WHITNEY, James Martin, 1936-
A STUDY OF THE EFFECTS OF COUPLING BETWEEN BENDING AND STRETCHING ON THE MECHANICAL BEHAVIOR OF LAYERED ANISOTROPIC COMPOSITE MATERIALS.

The Ohio State University, Ph.D., 1968
Engineering Mechanics

University Microfilms, Inc., Ann Arbor, Michigan
A STUDY OF THE EFFECTS OF COUPLING BETWEEN BENDING AND STRETCHING ON THE MECHANICAL BEHAVIOR OF LAYERED ANISOTROPIC COMPOSITE MATERIALS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of the Ohio State University

By

James Martin Whitney, B.A., B.T.E., M.S.T.E. and M.S.

*****

The Ohio State University
1968

Approved By

[Signature]
Advisor
Department of Engineering Mechanics
ACKNOWLEDGEMENTS

I wish to express my sincere appreciation to Dr. A. W. Leissa for his technical advice during this investigation.

In addition, I am indebted to Mr. R. T. Schwartz and Mr. R. G. Spain, Air Force Materials Laboratory, for their encouragement and support in allowing me the necessary time to complete this research study.

I also wish to thank Mr. J. P. Hudson, SESCD, for his assistance in the computer programming.

My wife Phyllis also deserves thanks for her encouragement and patience during the course of this work.
VITA

September 6, 1936  Born - Owosso, Michigan


1959  ..........  B.T.E., Georgia Institute of Technology, Atlanta, Georgia

1961  ..........  M.S.T.E., Georgia Institute of Technology, Atlanta, Georgia

1964  ..........  M.S., The Ohio State University, Columbus, Ohio

1961-1968  ......  Materials Research Engineer, Air Force Materials Laboratory, Wright-Patterson Air Force Base, Ohio

PUBLICATIONS


FIELDS OF STUDY

Major Field: Engineering Mechanics


# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Acknowledgments</th>
<th>ii</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vita</td>
<td>iii</td>
</tr>
<tr>
<td>Fields of Study</td>
<td>v</td>
</tr>
<tr>
<td>List of Tables</td>
<td>vi</td>
</tr>
<tr>
<td>List of Illustrations</td>
<td>ix</td>
</tr>
</tbody>
</table>

## 1. INTRODUCTION

1.1 Background ........................................ 1
1.2 Statement of Research Problem .......... 3

## 2. EQUATIONS OF AN ANISOTROPIC ELASTIC CONTINUUM

2.1 Introduction ...................................... 5
2.2 Stress Tensors of Lagrange and Kirchhoff .... 5
2.3 Equations of Motion ........................... 7
2.4 Strain-Displacement and Constitutive Relations ... 9
2.5 Compatibility .................................. 11

## 3. EQUATIONS OF A LAMINATED ANISOTROPIC THIN PLATE

3.1 Basic Assumptions ............................ 13
3.2 Strain-Displacement Relations ............ 14
3.3 Equations of Motion ......................... 17
3.4 Constitutive Equations ...................... 21
3.5 Equations of Motion in Terms of Displacements . 22
3.6 Equations of Equilibrium in Terms of a Stress Function and Transverse Displacement .... 25
3.7 Boundary Conditions ......................... 28
3.8 Application to Fiber Reinforced Composite Materials .................. 33

## 4. EFFECT OF COUPLING ON THE MECHANICAL BEHAVIOR OF FIBROUS COMPOSITES

4.1 Introduction .................................... 36
4.2 Transverse Loading of Simply-Supported Plate .................. 38
4.3 Buckling of Simply-Supported Angle-Ply Composites Under Uniform Compression .... 45
4.4 Dynamics of Simply-Supported Plates .......... 49
5. APPLICATION OF DOUBLE FOURIER SERIES TO THE SOLUTION OF COUPLED LAMINATED PLATES

5.1 Fourier Series Solution of Boundary Value Problems .......................................................... 53
5.2 Simply-Supported Angle-Ply Plate Under Uniform Loading ............................................. 57

6. SUMMARY AND RECOMMENDATIONS FOR FUTURE WORK ............................................ 68

APPENDIX ................................................................................................................................. 93
REFERENCES ............................................................................................................................ 96
LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Numerical Results for Simply-Supported Cross-Ply Plate Under Transverse Loading</td>
<td>72</td>
</tr>
<tr>
<td>2.</td>
<td>Numerical Convergence of Solution for Simply-Supported 45° Angle-Ply Plate Under Uniform Load ( b = a )</td>
<td>73</td>
</tr>
<tr>
<td>3.</td>
<td>Numerical Convergence of Solution for Simply-Supported 45° Angle-Ply Plate Under Uniform Load ( b \neq a )</td>
<td>74</td>
</tr>
<tr>
<td>4.</td>
<td>Numerical Convergence of Solution for Simply-Supported 45° Angle-Ply Plate Under Uniform Load ( R = 2 )</td>
<td>75</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>1.</td>
<td>Two-Layer Angle-ply Composite With Fibers Oriented at +θ and -θ to the Plate Axes</td>
<td>76</td>
</tr>
<tr>
<td>2.</td>
<td>Stress Nomenclature on Body in Deformed State</td>
<td>77</td>
</tr>
<tr>
<td>3.</td>
<td>Location of Particles on Undeformed and Deformed Bodies</td>
<td>78</td>
</tr>
<tr>
<td>4.</td>
<td>Coordinate System of Plate</td>
<td>79</td>
</tr>
<tr>
<td>5.</td>
<td>Resultant Stress Nomenclature</td>
<td>80</td>
</tr>
<tr>
<td>6.</td>
<td>Nomenclature for Moment and Transverse Shear Results</td>
<td>81</td>
</tr>
<tr>
<td>7.</td>
<td>Unidirectional Composite at Parallel and Angle-Ply Orientations, Respectively</td>
<td>82</td>
</tr>
<tr>
<td>8.</td>
<td>Cross-Ply Composite</td>
<td>83</td>
</tr>
<tr>
<td>9.</td>
<td>Maximum Deflection as a Function of Angle-Ply Orientation for Simply-Supported Graphite-Epoxysquare Plate Under Transverse Loading</td>
<td>84</td>
</tr>
<tr>
<td>10.</td>
<td>Maximum Deflection as a Function of Angle-Ply Orientation for Simply-Supported Glass-Epoxysquare Plate Under Transverse Loading</td>
<td>85</td>
</tr>
<tr>
<td>11.</td>
<td>Critical Buckling Load as a Function of Angle-Ply Orientation for Simply-Supported Graphite-Epoxysquare Plate Under Uniaxial Compression</td>
<td>86</td>
</tr>
<tr>
<td>12.</td>
<td>Critical Buckling Load as a Function of Angle-Ply Orientation for Simply-Supported Glass-Epoxysquare Plate Under Uniaxial Compression</td>
<td>87</td>
</tr>
<tr>
<td>13.</td>
<td>Critical Buckling Load as a Function of Angle-Ply Orientation for Simply-Supported Graphite-Epoxysquare Plate Under Biaxial Compression</td>
<td>88</td>
</tr>
<tr>
<td></td>
<td>Fundamental Vibration Frequency as a Function of Aspects Ratio for a Simply-Supported Graphite-Epoxy Cross-Ply Plate</td>
<td></td>
</tr>
<tr>
<td>---</td>
<td>-------------------------------------------------------------------------------------------------------------------</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>89</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Fundamental Vibration Frequency as a Function of Aspects Ratio for a Simply-Supported Glass-Epoxy Cross-Ply Plate</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>90</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Fundamental Vibration Frequency as a Function of Angle-Ply Orientation for a Simply-Supported Graphite-Epoxy Square Plate</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>91</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Fundamental Vibration Frequency as a Function of Angle-Ply Orientation for a Simply-Supported Glass-Epoxy Square Plate</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>92</td>
<td></td>
</tr>
</tbody>
</table>
1. INTRODUCTION

1.1 Background

With the development of high modulus, high strength, low density filaments (e.g., boron and graphite) for use in composite materials, there has been an increased interest in the analysis of anisotropic plates and shells. In particular, it has been shown (1, 2)\* that laminated plate theory provides an indispensable tool in analyzing the mechanical behavior of composite materials.

Many solutions can be found in the literature (3, 4, 5 and 6) for orthotropic plates in which the material axes of symmetry coincide with some convenient axes of the plate. Some of these solutions appear in closed form, while others have been obtained by use of energy considerations (Ritz method).

The mechanics of laminated anisotropic materials can be traced back to the 1940's when Forest Product Laboratories in Madison, Wisconsin began studies on the mechanical behavior of plywood. They were able to use many of the existing orthotropic solutions (7, 8) to describe the mechanical behavior of wood laminates. Hearmon (9, 10) studied the dynamic behavior of wood laminates.

The first attempt to analyze laminated materials in terms of classical plate theory was accomplished by Smith (11). In analyzing the bending behavior of a two layer rectangular plywood plate in

\*Numbers in parenthesis correspond to the references.
which the grains of the adjacent layers are oriented at angles of $+\theta$ and $-\theta$ (Figure 1) to the axes of the plate, Smith concluded that the material behaved in the same manner as a homogeneous orthotropic plate. Later work by Reissner and Stavsky (12) showed this conclusion to be in error. They demonstrated a coupling phenomenon between stretching and bending which was not included in Smith's paper. Smith reached his conclusions by using "average stresses" and assuming the middle plane to be free of strain under bending loads. Reissner and Stavsky assumed bending and stretching to be coupled, and used the actual stresses in the plies. Results showed that the coupling effect disappeared if the laminate was layed up symmetrically about the middle plane. Stavsky (13) discussed in detail the difference between the coupled theory and the results obtained by Smith.

Formulating the problem in terms of a stress function and transverse deflection, Reissner and Stavsky used the inverse approach to obtain solutions for laminated plates under transverse loading. That is, they found a solution to the differential equation then determined the boundary conditions which were satisfied. Solutions in the form of polynomials were also obtained for simple loadings (e.g. a combination of uniform tension and uniform bending moments). Dong, Matthieson, Pister, and Taylor (14) also formulated the bending problem in terms of a stress function and
transverse displacement. In addition to obtaining polynomial solutions to long narrow plates under simple loads, they outlined a perturbation method of solution for the general problem.

Using the Ritz method Ashton (15, 16, and 17) obtained natural frequencies, critical buckling loads, and deflections due to transverse loadings for symmetrically laminated anisotropic rectangular and skewed plates. The mode shapes were approximated by use of characteristic beam functions. Ashton's work also included a comprehensive experimental program.

1.2 Statement of Research Problem

To date, only the bending theory of laminated plates has been developed in detail. The first goal of this investigation is to develop a general theory of thin laminated rectangular plates which incorporates the classical nonlinear assumptions of von Karman (3). Thermal stresses and inertia terms are also considered. The second major goal is to investigate the effect of coupling between bending and stretching on the linear behavior of fibrous composite materials. In particular, the effect of coupling on plate bending, buckling, and fundamental vibration frequencies is determined.

The derivation of the general equations which describe a laminated anisotropic plate departs somewhat from standard procedures. That is, the theory is developed directly from the equations of anisotropic continuum mechanics rather than from strength of materials.
To provide the necessary background, a separate chapter discussing the equations of an anisotropic continuum is included.

Principles of Fourier analysis are used to obtain solutions to the coupled equations.
2. EQUATIONS OF AN ANISOTROPIC ELASTIC CONTINUUM

2.1. Introduction

Four relationships are necessary to describe the behavior of an anisotropic elastic continuum. In particular, the equations of motion, strain-displacement relations, constitutive law, and compatibility equations provide a complete set of field equations.

Finite deformations and the existence of Hooke's law are assumed. Wherever possible, the resulting equations are summarized as the equations of elasticity are well known. Much of the presentation in the next two sections is taken from the work of Fung (18) and Pearson (19).

2.2. Stress Tensors of Lagrange and Kirchhoff

Figure 2 shows the stress nomenclature in cartesian coordinates. In linear mechanics little or no distinction is made between the stresses with respect to the deformed and undeformed coordinates; the difference being a second order effect. However, in the development of a plate theory which includes nonlinear effects it is useful to relate stresses on the deformed body to its initial configuration. This can be accomplished in the following manner.

Consider a force vector $\vec{dF}$ acting on a deformed surface $dS$ and a corresponding force vector $\vec{dF}_0$ acting on the same surface in the undeformed state $dS_0$. The stress components in the deformed state are given by the Cauchy relationship

5.
where \( \tau_{ji} \) are components of the Eulerian stress tensor and \( n_j \) are direction cosines of outer normal to the deformed surface. The Lagrange and Kirchhoff stress tensors both refer to the original configuration. The components of these two tensors are defined as follows

\[
dF_{oi} = \tau_{ji} \quad dS_j \quad (2-1)
\]

where \( \tau_{ji} \) are components of the Lagrange and Kirchhoff stress tensors, respectively, \( n_{0j} \) are direction cosines of the outer normal to the undeformed surface, \( x_i \) are coordinates of the undeformed surface, and \( \nabla_i \) are coordinates of the deformed surface.

In order to make use of the Lagrange and Kirchhoff stress tensors it is necessary to find a relationship between \( \tau_{ij} \), \( T_{ij} \), and \( \sigma_{ij} \). The derivation of the desired relationships is carried out in detail by Fung and will not be repeated here. The resulting equations are

\[
T_{ji} = \frac{\rho_1}{\rho_i} \frac{\partial x_i}{\partial \xi_k} \tau_{ki} \quad (2-4)
\]

\[
\sigma_{ji} = \frac{\rho_1}{\rho_i} \frac{\partial x_j}{\partial \xi_k} \frac{\partial x_i}{\partial \xi_m} \tau_{km} \quad (2-5)
\]

where \( \rho_i \) and \( \rho_0 \) are densities of the deformed and undeformed sur-
faces, respectively. Combining (2-4) and (2-5) yields

$$T_{jl} = \frac{\partial \bar{x}_j}{\partial x_k} \sigma_{ik} \tag{2-6}$$

However,

$$\bar{x}_j = x_j + u_j \tag{2-7}$$

where $u_j$ are components of the displacement vector. Substituting (2-7) into (2-6) yields

$$T_{jl} = \left[ \sigma_{ik} \delta_{jk} + \frac{\partial u_j}{\partial x_k} \right] \tag{2-8}$$

where $\delta_{jk}$ is the kronecker $\delta$.

A cursory examination of equations (2-4) and (2-5) shows that the Lagrange stress tensor is not symmetric, while the Kirchhoff stress tensor is. Therefore, the Lagrange tensor cannot be used directly in a symmetric constitutive relationship such as Hooke's law.

2.3 Equations of Motion

Consider a region in the deformed surface $S$, including volume $V$, subjected to surface tractions $\tau_{ji} n$; and body force per unit volume $X_i$. The following relationships are also specified.

$$X_i dV = X_{i1} dV_0 \tag{2-9}$$

where $X_{01}$ is the body force associated with the undeformed shape.

$$\rho' dV = \rho dV_0 \tag{2-10}$$
The resultant force \( R \) is

\[
R = \int_{S} \tau_{ij} n_{j} dS + \int_{V} X_{i} dV
\]  
(2-11)

Using equations (2-1), (2-2), and (2-9) yields

\[
R = \int_{S_0} T_{ij} n_{j} dS + \int_{V_0} X_{0i} dV_0
\]  
(2-12)

Applying Gauss' theorem to change a surface integral to a volume integral yields

\[
R = \int_{V_0} \left( \frac{\partial T_{ij}}{\partial x_j} + X_{0i} \right) dV_0
\]  
(2-13)

Using Newton's second law in conjunction with (2-13) yields

\[
\int_{V_0} \left( \frac{\partial T_{ij}}{\partial x_j} + X_{0i} - \rho \frac{\partial^2 u_i}{\partial t^2} \right) dV_0 = 0
\]  
(2-14)

Since (2-14) must hold for any arbitrary volume the integrand must vanish.

\[
\frac{\partial T_{ij}}{\partial x_j} + X_{0i} = \rho \frac{\partial^2 u_i}{\partial t^2}
\]  
(2-15)

Substituting equation (2-8) into (2-15) results in the following equations of motion in terms of the Kirchhoff stress tensor.
\[
\frac{\partial}{\partial x_j} \left[ \sigma_{kj} \left( \delta_{lk} + \frac{\partial u_i}{\partial x_k} \right) \right] + X_{0i} = \rho \frac{\partial^2 u_i}{\partial t^2} \quad (2-16)
\]

2.4. Strain-Displacement and Constitutive Relations

For finite deformations a strain tensor is sought in terms of coordinates on the undeformed surface. Consider the body shown in Figure 3 with two particles having coordinates \((x_1), (x_1 + dx_1)\) at time \(t = 0\), and having coordinates \((\bar{x}_1), (\bar{x}_1 + d\bar{x}_1)\) at time \(t = t\). The deformation of this body is described in terms of the stretch \(s\) which is defined in the Lagrangian sense as

\[
S = \frac{|d\bar{r}| - |d\bar{r}_0|}{|d\bar{r}_0|} = \frac{|d\bar{r}|}{|d\bar{r}_0|} \quad (2-17)
\]

where

\[
|d\bar{r}| = \sqrt{\frac{\partial \bar{x}_1}{\partial x_1} \frac{\partial \bar{x}_1}{\partial x_1}}, \quad |d\bar{r}_0| = \sqrt{dx_1 \cdot dx_1} \quad (2-18)
\]

Using the tensorial properties of the vector \(d\bar{r}\) yields

\[
d\bar{x}_1 = \frac{\partial \bar{x}_1}{\partial x_j} dx_j \quad (2-19)
\]

Substituting equation (2-19) into (2-18), the following results are obtained

\[
\frac{|d\bar{r}|}{|d\bar{r}_0|} = \sqrt{\frac{\partial x_j \cdot \partial x_k \frac{\partial \bar{x}_1}{\partial x_k} \frac{\partial \bar{x}_1}{\partial x_j}}{|d\bar{r}_0|^2}} \quad (2-20)
\]
Taking equation (2-7) into account, equation (2-20) becomes

\[
\frac{|d\mathbf{r}|}{|d\mathbf{r}_0|} = \sqrt{\frac{n_0 j n_0 k}{\delta_j \delta_k} \left( \frac{\partial u_i}{\partial x_k} + \delta_{ik} \right) \left( \frac{\partial u_i}{\partial x_j} + \delta_{ij} \right)}
\]

Writing equation (2-21) in a binomial series and retaining only the first two terms yields the following results for the stretch

\[
S = n_0 j n_0 e_{ij}
\]

where \( e_{ij} \) is the Green strain tensor and is defined by

\[
e_{ij} = \frac{1}{2} \left[ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \left( \frac{\partial u_k}{\partial x_i} \right) \left( \frac{\partial u_k}{\partial x_j} \right) \right]
\]

Equation (2-23) provides the necessary strain-displacement relationship for finite deformations.

The generalized Hooke's law including thermal strains is given by

\[
e_{ij} = S_{ijkm} \sigma_{km} + a_{ij} T
\]

where \( S_{ijkm} \) are components of the compliance tensor, \( a_{ij} \) is the thermal expansion coefficient tensor, and \( T \) is the temperature field.

An inverted form of equation (2-24) is often used.

\[
\sigma_{ij} = C_{i\ell km} e_{\ell m} - C_{ijkm} a_{\ell m} T
\]
where $C_{ijklm}$ is the stiffness or elastic modulus tensor. Due to the symmetry of the stiffness tensor, the maximum number of independent elastic moduli in an anisotropic material is 21. Similarly, the maximum number of independent thermal expansion coefficients is 6.

2.5. **Compatibility**

Given a strain field the question arises as to how equation (2-23) can be integrated to determine the displacements. Since there are six strain equations in three unknown displacements, solutions will not be single-valued or continuous unless certain relations are satisfied. The compatibility equations for infinitesimal displacements are well known. The derivation of compatibility relations for finite deformations is more difficult. However, for purposes of this investigation the linear relations will suffice. For convenience they are listed here in expanded form.

\[
\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = \frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} \quad (2-26)
\]

\[
\frac{\partial^2 \varepsilon_{23}}{\partial x_2 \partial x_3} = \frac{\partial^2 \varepsilon_{22}}{\partial x_3^2} = \frac{\partial^2 \varepsilon_{33}}{\partial x_2^2} \quad (2-27)
\]

\[
\frac{\partial^2 \varepsilon_{31}}{\partial x_3 \partial x_1} = \frac{\partial^2 \varepsilon_{11}}{\partial x_3^2} + \frac{\partial^2 \varepsilon_{33}}{\partial x_1^2} \quad (2-28)
\]

\[
2 \frac{\partial^2 \varepsilon_{11}}{\partial x_2 \partial x_3} = \frac{\partial}{\partial x_1} \left( -\frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{31}}{\partial x_3} + \frac{\partial \varepsilon_{12}}{\partial x_3} \right) \quad (2-29)
\]

\[
2 \frac{\partial^2 \varepsilon_{22}}{\partial x_1 \partial x_3} = \frac{\partial}{\partial x_2} \left( \frac{\partial \varepsilon_{23}}{\partial x_2} - \frac{\partial \varepsilon_{23}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3} \right) \quad (2-30)
\]
3. EQUATIONS OF A LAMINATED ANISOTROPIC THIN PLATE

3.1. Basic Assumptions

In many practical applications of thin plates the magnitude of the tractions acting on the surface parallel to the middle plane are small compared to the bending and membrane stresses. For a thin plate this implies that the tractions on any surface parallel to the middle plane are relatively small. Thus, an approximate state of plane stress exists.

A standard \((x, y, z)\) coordinate system, as shown in Figure 4, is used in deriving the equations. The displacements are denoted by \(u = u_x\), \(v = u_y\), and \(w = u_z\). The following basic assumptions are made.

1. The plate is constructed of \(n\) layers of orthotropic sheets bonded together. However, the orthotropic axes of material symmetry of an individual layer need not coincide with the \(x - y\) axes of the plate.

2. The plate is thin, i.e., the thickness \(h\) is much smaller than the other physical dimensions.

3. Tangential displacements \(u\) and \(v\) are small compared to the plate thickness.

4. In-plane strains \(\varepsilon_x\), \(\varepsilon_y\), and \(\varepsilon_{xy}\) are small compared to unity.

5. The transverse deflection \(w\) is no greater than the plate thickness, and the slope of the deflected plate can be large compared
where $\varepsilon_{ij}$ denotes engineering strains.

Equations (2-16), (2-23), and (2-24) - (2-31) provide a complete set of field equations.
to the strains. Thus, in the development of the governing equations nonlinear terms involving plate slopes are retained. All other nonlinear terms are neglected.

6. Transverse shear strains $\varepsilon_{xz}$ and $\varepsilon_{yz}$ are negligible.

7. Tangential displacements $u$ and $v$ are linear functions of the $z$ coordinate.

8. The transverse normal strain $\varepsilon_z$ is negligible.


10. The plate has constant thickness.

It should be noted that assumption 6 is a direct consequence of plane stress. This has been shown by Lekhnitskii (20), the details of which are presented in the Appendix. Together, 6 and 7 constitute the classical assumptions of Kirchhoff. Shear deformations could be accounted for by assuming tangential displacements of the form used by Mindlin (21) in developing a higher order approximation for homogeneous, isotropic plates. However, this work is limited to thin plates where Kirchhoff type assumptions are sufficient. Assumptions 4 and 5 are essentially equivalent to those used by von Karman (18) in developing a nonlinear plate theory for homogeneous, isotropic materials. Assumption 8 allows the problem to be reduced to a two dimensional study of the middle plane.

3.2 Strain-Displacement Relations

Using assumption 7 the tangential displacements are of the form
\[ u = u^0(x, y, t) + zF_1(x, y, t) \]  
\[ v = v^0(x, y, t) + zF_2(x, y, t) \]

where \( u^0 \) and \( v^0 \) are tangential-displacements of the middle-plane.

Applying Green's strain tensor, equation (2-23), with nonlinear terms retained in accordance with assumption 5 yields the following strain-displacement relationships.

\[ \varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \]  
\[ \varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \]  
\[ \varepsilon_z = \frac{\partial w}{\partial z} \]  
\[ \varepsilon_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \]  
\[ \varepsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \]  
\[ \varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \left( \frac{\partial w}{\partial x} \right) \left( \frac{\partial w}{\partial y} \right) \]

where the shear strains are engineering strains. Substituting (3-1) and (3-2) into (3-6) and (3-7), and applying assumption 6 yields

\[ \varepsilon_{xz} = F_1 + \frac{\partial w}{\partial x} = 0 \]  
\[ \varepsilon_{yz} = F_2 + \frac{\partial w}{\partial y} = 0 \]
Therefore,

\[ F_1 = -\frac{\partial w}{\partial x}, \quad F_2 = -\frac{\partial w}{\partial y} \]  \hspace{1cm} (3-11)

Assumption 8 along with equation (3-5) implies

\[ w = w(x, y, t) \]  \hspace{1cm} (3-12)

Using (3-11) in conjunction with (3-1), (3-2), (3-3), (3-4) and (3-8) yields

\[ \varepsilon_x = \varepsilon_x^0 + zK \]  \hspace{1cm} (3-13)

\[ \varepsilon_y = \varepsilon_y^0 + zK \]  \hspace{1cm} (3-14)

\[ \varepsilon_{xy} = \varepsilon_{xy}^0 + zK_{xy} \]  \hspace{1cm} (3-15)

where

\[ \varepsilon_x^0 = \frac{\partial u_x^0}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \]  \hspace{1cm} (3-16)

\[ \varepsilon_y^0 = \frac{\partial u_y^0}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \]  \hspace{1cm} (3-17)

\[ \varepsilon_{xy}^0 = \frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x} + \left( \frac{\partial w}{\partial x} \right) \left( \frac{\partial w}{\partial y} \right) \]  \hspace{1cm} (3-18)

\[ K_x = -\frac{\partial^2 w}{\partial x^2}, \quad K_y = -\frac{\partial^2 w}{\partial y^2}, \quad K_{xy} = -2 \frac{\partial^2 w}{\partial x \partial y} \]  \hspace{1cm} (3-19)

Equations (3-13) - (3-19) coincide with those of classical homogeneous thin plate theory.
3.3 Equations of Motion

Assumption 6 cannot be satisfied unless the resultant shear vanishes. Obviously, this is not physically correct. This apparent inconsistency in classical plate theory is well recognized and accepted. It is discussed in detail by Fung (18).

Expanding the equations of motion (2-16) and retaining only those nonlinear terms which are consistent with assumption 5 yields the following relations for the kth layer of the laminate.

\[
\frac{\partial\sigma_{xx}^{(k)}}{\partial x} + \frac{\partial\sigma_{xy}^{(k)}}{\partial y} + \frac{\partial\sigma_{xz}^{(k)}}{\partial z} + X^{(k)} = \rho_o^{(k)} \frac{\partial^2 u}{\partial t^2} \tag{3-20}
\]

\[
\frac{\partial\sigma_{xy}^{(k)}}{\partial x} + \frac{\partial\sigma_{yy}^{(k)}}{\partial y} + \frac{\partial\sigma_{yz}^{(k)}}{\partial z} + Y^{(k)} = \rho_o^{(k)} \frac{\partial^2 v}{\partial t^2} \tag{3-21}
\]

\[
\frac{\partial}{\partial x} \left( \sigma_{xx}^{(k)} + \sigma_{xy}^{(k)} \frac{\partial w}{\partial x} + \sigma_{xz}^{(k)} \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial y} \left( \sigma_{xy}^{(k)} + \sigma_{yy}^{(k)} \frac{\partial w}{\partial x} + \sigma_{yz}^{(k)} \frac{\partial w}{\partial y} \right) + \\
\frac{\partial}{\partial z} \left( \sigma_{xz}^{(k)} + \sigma_{yz}^{(k)} \frac{\partial w}{\partial x} + \sigma_{zz}^{(k)} \frac{\partial w}{\partial z} \right) + Z^{(k)} = \rho_o^{(k)} \frac{\partial^2 w}{\partial t^2} \tag{3-22}
\]

where

\[ X = X_{01}, \quad Y = X_{02}, \quad Z = X_{03} \]

We now define stress and moment resultants as follows

\[
(N_x, N_y, N_{xy}) = \int_{-h/2}^{h/2} (\sigma_{xx}^{(k)}, \sigma_{xy}^{(k)}, \sigma_{xy}^{(k)}) \, dz \tag{3-23}
\]

\[
(Q_x, Q_y) = \int_{-h/2}^{h/2} (\sigma_{xz}^{(k)}, \sigma_{yz}^{(k)}) \, dz \tag{3-24}
\]
(M_x, M_y, M_{xy}) = \int_{-h/2}^{h/2} (\sigma_x^{(k)}, \sigma_y^{(k)}, \sigma_{xy}^{(k)}) z \, dz \quad (3-25)

These results are illustrated in Figures 5 and 6.

We now proceed by integrating equation (3-20) with respect to z

\[ \int_{-h/2}^{h/2} \frac{\partial \sigma^{(k)}}{\partial x} \, dz + \int_{-h/2}^{h/2} \frac{\partial \sigma^{(k)}}{\partial y} \, dz + \frac{\partial \sigma^{(k)}}{\partial z} (h/2) - \sigma_{xz} (-h/2) + \int_{-h/2}^{h/2} X^{(k)} \, dz = \int_{-h/2}^{h/2} \rho_0^{(k)} \frac{\partial^2 u}{\partial t^2} \, dz \]

Interchanging the order of differentiation and integration and using the definition of stress resultant yields

\[ \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + q = \int_{-h/2}^{h/2} \rho_0^{(k)} \frac{\partial^2 u}{\partial t^2} \, dz \quad (3-26) \]

where

\[ q_x = \sigma_{xz} (h/2) - \sigma_{xz} (-h/2) + \int_{-h/2}^{h/2} X^{(k)} \, dz \quad (3-27) \]

Using (3-1) and (3-11) in conjunction with the right side of (3-26) yields

\[ \int_{-h/2}^{h/2} \rho_0^{(k)} \frac{\partial^2 u}{\partial t^2} \, dz = \int_{-h/2}^{h/2} \rho_0^{(k)} \, dz - \int_{-h/2}^{h/2} \rho_0^{(k)} \, dz \]

The second term on the right side of (3-28) vanishes if \( \rho_0^{(k)} \) is an even function of z. In any event, it is small and can be neglected.

Compacting the notation, (3-26) becomes
\[
\frac{\partial N}{\partial x} + \frac{\partial N}{\partial y} + q = \frac{\partial^2 u}{\partial t^2} \quad (3-29)
\]

where
\[
\rho = \int_{-h/2}^{h/2} \rho \, dz \quad (3-30)
\]

In a similar manner equation (3-21) integrated with respect to \( z \)
becomes
\[
\frac{\partial N}{\partial x} + \frac{\partial N}{\partial y} + q = \frac{\partial^2 v}{\partial t^2} \quad (3-31)
\]

where
\[
q = \sigma (k) \left( \frac{\partial}{\partial z} \right) - \sigma (k) \left( \frac{\partial}{\partial z} \right) + \int_{-h/2}^{h/2} Y (k) \, dz
\]

Integrating equation (3-22) with respect to \( z \) and again inter-
changing the order of differentiation and integration yields
\[
\left( \frac{N}{x} \frac{\partial^2 w}{\partial x^2} + 2N \frac{\partial^2 w}{\partial x \partial y} + N \frac{\partial^2 w}{\partial y^2} \right) + \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial y} + q = \rho \frac{\partial^2 w}{\partial t^2} \quad (3-32)
\]

where
\[
q = \left( \sigma (k) + \sigma (k) \frac{\partial}{\partial x} + \sigma (k) \frac{\partial}{\partial y} \right) \left[ \int_{-h/2}^{h/2} Z (k) \, dz \right] + \left[ \int_{-h/2}^{h/2} Z (k) \, dz \right]
\]

Now multiplying equation (3-20) by \( z \) and integrating with respect
to \( z \) over the plate thickness yields
\[
\frac{\partial M}{\partial x} + \frac{\partial M}{\partial y} + \int_{-h/2}^{h/2} \frac{\partial \sigma (k)}{\partial z} \, dz + \int_{-h/2}^{h/2} X (k) \, dz = \int_{-h/2}^{h/2} \rho \, dz \quad (3-34)
\]
We note
\[
\frac{z \partial \sigma^{(k)}_{xz}}{\partial z} = \frac{\partial (z \sigma^{(k)}_{xz})}{\partial z} - \sigma^{(k)}_{xz}
\] (3-35)

Taking (3-24) into account, equation (3-35) becomes
\[
\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x + m_x = I \frac{\partial^3 w}{\partial x^2 \partial t^2}
\] (3-36)

where
\[
m_x = \frac{h}{2} \left[ \sigma_{xz}^{(k)} \left( \frac{h}{2} \right) + \sigma_{xz}^{(k)} \left( -\frac{h}{2} \right) \right] + \int_{-h/2}^{h/2} x^{(k)} z \, dz
\] (3-37)
\[
I = \int_{-h/2}^{h/2} \rho \left( \frac{k}{k} \right) z^2 \, dz
\] (3-38)

A similar procedure with equation (3-21) yields
\[
\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y + m_y = I \frac{\partial^3 w}{\partial y^2 \partial t^2}
\] (3-39)

where
\[
m_y = \frac{h}{2} \left[ \sigma_{yz}^{(k)} \left( \frac{h}{2} \right) + \sigma_{yz}^{(k)} \left( -\frac{h}{2} \right) \right] + \int_{-h/2}^{h/2} y^{(k)} z \, dz
\] (3-40)

Differentiating equations (3-36) and (3-39) with respect to \(x\) and \(y\) respectively yields
\[
\frac{\partial Q_x}{\partial x} = \frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial m_x}{\partial x} - I \frac{\partial^4 w}{\partial x^2 \partial t^2}
\] (3-41)
\[
\frac{\partial Q_y}{\partial y} = \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + \frac{\partial m_y}{\partial y} - I \frac{\partial^4 w}{\partial y^2 \partial t^2}
\] (3-42)

Putting (3-41) and (3-42) into (3-32), and taking into account (3-29) and (3-31) yields
\[
\frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} + \frac{\partial w}{\partial x} \left( \rho \frac{\partial^2 u^0}{\partial t^2} - q_x \right) + \frac{\partial w}{\partial y} \left( \rho \frac{\partial^2 v^0}{\partial t^2} - q_y \right) + \frac{\partial m_x}{\partial x} + \frac{\partial m_y}{\partial y} + q = \rho \frac{\partial^2 w}{\partial t^2} + I \frac{\partial^2}{\partial t^2} \nabla^2 w
\]  

(3-43)

where \(\nabla^2\) is the Laplacian operator \(\partial^2 /\partial x^2 + \partial^2 /\partial y^2\)

Equations (3-29), (3-31), and (3-43) constitute the equations of motion.

3.4. Constitutive Equations

Assuming a state of plane stress and applying assumption 8, equation (2-25) for the \(k\)th layer of the plate reduces from a 6 x 6 matrix to the following 3 x 3.

\[
\begin{bmatrix}
\sigma_x^{(k)} \\
\sigma_y^{(k)} \\
\sigma_{xy}^{(k)}
\end{bmatrix} =
\begin{bmatrix}
C_{11}^{(k)} & C_{12}^{(k)} & C_{16}^{(k)} \\
C_{12}^{(k)} & C_{22}^{(k)} & C_{26}^{(k)} \\
C_{16}^{(k)} & C_{26}^{(k)} & C_{66}^{(k)}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x - \alpha_1^{(k)} T \\
\varepsilon_y - \alpha_2^{(k)} T \\
\varepsilon_{xy} - \alpha_6^{(k)} T
\end{bmatrix}
\]  

(3-44)

We now define the following "equivalent thermal loads"

\[
N_T^{x^*} M_x^T = \int_{-h/2}^{h/2} (C_{11}^{(k)} a_1^{(k)} + C_{12}^{(k)} a_2^{(k)} + C_{16}^{(k)} a_6^{(k)}) T(1, z) dz 
\]  

(3-45)

\[
N_T^{y^*} M_y^T = \int_{-h/2}^{h/2} (C_{12}^{(k)} a_1^{(k)} + C_{22}^{(k)} a_2^{(k)} + C_{26}^{(k)} a_6^{(k)}) T(1, z) dz 
\]  

(3-46)

\[
N_T^{xy} M_{xy}^T = \int_{-h/2}^{h/2} (C_{16}^{(k)} a_1^{(k)} + C_{26}^{(k)} a_2^{(k)} + C_{66}^{(k)} a_6^{(k)}) T(1, z) dz 
\]  

(3-47)
Substituting (3-44) into the stress and moment resultants yields

\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy} \\
M_x \\
M_y \\
M_{xy}
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\
A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\
A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\
B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\
B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\
B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon^0_x \\
\varepsilon^0_y \\
\varepsilon^0_{xy} \\
K_x \\
K_y \\
K_{xy}
\end{bmatrix}
\begin{bmatrix}
N_x^T \\
N_y^T \\
N_{xy}^T \\
M_x^T \\
M_y^T \\
M_{xy}^T
\end{bmatrix}
\] (3-48)

where

\[
(A_{ij}, B_{ij}, D_{ij}) = \int_{h/2}^{h/2} C_{ij}^{(k)} (l, z, z^2) \, dz
\] (3-49)

In the general case there are 18 different plate stiffness constants arising from 6 sets of ply stiffnesses. The most important feature of (3-48) is the coupling phenomena which exists between stretching and bending. If \(C_{ij}^{(k)}\) is an even function of \(z\) (symmetric lay-up of the laminate) \(B_{ij} = 0\) and coupling is eliminated. One might conjecture that coupling could be eliminated by picking the coordinate system to be other than the middle surface of the plate. However, it can be shown (2) that in the general case, coupling cannot be completely eliminated.

It should be noted that variable thickness is accounted for by considering variable limits of integration \(i.e., h = h(x, y)\).

3.5 Equations of Motion in Terms of Displacements

Substituting the constitutive relations (3-48) into the equations of motion (3-29), (3-31), and (3-43), and using the strain-displacement
relations (3-16) - (3-19) yields

\[ \frac{\partial^2 u}{\partial x^2} + 2 A_{16} \frac{\partial^2 u}{\partial x \partial y} + A_6 \frac{\partial^2 u}{\partial y^2} + A_{16} \frac{\partial^2 v}{\partial x^2} + (A_{12} + A_{66}) \frac{\partial^2 v}{\partial x^2} + A_{26} \frac{\partial^2 v}{\partial y^2} - B_{11} \frac{\partial^3 w}{\partial x^3} - 3 B_{16} \frac{\partial^3 w}{\partial x^2 \partial y} - (B_{12} + 2B_{66}) \frac{\partial^3 w}{\partial x \partial y^2} - B_{26} \frac{\partial^3 w}{\partial x^3} \]

\[ + \frac{\partial w}{\partial x} \left( A_{11} \frac{\partial^2 w}{\partial x^2} + 4 A_{16} \frac{\partial^2 w}{\partial x \partial y} + A_6 \frac{\partial^2 w}{\partial y^2} \right) \]

\[ + \frac{\partial w}{\partial y} \left[ A_{16} \frac{\partial^2 w}{\partial x^2} + \left( A_{12} + A_{66} \right) \frac{\partial^2 w}{\partial x \partial y} + A_{26} \frac{\partial^2 w}{\partial y^2} \right] \]

\[ - \frac{\partial N^T_x}{\partial x} - \frac{\partial N^T_y}{\partial y} + q_x = \rho \frac{\partial^2 u}{\partial t^2} \]  

(3-50)

\[ A_{16} \frac{\partial^2 u}{\partial x^2} + (A_{12} + A_{66}) \frac{\partial^2 u}{\partial x \partial y} + A_{26} \frac{\partial^2 u}{\partial y^2} + A_{66} \frac{\partial^2 v}{\partial x^2} + 2 A_{26} \frac{\partial^2 v}{\partial x \partial y} + A_{22} \frac{\partial^2 v}{\partial y^2} - B_{16} \frac{\partial^3 w}{\partial x^3} - (B_{12} + 2B_{66}) \frac{\partial^3 w}{\partial x^2 \partial y} - 3 B_{26} \frac{\partial^3 w}{\partial x \partial y^2} - B_{22} \frac{\partial^3 w}{\partial x^3} \]

\[ - 3 B_{16} \frac{\partial^3 w}{\partial x^2 \partial y} - B_{22} \frac{\partial^3 w}{\partial x \partial y^2} + \frac{\partial w}{\partial x} \left[ A_{16} \frac{\partial^2 w}{\partial x^2} + (A_{12} + A_{66}) \frac{\partial^2 w}{\partial x \partial y} + A_{26} \frac{\partial^2 w}{\partial y^2} + 2 A_{22} \frac{\partial^2 w}{\partial x \partial y} + A_{22} \frac{\partial^2 w}{\partial y^2} \right] \]

\[ + \frac{\partial w}{\partial y} \left( A_{66} \frac{\partial^2 w}{\partial x^2} + 2 A_{26} \frac{\partial^2 w}{\partial x \partial y} + A_{22} \frac{\partial^2 w}{\partial y^2} \right) \]

\[ - \frac{\partial N^T_x}{\partial x} - \frac{\partial N^T_y}{\partial y} + q_y = \rho \frac{\partial^2 v}{\partial t^2} \]  

(3-51)

\[ D_{11} \frac{\partial^4 w}{\partial x^4} + 4 D_{16} \frac{\partial^4 w}{\partial x^3 \partial y} + 2 (D_{12} + 2 D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + 4 D_{26} \frac{\partial^4 w}{\partial x \partial y^3} + D_{22} \frac{\partial^4 w}{\partial y^4} - B_{11} \frac{\partial^3 u}{\partial x^3} - 3 B_{16} \frac{\partial^3 u}{\partial x^2 \partial y} - (B_{12} + 2B_{66}) \frac{\partial^3 u}{\partial x \partial y^2} - B_{26} \frac{\partial^3 u}{\partial y^3} - B_{16} \frac{\partial^3 v}{\partial x^3} \]
\[-(B_{12} + 2B_{66}) \frac{\partial^3 v^o}{\partial x^3 \partial y} - 3B_{26} \frac{\partial^3 v^o}{\partial x^2 \partial y^2} - B_{22} \frac{\partial^3 v^o}{\partial y^3} - B_{11} \left( \frac{\partial^2 w}{\partial x^2} \right)^2 \]

\[-3B_{16} \left( \frac{\partial^2 w}{\partial x^2} \right) \left( \frac{\partial^2 w}{\partial y^2} \right) - 2B_{66} \left( \frac{\partial^2 w}{\partial x^2} \right) \left( \frac{\partial^2 w}{\partial y^2} \right) - 2 \left( B_{12} + B_{66} \right) \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \]

\[-3B_{26} \left( \frac{\partial^2 w}{\partial x \partial y} \right) \left( \frac{\partial^2 w}{\partial y^2} \right) - B_{22} \left( \frac{\partial^2 w}{\partial y^2} \right)^2 - \frac{\partial w}{\partial x} \left[ \frac{B_{11}}{\partial x^2} \right] \]

\[+ 3B_{16} \frac{\partial^3 w}{\partial x^3 \partial y} + (B_{12} + 2B_{66}) \frac{\partial^3 w}{\partial x^2 \partial y^2} + B_{26} \frac{\partial^3 w}{\partial x \partial y^2} \]

\[-\frac{\partial w}{\partial y} \left[ \frac{B_{16} \frac{\partial^3 w}{\partial x^3 \partial y} + (B_{12} + 2B_{66}) \frac{\partial^3 w}{\partial x^2 \partial y^2} + 3B_{26} \frac{\partial^3 w}{\partial x \partial y^2} \right] \]

\[+ B_{22} \frac{\partial^3 w}{\partial y^3} \right] + \rho \frac{\partial^2 w}{\partial t^2} + I \left( \frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\partial^4 w}{\partial y^2 \partial t^2} \right) = \left( A_{11} \left[ \frac{\partial u^o}{\partial x} + 2 \left( \frac{\partial w}{\partial x} \right)^2 \right] + A_{12} \left[ \frac{\partial v^o}{\partial y} + 2 \left( \frac{\partial w}{\partial y} \right)^2 \right] \right)

\[+ A_{16} \left[ \frac{\partial v^o}{\partial y} + \frac{\partial u^o}{\partial y} + \frac{\partial w}{\partial x} \left( \frac{\partial w}{\partial y} \right) \right] - D_{11} \frac{\partial^2 w}{\partial x^2} \]

\[-D_{12} \frac{\partial^3 w}{\partial y^2} - 2D_{16} \frac{\partial^2 w}{\partial x \partial y} \right) \frac{\partial^2 w}{\partial x^2} + 2 \left( A_{12} \left[ \frac{\partial u^o}{\partial x} + 1 \left( \frac{\partial w}{\partial x} \right)^2 \right] \]

\[+ A_{22} \left[ \frac{\partial v^o}{\partial y} + 1 \left( \frac{\partial w}{\partial y} \right)^2 \right] + A_{26} \left[ \frac{\partial u^o}{\partial y} + \frac{\partial v^o}{\partial x} + \frac{\partial w}{\partial x} \left( \frac{\partial w}{\partial y} \right) \right] \]

\[-D_{12} \frac{\partial^2 w}{\partial x^2} - D_{22} \frac{\partial^2 w}{\partial y^2} - 2D_{26} \frac{\partial^2 w}{\partial x \partial y} \right) \frac{\partial^2 w}{\partial x^2} + \left( A_{16} \left[ \frac{\partial u^o}{\partial x} \right] \]

\[+ \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] + A_{26} \left[ \frac{\partial v^o}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right] + A_{66} \left[ \frac{\partial u^o}{\partial y} + \frac{\partial v^o}{\partial y} \right] \]

\[+ \frac{\partial w}{\partial x} \left( \rho \frac{\partial^2 u^o}{\partial t^2} - q_x \right) + \frac{\partial w}{\partial y} \left( \rho \frac{\partial^2 v^o}{\partial t^2} - q_y \right) + q \]

\[+ \frac{\partial m}{\partial x} + \frac{\partial m}{\partial y} - \frac{\partial^2 M^T}{\partial x^2} - 2 \frac{\partial^2 M^T}{\partial x \partial y} - \frac{\partial^2 M^T}{\partial y^2} \]
Obviously almost insurmountable difficulty will arise in solving equations (3-50) - (3-52) for the general case. Fortunately, for most physical cases these equations will simplify because of certain symmetry conditions of real materials. This is discussed in more detail in the next chapter.

3.6. Equations of Equilibrium in Terms of a Stress Function and Transverse Displacement

For certain static problems a stress function formulation of the in-plane problem often proves useful. We now define a stress function \( \phi \) such that

\[

N_x = \frac{\partial^2 \phi}{\partial y^2} - P, \quad N_y = \frac{\partial^2 \phi}{\partial x^2} - P, \quad N_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}
\]  

(3-53)

where \( P \) is a potential function given by the relation

\[

q_x = \frac{\partial P}{\partial x}, \quad q_y = \frac{\partial P}{\partial y}
\]  

(3-54)

Equations (3-29) and (3-31) are now identically satisfied.

Equation (3-48) can be written in the abbreviated form

\[

\begin{bmatrix}
N \\
M
\end{bmatrix} = \begin{bmatrix}
A & B \\
B & D
\end{bmatrix} \begin{bmatrix}
\varepsilon^0
\end{bmatrix} - \begin{bmatrix}
N^T \\
M^T
\end{bmatrix}
\]

(3-55)

or using matrix equations

\[

N = A\varepsilon^0 + BK - N^T
\]  

(3-56)

\[

M = B\varepsilon^0 + DK - M^T
\]  

(3-57)
Multiplying equation (3-56) by $A^{-1}$ we have

$$
\varepsilon^o = A^{-1} N - A^{-1} BK + A^{-1} N^T
$$

(3-58)

Putting (3-58) into (3-57) yields

$$
M = BA^{-1} N + (D - BA^{-1} B) K + BA^{-1} N^T - M^T
$$

(3-59)

Now (3-58) and (3-59) can be written in the form

$$
\begin{bmatrix}
\varepsilon^o \\
M
\end{bmatrix} = \begin{bmatrix}
A^* & B^* \\
- (B^*)^T & D^*
\end{bmatrix} \begin{bmatrix}
N \\
K
\end{bmatrix}
+ \begin{bmatrix}
A^* & O \\
- (B^*)^T & - I
\end{bmatrix} \begin{bmatrix}
N^T \\
M^T
\end{bmatrix}
$$

(3-60)

where

$$
A^* = A^{-1} \\
B^* = -A^{-1} B \\
D^* = D - BA^{-1} B
$$

In the general case $A^*$ and $D^*$ are symmetric while $B^*$ is not.

Substituting (3-60) into (3-43) and using (3-53) yields

$$
\begin{align*}
D_{11}^* \frac{\partial^4 w}{\partial x^4} + 4D_{16}^* \frac{\partial^4 w}{\partial x \partial y^3} + 2 (D_{12}^* + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} & \\
+ 4D_{26}^* \frac{\partial^4 w}{\partial x \partial y} + D_{22}^* \frac{\partial^4 w}{\partial y^4} + B_{21}^* \frac{\partial^4 \phi}{\partial x^4} & \\
+ (2B_{26}^* - B_{61}^*) \frac{\partial^4 \phi}{\partial x \partial y} + (B_{11}^* + B_{22}^* - 2B_{66}^*) \frac{\partial^4 \phi}{\partial x^2 \partial y^2}
\end{align*}
$$
\[ + (2B_{16}^* - B_{22}^*) \frac{\partial^4 \phi}{\partial x \partial y^3} + B_{12}^* \frac{\partial^4 \phi}{\partial y^4} + B_{11}^* \frac{\partial^2 N_x^T}{\partial x^2} \]

\[ + B_{12}^* \frac{\partial^2 N_x^T}{\partial y^2} + 2B_{16}^* \frac{\partial^2 N_x^T}{\partial x \partial y} + B_{21}^* \frac{\partial^2 N_y^T}{\partial x^2} + 2B_{26}^* \frac{\partial^2 N_y^T}{\partial x \partial y} \]

\[ + B_{22}^* \frac{\partial^2 N_y^T}{\partial y^2} + B_{61}^* \frac{\partial^2 N_x^T}{\partial x \partial y} + B_{62}^* \frac{\partial^2 N_y^T}{\partial x \partial y} + 2B_{66}^* \frac{\partial^2 N_x^T}{\partial y^2} \]

\[ = q + \left( \frac{\partial^2 \phi}{\partial y^2} - P \right) \left( \frac{\partial^2 w}{\partial x^2} \right) - 2 \left( \frac{\partial^2 \phi}{\partial x \partial y} \right) \left( \frac{\partial^2 w}{\partial x \partial y} \right) \]

\[ + \left( \frac{\partial^2 \phi}{\partial x^2} - P \right) \left( \frac{\partial^2 w}{\partial y^2} \right) - B_{11}^* \frac{\partial^2 P}{\partial x^2} - B_{12}^* \frac{\partial^2 P}{\partial y^2} \]

\[ - 2B_{16}^* \frac{\partial^2 P}{\partial x \partial y} - B_{21}^* \frac{\partial^2 P}{\partial x^2} - B_{22}^* \frac{\partial^2 P}{\partial y^2} - 2B_{26}^* \frac{\partial^2 P}{\partial x \partial y} + \frac{\partial m_x}{\partial x} + \frac{\partial m_y}{\partial y} \]

\[ \frac{\partial^2 \varepsilon_x^O}{\partial y^2} + \frac{\partial^2 \varepsilon_y^O}{\partial x^2} - \frac{\partial^2 \varepsilon_x^O}{\partial x \partial y} = \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 - \left( \frac{\partial^2 w}{\partial x^2} \right) \left( \frac{\partial^2 w}{\partial y^2} \right) \]

Equation (3-61) involves two unknowns, thus a second relationship is necessary. A cursory examination of the compatibility equations (2-27) - (2-31) reveals that they are exactly satisfied by the strain equations (3-13) - (3-15). The plane stress compatibility equation (2-26) becomes

\[ \frac{\partial^2 \varepsilon_x^O}{\partial y^2} + \frac{\partial^2 \varepsilon_y^O}{\partial x^2} - \frac{\partial^2 \varepsilon_x^O}{\partial x \partial y} = \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 - \left( \frac{\partial^2 w}{\partial x^2} \right) \left( \frac{\partial^2 w}{\partial y^2} \right) \]

Substituting (3-60) into (3-62) and using (3-53) yields

\[ A_{22}^* \frac{\partial^4 \phi}{\partial x^4} + 2A_{26}^* \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + (2A_{12}^* + A_{66}^*) \frac{\partial^4 \phi}{\partial x^2 \partial y^2} \]

\[ - 2A_{16}^* \frac{\partial^4 \phi}{\partial x \partial y^3} + A_{11}^* \frac{\partial^4 \phi}{\partial y^4} - B_{21}^* \frac{\partial^4 w}{\partial x^4} + (B_{61}^* - 2B_{26}^*) \frac{\partial^4 w}{\partial x^2 \partial y^2} \]

\[ + (2B_{66}^* - B_{11}^* - B_{22}^*) \frac{\partial^4 w}{\partial x \partial y^3} + (B_{62}^* - 2B_{16}^*) \frac{\partial^4 w}{\partial x \partial y^3} - B_{12}^* \frac{\partial^4 w}{\partial y^4} \]

\[ + A_{11}^* \frac{\partial^2 N_x^T}{\partial y^2} + A_{12}^* \frac{\partial^2 N_y^T}{\partial y^2} + A_{16}^* \frac{\partial^2 N_x^T}{\partial x \partial y} + A_{12}^* \frac{\partial^2 N_x^T}{\partial x^2} \]
Again considerable difficulty arises in solving the general case.

For symmetric laminates \((B_{ij} = 0)\), equations (3-61) and (3-63) for the linear case in which there are no body forces or thermal stresses become

\[
D_{ij} \frac{\partial^4 w}{\partial x^4} + 4D_{16} \frac{\partial^4 w}{\partial x^2 \partial y^2} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^4} + 4D_{26} \frac{\partial^4 w}{\partial x \partial y^5} + D_{22} \frac{\partial^4 w}{\partial y^4} = q
\]

\[
\frac{\partial^4 \phi}{\partial x^4} - 2A_{26} \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + (2A_{12} + A_{66}) \frac{\partial^4 \phi}{\partial x^2 \partial y^4} - 2A_{16} \frac{\partial^4 \phi}{\partial x \partial y^5} + A_{11} \frac{\partial^4 \phi}{\partial y^4} = 0
\]

which coincides with the equations of an anisotropic homogeneous plate. It should be noted, however, that this does not imply that the plate can be considered homogeneous. If the plate were homogeneous the following relationship would hold.

\[
D_{ij} = \frac{h^2}{12} A_{ij}
\]

3.7. **Boundary Conditions**

The appropriate boundary conditions are those which are sufficient to guarantee unique solutions to the governing differential
equations. The development here parallels very closely that of Mindlin (21). However, his work was applied to the dynamic analysis of homogeneous plates which included shear deformations. The argument presented is based on properties of linear systems. As a result, the proper boundary conditions are obtained for the linear bending problem only.

The strain energy density $W$ of a laminated plate is given by

$$2W = \sigma \varepsilon + \sigma \varepsilon + \sigma \varepsilon$$

Using equations (3-13) - (3-15) yields

$$2W = \sigma \varepsilon^0 + \sigma \varepsilon^0 + \sigma \varepsilon^0 + \sigma zK + \sigma zK + \sigma zK$$

Integrating equation (3-68) over the plate thickness, and applying the definition of force and moment resultants yields the analogous strain energy function $U$ in terms of plate nomenclature.

$$2U = N \varepsilon^0 + N \varepsilon^0 + N \varepsilon^0 + M K + M K + M K$$

The total strain-energy $U$ can be written as

$$U = \int_a^b \int_0^1 \bar{U} \; dx \; dy$$

Using equations (3-16) - (3-19), equation (3-69) becomes

$$\bar{U} = \left( N_x \frac{\partial u^0}{\partial x} + N_y \frac{\partial u^0}{\partial y} \right) + \left( N_y \frac{\partial v^0}{\partial y} + N_x \frac{\partial v^0}{\partial x} \right)$$

$$- \left( M_x \frac{\partial^2 w}{\partial x^2} + M_y \frac{\partial^2 w}{\partial y^2} + 2M \frac{\partial^2 w}{\partial x \partial y} \right)$$

Integrating the surface integral in equation (3-70) by parts and
applying Green's theorem (25) to change a surface integral to a line integral yields

\[
\int_0^b \int_0^a U \, dx \, dy = \int_c \left[ (N_x \ell + N_{xy} m) u^0 + (N_{xy} \ell + N_y m) v^0 \right. \\
- (M_x \ell + M_{xy} m) \frac{\partial w}{\partial x} - (M_{xy} \ell + M_y m) \frac{\partial w}{\partial y} \bigg| ds \\
+ \int_0^b \int_0^a \left[ \frac{\partial w}{\partial x} \left( \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} \right) + \frac{\partial w}{\partial y} \left( \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} \right) \\
- \left( \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} \right) u^0 - \left( \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} \right) v^0 \right] \, dx \, dy \quad (3-72)
\]

where \( \ell \) and \( m \) are sine and cosine, respectively, of the angle between normal to edge of plate and \( x \)-axis.

The line integral in (3-72) can be further simplified by changing to a coordinate system normal and tangential to the boundary and denoted, respectively, by \( n \) and \( s \). The following relations are used to obtain the desired transformation.

\[
u^0 = \frac{\ell u^0 - m v^0}{n} \quad (3-73)
\]

\[
u^0 = \frac{mu^0 + \ell u^0}{n} \quad (3-74)
\]

\[
\frac{\partial w}{\partial x} = m \frac{\partial w}{\partial n} + \ell \frac{\partial w}{\partial s} \quad (3-75)
\]

\[
\frac{\partial w}{\partial y} = m \frac{\partial w}{\partial n} + \ell \frac{\partial w}{\partial s} \quad (3-76)
\]

\[
(N_n, M_n) = (N_x, M_x) \ell^2 + (N_y, M_y)m^2 + 2(N_{xy}, M_{xy})m \ell \quad (3-77)
\]

\[
(N_{sn}, M_{sn}) = [(N_y, M_y) - (N_x, M_x)] m \ell + (N_{xy}, M_{xy})(\ell^2 - m^2) \quad (3-78)
\]
Applying equations (3-73) - (3-78), the line integral in equation (3-72) becomes

\[
\int_C \left( N_n \frac{u^0}{n} + N_{ns} \frac{u^0}{s} - M_n \frac{\partial w}{\partial n} - M_{ns} \frac{\partial w}{\partial s} \right) ds \tag{3-79}
\]

Applying the equations of motion (3-29), (3-31), (3-36), and (3-39), the surface integral in equation (3-72) becomes

\[
\int_0^b \int_0^a \left[ \frac{\partial^2 w}{\partial x \partial t} \left( \frac{Q_x}{x-m_x} \right) + \frac{\partial^2 w}{\partial y \partial t} \left( \frac{Q_y}{y-m_y} \right) + q_x u^0 + q_y v^0 \right] dx dy \tag{3-80}
\]

Again using Green's theorem and writing the results in terms of \(n-s\) coordinates, equation (3-80) becomes

\[
\int_C w Q_n ds + \int_0^b \int_0^a \left[ q_x u^0 + q_y v^0 - w \left( \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} \right) - m_x \frac{\partial w}{\partial x} - m_y \frac{\partial w}{\partial y} \right] dx dy \tag{3-81}
\]

Combining the line integrals in (3-79) and (3-81) yields

\[
\int \left( N_n \frac{u^0}{n} + N_{ns} \frac{u^0}{s} - M_n \frac{\partial w}{\partial n} - M_{ns} \frac{\partial w}{\partial s} + w Q_n \right) ds \tag{3-82}
\]

Now we note from integration by parts that

\[
- \int_C M_{ns} \frac{\partial w}{\partial s} ds = \int_C \frac{\partial M_{ns}}{\partial s} w ds \tag{3-83}
\]

Using (3-83) in (3-82), and applying the equation of motion (3-32) to the surface integral in equation (3-81), we arrive at the following
form of equation (3-70)

\[
U = \int_c \left[ N_n u_n^0 + N_{ns} u_s^0 - M_n \frac{\partial w}{\partial x} + \left( \frac{\partial M_{ns}}{\partial x} + Q_n \right) w \right] \, ds \\
\int_0^a \int_0^b \left( q_x u^0 + q_y v^0 + \left( q + \frac{\partial m_x}{\partial x} + \frac{\partial m_y}{\partial y} \right) w - m_x \frac{\partial w}{\partial x} \right) \, dx \, dy 
\]

(3-84)

Let us assume that two solutions exist which satisfy the differential equations of equilibrium and prescribed boundary conditions.

The difference between these two solutions will also be a solution to the linear equations with all prescribed boundary conditions vanishing. The strain energy calculated from the difference between the two solutions will satisfy an equation of the same form as (3-84).

If the right side of equation (3-84) vanishes, then \( U \) will vanish. Furthermore, if \( U \) vanishes then by equation (3-69), assuming the strain energy density \( U \) to be positive, the plate strains and curvatures must vanish. This in turn implies that the stress and moment resultants must vanish in order that the constitutive relations (3-48) be satisfied. Thus, for the difference of the two assumed solutions the vanishing of the right hand side of equation (3-84) guarantees the solutions to be identical, within a rigid-body displacement. This difference can be eliminated by requiring the initial displacements to be the same.
Necessary conditions for uniqueness can be summarized as follows:

1. On the edge of the plate: any combination which contains one member of each of the four pairs of terms in the line integral of Equation (3-84).

2. Throughout the plate: $q_x$, $q_y$, $q$, $m_x$, and $m_y$.

3.8. Application to Fiber Reinforced Composite Materials

As mentioned previously, laminated plate theory has direct application to fiber reinforced composites. Using various micromechanical studies one can predict the elastic coefficients of a unidirectional composite as a function of fiber and resin elastic properties and volume content of filament. Once these properties are determined, the unidirectional layer is treated as homogeneous and orthotropic.

If we consider a multidirectional composite as being made up of numerous unidirectional layers with different fiber orientations then the theory of laminated anisotropic plates is directly applicable.

Consider a unidirectional composite as shown in Figure 7a. Hooke's law in terms of engineering constants is given by (20)

$$
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_{xy}
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{E_x} & \frac{-\nu_{xy}}{E_x} & 0 \\
\frac{-\nu_{xy}}{E_x} & \frac{1}{E_y} & 0 \\
0 & 0 & \frac{1}{G_{xy}}
\end{bmatrix}
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_{xy}
\end{bmatrix}
$$

(3-85)
Inverting (3-85) yields the following components of the stiffness matrix

\[
C_{11} = \frac{E_x^2}{(E_x - E_y) v_{xy}^2} \quad (3-86)
\]

\[
C_{12} = \frac{E_x E_y v_{xy}}{(E_x - E_y) v_{xy}^2} \quad (3-87)
\]

\[
C_{22} = \frac{E_y^2}{(E_x - E_y) v_{xy}^2} \quad (3-88)
\]

\[
C_{66} = G_{xy} \quad (3-89)
\]

It has been shown (23) that \(E_x\) and \(v_{xy}\) can be determined by a simple rule-of-mixtures relationship

\[
E_x = E_f \lambda + E_m (1-\lambda) \quad (3-90)
\]

\[
v_{xy} = v_f \lambda + v_m (1-\lambda) \quad (3-91)
\]

where \(E_f\), \(E_m\), \(v_f\), \(v_m\), and \(\lambda\) are fiber modulus, matrix modulus, Poisson's ratio of fiber and matrix, and fiber volume content, respectively. The transverse modulus \(E_y\) and the shear modulus \(G_{xy}\) cannot be determined from elementary considerations. Various elasticity solutions are available in the literature. However, most of these are not in closed form. A very useful graphical representation of \(E_y\) and \(G_{xy}\) as a function of constituent material properties can be found in Reference 23. The solutions are based on a square array of circular filaments in an infinite media. Limited experimental data shows the analytical results to be reasonably reliable.
Let us now consider the case of a laminate which contains plies having fibers oriented at some angle $\theta$ to the axis of the plate. This is shown in Figure 7b. Knowing the elastic properties with respect to the $x$-$y$ axis it is necessary to find them in terms of the $x'$-$y'$ axes in order to apply laminated plate theory. This can be done simply by using the transformation properties of the stiffness tensor.

The details are discussed in the Appendix. A complete set of transformation equations have been derived by Love (24).

\begin{align}
C_{11} &= C_{11} \cos^4 \theta + C_{22} \sin^4 \theta + 2(C_{12} + 2C_{66}) \sin^2 \theta \cos^2 \theta \quad (3-92) \\
C_{12} &= (C_{11} + C_{22} - 4C_{66}) \cos^2 \theta \sin^2 \theta + C_{12} (\cos^4 \theta + \sin^4 \theta) \quad (3-93) \\
C_{22} &= C_{11} \sin^4 \theta + C_{22} \cos^4 \theta + 2(C_{12} + 2C_{66}) \sin^2 \theta \cos^2 \theta \quad (3-94) \\
C_{16} &= -C_{11} \cos^3 \theta \sin \theta + C_{22} \cos \theta \sin^3 \theta + (C_{12} + 2C_{66}) \\
&\quad (\cos^3 \sin \theta - \cos \theta \sin^3 \theta) \quad (3-95) \\
C_{26} &= -C_{11} \cos \theta \sin^3 \theta + C_{22} \cos^3 \theta \sin \theta + (C_{12} + 2C_{66}) \\
&\quad (\cos \theta \sin^3 \theta - \cos^3 \theta \sin \theta) \quad (3-96) \\
C_{66} &= (C_{11} + C_{22} - 2C_{12}) \cos^2 \theta \sin^2 \theta + C_{66} (\cos^2 \theta \sin^2 \theta)^2 \quad (3-97)
\end{align}

In terms of the $x'$-$y'$ axes we have six elastic coefficients indicating complete anisotropy.

In subsequent sections the subscripts $L$ and $T$ are used to denote properties along the direction of the filaments and transverse to the filaments, respectively, in a unidirectional composite.
4. EFFECT OF COUPLING ON THE MECHANICAL
BEHAVIOR OF FIBROUS COMPOSITES

4.1. Introduction

As pointed out in the previous chapter, the most significant difference between a laminated plate and a homogeneous plate is the existence of coupling between bending and stretching. We will now proceed to determine what effect this coupling has on the behavior of laminated plates. In particular, we choose examples of fiber reinforced composites for which closed form solutions can be obtained. Results will then be compared to the same material under the exact same loading with coupling neglected. That is, results will be compared to an analogous homogeneous problem. Linear, small deflection theory is used.

Two basic types of composite materials are considered, cross-ply materials and angle-ply materials. Cross-ply composites consist of an even number of unidirectional plies alternately oriented at 0° and 90° to the plate axes as shown in Figure 8. Each ply is of the same thickness and made of the same material. For this cross-ply laminate the constitutive relations (neglecting thermal effects) are given by
Using (4-1), equations (3-50) - (3-52) for the linear case with no body forces or moments and rotatory inertia neglected become

\[
A_1 \frac{\partial^2 u^0}{\partial x^2} + A_2 \frac{\partial^2 u^0}{\partial y^2} + A_3 \frac{\partial^2 v^0}{\partial x \partial y} - B_1 \frac{\partial^3 w}{\partial x^3} + q_x = \rho \frac{\partial^2 u^0}{\partial t^2} \tag{4-2}
\]

\[
A_3 \frac{\partial^2 u^0}{\partial x \partial y} + A_2 \frac{\partial^2 v^0}{\partial x^2} + A_1 \frac{\partial^2 v^0}{\partial y^2} + B_1 \frac{\partial^3 w}{\partial y^3} + q_y = \rho \frac{\partial^2 v^0}{\partial t^2} \tag{4-3}
\]

\[
D_1 \frac{\partial^4 w}{\partial x^4} + D_2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_3 \frac{\partial^4 w}{\partial y^4} - B_1 \frac{\partial^3 u^0}{\partial x^3} + B_1 \frac{\partial^3 v^0}{\partial y^3}
+ \rho \frac{\partial^2 w}{\partial t^2} = q \tag{4-4}
\]

where

\[
A_1 = A_{11}, \ A_2 = A_{66}, \ A_3 = A_{12} + A_{66}, \ B_1 = B_{11}, \ D_1 = D_{11},
\]

\[
D_2 = 2 \left( D_{12} + 2D_{66} \right), \ D_3 = D_{66}
\]

Angle-ply composites consist of an even number of unidirectional plies alternately oriented at angles of $+\theta$ and $-\theta$ to the plate axes similar to the two-ply laminate shown in Figure 1. Again each ply is of the same material. The constitutive relations (neglecting thermal effects) for the angle-ply composites are given by
Using (4-5), equations (3-50) - (3-52) for the linear case with thermal effects and rotatory inertia neglected become

\[ \begin{align*}
A_1 \frac{\partial^2 u^0}{\partial x^2} + A_2 \frac{\partial^2 u^0}{\partial y^2} + A_3 \frac{\partial^2 w^0}{\partial x \partial y} &- 3B_2 \frac{\partial^3 w}{\partial x^2 \partial y} - B_3 \frac{\partial^3 w}{\partial y^3} + q_x = \rho \frac{\partial^2 u^0}{\partial t^2} \\
A_3 \frac{\partial^2 u^0}{\partial x \partial y} + A_2 \frac{\partial^2 v^0}{\partial x^2} + A_4 \frac{\partial^2 v^0}{\partial y^2} - B_2 \frac{\partial^3 w}{\partial x^3} - 3B_3 \frac{\partial^3 w}{\partial x \partial y^2} &+ q_y = \rho \frac{\partial^2 v^0}{\partial t^2} \\
D_1 \frac{\partial^4 w}{\partial x^4} + D_2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_4 \frac{\partial^4 w}{\partial y^4} - 3B_2 \frac{\partial^3 u^0}{\partial x^3} - B_3 \frac{\partial^3 u^0}{\partial x \partial y^2} &- B_2 \frac{\partial^3 v^0}{\partial x^3} - 3B_3 \frac{\partial^3 v^0}{\partial x \partial y^2} + \rho \frac{\partial^2 w}{\partial t^2} = q
\end{align*} \]

(4-6) (4-7) (4-8)

where

\[ \begin{align*}
A_4 &= A_{22} , \quad B_2 = B_{16} , \quad B_3 = B_{26} , \quad D_4 = D_{22}
\end{align*} \]

3.2 Transverse loading of simply supported Plate

Consider a simply-supported cross-ply plate under the transverse load \( q = q_o \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \). Also, we prescribe \( q_x = q_y = 0 \).
The simple supports are of the type which allows normal displacements on the boundary, but prevents lateral (tangential) displacements.

Thus, we have the following boundary conditions

at \( x = o \) and \( x = a \):

\[
v^o = w = N_x = M_x = 0 \quad (4-9)
\]

at \( y = o \) and \( y = b \):

\[
u^o = w = N_y = M_y = 0 \quad (4-10)
\]

Assume:

\[
u^o = A \cos \frac{\pi x}{a} \sin \frac{\pi y}{b} \quad (4-11)
\]

\[
v^o = B \sin \frac{\pi x}{a} \cos \frac{\pi y}{b} \quad (4-12)
\]

\[
w = \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \quad (4-13)
\]

Equations (4-11) - (4-13) satisfy all of the boundary conditions in

(4-9) and (4-10). Substituting (4-11) - (4-13) into equations (4-2) -

(4-4) yields the following set of equations

\[
(A_1 + A_2 R^2) A + A_3 RB - \frac{B_1 \pi}{b} C = 0 \quad (4-14)
\]

\[
A_3 RA + (A_2 + A_1 R^2) B + B_1 \pi R^2 \frac{C}{b} = 0 \quad (4-15)
\]

\[
-\frac{B_1 \pi}{Rb} A + B_1 \pi R^2 \frac{B + \pi^2 R^2}{R^2 b^2} \left[ D_1 (1+R^4) + D_2 R^2 \right] C = \nonumber
\]

\[
\frac{q_o R^2 b^2}{\pi^2} \quad (4-16)
\]
where \( R = a/b \)

Solving equations (4-14) - (4-16) for \( A, B, \) and \( C \) yields the following results

\[
A = q_o R^3 b^2 B_1 \frac{(A_1 R^2 + A_2 + A_3 R^4)}{\pi^4 D} \tag{4-17}
\]

\[
B = - q_o R^3 b^3 B_1 \frac{(A_1 R^2 + A_2 R^4 + A_3)}{\pi^4 D} \tag{4-18}
\]

\[
C = q_o R^4 b^4 \frac{\left[ (A_1 + A_2 R^2)(A_2 + A_1 R^2) - A_3^2 R^2 \right]}{\pi^4 D} \tag{4-19}
\]

where

\[
D = \left[ (A_1 + A_2 R^2)(A_2 + A_1 R^2) - A_3^2 R^2 \right] \left[ D_1 (1 + R^4) + D_2 R^2 \right]
- B_1^2 \left[ A_1 (1 + R^4) + A_2 (1 + R^8) + 2A_3 R^4 \right] \tag{4-20}
\]

When \( B_1 = 0 \) it is easily seen that \( A = B = 0, \) and

\[
C = \frac{q_o R^4 b^4}{\pi^4 \left[ D_1 (1 + R^4) + D_2 R^2 \right]} \tag{4-21}
\]

which is the solution for a simply supported orthotropic plate.

Stress resultants and moments can be calculated from the constitutive relations (4-1) and are given by

\[
N_x = - \frac{\pi}{b} \left[ \frac{A_1}{R} \frac{A + (A_3 - A_2)}{a} + B - B_1 \frac{\pi}{R^2 b} \frac{C}{b} \right] \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \tag{4-22}
\]

\[
N_y = - \frac{\pi}{b} \left[ \frac{(A_3 - A_2)}{R} \frac{A + A_1 B + B_1 R^2}{a} \frac{C}{b} \right] \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \tag{4-23}
\]

\[
N_{xy} = \frac{A_2 \pi}{b} \left( \frac{A + \bar{B}}{R} \right) \cos \frac{\pi x}{a} \cos \frac{\pi y}{b} \tag{4-24}
\]
Results are shown in Table 1 for Graphite-Epoxy ($E_L = 30 \times 10^6$ psi, $E_T = 0.75 \times 10^6$ psi, $G_{LT} = 0.7 \times 10^6$ psi, and $v_{LT} = 0.25$) and Glass-epoxy ($E_L = 7.5 \times 10^6$ psi, $E_T = 2.6 \times 10^6$ psi, $G_{LT} = 1.25 \times 10^6$ psi, and $v_{LT} = 0.25$). These numbers are based on experimental data. The plate has an aspect ratio $R$ of 2 and a span-to-depth ratio $b/h$ of 50. As the number of layers are increased the effect of coupling is decreased quite rapidly. The glass-epoxy composite approaches a homogeneous solution (no. of layers $\rightarrow \infty$) more rapidly than a graphite-epoxy composite. This is due to the high degree of anisotropy (i.e. the ratio $E_L/E_T$ is high) of graphite-epoxy composites. It is also interesting to note that the in-plane forces and displacements introduced because of coupling are relatively small compared to externally applied in-plane forces and transverse displacements, respectively.

Now consider a simply-supported angle-ply plate under the same transverse loading as the cross-ply plate just discussed. Again $q_x = q_y = 0$. The simple supports are smooth pins. That is the normal displacements and in-plane shear resultants are zero. This

\[
M_x = -\frac{\pi}{b} \left\{ B_1 B - \frac{\pi}{b} C \left[ D_1 + \frac{(D_2 - 4D_3)}{2R} \right] \right\} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \quad (4-25)
\]

\[
M_y = \frac{\pi}{b} \left\{ B_1 B + \frac{\pi}{b} C \left[ D_1 + \frac{(D_2 - 4D_3)}{2R} \right] \right\} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \quad (4-26)
\]

\[
M_{xy} = -\frac{2\pi^2}{Rb^2} C \cos \frac{\pi x}{a} \cos \frac{\pi y}{b} \quad (4-27)
\]
results in the following set of boundary conditions

at \( x = 0 \) and \( x = a \):

\[
\begin{align*}
\varphi^0 &= w = N_{xy} = M_x = 0 \\
(4-28)
\end{align*}
\]

at \( y = 0 \) and \( y = b \):

\[
\begin{align*}
\varphi^0 &= w = N_{xy} = M_y = 0 \\
(4-29)
\end{align*}
\]

Assume:

\[
\begin{align*}
\varphi^0 &= E \sin \frac{\pi x}{a} \cos \frac{\pi y}{b} \quad (4-30) \\
\psi^0 &= F \cos \frac{\pi x}{a} \sin \frac{\pi y}{b} \quad (4-31) \\
\zeta^0 &= G \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \quad (4-32)
\end{align*}
\]

Equations (4-30) - (4-32) assure that the boundary conditions (4-28) and (4-29) are satisfied. Substituting (4-30) - (4-32) into equations (4-6) - (4-8) yields the following set of equations

\[
\begin{align*}
(A_1 + A_2 R^2) E + A_3 R F - \frac{\pi}{b} (3B_2 + B_3 R^2) G &= 0 \\
(4-33)
\end{align*}
\]

\[
\begin{align*}
A_3 R E + (A_2 + A_4 R^2) F - \frac{\pi}{Rb} (B_2 + 3B_3 R^2) G &= 0 \\
(4-34)
\end{align*}
\]

\[
\begin{align*}
- \frac{\pi}{b} (3B_2 + B_3 R^2) E - \frac{\pi}{Rb} (B_2 + 3B_3 R^2) F + \\
\frac{\pi}{R^2 b^2} (D_1 + D_2 R^2 + D_4 R^4) G &= \frac{\varphi_0 R^2 b^2}{\pi^2} \\
(4-35)
\end{align*}
\]

Solving equations (4-33) - (4-35) for \( E \), \( F \), and \( G \) yields the following results

\[
E = \frac{\varphi_0 R^2 b^3}{\pi^3 H} \left[ (3B_2 + B_3 R^2)(A_1 + A_2 R^2) - A_3 R^2 (B_2 + 3B_3 R^2) \right] \quad (4-36)
\]
\[ F = \frac{q_0 R^3 b^3}{\pi^3 H} \left[ (B_2 + 3B_3 R^2) (A_1 + A_2 R^2) - A_3 R^2 (3B_2 + B_3 R^2) \right] \quad (4-37) \]
\[ G = \frac{q_0 R^4 b^4}{\pi^4 H} \left[ (A_1 + A_2 R^2) (A_2 + A_4 R^2) - A_3^2 R^2 \right] \quad (4-38) \]

where
\[ H = \left[ (A_1 + A_2 R^2) (A_2 + A_4 R^2) - A_3^2 R^2 \right] (D_1 + D_2 R^2 + D_4 R^4) \]

\[ + 2A_3 R^2 (B_2 + 3B_3 R^2) (3B_2 + BR^2) - (A_1 + A_2 R^2)^2 \]

\[ - (A_2 + A_4 R^2) (3B_2 + B_3 R^2)^2 \quad (4-39) \]

When \( B_2 = B_3 = 0 \), we have \( E = F = 0 \), and
\[ G = \frac{q_0 R^4 b^4}{\pi^4 (D_1 + D_2 R^2 + D_4 R^4)} \quad (4-40) \]

which again is the solution for a simply supported orthotropic plate.

Using the constitutive relations (4-5), the following resultants for stresses and moments are obtained

\[ N_x = \frac{\pi}{b} \left[ \frac{A_1}{R} E + (A_3 - A_2) F - 2 \frac{B_2}{R} \frac{\pi}{b} G \cos \frac{\pi x}{a} \cos \frac{\pi y}{b} \right] \quad (4-41) \]
\[ N_y = \frac{\pi}{b} \left[ \frac{(A_3 - A_2)}{R} E + A_4 F - 2 \frac{B_3}{R} \frac{\pi}{b} G \cos \frac{\pi x}{a} \cos \frac{\pi y}{b} \right] \quad (4-42) \]
\[ N_{xy} = -\frac{\pi}{b} \left[ A_2 A + \frac{A_2}{R} B - \frac{\pi}{b} \left( \frac{B_2}{R} + B_3 \right) C \right] \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \quad (4-43) \]
\[ M_x = -\frac{\pi}{b} \left\{ B_2 \left( E + \frac{F}{R} \right) - \frac{\pi}{b} G \left[ \frac{D_1}{R} + \frac{(D_2 - 4D_3)}{2} \right] \right\} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \quad (4-44) \]
\[ M_y = -\frac{\pi}{b} \left\{ B_3 \left( E + \frac{F}{R} \right) - \frac{\pi}{b} G \left[ D_4 + \frac{(D_2 - 4D_3)}{2R} \right] \right\} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \quad (4-45) \]
\[ M_{xy} = \frac{\pi}{b} \left( B_2 \frac{E}{R} + B_3 F - 2 \frac{\pi}{Rb} G \right) \cos \frac{\pi x}{a} \cos \frac{\pi y}{b} \quad (4-46) \]

Results are shown in Figures 9 and 10 for the maximum
deflection of a square plate as a function of the angle of orientation \( \theta \) of the angle-ply composite. Again graphite-epoxy and glass-epoxy composites are used. For the highly anisotropic 2 layer graphite-epoxy system we see considerable difference between the coupled solution and the orthotropic solution (no. of layers \( \rightarrow \infty \)). As the layers are increased this difference disappears quite rapidly as displayed by the 6 layer composite. For the glass-epoxy system, the two layer composite does not differ to as great a degree from the orthotropic solution. As previously pointed out, glass-epoxy does not have as high a degree of anisotropy as graphite-epoxy.

For any other transverse loadings, solutions can be obtained by a Fourier series. The load and solutions for cross-ply plates are

\[
q(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}
\]

\[
u^0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b}
\]

\[
u^0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b}
\]

\[
w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}
\]

which can be obtained. Similarly for the angle-ply system solutions of the form

\[
u^0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} E_{mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b}
\]
\[
\psi = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} F_{mn} \frac{\cos \frac{m \pi x}{a} \sin \frac{n \pi y}{b}}{b}
\]

(4-52)

\[
w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} G_{mn} \frac{\sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b}}{b}
\]

(4-53)

can be obtained.

4.3 Buckling of Simply-Supported Angle-Ply Composites Under Uniform Compression

For a linear buckling analysis of a homogeneous plate, in-plane force effects are retained. In particular, terms in the equilibrium equation (3-32) involving products of the stress resultants and plate curvatures are retained, while all nonlinear strain-displacement terms are neglected. With the in-plane problem and bending problem uncoupled, the force resultants in the pre-buckled mode will coincide with those in the buckled mode. However, because of the coupling between bending and stretching, the pre-buckling stress resultants in a laminated plate may not be the same as those in the buckled configuration. Thus, as in the case of shell theory, the instability of a layered plate is calculated in terms of the initial load.

Using equation (3-52), the third equilibrium equation (4-8) for an angle-ply plate becomes

\[
D_1 \frac{\partial^4 w}{\partial x^4} + D_2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_4 \frac{\partial^4 w}{\partial y^4} - 3B_2 \frac{\partial^2 v^0}{\partial x \partial y} - 3B_3 \frac{\partial^2 u^0}{\partial y^2} - B_2 \frac{\partial^3 v^0}{\partial x^3} - 3B_3 \frac{\partial^3 v^0}{\partial x \partial y^2} = q + N^i \frac{\partial^2 w}{\partial x \partial y} + 2N^i \frac{\partial^2 w}{xy \partial x \partial y} + N^i \frac{\partial^2 w}{y \partial y^2}
\]

(4-54)
Now consider an angle-ply plate with pinned edges as discussed in the previous section. However, let us assume that the pin supports are such that they allow uniform displacements along two adjacent boundaries. The following boundary conditions are appropriate at \( x = 0 \):
\[
u^0 = w = N_{xy} = M_x = 0 \tag{4-55}\]
at \( x = a \):
\[
u^0 = c_1 = \text{constant}, \ w = N_{xy} = M_x = 0 \tag{4-56}\]
at \( y = 0 \):
\[
u^0 = w = N_{xy} = M_y = 0 \tag{4-57}\]
at \( y = b \):
\[
u^0 = c_2 = \text{constant}, \ w = N_{xy} = M_y = 0 \tag{4-58}\]

Assume:
\[
u^0 = c_1 x, \ \nu^0 = c_2 y, \ w = 0 \tag{4-59}\]

A cursory examination of equations (4-6) - (4-8) reveals that they are exactly satisfied by the assumed displacements (4-59). The constants \( c_1 \) and \( c_2 \) are now chosen such that
\[
N_x^1 = -N, \ N_y^1 = -KN \tag{4-60}\]

where \( K \) is a constant. Substituting (4-59) into the constitutive relations (4-5), we find
\[
c_1 = -\frac{(A_{22} - A_{12}K)N}{(A_{11}A_{22} - A_{12}^2)}, \ c_2 = -\frac{(KA_{11} - A_{12})N}{(A_{11}A_{22} - A_{12}^2)} \tag{4-61}\]
To obtain a critical value of \( N \) we assume the following displacements which assure that the boundary conditions (4-54) are satisfied.

\[
u^0 = - \frac{(A_{22} - A_{12} K)N x}{(A_{11} A_{22} - A_{12}^2)} + A_{\text{mn}} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \quad (4-62)
\]

\[
v^0 = - \frac{(K A_{11} - A_{12}) N y}{(A_{11} A_{22} - A_{12}^2)} + B_{\text{mn}} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (4-63)
\]

\[
w = C_{\text{mn}} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (4-64)
\]

Substituting (4-62) - (4-64) into the equilibrium equations (4-6), (4-7), and (4-54) with \( q = 0 \), yields the following set of equations

\[
(A_1 m^2 + A_2 n^2 R^2) A_{\text{mn}} + A_3 \text{mn} R B_{\text{mn}} = \frac{n\pi}{b} (3B_2 m^2 + B_3 n^2 R^2) C_{\text{mn}} \quad (4-65)
\]

\[
A_3 \text{mn} R A_{\text{mn}} + (A_2 m^2 + A_4 n^2 R^2) B_{\text{mn}} = \frac{m\pi}{Rb} (B_2 m^2 + 3B_3 n^2 R^2) C_{\text{mn}} \quad (4-66)
\]

\[
\frac{R^2 b n}{\pi} (3B_2 m^2 + B_3 n^2 R^2) A_{\text{mn}} - \frac{mrb}{\pi} (B_2 m^2 + 3B_3 n^2 R^2) B_{\text{mn}} +
(D_1 m^4 + D_2 m^2 n^2 R^2 + D_4 n^4 R^4) C_{\text{mn}} = \frac{Nb^2 R^2}{\pi^2} \left( m^2 + K n^2 R^2 \right) C_{\text{mn}} \quad (4-67)
\]

Solving (4-64) and (4-65) for \( A_{\text{mn}} \) and \( B_{\text{mn}} \) in terms of \( C_{\text{mn}} \) yields

\[
A_{\text{mn}} = \frac{\pi n E_{\text{mn}} C_{\text{mn}}}{b D_{\text{mn}}} \, , \, B_{\text{mn}} = \frac{\pi m F_{\text{mn}} C_{\text{mn}}}{R b D_{\text{mn}}} \quad (4-68)
\]

where

\[
D_{\text{mn}} = (A_1 m^2 + A_2 n^2 R^2)(A_2 m^2 + A_4 n^2 R^2) - A_3^2 m^2 n^2 R^2 \quad (4-69)
\]

\[
E_{\text{mn}} = (A_2 m^2 + A_4 n^2 R^2)(3B_2 m^2 + B_3 n^2 R^2) - A_3 n^2 R^2 \left( B_2 m^2 + 3B_3 n^2 R^2 \right) \quad (4-70)
\]

\[
F_{\text{mn}} = (A_1 m^2 + A_2 n^2 R^2)(B_2 m^2 + 3B_3 n^2 R^2) - A_3 n^2 R^2 \left( 3B_2 m^2 + B_3 n^2 R^2 \right) \quad (4-71)
\]

Substituting (4-67) into (4-66) yields the following relationship for the
critical buckling load $N_{cr}$

$$N_{cr} = \frac{\pi^2}{R^2 (m^2 + K n^2 R^2)^2} \left[ \left( D_1 m^4 + D_3 m^2 n^2 R^2 + D_4 n^4 R^4 \right) - \frac{E_{mn}}{D_{mn}} \left( 3B_1 m^2 + B_2 n^2 R^2 \right) - \frac{F_{mn}}{D_{mn}} \left( 3B_2 n^2 R^2 + B_1 m^2 \right) \right]$$

(4-72)

For symmetric composites $B_2 = B_3 = 0$, and

$$N_{cr} = \frac{\pi^2}{b^2 R^2 (m^2 + K n^2 R^2)^2} \left( D_1 m^4 + D_3 m^2 n^2 R^2 + D_4 n^4 R^4 \right)$$

(4-73)

which is the buckling relationship for a simply supported orthotropic plate under uniform biaxial compression.

Results are shown in Figures 11 thru 13 for a square plate.

Figures 11 and 12 illustrate the effect of coupling for graphite-epoxy and glass-epoxy composites with $K = 0$ (uniaxial compression).

These plots are analogous to those obtained in Figures 10 and 11 for transverse loading. Coupling tends to reduce the buckling load. In some cases the critical load is reduced by a factor of three. As the number of layers are increased, the buckling load approaches an orthotropic solution. Again we see that coupling has much less effect on the glass-epoxy system because of its relatively low degree of anisotropy.

Figure 13 shows critical loads for graphite-epoxy composites with $K = 1$ (biaxial compression). Similar trends as obtained for uniaxial compression are shown.

It should be noted that the lowest critical load occurred at $m = n = 1$ for the glass-epoxy system and is reflected by the symmetric
curves in Figure 12. The unsymmetric curves for graphite-epoxy as displayed in Figure 11 are due to the lowest buckling load occurring, for some orientation angles, at values of m and n above one.

4.4. Dynamics of Simply Supported Plates

The effect of coupling on the fundamental frequency of vibration for simply supported cross-ply and angle-ply plates can easily be ascertained. For the cross-ply plate discussed in section 4.1 the following displacements will be sufficient

\[ u^0 = (A \cos \frac{m \pi x}{a} \sin \frac{n \pi y}{b}) e^{i \omega t} \]  \hfill (4-74)

\[ v^0 = (B \sin \frac{m \pi x}{a} \cos \frac{n \pi y}{b}) e^{i \omega t} \]  \hfill (4-75)

\[ w = (C \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b}) e^{i \omega t} \]  \hfill (4-76)

It is obvious that the boundary conditions (4-9) and (4-10) will be satisfied by the above displacement field.

Putting equations (4-74) - (4-76) into the equations of motion (4-2) - (4-4) with \( q = 0 \) and neglecting in-plane inertia terms yields the following set of equations in matrix notation

\[
\begin{bmatrix}
A_{mn} & B_{mn} & C_{mn} \\
B_{mn} & D_{mn} & E_{mn} \\
C_{mn} & E_{mn} & (F_{mn} - \lambda)
\end{bmatrix}
\begin{bmatrix}
A \\
B \\
C
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]  \hfill (4-76)

where

\[ A_{mn} = A_1 m^2 + A_2 n^2 R^2 \]
\[ B_{mn} = A_3 mnR \]
\[ C_{mn} = -\frac{B_1 m^3 \pi}{Rb} \]
\[ D_{mn} = A_2 m^2 + A_1 n^2 R^2 \]
\[ E_{mn} = \frac{B_1 n^3 R^2 \pi}{b} \]
\[ F_{mn} = \frac{R^2}{b^2} \left[ D_1 (m^4 + n^4 R^4) + D_2 m^2 n^2 R^2 \right] \]
\[ \lambda = \frac{\rho \omega^2 R^2 b^2}{\pi^2} \]

For a non-trivial solution of (4-77) we determine \( \omega \) such that the coefficient matrix vanishes.

Similarly for the angle-ply composite in 4.1 we use the following displacements

\[ u^0 = (D \sin \frac{m \pi x}{a} \cos \frac{n \pi y}{b}) e^{i \omega t} \] (4-78)
\[ v^0 = (E \cos \frac{m \pi x}{a} \sin \frac{n \pi y}{b}) e^{i \omega t} \] (4-79)
\[ w = (F \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b}) e^{i \omega t} \] (4-80)

Substituting these displacements into the equations of motion (4-6) - (4-8), with \( q = 0 \), and again neglecting in-plane inertia yields the following equations

\[
\begin{bmatrix}
A_{mn} & B_{mn} & G_{mn} \\
B_{mn} & D_{mn} & H_{mn} \\
G_{mn} & H_{mn} & (J_{mn} - \lambda)
\end{bmatrix}
\begin{bmatrix}
D \\
E \\
F
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\] (4-81)
where

\[ G_{mn} = \frac{m\pi}{b} (3B_2 m^2 + B_3 n^2 R^2) \]

\[ H_{mn} = \frac{m\pi}{Rb} (B_2 m^2 + 3B_3 n^2 R^2) \]

\[ J_{mn} = \frac{\pi^2}{R^2 b^2} (D_1 m^4 + D_2 m^2 n^2 R^2 + D_4 n^4 R^4) \]

Again, \( w \) is determined such that the determinant of the coefficient matrix is zero.

For angle ply composites in which \( B_2 = B_3 = 0 \), the fundamental frequency is given by

\[ \omega^2 = \frac{\pi^4}{\rho R^4 b^4} (D_1 m^4 + D_2 m^2 n^2 R^2 + D_4 n^4 R^4) \]  \hspace{1cm} (4-82)

which is the frequency equation for the transverse vibration of a homogeneous orthotropic plate. Similar results are obtained for cross-ply composites in which \( B_1 = 0 \). Equation (4-82) with \( D_4 = D_1 \) is the resulting frequency relationship. It is obvious from equation (4-82) that the fundamental frequency for a homogeneous orthotropic plate always occurs for \( m = n = 1 \). However, for the coupled laminates this is not obvious. For the results presented in this investigation all fundamental frequencies do occur for \( m = n = 1 \).

Figures 14 and 15 show the fundamental frequency as a function of the aspect ratio \( R \) for cross-ply composites. The effect of coupling is to lower the vibration frequency. This is consistent with the larger static deflections and smaller buckling loads obtained previously. For the graphite-epoxy system, the effect is again much greater.
than in the glass-epoxy. In Figure 16 and 17 the fundamental frequencies are plotted as a function of ply angle $\theta$ for angle-ply composites. Results are analogous to those obtained for the cross-ply composites.

It should be noted that in shell problems and sandwich structures where there is coupling between transverse deflections and in-plane displacements, the in-plane inertia terms are very often neglected when determining fundamental frequencies for transverse vibrations. Thus, the neglect of in-plane inertia terms for transverse vibrations of laminated plates is not without precedent in similar problems of continuous media.
5. APPLICATION OF DOUBLE FOURIER SERIES TO THE SOLUTION OF COUPLED LAMINATED PLATES

5.1. Fourier Series Solution of Boundary Value Problems

In the previous chapter some very useful closed form solutions were obtained for coupled laminated plates. However, these solutions were generated for a limited number of boundary conditions. We now outline a procedure, which utilizes double Fourier series, to obtain solutions for a wider variety of boundary conditions. The method has been previously applied to rectangular homogeneous plate problems by Green (25), Fletcher and Thorne (26), and Claassen and Thorne (27). A related method has recently been used by Fromme (28) to analyze vibrations of a rectangular parallelepiped. The presentation here closely parallels the work of Green. Certain difficulties in applying this method have previously been overlooked and are discussed in detail.

When using double Fourier series to represent the solution of a boundary value problem, it is inherently assumed that the series is term by term differentiable. It has been shown by Green (25) that the validity of such an assumption depends on the boundary conditions satisfied by the function.

Let \( f(x, y) \) be a function which can be expanded in a double sine series over the region \( 0 < x < a, \ 0 < y < b \). If the partial derivative \( \frac{\partial f}{\partial x} \) can be expanded in a cosine-sine series then the coefficients can be related to the original series by a partial integration. Similar
statements apply when \( f \) is expanded in a sine-cosine series, a cosine-cosine series or a cosine-sine and correspondingly \( \partial f / \partial x \) is expanded in a cosine-cosine series, a sine-cosine series, or a sine-sine series.

The desired relationships are obtained in the following manner

\[
f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}
\]

\((0 < x < a, 0 < y < b)\)

\[
\frac{\partial f}{\partial x} = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} B_{mn} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b}
\]

\((0 < x < a, 0 < y < b)\)

Multiplying equation (5-2) by \( \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \), integrating the results with respect to \( x \) and \( y \), and using the orthogonality properties of the cosine and sine, we obtain the following results

\[
B_{mn} = \frac{2}{ab} \int_{a}^{b} \int_{0}^{b} \left[ f(a, y) - f(0, y) \right] \sin \frac{n\pi y}{b} \, dy
\]

\((5-3)\)

\[
B_{mn} = \frac{4}{ab} \int_{a}^{b} \int_{0}^{b} \left[ \int_{0}^{a} \frac{\partial f}{\partial x} \cos \frac{m\pi x}{a} \, dx \right] \sin \frac{n\pi y}{b} \, dy
\]

\((5-4)\)

Using integration by parts, we obtain the following relationship

\[
\int_{0}^{a} \frac{\partial f}{\partial x} \cos \frac{m\pi x}{a} \, dx = \left[ f(a, y) - f(0, y) \right] + \frac{m\pi}{a} \int_{0}^{a} f \sin \frac{m\pi x}{a} \, dx
\]

\((m \text{ even, } m \neq 0)\) \hspace{1cm} (5-5)

\[
\int_{0}^{a} \frac{\partial f}{\partial x} \cos \frac{m\pi x}{a} \, dx = -\left[ f(a, y) + f(0, y) \right] + \int_{0}^{a} f \sin \frac{m\pi x}{a} \, dx
\]

\((m \text{ odd})\) \hspace{1cm} (5-6)

Let us assume that the following functions can be expanded in a Fourier series
\[ 2 \left[ \frac{f(a, y) - f(o, y)}{a} \right] = \sum_{n=1}^{\infty} a_n \sin n\pi y \quad (5-7) \]

\[ (o < y < b) \]

\[ -2 \left[ \frac{f(a, y) + f(o, y)}{a} \right] = \sum_{n=1}^{\infty} b_n \sin n\pi y \quad (5-8) \]

\[ (o < y < b) \]

Then from the theory of Fourier series we have

\[ a_n = \frac{4}{ab} \int_{a}^{b} \left[ f(a, y) - f(o, y) \right] \sin \frac{n\pi y}{b} \, dy \quad (5-9) \]

\[ b_n = -\frac{4}{ab} \int_{o}^{b} \left[ f(a, y) + f(o, y) \right] \sin \frac{n\pi y}{b} \, dy \quad (5-10) \]

Similarly,

\[ A_{mn} = \frac{4}{ab} \int_{o}^{a} \int_{o}^{b} f \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx \, dy \quad (5-11) \]

Putting (5-5) and (5-6) into (5-4) and taking into account (5-9) - (5-11) yields

\[ B_{on} = \frac{a_n}{2} \quad (5-12) \]

\[ B_{mn} = \frac{m\pi}{a} A_{mn} + a_n \quad (m \text{ even, } m \neq 0) \]

\[ = \frac{m\pi}{a} A_{mn} + b_n \quad (m \text{ odd}) \quad (5-13) \]

Thus, term by term differentiability depends on the boundary conditions satisfied by \( f(x, y) \). In particular, if \( f(x, y) \) vanishes on the boundary then the series (5-1) is term by term differentiable.

Similarly, if
\[ f(x, y) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \]  

\[ (0 < x < a, 0 \leq y \leq b) \]

and

\[ \frac{\partial f}{\partial x} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} B_{mn} \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \]  

\[ (0 \leq x \leq a, 0 \leq y \leq b) \]

then

\[ B_{00} = \frac{1}{4} c_0 \]  

\[ B_{m0} = \frac{m\pi}{a} A_{m0} + \frac{1}{2} c_0 \quad (m \text{ even, } m \neq 0) \]

\[ = \frac{m\pi}{a} A_{m0} + \frac{1}{2} d_0 \quad (m \text{ odd}) \]  

\[ B_{on} = \frac{1}{2} c_n \quad (n \neq 0) \]  

\[ B_{mn} = \frac{m\pi}{a} A_{mn} + c_n \quad (m \text{ even, } m \neq 0) \]

\[ B_{mn} = \frac{m\pi}{a} A_{mn} + d_n \quad (m \text{ odd}) \]

where

\[ c_n = \frac{4}{a^2b^2} \int_0^b \left[ f(a, y) - f(0, y) \right] \cos \frac{n\pi y}{b} \, dy \]  

\[ d_n = -4 \int_0^b \left[ f(a, y) - f(0, y) \right] \cos \frac{n\pi y}{b} \, dy \]  

If

\[ f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m\pi}{a} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \]  

\[ (5-25) \]
(o<x<a, o<y<b)

Similar relationships can be developed for \( \frac{\partial f}{\partial y} \).

5.2. Simply Supported Angle-Ply Plate Under Uniform Loading

Consider an angle-ply composite on smooth supports subjected to a uniform load \( q = q_0 \). The following set of boundary conditions are applicable.

at \( x = 0 \) and \( x = a \):

\[
w = N_x = N_{xy} = M_x = 0 \quad (5-26)
\]

at \( y = 0 \) and \( y = b \):

\[
w = N_y = N_{xy} = M_y = 0 \quad (5-27)
\]

With the in-plane boundary conditions being stress conditions, it is simpler to write the governing equations in terms of a stress function and transverse displacement rather than in terms of displacements. Using equations (3-61) and (3-62), and taking into account a partially inverted form of the constitutive relations (4-5) of an angle-ply composite yields the following equations.

\[
A_7 \frac{\partial^4 \phi}{\partial x^4} + A_6 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + A_5 \frac{\partial^4 \phi}{\partial y^4} + B_5 \frac{\partial^4 w}{\partial x^3 \partial y} + B_6 \frac{\partial^4 w}{\partial x \partial y^3} = 0 \quad (5-28)
\]

\[
D_5 \frac{\partial^4 w}{\partial x^4} + D_6 \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_7 \frac{\partial^4 w}{\partial y^4} - B_5 \frac{\partial^4 \phi}{\partial x^3 \partial y} - B_6 \frac{\partial^4 \phi}{\partial x \partial y^3} = q \quad (5-29)
\]
where

\[ A_5 = A_{11}^*, \quad A_6 = (2A_{12}^* + A_6^*), \quad A_7 = A_{22}^*, \quad B_5 = (B_{61}^* - 2B_{26}^*), \]

\[ B_6 = (B_{62}^* - 2B_{16}^*), \quad D_5 = D_{11}^*, \quad D_6 = 2(D_{12}^* + 2D_6^*), \quad D_7 = D_{22}^* \]

Expanding \( q \) in a Fourier series yields

\[
q = \frac{16q_0}{\pi^2} \sum_{m=1, 3, \ldots}^{\infty} \sum_{n=1, 3, \ldots}^{\infty} \frac{1}{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{5-30}
\]

**Assume:**

\[
\phi = \sum_{m=1, 3, \ldots}^{\infty} \sum_{n=1, 3, \ldots}^{\infty} \frac{A_{mn}}{a} \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \tag{5-31}
\]

\[
(0 \leq x \leq a, \quad 0 \leq y \leq b)
\]

\[
w = \sum_{m=1, 3, \ldots}^{\infty} \sum_{n=1, 3, \ldots}^{\infty} \frac{B_{mn}}{a} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{5-32}
\]

Due to the symmetry of the problem only odd values of \( m \) and \( n \) are taken in (5-31) and (5-32). For the remainder of this section the summation sign implies odd values for \( m \) and \( n \) summed from one to infinity.

A cursory examination of equations (5-31) and (5-32) reveals that they satisfy all of the boundary conditions (5-26) and (5-27) except those involving the normal stresses \( N_x \) and \( N_y \). Using the results of the previous section, we proceed as follows

\[
\frac{\partial \phi}{\partial x} = -\sum_{m=1, 3, \ldots}^{\infty} \sum_{n=1, 3, \ldots}^{\infty} \frac{A_{mn}}{a} \sin \frac{m\pi x}{a} \cos \frac{m\pi y}{b} \tag{5-33}
\]

\[
(0 \leq x \leq a, \quad 0 \leq y \leq b)
\]

\[
\frac{\partial^2 \phi}{\partial x^2} = -\sum_{m=1, 3, \ldots}^{\infty} \sum_{n=1, 3, \ldots}^{\infty} \frac{A_{mn}}{a^2} \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \tag{5-34}
\]

\[
(0 \leq x \leq a, \quad 0 \leq y \leq b)
\]
\[
\frac{\partial^3 \phi}{\partial x^3} = - \sum \sum \frac{m^3 \pi^3}{a^3} A_{mn} \sin \frac{m \pi x}{a} \cos \frac{n \pi y}{b} \tag{5-35}
\]

\[(0 < x < a, \, 0 < y < b)\]

\[
\frac{\partial^4 \phi}{\partial x^4} = \sum \sum \left( \frac{m^4 \pi^4}{a^4} A_{mn} + a_n \right) \cos \frac{m \pi x}{a} \cos \frac{n \pi y}{b} \tag{5-36}
\]

Similarly, for the partial derivatives with respect to y the series (5-30) is term by term differentiable down to the third derivative.

Thus,

\[
\frac{\partial^4 \phi}{\partial y^4} = \sum \sum \left( \frac{n^4 \pi^4}{b^4} A_{mn} + b_m \right) \cos \frac{m \pi x}{a} \cos \frac{n \pi y}{b} \tag{5-37}
\]

The desired mixed partial derivatives are given by the usual term by term differentiation. The terms \(a_n\) and \(b_m\) are Fourier coefficients for the expansions

\[
-2 \left[ \frac{\partial^3 \phi(a, y)}{\partial x^3} + \frac{\partial^3 \phi(0, y)}{\partial x^3} \right] = \sum a_n \cos \frac{n \pi y}{b} \tag{5-37}
\]

\[
-2 \left[ \frac{\partial^3 \phi(x, b)}{\partial y^3} + \frac{\partial^3 \phi(x, 0)}{\partial y^3} \right] = \sum b_m \cos \frac{m \pi x}{a} \tag{5-38}
\]

For all of the desired partial derivatives of \(w\), term by term differentiation is justifiable, as (5-32) satisfies all boundary conditions.

Substituting (5-31) and (5-32) into the governing equations

(5-28) and (5-29) yields

\[
- (A_5 m^4 + A_6 m^2 n^2 R^2 + A_7 n^4 R^4) A_{mn} + (B_5 m^3 n R + B_6 m n^3 R^3),
\]

\[
B_{mn} = \frac{A_5 R^4 b^4}{\pi^4} a_n + \frac{A_7 R^4 b^4 b_m}{\pi^4} \tag{5-39}
\]

\[
(B_5 m^2 + B_6 n^2 R^2) mn R A_{mn} + (D_5 m^4 + D_6 m^2 n^2 R^2 + D_7 n^4 R^4),
\]

\[
B_{mn} = \frac{16 c_n R^4 b^4}{mn \pi^6} \tag{5-40}
\]
Solving equations (5-39) and (5-40) for $A_{mn}$ and $B_{mn}$ in terms of $a_n$ and $b_m$ yields

$$A_{mn} = \frac{R^4 b^4}{\pi^2 D_{mn}} \left[ 16 q_0 R (B_5 m^2 + B_6 n^2 R^2) - \pi^2 (D_5 m^2 n^2 R^2 + D_6 m^2 n^2 R^2)ight] + D_7 n^4 R^4 (A_{5 a_m} + A_{7 b_m})] \tag{5-41}$$

$$B_{mn} = \frac{R^4 b^4}{\pi^2 D_{mn}} \left[ \pi mn R (B_5 m^2 + B_6 n^2 R^2) (A_{5 a_n} + A_{7 b_m})ight] + \frac{16 q_0}{mn} (A_{5 m^2} + A_{6 m^2 n^2 R^2} + A_{7 n^4 R^4})] \tag{5-42}$$

where

$$D_{mn} = \left[ (D_5 m^4 + D_6 m^2 n^2 R^2 + D_7 n^4 R^4) (A_{5 m^4} + A_{6 m^2 n^2 R^2} + A_{7 n^4 R^4}) \right] m^2 n^2 R^2 (B_5 m^2 + B_6 n^2 R^2)^2$$

From equations (3-53) and (5-31) we have

$$N_x = -\frac{\pi^2}{b^2} \sum \sum n^2 A_{mn} \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \tag{5-43}$$

In order for $N_x$ to vanish on the edges $x = 0$ and $x = a$, we require that

$$\sum_{m=1,3,...}^\infty A_{mn} = 0 \tag{5-44}$$

for all values of $n$.

Similarly,

$$N_y = -\frac{\pi^2}{R^2 b^2} \sum \sum m^2 A_{mn} \cos \frac{m\pi x}{a} \cos \frac{m\pi y}{b} \tag{5-45}$$

and for $N_y$ to vanish on the edges $y = 0$ and $y = b$ must have
for all the values of $m$.

The $a_i$'s and $b_i$'s can be calculated from conditions (5-44) and (5-46). That is, an infinite set of simultaneous equations are generated. In practice this system is truncated (assuming convergence of the simultaneous equations) at some chosen value of $m$ and $n$, and the resulting finite system is solved.

As an example, let us consider a two layer composite in which the ply angle is $45^\circ$. For this orientation we find

$$A_7 = A_5, \quad B_6 = B_5, \quad D_7 = D_5$$

We now define the following Quantities

$$M_{mn} = \left[ D_5 (m^4 + n^4 R^4) + D_6 m^2 n^2 R^2 \right] A_5$$

$$R_{mn} = \frac{16 B_5 R (m^2 + n^2 R^2)}{\pi^2 D_{mn}}$$

Substituting (5-41) into (5-44) and (5-46), and taking (5-47) - (5-49) into account yields the following equations for determining $a_m$ and $b_m$.

$$\sum_{m=1,3,...}^{\infty} (M_{mn} a_m + M_{mn} b_m R_{mn}) = 0$$

(5-50)

for all values of $n$.

$$\sum_{n=1,3,...}^{\infty} (M_{mn} a_n + M_{mn} b_n R_{mn}) = 0$$

(5-51)

for all values of $m$.

Truncating equations (4-50) and (4-51) at $m = n = N$ leads to
the following set of equations

\[(M_{11} + M_{31} + \ldots M_{N1})a_1 + M_{11}b_1 + M_{31}b_3 + \ldots M_{N1}b_N = R_{11} + R_{31} + \ldots R_{N1}\]

\[(M_{13} + M_{33} + \ldots M_{N3})a_3 + M_{13}b_1 + M_{33}b_3 + \ldots M_{N3}b_N = R_{13} + R_{33} + \ldots R_{N3}\]

\[\vdots\]

\[\vdots\]

\[(M_{1N} + M_{3N} + \ldots M_{NN})a_N + M_{1N}b_1 + M_{3N}b_3 + \ldots M_{NN}b_N = R_{1N} + R_{3N} + \ldots R_{NN}\]

\(= R_{1N} + R_{3N} + \ldots R_{NN}\) \hspace{1cm} (5-52)

\[M_{11}a_1 + M_{13}a_3 + \ldots M_{1N}a_N + (M_{11} + M_{13} + \ldots M_{1N})b_1 = R_{11} + R_{13} + \ldots R_{1N}\]

\[M_{31}a_1 + M_{33}a_3 + \ldots M_{3N}a_N + (M_{31} + M_{33} + \ldots M_{3N})b_3 = R_{31} + R_{33} + \ldots R_{3N}\]

\[\vdots\]

\[\vdots\]

\[M_{N1}a_1 + M_{N3}a_3 + \ldots M_{NN}a_N + (M_{N1} + M_{N3} + \ldots M_{NN})b_N = R_{N1} + R_{N3} + \ldots R_{NN}\] \hspace{1cm} (5-53)

It should be noted that the sum of the equations in (5-52) are identical to the sum of the equations in (5-53). Thus, the system of linear equations for determining \(a_n\) and \(b_m\) are not independent. This means
that the solution is not unique. However, we know from section 3.7 that a unique solution for the boundary conditions (5-26) and (5-27) does exist. The solution to this apparent paradox lies in the fact that the $a_n$'s are not independent of the $b_m$'s. For example, in the case of a square plate ($R=1$) it can be shown that, due to symmetry, equations (5-50) and (5-51) are both satisfied by choosing $b_m = a_m$. For this case equation (4-50) becomes

$$\sum_{m=1, 3, \ldots}^{\infty} (M_{mn} a_n + M_{nm} a_m - R_{mn}) = 0$$

(5-54)

Interchanging the indices leads to the following form of (5-54)

$$\sum_{n=1, 3, \ldots}^{\infty} (M_{nm} a_m + M_{mn} a_n - R_{mn}) = 0$$

(5-55)

An inspection of equations (5-48) and (5-49) reveals that for $R=1$,

$$M_{nm} = M_{mn}, \quad R_{mn} = R_{nm}.$$  

Thus, equation (5-55) becomes

$$\sum_{n=1, 3, \ldots}^{\infty} (M_{mn} a_n + M_{nm} a_m - R_{mn}) = 0$$

(5-56)

which is equal to equation (4-51) for the case $b_m = a_m$.

For other values of the aspect ratio, one may encounter great difficulty in obtaining the proper relationship between $a_m$ and $b_m$.

As pointed out at the beginning of this chapter, other investigators have used this method for solving homogeneous plate problems. The question immediately arises as to whether they encountered the same problem of linear dependence of their system of simultaneous equations. The answer is yes, but they did not recognize the
Consider the problem of a homogeneous isotropic rectangular plate clamped on all sides and subjected to a uniform transverse load. This problem was discussed by Green (25). Using a double Fourier sine series for the transverse deflection \( w \) he was able to satisfy all of the prescribed boundary conditions except the vanishing of the slopes normal to the edges. Using the method discussed in this chapter he arrived at the following equations for satisfying the slope conditions

\[
\sum_{m=1, 3, \ldots}^{\infty} \frac{m^2}{(m^2+n^2R^2)^2} + R^4 \sum_{m=1, 3, \ldots}^{\infty} \frac{n^2}{(m^2+n^2R^2)^2} + \frac{16 a_n R^4 b^4}{hD\pi^6} \frac{1}{n^2} + \frac{16 R^4 b^4}{hD\pi^6 m} = 0
\]  

(5-57)

(for all values of \( n \))

\[
\sum_{n=1, 3, \ldots}^{\infty} \frac{n^2}{(m^2+n^2R^2)^2} + R^4 \sum_{m=1, 3, \ldots}^{\infty} \frac{n^2}{(m^2+n^2R^2)^2} + \frac{16 R^4 b^4}{hD\pi^6 m} \frac{1}{n^2} = 0
\]  

(5-58)

(for all values of \( m \))

where \( D \) is the plate stiffness and is given by the usual relationship for homogeneous isotropic plates. If one expands equation (5-57) and (5-58), a situation develops which is very similar to that found in equations (4-52) and (4-53). In particular, the set of equations is not linearly independent. However, this fact was never recognized for
the following reasons. In performing the actual numerical calculations, the following identities were used

\[ \sum_{M=1, 3, \ldots}^{\infty} \frac{1}{(m^2 + n^2 R^2)^2} = \pi \frac{(\sinh \pi n R - \pi n R)}{16 n^3 R^3 \cosh^2 \frac{\pi n R}{2}} \quad (5-59) \]

\[ \sum_{m=1, 3, \ldots} \frac{m^2}{(m^2 + n^2 R^2)^2} = \pi \frac{(\sinh \pi n R + \pi n R)}{16 n R \cosh^2 \frac{\pi n R}{2}} \quad (5-60) \]

With similar ones for summations on \( n \). As a result of using these identities, the problem of linear independence is circumvented. In particular, some terms of equations (5-57) and (5-58) were summed to infinity exactly, while other terms were truncated arbitrarily at \( m = n = N \). This procedure is analogous to expanding the system for \( m = n = N \), arbitrarily choosing a number \( M < N \), and then setting all values of \( a > a_M \) and \( b > b_M \) equal to zero. Then after removing the appropriate equations from the system, the remaining \( 2M \times 2M \) set of equations is solved for values of \( a_1, a_3, \ldots a_M \) and \( b_1, b_3, \ldots b_M \).

In fact, if we take this procedure to the limit as \( N \to \infty \), the results would be exactly equivalent to equations (5-57) and (5-58).

Returning now to the laminated plate problem, we see that terms in equations (5-50) and (5-51) involve the elastic compliances \( (A_i, B_i, D_i) \) and as a result identities similar to those of equations (5-59) and (5-60) cannot be obtained without great difficulty. As a result, the following alternate procedure is suggested to obtain a set of linear independent equations. The \( 2N \times 2N \) set of equations...
generated by (5-50) and (5-51) are of rank 2N-1 when we truncate at m = n = N. We now take \( b_N = 0 \), remove the last equation of the system, and solve the remaining 2N-1 set for \( a_1, a_3, \ldots, a_N, b_1, b_3, \ldots, b_{N-1} \). This is analogous to the method used by Green and discussed in previous paragraphs of this section.

To complete the 45° angle-ply problem, we write down the following relations obtained from the solutions and the constitutive relations.

\[
N_{xy} = \frac{\pi^2}{Rb} \sum \sum mn A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{5-61}
\]

\[
M_x = \frac{\pi^2}{2R^2 b^2} \sum \sum \left\{2B_7 mnR A_{mn} + \left[2D_5 m^2 + (D_6 - 4D_8) n^2 R^2 \right] B_{mn} \right\} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{5-62}
\]

\[
M_y = \frac{\pi^2}{2R^2 b^2} \sum \sum \left\{2B_7 mnR A_{mn} + \left[(D_6 - 4D_8)m^2 + 2D_8 n^2 R^2 \right] B_{mn} \right\} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{5-63}
\]

\[
M_{xy} = \frac{\pi^2}{2R^2 b^2} \sum \sum \left[ (B_5 + B_7)(m^2 + n^2 R^2) A_{mn} - 4mnR D_8 B_{mn} \right] \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \tag{5-64}
\]

where

\[
B_7 = -B_7^*, \quad D_8 = D_8^*
\]

We also define the following quantities

\[
w' = \frac{h^3}{q_0 b^4}, \quad N' = \frac{h^3}{q_0 b^4}, \quad w, \quad M' = \frac{1}{q_0 b^2} M_x \tag{5-65}
\]
Table 2 shows the results for a two layer 45° angle-ply plate of graphite-epoxy. These numbers were obtained for $b_m = a_m$. Rapid convergence is obtained for $w'$ and $M'$, while $N'$ has a relatively slow rate. Table 3 shows results for the same plate using the scheme outlined in the previous paragraph instead of the correct relationship between $a_m$ and $b_m$. The numbers obtained in this manner are reasonably close to those shown in Table 2. These excellent results give one a certain degree of confidence in the proposed method. Using this approach, solutions are shown for $R=2$ in Table 4. The pattern of convergence is very similar to that found for the square plate.

Obviously, one could use this general Fourier method to obtain solutions for various boundary conditions. For displacement conditions (e.g. a plate clamped along all four edges), in addition to having three partial differential equations, more equations similar to (5-44) and (5-46) would be necessary in order to satisfy all prescribed boundary conditions. For the same number of terms in the original series this would mean a larger system of simultaneous equations. Assuming reasonably rapid convergence, this should not cause any real difficulty. The method outlined here (or a similar scheme) should handle any problem of linear dependence which might arise.

All solutions presented in this chapter were obtained from an IBM 7094 digital computer. A standard Gaussian elimination procedure was used to solve the simultaneous equations.
6. SUMMARY AND RECOMMENDATIONS
FOR FUTURE RESEARCH

A general theory of anisotropic laminated plates based on the Kirchhoff hypotheses and incorporating the large deflection assumptions of von Karman has been developed. Inertia terms and thermal effects were also included. The resulting equations showed a coupling between bending and stretching of the plate which did not disappear when the governing equations were linearized. This coupling phenomena is not found in the linear theory of homogeneous plates.

The effect of this coupling phenomena on the transverse deflection, critical buckling load, and fundamental frequency of vibration was explored in some depth. Closed form solutions for the linearized equations were obtained and revealed that coupling increases transverse deflections, decreases critical buckling loads, and decreases fundamental vibration frequencies relative to analogous anisotropic homogeneous plates. The size of these effects depend on the degree of anisotropy of the individual layers and on the total number of plies in the laminate. For two layer highly anisotropic plates, coupling effects are quite severe.

The closed form solutions included in the present work have escaped others working in laminated plate theory because they failed to formulate the problem in terms of displacements rather than in terms of a stress function and transverse displacement. However, the importance of the original work by Reissner and Stavsky (10) cannot be
over emphasized. By displaying the existence of the coupling phenomena, they were able to show conclusively that the earlier work of Smith (11) was erroneous in concluding that anisotropic laminates behave as a homogeneous material. It is hoped that this dissertation has contributed to furthering the understanding of the mechanical behavior of laminated plates, and in particular has provided firmer grounds for determining the circumstances under which coupling must be accounted for. This takes on particular significance in light of the fact that practical applications exist for which it is virtually impossible to use a symmetric laminate (e.g. the skin of an aircraft structure).

The method outlined for the use of double Fourier series in solving coupled laminated plate problems provides the ground work for obtaining solutions to a wider variety of boundary conditions. Results in the present work show that the effects of coupling disappear rather rapidly with an increasing number of plies. However, this does not provide conclusive evidence that this will happen under other conditions. For example, consider buckling under pure shear. It is possible that many more layers may be required to remove the coupling effect for this type of loading.

Further work should be done in checking the validity of the scheme outlined for obtaining a linear set of equations in the Fourier analysis. Results indicate the method to be sound. However, it
should be subjected to further scrutiny before receiving final accept-
ance.

Various approximate methods for solving coupled laminated plate
problems should be given future consideration. The Ritz (energy)
method offers one possibility. Another interesting possibility has
been suggested by Kicher (29). The real effect of coupling is to re-
duce the effective stiffness of the laminate. The equivalent of this
can be done by using the partially inverted form of the constitutive
relations (3-60) and ignore the coupled terms due to $B^*$. This yields

$$
\begin{bmatrix}
\varepsilon^0 \\
 M
\end{bmatrix} =
\begin{bmatrix}
 A^* & 0 \\
 0 & D^*
\end{bmatrix}
\begin{bmatrix}
 N \\
 K
\end{bmatrix}
$$

Use of (6-1) would result in equations equivalent to a homogeneous
anisotropic plate with the stiffness terms $D_{ij}$ being replaced by the
coupled stiffness coefficients $D^*_{ij}$. Since $D^* = D - BA^{-1}B$, coupling
could be introduced in an indirect manner with the result being a re-
duced plate stiffness. This suggestion has not been explored to any
great extent. A complete study would reveal proper conditions for
which this might be a valid approximation.

Finally, we come to the most important area in which future work
should be concentrated. Any theoretical model is only as good as the
assumptions from which it is developed. For the present theory a
key assumption is that shear strains $\varepsilon_{xz}$ and $\varepsilon_{yz}$ are negligible. For most homogeneous isotropic materials we know this is a reasonable assumption if the ratio of the smallest width to the thickness is relatively large. However, Pagano (30) has shown for laminated cross-ply beams, that shear deformations can change the predicted value of the maximum deflection as much as 20% for a span-to-depth ratio as large as 30 to 1. This has also been indicated experimentally from the elastic modulus of unidirectional composites obtained from bending tests. When shear deformations are neglected in analyzing the data, the modulus is invariably lower than values obtained from a tension test. When beam equations are used which account for shear deflections, the flex modulus is very close to the modulus measured in a tension test. In addition to shear deformations, the effect of rotary inertia and in-plane inertia should also be ascertained. Obviously, any theory which includes the effects of shear deformation will be more complicated than the present theory. However, the equations should be formulated in hopes of obtaining solutions for very simple loadings. Informative results might be obtainable from such an approach.

Other areas of interest include thermal stress problems and application of the elastic-viscoelastic correspondence principle to the solution of plate problems which involve time dependent material response.
## TABLE 1

Numerical Results for Simply-Supported Cross-Ply Plate Under Transverse Loading

<table>
<thead>
<tr>
<th>N*</th>
<th>$u_{\text{max}}^0/q_0^b$</th>
<th>$w_{\text{max}}/q_0^b$</th>
<th>$N_{\text{max}}/q_0^b$</th>
<th>$N_{\text{ymax}}/q_0^b$</th>
<th>$M_{\text{max}}/q_0^b$</th>
<th>$M_{\text{ymax}}/q_0^b$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.000018</td>
<td>0.002499</td>
<td>0.063832</td>
<td>0.027244</td>
<td>1.134610</td>
<td>0.499064</td>
</tr>
<tr>
<td>6</td>
<td>0.000002</td>
<td>0.000966</td>
<td>0.007973</td>
<td>0.003454</td>
<td>1.189095</td>
<td>0.189822</td>
</tr>
<tr>
<td>10</td>
<td>0.000001</td>
<td>0.000900</td>
<td>0.002626</td>
<td>0.001944</td>
<td>1.191255</td>
<td>0.180856</td>
</tr>
<tr>
<td>$\infty$</td>
<td>0</td>
<td>0.000896</td>
<td>0</td>
<td>0</td>
<td>1.192385</td>
<td>0.176176</td>
</tr>
</tbody>
</table>

**Graphite-Epoxy ($E_L/E_T = 4.0$)**

| 2  | 0.000012               | 0.002559            | 0.363364            | 0.155075            | 1.403775            | 0.862270            |
| 6  | 0.000003               | 0.002231            | 0.105616            | 0.050744            | 1.410490            | 0.751890            |
| 10 | 0.000002               | 0.002209            | 0.062727            | 0.024093            | 1.410955            | 0.744265            |
| $\infty$ | 0               | 0.002200            | 0                   | 0                   | 1.411210            | 0.740045            |

**Glass-Epoxy ($E_L/E_T = 2.9$)**

$^*N$ indicates the number of layers in the composite.
TABLE 2

Numerical Convergence of Solution for Simply-Supported 45° Angle-Ply Plate Under Uniform Load (b_m = a_m)

<table>
<thead>
<tr>
<th>N*</th>
<th>( w^{(1/2,1/2)} \times 10^{-9} )</th>
<th>( N_x^{(1/4,1/4)} \times 10^{-4} )</th>
<th>( M_x^{(1/2,1/2)} \times 10^{-2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>9.761035</td>
<td>-0.906481</td>
<td>6.902698</td>
</tr>
<tr>
<td>9</td>
<td>9.790767</td>
<td>-1.386270</td>
<td>7.207140</td>
</tr>
<tr>
<td>16</td>
<td>9.784955</td>
<td>-1.162665</td>
<td>7.090776</td>
</tr>
<tr>
<td>25</td>
<td>9.786604</td>
<td>-1.090144</td>
<td>7.145097</td>
</tr>
<tr>
<td>36</td>
<td>9.785995</td>
<td>-1.136758</td>
<td>7.115070</td>
</tr>
<tr>
<td>49</td>
<td>9.786258</td>
<td>-1.189500</td>
<td>7.133162</td>
</tr>
<tr>
<td>64</td>
<td>9.786129</td>
<td>-1.148437</td>
<td>7.121334</td>
</tr>
<tr>
<td>81</td>
<td>9.786197</td>
<td>-1.124072</td>
<td>7.129433</td>
</tr>
<tr>
<td>100</td>
<td>9.786158</td>
<td>-1.142862</td>
<td>7.123617</td>
</tr>
<tr>
<td>121</td>
<td>9.786182</td>
<td>-1.161790</td>
<td>7.127916</td>
</tr>
</tbody>
</table>

*N indicates number of terms in double Fourier series.
TABLE 3

Numerical Convergence of Solution for Simply-Supported 45° Angle-Ply Plate Under Uniform Load ($b_m 
eq a_m$)

<table>
<thead>
<tr>
<th>N*</th>
<th>$w'(1/2, 1/2) \times 10^{-9}$</th>
<th>$N_x'(1/4, 1/4) \times 10^{-4}$</th>
<th>$M_x'(1/2, 1/2) \times 10^{-2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>9.761035</td>
<td>-0.906481</td>
<td>6.902698</td>
</tr>
<tr>
<td>9</td>
<td>9.790767</td>
<td>-1.386268</td>
<td>7.020714</td>
</tr>
<tr>
<td>16</td>
<td>9.784955</td>
<td>-1.162664</td>
<td>7.090776</td>
</tr>
<tr>
<td>25</td>
<td>9.786604</td>
<td>-1.090145</td>
<td>7.145097</td>
</tr>
<tr>
<td>36</td>
<td>9.785995</td>
<td>-1.136758</td>
<td>7.115070</td>
</tr>
<tr>
<td>49</td>
<td>9.786258</td>
<td>-1.189506</td>
<td>7.133162</td>
</tr>
<tr>
<td>64</td>
<td>9.786129</td>
<td>-1.148437</td>
<td>7.121344</td>
</tr>
<tr>
<td>81</td>
<td>9.786197</td>
<td>-1.124064</td>
<td>7.129433</td>
</tr>
<tr>
<td>100</td>
<td>9.786158</td>
<td>-1.142862</td>
<td>7.123617</td>
</tr>
<tr>
<td>121</td>
<td>9.786182</td>
<td>-1.161786</td>
<td>7.127916</td>
</tr>
</tbody>
</table>

*N indicates number of terms in double Fourier series.
TABLE 4

Numerical Convergence of Solution for Simply-Supported 45°
Angle-Ply Plate Under Uniform Load (r = 2)

<table>
<thead>
<tr>
<th>N*</th>
<th>w'($\frac{1}{2},\frac{1}{2}$) x 10^{-8}</th>
<th>N'$_x$($\frac{1}{4},\frac{1}{4}$) x 10^{-4}</th>
<th>M'$_x$($\frac{1}{2},\frac{1}{2}$) x 10^{-1}</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2.805732</td>
<td>-1.128112</td>
<td>1.576921</td>
</tr>
<tr>
<td>9</td>
<td>2.825028</td>
<td>-2.053988</td>
<td>1.633090</td>
</tr>
<tr>
<td>16</td>
<td>2.820684</td>
<td>-1.507646</td>
<td>1.610917</td>
</tr>
<tr>
<td>25</td>
<td>2.822012</td>
<td>-1.333843</td>
<td>1.621432</td>
</tr>
<tr>
<td>36</td>
<td>2.821508</td>
<td>-1.391192</td>
<td>1.615558</td>
</tr>
<tr>
<td>49</td>
<td>2.821729</td>
<td>-1.442193</td>
<td>1.619110</td>
</tr>
<tr>
<td>64</td>
<td>2.821620</td>
<td>-1.405654</td>
<td>1.616782</td>
</tr>
<tr>
<td>81</td>
<td>2.821679</td>
<td>-1.382472</td>
<td>1.618377</td>
</tr>
<tr>
<td>100</td>
<td>2.821645</td>
<td>-1.402857</td>
<td>1.617231</td>
</tr>
<tr>
<td>121</td>
<td>2.821665</td>
<td>-1.425105</td>
<td>1.618078</td>
</tr>
</tbody>
</table>

*N indicates number of terms in double Fourier series.
Figure 1. Two-Layer Angle-Ply Composite with Fibers Oriented at $+\theta$ and $-\theta$ to the Plate Axes.
Figure 2. Stress Nomenclature on Body in Deformed State.
Figure 3. Location of Particles on Undeformed and Deformed Bodies.
Figure 4. Coordinate System of Plate.
Figure 5. Resultant Stress Nomenclature.
Figure 6. Nomenclature for Moment and Transverse Shear Resultant.
Figure 7. Unidirectional Composite at Parallel and Angle-Ply Orientations, Respectively.
Figure 8. Cross-Ply Composite.
Figure 9. Maximum Deflection as a Function of Angle-Ply Orientation for Simply-Supported Graphite-Epoxy Square Plate Under Transverse Loading.
Figure 10. Maximum Deflection as a Function of Angle-Ply Orientation for Simply-Supported Glass-Epoxy Square Plate Under Transverse Loading.
Figure 11. Critical Buckling Load as a Function of Angle-Ply Orientation for Simply-Supported Graphite-Epoxy Square Plate Under Uniaxial Compression.
Figure 12. Critical Buckling Load as a Function of Angle-Ply Orientation for Simply-Supported Glass-Epoxy Square Plate Under Uniaxial Compression.
Figure 13. Critical Buckling Load as a Function of Angle-Ply Orientation for Simply-Supported Graphite-Epoxy Square Plate Under Biaxial Compression.
Figure 14. Fundamental Vibration Frequency as a Function of Aspect Ratio for a Simply-Supported Graphite-Epoxy Cross-Ply Plate.
Figure 15. Fundamental Vibration Frequency as a Function of Aspect Ratio for a Simply-Supported Glass-Epoxy Cross-Ply Plate.
Figure 16. Fundamental Vibration Frequency as a Function of Angle-Ply Orientation for a Simply-Supported Graphite-Epoxy Square Plate.
Figure 17. Fundamental Vibration Frequency as a Function of Angle-Ply Orientation for a Simply-Supported Glass-Epoxy Square Plate.
APPENDIX

TRANSFORMATION PROPERTIES OF AN ORTHOTROPIC STIFFNESS MATRIX

For the unidirectional composite shown in Figure 7b we have the following transformations from the theory of elasticity

\[
\begin{bmatrix}
    x' \\
    y' \\
    z'
\end{bmatrix} =
\begin{bmatrix}
    l & m & 0 \\
    -m & l & 0 \\
    0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
    x \\
    y \\
    z
\end{bmatrix}
\]  \hspace{1cm} (A-1)

\[
\sigma' = T_\sigma \sigma
\]  \hspace{1cm} (A-2)

\[
\epsilon' = T_\epsilon \epsilon
\]  \hspace{1cm} (A-3)

where

\[
T_\sigma =
\begin{bmatrix}
    l^2 & m^2 & 0 & 2lm & 0 & 0 \\
    m^2 & l^2 & 0 & -2lm & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & l & -m \\
    0 & 0 & 0 & 0 & m & l \\
    lm & lm & 0 & (l^2-m^2) & 0 & 0
\end{bmatrix}
\]

\[
T_\epsilon =
\begin{bmatrix}
    l^2 & m^2 & 0 & lm & 0 & 0 \\
    m^2 & l^2 & 0 & -lm & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & l & -m \\
    0 & 0 & 0 & 0 & m & l \\
    -2ml & 2ml & 0 & (l^2-m^2) & 0 & 0
\end{bmatrix}
\]

and

\[
l = \cos \theta, \quad m = \sin \theta
\]  \hspace{1cm} (A-4)

Hooke's law with respect to the x-y axes in Figure 7b is given by

93.
Again using Hooke's law, yields

\[ \sigma' = C' \epsilon' \]

(A-6)

Using equation (A-5) along with (A-3) yields

\[ \sigma = C \epsilon = CT^{-1} \epsilon' \]

(A-7)

Applying (A-2) to equation (A-7) gives the following results

\[ T_o^{-1} \sigma' = CT^{-1} \epsilon' \]

(A-8)

Multiplying (A-8) by \( T_o \) yields

\[ \sigma' = T_o CT^{-1} \epsilon' \]

(A-9)

Equations (A-6) and (A-9) result in the desired transformation relationship

\[ C' = T_o CT^{-1} \epsilon' \]

(A-10)

Expanding equation (A-10) yields the following results

\[ C_{11} = C_{11} l^4 + C_{22} m^4 + 2(C_{12} + 2C_{66}) l^2 m^2 \]

(A-11)

\[ C_{12} = C_{21} = (C_{11} + C_{22} - 4C_{66}) l^2 m^2 + C_{12}(l^4 + m^4) \]

(A-12)

\[ C_{13} = C_{41} = C_{13} l^2 + C_{23} m^2 \]

(A-13)
\[ C_{16} = C_{61} = -C_{11} \ell^3 m + C_{22} \ell m^3 + (C_{12} + 2C_{66})(\ell^3 m - \ell m^3) \] (A-14)

\[ C_{22} = C_{11} m^4 + 2(C_{12} + 2C_{66}) \ell^2 m^2 + C_{22} \ell^4 \] (A-15)

\[ C_{13} = C_{12} = C_{23} \ell^2 + C_{13} \ell^2 \] (A-16)

\[ C_{26} = C_{62} = -C_{11} \ell m^3 + C_{22} \ell^3 m + (C_{12} + 2C_{66})(\ell^3 m - \ell^3 m) \] (A-17)

\[ C_{33} = C_{33} \] (A-18)

\[ C_{36} = C_{63} = (C_{23} - C_{13}) \ell m \] (A-19)

\[ C_{44} = C_{44} \ell^2 + C_{55} \ell^2 \] (A-20)

\[ C_{45} = C_{54} = (C_{44} - C_{55}) \ell \] (A-21)

\[ C_{55} = C_{44} \ell^2 + C_{55} \ell^2 \] (A-22)

\[ C_{66} = (C_{11} + C_{22} - 2C_{12}) \ell^2 m^2 + C_{66} (\ell^2 - m^2) \] (A-23)

All other coefficients vanish. Thus, the stiffness matrix with respect to the \( x' \)-\( y' \) axes is given by

\[
C' = \begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\
C_{12} & C_{12} & C_{13} & 0 & 0 & C_{16} \\
C_{13} & C_{13} & C_{13} & 0 & 0 & C_{16} \\
0 & 0 & 0 & C_{44} & C_{45} & 0 \\
0 & 0 & 0 & C_{54} & C_{55} & 0 \\
C_{16} & C_{16} & C_{16} & 0 & 0 & C_{66}
\end{bmatrix}
\] (A-24)
REFERENCES


8. March, H. W., "Buckling of Flat Plywood Plates Under Compression, Shear, or Combined Compression and Shear," FOREST PRODUCTS LABORATORY REPORT NO. 1316, April 1942.

of Rectangular Wood and Plywood Plates," PROCEEDINGS OF

10. Hearmon, R. F. S., "The Frequency of Flexural Vibration of
Rectangular Orthotropic Plates with Clamped or Supported Edges,"

of Orthotropic Material," JOURNAL OF APPLIED MECHANICS,

12. Reissner, E. and Stavsky, Y., "Bending and Stretching of Cer-
tain Types of Heterogeneous Aeolotropic Elastic Plates,"
JOURNAL OF APPLIED MECHANICS, Vol. 28, September 1961,
pp. 402-408.

13. Stavsky, Y. and McGarry, F. J., "Investigation of Mechanics of
Reinforced Plastics," Air Force TECHNICAL DOCUMENTARY

14. Dong, S. B., Matthiesen, R. B., Pister, K. S., and Taylor,
ARL-76, September 1961.

15. Ashton, J. E., "Anisotropic Plate Analysis," GENERAL DYNAM-
ICS REPORT FZM-4899, October 1967.

NAMICS REPORT FZM-4992, February 1968.

17. Ashton, J. E., "Dynamic Response of Anisotropic Plates,"
GENERAL DYNAMICS REPORT FZM-5008, March 1968.


29. Kicher, T. P., Case Western Reserve University, Private Communication, June 1968.