H_a = \rho_c + \frac{1}{2} \dot{u}_c^2 = \rho_a w$$

It follows naturally that the temperature and the viscosity at the adiabatic wall will be constant providing the flow is isoenergetic and the total enthalpy is constant.

**Transformation.** It is discovered in the Boundary Layer Theory that the viscous, compressible flow problem can be more easily solved by transforming the original variables to a new density-distorted plane, in which the variation of the density could be temporarily absorbed inside the new coordinates. However, the choice of the reference condition for the transformation is both arbitrary and individual. For the present case, it seems that
the wall conditions are the best reference conditions to be used in the transformation, since both the density and the viscosity at the adiabatic wall are constant. Thus, assume a linear viscosity-temperature relationship with the reference at the wall, \( \rho M = \rho M_W = \) constant, then transform the system of equations (49), (50) and (51) into a new coordinate by employing the well-known Doronitz Transformations,

\[
X = \zeta, \quad R = \int \frac{\rho}{\rho_W} d\zeta.
\] (52)

where the capital letter represents the corresponding variable in the transformed plane. After rearranging, the system of the equations have the following form:

**Momentum integral equation**

\[
\frac{\partial}{\partial X} \int_0^{R_W} u^2 \, dR = \chi W \left( \frac{dW}{dR} \right)_{R=R_W} - \left( \frac{C_W}{R_W} \right) \chi W \left( R \frac{d^2 u}{dR^2} \right)_{R=R_W} (53)
\]

**Energy integral equation**

\[
\frac{\partial}{\partial X} \int_0^{R_W} u H \, dR = \chi W \left( \frac{dW}{dR} \right)_{R=R_W} = 0 \quad (54)
\]

**Mass flow integral equation**

\[
\dot{m} = 2 \rho W \int_0^{R_W} u \, dR = \text{constant.} \quad (55)
\]

where \( R_W \) is the fictitious radius of the channel.
contour on the transformed plane, by definition,

\[ R_w = \int_0^{2\pi} \frac{R}{\rho} d\theta \]  

System of equations (53), (54) and (55) has been successfully transformed into the new coordinates \( R \) and \( X \), except one parameter, \( \left( \frac{R_w}{R} \right) \) which is in the last term of the momentum integral equation. It is not difficult to recognize that this parameter \( \left( \frac{R_w}{R} \right) \) is the only difference between the transformed compressible integral equations and the corresponding incompressible integral equations. It is also because of this parameter \( \left( \frac{R_w}{R} \right) \), the momentum and the energy integral equations are coupled each other together. Thus the determination of the value of \( \left( \frac{R_w}{R} \right) \) becomes very important, and very difficult too, since \( r_w \) and \( R_w \) are in the different coordinate planes. However, the value of the ratio, \( \left( \frac{R_w}{R} \right) \) can be well correlated to the enthalpy profile \( \frac{\theta}{R_w} \) in the following manner; from eq. (52)

\[ r_w = \int_0^{R_w} \frac{R}{\rho} dR \]

\[ \therefore \quad \frac{R_w}{R} = \frac{1}{R_w} \int_0^{R_w} \frac{\rho}{\rho} dR = \int_0^{\frac{R}{R_w}} \frac{d\xi}{\xi} \]

\[ \Rightarrow \quad \frac{R}{R_w} = \frac{\xi}{\xi} = \frac{R}{R_w} \]  

where \( \xi = \frac{R}{R_w} \) is the non-dimensional parameter on the transformed plane.
Velocity Profile: Similar to the incompressible case, a power series velocity profile is assumed on the transformed plane,

\[ \frac{u}{u_c} = \sum_{i=0}^{n} a_i \xi^i \]

where \( \xi = \frac{R}{R_w} \)

The coefficients \( a_i \)'s of the velocity profile can be again determined by utilizing the boundary conditions and the compatibility condition.

Boundary conditions:

at the center-line \( (\xi = 0) \);

\[ \frac{u}{u_c} = 1 \quad \therefore a_0 = 1 \]

\[ \frac{d(u/u_c)}{d\xi} = 0 \quad \therefore a_1 = 0 \]

at channel wall \( (\xi = 1) \);

\[ \frac{u}{u_c} = 1 + a_2 + a_3 + a_4 = 1 + a_2 - a_3 + a_4 = 0 \]

\[ \therefore a_2 = 0, \quad a_4 = -(1 + a_2) \]

Compatibility condition;

\[ \frac{\partial p}{\partial x} = -u_0 \left( \frac{d^2 u}{dR^2} \right)_{R=R_w} = -u_0 \frac{u_c}{R_w^2} (2a_2 + 2a_4) \]

\[ \therefore a_2 = -\frac{1}{2} \left( \frac{R_w^2}{u_0 u_c} \frac{\partial p}{\partial x} + 12 \right) = \Lambda \]

(58)

where \( \Lambda \) is the pressure gradient parameter in the transform plane.
Thus

\[ \frac{u}{u_c} = 1 + A \xi^2 - (1 + A) \xi^4 \]  \hspace{1cm} (59)

This velocity profile appears exactly in the same form as in the incompressible flow case, however, it must be remembered that those variables such as \( \xi \) and \( A \) are all expressed in the transformed plane.

**Enthalpy Profile.** Assume again a fourth-order power series for the enthalpy profile in the transformed plane

\[ \frac{P}{\bar{n}_w} = b_0 + b_1 \xi + b_2 \xi^2 + b_3 \xi^3 + b_4 \xi^4 \]

where \( b_i \)'s are enthalpy coefficients which are determined from the temperature or heat transfer boundary conditions.

**Boundary conditions:**

at center line \( (\xi = 0) \);

\[ \frac{\dot{h}}{h_w} = \frac{h_c}{h_w} \quad \Rightarrow \quad b_0 = \frac{\dot{h}_c}{h_w} \]

\[ \left( \frac{\partial \frac{P}{\bar{n}_w}}{\partial \xi} \right)_{\xi=0} = 0 \quad \Rightarrow \quad b_1 = 0 \]

at channel wall \( (\xi = \pm 1) \),

\[ \left( \frac{\partial \frac{P}{\bar{n}_w}}{\partial \xi} \right)_{\xi=\pm 1} = 0 \quad \Rightarrow \quad b_3 = 0, \quad b_2 + 2b_4 = 0 \]

\[ \frac{\dot{h}}{h_w} = \frac{h_w}{\bar{n}_w} \quad \Rightarrow \quad b_2 = 2(1 - \frac{\dot{h}_c}{h_w}), \quad b_4 = -(1 - \frac{\dot{h}_c}{h_w}) \]
Finally, the enthalpy profile becomes

\[ \frac{\frac{\partial h}{\partial \nu}}{h_w} = \frac{h_0}{h_w} + 2(1-\frac{h_0}{h_w})\frac{\xi^2}{2} - \left(1-\frac{h_0}{h_w}\right)\frac{\xi^4}{4} \quad (60) \]

In the same way that the pressure gradient parameter, \( \lambda \), dominates the velocity profile, the enthalpy parameter \( \frac{h_0}{h_w} \) in the above expression is the only parameter which totally determines the enthalpy profile. The value of \( \frac{h_0}{h_w} \) is usually not known in advance, however, it does depend on the solution of the velocity field. Hence, the enthalpy profile on the transformed plane depends implicitly upon the velocity solution; on the other hand, the velocity profile on the transformed plane is apparently independent from the enthalpy solution, but its transformed coordinates depends on the enthalpy solution (i.e., density solution). Thus, the velocity field and the enthalpy field are actually still coupled to each other, and have to be solved together simultaneously.

**Momentum Integral Equation**

The momentum integral equation on the transformed plane coupled with the energy equation only through the term \( \frac{\sigma_{w}}{h_w} \) which, however, can be obtained now by using the enthalpy profile as below:

\[ \frac{\sigma_{w}}{h_w} = \int_0^1 \frac{\partial}{\partial \nu} d\xi = \frac{1}{16} \left( 7 + 8 \frac{h_0}{h_w} \right) \quad (61) \]
from the velocity profile, it is easy to obtain the following:

\[
\int_0^{R_w} u^2 dR = \frac{8R_w U_c^2}{315} (A^2 + 8A + 28)
\]

\[
\left( \frac{dU}{dR} \right)_{R=R_w} = -\frac{U_c}{R_w} (2A + 4)
\]

\[
\left( \frac{d^2U}{dR^2} \right)_{R=R_w} = -\frac{U_c}{R_w^2} (10A + 12)
\]

Thus, the momentum integral equation becomes

\[
\frac{d}{dX} \left[ (A^2 + 8A + 28) R_w U_c^2 \right] = \frac{21/2}{4R_w} U_c \left[ (7 + 8 \frac{h_a}{h_w} \frac{U_c}{R_w}) (5A + 6) - 15 (A + 2) \right] \tag{62}
\]

For incompressible flow case, \( R_w \) reduces to \( r_w, x \) to \( x, \lambda \) to \( \lambda \) and \( (\frac{h_a}{h_w}) = 1 \), then the above equation (62) automatically reduces to the incompressible flow case eq. (28).

**Mass Flow Integral Equation**

With the velocity profile, the mass flow integral equation can be easily integrated out

\[
\dot{m} = 2\rho_w \int_0^{R_w} \nu dR = \frac{4}{15} \left( \frac{\gamma}{\gamma - 1} \right) R_w U_c (A + 6) \tag{63}
\]
Eliminate $U_c$ from eq. (62) by using the relation eq. (63) it is obtained,

$$\frac{d}{dx} \left[ \frac{(A^2 + 8A + 28)}{(A + 6)^2} \right]$$

$$= \frac{2}{5} \left( \frac{h_0}{h} \right)^{1/2} \left[ (7 + 8 \frac{h_0}{h}) \left( \frac{5A + 6}{A + 6} \right) - 15 \left( \frac{A + 2}{A + 6} \right) \right] \tag{64}$$

Unlike the incompressible flow case, eq. (64) contains another parameter $\left( \frac{h_0}{h} \right)$ in addition to the two variables $A$ and $R_w$. Thus, eq. (64) can be either determined the velocity profile (i.e., $A$) or the wall contour for similar solution (i.e., $R_w$) only provides that the enthalpy ratio $\left( \frac{h_0}{h} \right)$ is known. Again, it is not difficult to see that for incompressible flow, $\frac{h_0}{h} = 1$, eq. (64) then reduces to eq. (30).

Energy Integral Equation

The enthalpy ratio $\left( \frac{h_0}{h} \right)$ can be usually determined from the energy integral equation by employing both the velocity and the enthalpy profiles;

$$\frac{d}{dx} \int_0^{R_w} uH dR = 0 \quad \text{(adiabatic wall)}$$

i.e.

$$\int_0^{R_w} \frac{1}{h} u dR + \frac{1}{2} \int_0^{R_w} u^2 dR = \text{constant}.$$  

However, there is a more simple and direct way to obtain the relation between $h_0$ and $h_w$ for this particular case, i.e., to utilize the previous relation, eq. (52), then

$$\frac{h_0}{h_w} = 1 - \frac{1}{2} \frac{U_c^2}{R_w} = \left[ 1 - \frac{1}{2} \frac{1}{R_w} \left( \frac{15}{A + 6} \frac{27}{R_w} \frac{1}{A + 6} \right)^2 \right] \tag{65}$$
substitute eq. (65) into eq. (64), it becomes

\[
\frac{d}{dx} \left[ \frac{(A^2+6A+28)}{(A+6)^2} \frac{1}{R_w} \right]
\]

\[
= \frac{\gamma}{4} \left( \frac{\rho_{nw}}{m} \right) \frac{1}{R_n} \left( \frac{1}{A+6} \right) \left[ 48(A+1) - \frac{\gamma}{8} \frac{\rho_n}{\rho_{nw}} \frac{1}{R_n} \left( \frac{5A+6}{(A+6)^2} \right) \right]
\]

Eq. (66) is the final equation for this compressible, two-dimensional flow, which contains only two variables \( A \) and \( R_w \). Thus it is possible to determine any one of the two variables providing the other one is given. After determining the value of \( A \) or \( R_w \), the other solutions of the problem such as the velocity profile, enthalpy profile and the center-line velocity can all be determined from the equations (59), (60), (63) and (65). However, these solutions are all on the density-distorted transformed plane. In order to obtain the real solutions on the original, physical plane, the solutions must be transformed back to the original plane. The transformation used is the inverse of the previous one, eq. (52), however, here the density solution is supposed to be known from the enthalpy profile:

\[
\chi = X, \quad \alpha = \int \frac{R_w}{\rho} dR = \int \frac{P}{\rho_{nw}} dR
\]
Similar Solutions. The similar solution for this flow case can be found in the same procedure as before by assuming a constant pressure gradient parameter, \( \Lambda \), in eq. (66), then solving it for the similar wall contour \( R_w \). From eq. (66),

\[
\frac{dR_w}{dX} = \frac{2 (\Lambda \omega)}{4 (\Lambda^2 + 8 \Lambda + 28)} \left[ 48 (\Lambda + 1) - 4 \epsilon \frac{2 \pi^2}{\rho^2 \nu^2} \frac{5 \Lambda + 6}{(\Lambda + 6)^2} R_w^2 \right]
\]

Since \( \Lambda \), \( \omega \), \( \rho \), \( \nu \), and \( hw \) are constants, it is more convenient to label them as

\[
K_1 = \frac{2 \Lambda \omega}{4 \omega (\Lambda^2 + 8 \Lambda + 28)}
\]

\[
K_2 = 4 \epsilon \frac{2 \pi^2}{\rho^2 \nu^2} \frac{(5 \Lambda + 6)}{(\Lambda + 6)^2}
\]

\[
K_3 = -48 (\Lambda + 1)
\]

Then eq. (68) reduces to

\[
\frac{dR_w}{dX} = K_1 \left[ K_2 \left( \frac{1}{R_w^2} \right) + K_3 \right]
\]

rearranged,

\[
\int \frac{R_w^2 dR_w}{(K_2 + K_3 R_w^2)} = \int K_1 dX = K_1 X + K_4.
\]
but,

\[ \int \frac{R_w^2 dR_w}{(K_2 + K_3 R_w^2)} = \frac{R_w}{K_3} - \frac{K_2}{K_3 \sqrt{K_2 K_3}} \tan^{-1} \frac{\sqrt{K_2 K_3} R_w}{K_2} \]

or

\[ \frac{R_w}{K_3} - \frac{K_2}{K_3 \sqrt{K_2 K_3}} \tan^{-1} \frac{\sqrt{K_2 K_3} R_w}{K_2} \]

hence,

\[ R_w - \frac{K_2}{K_3} \tan^{-1} \frac{\sqrt{K_2 K_3} R_w}{K_2} \]

\[ R_w - \frac{K_2}{K_3} \tanh^{-1} \frac{\sqrt{K_2 K_3} R_w}{K_2} \]

here \( \tan^{-1} \) is for the case \( K_2 K_3 > 0 \),

\( \tanh^{-1} \) is for the case \( K_2 K_3 < 0 \).

finally, it is obtained,

\[ R_w - a \tan^{-1} \left( \frac{R_w}{a} \right) \]

\[ R_w - a \tanh^{-1} \left( \frac{R_w}{a} \right) \]

\[ = b X + C \]

\[ \begin{cases} \text{for } a^2 > 0 \\ \text{for } a^2 < 0 \end{cases} \]

(69)

where

\[ a^2 = -\frac{\pi^2}{16} \left( \frac{\sin^2 \frac{\pi}{m}}{m^2 \rho_{1s}^2} \right) \frac{(\sigma A + 6)}{(A+1)(A+6)^2} \]

\[ b = -84 \frac{m A}{m A} \left( A+6 \right) \left( A+1 \right) \]

\[ c = -48 K (A+1) \]
Where $K$ is an integration constant.

Eq. (69) gives the wall contour shape, $R_w$ for the similar solution on the transformed plane. Unfortunately, $R_w$ is not a simple geometry of linear wall, but a complicated form of transcendental functions. Furthermore, the wall contour $R_w$ obtained in eq. (69) is on the density-distorted transformed plane which is not physically real. In order to obtain the real shape of the channel contour, it is necessary to transform the $R_w$ back to the original $r_w$ by using the inversed Doronitzn's transformation. Due to the various conditions of enthalpy profile, the transformation might not be unique in this case.

A) Special Case of $\lambda = -1$.

The general result of the similar solution in eq. (69) is doubtless quite complicated. However, there is a simple and interesting similar problem for the particular case of $\lambda = -1$. From eq. (68),

$$\frac{dR_w}{dX} = \frac{3}{4} \frac{\mu_x \dot{m}}{R_w^2} \frac{1}{R_w^2}$$

$$R_w = \left[\frac{6 \mu_x \dot{m}}{4 \rho_w^2 \dot{h_w}}\right]^{\frac{1}{3}} X^{\frac{1}{3}} + K \quad (70)$$

where $K$ is an arbitrary integration constant.
As recalled from the incompressible flow, the case of $A = -1$ is the one of the parallel flow in a straight channel. But for this compressible flow case, it is not a straight parallel channel again; instead, its wall contour is proportional to the one third power of the $X$. The velocity profile for this special case remains parabolic as in the incompressible flow, however, it is on the transformed plane.
PART II: FLOWS WITH SLIP AT THE WALL

(1) Introduction:

In Part I, the low Reynolds number, fully viscous flow inside the slender channel has been thoroughly studied under the condition that the flow density is very low but not low enough to produce any slip effects on the channel wall. However, if the gas density of the flow inside the channel becomes smaller and smaller so that its mean free path is of the order of 1 to 10 percent of the characteristic dimensional of the body (e.g., the inlet diameter), then the flow at the solid boundary starts to slip, and the flow problem falls immediately into the well-known "Slip Flow" region. In this case, the analytical method developed in Part I is no longer applicable since the boundary conditions will be different for the two. Thus it is necessary here to modify the previous work by including the slip effect in the boundary conditions.

The slip effect on the solid surface not only alters the boundary condition of the velocity at wall from zero to some finite value, but also increases the total mass flow rate passing through the channel. The resultant and significant features of this slip effect are, however, the
reduction of skin friction and heat transfer at the channel wall. Due to the unclear characteristics of the slip flow, together with the lack of a properly established set of equations of motion, the quantitative calculation of the above slip effects are usually much more difficult than the qualitative estimation.

The most suitable equations of motion for the slip flow have been under discussion and argument for quite a long time, yet there is no completely satisfactory result so far. Burnett and Grad have tried to derive the equations of motion for the slip flow approaching from the kinetic theory of gases. However, their resulting Burnett and thirteen moment equations are all based on monatomic gases, and seriously limit the possibility for aerodynamic applications. Besides, the solutions obtained from the Burnett and thirteen moment equations have shown relatively poor agreement with the experimental results. On the other hand, the solutions obtained from the Navier-Stokes equations subjected to the slip boundary condition at the wall show much better results, although the Navier-Stokes equations are neither developed from nor for the slip flow. Thus the most appropriate equations of motion for the slip flow are probably the Navier-Stokes equations together with the slip boundary conditions.
After selecting the equations of motion to be used for the slip flow analysis, the difficulties of the fully viscous slip flow inside the channel remain merely in the boundary conditions. In the non-slip flow case (Part I), there is only one unknown boundary condition at the center line, however, for this slip flow case, both the boundary conditions at wall (slip velocity) and at the center line (center line velocity) are unknown.

The principle method used here to solve the slip flow problem will be actually similar to those in Part I, except for a minor correction for the slip boundary condition. The unknown slip velocity at the wall can be technically eliminated by modifying the channel contour to certain extent so that the flow will result in zero velocity at the fictitious new wall, and simultaneously slip velocity at the original contour wall. The results of this new approach will be shown and compared to the non-slip case for some simple, particular problems.

(2) Equations of Motion:

From the previous discussion, there is no doubt that the best equations representing the slip flow are the Navier-Stokes equations together with the slip boundary conditions. Following the same technique of dimensional analysis as performed in Part I, the Navier-Stokes equations
for the slender channel can be reduced as below;

Continuity equation

\[
\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial z}(\rho v) = 0
\]  

(5)

x-momentum equation

\[
\rho u \frac{du}{dx} + \rho v \frac{dv}{dz} + \frac{1}{2} \frac{\partial P}{\partial x} = \frac{1}{\rho} \frac{\partial}{\partial x} \left( \rho \frac{\partial u}{\partial x} \right)
\]  

(6)

r-momentum equation

\[
\frac{\partial P}{\partial r} = 0
\]  

(7)

Energy equation

\[
\rho u \frac{dh}{dx} + \rho v \frac{dh}{dz} = u \frac{\partial h}{\partial x} + \frac{1}{2} \frac{\partial P}{\partial x} \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial x}^2
\]  

(8)

Equation of state

\[ p = \rho R T \]  

(9)

where 

\[ j = 0 \quad \text{for two-dimensional flow} \]

\[ j = 1 \quad \text{for axisymmetric flow} \]

The boundary conditions for this slip flow, however, will be different from those in Part I.

Boundary conditions;

at channel wall \((r = r_w);\)

\[ u = u_w \neq 0 \quad v = v_n \neq 0 \quad \rho = \rho_w \]  

(71)
at center line \((r = 0)\);

\[ u = u_c, \quad v = 0, \quad p = p_c. \]  

\[ \frac{\partial u}{\partial r} = 0, \quad \frac{\partial v}{\partial r} = 0. \]  

(3) **Slip Velocity, Temperature Jump and Wall Contour Modification:**

The solution of the above system of equations will not be determined completely as far as the boundary conditions are not known. There are quite a few proposed formulae for the slip velocity and the temperature jump at the wall for the slip flow, however, none of them is really accurate and satisfactory. For small velocity and temperature gradient, the slip velocity and the temperature jump at the wall can be approximated to \((5)\),

\[ u_s = \frac{2 - \sigma}{\sigma} \bar{\lambda} \frac{\partial u}{\partial r} + \frac{2}{\gamma} \frac{\rho}{\rho} \frac{\partial T}{\partial r} \]  

\[ (T_{gw} - T_w) = \frac{2 - \sigma}{\sigma} \frac{2}{\gamma + 1} \bar{\lambda} \frac{\partial T}{\partial r} \]  

where, \( \bar{\lambda} \) is the mean free path of the gas flow  
\( \sigma \) is the accommodation coefficient  
\( \tau \) is the reflection coefficient  
\( T_{gw} \) is the gas temperature at wall  
\( T_w \) is the wall temperature
The value of $\varphi^*$ is varied with the different surface conditions. According to experimental data\(^{(6)}\), $\varphi^*$ has a value ranging from 0.8 to 1.0 for different choices of surface conditions. For simplicity, it can be approximately considered

$$\frac{2 - \varphi^*}{\varphi^*} = \bar{\varphi} \approx 1$$

Neglecting the "Thermal Creep," i.e., the temperature gradient along the surface, then the slip velocity reduces to

$$u_w = \bar{\ell} \left( \frac{\partial u}{\partial z} \right)_w$$

Equation (75) implies two very simple formulas for $u_w$ and $(T_w - T_w)$ respectively, however, due to the undetermined velocity gradient and temperature gradient at the wall, the slip velocity and the temperature jump are still unknown. It is of interest to note that both the slip velocity and the temperature jump are linearly proportional to the velocity gradient and the temperature gradient at wall respectively, and
the proportional coefficients are identically the mean free path of the gas flow, $\bar{\ell}$. Thus the mean free path of the gas flow plays a very important role in the boundary conditions on this slip flow problem.

From the geometrical relationship shown in figure 22, it is not difficult to visualize that the mean free path, $\bar{\ell}$ in eqs (75) and (76) actually represents a length, which also could be interpreted as a virtual displacement distance backward from the wall so that the flow will result in zero velocity at this virtual wall. In other words, if the wall surface were displaced a distance, $\bar{\ell}$, then the velocity and the temperature profiles would extend linearly right up to zero velocity and wall temperature at the new, "virtual" wall, and the motion of this slip flow would be exactly the same as the motion of a non-slip flow with the new, "virtual" wall surface. Thus it is reasonable and promising to approach the slip flow problem by alternatively solving the corresponding non-slip case with the virtual, modified wall contour which is just larger than the original one by an increment of $\bar{\ell}$. (Please see fig. 23.)

The extending of the velocity and temperature profiles within the modified contour distance by linearly extrapolating the wall velocity and temperature gradient is approximate, but adequate, since the mean free path of
Figure 22: General Velocity Profile and Temperature Profile of the slip flow.

Figure 23: The Virtual Wall Contour Modification of a Channel under the Slip Flow Condition.
the gas flow is very small compared to the characteristic dimension (only about 1 to 10 percent) within the slip flow region. As the result of the above discussion, the determination of the mean free path of the gas flow becomes an essential part of the slip flow problems. From the kinetic theory of gases (7,8), the mean free path can be obtained through the expression,

\[ \frac{\mu}{p} \sqrt{\frac{nRT}{2}} = \frac{\mu}{\rho} \frac{1}{\sqrt{2\pi RT}} \]  (77)

Formula (77) implies that for both the incompressible and compressible flow, the mean free path will not be a constant since the temperature (for compressible case, density and viscosity, too) along the flow varies. Thus the modification of the channel wall as mentioned above will be non-uniform and varied from cross-section to cross-section. Fortunately, the change of the temperature along the channel for an incompressible flow is usually insignificant, and the mean free path for this case can be thus considered as a constant.

(4) Integral Equations:

As concluded from the preceding discussion, the fully viscous, slip flow problem inside the slender channel can be solved as a corresponding non-slip case with
the modified, virtual channel wall contour. In other words, the slip flow case can be treated as the non-slip one, as far as the slip boundary conditions have been properly replaced by a new non-slip boundary condition on the modified, virtual wall.

The slip flow boundary conditions in eq. (71) and (72) may be easily changed to the non-slip one providing the original wall is replaced by the virtual wall; thus, the new boundary conditions become:

at the virtual wall ($r = r_w = r_w + \delta$)

\[ u = 0, \quad v = 0, \quad p = p_w. \]  

(78)

at the center line ($r = 0$);

\[ u = u_c, \quad v = 0, \quad p = p_c \]

\[ \frac{du}{dr} = 0, \quad \frac{dp}{dr} = 0. \]  

(79)

**Momentum Integral Equation.** Integrating the momentum equation across the channel section in the same manner as in the Part I except that the outer boundary is not at the channel wall but at the modified, virtual wall, gives

\[ \frac{d}{dx} \int_{r_w}^{r_w} \rho u^2 r^2 dr = \left[ \nu \frac{d}{dr} \left( u \frac{du}{dr} \right) \right]_{r = r_w} - \frac{1}{r} \left[ \frac{d}{dr} \right] \left[ \frac{1}{r} \left( \frac{d}{dr} (ru^2) \right) \right]_{r = r_w} \]  

(80)
Energy Integral Equation. Similar to the momentum integral equation, the integration of the energy equation is also from the center line to the modified virtual wall. The result is found in the following:

\[
\frac{d}{dx} \int_0^{\tilde{z}_{\text{sw}}} (\rho u T \tilde{n}) \, dn = \tilde{n}_{\text{sw}} \left( \frac{\partial}{\partial \tilde{z}_{\text{sw}}} \frac{d T}{d \tilde{z}_{\text{sw}}} \right)_{\tilde{z}_{\text{sw}}} + H_{\text{sw}} \frac{d}{dx} \int_0^{\tilde{z}_{\text{sw}}} (\rho u n^2) \, dn
\]

(81)

where

\[
\left( \frac{\partial}{\partial \tilde{z}_{\text{sw}}} \frac{d T}{d \tilde{z}_{\text{sw}}} \right)_{\tilde{z}_{\text{sw}}} = \left( \frac{\partial}{\partial \tilde{z}_{\text{sw}}} \frac{d T}{d \tilde{z}_{\text{sw}}} \right)_{\tilde{z}_0} = \tilde{q}_{\text{sw}}
\]

since the heat transfer at the virtual wall will be the same as at the original wall if the temperature gradient is linearly extrapolated from the original wall to the modified virtual wall.

This energy integral equation for the slip flow contains an extra term in comparison to the case of non-slip flow. The reason for this is the fact that the mass flow rate between the original wall and the virtual wall of a channel is usually not constant, because

\[
\int_0^{\tilde{z}_{\text{sw}}} (\rho u n^2) \, dn = \int_0^{\tilde{z}_0} (\rho u n^2) \, dn + \int_{\tilde{z}_0}^{(\tilde{z}_0 + \tilde{e})} (\rho u n^2) \, dn
\]
The first term at the right-hand side of the above equation is always a constant, however, the second term is generally not a constant except in some very special cases such as the Hagan-Poiseuille flow which has both the constant $\bar{c}$, and the similar velocity profile.

**Mass Flow Integral Equation.** As shown in the above, the mass flow rate inside the modified virtual channel is not a constant in general, but the mass flow rate passing through the original, solid channel is a constant. Thus,

$$\dot{m} = 2 \int_{-\omega}^{\omega} \rho u (\pi \lambda) \, dl = \text{constant}$$  \hfill (82)

In which $\dot{m}$ is the actual mass flow rate, and the outer integration limit is $\omega$ instead of $2\omega$.

Thus so far, we have only shifted the unknown boundary conditions at the channel wall to another unknown length, $\bar{c}$ of the contour modification. Hence, the idea of contour modification has not gained anything practical unless further information or relationship of the mean free path can be obtained from the flow conditions.
(5) Incomparable, Two-dimensional Flow Case

In this case, j=0, \( P = \text{constant} \) and \( u = \text{constant} \). Consequently, the energy equation has the trivial solution, \( T = \text{constant} \). Thus the momentum integral equation and the mass flow rate equation are decoupled from the energy equation.

Momentum integral equation

\[
\frac{d}{dx} \int_0^{2\pi} u^2 \, dr = \frac{\nu}{\rho} \left( \frac{\partial u}{\partial r} \right)_r = 2\nu \frac{d^2 u}{dz^2} = 2\nu
\]

Mass flow integral equation

\[
\dot{m} = 2 \int_0^{2\pi} \rho u \, dr = \text{constant}
\]

Following the same procedure as performed in the non-slip case (Part I), a fourth order velocity profile is assumed. After applying the modified boundary conditions in eqs. (78) and (79), it is obtained,

\[
\frac{U}{U_c} = 1 + \lambda_s \eta_s^2 - (1 + \lambda_s) \eta_s^2
\]

where \( \eta_s = \frac{\nu}{2\nu} \) is a non-dimensional parameter

\[
\lambda_s = \frac{-1}{\lambda_s} \left( \frac{2\nu}{\rho u_c \frac{d^2 u}{dz^2} + 1.2} \right)
\]

is the pressure gradient parameter for the slip flow.
Although the velocity profile obtained here has exactly the same form as in the non-slip case, yet none of the parameters such as $\gamma_s$ and $\lambda_s$ were defined in the same way as in the non-slip case.

The momentum and the mass flow integral equations can be integrated out as in the following by using the velocity profile just obtained,

$$\frac{d}{dz} \left[ (\lambda_2^2 + 8\lambda_2 + 28) \sqrt{v_0} \frac{u_c^2}{\sqrt{v_0}} \right] = \frac{3\sqrt{5} u_c}{2v_0} (\lambda_2 + 1)$$

$$\dot{m} = 2 \rho u_c \sqrt{v_0} \left[ 1 + \frac{\lambda_2}{3} \frac{u_c^2}{v_0} - \frac{1 + \lambda_2}{5} \frac{u_c^2}{v_0^2} \right] = \text{constant}$$

eliminate the center line velocity, $u_c$ from the two, it results,

$$\frac{d}{dz} \left[ \frac{\sqrt{v_0} (\lambda_2^2 + 8\lambda_2 + 28)}{\sqrt{v_0} \left( 1 + \frac{\lambda_2}{3} \frac{u_c^2}{v_0} - \frac{1 + \lambda_2}{5} \frac{u_c^2}{v_0^2} \right)^2} \right]$$

$$= \frac{630}{\mu} \frac{(\lambda_2 + 1)}{\sqrt{v_0} \sqrt{v_0} \left( 1 + \frac{\lambda_2}{3} \frac{u_c^2}{v_0} - \frac{1 + \lambda_2}{5} \frac{u_c^2}{v_0^2} \right)}$$

(88)

In order to simplify the above equations, a local Knudsen number, $KN = \frac{L}{2v_0}$ for this slip channel flow is introduced. Then,
\[ \frac{U}{U_c} = 1 + \frac{\lambda_s}{(1 + Kn)^2} \eta^2 - \frac{(1 + \lambda_s)}{(1 + Kn)^2} \eta^4 \]  

(89)

where

\[ \eta = \frac{n}{n_w} \]

\[ \lambda_s = -\frac{1}{10} \left[ \frac{(1 + Kn)^2}{\mu U_c} \frac{d}{dz} + 12 \right] \]  

(90)

and

\[ \dot{m} = 2 \rho U_c n_w \left[ 1 + \frac{\lambda_s}{3(1 + Kn)^2} - \frac{(1 + \lambda_s)}{5(1 + Kn)^4} \right] \]  

(91)

\[ \frac{d}{dz} \left\{ \frac{(1 + Kn)(\lambda_s^2 + 8\lambda_s + 28)}{2w \left[ 1 + \frac{\lambda_s}{3(1 + Kn)^2} - \frac{(1 + \lambda_s)}{5(1 + Kn)^4} \right]^2} \right\} \]

\[ = \frac{630\mu}{\dot{m} n_w} \frac{(1 + \lambda_s)}{1 + \frac{\lambda_s}{3(1 + Kn)^2} - \frac{(1 + \lambda_s)}{5(1 + Kn)^4}} \]  

(92)

The above system of equations (from eq. 89 to 92) can be solved in the same way as in the non-slip case providing that the value of the local Knudsen number, \( Kn \) is already determined. By definition, the local Knudsen number of the flow is a function of both the flow conditions.
and the channel contour shape. Thus $K_n$ is generally subjected to vary along the channel. However, within the limitation of the slender channel and incompressible flow characteristics, the variation on $\bar{\omega}$ and $\lambda_s/2 \times r$ are quite small and intended to balance each other, since

$$\bar{\omega} = \sqrt{\frac{\pi}{2}} \frac{\lambda_s}{r} \approx \frac{1}{\sqrt{\rho}}$$

$$\lambda_s \approx A_{sc} \approx \frac{1}{\alpha} \approx \frac{1}{\sqrt{\rho}}$$

Hence, it is reasonable to consider the local Knudsen number, $K_n$ for the present case as a constant throughout the whole channel. The value of this constant can be then determined from the known initial conditions, (e.g., the flow conditions at the inlet of the channel).

After calculating the value of the local Knudsen number, eq. (92) contains then only two parameters; $\lambda_s$ and $\lambda_s$. Thus it is possible either to determine the wall contour shape for similar solutions, or to calculate the flow properties for any arbitrary non-similar problems.

**Similar Solutions.** For similar solution, the pressure gradient parameter must be a constant throughout the channel, thus equation (92) can be arranged as

$$\frac{d\lambda_s}{d\bar{\omega}} = \frac{630/\lambda_s (\lambda_s + 1) \left(1 + \frac{\lambda_s}{3(1+K_n)^2} - \frac{9 + \lambda_s}{8(1+K_n)^2}\right)}{\frac{1}{\frac{\lambda_s}{2} (1+K_n)^2 (\lambda_s^2 + 8\lambda_s + 28)}} = c_1 \quad \text{(constant)}$$
The above result of the similar solutions shows a quite familiar relationship of the linear channel wall contour, which is coincident with the non-slip case. The slip effect parameter, $K_s$ apparently has no influence on the shape of the channel contour for similar solutions, however, it does affect the constants $C_1'$ and $C_2'$ implicitly.

Considering the special case of $\lambda_s = 1$, from above it is not difficult to find that $C_1' = 0$, and $C_2' = C_2'$ = constant. Thus the slip parallel flow again forms a special case of the similar solutions.

A) **Slip Flow Inside Two Parallel Walls**

This flow, as mentioned above, is a special case of the similar solutions. With the values of $\lambda_s$ and $K_s$ equal constant, it is easy to obtain $\lambda_s = 1$. Then from eq. (89) and eq. (90), we obtain the velocity profile

$$\frac{U}{U_c} = \left[1 - \frac{1}{(1 + K_s)^2} \gamma^2\right]$$

Where

$$U_c = - (1 + K_s)^2 \frac{2\beta}{2u} \frac{d\beta}{d\lambda}.$$
\[ u = -\frac{1}{2\mu} \frac{dP}{dx} \left[ (1+K_n)^{\frac{2}{3} - \frac{2}{3} K_n} \right] \]

This is again a parabolic velocity profile which contains only a minor modification from the slip effect parameter, \( K_n \). It is important to notice here that the velocity at wall obtained from the above formula will no longer be zero, but equal to some finite value; i.e.,

\[ u_w = -\frac{2w}{2\mu} \frac{dP}{dx} (2K_n + K_n^2) \quad \text{(slip velocity)} \]

The mass flow rate can also be obtained from eq. (91)

\[ \dot{m} = -\frac{2w}{\mu} \frac{dP}{dx} \left[ \frac{2}{3} + 2K_n + K_n^2 \right] \]

It is clear that both the slip velocity and the mass flow rate are proportional to the Knudsen number. The larger the value of \( K_n \), the bigger the slip velocity and the mass flow rate. For zero value of \( K_n \), then \( u_w \) reduces to zero, and \( \dot{m} \) becomes the same as the non-slip flow case.

In order to show the degree of slip effect on the velocity profiles, two cases of different Knudsen number have been calculated and plotted in the figures (24).
with the comparison of non-slip case. For $K/N = 0.1$, which is roughly the upper limit of the slip flow, the flow has the larger slip effect; while for $K/N = 0.05$, the flow only displays a moderate slip effect. The increase of the mass flow rate due to the slip velocity at the channel wall is shown in fig. 25 for various values of the Knudsen number.

B) **Slip Flow Inside the Convergent and Divergent Channels.**

According to equation (93), any channel with a linear wall contour will result in a similar solution no matter if it is convergent or divergent.

Consider now the flow problem of a convergent (or divergent) channel with a $5^\circ$ half angle. Then from geometrical relationships, it is obtained

$$\frac{d\bar{w}}{d\bar{x}} = \pm 0.0875 \begin{cases} - & \text{sign for convergent} \\ + & \text{sign for divergent} \end{cases}$$

Eq. (92) becomes

$$\pm 0.0875(\lambda_s^2 + 8\lambda_s + 28) = \frac{-630\pi}{2\mu \kappa w} \frac{(\lambda_s + 1)}{(1 + K)^2}$$

introduce the Reynolds number

$$Re = \frac{ucL}{\nu} = \frac{ucL}{\kappa w \sin 5^\circ}$$

the above equation, finally we have
Figure 24: Velocity Profiles between the Parallel Walls for different degree of Slip.
Figure 25: Slip Mass flow Rate in Comparison to the Non-slip Case for various values of the Knudsen Number.
The above equation provides us a means to determine the value of $\lambda_s$ for any given $R_o$ and corresponding $K_n$. For example, if $R_o = 684$, then

for high subsonic ($M = 0.75$), $K_n = 0.03$

$\begin{cases} 
\lambda_s = -0.584 \text{ (convergent case)} \\
\lambda_s = -1.335 \text{ (divergent case)}
\end{cases}$

for low subsonic ($M = 0.4$), $K_n = 0.015$

$\begin{cases} 
\lambda_s = -0.598 \text{ (convergent case)} \\
\lambda_s = -1.326 \text{ (divergent case)}
\end{cases}$

The velocity profile of the above results are plotted in figures (26) and (27) with the comparison of non-slip case.

(6) Incompressible, Axisymmetric Flow Case,

For incompressible, axisymmetric flow, $j=1$, $\rho = \text{const}$, and $\alpha = \text{constant}$. The energy equation again has the trivial solution; $T = \text{constant}$.

Momentum integral equation becomes

$$\frac{d}{d\alpha} \int_0^{2\pi} \frac{u^2 r^2}{\rho} d\alpha = \frac{2}{\pi} \left[ (\frac{\partial u}{\partial \alpha})_{\alpha = \alpha_r} - (\frac{\partial u}{\partial \alpha})_{\alpha = \alpha_m} \right]$$

(94)

Mass flow integral equation becomes

$$m = 2\pi \int_0^{2\pi} \rho u r d\alpha = \text{constant}.$$  

(95)
Fig. 26: Velocity Profiles of the Slip Flow inside the Convergent Channel with Plane Walls.
K_n = 0 (no slip)

K_n = 0.015

K_n = 0.03

Re = 684

Fig. 27: Velocity Profiles of the Slip Flow inside the Divergent Channel with Plane Walls.
Assume a fourth order velocity profile and apply the boundary as well as the compatibility conditions as before, we obtain the velocity profile in the following:

\[ \frac{u}{u_c} = 1 + \lambda_{1s} \eta_s^2 - (1+\lambda_{1s}) \eta_s^4 \]  \hspace{1cm} (96)

where

\[ \eta_s = \frac{x}{x_{in}} \]

\[ \lambda_{1s} = -\frac{1}{12} \left( \frac{\rho_{in}}{\rho_u} \frac{d^2}{dx^2} + 16 \right) \]

Integrate eq. (94) and (95) by employing the velocity profile of eq. (96), it results

\[ \frac{d}{dx} \left[ (\lambda_{1s}^2 + 7\lambda_{1s} + 16) \rho_{in} u_c^2 \right] = 240 \pi u_c (\lambda_{1s} + 1) \]  \hspace{1cm} (97)

\[ \rho_i = \pi \rho u_c \rho_{in} \left( 1 + \frac{\lambda_{1s} \rho_{in}^2}{2 \rho_{in}} - \frac{(\lambda_{1s} + 1) \rho_{in}^2}{3 \rho_{in}} \right) \]  \hspace{1cm} (98)

Eliminate \( u_0 \) from eq. (97) by using the relation in eq. (98) a single final equation is obtained

\[ \frac{d}{dx} \left\{ \frac{\rho_{in}^2 (\lambda_{1s}^2 + 7\lambda_{1s} + 16)}{2 \rho_{in} \left( 1 + \frac{\lambda_{1s} \rho_{in}^2}{2 \rho_{in}} - \frac{(\lambda_{1s} + 1) \rho_{in}^2}{3 \rho_{in}} \right)^2} \right\} = \frac{240 \pi u_c}{\rho_i} \frac{(\lambda_{1s} + 1)}{\rho_{in}^2 \left( 1 + \frac{\lambda_{1s} \rho_{in}^2}{2 \rho_{in}} - \frac{(\lambda_{1s} + 1) \rho_{in}^2}{3 \rho_{in}} \right)} \]  \hspace{1cm} (99)
Similar to the previous section, a local Knudsen number, $K_n = \frac{\rho}{\rho w}$ is introduced into the above system of equations. Then

$$\frac{U}{u_c} = 1 + \frac{1}{(1 + K_n)^2} \eta^2 - \frac{1 + \frac{1}{2} \left(1 + K_n\right)^2}{(1 + K_n)^2} \eta^4$$

(100)

where

$$\begin{cases} \eta = \frac{\rho}{\rho w} \\ \lambda_{1s} = -\frac{1}{12} \left(1 + K_n\right)^2 \frac{\rho w^2}{u_c} \frac{d^2 P}{dz^2} + 16 \end{cases}$$

(101)

and

$$\Pi n = \Pi U_c \Pi w \left[1 + \frac{1}{2} \left(1 + K_n\right)^2 \frac{\rho w^2}{u_c} \frac{d^2 P}{dz^2} + 16 \right]$$

(102)

$$\frac{d}{dz} \left\{ \frac{1}{\lambda_{1s}} \left[1 + \frac{1}{2} \left(1 + K_n\right)^2 \frac{\rho w^2}{u_c} \frac{d^2 P}{dz^2} + 16 \right] \right\}$$

(103)

$$= \frac{2\rho U}{\Pi n} \left(\lambda_{1s} + 1\right)$$

$$\Pi n \left[1 + \frac{1}{2} \left(1 + K_n\right)^2 \frac{\rho w^2}{u_c} + 16 \right]$$

With the same argument as in the previous section, the local Knudsen number here could be considered as a constant throughout the whole channel. The value of this constant depends, of course, on the flow initial conditions. After determining the value of the local Knudsen number, equation (103) can be thus either determined $r_w$ or $\lambda_{1s}$ depending which one of the two is given.
**Similar Solutions.** The pressure gradient parameter, $\lambda_1$, of a similar solution must be a constant along all the $x$, thus equation (103) can be arranged as below:

\[
\frac{1}{\rho_r} \frac{d^2 \rho_r}{dx^2} = -\frac{120 \pi}{\sin (1+\lambda_1)} \left( 1 + \lambda_1 \right) \left( 1 + \frac{\lambda_1}{2(1+K_w)^2 - \lambda_1 + 1} \right)
\]

\[= C_3' \quad \text{(constant)} \]

integrate to obtain

\[\rho_r = C_4' e^{C_3' x} \quad (104)\]

The result of the above finding shows the same interesting characteristics as in the non-slip case that the similar solution exists only in the nozzle with the exponential wall contour shape.

There is also a special case of similar solution for this flow condition, i.e., when $\lambda_1 = -1$, $C_3' = 0$, then $\rho_r =$ constant. Obviously, this is the Hagan-Poiseuille pipe flow with slip velocity at the wall.
A) **Slip Flow inside the Circular Pipe.**

It is learned from above that this flow is a similar problem with pressure gradient parameter equal constant along all the $x$. The value of the $\lambda_{is}$ (which is $-1$ here), can be easily determined from eq. (103) with $K_n$ and $E_m$ equal constant. Then

$$\frac{U}{U_c} = 1 - \frac{1}{(1+K_n)^2} \lambda^2$$

$$U_c = -(1+K_n)^2 \frac{\lambda_{Is}^2}{\lambda} \frac{dJ}{dx}$$

$$U = -\frac{1}{\lambda} \frac{dJ}{dx} \left[(1+K_n)^2 \lambda_{Is}^2 - \lambda^2\right]$$

$$= -\frac{1}{\lambda} \frac{dJ}{dx} \left[\lambda_{Is}^2 - \lambda^2 + 2\lambda_{Is} \lambda + \lambda^2\right] \quad (105)$$

This result is nearly the same as the one obtained by Kennard(6) except an additional high order term on the mean free path, $\bar{\lambda}$ which can be considered as the second order correction for the slip effect.

The slip velocity at the wall and the slip mass flow rate are easily determined in the following:

$$U_{ws} = -\frac{\lambda_{Is}^2}{\lambda} \frac{dJ}{dx} \left[2K_n + \lambda_{Is}^2\right] \quad (106)$$
Here, the Knudsen number has the same effect on the slip velocity and the mass flow rate as in the two-dimensional case, i.e., the larger the Knudsen number, the greater the slip velocity at the wall, and the more the mass flow rate through the pipe. However, due to the three-dimensional circumference effect, the mass flow rate inside the pipe is much more sensitive to the Knudsen number than those in the two-dimensional case.

Similar to the two-dimensional case, two typical slip velocity profiles with the different values of Knudsen number have been computed and compared to the non-slip case in figure (28). One is for $K_N = 0.1$ which symbolizes the highly slip case; the other is for $K_N = 0.05$ which represents only the moderate slip case. The increase of the slip mass flow rate for various values of the Knudsen number is plotted in figure (25) with the comparison of the two-dimensional case. It is clearly shown that for the same Knudsen number the increase of the mass flow rate due to slip effect will be greater for the axisymmetric flow case than the two-dimensional flow case.
Fig. 28: Velocity Profiles of the Circular Pipe for different degree of Slip.
B) Slip Flow Inside the Nozzle with an Exponential Wall Contour.

For the convenience of comparison, let us consider the same exponential contour shape nozzle as solved in section 6B, Part I. The equation governed the nozzle wall contour is

\[
\frac{\rho u}{(\rho u)_0} = \pm 0.0804 \frac{x}{(\rho u)_0} \quad \{+ \text{ for divergent} \quad - \text{ for convergent} \}
\]

consequently,

\[
\frac{d(\rho u)}{dx} = \pm 0.0804 \frac{(\rho u)_0}{(\rho u)_0}.
\]

Since this is a similar problem, the pressure gradient parameter will be independent from the x. Thus eq. (103) can be rearranged as

\[
- \frac{120 \pi^4}{m^2} \frac{(\rho u)_0 (\lambda_{15} + 1)}{(1 + K\nu)^2 (\lambda_{15} + 7 \lambda_{15} + 16)} \left[1 + \frac{\lambda_{15}}{2(1 + K\nu)^2} - \frac{\lambda_{15} + 1}{3(1 + K\nu)^2}\right] = \pm 0.0804
\]

After introducing the Reynolds number at the inlet of the nozzle

\[
Re = \frac{(u_0) (\rho u)_0}{\nu} = \frac{m}{\pi \mu (\rho u)_0} \left[\frac{\lambda_{15}}{2(1 + K\nu)^2} - \frac{(\lambda_{15} + 1)}{3(1 + K\nu)^2}\right]
\]

the above equation can be reduced to the form

\[
(\lambda_{15} + 7 \lambda_{15} + 16) \pm \frac{1292.5}{Re (1 + K\nu)^2 (\lambda_{15} + 1)} = 0
\]
where \( + \) sign is for divergent case, and \( - \) sign is for convergent case.

This quadratic equation permits us to determine the value of \( \lambda_{is} \) for any given \( R^* \) and corresponding \( K^* \).

Four different cases for both the convergent and the divergent nozzles have been worked out. The results of these solutions are shown below, and the velocity profiles are plotted in figures (29) and (30).

For divergent case; let \( Re = 59 \),

then for \( K_n = 0.07 \) (\( M \sim 0.5 \)), \( \lambda_{is} = -1.375 \)

for \( K_n = 0.04 \) (\( M \sim 0.3 \)), \( \lambda_{is} = -1.357 \)

For convergent case; let \( Re = 49 \),

then for \( K_n = 0.07 \) (\( M \sim 0.5 \)), \( \lambda_{is} = -0.551 \)

for \( K_n = 0.04 \) (\( M \sim 0.3 \)), \( \lambda_{is} = -0.582 \)
Figure 29: Velocity Profiles of the Slip Flow inside the Convergent Nozzle with an Exponential Wall Contour.
Figure 30: Velocity Profiles of the Slip Flow inside the Divergent Nozzle with an Exponential Wall Contour.
PART III: EXPERIMENTAL WORK AND COMPARISON

(1) Introduction:

In the previous parts, an analytical method has been developed to solve both the similar and non-similar problems of a fully viscous flow inside the slender channels. Among those typical examples which have been examined, however, only the exact and the similar solutions could be compared with the existing results; the non-similar solution for the conical nozzle was then left without any comparison, since there were no other solutions or data available for comparison.

The main object of this phase of the study was an attempt to make an experimental comparison with the previous analytical solution for the non-similar problem of the conical nozzle flow. In addition, this experiment will also test the validity of the so-called "fully viscous" flow inside the slender nozzle if the flow condition is subjected to a considerable low Reynolds number range. The scope of the test is to obtain the velocity profiles at several different sections along the conical nozzle. In order to see the variation of the velocity profile with respect to the characteristic parameter \( \frac{2\pi r}{\nu} \), three different mass flow rates were used in the test.
The largest, $\dot{m}=6.7 \times 10^{-6}$ slug/sec, corresponded to the case of $\dot{m}/\mu = 214.59$ in. (\(\mu=37.5 \times 10^{-8}\) lb•sec/ft.\(^2\) for air); the next, $\dot{m}=5.36 \times 10^{-6}$ slug/sec, corresponded to the case of $\dot{m}/\mu = 171.67$ in.; the smallest, $\dot{m}=2.68 \times 10^{-6}$ slug/sec, corresponded to the case of $\dot{m}/\mu = 85.84$ in. The means employed in the velocity profile measurement was the usual pitot tube-transducer combination. However, due to the low Reynolds number effect, the total pressure was expected to be very small and difficult to measure.

Degassing procedures were necessary for this low density flow test; however, because of the temperature limitation on the vacuum pumps, the degassing time allowed was only around half an hour.

(2) Test Facility:

The flow system of the test facility can be divided into three components which are shown schematically in the flow diagram, figure 31.

A) The first component of the flow system is the high-pressure gas tank and the associated mass flow control panel. The air flow from the high-pressure tank flowed firstly into a fore pressure dome which could be regulated at a pressure range from 0 to 800 lb./in.\(^2\). Immediately down-stream was another similar dome for back pressure set up. Between these two domes there was a flow control
Fig. 31: Flow Diagram of the Test Facility.
throttle which could directly control the air mass flow rate back from the panel. The mass flow rate here was measured by a Foxboro, Type 13HA d/p cell transmitter, which provided a set of different sized orifices for measuring several different flow ranges.

B) The second component of the flow system is the conical nozzle. This nozzle was made with plastic and in accordance to the same shape as the one used in example (6G) of Part I (see Fig. 32). In order to compare the previous analytical, non-similar solution under exactly the same conditions, we thus chose the same four stations A, B, C and D along the testing nozzle where the velocity profiles are to be determined. The means employed to determine the velocity profile is an impact pressure probe plus a very sensitive 0.03 psid transducer which directly connects to an analog computer for output reading. Since the velocity is proportional to the square root of the differences between the total pressure and the static pressure, the reference side of the transducer was connected to the testing nozzle wall at the same station where the impact pressure probe is located.

c) The last component of the flow system was the vacuum pump station which was connected at the end of the testing nozzle. The vacuum pump station used in the Aeronautical and Astronautical Research Laboratory, OSU, is a high pumping capability, three-stage system. It can
Figure 32: Configuration of the Test Conical Nozzle.
accept 12,000 ft.\(^3\)/min. mass flow at the very low pressure of 2 mm Hg. At this low pressure, however, the pumping system tends to overheat, and operation is limited to about one hour. The actual testing time for one run is thus only around a half hour since it usually took about forty minutes to complete the degassing procedure.

(3) Instrumentation:

The principle instruments involved in this experimental work are the mass flow rate gauge and the impact pressure transducer. Because of the required low Reynolds number operation, both the instruments were operated in the lower end of their useful ranges, and, consequently, were difficult to control and to read accurately.

A) Calibration of the pressure transducer: The values of the impact pressures encountered in this low Reynolds number flow testing were expected to be quite small, hence the pressure transducer chosen for this experimental measurement was that having lowest range available in the Research Laboratory. It is a Carrier Demodulator 0.03 psid, transducer (Model CD 10) made by Pace Engineering Company. Due to its extremely sensitive characteristics, the calibration of this pressure transducer was difficult and had to be performed in a closed-door chamber without any kind of disturbance. The calibration curve is shown in Figure 33 and appears satisfactory.
1 Volt = 3x10^{-5} \text{ psi}

Figure 33: Calibration Curve of the Pressure Transducer.
B) Calibration of the mass flow rate gauge: In order to successfully simulate the low Reynolds number, fully viscous flow characteristics, the mass flow rate of the testing should be kept under certain low values, say between \(2.68 \times 10^{-6}\) slug/sec, to \(6.7 \times 10^{-6}\) slug/sec, in accordance with the previous analytical solutions. The Reynolds numbers based on the inlet diameter of the testing nozzle corresponding to those flow conditions are approximately within the range from \((Re)_p = 62.5\) to \((Re)_p = 156\).

As stated before, the mass flow rate of the present flow system depends totally on the size of the orifice inside the d/p cell transmitter, and the set values of the two dome pressures. (Actually, only the back dome pressure is important and critical). However, it is not possible to calculate the best combination of these three values which will give the most suitable mass flow range. Thus by trial and error, the following combination was selected:

\[
\begin{align*}
\text{Orifice size} &= 0.034 \text{ in}, \\
\text{Fore dome pressure set at} &= 50 \text{ psi}, \\
\text{Back dome pressure set at} &= 20 \text{ psi}.
\end{align*}
\]

The method employed here to calibrate the mass flow gauge is the common one. It only requires a solid, tight air tank whose volume is previously known or calculated. First of all the pressure inside the solid tank was recorded
and the mass flow rate gauge was set at a certain value, then the valve at the inlet was opened and the air allowed to flow into the tank (A stop-watch was started at the same time.) After a minute or so, the valve was closed (The stop-watch was stopped at the same time.), then the final pressure inside the solid tank was recorded. The mass flow rate for that gauge reading thus was calculated from the values of the pressure increase and the time consumed by using the equation,

$$\dot{m} = \frac{V}{RT} \left( \frac{\Delta P}{\Delta t} \right)$$  \hspace{1cm} (109)

where $V$ is the volume of the solid tank, $R$ is the gas constant and $T$ is the tank temperature. For the tank used and the environmental temperature maintained, it is found that the constant $\frac{V}{RT} = 1.0872 \times 10^{-6}$ ft$^3$.sec$^{-2}$.

The calibration curve for the mass flow rate gauge is shown in Figure 34. It may be noted that the three marked testing mass flow values are well located within the range of mass flow rate gauge. Figure 35 is the mass flow rate calibration data plotted against the square root of the gauge reading. It almost falls onto the straight line which should be the theoretical limit of those experimental data points.
Fig. 34: Calibration Curve of the Mass Flow Rate Gauge.
Fig. 35: Mass Flow Rate Calibration Data.
(4) Testing Procedure and Data Reduction:

The testing procedure of this experiment was quite simple and direct. However, due to the low Reynolds number and low density effects, there are two particular procedures which should be mentioned early in order to secure a good measurement. The first and the most important one is the degassing procedure, which requires the vacuum system operating for a certain period in order to eliminate the surface gas inside the flow system before taking any data. As proved by experience, this degassing procedure did give a much better pressure reading without the influence of the gas inside the system. The second one is the stability of the mass flow rate control. It was very difficult to keep the testing mass flow rate remaining constant for a relatively long period, hence, the pressure data had to be taken quickly as soon as the air flow became steady at the desired mass flow rate.

After starting the vacuum pumps stage by stage for about half an hour (degassing effort), the air was then allowed to flow into the testing nozzle at one of the selected mass flow rates. Once the flow became steady, the line connecting the pressure transducer to the pitot tube was opened quickly so that the impact pressure data for this particular mass flow rate and probe position could be recorded by the computer. This same procedure was repeated for the other mass flow rates and probe positions.
until sufficient data points were obtained across the test section.

The recorded pressure differences were used to determine the relative velocity distributions, \(u/u_c\), by using the Bernoulli equation for low speed flow,

\[
\frac{u}{u_c} = \sqrt{\frac{\Delta P'}{\gamma \Delta T}}
\]  

(110)

These profiles were then integrated to obtain the centerline velocity from

\[
\rho u_c = \frac{\pi}{2\pi \lambda \rho} \int_0^\infty (u_c) n d n
\]  

(111)

Their reduced data for the divergent case are tabulated in Table 2 and 3 respectively.
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(5) Comparison of Theory with Experiment:

The result of the velocity profile data, as well as the centerline velocity data are plotted in Figures 36, 37, 38, 39 and 40 with the comparison to the theoretical solutions from Part I.

It is clear that the experimental data are somewhat scattered and only fair in comparison to the theoretical curves. At the very low values of pressure for which the experiments were conducted, the instrumentation was at the limits of its capabilities and large errors were possible. It was conservatively estimated that an error of 10% in the relative velocity profile was not unlikely. (See Appendix.) However, the data do show consistency and verify the theoretical predictions in many respects:

a) The experimental velocity data clearly show fully developed velocity profiles which are definitely different from the boundary layer type velocity profiles. Thus the existence of the so-called fully viscous flow in the nozzle under the test flow conditions are verified.

b) From the theoretical prediction, the lower the mass flow rate, the fuller the velocity profile will be. The experimental data do show this trend in all the test stations.

c) At different test stations, there are theoretical limit values of \( \bar{m} \) for non-separated velocity profiles (See fig. 36 to fig. 39.) All the three testing mass flow rates are selected to be within the theoretical limit
of separation. The experimental results do show agreement with the theory that no separation or back flow phenomena appear at these test stations.

d) The theory shows that the velocity profiles of the three different mass flow rates are very similar at station A, and become widely separated at downstream Station C and D. The experimental data obtained in the test seem to be again in agreement with these theoretical characteristics.

(e) The centerline velocity distribution, according to the theory, is quite different from the inviscid solution. The experimental data obtained may be observed to support the theory in Part I.
Fig. 36: Comparison of the Velocity Profiles at Station A of the Divergent Nozzle
Fig. 37: Comparison of the Velocity Profiles at Station B of the Divergent Nozzle
Fig. 38: Comparison of the Velocity Profiles at Station C of the Divergent Nozzle
Fig. 39: Comparison of the Velocity Profiles at Station D of the Divergent Nozzle
Fig. 40: Comparison Of the Centerline Velocity along the Divergent Conical Nozzle
SUMMARY AND CONCLUSIONS

The principal object of the present study was to develop an integral method which could both determine and solve the similar solutions as well as the non-similar problems for fully viscous flow inside slender channels under various flow conditions. Due to the fact that fully viscous phenomena occurs only in the low Reynolds number flow region where the slip effect might become possible and significant, this unique integral method has been thus further developed and extended to include the slip flow region.

The integral method employed in this study is similar to the Karman-Pohlhausen technique used in boundary layer theory. However, the boundary conditions are more involved, especially in the slip flow case, for both the boundary conditions at the centerline and at the wall are unknown.

The finding of the similar solutions by using the present integral method are very successful for most of the flow conditions. Their results are also in good agreement with the previous existing exact solutions. The important findings and evidence of the similar solutions for different flow conditions are summarized in the following;
1) For two-dimensional, incompressible flow, similar solutions exist only in channels with plane walls. The channel can be either convergent, divergent, or parallel in accordance to the value of the pressure gradient parameter, \( \lambda \) whether it is respectively greater than, smaller than, or exactly equal to \(-1\).

2) For axisymmetric, incompressible flow, similar solutions exist only in nozzles with an exponential wall contour. The nozzle can also be either convergent or divergent depending upon the value of the pressure gradient parameter, \( \lambda \), whether it is greater or smaller than \(-1\) respectively. For the special case of \( \lambda = -1 \), \( r_w \) becomes constant, then the resultant solution reduces directly to the classical Hagan-Poiseuille's pipe solution.

3) For compressible, two-dimensional flow, the general similar solution results in a quite complicated geometric form of the transcendental functions. Even in the density-distorted transformed plane, only one simple particular similar solution can be obtained, which is the special case of \( \lambda = -1 \) and \( R_w \propto X^{1/2} \). This finding implies that if the wall contour shape, \( R_w \), of a channel is proportional to \( X^{1/2} \) in the transformed plane, then the velocity profiles of the whole channel will be similar and in a parabolic shape. However, it must be remembered here that the above particular similar solution is only
a fictitious shape in the transformed plane, but not the actual physical shape of the channel. The actual physical shape of the channel can be obtained by transforming the solution, \( R \omega \propto X^{\frac{3}{2}} \) back to the original plane. But due to the various possible density conditions, the inverted Doronitzn's transformation may generate many different solutions in the physical plane. Thus in compressible, two-dimensional flow, similar solutions are actually possible for a variety of wall contour shapes.

4) For slip flow, both the two-dimensional and axisymmetric cases have the same similar solutions as their corresponding non-slip cases, i.e., convergent or divergent channel with plane walls for the two-dimensional flow, and convergent or divergent nozzle with the exponential wall for the axisymmetric flow. However, the velocity profile of the slip flow is not a single-parameter function like the non-slip case, instead it is a two-parameter function which contains both the pressure gradient parameter and the Knudsen number of the flow. Thus a similar solution of a particular value of \( \lambda_s \) can have a family of similar velocity profiles corresponding to the different values of the Knudsen number of the flow.

The applications of the present integral method in the non-similar problems are also satisfactory for most of the cases. In the present scheme, it is possible to
integrate and combine the momentum, energy and mass flow integral equations to become a final first-order ordinary differential equation containing only the pressure gradient parameter, $\lambda$, and the channel wall radius, $r_w$. Once the flow condition is given, then the non-similar velocity profiles along the nozzle can be determined for any desired wall contour shape. One typical example of the non-similar problem, which concerns the flow in a convergent or divergent conical nozzle, has been completely worked out and illustrated. Some results of the solutions are quite interesting and worthy of mention below;

First, the most suitable characteristic parameter found in this internal flow problem is the parameter $(\frac{z}{u})$, which, in a certain sense, is equivalent to the Reynolds number of the external flow problem.

Second, the value of the pressure gradient parameter along the nozzle, both for convergent and divergent cases, is only sensitive to large values of the characteristic parameter, $(\frac{z}{u})$. For small values of $(\frac{z}{u})$, of the order of $21.46$ in, or less, the pressure gradient parameter changes insignificantly along the nozzle. Thus it is feasible to conclude that for extremely small $(\frac{z}{u})$, the pressure gradient parameter will be nearly constant throughout the nozzle, and the solution of the velocity profile becomes practically similar.
Third, the variation of the pressure gradient parameter as well as the centerline velocity, are very sharp and strange at the inlet of the nozzle, but become gradually flat near the end of the exhaust. This behavior indicates that the fully viscous effect causes the convergent flow to decelerate, and the divergent flow to accelerate near the vicinity of the nozzle inlet, which is obviously different from the inviscid solutions.

Due to the lack of any existing non-similar solutions to compare with the results by the integral method, a series of experiments were conducted to compare with the analytical non-similar solutions. The data obtained are generally in good agreement with the theory. The tests verified the existence of fully viscous flow inside the slender channel, and demonstrated the effectiveness of the present method to solve the flow problems in this category. Of course, further experimental investigations are strongly recommended and encouraged, especially in the slip flow region, so that the applicability of the present integral method can be further verified.
APPENDIX

ANALYSIS OF ERROR

Due to the very low pressure measurement in the present experiment, the impact pressure $\Delta p (= pt - p_{static})$ may have a rather large error, since the instrumentation was at the limits of its capabilities and accuracy. Besides, the environment temperature and the line degassing condition could also contribute some error. However, when the velocity profile data is reduced from the formula, $\frac{\mu}{\nu} = \sqrt{\frac{\Delta p}{\rho}}$
a large part of the error should be cancelled automatically within the ratio of the two measurements. Since all the velocity profile data were referred to the measurement at the centerline ($\Delta p_0$), then the farther away from the centerline, the more error involved in the data is expected. Estimating a relative error of $\pm 20\%$ in the measured pressure difference, then the possible maximum error in the velocity profile is determined as $\pm 10\%$.

Error enters the centerline velocity data, ($\rho \nu_c$) through two uncertainties; the mass flow rate and the integral, $\int_0^\gamma \frac{\rho}{\nu_c} d\gamma$. With the present mass flow rate calibration method, only $\pm 5\%$ error is expected. The error of the integral, $\int_0^\gamma \frac{\rho}{\nu_c} d\gamma$ can be calculated from the error of the velocity profile, ($\frac{\nu}{\nu_c}$) by using the following linear approximation:
Total error of $\int_0^1 \frac{d}{d\eta} \eta d\eta \approx$ shaded triangular area
\[ = \frac{1}{2} (0,1) (1) = 0.05 \text{ or } 5\% \]

Hence, the error of $(\mathcal{M}_c)$ will be around $\pm 10\%$. 
REFERENCES


