CONTRIBUTIONS TO THE ERGODIC THEORY
OF SEMI-MARKOVIAN OPERATORS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

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The Ohio State University
1969

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ACKNOWLEDGMENT

The author wishes to express his deep gratitude to his teacher, Professor Louis Sucheston, for the personal interest and attention given to the development of the author's mathematical career, secondly for suggesting the topics studied in this dissertation, and finally for the numerous discussions and constant encouragement.
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I. INTRODUCTION

A measurable space is a pair \((X, \mathcal{A})\) in which \(X\) is an abstract set and \(\mathcal{A}\) is a \(\sigma\)-algebra of subsets of \(X\): \(\mathcal{A}\) contains \(X\) and the empty set \(\emptyset\) and is closed under complementation and countable unions. A measure space is a triple \((X, \mathcal{A}, \mu)\) in which \((X, \mathcal{A})\) is a measurable space and \(\mu\) is a measure defined on \(\mathcal{A}\); i.e., \(\mu\) is a non-negative, \(\sigma\)-additive set function defined on \(\mathcal{A}\) such that \(\mu(\emptyset) = 0\). \(\mu\) is \(\sigma\)-finite if there are mutually disjoint sets \(X_i \in \mathcal{A}, i = 1, 2, \ldots\), such that \(\bigcup_{i=1}^{\infty} X_i = X\) and \(\mu(X_i) < \infty\) for each \(i\).

\(\mu\) is a probability measure if \(\mu(X) = 1\). A set \(A\) is \(\mu\)-null, or simply null, if \(\mu(A) = 0\). A measure \(\nu\) defined on \(\mathcal{A}\) is \(\mu\)-continuous, written \(\nu \ll \mu\), if every \(\mu\)-null set is \(\nu\)-null; \(\mu\) and \(\nu\) are equivalent if \(\mu \ll \nu\) and \(\nu \ll \mu\). For a \(\sigma\)-finite measure \(\mu\), the set function \(\varpi\) defined by

\[\varpi(A) = \sum_{n=1}^{\infty} \left[ 2^n \mu(X_n) \right]^{-1} \mu(A \cap X_n)\]

is easily seen to be a probability measure equivalent with \(\mu\). Thus a \(\sigma\)-finite measure always admits an equivalent probability measure.
All sets introduced in this paper are assumed measurable; all functions are measurable and extended real-valued. All relations are assumed to hold modulo sets of \( \mu \)-measure zero; for emphasis, we sometimes add the words a.e. (almost everywhere). The indicator function of a set \( A \) is written \( 1_A \); the function \( f \cdot 1_A \) is sometimes written \( f_A \). We write \( \text{supp} f \) for the set of points at which the function \( f \) is different from zero.

For a fixed number \( p \), \( 1 \leq p < \infty \), \( L_p(X, \mathcal{A}, \mu) \), or simply \( L_p \), denotes the Banach space of all measurable functions \( f \) such that \( |f|^p \) is \( \mu \)-integrable; \( L_\infty(X, \mathcal{A}, \mu) \) is the space of all \( \mu \)-essentially bounded measurable functions on \( X \). The \( L_p \)-norm of a function \( f \) is denoted by \( |f|_p \). \( L_p^+ \) is the class of non-negative, non-vanishing functions in \( L_p \). For a measurable set \( A \), \( L_p(A) \) is the class of functions \( f \) in \( L_p \) with \( \text{supp} f \) contained in \( A \).

The dual space of \( L_p \) is, by the Riesz representation theorem, isomorphic to the space \( L_q \) where \( \frac{1}{p} + \frac{1}{q} = 1 \). That is, for any continuous linear functional \( \gamma \) on \( L_p \), there is a unique function \( h \) in
such that

(1.1) \( Y(f) = \int f \cdot h \, d\mu \), \( f \in L_p \),

and conversely, for each \( h \in L_q \), relation (1.1) defines a continuous linear functional \( Y \) on \( L_p \); moreover, this correspondence is an isometry.

A linear operator \( T \) on \( L_p \) is a map from \( L_p \) into \( L_p \) such that for any real numbers \( a \), \( b \) and any \( f \), \( g \) in \( L_p \) we have \( T(af + bg) = aTf + bTg \). \( T \) is positive if \( Tf \geq 0 \) for every \( f \in L_p^+ \). The \( L_p \)-norm of a continuous (≡ bounded) operator \( T \) is denoted by \( |T|_p \), defined by

(1.2) \( |T|_p = \sup |Tf|_p \),

where the supremum is taken over all functions \( f \) with \( |f|_p \leq 1 \). The adjoint operator \( T^* \) acts on \( L_q \) according to the duality relation

(1.3) \( \int Tf \cdot h \, d\mu = \int f \cdot T^*h \, d\mu \), \( f \in L_p \), \( h \in L_q \).

It is well-known that \( |T|_p = |T^*|_q \) ([8], p. 478). The
potential operator associated with a positive operator $T$ is denoted by $T_\infty$; for each non-negative function $g$, $T_\infty g$ is the function $g + Tg + T^2g + \ldots$. A set $A$ is closed (under $T$) if $f \in L_p(A)$ implies $Tf \in L_p(A)$. A positive operator $T$ on $L_1$ is said to be conservative if for each non-negative function $g$, $T_\infty g = \infty$ or 0. A function $f$ is an invariant function for a linear operator $T$ if $Tf = f$.

We can now briefly describe the contents of the chapters to follow. Chapter II contains various well-known ergodic theorems which serve as important tools in the subsequent chapters. In Chapter III, we extend an ergodic theorem, obtained by Sucheston [20] for $L_1$-operators, to positive linear operators on $L_p$, $1 \leq p < \infty$. It has been shown in [21] and [14] that for a positive linear operator $T$ on $L_1$ such that $\sup_n |T^n|_1 < \infty$, the ratio theorem need not hold on the 'disappearing' part $Z^1$ (see Theorem 2.4); this result is also extended to the case $1 < p < \infty$ in Chapter III. Chapter IV contains a simple proof of a continuous parameter version of the Chacon-Ornstein theorem. It has been brought to our attention that this result has been obtained previously by M. Akcoglu and other authors; their papers however have
not been available to us and, as far as we know, have not yet been published.

In Chapter V we give necessary and sufficient conditions for the existence of positive invariant functions for a positive linear operator $T$ on $L_1$, satisfying the condition $\sup_n |T^n|_1 < \infty$ and the additional assumption that there is a positive bounded function invariant under $T^*$. This last assumption, apparently rather stringent, is, as we shall show in Chapter V, necessary in the important case when $|T|_1 \leq 1$.

Finally, in Chapter VI, we give some examples. It is known that the ratio theorem may fail if $|T|_1 > 1$; we give an example to show that also the Hopf decomposition theorem is not valid if $|T|_1 > 1$.

Most results in this paper are obtained by the method of Banach limits. It is recalled that **Banach limits**, or Banach-Mazur limits, are positive linear functionals on the space of bounded sequences of real numbers $(x_n)$, satisfying the axioms: (i) $L(1) = 1$; (ii) $L(x_n) \geq 0$ if $x_n \geq 0$, $n = 0, 1, 2, \ldots$; (iii) $L(x_n) = L(x_{n+1})$ (shift-invariance). The maximal value of Banach limits
on a sequence \((x_n)\) is (see e.g. [22])

\[
M(x_n) \overset{\text{def}}{=} \lim \left( \sup_n \frac{1}{n-1} \sum_{i=0}^{n-1} x_{i+j} \right),
\]

hence the minimal value is \(-M(-x_n)\), equal to

\[
m(x_n) \overset{\text{def}}{=} \lim \left( \inf_n \frac{1}{n-1} \sum_{i=0}^{n-1} x_{i+j} \right).
\]
II. ERGODIC THEOREMS

We begin by stating ergodic theorems basic to the results in Chapters III, IV, and V. Let \((X, \mathcal{A}, \mu)\) be a \(\mathcal{F}\)-finite measure space. A positive linear operator \(T\) on \(L_1\) is called semi-Markovian; \(T\) is sub-Markovian if \(|T|_1 \leq 1\). We state now the ergodic theorem of Chacon and Ornstein [3].

Theorem 2.1 Let \(T\) be a sub-Markovian operator on \(L_1\). If \(f \in L_1\), \(g \in L_1^+\), then the ratio

\[
D_n(T, f, g) = \frac{\sum_{i=0}^{n-1} T^i f}{\sum_{i=0}^{n-1} T^i g}
\]

(2.1)

converges to a finite limit a.e. on supp \(T_\infty g\).

Remarks. 1. We explain how this theorem applies to point transformations. A point transformation \(\tau\) from \(X\) into \(X\) is called measurable if \(\tau^{-1} \mathcal{A} \subseteq \mathcal{A}\); \(\tau\) is invertible if \(\tau\) is one-to-one, onto, and \(\tau^{-1}\)
is measurable; \( T \) is measure-preserving (with respect to \( \mu \)) if \( \mu(T^{-1}A) = \mu(A) \), \( A \in \mathcal{A} \). An invertible measure-preserving point transformation \( T \) generates an isometry \( T \) on \( L_p(X, \mathcal{A}, \mu) \) by the formula

\[
Tf(x) = f(Tx).
\]

This observation was first made by Koopman in 1931 for \( p = 2 \).

2. If the operator \( T \) in Theorem 2.1 is generated by an invertible measure-preserving point transformation \( T \) by means of formula (2.2), then Theorem 2.1 coincides with the Stepanoff-Hopf ratio ergodic theorem ([12], p. 49) which is itself a generalisation of the individual ergodic theorem of Birkhoff (see [10], p. 18).

2. Extending the individual ergodic theorem of Birkhoff, Hopf [13] proved that there is pointwise convergence of the Cesàro averages of \( T^n f \), \( f \in L_1 \), if it is assumed that \( T \) is a Markov operator with a finite invariant measure; i.e., \( \mu(X) < \infty \) and \( T \) is an integral-preserving positive linear operator on \( L_1(X, \mathcal{A}, \mu) \) such that \( T1 = 1 \). By letting \( g = 1 \) in (2.1), it is seen that Theorem 2.1 includes also Hopf's theorem.

The following decomposition theorem, proved by Hopf
in [13], describes the ergodic structure of a sub-Markovian operator $T$.

**Theorem 2.2** Let $T$ be a sub-Markovian operator on $L_1$. Then $X$ splits uniquely into disjoint sets $C$ and $D$ such that for every $f \in L_1^+$, $T_\infty f = 0$ or $\infty$ on $C$, and $T_\infty f < \infty$ on $D$. The class $\mathcal{C}$ of subsets $C_f = \{ T_\infty f = \infty \}$ of $C$ obtained as $f$ runs through $L_1^+$ forms a $\sigma$-algebra of subsets of $C$.

**Remarks.** 1. The sets $C$ and $D$ are called the conservative and dissipative parts of $T$. The decomposition $X = C + D$ with the prescribed properties may also be obtained as a simple consequence of Theorem 2.1. 2. The subsets $B$ of $C$ belonging to $\mathcal{C}$ may be described in a number of equivalent ways (cf. Neveu [17], p. 198): (i) $B \in \mathcal{C}$ if and only if $B = C_f$ for some $f \in L_1^+$; (ii) $B \in \mathcal{C}$ if and only if $T*1_B = 1_B$ on $C$; (iii) $B \in \mathcal{C}$ if and only if $T*1_B \leq 1_B$ on $C$; (iv) $B \in \mathcal{C}$ if and only if $B$ is $T$-closed.

Given two disjoint sets $A$ and $B$, in general functions with support in $B$ may enter $A$ under $T$; the operator $R(T,A,B)$ defined by
\[(2.4) \quad R(T,A,B) = f_A + (Tf_B)_A + \ldots + (T(T^n f_B)_B)_A + \ldots\]

adds to \(f_A\) the successive contributions of \(B\). Given a \(\sigma\)-algebra \(\mathcal{F} \subset \mathcal{A}\) and any function \(f \in L_1\), we shall denote by \(E(f | \mathcal{F})\) the (unique) Radon-Nikodým derivative of the measure \(\nu(A) = \int_A f \, d\mu\), \(A \in \mathcal{F}\), taken with respect to the restriction of \(\mu\) to \(\mathcal{F}\). The limit of the ratio \(D_n(T,f,g)\) in Theorem 2.1, identified by Chacon [2] and Neveu (see [17], p. 211), can now be stated.

**Theorem 2.3** For any \(f \in L_1\), \(g \in L_1^+\), the limit of \(D_n(T,f,g)\) is equal to \(\frac{E[R(T,C,D)f | \mathcal{F}]}{E[R(T,C,D)g | \mathcal{F}]}\) on \(C \cap \text{supp } T_\infty g\), and is equal to the quotient of the convergent series \(\sum_{i=0}^{\infty} T^i f\) and \(\sum_{i=0}^{\infty} T^i g\) on \(D \cap \text{supp } T_\infty g\).

The Chacon-Ornstein ratio ergodic theorem requires that \(T\) be sub-Markovian. Replacing the sub-Markovian assumption by a more general boundedness condition \((b_h)\), defined below, Sucheston [20] has obtained the following Theorems 2.4, 2.5, and Corollary 2.1. A semi-Markovian operator \(T\) on \(L_1\) is said to satisfy condition \((b_h)\).
for some fixed function \( h \in L_\infty^+ \) if

\[
(b_h) \quad \sup_n \int T^n f \cdot h \, d\mu < \infty, \quad f \in L_1^+.
\]

In the sequel, \( M \) and \( m \) are the functionals defined by (1.4) and (1.5).

**Theorem 2.4** Let \( T \) be a semi-Markovian operator on \( L_1 \), satisfying condition \((b_h)\) for some fixed function \( h \in L_\infty^+ \). Then the space \( X \) uniquely decomposes into sets \( Y^h \) and \( Z^h \) with the following properties. \( Z^h \) is \( T \)-closed. If \( f \in L_1^+(Y^h) \), then \( M \left[ \int T^n f \cdot h \, d\mu \right] > 0 \); if \( f \in L_1(Z^h) \), then \( M \left[ \int T^n |f| \cdot h \, d\mu \right] = 0 \). If \( h = 1 \), then \( f \in L_1^+(Y^1) \) implies \( \liminf_{n \to \infty} \int T^n f \, d\mu > 0 \) and \( f \in L_1(Z^1) \) implies \( \lim_{n \to \infty} \int T^n |f| \, d\mu = 0 \).

To state Theorem 2.5, we need the following definition: a subset \( A \) of \( B \) is \textit{closed on} \( B \) if \( f \in L_1(A) \) implies \( \int_B Tf \, d\mu \in L_1(A) \).

**Theorem 2.5** Under the same assumptions as in Theorem 2.4, the set \( Y^h \) decomposes into sets \( Y_C^h \) and
$YD^h$, such that for each $f \in L_1^+$, $T_\infty f = 0$ or $\infty$ on $YC^h$, $T_\infty f < \infty$ on $YD^h$. $YC^h$ is closed on $Y^h$; the subsets of $YC^h$ closed on $Y^h$ form a $\sigma$-algebra, $\mathcal{C}^h$.

If $X \neq Z^h$, then the equation in $e$

$$e \in L_\infty^+, \quad \text{supp } e = Y^h, \quad T*e = e$$

admits a solution which on $YC^h$ is unique modulo multiplication by a $\mathcal{C}^h$-measurable function. If $e$ is a solution of (2.5), $e \cdot f$, $f_Z$, $g_Z \in L_1$; $e \cdot g \in L_1^+$, then the ratio $D_n(T,f,g)$ converges to a finite limit a.e. on the set $Y \cap \text{supp } T_\omega g$.

The limit of $D_n(T,f,g)$ is identified in [20].

Assume (b_l) and $X = YC^1$, then according to Theorem 2.5, the equation in $e$

$$e \in L_\infty^+, \quad \text{supp } e = X, \quad T*e = e$$

admits a solution which is unique modulo multiplication by a $\mathcal{C}^1$-measurable function. If in addition $T$ is indecomposable; i.e., $\mathcal{C}^1 = \{X, \emptyset\}$, then the solution of (2.6) is unique up to multiplication by constants.
Corollary 2.1 Assume \((b_1)\) and \(X = YC^1\). If \(o.f \in L_1\), \(e.g \in L_1^+\), then the limit of \(D_n(T,f,g)\) is equal to \(\frac{E(o.f | \mathcal{E})}{E(e.g | \mathcal{E})}\) a.e. on the set \(\text{supp } T \omega e\).

We turn now our attention to some well-known results in the theory of invariant measures. A measurable point transformation \(\tau\) in \(X\) is null-preserving if \(\mu(A) = 0\) implies \(\mu(\tau^{-1}A) = 0\). A measure \(\nu\) on \((X, \mathcal{O})\) is called invariant if \(\nu(A) = \nu(\tau^{-1}A)\), \(A \in \mathcal{O}\); equivalent, if \(\nu\) and \(\mu\) have the same null sets. The existence of a finite equivalent measure has been of interest since many results in ergodic theory are proved under the assumption that the point transformation is measure-preserving.

We recall from Chapter I that a \(\sigma\)-finite measure always admits an equivalent probability measure; hence, for the purpose of finding conditions for the existence of a finite equivalent invariant measure, we may, and do, assume \(\mu(X) = 1\). The next theorem, which summarizes results obtained by various authors, assumes that \(\tau\) is a null-preserving measurable point transformation in \(X\).
Theorem 2.6 The following conditions are pairwise equivalent.

(o) There exists a finite equivalent invariant measure;

(i) \( \lambda_1(A) = \lim_{n \to \infty} \inf \mu(\tau^{-n}A) > 0 \) if \( \mu(A) > 0 \);

(ii) \( \lambda_2(A) = \lim_{n \to \infty} \inf \sum_{i=0}^{n-1} \mu(\tau^{-i}A) > 0 \) if \( \mu(A) > 0 \);

(iii) \( \lambda_3(A) = \lim_{n \to \infty} \sup \sum_{i=0}^{n-1} \mu(\tau^{-i}A) > 0 \) if \( \mu(A) > 0 \);

(iv) \( \lambda_4(A) = \lim_{n \to \infty} \sup \left[ \sum_{j=0}^{n+j-1} \sum_{i=0}^{n-1} \mu(\tau^{-i}A) \right] > 0 \) if \( \mu(A) > 0 \).

Mrs. Dowker [6] and, independently, Calderon [1] proved the equivalence of (o) and (i); Calderon [1] also showed that (o) is equivalent to (ii). Again, Mrs. Dowker [7] and, using a different method, Hajian and Kakutani [9], proved the equivalence of (o) and (iii). The equivalence of (o) and (iv) was established by Sucheston [19]. We note that \( \lambda_4(A) = M \left[ \mu(\tau^{-n}A) \right] \) where \( M \) is defined by relation (1.4). From the obvious inequalities \( \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4 \), it is seen that each result is an improvement over the previous one.

We turn now to an operator-theoretic generalisation of Theorem 2.6. Let \( T \) be a sub-Markovian operator on
A function \( f \in L_1 \) is called invariant if \( T_f = f \); positive, if \( f > 0 \) \( \mu \)-a.e. For each set \( A \) let

\[
\pi_n(A) = \int_A T^n 1 \, d\mu, \quad n = 0, 1, \ldots.
\]

The following theorem was proved by Dean and Sucheston [43] and, independently, by Neveu [18].

**Theorem 2.7** The following conditions are equivalent.

(i) There exists a positive invariant function \( f \in L_1 \);

(ii) \( \mu(A) > 0 \) implies \( \inf_n \pi_n(A) > 0 \);

(\( \mu(A) > 0 \) implies \( M[\pi_n(A)] > 0 \).

**Remarks.** By the Radon-Nikodym theorem, \( L_1(X, \mathcal{A}, \mu) \) is isometric to \( \Phi \), the space of finite \( \mu \)-continuous signed measures on \( (X, \mathcal{A}) \), under the correspondence

\( f \leftrightarrow \varphi \) where \( f \in L_1 \), \( \varphi \in \Phi \) and \( f = d\varphi/d\mu \). There is a one-to-one correspondence, defined by

\[
\Lambda \varphi(A) = \int_A T \frac{d\varphi}{d\mu} \, d\mu, \quad A \in \mathcal{A}, \quad \varphi \in \Phi
\]

between the sub-Markovian operators \( T \) on \( L_1(X, \mathcal{A}, \mu) \) and operators \( \Lambda \) on \( \Phi \) which are positive, linear and do not increase the total-variation norm in \( \Phi \). Thus
the measures $\pi_n$ in (2.8) are simply the iterates of $\mu$ under $\Lambda$. It is easy to see from relation (2.9) that a function $f = \frac{d\phi}{d\mu} \in L_1$ is a positive invariant function under $T$ if and only if $\Lambda\phi = \phi$.

The operator $\Lambda$ may be generated by a null-preserving measurable point transformation $\gamma$ by the relation

$$\Lambda \phi (A) = \phi (\gamma^{-1} A), \quad A \in \mathcal{A}, \quad \phi \in \Phi.$$  

Thus, it is seen that Theorem 2.6 is contained in Theorem 2.7.

3. $\Lambda$ may be generated by a Markov process $P(x, A)$ by the relation

$$\Lambda \phi (A) = \int P(x, A) \phi (dx), \quad A \in \mathcal{A}, \quad \phi \in \Phi;$$

a Markov process $P(x, A)$ is a real-valued function of two variables such that for every $x \in X$, $P(x, \cdot)$ is a probability measure on $\mathcal{A}$, and for every $A \in \mathcal{A}$, $P(\cdot, A)$ is a measurable function. To ensure the $\mu$-continuity of $\Lambda\phi$ for each $\phi$, the Markov process has to be assumed null-preserving: $\mu(A) = 0$ implies $P(x, A) = 0$ $\mu$-a.e.

Thus, Theorem 2.7 is seen to include also a solution to the problem of existence of finite equivalent measures invariant under a Markov process, studied by Ito in [15].
For an arbitrary sub-Markovian operator $T$ on $L_1$, the space $X$ decomposes into sets $P$ and $N$ such that $P$ is the largest set which supports a positive invariant function. This was shown by Krengel [18] after Dean and Sucheston ([4], Theorem 2) showed that $X = N$ in the particular case when $T$ is assumed conservative, ergodic, and has no positive invariant function.

Theorem 2.8 (Krengel) Let $T$ be a sub-Markovian operator on $L_1(X, \mathcal{C}, \mu)$. Then $X$ is the disjoint union of sets $P$ and $N$, the positive and null parts respectively, with the following properties.

a) $A \subseteq P$, $\mu(A) > 0$ implies $\mathbb{M} [\tau_n(A)] > 0$;

b) $N$ is the disjoint union of countably many sets $X_i$ such that $\mathbb{M} [\tau_n(X_i)] = 0$ for each $i$;

c) $P$ is closed under $T$;

d) There is a non-negative function $f$ in $L_1$, $f > 0$ on $P$ and $Tf = f$. 
III. RATIO ERGODIC THEOREM FOR $L_p$-OPERATORS

The object of this chapter is to extend some results valid for semi-Markovian operators on $L_1$ to operators which operate on $L_p$-spaces, $1 < p < \infty$. These extensions are straightforward and our proofs are analogous to those used in [20]. We assume throughout this chapter that $(X, \mathcal{A}, \mu)$ is a (fixed) $\sigma$-finite measure space. For a fixed $p$, set $1/p + 1/q = 1$; then, by the Riesz representation theorem, the dual space of $L_p$ can be identified with $L_q^*$. A positive linear operator $T$ on $L_p$ is said to satisfy condition $(b_h)$ if for some fixed function $h \in L_q^+$ we have

$$(b_h) \quad \sup_n \int T^n f \cdot h \, d\mu < \infty \quad \text{for each } f \in L_p^+.$$ 

$T$ is said to satisfy condition $(b)$ if

$$(b) \quad \sup_n \left| T^n \right|_p < \infty.$$ 

It follows immediately from the uniform boundedness principle that $T$ satisfies $(b)$ if and only if $T$ satisfies
(bh) for every h ∈ Lq+ . Note that for p = 1 (but not otherwise), (b) is (b1) ; i.e., (bh) with h = 1 .

The following theorem extends Theorem 2.4 to the case when 1 < p < ∞ .

**Theorem 3.1** Let T be a positive linear operator on Lp and assume that T satisfies (bh) for some fixed function h ∈ Lq+ . Then the space X uniquely decomposes into sets Yh and Zh with the following properties. The set Zh is T-closed. If f ∈ Lp+(Yh) , then γ*(f) def M[∫ Tnf·h dμ] > 0 ; if f ∈ Lp(Zh) , then γ*(|f|) = 0 . Moreover, if Yh ≠ ø , then the following equation in eh admits a solution:

(3.1) e^h ∈ Lq+ , supp e^h = Y^h , T*e^h = e^h .

**Proof.** The proof of Theorem 2.4 in [20] for the case p = 1 extends to our case. Let {Lβ : β ∈ B} be the collection of all Banach limits. For a fixed β , the relation

(3.2) γ_β(f) = L_β [∫ T^nf·h dμ] , f ∈ L_p ,
clearly defines a positive linear (and hence continuous) functional on $L_p$. By the Riesz representation theorem, there is a function $h_\beta \in L_q^+$ or $h_\beta = 0$ such that

$$(3.3) \quad \gamma_\beta(f) = \int f \cdot h_\beta \, d\mu, \quad f \in L_p.$$ 

Let $\pi$ be a probability measure equivalent with $\mu$ and let $Y_\beta = \text{supp } h_\beta$. Choose Banach limits $L_n$ such that

$$\lim_{n \to \infty} \pi(Y_n) = \sup_{\beta \in B} \pi(Y_\beta).$$ 

Then the set $Y_M$ corresponding to the Banach limit $L_M = \sum_{n=1}^{\infty} 2^{-n}L_n$ contains $Y_\beta$ for every $\beta \in B$. We now set $Y^h = Y_M$ and $Z^h = X - Y_M$. If $f \in L_p(Z^h)$, then for each $\beta \in B$, we have

$$\gamma_\beta(|f|) = \int |f| \cdot h_\beta \, d\mu = 0$$ 

since $Y_\beta \subset Y_M$; thus

$$\gamma^*(|f|) = 0.$$ 

If $f \in L_p^+(Y^h)$, then $\gamma^*(f) \geq \gamma_M(f) = \int f \cdot h_M \, d\mu > 0$ since $Y_M \cap \text{supp } f \neq \emptyset$. If $f \in L_p^+(Z^h)$, then $\gamma^*(Tf) \leq \gamma^*(Tf) = \gamma^*(f) = 0$ which implies $\text{supp } Tf \subset Z^h$; hence $Z^h$ is $T$-closed. To prove the last assertion of the theorem, let $e^h = h_M$, then $e^h$ clearly satisfies the first and second conditions of (3.1). That $T^*e^h = e^h$ is a consequence of the shift-invariance
Our next aim is to show that for an arbitrary positive linear operator $T$ on $L_{p}$, there is a function $\overline{h} \in L_{q}^{+}$ such that the set $Y$ corresponding to $\overline{h}$ is maximal.

For a positive linear operator $T$ on $L_{p}$, we let $H(T) = \left\{ h \in L_{q}^{+} : T \text{ satisfies condition } (b_{h}) \right\}$.

**Proposition 3.1** Let $T$ be a positive linear operator on $L_{p}$. If $H(T) \neq \emptyset$, then there is an $\overline{h} \in H(T)$ such that $Y_{\overline{h}} > Y^{h}$ for every $h \in H(T)$.

**Proof.** Assume $H(T) \neq \emptyset$. Let $\pi$ be a probability measure equivalent with $\mu$. Choose $h_{i} \in H(T)$, $i = 1, 2, \ldots$, such that $\lim_{i \to \infty} \pi(Y^{h_{i}}) = s$ defined $\sup_{h \in H(T)} \pi(Y^{h})$.

It is clear from the uniform boundedness principle that $T$ satisfies $(b_{h})$ if and only $\sup_{n} \left| T^{*n}h_{i} \right|_{q} < \infty$.

For each $i$, let $M_{i} = \sup_{n} \left| T^{*n}h_{i} \right|_{q}$, and set

$$\overline{h} = \sum_{i=1}^{\infty} 2^{-i} M_{i}^{-1} h_{i}; \text{ then } \overline{h} \in H(T).$$

For the remainder of the proof, we shall write $Y$ for $Y^{\overline{h}}$, $Z$ for $Z^{\overline{h}}$, $Y_{i}$ for $Y^{h_{i}}$, $Z_{i}$ for $Z^{h_{i}}$. It is easy to see that if...
Let \( h, g, h \in H(T) \) and \( g \geq h \), then \( Y^g \supseteq Y^h \). It thus follows from the definition of \( \overline{h} \) that \( \overline{Y} \) contains every \( Y_i \), and hence \( \pi(\overline{Y}) \geq s \); \( \pi(\overline{Y}) = s \) since \( \overline{h} \in H(T) \).

It remains to show that \( \overline{Y} \supset Y^h \) for every \( h \in H(T) \).

If not, then there is a function \( h \in H(T) \) such that \( Y^h \) is not contained in \( \overline{Y} \). Thus \( \pi(\overline{Y} \cup Y^h) > s \). Let \( g = \overline{h} + h \); since \( g \in H(T) \) and \( g \geq \overline{h}, g \geq h \), we have \( \pi(Y^g) \geq \pi(\overline{Y} \cup Y^h) > s \), a contradiction to the definition of \( s \).

Thus for an arbitrary positive linear operator on \( L_p \), the sets \( \overline{Y} \) and \( \overline{Z} \) introduced in the proof of Proposition 3.1 are uniquely defined. The next corollary states that \( \overline{Y} \) is the largest set which supports a \( T^* \)-invariant function.

**Corollary 3.1** Let \( T \) be a positive linear operator on \( L_p \). If \( X \neq \overline{Z} \), then the equation in \( e \)

\[
(3.4) \quad e \in L^+_q, \quad \operatorname{supp} e = \overline{Y}, \quad T^*e = e
\]

admits a solution. Moreover, if \( e' \) satisfies the first and third conditions of \( (3.4) \), then \( \operatorname{supp} e' \subseteq \overline{Y} \).
Proof. The first part of the corollary follows immediately from Proposition 3.1 and Theorem 3.1; to prove the second part, it suffices to note that if \( T^*e' = e' \), then \( e' \in H(T) \) and \( \text{supp } e' = Y^{e'} \).

Proposition 3.2 Let \( T \) be a positive linear operator on \( L_p \) and assume that \( T \) satisfies condition (b).

If \( f \in L_p(\mathbb{Z}) \), then \( \lim_{n \to \infty} n^{-1} \sum_{i=0}^{n-1} T^i f = 0 \) in \( L_p \).

Proof. We recall that condition (b) is equivalent to \( H(T) = L_q^+ \); hence, by Proposition 3.1, \( \mathbb{Z} \) is contained in \( Z^h \) for every \( h \in L_q^+ \). To prove the proposition, we may assume \( f \geq 0 \) since \( |Tf| \leq T|f| \). By Theorem 3.1, \( M \left[ \int T^n f \cdot h \, d\mu \right] = 0 \) and hence

\[
\lim_{n \to \infty} \int f_n \cdot h \, d\mu = 0 \quad \text{for every } h \in L_q^+, \text{ where }
\]

\( f_n = n^{-1} \sum_{i=0}^{n-1} T^i f \); i.e., \( f_n \) converges weakly to zero.

On the other hand, since \( \sup_n \left| T^n \right|_p < \infty \), a mean ergodic theorem for Banach spaces ([8], p. 661) asserts that the sequence \( (f_n) \), being weakly convergent, is convergent in the strong topology of \( L_p \), and hence its limit in \( L_p \) must be zero.
Our next theorem extends Theorem 2.5 to the case $1 < p < \infty$.

**Theorem 3.2** Let $T$ be a positive linear operator on $L_p$ and let $e$ be a solution of (3.4). If $f, g$ are measurable functions such that $g \geq 0; f \cdot e, g \cdot e \in L_1$, then the ratio

$$D_n(T; f, g) \overset{\text{def}}{=} \frac{\sum_{i=0}^{n-1} T^i f}{\sum_{i=0}^{n-1} T^i g}$$

converges to a finite limit a.e. on the set $\overline{Y} \cap \text{supp } T_0 e$.

**Proof.** It is well-known ([17], p. 188) that a positive linear operator $T$ on $L_p(X, \mathcal{A}, \mu)$ may be extended in a unique way to a positive linear map from $\mathcal{M}^+(X)$ into $\mathcal{M}^+(X)$, the cone of non-negative and extended real-valued measurable functions on $(X, \mathcal{A})$. $T^*$ may be similarly extended. Moreover, these extensions, still denoted by $T$ and $T^*$, are such that the duality relation remains valid for any $f, g \in \mathcal{M}^+(X)$:

$$\int T f \cdot g \, d\mu = \int f \cdot T^* g \, d\mu.$$  

To prove the theorem, we introduce an auxiliary operator $V$ from $\mathcal{M}^+(X)$ into $\mathcal{M}^+(X)$, defined by the relation
(3.6) \[ Vf = e \cdot T\left( \frac{f}{e + 1/\overline{Z}} \right) \]

It follows from \( T^* \alpha = \alpha \) and (3.5) that for each \( f \in \mathcal{M}(X) \)

(3.7) \[ \int Vf \, d\mu = \int e \cdot \frac{f}{e + 1/\overline{Z}} \, d\mu = \int \frac{f}{\overline{Y}} \, d\mu \leq \int f \, d\mu \]

The operator \( V \), extended to \( L_1(X, \alpha, \mu) \) by linearity, is therefore sub-Markovian. Since \( \text{supp } e = \overline{Y} \) and \( \overline{Z} \) is \( T \)-closed, the following relation holds for each \( f \in L_1 \) and each positive integer \( n \):

(3.8) \[ V^n f = e \cdot T^n\left( \frac{f}{e + 1/\overline{Z}} \right) \]

The Chacon-Ornstein theorem (Theorem 2.1) is now applied to the operator \( V \), yielding the convergence of the ratio \( D_n(V,f',g') \) \( \mu \)-a.e. on the set \( \text{supp } V_\alpha g' \) whenever \( f' \in L_1 \) and \( g' \in L_1^+ \). For arbitrary measurable functions \( f \) and \( g \), \( g \geq 0 \), we set \( f' = f \cdot e \) and \( g' = g \cdot e \).

Recalling that \( \text{supp } e = \overline{Y} \) and \( \overline{Z} \) is \( T \)-closed, we obtain from (3.8)

(3.9) \[ D_n(V,f',g') = \frac{e \cdot \sum_{i=0}^{n-1} T^i f}{e \cdot \sum_{i=0}^{n-1} T^i g} = D_n(T,f,g) \]
Thus the ratio \( D_n(T, f, g) \) converges to a finite limit on the set \( \text{supp } V_\infty g^* = \overline{Y} \cap \text{supp } T_\infty g \) whenever \( f \cdot e \in L_1, g \cdot e \in L_1^+ \).

For the remainder of this chapter, we shall restrict our attention to the case when \( T \) is a positive linear operator on \( L_p \), satisfying condition (b); i.e.,

\[
\sup_n |T^n|_p < \infty.
\]

We shall say that the ratio theorem holds (for \( T \)) on a subset \( A \) of \( X \) if whenever \( f \in L_p, g \in L_p^+ \), the ratio \( D_n(T, f, g) \) converges to a finite limit \( \mu \)-a.e. on the set \( A \cap \{ T_\infty g > 0 \} \); otherwise we say that the ratio theorem fails on \( A \). In particular, Theorem 3.2 asserts that the ratio theorem holds on \( \overline{Y} \); a natural question that arises is whether the ratio theorem holds also on \( \overline{Z} \). For \( p = 1 \), this question has been answered in the negative ([21], [14]). We conclude this chapter by showing that an analogous situation exists in the case \( 1 < p < \infty \). Our method is borrowed from [14].

Let \( M \) denote the space of real-valued measurable functions on \( (X, \mathcal{A}) \).
Lemma 3.1 Let $S$ be a positive linear operator from $L_p$ into $\mathcal{M}$. Let $f_n \in L_p$ be such that $\sum_{n=1}^{\infty} |f_n|_p < \infty$. Then $Sf_n(x) \to 0$ $\mu$-a.e.

Proof. Since $S$ is positive linear, we may assume with no loss of generality that each $f_n$ is non-negative. Let $f = \sum_{n=1}^{\infty} f_n$; then $f \in L_p$ because $|f|_p = \sum |f_n|_p$. We have $0 \leq \sum_{n=1}^{\infty} Sf_n \leq Sf$, and since $Sf(x) < \infty$ $\mu$-a.e., the lemma is proved. \qed

Theorem 3.3 Let $T$ be a positive linear operator on $L_p$ such that $\sup_n |T^n|_p < \infty$. If there is a function $g \in L_p^+(\mathbb{Z})$ such that the set $C_g = \{ T^\infty g = \infty \}$ is non-null, then the ratio theorem fails on every non-null subset of $C_g$.

Proof. Let $A \subset C_g$ and $\mu(A) > 0$. Assume that the ratio theorem holds on $A$; in particular, the ratio $D_n(T,f,g)$ converges to a finite limit $\mu$-a.e. on $A$ for every $f \in L_p$. Define an operator $S$ from $L_p$ into $\mathcal{M}$ by
Clearly, $S$ is linear and positive. Since $g \in L^+_p(\mathbb{Z})$ and Proposition 3.2 asserts that $\|g\|_p < \infty$. Hence we may choose a subsequence $(g_{n_i})$ such that 

$$\sum_{i=1}^{\infty} |g_{n_i}|_p < \infty,$$

and it follows from Lemma 3.1 that $Sg_{n_i}(x) \to 0$ $\mu$-a.e. On the other hand, since $\lim_{n \to \infty} g = \infty$ on $A$, it is clear that

$$Sg_n(x) = \lim_{m \to \infty} \frac{n^{-1} \sum_{i=0}^{m-1} T^i g(x)}{\sum_{i=0}^{m-1} T^i g(x)} \geq 1$$

on $A$. We have thus arrived at a contradiction. \[ \square \]

We can now characterize the behavior of the ratio theorem on $X$ for a positive linear operator $T$ on $L_p(X, \mathcal{A}, \mu)$ such that $\sup_n |T^n|_p < \infty$.

**Corollary 3.2** Let $T$ be a positive linear operator on $L_p(X, \mathcal{A}, \mu)$ such that $\sup_n |T^n|_p < \infty$. If for every
\[ f \in L_p^+(X, \mathcal{A}, \mu), \quad T_a f < \infty \quad \mu\text{-a.e. on } \mathcal{Z}, \] then the ratio theorem holds on \( X \). If there is a function \( f \in L_p^+(X, \mathcal{A}, \mu) \) such that \( C_f \cap \mathcal{Z} \) is non-null, then the ratio theorem fails on every non-null subset \( A \subset C_f \cap \mathcal{Z} \).

**Proof.** The first part of the corollary is obvious since the ratio theorem holds on \( \mathcal{Y} \). To prove the second part, let \( f \in L_p^+ \) be such that \( \mu(C_f \cap \mathcal{Z}) > 0 \). Then either \( \mu(C_g) = 0 \) for every \( g \in L_p^+(\mathcal{Z}) \), in which case the ratio theorem obviously fails, or there is a function \( g \in L_p^+(\mathcal{Z}) \) such that \( \mu(C_g) > 0 \), in which case the conclusion of the corollary follows from Theorem 3.3. \( \Box \)
In this chapter we shall extend some of the results in Chapter II to semi-groups of operators with continuous time parameter. A family \( \Gamma = \{ T_t : t \geq 0 \} \) of linear operators in \( L_1(X, \mathcal{A}, \mu) \) is a semi-group if

\begin{equation}
(4.1) \quad T_0 = I, \quad T_{t+s} = T_t T_s, \quad t, s \geq 0,
\end{equation}

where \( I \) denotes the identity operator. \( \Gamma \) is said to be strongly continuous if for each \( s \geq 0 \) and each \( f \in L_1 \), we have

\begin{equation}
(4.2) \quad \lim_{t \to s} \left| T_t f - T_s f \right|_1 = 0.
\end{equation}

It is known (see [8], p. 616) that if \( \Gamma \) is a strongly continuous semi-group of bounded linear operators in \( L_1(X, \mathcal{A}, \mu) \), then \( \Gamma \) is strongly integrable on every finite interval \([\alpha, \beta] \) of \([0, \infty) \); more precisely, for each \( f \in L_1 \), and \( 0 \leq \alpha < \beta < \infty \), the integral \( \int_\alpha^\beta T_t f \, dt \) is defined and is an element of \( L_1(X, \mathcal{A}, \mu) \).
Hence (cf. [8], p. 686) for each \( f \in L^1 \), there is a scalar function \( T_t f(x) \), measurable with respect to the product of Lebesgue measure and \( \mu \), such that for almost all \( t \), \( T_t f(x) \), as a function of \( x \), belongs to \( T_t f \); moreover, there is a set \( E(f) \), \( \mu(E(f)) = 0 \), dependent on \( f \) but independent of \( t \), such that if \( x \notin E(f) \) then \( T_t f(x) \) is integrable on every finite interval \([\alpha, \beta]\) and the integral \( \int_{\alpha}^{\beta} T_t f(x) \, dt \), as a function of \( x \), is equal to the vector \( \int_{\alpha}^{\beta} T_t f \, dt \) in \( L^1 \). Thus for each \( u > 0 \) and each \( f \in L^1 \), the integral

\[
S_u f(x) = \int_{0}^{u} T_t f(x) \, dt
\]

is defined for every \( x \notin E(f) \).

The following theorem is a continuous parameter analogue of the Chacon-Ornstein theorem. Our proof makes use of the standard reduction method of the continuous case to the discrete case.

**Theorem 4.1** Let \( \Gamma = \{ T_t : t \geq 0 \} \) be a strongly continuous semi-group of positive linear operators in \( L^1 \) such that \( |T_t|_1 \leq 1 \) for each \( t \geq 0 \). Let \( f, g \in L^1 \).
$c > 0$. Then, as $u \to \infty$, the ratio

\begin{equation}
D_u(f, g)(x) \overset{\text{def}}{=} \frac{S_u f(x)}{S_u g(x)}
\end{equation}

converges to a finite limit $\mu$-a.e. on the set $A(g) \overset{\text{def}}{=} \{ x : S_u g(x) > 0 \text{ for some } u > 0 \}$.

**Proof.** For each $f \in L_1$, let $\overline{f}(x) = S_1 f(x)$. For each $u > 0$, write $u = n + r$, where $n = \lfloor u \rfloor$ and $0 \leq r < 1$. Then

\[
S_u f = \int_0^u T_t f \, dt = \sum_{k=0}^{n-1} \int_k^{k+1} T_t f \, dt + \int_n^{n+r} T_t f \, dt
\]

\[
= \sum_{k=0}^{n-1} T_k \int_0^1 T_t f \, dt + T_n \int_0^r T_t f \, dt
\]

and hence

\begin{equation}
S_u f(x) = \sum_{k=0}^{n-1} T_k \overline{f}(x) + T_n (S_r f)(x).
\end{equation}

We may, and do, assume that $f$ and $g$ are non-negative. Since $T_t$ is positive, it follows that $0 \leq S_r f(x) \leq \overline{f}(x)$ and $0 \leq S_r g(x) \leq \overline{g}(x)$ a.e., $0 \leq r \leq 1$. Thus, for $u$ sufficiently large, we have on $A(g)$
Since \( f, g \in L_1^+ \), the Chaon-Ornstein theorem applied to \( T = T_t \) implies that the first and the third terms in (4.6) converge to the same finite limit on the set

\[
\left\{ x : \sum_{k=0}^{\infty} T_k g(x) > 0 \right\} = A(g). \text{ This completes the proof of the theorem. } \]

We now weaken the sub-Markovian assumption \( |T_t|_1 \leq 1 \) on the semi-group \( \Gamma \) to the following boundedness condition:

\[
(b) \quad \sup_{t \geq 0} |T_t|_1 < \infty.
\]

**Theorem 4.2** Let \( \Gamma = \{ T_t : t \geq 0 \} \) be a semi-group of positive linear operators in \( L_1 \) such that condition (b) holds. Then the space \( X \) decomposes into sets \( Y \) and \( Z \) with the following properties: \( Z \) is \( T_t \)-closed for each \( t \geq 0 \);

\[
(4.7) \quad \left\{ \begin{array}{l} f \in L_1^+(Y) \text{ implies } \liminf_{t \to \infty} \int T_t f \, d\mu > 0 ; \\ f \in L_1(Z) \text{ implies } \lim_{t \to \infty} \int T_t |f| \, d\mu = 0 . \end{array} \right.
\]
Proof. Suppose that \( \sup_{t \geq 0} |T_t|_1 = M < \infty \).

Let \( T = T_1 \); then \( T^n = T_n, n = 1, 2, \ldots \). Since

\[
\sup_n |T^n|_1 < \infty ,
\]

Theorem 2.4 applied to \( T \) implies that \( X \) decomposes into sets \( Y \) and \( Z \) such that

\[
\left\{
\begin{array}{l}
 f \in L_1^+(Y) \implies \liminf_{n \to \infty} \int T_n f \, d\mu > 0 ; \\
 f \in L_1(Z) \implies \lim_{n \to \infty} \int T_n |f| \, d\mu = 0 .
\end{array}
\right.
\]

(4.8)

Suppose that \( f \in L_1^+ \) and \( \liminf_{t \to \infty} \int T_t f \, d\mu = 0 \); then given \( \epsilon > 0 \), there is an \( s > 0 \) such that

\[ 0 < \int T_s f \, d\mu < \frac{\epsilon}{M} . \]

For \( t > s \), we have

\[ 0 \leq \int T_t f \, d\mu = \int T_{t-s} (T_s f) \, d\mu \leq |T_{t-s}|_1 \cdot |T_s f|_1 < \epsilon , \]

which shows that \( \lim_{t \to \infty} \int T_t f \, d\mu = 0 \); in view of (4.8),

(4.7) is now proved. That \( Z \) is \( T_t \)-closed for each \( t \geq 0 \) is an easy consequence of (4.7).

We recall (Theorem 2.5) that in the discrete parameter case the equation in \( e \)

\[
(4.9) \quad e \in L_\infty^+, \quad \text{supp} \, e = Y, \quad T^* e = e .
\]
admits a solution. The existence of a function $e$ satisfying (4.9) plays an important role in the proof of the ratio ergodic theorem (Theorems 2.5, 3.2) for semi-Markovian operators in the discrete case. In view of Theorem 4.1, the search for a function $e$ satisfying

$$(4.10) \quad e \in L^\infty_\text{c} , \quad \text{supp } e = Y , \quad T_t^* e = e \quad \text{for each } t \geq 0$$

is also of interest. Here we restrict our attention to the following special case.

We recall from Chapter II that for a semi-Markovian operator $T$ on $L_1$ satisfying

$$(4.11) \quad \sup_n |T_n|_1 < \infty , \quad X = YC_1 , \quad T \text{ is indecomposable}$$

there is a unique (modulo multiplication by constants) function $e$ satisfying (4.9) with $Y = X$. We shall say that a semi-group $\Gamma = \{ T_t : t \geq 0 \}$ satisfies condition (I) if for some fixed $s > 0$, $T = T_s$ satisfies equation (4.11).

Theorem 4.3 Let $\Gamma = \{ T_t : t \geq 0 \}$ be a strongly continuous semi-group of positive linear operators in $L_1$, satisfying conditions (b) and (I). Then equation
(4.10) admits a solution which is unique modulo multiplication by constants.

Proof. There is a fixed \( s > 0 \) and a function \( e \), unique up to multiplication by constants, such that

\[
(4.12) \quad e \in L^+_\infty, \quad \text{supp } e = X, \quad T_s^* e = e.
\]

Theorem 2.5 applied to \( T = T_{s/n} \) implies that for each positive integer \( n \), there is a function \( e_n \) satisfying

\[
(4.13) \quad e_n \in L^+_\infty, \quad \text{supp } e_n = X, \quad T_{s/n}^* e_n = e_n.
\]

Since \( T_{s/n}^* e_n = (T_{s/n})^n e_n = e_n \), it follows that \( e_n \) is a constant multiple of \( e \); hence \( T_r^* e = e \) for each \( r = s/n, \ n = 1, 2, 3, \ldots \). Let

\[
D = \left\{ \frac{ks}{n} : k, \ n = 1, 2, \ldots, \ 1 \leq k \leq n \right\}.
\]

Then \( D \) is dense in \( [0, s] \); moreover, it is clear that the equation \( T_r^* e = e \) holds for each \( r \in D \). For any fixed \( t \in [0, s] \) and any \( r \in D \), we have
The strong continuity of $\Gamma$ implies that the majorant in (4.14) tends to zero as $r$ tends to $t$ through values in $D$. Thus $\int f(T_t^* e - e) \, d\mu = 0$ for each $f \in L_1^+$. This implies $T_t^* e = e$ for every $t \in [0, s]$ and hence for every $t \geq 0$. The uniqueness of $e$ follows because $T_s$ is indecomposable.

Again assume that $\Gamma = \{ T_t : t \geq 0 \}$ satisfies the hypothesis of Theorem 4.3. Let $e$ be the unique solution of equation (4.10); we may, and do, assume that $0 < e \leq 1$. Since each $T_t$ is positive linear on $L_1^+$, each $T_t$ may be extended to a positive linear operator on $\mathcal{M}^+$, the cone of non-negative measurable functions on $(X, \mathcal{A})$; for each fixed $t \geq 0$, if $f \in \mathcal{M}^+$, then $T_t f$ is defined to be the almost everywhere limit of any sequence $(T_t f_n)_{n=1}^{\infty}$ where $f_n \in L_1^+$ and $f_n \uparrow f$ a.e. (see Chapter III, Theorem 3.2). We assert that the

(4.14) $\left| \int f(T_t^* e - e) \, d\mu \right| = \left| \int f(T_t^* e - T_r^* e) \, d\mu \right|

= \left| \int (T_t f - T_r f) \cdot e \, d\mu \right|

\leq |e|_{\infty} \cdot |T_t f - T_r f|_1$. 
extended operators $T_t$ also satisfy the semi-group property on $\mathcal{M}^+$; i.e.,

$$(4.15) \quad T_{t+s}f = T_t(T_sf), \quad f \in \mathcal{M}^+, \quad t, s \geq 0.$$ \]

Indeed, let $f \in \mathcal{M}^+$ and $f_n \in L_1^+$, $f_n \uparrow f$; then

$$T_{t+s}f = \lim_{n \to \infty} T_{t+s}f_n = \lim_{n \to \infty} T_t(T_sf_n) = T_t(T_sf),$$

where the second equality holds since $f_n \in L_1$ and the last equality holds because $T_sf_n \uparrow T_sf$. We shall assume for the remainder of this chapter that each $T_t$ has been so extended.

For each fixed $t \geq 0$, we define an operator $V_t$ on $L_1^+$ by the relation

$$(4.16) \quad V_tf = e \cdot T_t(f/e),$$

and extend $V_t$ by linearity to $L_1$. The proof of Theorem 3.2 shows that $\mathcal{P}^* = \{V_t : t \geq 0\}$ is a family of sub-Markovian operators in $L_1$. Our aim is to show that $\mathcal{P}^*$ is again a strongly continuous semi-group.

Let $f \in L_1^+$. Then $f/e \in \mathcal{M}^+$ and it follows
from (4.15) and (4.16) that

\[(4.17) \quad V_{t+s}f = e \cdot T_{t+s}(f/e) = e \cdot T_t(T_s(f/e)) = e \cdot T_t(e \cdot T_s(f/e)) = V_t(V_s f).\]

Thus \( \mathcal{P} \) is a semi-group. To prove the strong continuity of \( \mathcal{P} \), let \( L = \{ g ; \ g = f \cdot e , \ f \in L_1 \} \). Since \( 0 < e = 1 \), it is clear that \( L \) is a dense subspace of \( L_1 \). Let \( s \geq 0 \) be fixed and \( g = f \cdot e \in L \), \( f \in L_1 \). Then

\[(4.18) \quad |V_t g - V_s g|_1 = \left| e \cdot T_t(e/g) - e \cdot T_s(e/g) \right|_1 \leq |e|_\infty \cdot |T_t f - T_s f|_1 \]

which, by the strong continuity of \( \mathcal{P} \) on \( L_1 \), tends to zero as \( t \to s \). The case of a general \( g \in L_1 \) follows by approximation since \( |V_t|_1 \leq 1 \). Thus \( \mathcal{P} \) is a strongly continuous semi-group of sub-Markovian operators in \( L_1 \).

Theorem 4.1 can now be applied to \( \mathcal{P} \); if \( f' \in L_1 \), \( g' \in L_1^+ \), then the ratio
\[ \frac{\int_{0}^{u} V_t f'(x) \, dt}{\int_{0}^{u} V_t g'(x) \, dt} \]

converges to a finite limit a.e. on the set
\[ \{ \int_{0}^{u} V_t g'(x) \, dt > 0 \text{ for some } u > 0 \} \]. For arbitrary measurable functions \( f \) and \( g \), we write \( f' = f^e \), \( g' = g^e \). If \( f' \in L_1 \), \( g' \in L_1^+ \), then

\[ (4.19) \quad \frac{\int_{0}^{u} V_t f'(x) \, dt}{\int_{0}^{u} V_t g'(x) \, dt} = \frac{\int_{0}^{u} e(x) \cdot T_t f(x) \, dt}{\int_{0}^{u} e(x) \cdot T_t g(x) \, dt} = \frac{\int_{0}^{u} T_t f(x) \, dt}{\int_{0}^{u} T_t g(x) \, dt} . \]

Hence the last ratio in (4.19) converges on the set
\[ A(g) \overset{\text{def}}{=} \{ \int_{0}^{u} T_t g(x) \, dt > 0 \text{ for some } u > 0 \} \].

The above discussion is now summarized into

**Theorem 4.4** Let \( \Gamma = \{ T_t : t \geq 0 \} \) be a strongly continuous semi-group of positive linear operators in \( L_1 \). Assume that \( \Gamma \) satisfies (i) and (I). If \( f, g \) are measurable functions such that \( f^e \in L_1 \), \( g^e \in L_1^+ \), then
(4.20) \[ \lim_{u \to \infty} \frac{\int_0^u T_t f(x) \, dt}{\int_0^u T_t g(x) \, dt} \]

exists and is finite on the set \( A(g) \).
V. INVARIANT FUNCTIONS FOR SEMI-MARKOVIAN OPERATORS

We give in this chapter necessary and sufficient conditions for the existence of positive invariant functions for a semi-Markovian operator $T$ on $L_1$, satisfying

\[ \sup_n |T^n|_1 < \infty, \]

and the additional assumption that $X = Y^1$ (for the definition of $Y^1$ see Chapter II). We mentioned in the introduction that this last assumption, apparently rather stringent, can be shown to be necessary in the case $|T|_1 \leq 1$. Indeed, assume $|T|_1 \leq 1$ and let $X = C + D$ be the Hopf decomposition of $T$ (Theorem 2.2). We recall from Chapter II that $T^*1_C \geq 1_C$; hence for each $f \in L_1^+$ and each $n$, we have

\[ \int T^n f \, d\mu \geq \int f \cdot T^n 1_C \, d\mu \geq \int f \cdot 1_C \, d\mu. \]

If $f \in L_1^+(Z)$, then according to Theorem 2.4, \[ \lim_{n \to \infty} \int T^n f \, d\mu = 0 \] which implies \[ \int_C f \, d\mu = 0. \] Hence we have $Z \subset D$. Thus if $T$ has a positive invariant function $f$, then $X = C$; consequently, $Z = \emptyset$.
For the purpose of finding conditions for the existence of invariant functions we may, and do, assume \( \mu(X) = 1 \) throughout this chapter (see Chapter II). The operator \( T \) admits an adjoint operator \( T^* \) which acts on \( L_\infty \); the adjoint \( T^{**} \) of \( T^* \) operates on the space \( \Psi \) of signed finite finitely additive set functions which vanish on \( \mu \)-null sets (cf. [8], p. 296). Under the natural embedding of the Banach space \( L_1 \) in its second conjugate \( \Psi^* \), \( L_1 \) is mapped on \( \Phi \), the space of finite signed \( \mu \)-continuous measures. If \( \nu \in \Phi \), then \( T^{**}\nu \in \Phi \) and

\[
(5.1) \quad T^{**}\nu(A) = \int_A T \frac{d\nu}{d\mu} \, d\mu, \quad A \in \mathcal{A}.
\]

We shall often write \( T\nu \) for \( T^{**}\nu \) if \( \nu \in \Phi \). If \( \nu = \mu \), then \( d\nu/d\mu = 1 \), and we have for each \( n \geq 0 \)

\[
(5.2) \quad T^n \mu(A) = \int_A T^n1 \, d\mu = \int T^{*n}1_A \, d\mu, \quad A \in \mathcal{A}.
\]

The following proposition is a partial extension of Theorem 2.8; if in addition \( X = Y^1 \), then we obtain a complete generalisation (Proposition 5.2).
Proposition 5.1 Let \( T \) be a semi-Markovian operator on \( L_1 \), satisfying condition (b). Then \( X \) is the disjoint union of two uniquely determined sets \( P \) and \( N \) with the following properties:

a) \( A \subset P \), \( \mu(A) > 0 \) implies \( M \left[ T^n \mu(A) \right] > 0 \);

b) \( N \) is the disjoint union of countably many sets \( X_i \) with \( M \left[ T^n \mu(X_i) \right] = 0 \) for each \( i \);

c) \( P \) is closed under \( T \).

Proof. Let

\[ \gamma(A) = M \left[ T^n \mu(A) \right], \quad A \in \mathcal{A}. \]

It is clear that \( \gamma(\emptyset) = 0 \) and that if \( A \subset B \), then \( \gamma(A) \leq \gamma(B) \). Let \( X_i, i = 1, 2, \ldots \) be a disjoint sequence of sets such that \( \gamma(X_i) = 0 \) for each \( i \)

and \( \lim_{n \to \infty} \mu\left( \bigcup_{i=1}^{n} X_i \right) = \sup_{\gamma(A) = 0} \mu(A) \). Set \( N = \bigcup_{i=1}^{\infty} X_i \) and \( P = X - N \). It is then easy to verify that \( P \) and \( N \) satisfy a) and b). We now prove c); our argument is simpler than Krengel's in [16]. Since \( P \) is \( T \)-closed if and only if its complement \( N \) is \( T^* \)-closed, we need only to show that \( T^*1_N = 0 \) on \( \mu \). By the monotone continuity property of \( T^* \) (cf. Neveu [17], p. 187), we
have $T^*1_N = \lim_{n \to \infty} T^*(\sum_{i=1}^{n} 1_{X_i})$; thus it is sufficient to show that $T^*1_{X_i} = 0$ on $P$ for each $i$. Assume $T^*1_{X_i} \neq 0$ on $P$ for some $i$. Then there is an $\epsilon > 0$ and a set $A \subset P$ with $\mu(A) > 0$ such that $T^*1_{X_i} \geq \epsilon$ on $A$. It follows that $T^{n+1}1_{X_i} \geq \epsilon \cdot T^*1_{A}$ for every $n$.

This yields the contradiction $M \left[ T^n \mu(X_i) \right] \geq \epsilon \cdot M \left[ T^n \mu(A) \right] > 0$. Thus $P$ is $T$-closed. 

We now assume $X = Y$. By Theorem 2.5, there is a bounded function $e$, $0 < e \leq 1$, such that $T^*e = e$; in the sequel, $e$ shall be assumed chosen and fixed. We introduce an auxiliary operator $V$ on $L_1$ by the relation

$$(5.3) \quad Vf = e \cdot T(f/e), \quad f \in L_1^+,$$

and define $Vf$ by linearity for $f \in L_1$. $V$ is then a sub-Markovian operator on $L_1$, and for each $n$,

$V^nf = e \cdot T^n(f/e)$ (see Theorem 3.2). Proposition 5.1 applied to $T$ and $V$ gives the decompositions $X = P_T + N_T$ and $X = P_V + N_V$. 
Lemma 5.1 Assume \( X = Y \) and let \( V \) be defined by (5.3). Then the decompositions \( X = P_T + N_T \) and \( X = P_V + N_V \) coincide.

Proof. For each \( \epsilon > 0 \), let \( E_\epsilon = \{ \epsilon < \epsilon \} \). Let \( A \subseteq P_V \) with \( \mu(A) > 0 \). Since \( |V|_1 \leq 1 \) and \( 0 < \epsilon \leq 1 \), we have

\[
v^n \mu(A) = \int_A v^n d\mu = \int_A v^n_{E_\epsilon} d\mu + \int_A v^n_{E_\epsilon^c} d\mu \\
\leq \int_X v^n_{E_\epsilon} d\mu + \int_A \epsilon \cdot T^n (1_{E_\epsilon^c/\epsilon}) d\mu \\
\leq \mu(E_\epsilon) + (1/\epsilon) \cdot \int_A T^n_{E_\epsilon^c} d\mu \\
\leq \mu(E_\epsilon) + (1/\epsilon) \cdot \int_A T^n d\mu
\]

for each \( \epsilon > 0 \) and each \( n \). Since \( \epsilon > 0 \), for each \( \delta > 0 \) there is an \( \epsilon > 0 \) such that \( \mu(E_\epsilon) < \delta \), and hence

\[
(5.4) \quad v^n \mu(A) < \delta + (1/\epsilon) \cdot T^n \mu(A)
\]

which implies \( M[T^n \mu(A)] > 0 \). It follows easily that \( A \subseteq P_T \) and \( P_V \subseteq P_T \).
We next let \( A \subseteq \mathcal{P}_T \) with \( \mu(A) > 0 \). Since \( \mu(B_\varepsilon) \to 0 \) as \( \varepsilon \to 0 \), there exists an \( \varepsilon > 0 \) such that

\[
\mu(A \cap E_\varepsilon^c) > 0 ,
\]

and

\[
\forall n \mu(A) \geq \mu(A \cap E_\varepsilon^c) = \int_{A \cap E_\varepsilon^c} e \cdot T^n(1/\varepsilon) \, d\mu \geq \varepsilon \int_{A \cap E_\varepsilon^c} T^n 1 \, d\mu = \varepsilon \cdot T^n \mu(A \cap E_\varepsilon^c)
\]

for each \( n \). It follows that \( M[\forall n \mu(A)] \geq \varepsilon \cdot M[T^n \mu(A \cap E_\varepsilon^c)] > 0 \) and \( A \subseteq \mathcal{P}_V \). Hence \( \mathcal{P}_T \subseteq \mathcal{P}_V \).

We recall that a function \( f \in L_1 \) is called invariant if \( Tf = f \); positive, if \( f > 0 \) \( \mu \)-a.e.. The next theorem extends Theorem 2.7 by weakening \( |T|_1 \leq 1 \) to condition (b).

**Theorem 5.1** Let \( T \) be a semi-Markovian operator on \( L_1 \), satisfying condition (b), and assume \( X = Y \). Then the following conditions are equivalent:

(o) There exists a positive invariant function \( f \in L_1 \).

(i) \( \mu(A) > 0 \) implies \( \inf_n T^n \mu(A) > 0 , A \in \mathcal{C}_L \).

(ii) \( \mu(A) > 0 \) implies \( M[T^n \mu(A)] > 0 , A \in \mathcal{C}_L \).
Proof. We introduce a third condition:

(iii) $\mu(A) > 0$ implies $m \left[ T^n \mu(A) \right] > 0$, $A \in \mathcal{A}$.

Our proof follows the scheme $(o) \Rightarrow (ii) \Rightarrow (i) \Rightarrow (iii) \Rightarrow (o)$. The implication $(i) \Rightarrow (iii)$ is obvious.

Part I: $(o) \Rightarrow (ii)$. Assume that there is a positive invariant function $f \in L^1$. Let $g = f \cdot e$; then $g \in L^1$, $g > 0$, and $Vg = g$. Since $V$ is sub-Markovian, by Theorem 2.7, $X = P_V$; thus, by Lemma 5.1, $X = P_T$.

Part II: $(ii) \Rightarrow (i)$. Assume condition $(ii)$; i.e., $X = P_T$. By Lemma 5.1, $X = P_V$; since $|V| \leq 1$, the condition $X = P_V$ is equivalent, by Theorem 2.7, to

$$ (5.5) \quad \mu(A) > 0 \ implies \ inf \ V^n \mu(A) > 0, \ A \in \mathcal{A}. $$

Relations $(5.4)$ and $(5.5)$ together imply that condition $(i)$ holds.

Part III: $(iii) \Rightarrow (o)$. We imitate in part the argument in [4]. Let $L$ be a Banach limit. The relation

$$ (5.6) \quad \lambda(h) = L \left[ \int T^n h \, d\mu \right], \ h \in L^\infty $$

clearly defines a positive linear functional on $L^\infty$; hence
\[ \lambda \in \mathcal{V} \text{ and } \lambda \geq 0 \text{ (for the definition of } \mathcal{V} \text{ see the beginning of the chapter). Moreover, it follows from the shift-invariance of } L \text{ that } T^*T \lambda = \lambda \text{. It is recalled that any non-negative element in } \mathcal{V} \text{ may be uniquely decomposed into a measure and a pure charge; hence we have } \lambda = \nu + \eta \text{ where } \nu \geq 0 \text{ is a measure and } \eta \text{ is a pure charge; i.e., } \eta \text{ does not dominate any non-trivial measure (cf. [11], p. 52). From } T^*T \nu + T^*T \eta = \nu + \eta \text{, we obtain } (T^*T \nu - \nu)^+ \leq \eta \text{. Since } \eta \text{ is a pure charge and } (T^*T \nu - \nu)^+ \text{ is a measure, we have } T^*T \nu \leq \nu \text{. We next show that } \nu \text{ is equivalent to } \mu \text{. Assume the contrary, then there is a set } B \text{ such that } \nu(B) = 0 \text{ and } \mu(B) > 0 \text{. Restricted to the } \mathcal{F}\text{-algebra of subsets of } B \text{, } \eta \text{ is a pure charge and } \mu \text{ is a non-trivial measure. Since a pure charge is 'nearly orthogonal' to every measure ([11], p. 50), for each } \mathcal{G} \text{ with } 0 < \varepsilon < \mu(B) \text{, there is a subset } A \text{ of } B \text{ such that } \mu(A) > \mu(B) - \varepsilon > 0 \text{ and } \eta(A) = 0 \text{. Thus } \lambda(A) = \nu(A) + \eta(A) = 0 \text{ which is a contradiction since } \lambda(A) = \int \left[ T^n \mu(A) \right] \geq m \left[ T^n \mu(A) \right] > 0 \text{. Let } f = d\nu/d\mu \text{; then } f \in L_1, f > 0 \text{, and } Tf \preceq f \text{. The following standard argument shows that if } T \text{ is conservative on } L_1 \text{, then } Tf = f \text{. Indeed, let } g = f - Tf \text{; we have } g \in L_1, g \geq 0 \text{, and} \]
\[ \int \sum_{i=0}^{n-1} T_i^* \, d\mu = \int f \, d\mu - \int T^n f \, d\mu \leq \int f \, d\mu < \infty \]

for every \( n \). Thus \( g = 0 \). It remains only to show that \( T \) is conservative. Since Hopf's decomposition is valid on \( Y \), hence on \( X \) (see Theorem 2.5), it is sufficient to show that \( T^n 1 = \infty \) on \( X \). Since (iii) \( \Rightarrow \) (ii), for each set \( E \) with \( \mu(E) > 0 \), we have

\[ \int M[T^n 1_E] \, d\mu \geq M\left[\int T^n 1_E \, d\mu\right] = M\left[\int E \, T^n 1 \, d\mu\right] > 0. \]

It follows that \( \sum_{n=0}^{\infty} T^n 1_E = \infty \) on a set of positive \( \mu \)-measure. If \( \{ T^n 1 < \infty \} \neq \emptyset \), then there is a constant \( \alpha \geq 0 \) such that \( \mu\{ T^n 1 \leq \alpha \} > 0 \). Set \( E = \{ T^n 1 \leq \alpha \} \); then

\[ (5.7) \int \sum_{i=0}^{n} T_i^* 1_E \, d\mu = \int_{E} \sum_{i=0}^{n} T_i^* 1 \, d\mu \leq \alpha \cdot \mu(E) < \infty \]

for every \( n \). This is a contradiction since the first term in (5.7) tends to infinity as \( n \to \infty \). \( \square \)

**Proposition 5.2** Let \( T \) be a semi-Markovian operator on \( L_1 \), satisfying condition (b). Assume \( X = Y \).

Then there is a non-negative function \( f \in L_1 \) with \( f > 0 \) on \( P \) and \( Tf = f \).
Proof. Assume $X = Y$; by Lemma 5.1, $P_T = P_V$, which we denote simply by $P$. Define an operator $T'$ on $L_1(P, P \cap \mathcal{A}, \mu)$ by the relation

\begin{equation}
T'f = Tf, \quad f \in L_1(P).
\end{equation}

Clearly, $T'$ satisfies condition (b) on $L_1(P)$. The adjoint $T'^*$ of $T'$ is given by the relation

\begin{equation}
T'^*h = l_p \cdot T^*h, \quad h \in L_\infty(P).
\end{equation}

$V'$ and $V'^*$ are defined similarly. Proposition 5.1 applied to $T'$ and $V'$ gives the decompositions $P = P_{T'} + N_{T'}$ and $P = P_{V'} + N_{V'}$. Since $|V|_1 \leq 1$ on $L_1$, by Theorem 2.3 there is a non-negative function $f_o \in L_1$ with $f_o > 0$ on $P$ and $Vf_o = f_o$; equivalently, $f_o$ is a positive invariant function in $L_1(P)$ for the sub-Markovian operator $V'$. Hence $P = P_{V'}$. Recalling that $T^*e_p = T^*(e_p + e_N) = e_p + e_N$ and that $N$ is $T^*$-closed, we have $T'^*e_p = l_p \cdot T^*e_p = e_p$. Thus the space $P$ is seen to be $Y$ for $T'$. Moreover, since

\[ V'f = Vf = e \cdot T(f/e) = e_p \cdot T'(f/e_p), \quad f \in L_1^+(P), \]
Lemma 5.1 shows that \( T' \) and \( V' \) give rise to identical decompositions: \( P = P_{V'} = P_{T'} \). Theorem 5.1 applied to \( T' \) shows that there is a function \( f \in L_1(P) \) with \( f > 0 \) and \( T'f = f \). This proves the proposition. 

The next theorem, due in the sub-Markovian case to Dean and Sucheston [4], relates the existence of \( T \)-invariant functions to the uniqueness of Banach limits on sequences \( T^n \mu(A) \). To prove the theorem, we need the following proposition from [4].

**Proposition A** Let \( T \) be a semi-Markovian operator on \( L_1 \) such that condition \( (b) \) holds. Let \( 0 \leq g \in L_1 \) and set

\[
(5.10) \quad Y_n(A) = \int_A T^n g \, d\mu, \quad A \in \mathcal{A}.
\]

If \( T \) has a positive invariant function, then the measures \( Y_n \) are uniformly \( \mu \)-continuous.

**Theorem 5.2** Let \( T \) be a semi-Markovian operator on \( L_1 \), satisfying condition \( (b) \). Assume \( X = Y \). If \( T \) has a positive invariant function, then for each set \( A \) all Banach limits on the sequence \( T^n \mu(A) \) coincide; if \( \lambda(A) \) is their common value, then
and $d\lambda/d\mu$ is a positive invariant function. Conversely, if for each set $A$, all Banach limits on the sequence $T^n \mu(A)$ coincide and if $T$ is conservative, then $T$ has a positive invariant function.

**Proof.** Assume that there is a positive invariant function $f$. By Theorem 2.5, Hopf's decomposition holds on $Y$ and hence on $X$; thus $T$ is conservative since \{ $T_\infty f = \infty$ \} = $X$. Let $L$ be a Banach limit and set

$$\lambda(h) = L \left[ \int T^n h \, d\mu \right], \quad h \in L_\infty.$$  

It is clear that $\lambda$ defines a positive (hence continuous) linear functional on $L_\infty$. Writing $\lambda(A)$ for $\lambda(1_A)$, we obtain

$$\lambda(A) = L \left[ \int T^n 1_A \, d\mu \right] = L \left[ \int A T^n 1 \, d\mu \right] = L \left[ T^n \mu(A) \right].$$  

Applying Proposition A to $T$ with $g = 1$, we conclude that $\lambda$ is a finite $\mu$-continuous measure. Let $f = d\lambda/d\mu$; then $f \in L_1$, $f \geq 0$,

$$\lambda(h) = \int f \cdot h \, d\mu, \quad h \in L_\infty;$$
and it follows from the shift-invariance of $L$ that
$Tf = f$. Set $F = \{f = 0\}$; then $\mu(F) = 0$, for otherwise it would follow from Theorem 5.1 that $\lambda(F) > 0$. Hence $f$ is a positive invariant function. If $L'$ is another Banach limit, then the same argument shows that

\[(5.15) \quad \lambda'(h) = L' \left[ \int T^* n h \, d\mu \right] = \int f' \cdot h \, d\mu, \quad h \in L_\infty \]

where $f' \in L_1$, $f' > 0$, and $Tf' = f'$. Since $X = Y$ and $T$ is conservative, Corollary 2.1 implies

\[(5.16) \quad f = \lim_{n \to \infty} n^{-1} \sum_{i=0}^{n-1} Tf = \frac{E(f \cdot e | \mathcal{C})}{E(f' \cdot e | \mathcal{C})} \cdot f', \]

where $\mathcal{C}$ is the $\sigma$-algebra of $T$-closed, and hence $T^*$-closed, sets. Thus $f = g \cdot f'$ where $g$ is a $\mathcal{C}$-measurable function. If $A \in \mathcal{C}$, then both $A$ and $A^c$ are $T^*$-closed; thus $T^* e_A = e_A$ and

\[(5.17) \quad \int_A g \cdot f' \cdot e \, d\mu = \lambda(e_A) = \lambda'(e_A) = \lambda'(e_A) = \int_A f' \cdot e \, d\mu \]

for each $A \in \mathcal{C}$. Since $e \cdot f' > 0$ and $g$ is $\mathcal{C}$-measurable, (5.17) shows that $g = 1$ and $f = f'$. This proves the first part of the theorem. To prove the second part, we note that since all Banach limits on the sequence $T^n \mu(A)$ coincide, the sequence $T^n \mu(A)$ converges Cesàro.
By the Vitali-Hahn-Saks theorem (cf. [8], p. 159), the set function \( \lambda \) defined by

\[
(5.11) \quad \lambda(A) = \lim_{n \to \infty} n^{-1} \sum_{i=0}^{n-1} T^i \mu(A), \quad A \in \mathcal{C},
\]

is a measure on \( (X, \mathcal{C}) \); \( \lambda \) is clearly \( \mu \)-continuous; moreover, \( \lambda \) is finite since \( \sup_n |T^n|_1 < \infty \); finally, it is also easy to see that \( T\lambda = \lambda \). Thus, letting \( f = \frac{d\lambda}{d\mu} \), we have \( f \in L_1 \), \( f \geq 0 \), and \( Tf = f \).

Set \( F = \{ f = 0 \} \); then both \( F \) and \( F^c \) are \( T^* \)-closed, and thus \( T^*e_F = e_F \). If \( \mu(F) > 0 \), then we arrive at the contradiction

\[
\lambda(F) \geq \lim inf_{n} n^{-1} \sum_{i=0}^{n-1} \int T^i e_F \, d\mu = \int e_F \, d\mu > 0.
\]

Thus \( f \) is a positive invariant function of \( T \). \( \square \)
VI. EXAMPLES

In our examples below, the underlying measure space (X, \mathcal{A}, \mu) is the following one: X = \{0, 1, 2, \ldots\}; \mathcal{A} is the σ-algebra of all subsets of X; \mu is the counting measure on \mathcal{A}. Hence the L^p-spaces in this case are simply the familiar little \ell_p's: for

\[ 1 \leq p < \infty, \quad \ell_p \text{ is the space of all sequences of real numbers } f = (f_0, f_1, \ldots) \text{ such that } \sum_{i=0}^{\infty} |f_i|^p < \infty, \]

and \ell_\infty is the space of all bounded sequences \( h = (h_0, h_1, \ldots) \).

**Example 1** Theorem 2.4 asserts that if T is a positive linear operator on L_1, satisfying condition (b), and if \( f \in L_1^+(\mathbb{Z}) \), then \( \lim_{n} \|T^n f\|_1 = 0 \). For \( 1 < p < \infty \), Proposition 3.2 asserts that for an operator T on L_p satisfying condition (b), the iterates \( T^n f \) converge Cesàro to zero in L_p if \( f \in L_p(\mathbb{Z}) \). Here we give an example to show that this conclusion cannot be strengthened to read \( \lim_{n \to \infty} \inf_p \|T^n f\|_p = 0 \).
For \( f = (f_0, f_1, f_2, \ldots) \in \ell_p \) for some \( p \), \( 1 \leq p < \infty \), we let \( T f = (0, f_0, f_1, \ldots) \). Clearly, \( T \) is a positive linear isometry on \( \ell_p \). If \( 1 < p < \infty \) and \( h \in \ell_q^+ \), then \( |T^*n h|_q \to 0 \) as \( n \to \infty \). Thus

\[
\lim_{n \to \infty} \int T^*n f \cdot h \, d\mu = \lim_{n \to \infty} \int f \cdot T^*n h \, d\mu = 0 \quad \text{for each } f \in \ell_p
\]

and each \( h \in \ell_q^+ \). This shows \( X = \mathbb{Z} \). Proposition 3.2 implies that \( n^{-1} \sum_{i=0}^{n-1} T^i f \to 0 \) in \( \ell_p \); on the other hand, we have \( |T^n f|_p = |f|_p \) for each \( f \) and each \( n \).

**Example 2** The authors in [21] and [14] have shown that for a semi-Markovian operator on \( L_1 \), the ratio ergodic theorem need not hold on \( \mathbb{Z} \) (see also Theorem 3.3). Here we give an example to show that Hopf's decomposition (see Theorem 2.2), which is an easy consequence of the ratio ergodic theorem, also ceases to be valid on \( \mathbb{Z} \).

Let \( T \) be defined on \( \ell_1 \) by the relation

\[
(Tf)_j = \begin{cases} 
\sum_{i=1}^{\infty} f_i, & j = 0 \\
0, & j \geq 1 
\end{cases}
\]

It follows that for each \( n \geq 1 \),
\[(T^n f)_j = \begin{cases} 
\sum_{i=n}^{\infty} f_i, & j = 0 \\
f_{j+n}, & j \geq 1
\end{cases} \]

and that \(|T^n f|_1 \leq 2 \cdot \sum_{i=n}^{\infty} |f_i| \leq 2 \cdot |f|_1\). Thus \(\sup_n |T^n|_1 \leq 2\)

and \(\lim_{n} |T^n f|_1 = 0\) for every \(f \in \ell_1\); hence \(X = Z\).

Consider the function \(g = (g_i)\) with \(g_i = 1/i^2\), \(i \geq 1\), and \(g_0 = 0\). Then \(g \in \ell_1^+\), and \((T_\omega g)_0 = \infty\). On the other hand, for every non-negative, non-vanishing function \(f\) which vanishes on all but a finite number of points we have \(0 < (T_\omega f)_0 < \infty\). Hence Hopf's decomposition does not hold on the singleton set \(\{0\}\).
REFERENCES


