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DISSERTATION

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By


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Approved by

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CHAPTER I
INTRODUCTION

Recent years have seen significant advances in satellite communication technology and space communication systems in general. The reader is probably well aware of the fact that the potential of this relatively new field is almost unlimited.

A major constraint on communication reliability in this relatively new field of communication is the limited available power and antenna gain. The engineer is constantly seeking new methods to increase both of these quantities in an effort to improve communication reliability. In the antenna area, the classical example of recent vintage is the adaptively phased array. Although the literature is full of important contributions, considerable work still remains before the phase array capabilities are fully realized. This is especially true in the adaptive transmitting array concept for communication purposes.

This treatise presents and analyzes a new technique for achieving an adaptive transmitting array. The approach is superior in many important respects to previously proposed schemes of its kind.

To start with, the basic concepts will be reviewed. The fundamental objective of the adaptive transmitting array for communication purposes is to achieve the antenna gain realizable from a single aperture whose area is equal to the entire array without increasing
the individual aperture size. This can be accomplished if the antennas of the array are radiating signals that are in phase (coherent) at the desired receiving point. An example of this is a system of $N$ identical antennas, each having an effective aperture $A_e$ and radiating a power $P$, phased in the above manner. The electric field intensity at the receiving point resulting from this phased array would then be:

\[(1) \quad E_N = NE_1\]

where $E_1$ is the electric field from one antenna radiating by itself. Since the power density at a point is proportional to the square of the electric field, the power density due to one antenna would be

\[(2) \quad S_1 = aE_1^2\]

and that due to $N$ elements would be

\[(3) \quad S_N = aE_N^2 = aN^2E_1^2 = N^2S_1\]

where $a$ is a constant of proportionality. This $S_N$ is equivalent to one antenna with effective aperture $NA_e$ radiating a power $NP$. An antenna gain is achieved without increasing the individual aperture size.

The phase coherent transmission problem involves measuring or calculating the differential path-length between each antenna's phase center and the target so that the necessary phase adjustment of each transmitter can be made. The receiving point may not be fixed with
respect to the transmitters, such as in the case of satellite communication, which is of primary concern here. Consequently, the electrical phase of the signal at each array element must be continuously or at least periodically adjusted. The problem lies in obtaining the necessary information to make this phase adjustment properly.

Past research on the coherent transmission system has indicated that three general procedures are available for accomplishing this phase adjustment: 1) pure prediction or computer control techniques, 2) phase conjugation or retrodirection techniques, and 3) beam tagging techniques. [1,2,3,4] Each technique has its own advantages and disadvantages with respect to a given application. The different approaches will now be briefly discussed with an emphasis on the principal advantages and limitations of each.

Pure prediction or computer control techniques, as the name implies, involve the calculation of the necessary phase control information based on the geometry of the application. Its effectiveness depends on the a priori knowledge of both the target position and the location of the phase center of each antenna. Even if one has acceptable target position information, accurate knowledge of the location of the phase center of a narrow-beam antenna is difficult to obtain and due to mechanical stress will actually change for large parabolic antennas as they are steered. This method also suffers from the standpoint that it cannot compensate for differential changes in the propagation medium, i.e., for different effects of
the medium on each transmitted signal. The technique's advantage is the ability to compensate for the propagation time delay.

In phase conjugation systems, the required phase control information is determined by processing a signal that is received via the target (satellite). The essence of this method is simply to measure the received phase at each element of the array relative to some reference and then use the conjugate of this measured phase difference to correct the phase of the transmitted signal. To be more specific, if the $i^{th}$ element has phase delay $e^{-j\phi_i}$ on reception, then the conjugate phase delay for transmission is $e^{j\phi_i}$. When phased in this manner, the resultant transmitted signal will be focused at the satellite. A basic problem encountered in this system is that the received signal frequency sometimes differs from the transmitted signal frequency, e.g., when frequency isolation is used, and at the same time the elements of the array are separated by many wavelengths. Unfortunately under these conditions, measuring the principal value of the phase relative to some reference, $(\phi^R)p$, does not provide sufficient information for determining the required phase for transmission.[5,6] Should the value of $(\phi^R)p$ be used to provide correction to the transmitter phasing, a beam offset error results which in some cases is much larger than the beamwidth of individual elements. This is intolerable. The phase control accuracy of this technique, other than the frequency offset problem, is a function of the signal-to-noise ratio at each element and is independent of the number of elements in the array. This effect is not present with the beam tagging technique.
With beam tagging techniques, the signal radiated from each element of the array is tagged with some identifying modulation, such that phase control information may be obtained by properly processing the resultant signal of the array.\[7,8,9\] For example, if the transmitted signal from one antenna is AM modulated at frequency $\rho$, phase control information is available by measuring the PM of the resultant signal at frequency $\rho$. Beam tagging control has an important limitation in that the accuracy of the method decreases as the number of array elements increases. This results from the fact that the method involves measuring how an individual signal affects the resultant signal. Consequently, the more elements there are, the less effect any one signal has on the resultant. On the other hand, the beam tagging method is a closed-loop phase control system and thus will automatically correct for the phase instabilities which are always present in practical systems. The other two techniques are open-loop systems and thus cannot compensate for phase changes that are random in nature.

It has been observed that a combined scheme involving either phase conjugation or computer control techniques in conjunction with beam tagging is a better approach for satellite communications than any one used by itself.\[10,11\] This is especially true for low-orbit satellites where its angular rate with respect to the array is large. The purpose of the combined technique is to take advantage of the desirable characteristics of each method while de-emphasizing the undesirable ones. Phase conjugation or computer control techniques
provide an open-loop correction while the beam tagging technique provides the closed-loop corrections for errors not completely removed by the open-loop operation. Such open-loop errors arise from phase instabilities and offset errors, if phase conjugation is used alone or phase instabilities and phase center changes, if computer control techniques are used unaided. The rate of change of these uncorrected phase errors when open-loop operation is used is ordinarily much smaller than the over-all phase correction rate. A combined approach is used quite frequently in conventional control systems. As an example, consider the master-slave antenna problem where it is required that the slave antenna follows the master antenna with the minimum amount of angular offset. This is accomplished by providing the slave antenna with two control mechanisms. One results in an open-loop correction which is proportional to the velocity of the master and the second provides a closed-loop correction based on the position of the slave relative to the master. Hence if open-loop correction is used, the requirements on the closed-loop system can be significantly reduced.

In the case of the adaptive array, the reduction in the required phase correction rate for the closed-loop system here led to the development of the sequential beam tagging scheme. This sequential scheme provides phase correction for individual transmitters in sequence rather than simultaneous correction of these elements. Simultaneous tagging requires that a different modulation
be assigned to each transmitted signal which results in an interaction problem that may be intolerable.\[12,13\] Sequential correction avoids this problem since only one sequentially applied modulation is necessary.

It should be emphasized that there are two major reasons why beam tagging is attractive for phase control operation; first, because of its closed-loop nature and the desirable type of correction a closed-loop system can provide and secondly, because of its compatibility with conventional open-loop techniques for controlling adaptive arrays. However, for beam tagging to be an effective tool in satellite communications, it must possess other characteristics. Among these are: 1) it must be able to provide accurate phase control information by using only a small fraction of the total available power, 2) it must be minimally influenced by the data transmission, 3) it must have an insignificant effect on the data transmission, and 4) it must be practically implementable. The author feels that work to date has not produced a system with all these important characteristics. It is believed that the system developed herein possesses all these important characteristics. The beam tagging system used here is a sequential correction scheme. The beam tagging modulation is a simple $0^0, 180^0$, phase modulation applied to the signal transmitted by each element in sequence and the rate of the beam tagging phase modulation is much lower than the data rate. Also, the phase error detection receiver that is developed has the ability to obtain the phase error information, that
is necessary for the adaptive phase control, in the presence of the data modulation. How this is all accomplished will become clear as the discussion progresses.

First of all the standard technique for eliminating the interaction between data and phase control information is to provide a separate frequency band for each of these operations. This will work for linear satellite repeaters but may not work for the ideal hard-limiting variety because of the intermodulation products that are generated in the limiting process. Since the hard-limiting type of satellite has been used extensively to provide maximum utilization of down-link power, means would have to be provided to minimize the effect of this interaction. In addition, the frequency separation technique requires that a percentage of the available power be used exclusively for the phase control operation. This power is then not available for data transmission. A self-synchronizing technique can be employed to avoid the assignment of power exclusively to the phase tracking function. The squaring-loop technique for PSK synchronization is an example of a self-synchronized system.\[14,15,16\] In fact Van Trees has stated,

That under a wide variety of conditions, the best strategy will be to devote all of the available power to modulation and to obtain the synchronizing information by performing some operation on the incoming signal which removes its dependence on the data.\[17\]

It seems reasonable to assume that this strategy will also be quite effective for a beam tagging scheme. This strategy forms the basis of the self-synchronized phase control system that is presented here.
In this new approach, no power will be made available exclusively for phase control. Instead, the beam tagging modulation is sequentially applied to the transmitted signals without interrupting the data transmission from any of the elements. The resultant signal is processed in a particular manner which removes its dependence on the data modulation. In addition it will be seen that means are available to minimize the effect of the beam tagging on the data modulation. Also if the frequency separation technique is used to eliminate the interaction of the data and beam tagging, it is implied that the propagation medium will produce identical phase effects in the separate frequency bands. This has no bearing on the approach used here.

To be more specific, a sequential, phase reversal beam tagging technique is proposed along with a new error detection receiver that employs a squaring loop to remove the data which is assumed to be in PSK form. The restriction to PSK data modulation is not very severe for satellite communication purposes since it is anticipated that this form of modulation will be used extensively in future satellite communication systems. The proposed technique will work with either passive or active satellites but the main emphasis is placed on the latter. It has the added feature that the beam tagging modulation can be added mod 2 to the data which facilitates the implementation of the method.

This report begins with a signal analysis of the proposed scheme to demonstrate its theoretical capabilities. Then a noise analysis is performed on the phase error detection system. This
analysis relates phase error measurement accuracy to the system parameters. Among these parameters are: 1) data rate, 2) number of array elements, 3) signal-to-noise ratio, and 4) total estimation time of the phase error. Next, a closed-loop model is developed that adequately represents the physical situation, i.e., the phase control operation. The important statistics related to this model are developed and these statistics are then related to over-all phase control accuracy which is defined in terms of a performance index called the "coherence factor." This index is a measure of how efficiently the signals from the array are adding on a power basis. The study concludes with two design examples that illustrate the use of the derived results. The results are presented in graphical form such that for a given application and a desired degree of performance the system designer can easily specify the necessary requirements on the various parameters. He will also be able to tell if an open-loop system is necessary to assist the proposed beam tagging system.

It is worth noting that some of the statistical results derived in the Appendixes are also useful in evaluating the performance of receiving arrays.
CHAPTER II
SIGNAL ANALYSIS

The objective of this chapter is to describe and analyze the sequential phase coding scheme, the phase-error detection receiver, and certain important constraints that must be imposed for successful phase coherent operation of the system.

The output of each element of the transmitting array is examined and an expression for the total signal at the satellite is derived. This expression is used to establish the maximum data rate that the transmitting array can accommodate. An expression for the signal returned from the satellite is derived and a signal analysis of the phase-error detection receiver is presented. This analysis shows how the phase-error estimate can be obtained in the presence of the data modulation. Also, certain constraints on the error-detection receiver are established.

As a first step in the analysis consider the N-element transmitting array as shown in Fig. 1. Suppose at time $t = t'$, the $i^{th}$ element is tagged, i.e., some form of modulation is applied to the transmitted signal concurrent with the data modulation. Here, the tagging technique that is being considered is a simple $0^\circ, 180^\circ$, low-frequency phase-modulation applied to each element in sequence. The tagging modulation frequency is much less than the data modulation frequency. The output of this $i^{th}$ element at time $t'' > t'$ can be represented mathematically as
Fig. 1--N element transmitting array.
(4) \[ s_{oi}(t'') = \sqrt{2P_{oi}} \ m(t'') \ S(t'') \ \cos [\omega_Tt'' + \phi_i(t'')] \]

where;

- \( P_{oi} \) = average power output of \( i \)th element during tagging period,
- \( m(t) = \pm 1 \) \( T_b \leq t \leq (v+1)T_b \) \( v = 1, 2 \) (PSK data),
- \( T_b \) = data bit length,
- \( S(t) = \) square wave of value 1, -1, with period \( T_c \) \( (T_b \ll T_c) \),
- \( \omega_T \) = carrier frequency in radian/second, and
- \( \phi_i(t) \) = controllable phase adjustment.

The output of the \( j \)th element during the time the \( i \)th element is tagged will thus be

(5) \[ s_{oj}(t'') = \sqrt{2P_{oj}} \ m(t'') \ \cos [\omega_Tt'' + \phi_j(t'')] \]

\( (j = 1, 2, \ldots, N, j \neq i) \)

Where the terms are defined in a similar manner as above. Note that power \( P_{oi} \) can be different from the power output of the rest of the elements. Assuming that the only effect of the medium is to attenuate the signals, the total signal \( S_{si}(t'') \) at the satellite is (see Fig. 2 for further clarification)

(6) \[ S_{si}(t'') = \sqrt{2P_{si}} \ m\left(t'' - \frac{\Delta T_i}{2}\right) \ S\left(t'' - \frac{\Delta T_i}{2}\right) \times \]
\[ \cos \left[ \omega_T\left(t'' - \frac{\Delta T_i}{2}\right) + \phi_i\left(t'' - \frac{\Delta T_i}{2}\right)\right] \]
\[ + \sum_{j=1}^{N} \sqrt{2P_{sj}} \ m(t'' - \Delta T_j/2) \ \cos \left[ \omega_T\left(t'' - \frac{\Delta T_j}{2}\right) \right] \]
\[ + \phi_j(t'' - \Delta T_j/2), \left(t'' > t' + \frac{\Delta T_i}{2}\right) \]
\[
\begin{align*}
\Delta T_i &= \text{UP-LINK TIME DELAY FOR } i^{th} \text{ ELEMENT} \\
\Delta T &= \text{DOWN-LINK TIME DELAY FOR RECEIVING} \\
&\quad \text{ELEMENT (POSSIBLY THE ENTIRE ARRAY)} \\
T_T &= \text{TOTAL CODING TIME}
\end{align*}
\]

Fig. 2--Time scale during coding of \( i^{th} \) element.
where;

\[ \frac{T_j}{2} = \text{delay time from } j^{th} \text{ element to satellite}, \]

and

\[ P_{sj} = \text{average power at satellite from } j^{th} \text{ element.} \]

It will be assumed that the output power is the same for each element except \( j \)th element where power reductions may be used and the path attenuation is the same for all signals, so that

\[ P_{s1} = P_{s2} = \cdots P_{sN} = k^2 P_{si} \]

where,

\( k^2 \) is the factor by which the power in the tagged beam is less than the other beams. For effective communication transfer, the data modulation must be essentially in time alignment, i.e.,

\[ m\left(t'' - \frac{\Delta T_1}{2}\right) = m\left(t'' - \frac{\Delta T_2}{2}\right) = \cdots m\left(t'' - \frac{\Delta T_N}{2}\right) . \]

The last statement is approximately correct if

\[ |\frac{\Delta T_k}{2} - \frac{\Delta T_2}{2}|_{\max} \ll T_b . \]

The implication of Eq. (9) is that the maximum differential time delay should be much less than the data bit length. For the sake of argument, the maximum differential time delay should be at most one-tenth of the data bit length. This defines the so called
"Array bandwidth."[19] The array bandwidth can be improved by time delay control of the data on the individual transmitted signals. This control can be accomplished by obtaining timing information from satellite position information or the timing information that will be available if bit alignment is used in the receiving mode of the phased array. Of course, the ultimate bandwidth (data bit rate) will depend on the accuracy with which the above control techniques can be implemented.

By use of Eqs. (7) and (8) the total signal is

\[ S_{Si}(t'') = \sqrt{2P_{s1}} m(t'' \pm \frac{\Delta T_i}{2}) \{ \left( \frac{1}{k} \right) S(t'' \pm \frac{\Delta T_i}{2}) \} \]

\[ \cos \left[ \omega_T \left( t'' \pm \frac{\Delta T_i}{2} \right) + \phi_i \left( t'' \pm \frac{\Delta T_i}{2} \right) \right] + \]

\[ \sum_{j=1}^{N} \cos \left[ \omega_T \left( t'' \pm \frac{\Delta T_j}{2} \right) + \phi_j \left( t'' \pm \frac{\Delta T_j}{2} \right) \right] \]

Perfect phase coherence will occur if

\[ -\omega_T \frac{\Delta T_i}{2} + \phi_i \left( t'' \pm \frac{\Delta T_i}{2} \right) = \cdots -\omega_T \frac{\Delta T_N}{2} + \phi_N \left( t'' \pm \frac{\Delta T_N}{2} \right). \]

Perfect phase alignment will not be possible in general because of: 1) satellite motion since phase correction is based on satellite position one signal round trip time delay earlier, and 2) inaccurate phase correction due to noise. The phase of each signal will be defined as:
These phases will be assumed constant during the tagging time $T_T$. For a given application, this assumption fixes the maximum permissible tagging time. This point will be discussed further in the analysis. Thus using Eq. (12), Eq. (10) reduces to

$$S_{s_1}(t'') = \sqrt{2p_{s_1}} \ m \left( t'' - \frac{\Delta T_i}{2} \right) \left\{ \left( \frac{1}{k} \right) S \left( t'' - \frac{\Delta T_i}{2} \right) \times \cos [\omega_i t'' + \xi_j] + \sum_{j \neq i}^{N} \cos [\omega_i t'' + \xi_j] \right\}.$$  

The summation in Eq. (13) can be written as

$$\sum_{j=1}^{N} \cos [\omega_i t'' + \xi_j] = (N-1) \beta \cos [\omega_i t'' + \xi_{N-1}]$$

where by definition:

$\xi_{N-1}$ is the resultant phase angle associated with the sum of the $N-1$ signals,

and

$$\beta^2 = \frac{\sum_{k=1}^{N-1} \sum_{p=1}^{N-1} e^{i(\xi_k - \xi_p)}}{(N-1)^2}$$

is defined as the "coherence factor."
Note that $\beta^2 = 1$ when all phase angles are equal, i.e., perfect coherence. The statistical properties of this "coherence factor" will be studied in a subsequent analysis. For the present assume that $\beta^2$ is known. By use of Eq. (14), Eq. (13) becomes

$$S_{s1}(t^n) = \sqrt{2P_{s1}} m \left( t^n - \frac{\Delta T_i}{2} \right) \left\{ \left( \frac{1}{k} \right) s \left( t^n - \frac{\Delta T_i}{2} \right) \cos [\omega t^n + \xi_i] + (N-1)\beta \cos [\omega t^n + \xi_{N-1}] \right\}. \tag{16}$$

Active satellites such as the IDCSP (Initial Defense Communication Satellite Project) are of primary concern here. The model usually used for these satellites is a bandpass amplifier followed by an ideal hard-limiter. These satellites have constant total output power regardless of the input signal power because the satellite is composed of a traveling wave amplifier that is saturated by the effective input thermal noise alone. Also if the input signal-to-noise ratio is greater than 10 dB, the output power of the satellite will essentially be all signal power.[20] The phased array under consideration will be designed so that the minimum signal power of $S_{s1}(t)$ is greater than 10 dB over the effective input noise to the satellite, i.e.,

$$\frac{\left[ \sqrt{2P_{s1}} ((N-1)\beta - 1/k) \right]^2}{2} = P_{s1} [(N-1)\beta - 1/k]^2 \geq 10 N_T \tag{17}$$
where \( N_i \) is the effective input noise power to the satellite.

Equation (17) is derived from Eq. (16) for the conditions \( \xi_i = \xi_{N-1} \) and \( S(t) = -1 \) or \( \xi_i = 180^\circ + \xi_{N-1} \) and \( S(t) = +1 \). Of course this assumes that there is sufficient phase-coherence among the signals at the satellite to produce this condition \( \beta^2 \geq 0.9 \) for efficient operation of the active array). This condition may not be the case during the acquisition phase of beam tagging for phase control, for example, offset errors may be large enough to produce phase cancellation. Consequently an acquisition mode will be necessary in the over-all phase control system, such as step-phase adjustments of individual elements to produce sufficient coherent power for the beam tagging control to function. This aspect of the system will not be considered here.

Thus, assuming that the signal power is sufficient to "capture" the satellite and the bandwidth of the satellite repeater is sufficiently wide to pass the signal with no phase distortion, the output signal of the satellite is

\[
S_{so}(t^n) = \sqrt{2P_s(s(out)} \cos \left[ \omega_R t^n + \xi_{N-1} + \gamma \right] + \phi \left( t^n - \frac{\Delta T_i}{2} \right) \]

where;

\( \omega_R = \) output carrier frequency of satellite,
\( \gamma = \) arbitrary phase shift introduced by the satellite,
\( P_{s(\text{out})} = \text{total output power of satellite} \)

(approximately 2 watts for IDCSP),

and

\[
\phi(t) = 2 \tan^{-1}\left[ \frac{S(t) \sin(\xi_i - \xi_{N-1})}{\mu + S(t) \cos(\xi_i - \xi_{N-1})} \right]
\]

where \( \mu \) is defined as

\[
\mu = k(N-1)\beta.
\]

Again assuming that the transmission medium simply attenuates the signal, the return signal at the array is

\[
S_R(t') = \sqrt{2P_R} \cos\left[ \omega_R t' - \frac{\Delta T_i}{2} + \frac{\Delta T}{2} + \phi(t' - \frac{\Delta T_i}{2} - \frac{\Delta T}{2})/2 \right]
\]

where,

\( P_R = \text{total power received.} \)

\( S_R(t) \) is the signal that the error-detection system (Beam-coding receiver) must process in order to obtain an estimate of \( (\xi_i - \xi_{N-1}) \) during the coding time \( T_T \). The function \( \phi(t) \) is a phase-modulation at the beam tagging frequency that contains the information on the phase-error \( (\xi_i - \xi_{N-1}) \). This phase-modulation will also be present in the data demodulation process at the particular receiving site.
for the data. Since $\mu$ will be adjusted to be at least 10 under normal operating conditions, the maximum phase swing that $\phi(t)$ produces is $\pm 5.7^\circ$ ($\pm \tan^{-1}1/10$) which occurs when $\xi_i - \xi_{N-1} = 90^\circ$.

Because the maximum phase swing is small and the fact that the tagging frequency is much less than the data rate, the beam tagging effect on the data reception operation should be insignificant.

To simplify the expression for the return signal one can arbitrarily set $t' = 0$ and make the following definitions:

\begin{align}
(22) & \quad t_1 = \frac{\Delta T_i}{2} + \frac{\Delta T}{2}, \\
(23) & \quad t_2 = t_1 + T_T, \\
(24) & \quad t'' = t.
\end{align}

This reduces the signal $S_R(t)$ to

\begin{align}
(25) & \quad S_R(t) = \sqrt{2P_R} m(t-t_1) \cos \left[ \omega_0 t - \frac{\Delta T \omega_R}{2} + \xi_{N-1} + \gamma \\
& \quad + \phi(t-t_1)/2 \right] \\
& \quad (t_1 \leq t \leq t_2)
\end{align}

after it has been down-converted to the RF frequency $\omega_0$. Further simplifications will be made through use of the following series of definitions.
Let

\[ \Delta \xi_i = \xi_i - \xi_{i-1} \]  

then

\[ \phi(t) = 2 \tan^{-1} \left[ \frac{S(t) \sin \Delta \xi_i}{\mu + S(t) \cos \Delta \xi_i} \right] \]

which is illustrated in Fig. 3. Also define

\[ f_1(\Delta \xi_i) = \tan^{-1} \left[ \frac{\sin \Delta \xi_i}{\mu + \cos \Delta \xi_i} \right] + \tan^{-1} \left[ \frac{\sin \Delta \xi_i}{\mu - \cos \Delta \xi_i} \right] \]

and

\[ f_2(\Delta \xi_i) = \tan^{-1} \left[ \frac{\sin \Delta \xi_i}{\mu + \cos \Delta \xi_i} \right] - \tan^{-1} \left[ \frac{\sin \Delta \xi_i}{\mu - \cos \Delta \xi_i} \right] \]

then

\[ \phi(t) = f_2(\Delta \xi_i) + f_1(\Delta \xi_i) S(t) \]

---

**Fig. 3**—Phase variation of the resultant signal due to the beam-tagging operation.
Finally let

\[ \Theta/2 = -\frac{\Delta T \omega_R}{2} + \xi_N - 1 + \gamma + f_2(\Delta \xi_1)/2. \]  

Through the use of the preceding definitions one arrives at

\[ S_R(t) = \sqrt{2P_R} m(t-t_1) \cos \left[ \omega_0 t + \Theta/2 + \frac{f_1(\Delta \xi_1)}{2} \right] S(t-t_1) \]

\[ t_1 \leq t \leq t_2. \]

The signal \( S_R(t) \) can also be written in the following form

\[ S_R(t) = a(t-t_1) \cos \omega_0 t - n(t-t_1) \sin \omega_0 t \]

where

\[ a(t) = \sqrt{2P_R} m(t) \left[ \left( \cos \frac{\Theta}{2} \right) \left( \cos \frac{f_1(\Delta \xi_1)}{2} \right) - \left( \sin \frac{\Theta}{2} \right) \left( \sin \frac{f_1(\Delta \xi_1)}{2} \right) S(t) \right] \]

and

\[ n(t) = \sqrt{2P_R} m(t) \left[ \left( \sin \frac{\Theta}{2} \right) \left( \cos \frac{f_1(\Delta \xi_1)}{2} \right) + \left( \cos \frac{\Theta}{2} \right) \left( \sin \frac{f_1(\Delta \xi_1)}{2} \right) S(t) \right]. \]

This form of \( S_R(t) \) will be useful in the subsequent noise analysis.

The sine and cosine of \( f_1(\Delta \xi_1) \) are two important functions and are easily shown to be

\[ \sin[f_1(\Delta \xi_1)] = \frac{2\mu \sin(\Delta \xi_1)}{\left(\mu^2 + 1\right)^2 - 4\mu^2 \cos^2(\Delta \xi_1)} \]
and

\[
\cos[f_1(\Delta \xi_i)] = \frac{\mu^2 - 1}{[(\mu^2 + 1)^2 - 4\mu^2 \cos^2(\Delta \xi_i)]^{\frac{1}{2}}}.
\]

These functions are easily derived by elementary trigonometric operations.

The function \(\sin[f(\Delta \xi_i)]\) is shown in Fig. 4 for various values of \(\mu\), along with the linear approximation

\[
\sin[f_1(\Delta \xi_i)] \approx \frac{2\Delta \xi_i}{\mu}.
\]

This approximation is very good for the range \(|\Delta \xi_i| \leq 30^\circ\), and \(\mu \geq 4\). The maximum absolute error in the linear approximation is 0.006 which occurs at \(\Delta \xi_i = 30^\circ\) and corresponds to about 4% error.

\(\cos f_1(\Delta \xi_i)\) is limited to the following range

\[
\frac{\mu^2 - 1}{\mu^2 + 1} \leq \cos[f_1(\Delta \xi_i)] \leq 1.
\]

The importance of these two functions will be apparent later in the analysis.

The proposed error detection system for the measurement of \(\Delta \xi_i\) is shown in block diagram form in Fig. 5. The purpose of the square-law device is to remove the data modulation \(m(t)\). The function of the other devices will be explained as the signal analysis progresses.
Fig. 4—Sin[\(f_1(\Delta \xi_1)\)] and its linear approximation.
In the signal analysis of the error detection system only steady-state conditions will be considered. The error introduced by neglecting the transient time will be small if

\[ T_T = pT_c \gg \tau_t \quad (p - \text{integer}) \]

where \( \tau_t \) is time required for the error detection system to reach a steady-state. Further remarks on this time will be forthcoming.

The bandwidth \( 2B_1 \) will be assumed sufficiently wide to pass \( S_R(t) \) with no distortion. A partial compensation for the filter effect will be to reduce the signal power by a factor \( r^2 \), i.e.,
\[ S_{R1}(t) = \sqrt{2PR} \cdot \text{ratio m}(t-t_1) \cdot \cos(\omega_0 t + \frac{\Delta \xi}{2} + \frac{f_1(\Delta \xi)}{2} \cdot S(t-t_1)) \]

where \( r^2 \) is the ratio of filtered signal power to \( P_R \) (to be evaluated later). Hence, \( S^2_{R1}(t) \) is

\[ S^2_{R1}(t) = P_R \cdot r^2 \cdot [1 + \cos(2\omega_0 t + \Theta + f_1(\Delta \xi) \cdot S(t-t_1))] \]

Notice, that the low-frequency component of \( S^2_{R1}(t) \) is proportional to the input power. This information can be used to produce an adaptive phase-control system as shown in the dashed portion of Fig. 5. The system will be adaptive in the sense that a phase control cycle, e.g., sequential phase correction of the individual elements will only be initiated if the signal power prescribed threshold. The relative merit of this adaptive feature will best be determined from experimental observations and therefore will not be pursued further here.

The second filter is a bandpass filter with the pass-band centered on \( 2\omega_0 \) and bandwidth \( 2B_2 \) sufficiently large to pass \( S(t) \) with little distortion. The output of the second bandpass filter is then

\[ S_{R2}(t) = r^2 \cdot P_R \cdot \cos(2\omega_0 t + \Theta + f_1(\Delta \xi) \cdot S(t-t_1)) \]

The phase-lock-demodulator down-converts \( S_{R2}(t) \) to baseband. The essential features of its operation are shown in Fig. 6. In Fig. 6, \( A_r \) is the amplitude of the reference signal and \( \hat{\Theta} \) is the estimate of the phase \( \Theta \) via operation of the phase-lock-loop.
$S_{R2}(t)$ \[\xrightarrow{\text{L.P. FILTER}}\] $S_{R3}(t)$

$\text{Ar Sin } [2\omega_0 t + \theta]$

Fig. 6--Operation of the phase-lock-demodulator.

$S_{R3}(t)$ is easily written as

$$S_{R3}(t) = -\frac{A_r P_R r^2}{2} \left[ \sin[\phi + f_1(\Delta \xi_i) S(t-t_1)] \right]$$

where,

$$\phi = \theta - \hat{\theta}$$

From Eq. (44) and the following equations

$$\cos[f_1(\Delta \xi_i) S(t-t_1)] \equiv \cos[f_1(\Delta \xi_i)]$$

and

$$\sin[f_1(\Delta \xi_i) S(t-t_1)] \equiv \sin[f_1(\Delta \xi_i)] S(t-t_1),$$

the signal $S_{R4}(t)$ is found to be

$$S_{R4}(t) = -\frac{A_r P_R r^2}{2} \left\{ \sin \phi \cos[f_1(\Delta \xi_i)] S(t-t_1) + \cos \phi \sin[f_1(\Delta \xi_i)] S(t-t_1) S(t-t_1) \right\}$$
where $\hat{t}_1$ is the time estimate of $t_1$ obtained either from ranging or prediction techniques.

Experience gained at the ElectroScience Laboratory's Satellite Communication Facility has indicated that the range of a satellite can be predicted with an accuracy of $\pm 1$ KM. Therefore,

\begin{equation}
|t_1 - \hat{t}_1|_{\text{max}} = \frac{2\Delta R}{c} = 0.00667 \text{ m sec}.
\end{equation}

where $\Delta R$ is the range inaccuracy and $c$ is the velocity of propagation. If $T_c > 1$ m sec ($f_c \leq \text{KHz}$), then

\begin{equation}
\left[ \frac{|t_1 - \hat{t}_1| \times 100}{T_c} \right]_{\text{max}} = 0.667\%.
\end{equation}

Since the time delay can be found to such a small percentage of $T_c$, one has

\begin{equation}
\frac{1}{p_{t_1}} \int_{t_1}^{t_1 + pT_c} S(t-t_1) S(t-\hat{t}_1) dt \approx 1
\end{equation}

and because

\begin{equation}
\int_{t_1}^{t_1 + pT_c} S(t-t_1) dt = 0
\end{equation}

the output signal is
From Eqs. (20) and (38), \( S_{RO}(t) \) is approximately

\[
S_{RO}(t) = \left[ -A_r \frac{r^2 p R \cos \phi}{k(N-1) \beta} \right] \Delta \xi_1 \quad \text{for} \quad |\Delta \xi_1| \leq 30^\circ.
\]

Since \( A_r \) is an arbitrary gain factor, it can be set to

\[
A_r = \frac{-k(N-1)}{r^2 P_{RE}}
\]

where \( P_{RE} \) is the designed value of the received power. Thus the output signal can be written as

\[
S_{RO}(t) = K_1 \left[ \frac{\cos \phi}{\beta} \right] \Delta \xi_1
\]

where \( K_1 = \frac{P_R}{P_{RE}} \).

The form of Eq. (56) will be useful when the over-all phase control stability and accuracy is investigated as the factor \( K_1 \frac{\cos \phi}{\beta} \) deviates from one. Notice that for perfect phase demodulation (\( \cos \phi = 1 \)), perfect phase coherence (\( \beta = 1 \)), and perfect error-detection system gain calibration (\( K_1 = 1 \), \( K_1 \frac{\cos \phi}{\beta} \) is one).

Under these conditions, the output signal has both the correct magnitude and sense for proper adjustment of the phase of the \( i \)th transmitted signal.
A brief discussion will now be presented about the relationship among the various parameters in the error detection system. Since the first filter is designed to pass \( m(t) \) with little distortion and the second filter to pass \( S(t) \) with little distortion and \( T_c \gg T_b \), one may conclude that

\[
B_1 \gg B_2 .
\]

The phase-lock-loop is designed to track \( \theta \) which varies at a rate much less than \( 1/T_c \). Thus

\[
B_L \ll 1/T_c \ll B_2
\]

\((B_L = \text{phase-lock-loop bandwidth})\).

Since \( B_L \) is the smallest bandwidth in the system, the phase-lock-loop will essentially determine the duration of the system transient response. Hence,

\[
B_L \gg 1/pT_c
\]

since the assumption was made earlier that the system transient time \( \tau_t \ll pT_c \). It should be pointed out that \( \theta \) will change abruptly at the start of a new phase estimation period (see Eqs. (29), (31)). This sudden change will be small (at most a few degrees) under normal operating conditions and therefore the phase-lock-loop will remain in the locked mode. Combining Eqs. (57), (58) and (59), the constraints on the system parameters are
In practice, a factor of 10 between inequalities is usually sufficient. These relationships are very important in the forthcoming noise analysis.

The next chapter will consider the operation of the phase detection receiver in the presence of noise. A determination of the accuracy of the estimation of the phase $\Delta \varepsilon_i$ will be made.
CHAPTER III
NOISE ANALYSIS OF ERROR DETECTION SYSTEM

Here the accuracy of the measurement of the phase error $\Delta \xi_1$ is investigated when the received signal is contaminated by additive noise. The principal emphasis is placed on the standard deviation of the estimate. This standard deviation is shown to be a function of the input signal-to-noise ratio, number of array elements, data bit length, and total tagging time. It is also shown that the measurement error is independent of phase error $\Delta \xi_1$ and the beam tagging modulation $S(t)$.

Later it will be shown that the absolute estimate $\hat{\Delta \xi}_1$ of $\Delta \xi_1$ depends on two factors. One factor is an unknown scale factor that is a function of: (1) the value of the parameter $\beta$ at the time of the estimate, (2) the phase error $\phi$ in the phase-demodulation process, and (3) the system gain calibration. The other factor is the additive error due to the input noise.

The analysis begins by assuming that the dominant input noise to the phase error detection system is generated in the pre-amplifier of the receiving antenna. This noise is usually assumed to be a sample function from a stationary, zero mean, Gaussian process with flat, two-sided spectral density $N_0/2$ (watts/Hz). Therefore, the output of the first filter (this chapter refers to Fig. 5 throughout) can be represented as

33
(61) \[ n_1(t) = x(t) \cos \omega_0 t - y(t) \sin \omega_0 t \quad (2B_1 \ll f_0) \]

where \( x(t) \) and \( y(t) \) are slowly varying (relative to \( \omega_0 \)) sample functions from independent, identically distributed, zero mean, Gaussian processes. It is shown in Appendix I that the output noise of the second filter is approximately Gaussian for \( B_1 \ll B_2 \).

The noise \( n_2(t) \) can be represented as

(62) \[ n_2(t) = X(t) \cos 2\omega_0 t - Y(t) \sin 2\omega_0 t \]

where \( X(t) \) and \( Y(t) \) are uncorrelated, zero mean processes with identical correlation functions. This representation includes both the interaction of the data signal with noise and noise with itself. This interaction is generated in the squaring operation.

The autocorrelation function of \( X(t) \) or \( Y(t) \) is as derived in Appendix I, Eq. (187),

(63) \[ R_X(t,t+\tau) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h^2_\chi(t-\mu) h^2_\chi(t+\tau-\rho) R_X(\rho-\mu) \]

\[ \cdot E \{ \alpha(\mu) \alpha(\rho) \} + E \{ \eta(\mu) \eta(\rho) \} + R_X(\rho-\mu) \] \[ d\mu \ d\rho \]

where

\( h^2_\chi(t) = \) low-pass equivalent impulse response of the second filter,

\( R_X(\tau) = \) autocorrelation function of \( x(t) \),
and

\( a(t) \) and \( n(t) \) are defined in Eqs. (34) and (35).

Appendix II shows that the output noise \( n_3(t) \) of the phase-lock-demodulator is also approximately a sample function from a Gaussian process when \( B_L << B_2 \). In addition, Appendix II shows that the autocorrelation function of \( n_3(t) \) is Eq. (63) multiplied by the constant \( A_r^2/4 \). This autocorrelation function will now be derived.

Starting with Eqs. (34) and (35), the expression \( a(\mu)a(\rho) + n(\mu)n(\rho) \) in Eq. (63) can be simplified to

\[
(64) \quad a(\mu)a(\rho) + n(\mu)n(\rho) = 2P_R m(\mu)m(\rho) \left[ \cos^2 \left( \frac{f_1(\Delta\xi)}{2} \right) + \sin^2 \left( \frac{f_1(\Delta\xi)}{2} \right) S(\mu) S(\rho) \right].
\]

Before proceeding further, a model for \( m(t) \) and \( S(t) \) must be selected. The model usually used for PSK data modulation is that \( m(t) \) can be represented as a sample function from a random binary train. Then the \( E\{m(\mu)m(\rho)\} \) is easily evaluated and the result is shown in Fig. 7. The function \( S(t) \) is a square-wave of value +1 and -1 and period \( T_c \). Since the time origin of \( S(t) \) is arbitrary, \( S(t) \) is assumed to be a sample function from an ensemble of square-waves with a random epoch that is uniformly distributed over the period \( T_c \). Under this condition, the \( E\{S(\mu)S(\rho)\} \) is also easily evaluated and the result is shown in Fig. 7 for the case
Fig. 7--Autocorrelation function of $m(t)$ (PSK data) and $S(t)$ (beam coding function).

$T_c = 40T_b$. It should be clear from Fig. 7 that when $T_c >> T_b$, the case in point here, that

$$E\{m(t) m(t')\} E\{S(t) S(t')\} \approx E\{m(t) m(t')\}.$$  

By use of Eq. (65), the expectation of the function given in Eq. (64) can be approximated by

$$E\{\alpha(t) \alpha(t') + \eta(t) \eta(t')\} \approx 2PR_m(\rho - \mu)$$

where $R_m(t)$ is the autocorrelation function of $m(t)$. Therefore Eq. (63) can be written as
With this engineering approximation, the noise power associated with
\( X(t) \) or \( n_3(t) \) is independent of \( S(t) \) and \( \Delta \xi_1 \). Since the noise
power is a second-order statistic and because \( n_3(t) \) is approximately
Gaussian one is then justified in assuming that \( n_3(t) \) is statistically
independent of \( \Delta \xi_1 \) and \( S(t) \). This point will be very useful in the
next chapter.

The power spectral density function of \( X(t) \) is now derived
using Eq. (67). This will then be used in evaluating the standard
deviation of the measurement error.

Making a change of variables, Eq. (67) can be written as

\[
(68) \quad R_X(\tau) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h_L(t-y) h_L(t+z) R(t+y-z) dy \, dz
\]

where

\[
(69) \quad R(\tau) = [2P_R R_m(\tau) + R_X(\tau)] R_X(\tau).
\]

The Fourier transform of Eq. (68) is therefore [22]

\[
(70) \quad S_X(\omega) = |H_L(\omega)|^2 S(\omega)
\]

where
$S_X(\omega) = \text{power spectral density function of } X(t)$,
$H_2(\omega) = \text{equivalent low-frequency response of second filter}$
and
\begin{equation}
S(\omega) = \int_{-\infty}^{+\infty} R(\tau) e^{-j\omega\tau} d\tau.
\end{equation}

The first and second filters are assumed ideal bandpass filters, i.e.,
\begin{equation}
H_1(j\omega) = p_{2\pi B_1}(\omega-\omega_0) e^{-j(\omega-\omega_0)t_1} + p_{2\pi B_1}(\omega+\omega_0)e^{-j(\omega+\omega_0)t_1}
\end{equation}
and
\begin{equation}
H_2(j\omega) = p_{2\pi B_2}(\omega-2\omega_0) e^{-j(\omega-2\omega_0)t_2} + p_{2\pi B_2}(\omega+2\omega_0)e^{-j(\omega+2\omega_0)t_2}
\end{equation}
where
\begin{equation}
p_{2\pi B}(\omega) = \begin{cases} 1 & \text{for } |\omega| \leq 2\pi B \\ 0 & \text{for } |\omega| > 2\pi B \end{cases}
\end{equation}
The magnitude of the various filter transfer functions are illustrated in Fig. 8.

The power spectral density function $S(\omega)$ can be written as
\begin{equation}
S(\omega) = S_{x.x}(\omega) + S_{m.x}(\omega)
\end{equation}
Fig. 8—Ideal bandpass characteristics of the first and second filter.
where

\begin{equation}
S_{\text{x,x}}(\omega) = \int_{-\infty}^{+\infty} R_x(\tau) R_x(\tau) e^{-j\omega \tau} \, d\tau
\end{equation}

and

\begin{equation}
S_{\text{x,m}}(\omega) = 2P_R \int_{-\infty}^{+\infty} R_m(\tau) R_x(\tau) e^{-j\omega \tau} \, d\tau .
\end{equation}

$S_{\text{x,x}}(\omega)$ represents the interaction of noise with itself and $S_{\text{x,m}}(\omega)$ represents the interaction of noise with $m(t)$.

Using the convolution theorem[23] and the ideal filter assumption, Eq. (76) reduces to

\begin{equation}
S_{\text{x,x}}(\omega) = N_0^2 \frac{1}{2\pi} \int_{-\infty}^{+\infty} P_{2\pi B_1}(y) P_{2\pi B_1}(\omega-y) \, dy
\end{equation}

which is illustrated in Fig. 9.
Similar procedure gives

\begin{equation}
S_{x \cdot m}(\omega) = \frac{P_R N_0}{\pi} \int_{\omega-2\pi B_1}^{\omega+2\pi B_1} \frac{4 \sin^2 \left[ y T_b/2 \right]}{y^2} \, dy = \\
\frac{T_b}{2} \left( \omega + 2\pi B_1 \right) \frac{\sin^2 x}{x^2} \, dx.
\end{equation}

Equation (79) was evaluated numerically and the results are shown in Fig. 10 for \( \omega > 0 \) and various values of the product \( T_b B_1 \).

Figures 9 and 10 show that \( S(\omega) \) is essentially flat for \( |\omega| \leq 0.1 \) \((2\pi B_1)\), i.e.,

\begin{equation}
S(\omega) \approx 2B_1 N_0^2 + 2P_R N_0 r^2 \quad (|\omega| \leq 0.1 \cdot 2\pi B_1)
\end{equation}

where

\begin{equation}
r^2 = \frac{2}{\pi} \int_{0}^{\pi T_b B_1} \frac{\sin^2 x}{x^2} \, dx.
\end{equation}

Using Eq. (80), Eq. (70) is evaluated approximately as

\begin{equation}
S_X(\omega) = \begin{cases} 
2B_1 N_0^2 + 2P_R N_0 r^2 & |\omega| \leq 2\pi B_2 \\
0 & |\omega| > 2\pi B_2 
\end{cases} 
\quad (B_2 \leq 0.1B_1).
\end{equation}

Use will be made of Eq. (82) when the standard deviation of the output noise is derived which is the standard deviation of the measurement error.
The output noise of the error detection system is

$$\Delta = \frac{1}{pT_C} \int_{\hat{t}_1}^{\hat{t}_1+P_T} n_3(t) S(t-\hat{t}_1) dt = \frac{1}{pT_C} \int_0^{P_T} n_3(y+\hat{t}_1) S(y) dy.$$  

Since $\Delta$ is a linear functional of $n_3(t)$, $\Delta$ is a Gaussian random variable. [24] The noise $\Delta$ has zero mean since $n_3(t)$ has zero mean and its variance can be written as
\[ E(\Delta^2) = \frac{1}{(pT_c)^2} \int_0^{pT_c} \int_0^{pT_c} S(x)S(y) R_{n3}(x-y) \, dx \, dy = \sigma^2 \]

where \( R_{n3}(\tau) \) is the autocorrelation function of \( n_3(t) \). The double integration in Eq. (84) can be written as

\[ \int_0^{pT_c} \int_0^{pT_c} S(x)S(y) R_{n3}(x-y) \, dx \, dy = \]

\[ \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{n3}(\omega) \, d\omega \int_0^{pT_c} \int_0^{pT_c} S(x)S(y)e^{j\omega(x-y)} \, dx \, dy \]

where

\[ R_{n3}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{n3}(\omega)e^{j\omega\tau} \, d\omega \]

It can be easily shown that

\[ \int_0^{pT_c} \int_0^{pT_c} S(x)S(y)e^{j\omega(x-y)} \, dx \, dy = \frac{16 \sin^4 \left( \frac{\omega T_c}{4} \right)}{\omega^2} \left[ \frac{\sin \frac{p\omega T_c}{2}}{\sin \frac{\omega T_c}{2}} \right]^2 \]

Using Eq. (87) and the value of \( S_{n3}(\omega) \) (Eq. (82) multiplied by \( A_r^2 / 4 \)), Eq. (85) reduces to
\[ S(x)S(y) R_n^3 (x-y) dx \, dy = \]
\[ \frac{A_r^2}{4} \left[ \frac{2B_1N_0^2 + 2P_Nr^2}{1} \right] \frac{16}{2\pi} \int_{-2\pi B_2}^{2\pi B_2} \frac{\sin 4 \frac{\omega T_c}{2}}{2 \sin \frac{\omega T_c}{2}} \left( \frac{\sin \frac{p\omega T_c}{2}}{\sin \frac{\omega T_c}{2}} \right)^2 d\omega \]
\[ = \frac{A_r^2}{4} \left[ \frac{2B_1N_0^2 + 2P_Nr^2}{1} \right] \frac{4T_c}{\pi} \int_{-\pi T_c B_2}^{\pi T_c B_2} \frac{\sin 4 \left( \frac{x}{2} \right)}{x^2} \left( \frac{\sin px}{\sin x} \right)^2 dx \]

where the change of variables \( \frac{\omega T_c}{2} = x \) has been made in the last step. Since the product \( T_cB_2 \) has been assumed large, at least 10, the limits of the last integral can be replaced by +\( \infty \) and -\( \infty \) with negligible error. The results of Appendix III can be used to simplify Eq. (88) to

\[ \oint_{pT_c} S(x)S(y) R_n^3 (x-y) dx \, dy = \frac{A_r^2}{4} \left[ pT_c \left( 2B_1N_0^2 + 2P_Nr^2 \right) \right] . \]

Substituting Eq. (55) and Eq. (89) into Eq. (84), one finally has

\[ \sigma_\Delta^2 = \frac{k^2 k^2 (N-1)^2}{4 \pi^2} \left[ \frac{2B_1N_0^2 + 2P_Nr^2}{pT_c} \right] . \]
The normalized standard deviation of the phase measurement is defined as

\[ \sigma_{\Delta_1} = \frac{\sigma_{\Delta}}{K_1 k(N-1)} = \frac{1}{\sqrt{2\pi}} \left( 1 + \frac{k_1 \left( \frac{pT_c}{T_B} \right)}{r^2 \left[ \frac{P_{R\Delta} T_c}{N_0} \right]} \right)^{-\frac{1}{2}} \]

where \( \Delta_1 \) is the normalized output noise and \( r^2 \) and \( k_1 \) have paired values. A few of these values are given in Table I as evaluated from Eq. (81). Notice that \( \sigma_{\Delta_1} \) is a function of the ratio of the energy received in the period \( pT_c \) \( (P_{R\Delta} T_c) \) to the noise density \( N_0 \). A graph of \( \sigma_{\Delta_1} \) is presented in Fig. 11. One now has the standard deviation of the measurement error as a function of all the significant parameters.

**TABLE I**

Ratio of filtered signal power to input power, i.e., \( r^2 \) for different values of \( k_1 = B T_B \)

<table>
<thead>
<tr>
<th>( k_1 )</th>
<th>( r^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.902</td>
</tr>
<tr>
<td>2</td>
<td>0.950</td>
</tr>
<tr>
<td>3</td>
<td>0.965</td>
</tr>
<tr>
<td>4</td>
<td>0.975</td>
</tr>
<tr>
<td>5</td>
<td>0.980</td>
</tr>
</tbody>
</table>

The total output signal \( S_{R\Delta}(t) + \Delta \) of the beam tagging receiver represents the estimate \( \hat{\Delta_1} \) of the phase \( \Delta_1 \). This estimate is given in Eq. (92),
Fig. 11—Normalized standard deviation of the phase error measurement.
\begin{align}
\hat{\Delta} \xi_i &= S_{Ro}(t) + \Delta = K_1 \left( \frac{\cos \phi}{\beta} \right) \Delta \xi_i + K_1 k(N-1) \Delta_1 \\
&= K_1 \left[ \Delta \xi_i + \left( \frac{\cos \phi}{\beta} - 1 \right) \Delta \xi_i + k(N-1) \Delta_1 \right].
\end{align}

A brief review of the various terms in this estimate will now be given.

\(K_1\) is a number whose value is one when the received power \(P_R\) is equal to the value that the system is designed for. In practice, if input power variations are too severe, a hard-limiter would be employed in the error detection system, such as in the phase-lock-loop, and thus forcing \(K_1\) to be a constant independent of \(P_R\). It is felt that this modification to the system would only serve to complicate the mathematics here and would not significantly change the results.

The value of \(\cos \phi\) depends on how well the double-frequency carrier phase \(\phi\) can be tracked by the phase-lock-loop. By employing the linear model for the phase-lock-loop, (25) the standard deviation of \(\phi\) is easily derived from previous results and is

\begin{align}
(93) \quad \sigma_\phi &= \left[ \frac{\text{Noise power in bandwidth } 2B_L \text{ about } 2\omega_0}{\text{carrier power at } 2\omega_0} \right]^{\frac{1}{2}} \\
&= \left[ \frac{2B_L [2B_1 N_0^2 + 2P_R N_0 r^2]}{(r^2 P_R/2) \cos f_1(\Delta \xi_i)} \right]^{\frac{1}{2}} \\
&= \frac{2 \sqrt{B_L [2B_1 N_0^2 + 2P_R N_0 r^2]}}{\cos f_1(\Delta \xi_i) r^2 P_R} \frac{1}{\frac{1}{2}}.
\end{align}
This result is derived by using Eqs. (43) and (82). It has been shown previously in Eq. (39) that

\[
\frac{(k(N-1)N)^2 - 1}{(k(N-1)N)^2 + 1} \leq \cos f_1(\Delta \xi_i) \leq 1.
\]

As was mentioned previously, \(k(N-1)\) is normally adjusted to be at least 10, then

\[
\frac{100B^2 - 1}{100B^2 + 1} \leq \cos f_1(\Delta \xi_i) \leq 1.
\]

Let \(B^2 = 1/2\) which is far below any reasonable value, then

\[
\frac{49}{51} \leq \cos f_1(\Delta \xi_i) \leq 1.
\]

Thus from a practical standpoint, \(\cos f_1(\Delta \xi_i)\) can be assumed to be equal to one. With this approximation, the tracking ability of the loop is also independent of \(\Delta \xi_i\). Therefore,

\[
\sigma_\phi = \frac{2\sqrt{B_L[2B_1N_0^2 + 2P_NR_rN_r]}^{1/2}}{r_2^2p_R} = 4\sqrt{B_Lp_Tc} \sigma_{\Delta \xi_i}
\]

where Eq. (91) was used in the last step above.

The factor \(B\) in the estimate of \(\Delta \xi_i\) depends on the degree of coherence of the N-1 signals at the time of the estimate. More will be said about this important point in the next chapter.

\(\Delta\) is the error in the estimate of \(\Delta \xi_i\) due to the input noise \(n_i(t)\). It has been shown to be independent of the phase error \(\Delta \xi_i\).
Some appropriate questions should now be asked; namely, "What is an acceptable upper limit to the standard deviation of the phase measurement for adequate phase control operation?" and "Will the variations in the scale factor \( K_1 \cos \phi/\beta \) be detrimental to the phase control operation?" The next chapter will answer these equations when a closed-loop system is developed that adequately represents the physical situation, i.e., the phase control for achievement of the adaptive array concept.
In order to answer the questions posed at the end of Chapter III, a specific closed-loop phase control system must be postulated. It will be assumed that a constant multiple of the phase estimate $\Delta \hat{\xi}_i$ is used to correct the phase of the $i^{th}$ transmitted signal. The approach taken here is to examine the phase control operation of the $i^{th}$ phase $\xi_i(t)$ as a function of time with respect to the phase $\xi_{N-1}(t)$. The latter phase is the phase associated with the resultant signal due to the remaining N-1 transmitting elements and herein referred to as the reference phase.

The closed-loop operation of the $i^{th}$ phase $\xi_i(t)$ is shown in block diagram form in Fig. 12. It should be emphasized that this is strictly a mathematical representation of the phase control operation. The only quantity in this model that is available in the physical sense is the phase estimate $\Delta \hat{\xi}_i$. The function $\Delta \xi_i(t)$ at the satellite represents the controlled phase difference between the phase $\xi_i(t)$ and the reference phase $\xi_{N-1}(t)$. The time delay blocks take into account the up-link and down-link propagation times. The phase error detection system that provides the phase control information is represented by its equivalent operation as defined by Eq. (92). Recall that Eq. (92) represents the form of the phase estimate $\Delta \hat{\xi}_i$ of the phase $\Delta \xi_i$. This estimate is obtained every $T_d$
seconds and applied after multiplied by the constant $K_2$ as a phase correction to the $i^{th}$ transmitted signal. This correction will abruptly change the phase $\Delta \xi_i(t)$ at the satellite after the up-link time delay.

$$A = K_1K_2$$
$$C = \left( \frac{\cos \phi - 1}{\beta} \right)$$
$$T_d = \text{TIME BETWEEN CORRECTIONS}$$

Fig. 12--Block diagram of the closed-loop phase control operation.

It will be seen that the function $\Delta \xi_i(t)$ can be interpreted as a sample function from a stochastic process. The mean and standard deviation of this process are derived. These statistics give a measure of any one signal's phase alignment with the phase associated with the sum of the remaining signals. Relationships between these statistics and those of the individual phases as defined by Eq. (12) are then developed. Finally, the mean and standard deviation of the coherence factor are studied as a function of the
statistics of the individual phases. One may then specify the system parameters to achieve a desired level of performance and answer the question at the end of Chapter III.

In control system terminology, the closed-loop system represented by Fig. 12 is a sample-data feedback control system with the added complications of time delays and variable gain. The variable gain arises from the factors discussed in the last chapter. Actually, $C$ as defined in Fig. 12 is a random variable which further complicates the problem. To simplify the problem, the approach used here is to assume that the gain is constant and later study the effects of changing gain for some extreme cases. This approach will be justified by calculations indicating that this approximation represents a reasonable engineering lower bound for system performance.

Before any calculations on the statistics of $\Delta \xi_1(t)$ can be made, one must specify the form of the input to the sample-data control loop. In tracking a moving satellite it has been observed that the phase difference between two elements of an array is actually very close to being a ramp signal.\[26\] This will be true for a length of time equal to many phase cycle corrections. Thus, it is natural to assume that the phase angle $\Delta \xi_1(t)$ varies linearly between corrections, i.e.,

\[
(98) \quad \xi_{N-1}(t) = \Delta f t + \xi_0
\]
where,
\[ \xi_0 = \text{arbitrary phase angle}, \]
and
\[ \Delta f = \text{rate of change of phase}. \]

Appendix IV gives practical upper-bounds for \( \Delta f \) for different types of satellites and with or without open-loop control of the array (retrodirective or computer control). These values of \( \Delta f \) will be used in the next chapter when some design examples are given.

We are now ready to derive the mean and standard deviation of \( \Delta \xi_i(t) \) which can be thought of as a random process. Suppose at time \( t = T_d \), a phase error estimation is made. Recall that the tagging time per-element \( \tau \) is selected such that any phase changes during this time is small, i.e.,
\[ (99) \quad \Delta f \tau < 1^\circ \text{ or } 2^\circ. \]

Consequently,
\[ (100) \quad \Delta \xi_i(T_d) = K_1 \left[ (1+C) \Delta \xi_i(T_d - \frac{\Delta T}{2}) + k(N-1)\Delta_{11} \right] \]

where \( \Delta_{11} \) is defined as \( \Delta_1(T_d) \). After multiplied by \( K_2 \), this estimate is used to correct the phase of the \( i \)th transmitted signal. At time \( t = T_d + \Delta T \), the value of \( \Delta \xi_i(t) \) at the satellite is
\[ (101) \quad \Delta \xi_i(T_d + \frac{\Delta T}{2}) = \Delta f \Delta T + \Delta \xi_i(T_d - \frac{\Delta T}{2}) - K_2 \Delta \xi_i(T_d) \]
\[ = \Delta f \Delta T + \Delta \xi_i(T_d - \frac{\Delta T}{2}) \left[ 1 - A(1+C) \right] \]
\[ - Ak(N-1) \Delta_{11} \]
where $A = K_1 K_2$. At time $t = 2T_d$, the system again produces an estimate, i.e.,

$$
\Delta \hat{\xi}_i(2T_d) = K_1 \left[ (1+C) \Delta \xi_i \left( 2T_d - \frac{\Delta T}{2} \right) + k(N-1) \Delta_{12} \right]
$$

where $\Delta_{12} = \Delta_1(2T_d)$. As before, at $t = 2T_d + \Delta T/2$

$$
\Delta \xi_i \left( 2T_d + \frac{\Delta T}{2} \right) = \Delta f \Delta T + \Delta \xi_i \left( 2T_d - \frac{\Delta T}{2} \right) - K_2 \Delta \hat{\xi}_i(2T_d).
$$

Substituting Eqs. (102), (101) and (98) into Eq. (103), the result is

$$
\Delta \xi_i \left( 2T_d + \frac{\Delta T}{2} \right) = \Delta f \Delta T + \left[ 1 - A(1+C) \right]^2 \Delta \xi_i \left( T_d - \frac{\Delta T}{2} \right)
$$

$$
+ \left[ 1 - A(1+C) \right] \Delta f T_d
$$

$$
- k(N-1) A \left[ (1-A(1+C)) \Delta_{11} + \Delta_{12} \right]
$$

Proceeding in the same manner as above, it can be shown that

$$
\Delta \xi_i \left( \frac{\Delta T}{2} \right) = \Delta f \Delta T + \left[ 1 - A(1+C) \right]^2 \Delta \xi_i \left( T_d - \frac{\Delta T}{2} \right)
$$

$$
+ \Delta f T_d \sum_{p=1}^{p-1} \left[ 1 - A(1+C) \right]^p
$$

$$
- Ak(N-1) \sum_{p=0}^{p-1} \left[ 1 - A(1+C) \right]^p \Delta_1(\epsilon-p)
$$

($\epsilon = 1, 2, 3 \ldots$)

A typical sample function for $\Delta \xi_i(t)$ is shown in Fig. 13.
As an intermediate step, we now calculate the mean value and the mean square value of $\Delta \xi_i(t)$ after the $l^{th}$ correction. Recalling that

\begin{equation}
E\{\Delta_{1p}\} = 0 \quad (p = 1, 2, 3 \ldots \infty)
\end{equation}

one has

\begin{equation}
E\left\{\Delta \xi_i \left( T_d + \frac{\Delta T}{2} \right) \right\} = \Delta f \Delta T + \left[ 1 - A(1+C) \right] \Delta \xi_i \left( T_d - \frac{\Delta T}{2} \right)
\end{equation}

\[ + \Delta f T_d \sum_{p=1}^{p-1} \left[ 1 - A(1+C) \right]^p \]
To derive the mean-square value of $\Delta \xi_i(t)$ after the $\ell$th correction, certain other facts must be recalled. Since $T_d$ will be much larger than the correlation time of the noise, one has that

\begin{equation}
E\{\Delta_{1j}\Delta_{1k}\} = 0 \quad (j\neq k)
\end{equation}

also the statistics of $\Delta_1$ are stationary, i.e.,

\begin{equation}
E\{\Delta_{1j}\} = E\{\Delta_{1k}\} = \sigma_{\Delta_1}^2.
\end{equation}

Using Eqs. (106), (108), (109) and the fact that $\Delta_1$ and $\Delta \xi_i$ have been shown to be independent, one can immediately write

\begin{equation}
E\left\{ \Delta \xi_i \left( \ell T_d + \frac{\Delta T}{2} \right) \right\} =
\left( \Delta f \Delta T + (1-A(1+C)) \right)^{\frac{\ell}{2}} \Delta \xi_i \left( T_d + \frac{\Delta T}{2} \right) + \Delta f T_d \sum_{p=1}^{\frac{\ell}{2}-1} \left[ 1 - A(1+C) \right]^p \sigma_{\Delta_1}^2
+ A^2k^2(N-1)^2 \sum_{p=0}^{\frac{\ell}{2}-1} [1 - A(1+C)]^{2p} \sigma_{\Delta_1}^2.
\end{equation}

Steady-state statistics are defined in terms of a limiting process, that is,

\begin{equation}
\lim_{\ell \to \infty} E\{\Delta \xi_i (\ell T_d + \frac{\Delta T}{2})\} =
\Delta f \Delta T + \Delta f T_d \left[ \frac{1 - A(1+C)}{A(1+C)} \right]
\end{equation}

and
Equations (111) and (112) were derived using standard geometric series. Note that the gain for stable operation is restricted to the range given by Eq. (113).

Figures 14, 15 and 16 illustrate a few of the key series and terms in Eqs. (107) and (110) as \( \varepsilon \) is increased for various values of the gain \( A(1+C) \). It is apparent from these results that for

\[
(114) \quad 0.5 < A(1+C) < 1.5
\]

steady state conditions are practically reached after four or five corrections. Values for the gain outside this range will give unsatisfactory results.

One now has the necessary preliminaries required to calculate the mean and standard deviation of the process \( \Delta \varepsilon_i(t) \). This process is defined in terms of the sample function shown in Fig. 13. Additional sample functions are obtained by changing the time at which the abrupt changes occur. This time will be assumed to be a random variable with a uniform p.d.f. over the time \( T_d \). This assumption is realistic because the absolute time of a particular correction is arbitrary. Thus by using Eq. (111), the mean value of
Fig. 14—The value of the series
\[ \sum_{p=1}^{\ell-1} \left( 1 - A(1+C) \right)^p \] vs \( \ell \).

Fig. 15—The term \( [1-A(1+C)]^\ell \) vs \( \ell \).

Fig. 16—The value of the series
\[ \sum_{p=1}^{\ell-1} \left( 1 - A(1+C) \right)^p \] vs \( \ell \).
the process is

\[ E\{\Delta \xi_i(t)\} = \lim_{\lambda \to \infty} E \left\{ \frac{1}{T_d} \int_0^T \left[ \Delta \xi_i \left( \lambda T_d + \frac{\Delta T}{2} \right) + \Delta f t \right] dt \right\} \]

\[ = \Delta f \Delta T + \Delta f T_d \left[ \frac{2 - A(1+C)}{2A(1+C)} \right] . \]

The time average occurs in Eq. (115) because of the uniform distribution of the correction time. The mean-square value of the process can be written as

\[ E\{\Delta^2 \xi_i(t)\} = \lim_{\lambda \to \infty} E \left\{ \frac{1}{T_d} \int_0^T \left[ \Delta \xi_i \left( \lambda T_d + \frac{\Delta T}{2} \right) + \Delta f t \right]^2 dt \right\} \]

\[ = \lim_{\lambda \to \infty} \left\{ \Delta \xi_i \left( \lambda T_d + \frac{\Delta T}{2} \right) \right\}^2 + \Delta f T_d \lim_{\lambda \to \infty} E \left\{ \Delta \xi_i \left( \lambda T_d + \frac{\Delta T}{2} \right) \right\} + \frac{(\Delta f T_d)^2}{3} . \]

Using Eqs. (111), (112), (115) and (116) and after a little algebra, one obtains the variance of \( \Delta \xi_i(t) \)

\[ \sigma_{\Delta \xi_i}^2 = E\{\Delta^2 \xi_i(t)\} - [E\{\Delta \xi_i(t)\}]^2 = \]

\[ \frac{(\Delta f T_d)^2}{12} + \frac{A^2 k^2 (N-1)^2 \sigma_{\Delta_1}^2}{A(1+C) [2-A(1+C)]} . \]

Since there is nothing unique about the \( i \)th element, the statistics represented in Eqs. (115) and (117) are a measure of the degree of phase alignment between any one signal and the sum of the rest of the signals. It is natural to expect that there are relationships between these statistics and the statistics of
the individual phases as defined in Eq. (12). These relationships will now be derived.

To begin with consider N phasors with the following properties. Let the phase angle associated with the k^{th} phasor be

\[ \xi_k = \xi_k + \gamma_k \quad (k = 1, 2, 3 \ldots N) \]

where:

- \( \xi_k \) = average phase offset of k^{th} phasor from arbitrary zero degree reference,
- \( \gamma_k \) = random component of k^{th} phasor.

A typical phasor is shown in Fig. 17. Because of the way in which the problem is formulated there is no loss of generality in assuming that the \( \gamma_k \)'s have zero means. In the physical situation, the random phase components are produced by inaccurate, independent measurements and corrections. Therefore, they will be considered statistically independent. Since the same phase estimator as computed in the beam-tagging receiver is used to align each signal phase with the

![Diagram of a typical phasor](image)

**Fig. 17--Typical phasor**
phase of the sum of the rest of the signals, there is reason to believe that the $\gamma_k$'s ($k = 1, 2, \cdots N$) will have similar statistical properties. In addition it is likely that a measurement error in either direction will be obtained with equal frequency. This leads one to assume that the $\gamma_k$'s should have a symmetrical probability density function, with respect to zero degrees reference. In summary, the $\gamma_k$'s ($k = 1, 2, \cdots N$) will be assumed to be independent random variables each with zero mean and the same probability density function which is symmetrical with respect to the zero degree reference.

Now select any one of the phasors, for example the $i^{th}$ one, and define

$$\Delta \xi_i = \xi_i - \xi_{N-1}$$

where $\xi_{N-1}$ is the phase angle associated with the resultant phasor which is obtained by adding the remaining $N-1$ phasors. The phase angle $\xi_{N-1}$ is given in general by

$$\tan \xi_{N-1} = \frac{\sum' \sin \xi_p}{\sum' \cos \xi_p}$$

where the $\sum'$ means omit $i$ in the summation. This expression is too complicated to be of any practical use here. But in the steady-state behavior of the array the phase angles are small and therefore Eq. (120) can be replaced by the more useful approximate form.
where the small angle trigonometrical approximations have been used.

Hence, Eq. (119) can be written as

\[
\Delta \xi_i = \xi_i - \frac{\sum_{p=1}^{N'} \xi_p}{(N-1)}
\]

Using the assumed statistical properties of the phase angles, one has that

\[
E(\Delta \xi_i) = \overline{\xi}_i - \frac{\sum_{p=1}^{N'} \overline{\xi}_p}{N-1} = \left[ \frac{N}{N-1} \right] \overline{\xi}_i
\]

because

\[
\sum_{p=1}^{N} \overline{\xi}_p = 0
\]

which is implied by the zero degree phase reference. It should be emphasized that this reference is arbitrary and its choice does not influence the results. By using the statistical properties of the \( \gamma_k \)'s, it can be shown that

\[
E(\Delta \xi_i^2) = \left[ \overline{\xi}_i - \frac{\sum_{p=1}^{N'} \overline{\xi}_p}{N-1} \right]^2 + E\left\{ \left[ \gamma_i - \frac{\sum_{p=1}^{N'} \gamma_p}{N-1} \right]^2 \right\}
\]

\[
= \left[ \overline{\xi}_i - \frac{\sum_{p=1}^{N'} \overline{\xi}_p}{N-1} \right]^2 + \left( \frac{N}{N-1} \right) \sigma_y^2
\]
where

\[(126) \quad E(\gamma_p^2) = \sigma_{\gamma}^2 \quad (p = 1, 2, \ldots, N).\]

Consequently,

\[(127) \quad \sigma^2_{\Delta \xi_i} = \frac{N}{N-1} \sigma_{\gamma}^2 .\]

Equations (123) and (127) are the two important relationships that can be used to relate the previously derived statistics concerning the closed-loop model and the statistics of the individual phases.

The foregoing results can now be incorporated into a design criteria to evaluate system performance. The coherence factor is selected as an important design criteria because it is a measure of the power of the resultant signal at the satellite. This coherence factor for \(N\) signals is

\[(128) \quad \beta^2 = \frac{\sum_{k=1}^{N} \sum_{p=1}^{N} e^{j(\xi_k - \xi_p)}}{N^2} .\]

One is usually interested in the average power and the variability of the power about the average. The mean and standard deviation of \(\beta^2\) are used to evaluate these quantities. The average of \(\beta^2\) is

\[(129) \quad E(\beta^2) = E\left\{ \frac{\sum_{k=1}^{N} \sum_{p=1}^{N} e^{j(\xi_k - \xi_p)} e^{j(\gamma_k - \gamma_p)}}{N^2} \right\} = \frac{1}{N} + \frac{\left[ E\{\cos \gamma\}\right]^2 \sum_{k=1}^{N} \sum_{p=1}^{N} e^{j(\xi_k - \xi_p)}}{N^2} .\]
because the assumed properties of the phases permits one to write

\[ E\{e^{j(\gamma_k - \gamma_p)}\} = E\{e^{j\gamma_k}\} E\{e^{-j\gamma_p}\} = [E\{\cos \gamma\}]^2 (k \neq p) \]

where \( \sum' \) means \( k \neq p \). The double summation in Eq. (129) can be written as

\[ \sum_{k=1}^{N} \sum_{p=1}^{N'} \cos(\xi_k - \xi_p) + j \sin(\xi_k - \xi_p) \]

\[ = \sum_{k=1}^{N} \sum_{p=1}^{N'} \cos(\xi_k - \xi_p) \]

since the imaginary component sums to zero. It is obvious that

\[ \sum_{k=1}^{N} \sum_{p=1}^{N'} \cos(\xi_k - \xi_p) \geq (N^2 - N) \cos(2\xi_{\text{max}}) \]

where \( \xi_{\text{max}} \) is the maximum offset. Although this expression is valid, it is quite pessimistic. A better lower bound can be obtained by observing that approximately one-half of the terms in the summation are such that

\[ \cos(\xi_k - \xi_p) \approx 1 \]

because of the reference choice. Hence

\[ \sum_{k=1}^{N} \sum_{p=1}^{N'} \cos(\xi_k - \xi_p) \geq \frac{N^2 - N}{2} \left[ 1 + \cos(2\xi_{\text{max}}) \right] \]

and using Eq. (134), a lower bound to the average coherence is
This result can be written as

\[
E\{\theta^2\} \geq \left[ \frac{1}{N} + \frac{[E\{\cos \gamma\}]^2}{2N^2} (N^2-N) \right] (1 + \cos 2 \bar{\tau}_{\max}) \geq \left[ \frac{1}{N} + \frac{(N^2-N) E\{\cos \gamma\}^2}{N^2} \right] \left[ \frac{1 + \cos 2 \bar{\tau}_{\max}}{2} \right].
\]

which delineates the effect of the random phase components and offset on the average coherence. For the no offset case, Appendix V evaluates the mean and standard deviation of \( \theta^2 \) and other useful statistics as a function of the standard deviation of the random phase component \( \gamma \). The details are not included here because they are the results of standard statistical techniques. The results of Appendix V will be used extensively to evaluate system performance. A word of caution is appropriate here. The coherence is defined in Appendix V in terms of the addition of \( N-1 \) signals as it first appeared in the (Eq. (15)).

When we combine Eqs. (115) and (123), the important result is

\[
\left[ \frac{N}{N-1} \right] \bar{\tau}_{\max} = \Delta f \Delta T + \Delta f T_d \left[ \frac{2 - A(1+C)}{2A(1+C)} \right]
\]

where \( \Delta f \) is now the maximum rate of change of phase. As was stated previously, the maximum rate of change of phase is given in Appendix IV for various applications and with or without open-loop control.
of the array. Also, combining Eqs. (117) and (127), an equally important result is

$$\sigma^2 = \frac{(\Delta f T_d)^2}{12} + \frac{A^2 k^2 (N-1)^2 \sigma_{\Delta_1}^2}{A(1+C)(2 - A(1+C))} \tag{138}$$

Eqs. (136), (137) and (138) are the key expressions that are used to determine phase control accuracy. They will also be used to specify the system parameters necessary to obtain a desired level of performance.

The standard deviation of the phase (Eq. (138)) is shown in Figs. 18 and 19 as a function of the loop gain ($A$) for constant measurement accuracy ($k(N-1)\sigma_{\Delta_1}$) and maximum phase change between corrections ($\Delta f T_d$). The lower bound to the average coherence vs loop gain (combining Eqs. (136), (137), (138) and the results in Appendix V) is illustrated in Figs. 20, 21, 22 and 23 with the same parameters as above held fixed. Before discussing the significance of these results a brief justification is necessary concerning the particular values of $C$ which were selected.

As was mentioned at the beginning of this chapter we are interested in studying the system behavior for extreme cases. Thus if one can show that the system will perform adequately under adverse conditions, he can be assured of accurate, stable phase control operation under more normal conditions. The following heuristic argument will show that the probability of $C$ being outside the range $-0.2 \leq C \leq 0.2$ is very small. Repeating the definition of $C$ here for convenience,
(139) \[ C = \left[ \frac{\cos \phi - 1}{\beta} \right] \]

where \( \cos \phi \) and \( \beta \) are both restricted to the range \( 0 \leq \cos \phi \), \( \beta \leq 1 \). For normal operation of the P.L.L., the probability that \( |\phi| > 90^\circ \) is practically zero. [27] Hence, one can write

(140) \[ P[C > 0.2] = P\left[\frac{\cos \phi}{\beta} > 1.2 \right] \leq P[1/\beta > 1.2] = P[\beta < 0.835] \]

where \( P[---] \) stands for probability of occurrence. The system will be designed such that the average value of \( \beta \) is at least 0.96 and the maximum standard deviation of \( \beta \) is 0.02 (Appendix V, Fig. 35, \( \sigma_y \leq 0.2 \)). Consequently by using Tchebycheff's inequality, [28] one has

(141) \[ P[C > 0.2] \leq P[\beta < 0.835] \leq \frac{1}{\left(\frac{0.96 - 0.835}{0.02}\right)^2} < 0.028 \]

which is small even if one does not consider that the inequality of Tchebycheff is usually quite pessimistic. Furthermore,

(142) \[ P[C < -0.2] = P\left[\frac{\cos \phi}{\beta} < 0.8 \right] \leq P[\cos \phi < 0.8] = P[|\phi| \geq 37^\circ] \]

which by assuming that \( \sigma_\phi = 0.1 \) radian, a standard design criteria, is

(143) \[ P[C < -0.2] \leq \frac{1}{\left(\frac{37}{5.7}\right)^2} \leq \frac{1}{36} = 0.028 \]

Hence with proper design, one is assured that

(144) \[ P[|C| \leq 0.2] \geq 0.944 \]
The selection of values of $C$ in the range $|C| \leq 0.2$ has been reasonably justified.

One can now discuss the phase control performance as a function of the various parameters. First consider the standard deviation of the phase ($\sigma_\gamma$) which is related to the standard deviation of the coherence (Fig. 31, Appendix V). Notice in Figs. 18 and 19 that $\sigma_\gamma$ is somewhat insensitive to variations in $C$ for a loop gain near one. Also, a variation in this parameter for a change in $A$, caused by a change in $K_1$ due to inaccurate gain calibration, is reasonable. For the sake of illustration, suppose that $A = 1$ is the desired loop gain but the actual gain is $1.1$ (10% change). Then from Fig. 18 with $C = 0$ and $k(N-1)\sigma_{A_1} = 0.1$, the percent change in the standard deviation is

\begin{equation}
\% \text{ change} = \left[ \frac{0.120 - 0.114}{0.114} \right] \times 100 = 5.26\%.
\end{equation}

This change has an insignificant effect on the standard deviation of the coherence which is a measure of the power fluctuation of the resultant signal at the satellite (see Fig. 31). From a stability standpoint, the situation just discussed is highly desirable. If the power fluctuations were highly sensitive to changes in the coherence an unstable system would result.

The average coherence is illustrated in Figs. 20, 21, 22 and 23 for the same conditions as above. In the interest of economy only the case $N = 4$ is shown. Everything else being equal these curves
Fig. 18—Standard deviation of the phase $\sigma_\gamma$ vs loop gain for constant measurement accuracy $k(N-1)\sigma_\Delta_1$ and maximum phase change between corrections $\Delta f T_d$.

Fig. 19—Standard deviation of the phase $\sigma_\gamma$ vs loop gain for constant measurement accuracy $k(N-1)\sigma_\Delta_1$ and maximum phase change between corrections $\Delta f T_d$. 

have approximately the same shape for any \( N \). The curves would be somewhat lower for a larger \( N \). This change will amount to only a few percent even as \( N \) approaches infinity. This is not to be construed to mean that the phase control operation is independent of \( N \). Recall that these results are for constant \( k(N-1)\sigma_{\Delta_1} \) which is the actual standard deviation of the phase estimate. To maintain a given degree of measurement accuracy as \( N \) increases, \( \sigma_{\Delta_1} \) must be decreased at the same rate. This requires an increase in signal-to-noise ratio \( (P_R/N_0) \) or integration time \( (pT_c) \) or both. The significance of these results is that the average coherence is reasonably flat for the gain near one and the average performance changes only a few percent for the extreme range of \( C \). Again this is indicative of a stable system. Notice that the average coherence has a maximum value which usually does not occur for a loop gain of one. It is felt that the small possible increase in average coherence does not warrant that the system be designed for maximum average performance. More important, it is better to take a small loss in average performance and obtain a significant reduction in power fluctuations.

Based on these observations it is recommended that the system be designed with a loop gain of one. Also, the measurement accuracy \( (k(N-1)\sigma_{\Delta_1}) \) should be close to 0.15 radians (8.58 degrees r.m.s.). This value was selected as a compromise between 0.2 radians which is not quite accurate enough and 0.1 radians which is somewhat of an over-design. Reducing the accuracy of the measurement beyond this point will not produce any significant change in system response.
Figs. 20-23--Lower bound to average coherence $E(\beta^2)$ vs loop gain for constant measurement accuracy $k(N-1) \sigma_0^2$ and maximum phase change between corrections $\Delta f T_d$. 
In addition, the value of the phase change $\Delta f(\Delta T + T_d)$ should be no larger than $20^\circ$. If the system is designed to meet these specifications, the results indicate that a highly stable, quite accurate phase control system can be obtained. We have thus answered the questions that were posed at the end of Chapter III.

The next chapter presents two design examples that illustrate the use of the criteria that have been developed.
The first design example concerns the ElectroScience Laboratory's Satellite Communication Facility[29] and the IDCSP satellites. Appendix VI presents typical signal strength and signal-to-noise ratio calculations for such a communication link. These values will be used in the design examples to follow.

The ElectroScience Laboratory's array consists of four thirty-foot parabolic antennas located on the corners of a sixty-foot square. Hence with a transmitting frequency of 7.9 GHz, the maximum antenna separation in wavelengths is

\[
(146) \quad \frac{d_m}{\lambda} = \sqrt{2} \cdot \frac{60 \cdot 12 \cdot 2.54}{3.8} = 682.
\]

Using the values presented in Appendix IV, no open-loop correction, the maximum rate of change of phase is

\[
(147) \quad \Delta f = \frac{d_m}{\lambda} \cdot 5.6 \cdot 10^{-4} = 682 \cdot 5.6 \cdot 10^{-4} = 0.382 \text{ degrees/sec}.
\]

The maximum round-trip delay time is 0.253 seconds. Therefore

\[
(148) \quad \Delta f \Delta T = 0.382 \cdot 0.253 = 0.0966 \text{ degrees}
\]

which for all practical purposes can be considered zero. For the maximum phase change between corrections of an individual element to be close to 10°, the required \(T_d\) is
A value of 30 seconds is selected for convenience.

Since this phase correction technique is to be tested over the ElectroScience Laboratory's communication link, it is desirable to make the design as flexible as possible so that the theory may be experimentally checked. It is of interest to measure the effect, if any, of the beam tagging on the data reception operation. To accomplish this with one of the antennas at the site it is necessary that

\[ \frac{P_{RT_b}}{N_o} \geq 10 \]

which will guarantee sufficient signal energy in the data bit for adequate error measurements. Consequently the minimum bit length is

\[ T_b = \frac{10}{P_{R/N_o}} = \frac{10}{5.15 \cdot 10^5} = 19.4 \mu \text{sec} \]

The value of \( P_{R/N_o} \) was taken from Table VI, Appendix VI. A value of 20 \( \mu \)seconds for the bit length is selected.

The tagging period \( T_c \) is required to be much larger than \( T_b \) and the value selected is 1 m second. Also to satisfy the constraints on the parameters in the error detection system, a "p" of 100 is selected (see Eq. (60)). Note that the maximum phase change in the integration time (\( pT_c \)) is

\[ \Delta f pT_c = 0.0382 \text{ degrees} \]

which is consistent with the assumption that the phase angles, Eq. (12), are constant over this time period.
The bandwidth $B_1$ is assumed to be $3/T_b$. This should be sufficient to insure that the first filter in the error detection system can pass the data with negligible distortion. Hence from Eq. (91) with $k_1 = 3$, the normalized standard deviation of the measurement error is

$$\sigma_{\Delta 1} = \frac{1}{\sqrt{2(0.982)}} \left[ 1 + \frac{3}{(0.965)10.3} \right]^{\frac{1}{2}} \left[ 5.15 \cdot 10^4 \right]^{-\frac{1}{2}}$$

$$= 0.00362 \text{ radians.}$$

The actual standard deviation of the measurement error is $k(N-1)\sigma_{\Delta 1}$ and for this to be 0.15 radians requires that

$$k = \frac{0.15}{3 \cdot 0.00362} = 13.8.$$ 

This implies that the power output of the element being tagged be reduced by the factor $1/191$. This suggests a convenient method of changing the measurement accuracy for the experimental purpose of measuring system performance as a function of the measurement accuracy. To increase the standard deviation of the measurement, all that is required is to reduce further the power output of the phase tagged element. The only change required in the error detection receiver is a readjustment of the gain (see Eq. (55)).

The error detection system parameters are:

- $2B_1 = 300 \text{ KHz} = \text{bandwidth of first filter}$,
- $2B_2 = 30 \text{ KHz} = \text{bandwidth of second filter (this will pass eight harmonics of tagging frequency)}$. 
\[ 2B_L = 200 \, \text{Hz} = \text{bandwidth of PLL}, \]
\[ pT_c = 0.1 \, \text{sec} = \text{integration time}, \text{and} \]
\[ k = 13.8. \]

From Eq. (97), the standard deviation in the PLL is
\[ \sigma_{\phi} = 4 \sqrt{B_L pT_c} \sigma_{\Delta_1} = 4 \sqrt{10 \times 0.00362} = 0.0458 \, \text{radians} \]
which is more than sufficient. The selection of the PLL bandwidth was based on experimental observations of signals received over the IDCSP communication link. Its value is not critical provided that the response time of the loop is small compared to the integration time \( pT_c \).

The mean and standard deviation of the coherence factor for this system can now be calculated. Since these values are not very sensitive to \( C \) when \( A = 1 \), they are calculated for \( A = 1, C = 0 \) only. From Eq. (137), the maximum phase offset is
\[ \bar{\xi}_{\max} = 3/4 \left[ 0.0966 + \frac{30 \cdot 0.382}{2} \right] = 4.36^\circ. \]
Using Eq. (138), the standard deviation of the phase of the individual signals as defined by Eq. (12), is
\[ \sigma_{\gamma} = \sqrt{0.75 \left[ \frac{1}{12} \left( \frac{11.46}{57.3} \right)^2 + 0.0225 \right]^{1/2}} = 0.14 \, \text{rad}. \]
Hence using Eq. (136) and Fig. 30 in Appendix V, the lower-bound to the average coherence is
From Fig. 31 in Appendix V, the standard deviation of the coherence factor is

\[(159) \quad \sigma_{\beta^2} = 0.012\]

The next design example considers nine ten-foot parabolic antennas arranged as shown in Fig. 24. This array will be designed for the transmission of PSK data to a receiving site via an IDCSP satellite. The array is to operate with a carrier frequency of 8 GHz and a data rate of \(2 \times 10^5\) bits/sec \((T_b = 5\mu\text{sec})\). One antenna of the array will be used to receive the return signal for the processing of the phase control information. Assuming a 200°K receiver with 55 percent antenna efficiency, the signal power-to-noise density ratio is

\[(160) \quad \frac{P_R}{N_0} = 5.73 \times 10^5 \left[\frac{10}{15}\right]^2 = 2.55 \times 10^5 \text{ Hz}\]

where the appropriate values are obtained from Appendix VI. For this case, the maximum rate of change of phase is

\[(161) \quad \Delta f = \left(\frac{dm}{\lambda}\right) 5.6 \times 10^{-4} = \frac{56.6 \times 12 \times 2.54 \times 5.6 \times 10^{-4}}{3.75} = 0.258 \text{ degrees/sec} .\]

Therefore,

\[(162) \quad \Delta f\Delta T = 0.258 \times 0.253 = 0.0653 \text{ degrees} .\]
Selecting a correction cycle time $T_d$ of 60 seconds, the maximum phase change between corrections of individual elements is

$$\Delta fT_d = 60 \cdot 0.258 = 15.5^\circ.$$  \hspace{1cm} (163)

Again assuming a $T_c$ of 1 m second and selecting an integration time of 0.2 seconds ($p = 200$), the normalized standard deviation is

$$\sigma_{\Delta 1} = \frac{1}{\sqrt{2 \cdot 0.982}} \left[ 1 + \frac{3}{0.965 \cdot 1.275} \right]^{\frac{1}{2}} \left[ 5.1 \cdot 10^4 \right]^{\frac{1}{2}} = 0.00591 \text{ rad.}$$  \hspace{1cm} (164)

---

**Fig. 24**—Nine ten-foot parabolic antennas.
where the bandwidth $B_1$ is again selected as $3/T_b$. Adjusting the actual standard deviation of the measurement error to be 0.15 radians requires that

$$k = \frac{0.15}{8 \cdot 0.00591} = 3.17.$$  

This implies that the output power of the beam tagged element is to be reduced by a factor of $1/10.05$.

For this adaptive transmitting array, the error detection system parameters are:

- $2B_1 = 1200$ KHz,
- $2B_2 = 30$ KHz,
- $2B_L = 200$ Hz,
- $pT_c = 0.2$ sec = integration time, and
- $k = 3.17$.

The standard deviation in the PLL is

$$\sigma_{\phi} = 4\sqrt{20} \cdot 0.00591 = 0.105 \text{ rad.}$$

Following the same procedure as before, the average coherence is

$$E\{\beta^2\} = 0.96$$

and the standard deviation of the coherence factor is

$$\sigma_{\beta^2} = 0.012.$$
These two design examples are not intended to be a comprehensive study of all possibilities. They were presented here simply as an illustration of the use of the results that were derived in this report.
An adaptive transmitting array for communication purposes is an array in which the phase of each signal radiated is automatically adjusted to produce an in-phase condition among the signals at the receiving point. In this report a sequential, self-synchronized, beam tagging technique has been developed and shown to provide the necessary information for proper phase adjustment of the transmitted signals. Specifically, the beam tagging modulation considered is a 0°, 180° low frequency (relative to the data rate) phase modulation applied sequentially to each signal transmitted from the array. This tagging can be accomplished without interrupting the data transmission from any of the elements. The down-link signal is processed by a nonlinear device to remove the data modulation but retain the phase error information. The particular technique analyzed is a squaring device to remove the assumed PSK data modulation. For other types of data modulation, it may be possible to use different nonlinear operations to accomplish the same objective. It was shown that this beam tagging scheme possesses the characteristics necessary for coherent transmission to be effectively utilized in communication systems. These characteristics have been mentioned previously and therefore are not repeated here.
A signal and noise analysis of the phase error detection receiver was performed. An expression relating the rms error in the phase measurement to the signal-to-noise ratio, the data transmission rate, the estimation time for the phase error, and the number of array elements was derived. The rms error was shown to be directly proportional to the number of array elements. Thus to maintain the same measurement accuracy as the number of array elements is increased, it will be necessary to increase either the received power or total tagging time or both. Depending on the application, this will establish an upper limit to the number of elements that the phase correction technique can accommodate.

A particular closed-loop model for the phase control operation was developed. It was shown that this model is equivalent to a sample data feedback control system with the added complication of a random closed-loop gain. An approximate analysis of the closed-loop performance was done which assumed that the loop gain was constant. The phase control performance was evaluated in terms of a design criteria called the coherence factor. This factor is a measure of the effective power transmitted from the array in the direction of the "target." System performance was then investigated for extreme loop gain deviations. The results of the closed loop study indicated that adequate performance is obtained if the rms error in the phase error measurement is at most 0.15 radians and the correction rate is such that the maximum phase change between corrections of an individual element is no more than 20 degrees.
Two simple design examples were presented to illustrate the use of the results derived. While the design examples considered only adaptive arrays located on the ground, it should be pointed out that the array could be located on moving vehicles such as aircraft or deep space probes. Also, it is not a requirement that the array or one of the antennas of the array process the down-link signal to obtain the phase control information. For example, a signal processing satellite could process the up-link signal and obtain the phase control information. Of course this information would have to be made available to the transmitting array.

The following will briefly discuss possible extensions to the present work.* First for a small number of transmitting elements, it may be undesirable to considerably reduce the power output of an element during its beam tagging period. A serious decrease in array efficiency can be prevented by transmitting the available "uncoded power" from the tagged element. If this is done, the beam tagging modulation is equivalent to a square-wave AM modulation. There is a subtle difference between the AM beam tagging technique and the technique investigated in this report, i.e., power reduction and 0°, 180° PM beam tagging. For each approach, the PM of the down-link signal at the beam tagging frequency is a result of a phase mis-alignment between the tagged and the reference signal.

*These ideas were generated from technical discussions with Mr. R.J. Huff, the author's colleague at the ElectroScience Laboratory.
For the technique studied the reference signal is the resultant signal transmitted from the N-1 untagged elements. For the AM approach, the reference signal is the resultant signal transmitted from the N-1 untagged elements plus the signal component from the tagged element that has no beam tagging on it. For arrays with greater than 4 or 5 elements, this subtle difference may be insignificant. This point should be investigated further mathematically. An experimental study will be performed at the ElectroScience Laboratory's Satellite Communication Facility.

It has been suggested by R.J. Huff that it may be possible to replace S(t) by a pseudo-noise (PN) code. The goal would be to obtain the phase error estimate and the required range information from a common waveform. The range information would be obtained by tracking the PN code in a delay-lock loop. This approach is attractive and should be theoretically investigated to determine its potential. A third extension would be a study to determine what modifications are necessary to permit the coherent transmission of pulsed envelope signals such as employed in time division multiple access (TDMA) systems.
APPENDIX I

CHARACTERISTICS OF THE OUTPUT NOISE OF A SYSTEM CONSISTING OF A BAND-PASS FILTER, ZERO-MEMORY QUADRATIC DEVICE, AND A NARROW BAND-PASS FILTER CENTERED ON THE DOUBLE FREQUENCY OF THE INPUT

Consider the following system shown in Fig. 25, where $2B_1$ and $2B_2$ are the bandwidth of the first and second filters respectively.

\[ S_R(t) + n_1(t) \xrightarrow{\text{BAND-PASS FILTER- } \omega_0} S_{R1}(t) + n_1(t) \xrightarrow{\text{SQUARE-LAW DEVICE}} S_{R2}(t) + n_2(t) \]

Fig. 25--Block diagram of bandpass filter, square-law device and band-pass filter centered on double frequency of input.

The input noise $n_1(t)$ is assumed to be a sample function from a stationary, zero mean Gaussian process with flat power spectrum. If $2B_1 \ll \omega_0$, then $n_1(t)$ can be represented as

\[ (169) \quad n_1(t) = x(t) \cos \omega_0 t - y(t) \sin \omega_0 t \]

where $x(t)$ and $y(t)$ are slowly varying (relative to $\omega_0$) sample functions from independent stationary Gaussian processes each with zero mean and identical correlation functions. The signal $S_{R1}(t)$ will be represented in the form of a modulated sine wave, i.e.,
(170) \[ S_{R1}(t) = \alpha(t) \cos \omega_0 t - \eta(t) \sin \omega_0 t \]

where \( \alpha(t) \) and \( \eta(t) \) are also slowly varying compared to \( \omega_0 \). Thus the input to the square-law device is

(171) \[ S_{R1}(t) + n_1(t) = [\alpha(t) + x(t)] \cos \omega_0 t - [y(t) + \eta(t)] \sin \omega_0 t. \]

The output of the square-law device can easily be shown to be

(172) \[ [S_{R1}(t) + n_1(t)]^2 = \frac{1}{2} \{(\alpha(t) + x(t))^2 + (\eta(t) + y(t))^2\} \]
\[ + \frac{1}{2} \{(\alpha(t) + x(t))^2 - (\eta(t) + y(t))^2\} \]
\[ \times \cos 2\omega_0 t \]
\[ - (\alpha(t) + x(t))(\eta(t) + y(t)) \sin 2\omega_0 t. \]

Notice that this output contains two terms, one term whose spectrum is centered around zero frequency and the other whose spectrum is centered around \( 2\omega_0 \). These two terms are expressed in equation form as

(173) \[ [S_{R1}(t) + n_1(t)]^2_0 = \frac{1}{2} \{(\alpha(t) + x(t))^2 + (\eta(t) + y(t))^2\} \]

and

(174) \[ [S_{R1}(t) + n_1(t)]^2_{2\omega_0} = \frac{1}{2} \{(\alpha(t) + x(t))^2 - (\eta(t) + y(t))^2\} \]
\[ \times \cos 2\omega_0 t \]
\[ - (\alpha(t) + x(t))(\eta(t) + y(t)) \sin 2\omega_0 t. \]
Before proceeding further it should be pointed out that a classical problem of this form which has been solved by many authors is concerned with the filtering of signals of the form given in Eq. (173). [30,31,32] This form of the problem has the output filter centered on zero frequency. Of interest here is the filtering of signals of the form given in Eq. (174). This form of the problem is sufficiently different from the classical problem to warrant detailed consideration. To the author's knowledge this problem has not been completely solved by anyone. The reason is probably due to the fact that past interest has centered around the detection problem where the double frequency terms are discarded in favor of the low-pass terms. Here, the double frequency terms are of primary interest and therefore the output filter passband will be centered on $2\omega_0$. The effective input to this bandpass filter is given by Eq. (174).

Equation (174) can be written in the following form

\[
\begin{align*}
\left[S_{R1}(t) + n_1(t)\right] \frac{2}{2\omega_0} & = \frac{1}{2} \left(\alpha^2(t) - n^2(t)\right) \cos 2\omega_0 t - \alpha(t) n(t) \sin 2\omega_0 t \\
+ \left\{\alpha(t) x(t) - n(t) y(t) + \frac{1}{2} \left( x^2(t) - y^2(t) \right) \right\} \cos 2\omega_0 t \\
- \left\{ x(t) n(t) + \alpha(t) y(t) + x(t) y(t) \right\} \sin 2\omega_0 t
\end{align*}
\]

where the first two terms contain the signal of interest and the last two terms contain unwanted noise. Assuming that the output filter is a symmetrical bandpass filter centered on $2\omega_0$, the output noise $n_2(t)$ can be represented as:[33]
\[ n_2(t) = X(t) \cos 2\omega_0 t - Y(t) \sin 2\omega_0 t \]

where;

\[ X(t) = \int_{-\infty}^{+\infty} h_{\lambda}(t-\mu) [\alpha(\mu) x(\mu) - n(\mu) y(\mu) + \frac{1}{2}(x^2(\mu) - y^2(\mu))] \, d\mu, \]

\[ Y(t) = \int_{-\infty}^{+\infty} h_{\lambda}(t-\mu) [x(\mu) n(\mu) + \alpha(\mu) y(\mu) + x(\mu) y(\mu)] \, d\mu \]

and \( h_{\lambda}(\xi) \) is the low-pass equivalent impulse response of the output filter. The investigation of the statistical properties of \( n_2(t) \) will be the primary concern of the remainder of this section.

Assuming that statistical averaging and integration are inter-changeable (this assumption will be made throughout this section), the expectation of \( X(t) \) and \( Y(t) \) are

\[ \mathbb{E}\{X(t)\} = \int_{-\infty}^{+\infty} h_{\lambda}(t-\mu) \mathbb{E}\{\alpha(\mu) x(\mu) - n(\mu) y(\mu) + \frac{1}{2}(x^2(\mu) - y^2(\mu))\} \, d\mu, \]

and

\[ \mathbb{E}\{Y(t)\} = \int_{-\infty}^{+\infty} h_{\lambda}(t-\mu) \mathbb{E}\{x(\mu) n(\mu) + \alpha(\mu) y(\mu) + x(\mu) y(\mu)\} \, d\mu. \]

Also it is assumed that the signals \( \alpha(t) \) and \( n(t) \) are independent of \( x(t) \) and \( y(t) \), in addition we recall that \( x(t) \) and \( y(t) \) are independent with the following properties

\[ \mathbb{E}\{x(t)\} = \mathbb{E}\{y(t)\} = 0, \quad \mathbb{E}\{x(t) x(t+\tau)\} = \mathbb{E}\{y(t) y(t+\tau)\}. \]
It follows immediately that

\[(182) \quad E\{X(t)\} = E\{Y(t)\} = 0.\]

The cross-correlation between \(X(t)\) and \(Y(t)\) is

\[(183) \quad E\{X(t) Y(t+\tau)\} = \]

\[\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h_2(t-\mu) h_2(t+\tau-\rho) E\{[\alpha(\mu)x(\mu) - \eta(\mu)y(\mu)]

+ \frac{1}{2} (x^2(\mu) - y^2(\mu))] [x(\rho)x(\rho) + \alpha(\rho)y(\rho) + x(\rho)y(\rho)] \} d\mu d\rho.\]

Using Eq. (181) and the fact that for a stationary, zero mean, Gaussian process

\[(184) \quad E\{\alpha(\mu) \eta(\rho)\} = E\{\eta(\mu) \alpha(\rho)\} \equiv 0,\]

Eq. (183) reduces to

\[(185) \quad E\{X(t) Y(t+\tau)\} = 0\]

provided that

\[(186) \quad E\{\alpha(\mu) \eta(\rho)\} = E\{\eta(\mu) \alpha(\rho)\} .\]

The last statement is not true in general. However, it is true for the mathematical representation of the signal in this report. Therefore \(X(t)\) and \(Y(t)\) are uncorrelated.

The correlation function of \(X(t)\) is

\[(187) \quad E\{X(t) X(t+\tau)\} = \]

\[\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h_2(t-\mu) h_2(t+\tau-\rho) E\{[\alpha(\mu)x(\mu) - \eta(\mu)y(\mu)]

+ \frac{1}{2} (x^2(\mu) - y^2(\mu))] [x(\rho)x(\rho) - \eta(\rho)y(\rho)

+ \frac{1}{2} (x^2(\rho) - y^2(\rho))] \} d\mu d\rho .\]
For the same general reasons as before, the expectation of all cross terms in Eq. (187) are zero and therefore

\begin{align}
E\{X(t)X(t+\tau)\} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_\alpha(t-\mu)h_\alpha(t+\tau-\rho) \\
&\quad \left[ E\{\alpha(\mu)\alpha(\rho)\}E\{X(\mu)X(\rho)\} + E\{\eta(\mu)\eta(\rho)\}E\{y(\mu)y(\rho)\} \right. \\
&\quad \left. + \frac{1}{4} E\{(x^2(\mu)-\tau^2(\mu))(x^2(\rho)-\tau^2(\rho))\} \right] d\mu d\rho.
\end{align}

The following fact from Davenport and Root will be used in evaluating the last expectation. [34] If \(x_1, x_2, x_3,\) and \(x_4\) are real random variables, jointly Gaussian, and each with zero mean, then

\begin{equation}
E\{x_1 x_2 x_3 x_4\} = E\{x_1 x_2\} E\{x_3 x_4\} + E\{x_1 x_3\} E\{x_2 x_4\} \\
+ E\{x_1 x_4\} E\{x_2 x_3\}.
\end{equation}

Using Eq. (189), one obtains from Eq. (188)

\begin{align}
E\{[x^2(\mu)-\tau^2(\mu)] [x^2(\rho)-\tau^2(\rho)]\} &= 2E\{x^2(\mu)x^2(\rho)\} - 2[E\{x^2\}]^2 \\
&= 2[E\{x^2(\mu)\}]^2 + 2R_x^2(\rho-\mu) + 2R_x^2(\rho-\mu) - 2[E\{x^2(\mu)\}]^2 \\
&= 4R_x^2(\rho-\mu)
\end{align}

where \(R_x(\tau)\) is the correlation function of \(x(t)\). The substitution of Eq. (190) into Eq. (188) gives

\begin{equation}
E\{X(t)X(t+\tau)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_\alpha(t-\mu)h_\alpha(t+\tau-\rho)R_x(\rho-\mu) \\
E\{\alpha(\mu)\alpha(\rho)\} + E\{\eta(\mu)\eta(\rho)\} + R_x(\rho-\mu) d\mu d\rho.
\end{equation}

Also,
\begin{equation}
E\{Y(t)Y(t+\tau)\} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h_x(t-\mu)h_x(t+\tau-\rho)E\{x(\mu)n(\mu) + x(\rho)y(\rho) \} d\mu d\rho 
\end{equation}

which reduces immediately to
\begin{equation}
E\{Y(t)Y(t+\tau)\} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h_x(t-\mu)h_x(t+\tau-\rho)x(\rho-\mu)E(\alpha(\mu)\alpha(\rho)) + E(n(\mu)n(\rho)) + R_x(\rho-\mu) d\mu d\rho 
\end{equation}

and therefore the following important result is obtained
\begin{equation}
E\{X(t)X(t+\tau)\} = E\{Y(t)Y(t+\tau)\} .
\end{equation}

It is important to note that if \(\alpha(t)\) and \(n(t)\) are both wide-sense stationary, then \(X(t)\) and \(Y(t)\) are also wide-sense stationary. This is easily shown by a change of variables in Eq. (191) or Eq. (193). Equations (182), (185), (194) show that \(n_x(t)\) is wide-sense stationary if \(X(t)\) and \(Y(t)\) are wide-sense stationary.[35] If \(X(t)\) and \(Y(t)\) are both Gaussian, then \(n_x(t)\) is Gaussian since \(X(t)\) and \(Y(t)\) are uncorrelated.[36] It is obvious that in general neither \(X(t)\) nor \(Y(t)\) are Gaussian, but under certain assumptions they may be considered approximately Gaussian. The following will show what assumptions are sufficient to guarantee this property.

Toward this end, notice that \(X(t)\) and \(Y(t)\) can be written as the sum of three uncorrelated, zero mean, processes, i.e.,
\( \begin{align*} 
(195) \quad X(t) &= X_1(t) + X_2(t) + X_3(t), \\
\text{and} \quad Y(t) &= Y_1(t) + Y_2(t) + Y_3(t) \\
\text{where;} \\
(197) \quad X_1(t) &= \int_{-\infty}^{+\infty} h_x(t-\mu) \alpha(\mu) x(\mu) \, d\mu, \\
(198) \quad X_2(t) &= -\int_{-\infty}^{+\infty} h(t-\mu) n(\mu) y(\mu) \, d\mu, \\
(199) \quad X_3(t) &= \frac{1}{2} \int_{-\infty}^{+\infty} h_x(t-\mu) \left[ x^2(\mu) - y^2(\mu) \right] \, d\mu, \\
(200) \quad Y_1(t) &= \int_{-\infty}^{+\infty} h_x(t-\mu) n(\mu) x(\mu) \, d\mu, \\
(201) \quad Y_2(t) &= \int_{-\infty}^{+\infty} h_x(t-\mu) \alpha(\mu) y(\mu) \, d\mu \\
\text{and} \quad Y_3(t) &= \int_{-\infty}^{+\infty} h_x(t-\mu) x(\mu) y(\mu) \, d\mu.
\end{align*} \)

\( X_3(t) \) and \( Y_3(t) \) represent the zero signal input case. Signals of the form of \( X_3(t) \) have been investigated by Arthur[37] and the significant results are contained in a problem in Middleton.[38] The point of interest here is that Arthur showed that the first order statistics of \( X_3(t) \) are asymptotically normal for large values of
Signals of the form of $Y^3(t)$ were investigated by Lampard \cite{39} and his investigation is also summarized in a problem in Middleton. \cite{40} Lampard also concluded that the first order statistics of $Y^3(t)$ are asymptotically normal for large $B_1/B_2$. The signals $X_1(t)$, $X_2(t)$, $Y_1(t)$ and $Y_2(t)$ have the same mathematical form. If $a(t)$ and $n(t)$ are themselves stochastic processes, no detailed statements can be made concerning statistics of $X_1(t)$ ... $Y_2(t)$ unless the statistics of $a(t)$ and $n(t)$ are known. Obviously if $a(t)$ and $n(t)$ are Gaussian processes, then Lampard's work applies. Of interest here is the case where $a(t)$ and $n(t)$ are deterministic in nature. Middleton \cite{41} and others have shown that if

$$y(t) = \int_{a(t)}^{b(t)} A(t,\tau) x(\tau) \, d\tau$$

and if $x(t)$ is a Gaussian process, then $y(t)$ is also Gaussian. One can see that Eq. (203) is the same as Eqs. (197), (198), (200), (201) with

$$A(t,\tau) = h(t) a(\tau)$$

or

$$A(t,\tau) = h(t) n(\tau)$$

and $b(t) = \infty$ and $a(t) = -\infty$. Since the linear combination of independent Gaussian processes is also Gaussian, the conclusion is made that both $X(t)$ and $Y(t)$ are Gaussian if the value of $B_1/B_2$ is large and the input signal is deterministic in nature. It should be pointed out that only the first order statistics of $X_3(t)$ and $Y_3(t)$
have been shown to be Gaussian for a large bandwidth ratio. In general this does not imply that the $N^{th}$ order statistics are Gaussian. Although an important point, this question will not be considered here.

Based on the preceding analysis, $n_2(t)$ will be assumed to be a Gaussian process for sufficiently large values of $B_1/B_2$. This is an important assumption in the analysis of the error detection system.
APPENDIX II

OUTPUT NOISE OF A PHASE-LOCK DEMODULATOR WHOSE INPUT IS THE SUM OF A PHASE-MODULATED CARRIER AND STATIONARY ZERO MEAN, GAUSSIAN NOISE

The operation of a phase-lock demodulator can be represented as in Fig. 26. In Fig. 26,

\[(206) \quad S_{R2}(t) = r^2 p_R \cos[2\omega_0 t + z(t) + \theta]\]

is the phase-modulated signal and

\[(207) \quad n_2(t) = X(t) \cos 2\omega_0 t - Y(t) \sin 2\omega_0 t\]

is the input stationary, zero mean, Gaussian noise. Also;

- \(z(t)\) = desired information (\(f_1(\Delta t_i) S(t-t_1)\) in text),
- \(\theta\) = carrier phase at carrier frequency \(2\omega_0\),
- \(\hat{\theta}\) = estimate of carrier phase via operation of the phase-lock-loop.

The other terms are self-explanatory.

\[A_r \sin [2\omega_0 t + \hat{\theta}]\]

Fig. 26--Operation of a phase-lock-demodulator in noise.
Assuming that the low-pass filter will reject the $4\omega_0$ terms and pass the other terms with no distortion, it can easily be shown that

\begin{equation}
S_{R3}(t) + n_3(t) = \frac{A_r}{2} \left[ -r^2 p_R \sin(z(t) + \phi) + X(t) \sin(\theta - \phi) - Y(t) \cos(\theta - \phi) \right]
\end{equation}

where,

\begin{equation}
\phi = \theta - \hat{\theta}
\end{equation}

is a random variable that represents the phase error in the demodulation process. Therefore, the output noise is

\begin{equation}
n_3(t) = X(t) \sin(\theta - \phi) - Y(t) \cos(\theta - \phi)
\end{equation}

where the $A_r/2$ factor has been dropped for convenience.

Before any investigation on the statistical nature of $n_3(t)$ can be made, some discussion is necessary concerning the statistical relationship between $X(t)$, $Y(t)$ and $\phi$. If the phase estimate $\hat{\theta}$ of $\theta$ is obtained via a separate channel, for example, if a sub-carrier with phase characteristics identical with the phase characteristics of the carrier is also available for processing, then $X(t)$, $Y(t)$, and $\phi$ will be statistically independent. If the phase estimate $\hat{\theta}$ is obtained by processing $S_{R2}(t) + n_2(t)$, this is the function of the phase-lock-loop, then $\phi$, $X(t)$ and $Y(t)$ are theoretically not independent. However, in the case of practical interest the bandwidth of the phase-lock-loop is usually much narrower than the bandwidth of $n_2(t)$ ($B_L \ll B_2$). This condition implies a considerable
amount of smoothing of the input noise in the phase-lock-loop operation. Thus the phase estimate \( \hat{\phi} \) at a particular time is minimally influenced by the recent history of the input noise. With the above consideration in mind, \( X(t) \), and \( Y(t) \) are assumed to be statistically independent.

The conditional \( N \)th order characteristic function where

\[
\begin{bmatrix}
  n_3(t_1) \\
  n_3(t_2) \\
  \vdots \\
  n_3(t_N)
\end{bmatrix}
\]

is the \( N \)th order column vector will not be derived. By use of the following correspondence of notation in Middleton

\[
X_n \Leftarrow n_3
\]

\[
x_1 \Leftarrow X
\]

\[
x_2 \Leftarrow Y
\]

\[
a \Leftarrow \cos(\phi - \phi)
\]

\[
b \Leftarrow \sin(\phi - \phi)
\]

one can immediately write

\[
F_{n_3, \phi-\phi}(i\xi) = \exp^{-\frac{1}{2} \xi^T [K_{n_3}] \xi}
\]

where \([K_{n_3}]\) is the \( N \times N \) covariance matrix of \( n_3(t) \) and is

\[
[K_{n_3}] = \cos^2(\phi-\phi) [K_{XX}] + \sin^2(\phi-\phi) [K_{YY}]
\]

\[ - \cos(\phi-\phi) \sin(\phi-\phi) [K_{XY}] + [K_{XY}]^T \] .

Because of the nature of the input noise
Using Eqs. (215) and (216), Eq. (214) reduces to

\[(217) \quad [K_{n3}] = [K_{XX}]\]

and then

\[(218) \quad F_{n3,0-\phi} (i\xi) = \exp^{-\frac{1}{2} [\xi^T [K_{XX}] \xi]}.\]

Therefore, \(n_3(t)\) is a zero mean Gaussian process with identical statistics as \(X(t)\) regardless of the distribution of \(\phi\). That is, provided \(\phi\), \(X(t)\), and \(Y(t)\) are statistically independent.
This appendix evaluated the integral

\[ g(p) = \int_{-\infty}^{+\infty} \frac{\sin^4 \left( \frac{\omega}{2} \right)}{\omega^2} \left[ \frac{\sin \frac{p\omega}{2}}{\sin \omega} \right]^2 d\omega. \]  

Defining \( p = 2M + 1 \), a useful identity for evaluating the above integral is [44]

\[ \frac{\sin \frac{p\omega}{2}}{\sin \omega} = \frac{\sin \left(2M+1\right)\omega}{\sin \omega} = \sum_{-M}^{+M} e^{\pm 2k\omega}. \]

Assuming that integration and summations can be interchanged, \( g(p) \) can be written as

\[ g(p) = \sum_{l=-M}^{M} \sum_{k=-M}^{M} \int_{-\infty}^{+\infty} \frac{\sin^4 \left( \frac{\omega}{2} \right)}{\omega^2} e^{\pm (2k+2l)\omega} d\omega. \]

The integral in Eq. (221) is easily evaluated and is

\[ \int_{-\infty}^{+\infty} \frac{\sin^4 \left( \frac{\omega}{2} \right)}{\omega^2} e^{\pm (2k+2l)\omega} d\omega = 2\pi f(2k+2l). \]

where \( f(t) \) is

\[ f(t) = \mathcal{F}^{-1} \left\{ \frac{\sin^4 \left( \frac{\omega}{2} \right)}{\omega^2} \right\} \]

and \( \mathcal{F}^{-1} \{ \ldots \} \) denotes the inverse Fourier transform. Defining
\[(224) \quad F_1(\omega) = \sin^2 \left(\frac{\omega}{2}\right) = \frac{1}{2} \left[ 1 - \cos \omega \right] = \mathcal{L} \left\{ \frac{\delta(t)}{2} - \frac{\delta(t-1)}{4} - \frac{\delta(t+1)}{4} \right\}\]

and

\[(225) \quad F_2(\omega) = \frac{\sin^2 \left(\frac{\omega}{2}\right)}{\omega^2} = \mathcal{L} \left\{ q_1(t) \right\}\]

where

\[(226) \quad q_1(t) = \begin{cases} 1-|t| & \text{if } |t| \leq 1 \\ 0 & \text{if } |t| > 1 \end{cases}\]

then by the time convolution theorem

\[(227) \quad f(t) = \int_{-\infty}^{\infty} \left\{ \frac{\delta(\tau)}{2} - \frac{\delta(\tau-1)}{4} - \frac{\delta(\tau+1)}{4} \right\} \frac{q_1(t-\tau)}{4} \, d\tau = \frac{1}{16} \left[ 2q_1(t) - q_1(t-1) - q_1(t+1) \right]\]

Therefore,

\[(228) \quad g(p) = 2\pi \sum_{\lambda=-M}^{M} \sum_{k=-M}^{M} \frac{1}{16} \left[ 2q_1(2k+2\lambda) - q_1(2k+2\lambda-1) - q_1(2k+2\lambda+1) \right]\]

In the above summation, the second and third terms contribute zero for any \(\lambda\) and \(k\). The first term has a nonzero contribution only when \(k = -\lambda\), and therefore

\[(229) \quad g(p) = \frac{2\pi}{8} \left[ q_1(0) \right] \left[ 2^{M+1} \right] = \frac{\pi}{4} \quad P.\]
APPENDIX IV
MAXIMUM RATE OF CHANGE OF PHASE ($\Delta f$)

If beam coding is used exclusively for phase control, then the maximum rate of change of phase that the closed-loop system has to accommodate will be the maximum differential doppler frequency, i.e., the rate of change of interelement phase. In some cases, the differential doppler frequency is too high for adequate phase-control operation.[45] The purpose of this section is to determine the maximum rate of change of phase for the closed-loop operation if retro-direction control is used in combination with beam coding. The following simple analysis will give practical numbers for this situation. Consider the two transmitting elements depicted in Fig. 27. It has been shown[46] that if retrodirective control is used (with at least a 2nd order phase-lock-loop), the frequency difference of the two signals at the satellite is given to a good approximation by

\[(230) \quad \Delta f = (\Delta D) \Delta T + \frac{2(\Delta D)D}{f_T} \]

where:

- $f_T =$ transmitting frequency,
- $\Delta T =$ round trip propagation time,
- $D =$ average doppler frequency,
- $(\Delta D) =$ differential doppler frequency, and
Fig. 27--Two element transmitting array.
\( \Delta \dot{D} \) = rate of change of differential doppler.

Differential doppler frequency is defined as the rate of change of differential path length divided by the wave length, i.e.,

\[
\Delta D = \frac{1}{\lambda} \frac{d}{dt} \left[ d_m \sin \psi \right] = \left[ \frac{d_m}{\lambda} \right] \left( \cos \psi \right) \dot{\psi}.
\]

Therefore \( \Delta \dot{D} \) is given by

\[
\Delta \dot{D} = \frac{d_m}{\lambda} \left[ -(\sin \psi) (\dot{\psi})^2 + (\cos \psi) (\ddot{\psi}) \right].
\]

Also,

\[
\Delta \dot{T} = \frac{2R}{C}
\]

and

\[
\Delta \dot{(D)} = \frac{\dot{R}_\perp}{\lambda}
\]

where;

- \( R \) = range,
- \( \dot{R}_\perp \) = range rate in direction of propagation, and
- \( C \) = velocity of propagation in medium.

Using Eqs. (231), (232), (233), and (234), Eq. (230) is transformed to

\[
\Delta f = \left[ \frac{d_m}{\lambda} \right] \left[ \frac{2}{C} \right] \left[ \left( -(\sin \psi)(\dot{\psi})^2 + (\cos \psi)(\ddot{\psi}) \right) R 
+ \dot{R}_\perp (\cos \psi) \dot{\psi} \right].
\]
The interest here is not in the exact value of $\Delta f$ for a given orbit configuration but in a practical upper limit for typical satellite orbits. It is clear that $(\Delta f)_{\text{max}}$ is such that

$$
(\Delta f)_{\text{max}} < \left[ \frac{d_m}{\lambda} \right] \left[ \frac{2}{C} \right] \left[ (\dot{\psi}^2 + |\dot{\psi}|) R + |R_L \dot{\psi}| \right].
$$

Also for most satellite orbits the angular acceleration is approximately zero except for the unusual case of a satellite passing almost directly over the transmitting terminal. Therefore Eq. (236) reduces to

$$
(\Delta f)_{\text{max}} < \left[ \frac{d_m}{\lambda} \right] \left[ \frac{2}{C} \right] \left[ |R \dot{\psi}|^2 + |R_L \dot{\psi}| \right].
$$

Separating orbit parameters from terminal parameters, Eq. (237) can be written as

$$
\frac{(\Delta f)_{\text{max}}}{d_m/\lambda} < \frac{2}{C} |\dot{\psi}| \left[ |R| \dot{\psi} | + |R_L| \right]
$$

where,

$$
d_m/\lambda = \text{maximum element separation in wavelengths}. \ \ \ \ \text{Equation (238)}
$$

in the units ordinarily used for satellite orbits is

$$
\frac{\Delta f_{\text{max}}}{d_m/\lambda} < \frac{2 |\dot{\psi}|}{3 \cdot 10^5} \left[ \frac{R |\dot{\psi}|}{57.3} + |R_L| \right]
$$

where;

$$
\left[ \frac{\Delta f_{\text{max}}}{d_m/\lambda} \right] \text{is in degrees/sec}
$$

if,
\( \psi \) is in degrees/sec,
\( R \) in kilometers, and
\( \dot{R}_L \) in kilometers/sec.

Table II shows calculated values of \( \Delta f_{\text{max}} / (d_m / \lambda) \) and \( (\Delta D)_{\text{max}} / (d_m / \lambda) \) for some typical satellite orbits. In each case, maximum values of the various parameters were chosen in order to obtain an upper limit to the maximum rate of change of phase.

**TABLE II**

Maximum rate of change of phase for typical satellite orbits

| \( |\dot{\psi}| \) degrees/sec | \( R \)-kilometers | \( \dot{R}_L \) kilometers/sec | \( \Delta f_{\text{max}} \) (degrees/sec) | \( (\Delta D)_{\text{max}} \) degrees/sec |
|-------------------------------|-------------------|-----------------------------|---------------------------------|---------------------------------|
| 2.0                           | 1200              | 0.1                         | \( 5.6 \times 10^{-4} \)        | 2.0                             |
| 1.0                           | 2000              | 8.0                         | \( 2.9 \times 10^{-4} \)        | 1.0                             |

*Echo Satellites (nominal height 600 miles)*

| \( |\dot{\psi}| \) degrees/sec | \( R \)-kilometers | \( \dot{R}_L \) kilometers/sec | \( \Delta f_{\text{max}} \) (degrees/sec) | \( (\Delta D)_{\text{max}} \) degrees/sec |
|-------------------------------|-------------------|-----------------------------|---------------------------------|---------------------------------|
| \( 5.6 \times 10^{-4} \)    | 35,500            | 0.02                        | \( 1.35 \times 10^{-9} \)      | \( 5.6 \times 10^{-4} \)      |
| \( 4.2 \times 10^{-4} \)    | 38,000            | 0.05                        | \( 0.92 \times 10^{-9} \)      | \( 4.2 \times 10^{-4} \)      |

*IDCSP Satellites (nominal height 19,000 miles)*

<table>
<thead>
<tr>
<th>( (\Delta f)_{\text{max}} ) (degrees/sec)</th>
<th>( (\Delta D)_{\text{max}} ) (degrees/sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1.18 \times 10^{-6} )</td>
<td>( 6.0 \times 10^{-2} )</td>
</tr>
</tbody>
</table>

*Medium Range (6000 miles)*

*Data from Reference 46*
Note that in each case, the rate of change of phase for the closed-loop system is reduced by a factor of about $10^4$ when retro-direction control is used ($\Delta f_{\text{max}}$ compared with $\Delta D_{\text{max}}$). This represents a tremendous improvement.

As an example of the use of Table II, suppose it is desired that the maximum phase change during an individual coding period be no larger than $1^\circ$. Take $d_m/\lambda = 1000$ and assume an echo orbit. The maximum coding time $T_T$ is found to be

\begin{equation}
T_T = \frac{1^\circ}{(5.6 \cdot 10^{-4}) \cdot 10^3} = 1.8 \text{ sec} \left\{ \begin{array}{c}
\text{with retrodirective control} \\
\text{without retrodirective control}
\end{array} \right.
\end{equation}

and

\begin{equation}
T_T = \frac{1^\circ}{2 \cdot 10^3} = 5 \cdot 10^{-4} \text{ sec} \left\{ \begin{array}{c}
\text{with retrodirective control} \\
\text{without retrodirective control}
\end{array} \right.
\end{equation}
APPENDIX V
IMPORTANT PROPERTIES OF THE PARAMETERS $\beta^2$ AND $\beta$

This Appendix investigates some of the important properties of the parameter

$$\beta^2 = \frac{\sum_{k=1}^{N-1} \sum_{p=1}^{N-1} e^{j(y_k - y_p)}}{(N-1)^2}$$

(Addition of N-1 Signals)

and the corresponding amplitude factor

$$\beta = \left[\frac{\sum_{k=1}^{N-1} \sum_{p=1}^{N-1} e^{j(y_k - y_p)}}{(N-1)}\right]^{1/2}$$

(243)

where the "+" sign means the positive square-root. The parameter $\beta^2$ as defined here is a measure of how well the signals are adding power-wise at the satellite.

Without loss of generality, the $y_j$'s can be considered limited to the range, $-\pi \leq y \leq \pi$, where perfect coherence occurs when the $y_j$'s all are equal. Of interest here, is how random differences in the $y_j$'s affect the parameters $\beta^2$ and $\beta$. Before proceeding further certain assumptions must be made concerning these random variations. The $y_j$'s are assumed to be statistically independent. This is a reasonable assumption if the main phase control is the result of retrodirection action which is controlled from measurements taken in independent channels and the beam coding corrections are separated in time sufficiently to avoid any significant correlations.
Also it will be assumed that all $\gamma_j$'s have the same probability density function (p.d.f.) symmetrical with respect to the value $\gamma_j = 0$.

With these two assumptions, certain statistical properties of $\beta^2$ and $\beta$ will now be derived. It is important to note that the results will also be applicable to the sum of signals whose phases are aligned by independent phase-lock-loops, e.g., as the case of a receiving array.

Concentrating first on the parameter $\beta^2$, its expected value is

$$\mathbb{E}\{\beta^2\} = \mathbb{E}\left\{ \frac{\sum_{k=1}^{N-1} \sum_{p=1}^{N-1} e^{j(\gamma_k - \gamma_p)}}{(N-1)^2} \right\} = \frac{1}{(N-1)} \left[ 1 + (N-2) \mathbb{E}\{\cos \gamma\} \right]$$

where use has been made of both the assumption of statistical independence and the identical symmetrical p.d.f. as detailed in the following equation

$$\mathbb{E}\{e^{j(\gamma_k - \gamma_p)}\} = \mathbb{E}\{e^{j\gamma_k}\} \mathbb{E}\{e^{j\gamma_p}\} = \mathbb{E}\{\cos \gamma_k - j \sin \gamma_k\} \mathbb{E}\{\cos \gamma_p - j \sin \gamma_p\} = E^2 \{\cos \gamma\} (k \neq p).$$

Another important statistic of $\beta^2$ is its variance. This will be useful in the estimation of power fluctuations at the satellite.
In order to evaluate the variance of $\beta^2$, the expectation of $\beta^4$ is needed, i.e.,

$$
E\{\beta^4\} = \frac{1}{(N-1)^4} \left\{ \sum_{k=1}^{N-1} \sum_{p=1}^{N-1} \sum_{m=1}^{N-1} \sum_{l=1}^{N-1} e^{j[(\gamma_k - \gamma_p) + (\gamma_m - \gamma_l)]} \right\}
$$

(246)

Table III is helpful in evaluating the above expectation. Thus,

$$
E\{\beta^4\} = \frac{1}{(N-1)^4} \left[ (N-1) + (N-1)(N-2) + (N-1)(N-2) \right.
$$

$$
+ \left. (N-1)(N-2)(N-3)(N-4) E^4 \{ \cos \gamma \} \right]
$$

(247)

$$
+ (N-1)(N-2) E^2 \{ \cos 2\gamma \}
$$

$$
+ 4(N-1)(N-2)(N-3) E^2 \{ \cos \gamma \} E \{ \cos 2\gamma \}
$$

$$
+ 2(N-1)(N-2)(N-3) E^2 \{ \cos \gamma \} E \{ \cos 2\gamma \}
$$

$$
+ 4(N-1)(N-2) E^2 \{ \cos \gamma \} \]
$$

**TABLE III**

<table>
<thead>
<tr>
<th>Different cases and the number of terms in the evaluation of the expectation of $\beta^4$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Cases</strong></td>
</tr>
<tr>
<td>1. $k = p = \ell = m$</td>
</tr>
<tr>
<td>2. $k \neq p \neq \ell \neq m$</td>
</tr>
<tr>
<td>3. $(k=p) \neq (\ell=m)$</td>
</tr>
<tr>
<td>4. $(k=\ell) \neq (p=m)$</td>
</tr>
<tr>
<td>5. $(k=m) \neq (p=\ell)$</td>
</tr>
<tr>
<td>6. $(k=p) \neq \ell \neq m$</td>
</tr>
<tr>
<td>7. $(k=\ell) \neq p \neq m$</td>
</tr>
<tr>
<td>8. $(k=m) \neq p \neq \ell$</td>
</tr>
<tr>
<td>9. $(p=\ell) \neq k \neq m$</td>
</tr>
<tr>
<td>10. $(p=m) \neq k \neq \ell$</td>
</tr>
<tr>
<td>11. $(\ell=m) \neq k \neq p$</td>
</tr>
<tr>
<td>12. $(k=p=\ell) \neq m$</td>
</tr>
<tr>
<td>13. $(k=p=m) \neq \ell$</td>
</tr>
<tr>
<td>14. $(p=\ell=m) \neq k$</td>
</tr>
<tr>
<td>15. $(k=\ell=m) \neq p$</td>
</tr>
</tbody>
</table>
where the first three terms arise from cases (1,3,5), the fourth
term from case (2), the fifth term from case (4), the sixth term
from cases (6,8,9,11), the seventh term cases (7,10) and the eighth
term from cases (12,13,14,15). Equation (247) can be written in
a more compact form as

\[
E\{g^4\} = \frac{1}{(N-1)^3} \left[ (2N-3) + (N-2)(N-3)(N-4) E^4 \cos \left(\frac{\pi}{N-1}\right) \right. \\
+ (N-2) E^2 \{\cos 2\gamma\} + 4(N-2)^2 E^2 \{\cos 2\gamma\} \\
+ 2(N-2)(N-3) E^2 \{\cos \gamma\} E\{\cos 2\gamma\} \right].
\]

In order to evaluate the various expectations appearing in
Eqs. (244) and (248) in terms of the statistics of \( \gamma \), the proba-
bility density function of \( \gamma \) must be known. The following proba-
bility density function will be assumed

\[
p(\gamma) = \frac{\exp[\alpha \cos \gamma]}{2\pi I_0(\alpha)} \quad |\gamma| \leq \pi
\]

where;

\( \alpha \) is a parameter which represents the degree of coherence, and
\( I_0(\alpha) \) is the zeroth-order modified Bessel function. There are
several excellent reasons for this choice. First of all, it takes
into account both the perfectly coherent case (\( \alpha = \infty \)) and the in-
coherent case (\( \alpha = 0 \)) with a smooth transition in between. Second,
the p.d.f. chosen is the p.d.f. of the phase error associated with
a phase-lock-loop operating in the presence of Gaussian noise.[47]
Thus the results will be applicable to the receiving array proble-
where the first three terms arise from cases (1,3,5), the fourth term from case (2), the fifth term from case (4), the sixth term from cases (6,8,9,11), the seventh term cases (7,10) and the eightth term from cases (12,13,14,15). Equation (247) can be written in a more compact form as

\[
E(\beta^4) = \frac{1}{(N-1)^3} \left[ (2N-3) + (N-2)(N-3)(N-4) E^4 \{\cos \gamma\} \right. \\
+ (N-2) E^2 \{\cos 2\gamma\} + 4(N-2)^2 E^2 \{\cos \gamma\} \\
+ 2(N-2)(N-3) E^2 \{\cos \gamma\} E\{\cos 2\gamma\} \right].
\]

In order to evaluate the various expectations appearing in Eqs. (244) and (248) in terms of the statistics of \( \gamma \), the probability density function of \( \gamma \) must be known. The following probability density function will be assumed

\[
p(\gamma) = \frac{\exp[\alpha \cdot \cos \gamma]}{2\pi I_0(\alpha)} \quad |\gamma| \leq \pi
\]

where;

\( \alpha \) is a parameter which represents the degree of coherence, and
\( I_0(\alpha) \) is the zeroth-order modified Bessel function. There are several excellent reasons for this choice. First of all, it takes into account both the perfectly coherent case (\( \alpha = \infty \)) and the incoherent case (\( \alpha = 0 \)) with a smooth transition in between. Second, the p.d.f. chosen is the p.d.f. of the phase error associated with a phase-lock-loop operating in the presence of Gaussian noise.[47] Thus the results will be applicable to the receiving array problem.
where the individual signals are brought into phase alignment by the operation of independent phase-lock-loops. Third, since phase-conjugation control usually obtains phase control information from the phase-lock-loops in the receiving mode of the array, it is felt that this will also be a reasonable assumption for the active array problem. In addition the principal statistics of interest here will be practically independent of the shape of the p.d.f. for small phase deviations \((\sigma_\gamma < 0.3 \text{ radians})\) provided that the p.d.f. is a reasonably smooth function.

Therefore using the p.d.f. as defined by Eq. (249), the expectation of \(\cos \gamma\) is

\[
E\{\cos \gamma\} = \frac{1}{2\pi I_0(\alpha)} \int_{-\pi}^{+\pi} \exp[\alpha \cos \gamma] \cos \gamma \, d\gamma
\]

\[
= \frac{I_1(\alpha)}{I_0(\alpha)}
\]

where \(I_N(\alpha)\) is by definition \([48]\)

\[
I_N(\alpha) = \frac{1}{\pi} \int_{0}^{\pi} \exp[\alpha \cos \gamma] \cos N\gamma \, d\gamma.
\]

Also,

\[
E\{\cos 2\gamma\} = \frac{I_2(\alpha)}{I_0(\alpha)} = -\frac{2I_1(\alpha)}{\alpha I_0(\alpha)} + 1
\]

where use has been made of the following identity \([49]\)

\[
I_{N+1}(\alpha) = \left[\frac{-2N}{\alpha}\right] I_N(\alpha) + I_{N-1}(\alpha).
\]
Using Eqs. (250) and (252), Eqs. (244) and (248) become

\begin{equation}
E(\beta^2) = \frac{1}{(N-1)} \left[ 1 + (N-2) \left( \frac{I_1(\alpha)}{I_0(\alpha)} \right)^2 \right]
\end{equation}

and

\begin{equation}
E(\beta^4) = \frac{1}{(N-1)^3} \left[ (3N-5) - \frac{4(N-2)}{\alpha} \left( \frac{I_1(\alpha)}{I_0(\alpha)} \right) + \left(2(N-2)(3N-7) + \frac{4(N-2)}{\alpha^2} \right) \left( \frac{I_1(\alpha)}{I_0(\alpha)} \right)^2 - \frac{4(N-2)(N-3)}{\alpha} \left( \frac{I_1(\alpha)}{I_0(\alpha)} \right)^3 + (N-2)(N-3)(N-4) \left( \frac{I_1(\alpha)}{I_0(\alpha)} \right)^4 \right].
\end{equation}

\begin{equation}
E(\beta^2) \text{ and}
\end{equation}

\begin{equation}
\sigma_{\beta^2} = \sqrt{E(\beta^4) - E(\beta^2)^2}
\end{equation}

were calculated as a function of the standard deviation \( \sigma_Y \), i.e.,

\begin{equation}
\sigma_Y = \sqrt{\frac{1}{2\pi I_0(\alpha)} \int_{-\pi}^{+\pi} Y^2 \exp \left[ \alpha \cos \gamma \right] \, d\gamma}
\end{equation}

and the results of these calculations are illustrated in Figs. 28-31.

Similar statistics concerning the amplitude factor \( \beta \) are not so easily derived because of the square-root factor. However, excellent upper and lower bounds can be obtained for these statistics in terms of the results derived for \( \beta^2 \). Toward this end consider the binomial series of \( \beta \) as
By terminating the series after three terms an upper bound for $\beta$ is obtained because each additional term is negative, i.e.,

$$
(259) \quad \beta_1 = 1 + \frac{1}{2} (\beta^2 - 1) - \frac{1}{8} (\beta^2 - 1)^2 = \frac{1}{8} (3+6\beta^2-\beta^4) \geq \beta \\
0 \leq \beta^2 \leq 1.
$$

A lower bound to $\beta$ over the restricted range $0.172 \leq \beta^2 \leq 1$ can be obtained by changing the $1/8$ factor to $1/4$, i.e.,

$$
(260) \quad \beta_2 = 1 + \frac{1}{2} (\beta^2 - 1) - \frac{1}{4} (\beta^2 - 1)^2 = \frac{1}{4} (1+4\beta^2-\beta^4) \leq \beta \\
0.172 \leq \beta^2 \leq 1.
$$

A lower bound over the whole interval can be obtained by changing the $1/8$ factor to $1/2$ but the results will not be as good as $\beta_2$, i.e.,

$$
(261) \quad \beta_3 = 1 + \frac{1}{2} (\beta^2 - 1) - \frac{1}{2} (\beta^2 - 1)^2 = \frac{1}{2} (3\beta^2-\beta^4) \leq \beta \\
0 \leq \beta^2 \leq 1.
$$

$\beta_1$, $\beta_2$, and $\beta_3$ are tabulated as a function of $\beta^2$ and $\beta$ in Table IV in order to show the accuracy of the approximations. Equations (259), (260), and (261) are used to evaluate upper and lower bounds to the mean and standard deviation of $\beta$. 

### Table IV
Tabulation of the various binomial series approximations concerning the amplitude parameter $\beta$

<table>
<thead>
<tr>
<th>$\beta^2$</th>
<th>$\beta$</th>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
<th>$\beta_3$</th>
<th>$\beta_1^{-\beta}$</th>
<th>$\beta_2^{-\beta}$</th>
<th>$\beta_3^{-\beta}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0000</td>
<td>0.3750</td>
<td>0.2500</td>
<td>0.0000</td>
<td>0.3750</td>
<td>0.2500</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.1</td>
<td>0.3162</td>
<td>0.4488</td>
<td>0.3475</td>
<td>0.1450</td>
<td>0.1325</td>
<td>0.0313</td>
<td>-0.1712</td>
</tr>
<tr>
<td>0.2</td>
<td>0.4472</td>
<td>0.5200</td>
<td>0.4400</td>
<td>0.2800</td>
<td>0.0728</td>
<td>-0.0072</td>
<td>-0.1672</td>
</tr>
<tr>
<td>0.3</td>
<td>0.5477</td>
<td>0.5888</td>
<td>0.5275</td>
<td>0.4050</td>
<td>0.0410</td>
<td>-0.0202</td>
<td>-0.1427</td>
</tr>
<tr>
<td>0.4</td>
<td>0.6325</td>
<td>0.6550</td>
<td>0.6100</td>
<td>0.5200</td>
<td>0.0225</td>
<td>-0.0225</td>
<td>-0.1125</td>
</tr>
<tr>
<td>0.5</td>
<td>0.7071</td>
<td>0.7188</td>
<td>0.6875</td>
<td>0.6250</td>
<td>0.0116</td>
<td>-0.0196</td>
<td>-0.0821</td>
</tr>
<tr>
<td>0.6</td>
<td>0.7746</td>
<td>0.7800</td>
<td>0.7600</td>
<td>0.7200</td>
<td>0.0054</td>
<td>-0.0146</td>
<td>-0.0546</td>
</tr>
<tr>
<td>0.7</td>
<td>0.8367</td>
<td>0.8367</td>
<td>0.8275</td>
<td>0.8050</td>
<td>0.0021</td>
<td>-0.0092</td>
<td>-0.0317</td>
</tr>
<tr>
<td>0.8</td>
<td>0.8944</td>
<td>0.8950</td>
<td>0.8900</td>
<td>0.8800</td>
<td>0.0006</td>
<td>-0.0044</td>
<td>-0.0144</td>
</tr>
<tr>
<td>0.9</td>
<td>0.9487</td>
<td>0.9487</td>
<td>0.9475</td>
<td>0.9450</td>
<td>0.0001</td>
<td>-0.0012</td>
<td>-0.0037</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

Using Eq. (259), one obtains an upper bound to the mean as

$$E\{\beta_1\} = \frac{1}{8} [3 + 6E\{\beta^2\} - E\{\beta^4\}] \geq E\{\beta\} \ .$$

As far as the mean is concerned a practical lower bound is obtained over the range of interest of $\sigma_\gamma$ by using Eq. (261),

$$E\{\beta_3\} = \frac{1}{2} [3E\{\beta^2\} - E\{\beta^4\}] \leq E\{\beta\} \ .$$

$E\{\beta_1\}$ and $E\{\beta_3\}$ are shown in Figs. 32 and 33 as a function of $\sigma_\gamma$.

A lower bound to the standard deviation of $\beta$ can be obtained by using Eq. (259),

$$\sigma_\beta = [E\{\beta^2\} - E^2\{\beta\}]^{1/2} \geq [E\{\beta^2\} - E^2\{\beta_1\}]^{1/2} = \sigma_{\beta_1} \ .$$
Unfortunately, the nature of the upper bound $E(\beta_1)$ is such that the quantity in brackets in Eq. (264) can be negative for some values of $\sigma_Y$. This makes the lower bound useless. $\sigma_{\beta_1}$ is illustrated in Fig. 34 for the cases where meaningful results were obtained.

An upper bound to $\sigma_{\beta}$ can be obtained by using Eq. (260) provided that the contribution to the expectation of $\beta_2$ is negligible for $\beta^2$ in the range $0 \leq \beta^2 \leq 0.172$. This should be the case for $\sigma_Y$ in the range $0 \leq \sigma_Y \leq 0.5$ and thus

$$\sigma_\beta = \left[ E\{\beta^2\} - E^2\{\beta\} \right]^{1/2} \leq \left[ E\{\beta^2\} - E^2\{\beta_2\} \right]^{1/2} = \sigma_{\beta_2}. \tag{265}$$

This result is shown in Fig. 35.

Thus, both the average power and the average amplitude of the resultant signal are available as a function of the rms phase deviation. Also, means of estimating the spread of these quantities about their mean are available from their standard deviations.
Fig. 28—Average coherence \( E(\beta^2) \) vs the standard deviation of the phase \( \sigma_\gamma \). \((N-1)\) equals the number of signals.
Fig. 29--Standard deviation of $\beta^2$ vs the standard deviation of the phase $\sigma_{\gamma}$. $(N-1)$ equals the number of signals.
Fig. 30--Average coherence $E[\beta^2]$ vs the standard deviation of the phase $\sigma_\gamma$. $(N-1)$ equals the number of signals.
Fig. 31--Standard deviation of $\beta^2$ vs the standard deviation of the phase $\sigma_\gamma$. (N-1) equals the number of signals.
Fig. 32--Upper bound to the expectation of $\beta$ vs the standard deviation of the phase $\sigma_y$. $(N-1)$ equals the number of signals.

Fig. 33--Lower bound to the expectation of $\beta$ vs the standard deviation of the phase $\sigma_y$. $(N-1)$ equals the number of signals.
Fig. 34—Lower bound to the standard deviation of $\beta$ vs the standard deviation of the phase $\sigma_\gamma$. $(N-1)$ equals the number of signals.
Fig. 35--Upper bound to the standard deviation of $\beta$ vs the standard deviation of the phase $\sigma_y$. $(N-1)$ equals the number of signals.
The purpose of this section is to determine the signal strengths and signal-to-noise ratios on a typical satellite communication link.

The one-way path loss, $P_L$, between two isotropic antennas is simply the ratio of transmitted to received power

\[
P_L = \frac{\text{Transmitter power}}{\text{Received power}} = \left[\frac{4\pi d}{\lambda}\right]^2,
\]

or

\[
P_L (\text{dB}) = 32.45 + 20 \log_{10} f + 20 \log_{10} d
\]

where $d$ is distance (km), $f$ is frequency (MHz), and $\lambda$ is the wavelength. For the frequencies and distances encountered with the satellites, it is simple to construct a graph based on Eq. (226). This graph is shown in Fig. 36. It can be seen that the minimum value for path loss is 200 dB, while the maximum is slightly less than 206 dB. This last statement is valid, of course, only for a station at latitude 40° N, the latitude of the OSU Satellite Communication Facility.

Next, a typical link calculation is included.

*Taken by permission of author from Reference 20.
Up-Link

Power output  
Antenna gain, transmitter (approximate)  
Effective transmitted power  
Path loss (from Fig. 36, average)  
Atmospheric attenuation  
Tracking error  
Polarization mismatch (estimated)  
Incident power  
Antenna gain, satellite (assumed)  
Effective coherent power input  
to satellite

If the satellite input signal-to-noise ratio, $(S/N)_{in}$ is known, Davenport's results can be used to determine the signal-to-noise ratio at the output, $(S/N)_{out}$, and the coherent power output of a hard-limiting satellite. For the limiting cases of $(S/N)_{in} \rightarrow 0$, and $(S/N)_{in} \rightarrow \infty$, the ratios of $(S/N)_{out}$ to $(S/N)_{in}$ approach $-1$ dB and $+3$ dB respectively. That is, for low $(S/N)_{in}$ there results a degradation and for high $(S/N)_{in}$ there results an improvement; the cross-over point, i.e., $(S/N)_{out} = (S/N)_{in}$, occurring for $(S/N)_{in} \approx 0$ dB.
Fig. 36--Graph of calculated free-space path loss vs frequency and distance for IDCSP satellite applications.

The noise figure of the satellite is taken as 10 dB. Converting this to noise power density, $P_N$, gives

\[
(268) \quad P_N = k(F-1)T = 1.38 \times 10^{-23} \times (10-1) 290
\]

\[
= -194.3 \text{ dBW/cps} = -164.3 \text{ dBm/cps}.
\]
### At the Satellite

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Effective coherent power input to satellite</td>
<td>-79.9 dBm</td>
</tr>
<tr>
<td>Satellite noise power density</td>
<td>-164.3 dBm/cps</td>
</tr>
<tr>
<td>Satellite noise bandwidth (30 MHz)</td>
<td>+74.8 dB</td>
</tr>
<tr>
<td>Satellite noise power</td>
<td>-89.5 dB</td>
</tr>
<tr>
<td>Signal-to-noise ratio, input</td>
<td>+9.6 dB</td>
</tr>
</tbody>
</table>

### Down Link

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Power output (design)</td>
<td>+34 dBm</td>
</tr>
<tr>
<td>Antenna gain, satellite (design)</td>
<td>+3 dB</td>
</tr>
<tr>
<td>Effective radiated power (design)</td>
<td>+37 dBm</td>
</tr>
<tr>
<td>Path loss, average</td>
<td>-200.5 dB</td>
</tr>
<tr>
<td>Atmospheric attenuation</td>
<td>-0.2 dB</td>
</tr>
<tr>
<td>Tracking errors</td>
<td>-0.1 dB</td>
</tr>
<tr>
<td>Antenna gain, receiver</td>
<td>+53.4 dB</td>
</tr>
<tr>
<td>Plumbing losses</td>
<td>-1.1 dB</td>
</tr>
<tr>
<td>Polarization mismatch</td>
<td>-0.1 dB</td>
</tr>
<tr>
<td>Received power level</td>
<td>-111.6 dBm</td>
</tr>
</tbody>
</table>

Using Davenport's [50] results, entries to Table V are computed for the average figures given in the up-link computation above.

The above calculated figure of -111.6 dBm for received power level can be converted to received signal-to-noise ratio at the ground terminal by determining the noise threshold level of the receiver system. This noise power, $P_{\text{noise}}$, is given by
where \( k \) is Boltzman's constant, \( F \) is the receiver noise figure (numeric), \( T_a \) is the antenna temperature, and \( B_n \) is the IF noise bandwidth. For the case of interest,

\[
(269) \quad P_{\text{noise}} = k T_e B = k[(F-1) 290 + T_a] \cdot B_n,
\]

\[
= 1.38 \times 10^{-23} \times (700) \times 187
\]

\[
= -177.4 \text{ dBiW} = -147.4 \text{ dBm}.
\]

Using the values of calculated received power level and noise threshold level, a typical figure for the received signal-to-noise ratio on a communication satellite link for the Satellite Communication Facility is calculated below:

\[
(271) \quad (\text{Received} \frac{S}{N})_{\text{calculated}} = -(-147.4 + 111.6)dB = +35.8 \text{ dB}.
\]
### TABLE V
Nominal up-link power computations

<table>
<thead>
<tr>
<th>Transmitter Power Output</th>
<th>Satellite (S/N) Input</th>
<th>Satellite (S/N) Output</th>
<th>Satellite Coherent Power Output</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Per cent of Maximum</td>
</tr>
<tr>
<td>4.00 KW</td>
<td>+ 9.6 dB</td>
<td>12.2 dB</td>
<td>94.4</td>
</tr>
<tr>
<td>2.00</td>
<td>6.6</td>
<td>8.8</td>
<td>88.5</td>
</tr>
<tr>
<td>1.60</td>
<td>5.6</td>
<td>7.6</td>
<td>85.1</td>
</tr>
<tr>
<td>1.00</td>
<td>3.6</td>
<td>5.0</td>
<td>76.0</td>
</tr>
<tr>
<td>800 W</td>
<td>2.6</td>
<td>3.7</td>
<td>70.0</td>
</tr>
<tr>
<td>500</td>
<td>0.6</td>
<td>1.1</td>
<td>56.2</td>
</tr>
<tr>
<td>400</td>
<td>- 0.4</td>
<td>0.0</td>
<td>50.0</td>
</tr>
<tr>
<td>250</td>
<td>- 2.4</td>
<td>- 2.3</td>
<td>37.1</td>
</tr>
<tr>
<td>200</td>
<td>- 3.4</td>
<td>- 3.4</td>
<td>31.4</td>
</tr>
<tr>
<td>125</td>
<td>- 5.4</td>
<td>- 5.6</td>
<td>21.6</td>
</tr>
<tr>
<td>100</td>
<td>- 6.4</td>
<td>- 6.7</td>
<td>17.6</td>
</tr>
<tr>
<td>63</td>
<td>- 8.4</td>
<td>- 8.8</td>
<td>11.7</td>
</tr>
<tr>
<td>50</td>
<td>- 9.4</td>
<td>- 9.9</td>
<td>9.3</td>
</tr>
<tr>
<td>31</td>
<td>-11.5</td>
<td>-12.1</td>
<td>5.8</td>
</tr>
</tbody>
</table>
### TABLE VI
Nominal down-link power summary.

<table>
<thead>
<tr>
<th>Station Type</th>
<th>Satellite Power</th>
<th>Down-Link Loss</th>
<th>Antenna Gain</th>
<th>Misc. Loss</th>
<th>Received Power</th>
<th>Noise Spectral Density-$N_0$</th>
<th>$\frac{P_r}{N_0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>&quot;Large&quot; Station</td>
<td>+6.0 dBw</td>
<td>-201.0 dB</td>
<td>+59.4 dB</td>
<td>-1.0 dB</td>
<td>-136.6 dBw</td>
<td>$2.76 \times 10^{-21}$ watts/Hz</td>
<td>$8.25 \times 10^6$ Hz</td>
</tr>
<tr>
<td>&quot;Small&quot; Station</td>
<td>+6.0 dBw</td>
<td>-201.0 dB</td>
<td>+48.0 dB</td>
<td>-1.0 dB</td>
<td>-148.0 dBw</td>
<td>$2.76 \times 10^{-21}$ watts/Hz</td>
<td>$5.73 \times 10^5$ Hz</td>
</tr>
<tr>
<td>O.S.U. Station</td>
<td>+6.0 dBw</td>
<td>-201.0 dB</td>
<td>+53.3 dB</td>
<td>-1.0 dB</td>
<td>-142.7 dBw</td>
<td>$1.04 \times 10^{-20}$ watts/Hz</td>
<td>$5.15 \times 10^5$ Hz</td>
</tr>
</tbody>
</table>
REFERENCES


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34. Davenport and Root, *op. cit.*, p. 168, Eq. 8-121.


43. Middleton, D., op. cit., pp. 347, 348, Eqs. 7.43a, 7.43b.
44. Papoulis, A., II, op. cit., p. 78.
46. Barbiere, D., op. cit.