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By


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The Ohio State University
1968

Approved by

Adviser
Department of Electrical Engineering
The author wishes to acknowledge the guidance of his reading committee. Professor C.H. Walter, my advisor, made many valuable contributions. Professor L. Peters, Jr. spent many long hours and provided many of the motivating ideas for the investigation. Professor R.C. Rudduck carefully read the manuscript. Special thanks are due to Professor W.H. Peake both for the development of Appendix III and for "pinchhitting" on the oral exam.

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CHAPTER I
INTRODUCTION

The guided transmission of electromagnetic energy at microwave frequencies requires the use of a waveguide transmission line system. Desirable properties of a waveguide transmission system are the transfer of the energy with low loss, a single mode of operation, and the ability to transfer large amounts of power. Much effort has been devoted to waveguide studies; an excellent historical summary of the progress of waveguide systems has been written by Southworth[1]. To date primary emphasis has been placed on waveguides constructed from conducting surfaces or dielectric materials. The purpose of this dissertation is to consider the incorporation of more general impedance surfaces in a waveguide configuration which provides additional constraints on the modal structure of the waveguide. The proper utilization of these additional constraints provides new modal structures with more desirable properties than ordinary waveguide modes. Particular emphasis of this research effort is placed on developing a low attenuation configuration by utilizing these additional constraints.
The most obvious detriment of waveguide attenuation is the loss of signal energy. This energy loss becomes increasingly important at the higher microwave frequency range. For a uniform structure the energy loss is proportional to $e^{-2\alpha l}$ where $\alpha$ is the attenuation constant in nepers per unit length and $l$ is the length of the waveguide section. Reducing the attenuation constant can significantly improve the system performance.

Another limitation imposed by waveguide attenuation concerns the power handling capability of a waveguide. The peak power handling capacity of a waveguide is determined by voltage breakdown. In addition to this limitation, the average power handling capacity of a waveguide must be considered. The average power handling capacity of a waveguide is limited by the amount of heat dissipated by the lossy waveguide. The heat dissipated in a waveguide is a function of the attenuation constant and reducing the attenuation constant increases the average power handling capacity. King[2] has analyzed the heat transfer equations for the rectangular waveguide and has published curves for the temperature increase as a function of the average power level.

Finally, a third limitation imposed by waveguide attenuation is the contribution of the loss mechanism to the noise temperature of a low noise system. The waveguide losses contribute $7^\circ K$ per $0.1 \text{ dB}$
attenuation for a low loss system. With available antennas and receivers of steadily improved noise performance, the contribution of the transmission loss to the system noise becomes quite significant.

Previous methods of reducing waveguide attenuation have been primarily concerned with open lines and over-modeled waveguide configurations. Open lines, such as the Goubau line [3] and periodic arrays of elements [4], provide inherently low loss transmission lines. Open lines have fields existing exterior to the waveguide structure. These fields may radiate at discontinuities, however, and thus reduce the transmission efficiency. In addition these fields result in a potential interference problem.

Overmoding a given closed waveguide geometry offers a potential means of reducing waveguide attenuation. Two modes, the infinite \( \text{TE}_{01} \) parallel-plate waveguide mode and the \( \text{TE}_{01} \) circular waveguide mode, have received particular attention since both of these modes are characterized by theoretical attenuation constants that decrease indefinitely with increasing frequency. Another advantage of an over-modeled waveguide configuration is the increased bandwidth handling capability offered by overmoding. The penalty of overmoding is mode conversion problems that arise at junctions with system components and at inherent imperfections in the mechanical construction of a
practical waveguide system. Mode conversion in an overmoded configuration results in two or more propagating modes. These modes may interact in such a manner as to reduce the output energy, a phenomenon known as resonant absorption. A transmission line analysis has been developed[5] to analyze the energy reduction.

The overmoded circular waveguide operating in the TE$_{01}$ mode has received much attention as a low loss waveguide structure. The TE$_{01}$ mode is not the dominant mode of this geometry and considerable oversizing is required in order that the attenuation constant for the TE$_{01}$ mode be less than that of the dominant TE$_{11}$ mode. Typically, a low loss system using this mode allows 100 to 200 forward propagating modes. A more serious disadvantage of this mode is that the low loss TE$_{01}$ mode is degenerate with the higher loss TM$_{11}$ mode and as a result mode conversion is a serious practical problem. Two modes are said to be degenerate if the phase velocities of both modes are identical. In a waveguide system capable of supporting degenerate modes, usually a hybrid mode, consisting of a linear combination of the degenerate modes, exists. Much consideration has been given to mode conversion losses in overmoded circular waveguides. A major problem, primarily attributed to the degenerate TM$_{11}$ mode, is mode conversion caused by bends. An analysis has been developed[6] that indicates bends whose radii are on the order of kilometers substantially
increases the effective attenuation and that dielectric linings may be employed in the design of bends to reduce the bend radius. The bend radius, as pointed out in Reference 6, may also be used to determine a RMS straightness tolerance. In addition for the circular cross section deformations in the cross section convert the circular modes into elliptical modes. As pointed out by Chu[7], these elliptical modes have attenuation constants that do not decrease indefinitely with increasing frequency as the TE$_{01}$ circular waveguide modes do. Thus the tolerance of the waveguide cross section must also be considered.

For the circular waveguide the problem of modal control has been considered using both the previously mentioned dielectric lining and the Helix waveguide[8]. The geometry of the Helix waveguide is circular having the outer walls formed from a tightly wound helix embedded in a lossy dielectric. This structure is then enclosed in a circular conducting container. The desired TE$_{01}$ mode, having only transverse directed currents is not affected since the helix provides a conducting path for the current. The other modes in this geometry have longitudinal current components which are attenuated by the lossy dielectric. In both the Helix waveguide and the dielectric lined circular waveguide the degeneracy of the TE$_{01}$ and TM$_{11}$ modes is destroyed and as stated by Karbowiak[6], "any modification of the
waveguide which removes the degeneracy between the $H_{01}$ and $E_{11}$ modes is a step in the right direction."

The problems presented by waveguide attenuation are indeed complex. The exact configuration to be used is dictated by system requirements. Such factors as range, bandwidth requirement, attenuation levels, noise limitations, availability and efficiency of microwave components required by the system, power handling capability, and economics must be considered. These factors are function of frequency, materials, and mechanical tolerance. In this dissertation the rectangular geometry will be considered. This geometry is suitable for medium lengths of lines; a wide variety of components have already been developed; polarization properties are desirable for many transmission line to antenna applications; and $H$ plane bends may be readily achieved without mode conversion problems. In cases in which mode conversion presents serious enough problems to warrant restricting the waveguide to dominant mode operation, the rectangular geometry has less attenuation than the circular waveguide as is demonstrated in Appendix I.

Impedance surfaces will be used for modal control. Impedance surfaces will be utilized in both their cutoff and propagating modes of operation since it will be desirable in some cases to bind and in other cases to reject modal components. An impedance surface is said to
be propagating when it supports a surface wave mode and cutoff when it does not. One of the earliest attempts at incorporating an impedance surface in a waveguide was the work of Cutler[9]. This work considered corrugated surfaces in their propagating mode to guide energy.

In this dissertation the effect of impedance surfaces on the modal structure will be viewed from both a scattering point of view and a boundary value solution. The interaction of such surfaces with plane waves may be used to assess the effects within a waveguide. Both the rectangular and circular waveguide modes may be constructed from plane wave components as was demonstrated by Schelkunoff[10] and quite recently this point of view was extended to more general geometries[11]. Thus, by knowing the scattering properties of an impedance surface, its effect within a waveguide may be assessed.

In this dissertation the propagation between equal impedance parallel planes is analyzed and an impedance loci is developed for both polarizations that enables the determination of the eigenvalues of the problem. This analysis may be extended for the three dimensional rectangular waveguide. An impedance compatibility relation is developed for the rectangular waveguide to determine which impedance configurations yield a modal solution. Examples of waveguide that satisfy this relation and that are discussed in detail are a tall waveguide with cutoff impedance vertical walls designed to reject
cross polarized modes, a waveguide with propagating horizontal walls, and a waveguide called the "E-guide" with longitudinal impedance surfaces that only supports TM modes. The use of waveguide configurations whose impedance values do not yield a modal solution has been proposed as a means of eliminating higher order mode propagation in a rectangular coax and also as a container for a surface wave structure.
CHAPTER II
WAVE INTERACTION WITH IMPEDANCE SURFACES

The interaction of electromagnetic energy with a material body has received considerable study. In this dissertation the interaction of electromagnetic energy with an impedance surface is considered. Historically this problem was first considered theoretically by Zenneck[12] who sought a solution to the problem of electromagnetic waves guided by the Earth's surface before the reflection mechanism of the ionosphere was understood. Sommerfeld[13] later obtained a solution for a vertical dipole radiating in the presence of a conducting plane. This solution separated the field into a "space wave" and a "surface wave" of the same functional form as the Zenneck wave. Since this time much progress has been made in problems of this type. The nomenclature applied to these wave solutions is not standard since the wave solutions have been applied to differing problems by a multitude of authors. Several tutorial summaries have been written to explain the interaction mechanisms involved[14, 15, 16, 17, 18].

A. Radiation of a Line Source in the Presence of an Impedance Surface

To illustrate the characteristics of problems of this type and methods of analyzing such problems, consider the problem of a
y-directed line source located above an interface characterized by a surface impedance $Z_s$ as illustrated in Fig. 1. The surface impedance $Z_s$ is defined by $E = Z_s \hat{n} \times H$, where $E$ and $H$ are the tangential field components evaluated at the surface and $\hat{n}$ is the outward directed normal to the surface.

![Diagram of line source and impedance plane]

For an electric line source Zucker\cite{14} has obtained the formal solution:

$$E_y(x, z) = \frac{i}{4\pi} \int_{-\infty}^{\infty} \frac{1}{k_1} \left[ e^{-jk_1|x-h|} + \Gamma(\beta_z) e^{-jk_1(x+h)} \right] e^{-j\beta_z z} d\beta_z$$

$$k_1^2 = \omega^2 \mu_o \varepsilon_o - \beta_z^2 \quad ; \quad k_o = \omega \sqrt{\mu_o \varepsilon_o}$$

$$\Gamma(\beta_z) = \frac{Z_s - Z_o}{Z_s + Z_o} \quad ; \quad Z_o = \sqrt{\frac{\mu_o}{\varepsilon_o}}$$

The physical interpretation of this expression is the following. The first term in brackets is due to the source and the second term is due
to the image of the source as modified by the reflection coefficient \( \Gamma(\beta_z) \) of the impedance surface. A very useful interpretation of the quantities in the brackets is that of visualizing them in terms of plane components.

By using duality similar expressions involving a magnetic line source and the magnetic field components may be obtained. The solution to the electric line source case is suitable for the TE or parallel polarized case while the magnetic line source case provides a solution for the TM or perpendicular polarized case. Both of these solutions are based on the solution of excitation by a line source and may be viewed as the Green's function for each polarization. Solutions for more general forms of excitation sources may be obtained by the usual method of superposition.

The evaluation of the formal solution presented in Eq. (1) is quite involved. No exact solution exists and the integral is customarily evaluated using asymptotic methods. In evaluating the integral, contributions are obtained from a branch cut (i.e., \( k_1 = 0 \)) and from the residues contributed from \( \Gamma(\beta_z) \). The integration in the analytically continued plane \( k_z = \beta_z - j\alpha_z \) is pictured in Fig. 2. The branch cut contributions are associated with normal radiation from the source and its image. This field, evaluated by the saddle point method\(^{14}\), is given by
The contribution from the branch cut portion of the integration is termed the space wave by Sommerfeld. The residue contributed by \( \Gamma(\beta_z) \) or its analytic continuation results in the propagation of an inhomogeneous plane wave, i.e., one that is evanescent in one direction. One can obtain the form of propagation by noting the character of the wave number in each direction of propagation.

The waves arising from the residue contribution to the field are generally classified in two different categories, surface waves and

\[
E(x, \phi) \sim \frac{1}{\sqrt{8\pi kr}} e^{-j\left(\frac{\pi}{4} - k_0r\right)} \left[ e^{jk_0h \cos \phi} + \Gamma(\psi)e^{-jk_0h \cos \phi} \right]
\]

\( k_0r \gg 1 \)

\( \phi \neq \pm \frac{\pi}{2} \)
leaky waves. Notice here that, as mentioned previously, surface wave is used in a different context than the surface wave of Sommerfeld. The distinction between the leaky wave and the surface wave is the character of the wave number directed along the surface. The surface wave is characterized by pure propagation along a lossless surface, i.e., $k_z = \beta_z^S$ and exponential decay from the surface, while the leaky wave is characterized by a complex propagation constant along the surface, i.e., $k_z^L = \beta_z - j\alpha_z$. Typical examples are the poles indicated in the $k_z$ plane depicted in Fig. 2.

Surface waves have received much attention both for their transmission line applications as well as traveling wave antenna applications. A unified summary of this wave type is contained in References 17 and 18. Generally surface waves may have either of two polarizations depending on the reactive nature of the surface impedance. When the surface is inductive, it will support a perpendicular polarized surface wave. This polarization is rejected when the surface impedance is capacitive. The field components for perpendicular polarization are given by

$$H_y = A e^{-k_o \left(\frac{X_L}{Z_o}\right) z} e^{-j k_o \left(1 + \left(\frac{X_L}{Z_o}\right)^2\right)^{1/2} z}$$

$$E_z = j X_L A e^{-k_o \left(\frac{X_L}{Z_o}\right) z} e^{-j k_o \left(1 + \left(\frac{X_L}{Z_o}\right)^2\right)^{1/2} z}$$
The coordinate system used here is that of Fig. 1 and the surface impedance is assumed $Z_s = j X_L$.

When the surface impedance is capacitive it will support a parallel polarized surface wave. This polarization is rejected when the surface impedance is inductive. The field components for parallel polarization are given by

\[
E_x = \frac{k_0 \left( 1 + \left( \frac{X_L}{Z_0} \right)^2 \right)^{1/2}}{\omega \varepsilon_0} \quad A e^{-k_0 \left( 1 + \left( \frac{X_L}{Z_0} \right)^2 \right)^{1/2} z} e^{-j k_0 \left( 1 + \left( \frac{X_L}{Z_0} \right)^2 \right)^{1/2} z}
\]

\[
E_y = B e^{-j k_0 \left( 1 + \left( \frac{X_L}{Z_0} \right)^2 \right)^{1/2} z}
\]

\[
H_x = -\frac{k_0 \left( 1 + \left( \frac{Z_0}{X_c} \right)^2 \right)^{1/2}}{\omega \mu_0} \quad B e^{-k_0 \left( 1 + \left( \frac{Z_0}{X_c} \right)^2 \right)^{1/2} z} e^{-j k_0 \left( 1 + \left( \frac{Z_0}{X_c} \right)^2 \right)^{1/2} z}
\]

\[
H_z = -\frac{j}{X_c} B e^{-j k_0 \left( 1 + \left( \frac{Z_0}{X_c} \right)^2 \right)^{1/2} z}
\]

where the surface impedance $Z_s = -j X_c$ is assumed.

The rejection of a surface wave by a surface is defined as the cutoff region of operation for the surface wave mode. This terminology is unfortunate since it connotes evanescence as is used in waveguide terminology. Zucker\textsuperscript{[14]} suggests "kill-off" might be a more appropriate and descriptive term. However, "cutoff" is commonly
used and will be used here. In the cutoff mode of operation propagation in the surface wave mode of operation simply ceases and the propagation takes place via other radiation mechanisms. The cutoff region of operation has been previously utilized in reducing sidelobe levels of horn antennas [19] and as a electromagnetic screening fence [20]. Examination of Eqs. (3) and (4) shows that if a surface wave form of solution is attempted in the cutoff mode of operation, the fields grow exponentially away from the interface thus violating the radiation condition.

The leaky waves supported by an impedance surface have received both theoretical and experimental attention. A descriptive formalism of the leaky wave representation is contained in Reference 21. Leaky waves have practical importance in traveling wave antennas [22, 23]. This wave type travels along a lossless interface with a complex propagation constant thus providing an energy leakage away from the surface. The radiation of leaky waves is restricted to certain angular regions determined by the encounter of the steepest descent path and the leaky wave pole [14].

Both the leaky wave and surface wave properties of a structure may be analyzed using a network approach called transverse resonance [24]. The basis of the method is a transmission line
analysis. The surface wave and leaky wave poles are determined from the poles of $\Gamma(\beta_z)$ defined in Eq. (1). The transverse resonance condition is

$$Z_D + Z_s = 0 \tag{5}$$

which is a pole of $\Gamma(\beta_z)$. Propagation away from the interface may be viewed as a transmission line problem as justified by Marcuvitz and Schwinger[25]. A summary of this method and its application to traveling wave structures is found in Reference 24 and a useful table of discontinuity relations is found in Reference 23. This method provides an easy means of determining the phase velocities along a structure. The cutoff region can be likewise ascertained since it is in the range for which no solution to the resonance relation (Eq. (5)) exists.

This completes the discussion of the radiation of a line source in the presence of an impedance surface. In this dissertation impedance surfaces are incorporated in waveguide geometries. The problem of incorporating impedance surfaces in a waveguide is considered first for the parallel-plate waveguide. This problem will be investigated in detail in the next chapter. This problem was first discussed by Barlow and Cullen[17] and later in more detail by Barlow[26] and Wait[27]. The modes in a rectangular waveguide may
be represented in terms of plane waves\cite{10}. Hence the plane wave scattering properties of the impedance surface must be considered. In addition, it will be found desirable to both bind or reject waves from the waveguide walls and thus the surface and leaky wave properties of the impedance surface must be known. Finally, since the attenuation of the waveguide is of interest, the loss mechanism of the impedance surface must be considered. Several examples of impedance surfaces will be considered now in some detail.

B. Corrugated Surfaces

The surface wave properties of a corrugated surface have received much attention\cite{9}. This surface has been used as an example for the modified waveguide of Chapter IV and will be discussed now. The geometry of this surface is depicted in Fig. 3.

![Fig. 3--Geometry of the corrugated surface.](image)
This surface supports a perpendicular polarized surface wave and in this mode of operation exhibits pass band and stop band characteristics. The analysis of this mode of operation has been formulated using two different methods; the function theoretic and transverse resonance.

The function theoretic method was first used by Hurd[28] to obtain the solution of TM surface wave propagation in the direction normal to the corrugations. This solution was later extended by Hougardy and Hansen[29] for the case of surface wave propagation obliquely across the corrugations. The function theoretic method may be outlined briefly as follows. The fields above the corrugations are expanded in terms of the incident field and the diffracted fields satisfying the Floquet Theorem requirement[29]. The fields in the corrugations may be thought of as waveguide fields and are expanded in terms of TEM, TE, and TM waveguide modes. In the geometries considered in this work, the gap in the corrugations are less than one-half wavelength and hence the TE and TM mode representations of the gap are evanescent. Using these modal expansions above and within the corrugations, the tangential components of the electric and magnetic field are matched across the gaps of the corrugations. The coefficients of the resulting series obtained from matching these components are then evaluated using a calculus of residues technique.
The mode amplitudes and the phase velocities of the individual modes may be evaluated in this manner.

The transverse resonance technique is less involved than the function theoretic method. Essentially only the dominant mode of the function theoretic method is retained. The surface impedance used[30] is the average of the zero impedance of the teeth and the dominant TEM impedance presented by the gap and is given by

\[ Z_{\text{cor}} = j Z_o \frac{b-a}{b} \tan k_0 h \]

The above result holds for an infinitely wide surface. If a finite width (say w) is used, then the $TE_{01}$ mode is used giving

\[ Z_{\text{cor}} = j \frac{b-a}{b} Z_{TE_{01}} \tan \beta_{TE_{01}} h \]

\[ = j \frac{b-a}{b} \frac{Z_o}{1 - \left(\frac{\lambda}{2w}\right)^2} \tan k_0 \left[1 - \left(\frac{\lambda}{2w}\right)^2\right]^{1/2} h \]

From these values, the field components may be obtained from Eq. (3).

The more general plane wave scattering properties of the corrugated surface have received much less attention. The scattering properties in the principal plane (the $xz$ plane in Fig. 3) will be considered. Two polarizations are of interest, the $TE$ ($E^i = E_y$) and the $TM$ ($H^i = H_y$). Simple transmission line representations of the surface
may be used in order to obtain a qualitative description of the plane wave scattering.

The TE scattering behavior is the easier polarization to treat. The fields in the corrugations may be expanded in terms of TE waveguide modes for this polarization. Since the plates are spaced less than one-half wavelength, the TE modes are evanescent. In this analysis, the tooth height (h in Fig. 3) will be assumed to be of sufficient height so that the reflection from the terminating plane (y = -h in Fig. 3) will be negligible. This problem is similar to the scattering by an infinite stack of parallel plates. The infinite stack of parallel plates scatters like a conducting plane since the evanescent waveguide fields representing the modal structure of the gaps can transfer no energy. For the lossless case the reflection coefficient has a unit magnitude. The surface will thus scatter much like a conducting plane. A more exact analysis[31], taking into account the current distributions on the teeth and the evanescent fields in the gap, yields a phase angle correction to the unit amplitude reflection coefficient. This correction is given by $e^{-j\frac{4\pi d_1}{\lambda}}$ where

\[
\frac{2\pi d_1}{\lambda} = 2x \ln 2 + \sin^{-1} \frac{2x}{\sqrt{1 - 4y^2}} - \sin^{-1} \frac{x}{\sqrt{1 - 2y}} - \sin^{-1} \frac{x}{\sqrt{1 + 2y}}
\]

\[
\cos \theta
\]
For the geometries used here, this phase correction is negligible.

The scattering by an incident TM polarized wave is more complicated. For this polarization, a propagating TEM field is induced in the corrugations and the effect of the terminating plane must be taken into account. In order to obtain an approximate model to study the scattering behavior, the corrugated surface will be approximated as a homogeneous impedance plane. The scattering of this impedance plane is evaluated using the impedance boundary condition

\begin{align}
E_z &= Z_s H_y \\
\text{(9)}
\end{align}

It is to be noted that this surface impedance is anisotropic, i.e., for the field components \(E_y\) and \(H_z\), the surface impedance is zero. The total fields on the surface are used in this formulation. The reflection coefficient from this surface defined in terms of the magnetic field is given by

\begin{align}
R &= \frac{Z_o \cos \theta - Z_s}{Z_o \cos \theta + Z_s} \\
\text{(10)}
\end{align}
where \( Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \) is the impedance of free space. The surface impedance will be approximated in the following manner. The fields in the corrugations will be assumed to be principally composed of the propagating TEM fields, that is, the effect of the evanescent TM fields will be neglected. This is a reasonable assumption for small tooth widths and many corrugations per wavelength. The surface impedance, averaged as in the transverse resonance case, is given by Eq. (6). Since the surface impedance is assumed to be pure imaginary, the reflection coefficient has unit amplitude and is given by

\[
R = 1 / \beta
\]

where

\[
\beta = 2 \tan^{-1} \left( - \frac{(b-a) \tan \sqrt{\frac{\mu_0 \varepsilon_0}{b \cos \theta}} h}{b \cos \theta} \right)
\]

The loss mechanism of a surface is a very fundamental consideration when its incorporation in a low loss configuration is being considered. Collin[32] has considered the loss mechanism of the corrugated surface in its surface wave mode of operation. In his analysis, the dominant TEM mode fields are assumed in the corrugations and the dissipation loss in the corrugations is obtained by integrating \( \frac{1}{2} R_s |H_T|^2 \) over the corrugations. The parameter \( R_s \) is the surface resistance equal to \( \sqrt{\omega \mu / 2\sigma} \) where \( \sigma \) is the conductivity of
the material and $H_T$ is the tangential component of the magnetic field intensity. This analysis is valid only in the pass band of operation.

The dissipation loss in watts per unit width per meter, as obtained by Collin, is

$$D_L = \frac{1}{2} R_s A A^* \left( 1 + \frac{h}{b} + \frac{\sin 2k_0h}{2k_0b} \right)$$

where $A$ is proportional to the magnetic field at the surface. Taking a ratio of this quantity with the dissipation loss for a plane conducting surface, which equals $\frac{1}{2} R_s |H_T|^2$ defines a figure of merit for the corrugated surface

$$\delta_{SUR} = 1 + \frac{h}{b} + \frac{\sin 2k_0h}{2k_0b}.$$  \hspace{1cm} (12)$$

It is to be noted that this quantity always exceeds unity and the larger its value, the greater the loss.

For the more general plane wave scattering modes, the dissipation losses may be similarly calculated. Consider the dissipation loss for the TM polarized wave first. The current distribution will be assumed to be that induced on the equivalent impedance sheet. Using the definition of the magnetic field of Eq. (11):

$$|H_T| = H^i \left[ (1 + \cos \beta)^2 + \sin^2 \beta \right]^\frac{1}{2}$$

$$= H^i \left[ 2(1 + \cos \beta) \right]^\frac{1}{2}$$

where $H^i$ is the incident magnetic field intensity.
Consider a segment of surface containing one corrugation and extending a width d. The dissipation losses obtained by integrating \( \frac{1}{2} R_s |H_T|^2 \) over this surface is

\[
D_L = \frac{R_s}{2} H^{i2} \left[ (1 + \cos \beta) da + 2(1 + \cos \beta) \left( 1 + \frac{\sin 2k_0h}{2 k_0b} \right) dh \\
+ (1 + \cos \beta) d(b-a) \right]
\]

\[
= R_s H^{i2} [(1 + \cos \beta)d] \left[ b + 2h \left( 1 + \frac{\sin 2k_0h}{2 k_0b} \right) \right]
\]

The first term in the brackets is the dissipation contribution from the top of the tooth, the second term is the dissipation contribution from the sides of the teeth and the third term is the dissipation contribution from the terminating plane. Comparing this quantity with that of the ordinary conducting plane surface which equals \( \frac{1}{2} R_s |H|^2 \) db and again taking a ratio of the two quantities yields

\[
(14) \quad \delta_{TM} = \frac{1}{2} (1 + \cos \beta) + \frac{h}{b} \left( 1 + \frac{\sin 2k_0h}{2 k_0b} \right)
\]

The parameter \( \delta_{TM} \) defines a figure of merit for the TM polarized plane wave. It is to be noted that this figure is always greater than unity. When the corrugation height h approaches zero, \( \cos \beta \to 1 \) and \( \delta_{TM} \) equals unity as it should.

For the TE polarized plane wave no propagating fields are introduced in the corrugations. In this case the dissipation loss
appears only on the teeth tops since the evanescent field representation in the corrugations transport zero energy. For example, with teeth spaced $\lambda/10$, the field drops to its $1/e$ value in $\lambda/30$. The figure of merit in this case is equal to

\begin{equation}
\delta_{TE} = \frac{a}{b}
\end{equation}

This ratio is always less than unity.

These loss calculations are based on using a uniform current distribution as obtained from the propagating modes of the configuration in Fig. 3. The evanescent fields as used in the field representation of this configuration transport no energy for the perfectly conducting case and very little for the imperfectly conducting case since the waveguide modes comprising the field representations are well beyond cutoff for the geometrics considered. The assumption of a uniform current distribution on the corrugations seems to violate the well known edge conditions\[33\] as used in diffraction theory. Hansen\[34\], in a similar problem, has shown that the higher order modes add in such a manner to satisfy these edge conditions. Thus, it is expected that the first order results used here will give excellent results.

In designing a corrugated surface, several criteria must be observed in order that the analysis and approximations presented here are valid. First, there should be several corrugations per wavelength
in order that the corrugations function as a surface; in practice, ten corrugations per wavelength give excellent results. The ratio of the tooth width to corrugation width should not exceed one-half in order that the averaged impedance as defined in Eq. (6) is valid. Another assumption that has been made is that the higher order evanescent modes in the modal representation of the corrugation gap are attenuated to a negligible amount so that reflection from the terminating plane may be neglected. The attenuation factor for the next higher order mode is

$$\text{(16)} \quad -k_0 \left[ \left( \frac{\lambda}{2(b-a)} \right)^2 + \left( \frac{\lambda}{2d} \right)^2 - 1 \right] \frac{1}{\xi} - k_0 \left( \frac{\lambda}{2(b-a)} \right) \xi$$

for small $\xi$, where $d$ is the width of the corrugated surface which will be approximately $\lambda/2$ for a practical waveguide, $b-a$ is the width of the corrugation gap, and $\xi$ is the distance measured into the gap. The next higher order mode is attenuated to its $1/e$ value in the distance listed in Table I.

**TABLE I**

<table>
<thead>
<tr>
<th>Width of Gap</th>
<th>Distance to Decay to $1/e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda/10$</td>
<td>$0.0318 \lambda$</td>
</tr>
<tr>
<td>$\lambda/20$</td>
<td>$0.0159 \lambda$</td>
</tr>
<tr>
<td>$\lambda/30$</td>
<td>$0.0106 \lambda$</td>
</tr>
</tbody>
</table>
The surface impedance for the perpendicular polarized surface wave of the corrugated surface in Fig. 3 is anisotropic. One way of making the surface impedance isotropic is to doubly corrugate it or to use an array of pins embedded in a ground plane. The surface wave behavior of such structures was investigated both theoretically and experimentally by Querido[35]. For the square post case the phase velocity was found to be

\[ \frac{c}{v} = \left[ 1 + \left( \frac{b-a}{b} \right)^2 \tan^2 1.05 kh \right]^{\frac{1}{2}} \tag{18} \]

and for the circular post case

\[ \frac{c}{v} = \left[ 1 + \left( 1 - \frac{\pi a}{2s} \right)^2 \tan^2 1.05 kh \right]^{\frac{1}{2}} \tag{19} \]

where \( a \) is the post radius and \( s \) is the center to center spacing. These structures also exhibit the same pass band and stop band behavior as the ordinary corrugated surface.

The scattering behavior of the doubly corrugated surface differs from the behavior of the ordinary corrugated surface. Consider scattering in the principal planes and again the transmission line analysis will be employed. For the TM scattering the reflection coefficient will have unit amplitude for the lossless case. The phase of the reflection coefficient, defined in terms of the magnetic field,
may be obtained from Eq. (10). The appropriate impedances, obtained from the velocity ratios in Eqs. (18) and (19), are

\[
Z_{\text{sq}} = jZ_0 \left( \frac{b-a}{b} \right) \tan 1.05 \, kh \quad \text{square post}
\]

\[
Z_{\text{cir}} = jZ_0 \left( 1 - \frac{\pi a}{Z_s} \right) \tan 1.05 \, kh \quad \text{circular post}
\]

The phase of the reflection coefficient for the TM polarized scattering is given by

\[
\beta_{\text{sq}} = 2 \tan^{-1} \left( \frac{(b-a)\tan 1.05 \, kh}{b \cos \theta} \right) \quad \text{square post}
\]

\[
\beta_{\text{cir}} = 2 \tan^{-1} \left( \frac{\frac{\pi a}{Z_s} - 1}{\frac{Z_s}{\cos \theta}} \tan 1.05 \, kh \right) \quad \text{circular post}
\]

The TE polarization is somewhat different in this case than the corrugated surface. The scattering takes place from the terminating planes for this polarization since the scattering from the pins is small. Thus only a phase translation is required for this polarization. This scattering analysis has been used and yields approximate but useful results[36].

C. Dielectric Slabs

Another example of an impedance surface that is commonly used is a dielectric slab as depicted in Fig. 4. This surface is used both by itself or mounted on a ground plane. The surface wave behavior of
Fig. 4--Geometry of dielectric slab.

this surface is treated in detail in References 37 and 38. The modal structure of the surface wave behavior is described in Reference 37. The modes within the dielectric may be expanded in terms of sinusoidal functions having even or odd functional symmetry with respect to the center of the dielectric. In Reference 37, the characteristic equations for both polarizations are derived and a graphical method of determining the modal eigenvalues is developed. The dielectric slab without ground plane supports surface waves of both polarizations having no cutoff frequency and with higher order modes appearing each time the effective width of the corrugated surface reaches an integer
multiple of a half wavelength. For the case of a dielectric slab mounted on a ground plane, the parallel polarized modes support only the odd symmetric modes in the dielectric and the perpendicular polarized modes support even symmetry modes. Thus, the parallel polarized modes have a cutoff frequency whereas the perpendicular polarized modes do not. In addition the surface wave behavior of a dielectric covered ground plane exhibit pass band and stop band characteristics.

The problem may be also discussed from an impedance point of view. For the perpendicular polarized modes the even symmetry modes have an equivalent circuit consisting of a TEM transmission line based on the dielectric parameters terminated in a short circuit and the odd symmetry modes have an equivalent circuit consisting of the same TEM transmission line terminated in an open circuit. By duality the parallel polarized modes have the short and open circuit terminations reversed[37]. These impedances are listed in Table II where t is defined in Fig. 4. These impedances hold for a single surface wave mode. For configurations that are thick enough to support several surface wave modes, a more general impedance must be used. This problem has been considered in a recent paper[39].
TABLE II
IMPEDANCE RELATIONS FOR DIELECTRIC SLABS

<table>
<thead>
<tr>
<th>Polarization</th>
<th>Symmetry</th>
<th>Impedance value</th>
</tr>
</thead>
<tbody>
<tr>
<td>parallel</td>
<td>even</td>
<td>(-j \sqrt{\mu/\epsilon} \cot \omega \sqrt{\mu/\epsilon} t)</td>
</tr>
<tr>
<td>parallel</td>
<td>odd</td>
<td>(j \sqrt{\mu/\epsilon} \tan \omega \sqrt{\mu/\epsilon} t)</td>
</tr>
<tr>
<td>perpendicular</td>
<td>even</td>
<td>(j \sqrt{\mu/\epsilon} \tan \omega \sqrt{\mu/\epsilon} t)</td>
</tr>
<tr>
<td>perpendicular</td>
<td>odd</td>
<td>(-j \sqrt{\mu/\epsilon} \cot \omega \sqrt{\mu/\epsilon} t)</td>
</tr>
</tbody>
</table>

An expression for the attenuation constant of surface waves propagating on a thinly coated ground plane has been developed by Collin[40]. The attenuation constant is

\[
\alpha = k_o \left(R_s + \frac{k''k_{ot}}{k'} \right) \left(X_s + k_{ot} \frac{k' - 1}{k'} \right) \text{ nepers/length}
\]

where \(R_s + jX_s = (1 + j) \left( \frac{k_o}{Z_0 Z_o} \right) \) is the normalized surface impedance of the ground plane with conductivity \(\sigma\) and \(\epsilon_r = \kappa' - j\kappa''\) is the relative dielectric constant of the slab. For thick slabs having several modes propagating in the dielectric a perturbational form of solution, using the field values of the lossless configuration, may be used to determine the loss mechanism.

The dielectric slab is capable of supporting a leaky wave. A detailed study of these properties is contained in References 41 and 42. A transmission line analysis may be used to study this behavior.
When the slab is sufficiently thick to support two surface wave modes, it is also possible to propagate a leaky wave mode. Thus for a thick dielectric slab, the existence of both surface wave and leaky wave modes of operation must be considered.

D. A Leaky Wave Grid Surface

In this section a leaky wave grid structure depicted in Fig. 5 will be considered. This structure has found application for traveling wave antennas[22]. Leaky wave structures have been utilized within waveguide structures for filter applications[43].

![Diagram of leaky wave grid](image)

Fig. 5--Geometry of leaky wave grid.

A detailed discussion of the transverse resonance solution for this structure may be found in Reference 44 and propagation as a leaky wave structure in both the z and y directions in Fig. 5 is considered. In this mode of operation energy is incident on the structure from the \(-h < x < 0\) region and "leaks" into the space \(x > 0\).
via the slots in the surface. As previously discussed in Section A of this chapter the radiation contribution of a leaky wave pole is restricted to the angular region of space determined analytically by the encounter of the steepest descent path and the leaky wave pole[14]. In the previously mentioned waveguide filter application this property is used to radiate the undesired frequency components into an absorber region. By properly choosing the parameters of the configuration the desired frequency behavior of the filter may be obtained.

Quite recently the plane wave scattering behavior of the structure depicted in Fig. 5 has been considered[45]. The scattering properties of this surface have been considered for both the TE and TM polarized components using a transverse resonance method of analysis. The scattering properties are characterized by a unit amplitude reflection coefficient for the lossless structure. The phase of the reflection coefficient for the magnetic field varies extremely rapidly when the plane wave is incident at approximately the leaky wave angle. This behavior is illustrated in Fig. 6. This sharp change in the reflection properties of this surface suggests that this surface may be used as a scatterer in a filter application.
Fig. 6--Phase behavior of reflection coefficient for magnetic field for wave incident on leaky wave structure.

E. Application to Other Structures

The analysis of other structures has been considered using the concepts of propagation on an "impedance" surface. Perhaps the earliest applications of this analysis was the use of surface wave concepts by Serracchioli and Levis[46]. They used surface wave concepts to analyze the phase velocity along an infinite array of dipoles in order to obtain the phase velocity. This analysis was verified experimentally by Damon[47]. A unified summary of both
this method and conventional antenna analysis using the vector potential method was written by Mailloux[48]. The surface wave concept provides a useful method of analyzing such structures. The array of dipoles exhibits pass and stop band of operation in the end-fire mode of operation predicted by this analysis. This structure is also of interest for its already mentioned low loss properties[4]. Use of such a structure in a waveguide will be discussed later.
CHAPTER III 
PROPAGATION BETWEEN PARALLEL IMPEDANCE SURFACES 

The effect of electromagnetic energy incident on an impedance surface and the properties of several of these surfaces has been considered in the preceding chapter. In the present chapter consideration will be given to propagation between two parallel impedance surfaces. It appears that this problem was first considered by Barlow and Cullen[17] in 1953 and Barlow later discussed this problem in greater detail in another paper[26]. Wait[27] has also given theoretical consideration to this problem.

A. Formal Solution

The geometry of this problem is depicted in Fig. 7. Two polarizations will be considered here: perpendicular, which has $H_x$. 

![Diagram of parallel plate configuration](image)

Fig. 7--Geometry of parallel plate configuration.
$E_y$ and $E_z$ field components, and parallel, which has $E_x$, $H_y$ and $H_z$ field components. In the case when the impedance surfaces are conducting surfaces, the modal structure consists of a perpendicularly polarized TEM wave and higher order TM and TE waveguide bounce modes. Here the term "bounce modes" is used to refer to energy traveling in plane wave components bouncing between the plates. For the case in which the surface impedances are reactive, the modal structure would intuitively consist of surface waves bound to the impedance surfaces and the additional possibility of modal components bouncing between the parallel planes. Indeed, this is the case as further analysis will show. Each of these modal components satisfy the impedance boundary conditions; hence, as discussed in the case of the isolated impedance plane, it is expected that pass band and stop band modes of operation, dependent on the nature of the impedance, will exist in this case. It will be shown that this is the case.

The modes in this structure may be formulated in terms of a Hertzian vector[49]. The perpendicular polarized modes may be expanded as

$$\overline{E} = -j\omega\mu \nabla \times \overline{\pi_H}$$

$$\overline{H} = k^2 \overline{\pi_H} + \nabla \nabla \cdot \overline{\pi_H}$$

$$(23)$$

$$\nabla^2 \overline{\pi_H} + k^2 \overline{\pi_H} = 0$$

$$\overline{\pi_H} = \hat{\Lambda} \psi_H(y) e^{-\Gamma z}$$
and in a similar manner the parallel polarized modes may be expanded as

\[ \overline{H} = j\omega \varepsilon \nabla \times \overline{\pi}_E \]

\[ \overline{E} = k^2 \overline{\pi}_E + \nabla \nabla \cdot \overline{\pi}_E \]

\[ \nabla^2 \overline{\pi}_E + k^2 \overline{\pi}_E = 0 \]

\[ \overline{\pi}_E = \overline{\psi}_E(y) e^{-\Gamma(z)} \]

where \( k = \omega \sqrt{\mu \varepsilon} \) and \( \psi_H \) and \( \psi_E \) are scalar potential functions appropriate to the problem. Expanding these quantities, the field components are obtained for the perpendicular polarized mode

\[ H_x = k^2 \psi_H(y) \]

\[ E_y = j\omega \mu \Gamma \psi_H(y) \]

\[ E_z = j\omega \mu \frac{\partial \psi_H(y)}{\partial y} \]

\[ \frac{\partial^2 \psi_H(y)}{\partial y^2} + (\Gamma^2 + k^2) \psi_H(y) = 0 \]

and for the parallel polarized mode

\[ E_x = k^2 \psi_E(y) \]

\[ H_y = -j\omega \varepsilon \Gamma \psi_E(y) \]

\[ H_z = -j\omega \varepsilon \frac{\partial \psi_E(y)}{\partial y} \]

\[ \frac{\partial^2 \psi_E(y)}{\partial y^2} + (\Gamma^2 + k^2) \psi_E(y) = 0 \]
It now remains to obtain suitable potential functions $\psi_H$ and $\psi_E$.

The potential functions $\psi_H$ and $\psi_E$ must be chosen in such a manner that the impedance boundary conditions imposed by the surfaces will be satisfied. Considering the propagation mechanisms involved, it would seem that one type of mode would consist of surface waves bound to the parallel planes and another type of mode would consist of bounce modes. Accordingly, the potential functions

$$
\psi_H = e^{-p_0 y} + R_H e^{-p_0 (H-y)}
$$

and

$$
\psi_E = e^{-p_0 y} + R_E e^{-p_0 (H-y)}
$$

where chosen. Neglecting losses, real values of $p_0$ correspond to waves bound to the surfaces, while imaginary values of $p_0$ correspond to bounce modes. If losses were included, the values of $p_0$ would be complex to account for energy losses. The first term in the potential functions is a wave associated with the surface $y = 0$ and the second term is a wave associated with the surface $y = H$. Both potential functions yield the following eigenvalue relation

$$
 p_0^2 + \Gamma^2 + k^2 = 0 .
$$
The boundary conditions imposed by the impedance surfaces are

\[(30) \quad Z_1 = -\frac{E_y}{H_x} \bigg|_{y=0} \]

\[(31) \quad Z_2 = \frac{E_y}{H_x} \bigg|_{y=H} \]

for the perpendicular polarized modes and

\[(32) \quad Z_1 = \frac{E_x}{H_z} \bigg|_{y=0} \]

\[(33) \quad Z_2 = -\frac{E_x}{H_z} \bigg|_{y=H} \]

for the parallel polarized modes. It is to be noted that the field components used in these boundary conditions are orthogonal with respect to the two polarizations. Using the field components defined in Eqs. (25) and (26) and the potential functions defined in Eqs. (27) and (28), the following results are obtained

\[(32) \quad Z_1 = j \frac{P_0}{\omega \varepsilon} \left( \frac{1 - R_H e^{-p_0 H}}{1 + R_H e^{-p_0 H}} \right) \]

\[(33) \quad Z_2 = j \frac{P_0}{\omega \varepsilon} \left( \frac{R_H - e^{-p_0 H}}{R_H + e^{-p_0 H}} \right) \]
for the perpendicular polarized modes and

\[
(33) \quad Z_1 = -j \frac{\omega_L}{\omega_P} \left( \frac{1 + R_E e^{-\omega_P H}}{1 - R_E e^{-\omega_P H}} \right)
\]

\[
Z_2 = -j \frac{\omega_L}{\omega_P} \left( \frac{R_E + e^{-\omega_P H}}{R_E - e^{-\omega_P H}} \right)
\]

for the parallel polarized modes.

Equations (32) and (33) are two equations in two unknowns for each polarization. Solutions of these equations will complete the modal solutions.

Equations (32) and (33) may be separated in order to allow solution for \(\omega_P\) and \(R_H\) and \(R_E\). Solving each of the two equations for either \(R_E\) or \(R_H\) results in the equations

\[
(34) \quad \frac{1 - Z_1 \left( \frac{\omega_P}{j \omega_P} \right)}{1 + Z_1 \left( \frac{\omega_P}{j \omega_P} \right)} = e^{2 \omega_P H}
\]

\[
\frac{1 - Z_2 \left( \frac{\omega_P}{j \omega_P} \right)}{1 + Z_2 \left( \frac{\omega_P}{j \omega_P} \right)} = e^{2 \omega_P H}
\]

for the perpendicular polarized mode and

\[
(35) \quad \frac{Z_1 \left( \frac{j \omega_P}{\omega_L} \right) - 1}{Z_1 \left( \frac{j \omega_P}{\omega_L} \right) + 1} = e^{-2 \omega_P H}
\]

\[
\frac{Z_2 \left( \frac{j \omega_P}{\omega_L} \right) - 1}{Z_2 \left( \frac{j \omega_P}{\omega_L} \right) + 1} = e^{-2 \omega_P H}
\]

for the parallel polarized modes. Equation (34) is identical to the result obtained by Wait[27]. Each equation involves only the unknown
However, a general solution of the equations is a formidable task. Wait[27] considered some approximate solutions to this equation.

The quantities $R_H$ and $R_E$ may also be obtained as follows. For $R_H$ define

$$K_H = \frac{Z_1}{Z_2}$$  \hspace{1cm} \text{(36)}$$

Using this definition and Eq. (32), $R_H$ is obtained

$$R_H = \frac{1}{2} \left[ \frac{1 - K_H}{1 + K_H} \right] \left( e^{P_0H} - e^{-P_0H} \right)$$

$$\pm \left( \left( \frac{1 - K_H}{1 + K_H} \right)^2 \left( e^{P_0H} - e^{-P_0H} \right)^2 + 4 \right)^{\frac{1}{2}}.$$  \hspace{1cm} \text{(37)}$$

In a similar manner, define

$$K_E = \frac{Z_2}{Z_1}$$  \hspace{1cm} \text{(38)}$$

Then

$$R_E = \frac{1}{2} \left[ \frac{1 - K_E}{1 + K_E} \right] \left( e^{P_0H} - e^{-P_0H} \right)$$

$$\pm \left( \left( \frac{1 - K_E}{1 + K_E} \right)^2 \left( e^{P_0H} - e^{-P_0H} \right)^2 + 4 \right)^{\frac{1}{2}}.$$  \hspace{1cm} \text{(39)}$$

Substituting the values of $P_0$ as obtained from Eqs. (34) and (35) respectively, the values of $R_H$ and $R_E$ may be obtained.
B. **Equal Impedance Case**

The formulation is simplified when the impedances on the parallel planes are identical. This solution has not been previously considered. However, it will be utilized later. The values of $R_H$ and $R_E$ as given in Eqs. (37) and (39), both reduce to the values $\pm 1$ for the case when the impedances are equal. This is not surprising since physically the symmetry of the geometry requires equal amplitude waves and one would expect both a symmetric and an anti-symmetric distribution.

The impedances then reduce to the following values:

\[
Z_H = Z_1 = Z_2 = j \frac{P_0}{\omega \epsilon} \tanh \frac{P_0H}{2} \quad R_H = +1
\]

\[
= j \frac{P_0}{\omega \epsilon} \coth \frac{P_0H}{2} \quad R_H = -1
\]

and

\[
Z_E = Z_1 = Z_2 = - j \frac{\omega \mu}{P_0} \coth \frac{P_0H}{2} \quad R_E = +1
\]

\[
= - j \frac{\omega \mu}{P_0} \tanh \frac{P_0H}{2} \quad R_E = -1
\]

The problem may be simplified if it is assumed the impedances are pure reactive. Under this assumption, no loss mechanism is present and hence the propagation constant $\Gamma$ will be pure real, corresponding to evanescent waves, or pure imaginary, corresponding to a
propagating solution with no attenuation. As a consequence of the
eigenvalue Eq. (29), the value $p_0$ will be either pure real or pure
imaginary. Real values of $p_0$ correspond to bound surface waves on
the planes while imaginary values of $p_0$ correspond to bounce modes.
Table III summarizes the properties of the equal impedance, lossless,
parallel-plate waveguide.

For the isolated impedance plane, surface waves exhibit pass
band and stop band behavior depending on the reactive nature of the
impedance. Perpendicular polarized surface waves are cutoff for
capacitive surfaces and parallel polarized surface waves are cutoff
for inductive surfaces. The same behavior for the parallel plate
configuration is indicated by Table III. Perpendicular polarized
waves require inductive reactances in order to bind energy to the
surfaces and parallel polarized waves require capacitive reactances
to bind energy to the surface.

Each of the wave types illustrated in Table III possesses a
different impedance boundary condition; hence, for a given impedance,
each mode is represented by a different value of $p_0$. The impedance
loci for the perpendicular polarized case is given in Fig. 8. To find
the value $p_0$ for a given mode, plot the reactive value of the given
surface as a horizontal line. The intersection of this horizontal line
with the impedance loci yields the value of $p_0$. The bound modes give
<table>
<thead>
<tr>
<th>Mode</th>
<th>Polarization</th>
<th>$\text{RE or RH}$</th>
<th>$\text{Nature of } P_0^*$</th>
<th>Impedance Boundary Condition</th>
<th>Nature of Impedance</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>perpendicular</td>
<td>1, real</td>
<td>$j \frac{</td>
<td>P_0</td>
<td>}{\omega \varepsilon} \tanh \frac{</td>
</tr>
<tr>
<td>B</td>
<td>perpendicular</td>
<td>1, imaginary</td>
<td>$-j \frac{</td>
<td>P_0</td>
<td>}{\omega \varepsilon} \tan \frac{</td>
</tr>
<tr>
<td>C</td>
<td>perpendicular</td>
<td>-1, real</td>
<td>$j \frac{</td>
<td>P_0</td>
<td>}{\omega \varepsilon} \coth \frac{</td>
</tr>
<tr>
<td>D</td>
<td>perpendicular</td>
<td>-1, imaginary</td>
<td>$j \frac{</td>
<td>P_0</td>
<td>}{\omega \varepsilon} \cot \frac{</td>
</tr>
<tr>
<td>E</td>
<td>parallel</td>
<td>1, real</td>
<td>$-j \frac{\omega u}{</td>
<td>P_0</td>
<td>} \coth \frac{</td>
</tr>
<tr>
<td>F</td>
<td>parallel</td>
<td>1, imaginary</td>
<td>$j \frac{\omega u}{</td>
<td>P_0</td>
<td>} \cot \frac{</td>
</tr>
<tr>
<td>G</td>
<td>parallel</td>
<td>-1, real</td>
<td>$-j \frac{\omega u}{</td>
<td>P_0</td>
<td>} \tanh \frac{</td>
</tr>
<tr>
<td>H</td>
<td>parallel</td>
<td>-1, imaginary</td>
<td>$-j \frac{\omega u}{</td>
<td>P_0</td>
<td>} \tan \frac{</td>
</tr>
</tbody>
</table>

* real values of $P_0$ correspond to bound modes; imaginary values, to bounce modes
Fig. 8.-Impedance loci for perpendicular polarized, equal impedance, lossless parallel-plate waveguide.

several values of \( p_0 \) and the cutoff condition for a bounce mode, as obtained from the eigenvalue Eq. (29), is \( p_0 = k_0 \). For \( p_0 > k_0 \) the bounce modes are evanescent in the \( z \) direction. An interesting point to be noted about mode \( C \), the antisymmetric bound mode, is the rather high impedance required. This mode has a minimum impedance value of \( \frac{2}{H \omega \varepsilon} \) where \( \varepsilon_0 = 8.85 \times 10^{-12} \) farads per meter. For an impedance surface with a small reactive value, this mode is cutoff. A similar
The situation exists for the parallel polarized modes. The impedance loci for the parallel polarized case is given in Fig. 9.

![Impedance loci](image)

**Fig. 9**--Impedance loci for parallel polarized, equal impedance, lossless parallel-plate waveguide.

C. Power Flow

The power flow per unit width of parallel plate waveguide for each individual mode may also be computed. For the case of equal surface impedances $R_H$ and $R_E$ are equal to $\pm 1$, which is a real number, and $P_o$ is, for the lossless case, pure real or pure imaginary. The power flow per unit width is given by
(47) \[ P = \frac{1}{2} \text{Re} \int_0^H \mathbf{E} \times \mathbf{H}^* \cdot \mathbf{\hat{z}} \, dy \]

Assume the mode is propagating, i.e., \( \Gamma = j \beta \). For the perpendicular polarized modes

(48) \[ P = -\frac{1}{2} \text{Re} \int_0^H E_y H_x^* \, dy \]

\[ = \frac{1}{2} \omega \mu \beta k^2 \text{Re} \int_0^H \psi_H(y) \psi_H^*(y) \, dy \]

and for the parallel polarized modes

(49) \[ P = \frac{1}{2} \text{Re} \int_0^H E_x H_y^* \, dy \]

\[ = \frac{1}{2} \omega \varepsilon \beta k^2 \text{Re} \int_0^H \psi_E(y) \psi_E^*(y) \, dy . \]

Since the potential functions are of the same functional form, the integrals take the same form. For real values of \( p_o \), define

(50) \[ I_1 = \int_0^H (e^{-p_o y} + R e^{-p_o (H-y)}) (e^{-p_o y} + R e^{-p_o (H-y)}) \, dy \]

\[ = 1 - \frac{e^{-2p_o H}}{p_o} + 2 R H e^{-p_o H} . \]

For imaginary values of \( p_o \), let \( p_o = j \gamma \) and define

(51) \[ I_2 = \int_0^H (e^{-j\gamma y} + R e^{-j\gamma (H-y)}) (e^{j\gamma y} + R e^{j\gamma (H-y)}) \, dy \]

\[ = 2 H \left( 1 + R \frac{\sin \gamma H}{\gamma H} \right) . \]
Here use has been made of the fact that $R^2 = 1$. The power flow then becomes

\[
P = \frac{1}{2} \omega \mu \beta k^2 \left[ \frac{1 - e^{-2p_0H}}{p_0H} + R_H H e^{-p_0H} \right] \quad p_0 \text{ real}
\]

\[
= \omega \mu \beta k^2 H \left[ 1 + R_H \frac{\sin \gamma H}{\gamma H} \right] \quad p_0 = j \gamma \text{ imaginary}
\]

for the perpendicular polarized modes and

\[
P = \frac{1}{2} \omega \varepsilon \beta k^2 \left[ \frac{1 - e^{-2p_0H}}{p_0H} + 2 R_E H e^{-p_0H} \right] \quad p_0 \text{ real}
\]

\[
= \omega \varepsilon k^2 H \left[ 1 + R_E \frac{\sin \gamma H}{\gamma H} \right] \quad p_0 = j \gamma \text{ imaginary}
\]

for the parallel polarized case.

The possibility of generating backward waves on this structure was also considered. A backward wave is one in which the phase velocity and group velocity are oppositely directed. One method of determining whether backward waves can exist is to plot the phase velocity as a function of frequency, i.e., a $k$-$\beta$ diagram\[50\]. With the transcendental relations involved, this is a difficult task. For the parallel plate case, notice that $\Gamma$ has been chosen as $+j \beta$, i.e., propagation in the $+z$ direction. The power flow given in Eqs. (52) and (53) are always positive indicating a power flow in the $+z$ direction.
Hence the phase velocity and energy are both directed in the $+z$ direction and no backward wave can exist.

D. Orthogonality Relations

The orthogonality of these modes may be considered by using the orthogonality relationship

$$\int_0^H \mathbf{E}_{tn} \times \mathbf{H}_{tm} \cdot \hat{z} \, dy = 0 \quad n \neq m$$

where the subscript $t$ denotes the transverse field components and $n$ and $m$ refer to the individual modes. From the vector relationships of the field components the perpendicular and parallel polarized modes are seen to be orthogonal with respect to one another since the Poynting vector for a mixture of perpendicular and parallel modes has no component in the $z$ direction. For combinations of parallel or perpendicular polarized modes, the field components defined in Eqs. (25) and (26) yield the following integrals

$$I_H = -j \omega \mu l_1 k^2 \int_0^H \psi_{H1} \psi_{H2} \, dy \quad \text{perpendicular polarized}$$

$$I_E = -j \omega \epsilon l_1 k^2 \int_0^H \psi_{E1} \psi_{E2} \, dy \quad \text{parallel polarized}$$

where $\Gamma_1$ is the propagation constant associated with mode 1.
As shown in Appendix II, these modes are indeed orthogonal. This orthogonality is important, it means physically that a modal expansion using Fourier coefficients may be obtained. Suppose, for example, that the electric or magnetic field is specified on the plane $z = 0$, i.e.,

\begin{align}
H_x &= g_1(y) \text{ perpendicular polarized} \\
E_x &= g_2(y) \text{ parallel polarized}
\end{align}

The modal components may then be written as

\begin{align}
H_x &= \sum_n a_n k^2 \psi_{H_n}(y) e^{-\Gamma n z} \text{ perpendicular polarized} \\
E_x &= \sum_n b_n k^2 \psi_{E_n}(y) e^{-\Gamma n z} \text{ parallel polarized}
\end{align}

The Fourier coefficients may be obtained in the usual manner and are equal to

\begin{align}
a_n &= \frac{\int_0^H g_1(y) \psi_{H_n}(y) \, dy}{k^2 \left(1 + R_{H_n}^2\right) \left(1 - \frac{e^{-2p_n H}}{2p_n}\right) + 2 R_{H_n} H e^{-p_n H}}
\end{align}
Another orthogonality condition that holds only for the lossless case, i.e., the impedance surfaces are pure reactive, is

\[ b_n = \frac{\int_0^H g_2(y) \psi_{E_n}(y) \, dy}{k^2 \left( 1 + R_{E_n}^2 \right) \left( \frac{1 - e^{-2R_{E_n}H}}{2R_{E_n}} \right) + 2 R_{E_n} H e^{-P_{E_n}H}} \]

(58) (cont.)

For the lossless case this relation leads to an orthogonality relation also developed in Appendix II in a manner similar to the previous orthogonality relation. Orthogonality does not hold for the lossy case. However, from the vector relationships the parallel and perpendicular modes are orthogonal with respect to one another. A consequence of the non-orthogonality of the lossy case is that the power flow for a general excitation can no longer be written as a sum of the power flows of the individual modes. Physically this implies that coupling exists between the modes.

Wait[27] considered the coupling of a slit source into this waveguide. His analysis indicated that for a slit located towards one of the impedance surfaces of a fairly tall waveguide the majority of the
energy was launched into the surface wave mode. Wait also considered the modal attenuation of both the surface wave mode and the leaky modes. He found the waveguide modes attenuated more rapidly than the surface wave modes. This result is not surprising since the same holds for the perfectly conducting case.
CHAPTER IV
INCORPORATION OF IMPEDANCE SURFACES IN A RECTANGULAR GEOMETRY

In this chapter impedance surfaces will be incorporated within the rectangular waveguide structure. For waveguides with homogeneous isotropic cross sections terminated by impedance surfaces, an impedance compatibility condition is developed. The choice of impedance boundary conditions may not be made independently and several examples of waveguide satisfying this criteria will be discussed. The incorporation of an impedance surface within the waveguide cross section will also be considered.

A. Impedance Compatibility Condition

The geometry of the rectangular waveguide used here is depicted in Fig. 10. The impedance boundary conditions considered here are

---

Fig. 10--Geometry of rectangular waveguide cross section.
In Appendix III the most separable form of solution in rectangular coordinates is considered for this configuration. In this appendix it is shown that this form of solution cannot be achieved for a completely independent choice of impedances and the relation between the impedances is derived. The condition for a separable form of solution with an \( e^{-\Gamma z} \) dependence requires the following impedance relationship

\[
Z_1 Z_3 - Z_2 Z_3 + Z_2 Z_4 = 0
\]

This relation will be used to determine which waveguides have a modal solution.

The ordinary rectangular waveguide is characterized by zero value impedance walls; thus, Eq. (61) is satisfied but this is a trivial case. The modes in the ordinary waveguide are conventionally
classified as TE and TM modes. These modes may be constructed from plane wave components. The modal structure of a waveguide with impedance surfaces, as will be discussed in detail, may be obtained by considering the effect of the impedance surface on the plane wave construction.

In order to visualize the use of impedance surfaces in a rectangular geometry it is necessary to understand the modal structure of the waveguide. The modes in a perfectly conducting rectangular waveguide may be constructed from plane wave components \[10\]. For the TE\(_{01}\) rectangular waveguide mode the elemental plane waves comprising this mode travel in the waveguide as illustrated in Fig. 11.

![Fig. 11 - Elemental plane waves for rectangular waveguide operating in the TE\(_{01}\) mode.](image)

The structure and polarization of these elemental plane waves is to be noted for later use. The waves impinging on the vertical walls are parallel polarized and are incident at the angle \(\phi\) defined by
The waves on the horizontal walls travel diagonally across the surface and are polarized perpendicular to the surface.

B. Rectangular Waveguide with Cutoff Walls

One of the first configurations considered was a rectangular waveguide with anisotropic impedance surfaces in their cutoff mode of operation on the vertical walls [51]. The anisotropic surfaces are needed in order that the dominant mode is unaffected. An example of a suitable surface is the corrugated surface discussed in Chapter II. In terms of the impedance relations given in Eq. (60), $Z_1$, $Z_3$ and $Z_4$ are zero and $Z_2$ is non-zero. Thus this configuration satisfies the constraint imposed by Eq. (61). The modal structure of this configuration may be obtained by considering the effect of the impedance surface on the plane wave modal structure of this waveguide.

For the waveguide with cutoff vertical walls, the ordinary modal structure is modified. The ordinary $\text{TE}_{01}$ mode is modified by only the small phase shift given in Eq. (8) and thus propagates in a similar manner as the ordinary mode. The cross polarized mode, $\text{TE}_{10}$, is considerably modified. The elemental plane waves forming this mode would have to travel across a cutoff impedance surface. Such a surface rejects a wave traveling across it. The cross
polarized mode is thus rejected. The higher order cross polarized modes consist therefore of modes bouncing between the vertical walls. Thus the next higher mode is similar to a $\text{TE}_{11}$ mode. This mode has to bounce in the $y$ direction of Fig. 10 and the eigenvalue for the $x$ direction may be obtained from the parallel plate analysis. Fig. 8 may be used for a graphical solution for the eigenvalue. This problem will be considered in the next section. This mode may be made evanescent by a proper choice of waveguide dimensions.

The modified waveguide propagates a $\text{TE}_{01}$ dominant mode. The higher order $\text{TE}_{10}$ cross polarized mode is rejected by the cutoff corrugated surface. The rejection of this mode is important in that it allows a height increase while preserving a dominant mode of operation. As will be discussed in more detail, a height increase reduces the attenuation constant for a rectangular waveguide. The rejection of the cross polarized field component also offers potential applications for feed structures used with radiating systems.

It has long been known [52] that the attenuation of a rectangular waveguide may be reduced by increasing the height of the rectangular waveguide. This reduction in attenuation may be seen analytically by examining the expression for the attenuation constant of the rectangular waveguide. For the geometry in Fig. 10, the attenuation constant [52] for the dominant $\text{TE}_{01}$ rectangular waveguide mode is given by
\( \alpha = \frac{R_s}{Z_0 \sqrt{1-(f_c/f)^2}} \left[ \frac{1}{H} + \frac{2}{W} \left(\frac{f_c}{f}\right)^2 \right] \)

where \( f \) is the operating frequency, \( f_c \) is the waveguide cutoff frequency, \( R_s \) is the surface resistance equal to \( \frac{\omega}{\sqrt{2\sigma}} \) for a conductor of conductivity \( \sigma \), and \( Z_0 \) is the free space impedance. The first term in the bracket is the contribution from the horizontal walls and the second term in the bracket is the contribution from the vertical walls. Consider the functional form of these contributions and note in addition that the surface resistance is proportional to \( f^{1/2} \). The contribution from the vertical walls decreases with increasing frequency and asymptotically approaches \( K_2 f^{-3/2} \). The contribution from the horizontal walls increases with increasing frequency and asymptotically approaches \( K_2 f^{1/2} \). It is to be noted that the contributions from the horizontal walls results in the eventual increase in attenuation constant and in addition, the constant \( K_2 \) is inversely proportional to the waveguide height. The disadvantage in increasing the height of an ordinary waveguide is the possibility of propagating undesired higher order modes. For a waveguide with cutoff vertical impedance walls, the cross polarized component is rejected and the height may be increased while retaining dominant mode operation.

The reduction in the attenuation constant possible by a height increase in a rectangular waveguide may be considered by examining the expression for the attenuation constant given in Eq. (63). Consider
the reduction in the attenuation constant as a function of the height increase in comparison to a conventional rectangular waveguide with a width to height ratio of 2:1. Denote the parameters of the normal 2:1 waveguide by the subscript N and the parameters of the taller waveguide by the subscript T. The two waveguide are assumed to have the same width W and in the TE₀₁ mode of propagation will thus have the same cutoff frequency \( f_c \). Taking a ratio of the attenuation constants for these two waveguides

\[
\frac{\alpha_T}{\alpha_N} = \frac{\frac{1}{H_T} + \frac{2}{W} \left( \frac{f_c}{f} \right)^2}{\frac{1}{H_N} + \frac{2}{W} \left( \frac{f_c}{f} \right)^2}
\]

\[
= \frac{H_N}{H_T} \frac{1 + \frac{H_T}{H_N} \frac{2H_N}{W} \left( \frac{f_c}{f} \right)^2}{1 + \frac{2H_N}{W} \left( \frac{f_c}{f} \right)^2}
\]

When the frequency is much larger than the cutoff frequency, \( \alpha_T/\alpha_N \) asymptotically approaches the ratio of the waveguide heights. For a waveguide with the width restricted such that only a single bounce mode in the vertical plane propagates, the ratio of the attenuation constants of the tall guide to the normal guide is plotted in Fig. 12 as a function of \( f_c/f \) with a 2:1 (i.e. \( H_N/W = 1/2 \)) waveguide used as a basis of comparison.
Fig. 12--Ratio of attenuation constants as a function of height increase.
The above attenuation constants have been computed assuming that the loss mechanism is identical on all surfaces. For a tall waveguide having cutoff corrugated surfaces on the vertical walls, the loss mechanism of the vertical walls is less than that for the conducting surface. For the dominant $TE_{01}$ mode the elemental plane waves comprising this mode are polarized $TE$ with respect to the vertical walls. The ratio of the loss mechanism for the corrugated surfaces and a conducting surface has already been investigated and defined as $\delta_{TE}$ in Eq. (15). In order that the impedance approximation for the corrugated surface be valid, the tooth width of corrugation width should be less than one-half and thus an upper bound for $\delta_{TE}$ is one-half. The attenuation contribution of the vertical walls (the second term in the brackets of Eq. (63)) should be multiplied by $\delta_{TE}$ in order to calculate the attenuation constant for a tall waveguide having corrugated vertical walls. The attenuation constant will thus be still lower.

The dominant mode of this waveguide is $TE_{01}$. As long as the guide width remains less than one wavelength, the $TE_{0n}$, $n > 1$, modes will be cutoff. The higher order modes bouncing between all four walls are required to satisfy the boundary conditions imposed by the vertical corrugated surfaces. The higher order rectangular
waveguide modes do not satisfy these boundary conditions and the presence of the impedance boundary condition necessitates a hybrid mode formulation. Such a formulation will be given in the next section.

A one foot section of a corrugated waveguide with a square cross section has been constructed. This waveguide operates at X-band frequencies and appears in the photographs of Fig. 13.

(a) Experimental model

(b) Experimental system

Fig. 13—Experimental model and system.
Measurements on this section of waveguide have been made using the experimental system shown in the photographs. A direct measure of the attenuation is not feasible on this short section. The measurements made do confirm two fundamental premises of this analysis; first, substitution of the corrugated surfaces for conducting ones do not appear to increase the attenuation and second, the cutoff corrugated surfaces effectively reject the cross polarized mode components.

Measurements were performed with the corrugations both uncovered and covered with conducting tape. No discernable change in the output power level was noted thus confirming that the corrugations do not contribute a substantially increased loss mechanism when the modal energy is polarized parallel with respect to the corrugations.

The section of waveguide was then rotated by ninety degrees and an attempt was made to force the energy to propagate in the cross polarized configuration. The output power level was decreased by a nominal 15 dB over the operating frequency band. A sizable amount of this energy was oriented in the desired polarization indicating a mode conversion to the desired polarization. Above 9.4 GHz it was noted that some of the energy was propagating in a hybrid mode when the waveguide was operating cross polarized.
C. Rectangular Waveguide with Propagating Walls

In Chapter III propagation between parallel impedance planes was discussed in detail. In this section, a three dimensional waveguide will be considered and the equal impedance case analyzed in Section B of Chapter III will be modified to include the third dimension. Suppose conducting planes are used for the vertical walls of Fig. 10 and impedances surfaces are used as the horizontal walls. The impedance boundary conditions for this configuration in terms of the definitions in Eq. (60) become \( Z_3 \) and \( Z_4 \) equal zero while \( Z_1 \) equals \( Z_2 \) and both values non-zero for an isotropic impedance surface and \( Z_1 \) equals zero with \( Z_2 \) non-zero for the anisotropic surface used here. These configurations both satisfy the compatibility condition in Eq. (61).

Consider the modal structure of this modified waveguide from the point of view of the elemental plane waves comprising the ordinary TE\(_{01}\) waveguide mode. The effect of the propagating surface is to convert the homogeneous elemental plane waves into inhomogeneous surface waves bound to each propagating surface. The surface would produce in addition an "E-field tilt" to satisfy the impedance boundary conditions imposed by the surface. This field component modifies the TE\(_{01}\) mode and necessitates a hybrid mode formulation of the problem since there is a \( z \) component of both E and H.
The hybrid modes in rectangular geometry have been formulated by Collin[53] in terms of a Hertzian potential. The hybrid modes may be classified as longitudinal-section electric (LSE) and longitudinal-section magnetic (LSM). These modes are defined by

\begin{align*}
\text{(65) LSE modes} \\
\bar{\pi}_h &= \hat{x} \psi_h (x, y) e^{-\Gamma z} \\
\bar{E} &= -j\omega \mu \nabla \times \bar{\pi}_h \\
\bar{H} &= k^2 \bar{\pi}_h + \nabla \nabla \cdot \bar{\pi}_h \\
\nabla^2 \bar{\pi}_h + [\Gamma^2 + k^2] \bar{\pi}_h &= 0
\end{align*}

\begin{align*}
\text{(66) LSM modes} \\
\bar{\pi}_e &= \hat{x} \psi_e (x, y) e^{-\Gamma z} \\
\bar{H} &= j\omega \epsilon \nabla \times \bar{\pi}_e \\
\bar{E} &= k^2 \bar{\pi}_e + \nabla \nabla \cdot \bar{\pi}_e \\
\nabla^2 \bar{\pi}_e + [\Gamma^2 + k^2] \bar{\pi}_e &= 0
\end{align*}

Consider the rectangular geometry depicted in Fig. 10. The orientation of the x and y coordinates in this figure is arbitrary; thus two spatial orientations orthogonal to each other must be considered. Alternatively, the potential functions may be defined
oriented in the $y$ direction. Therefore four mode types, two orthogonally oriented modes each for the LSE and LSM modes, must be considered. In terms of the geometry of Fig. 10 and the scalar potentials $\psi_h$ and $\psi_e$, the field components are given by

\begin{align}
\text{(67) LSE modes} & \quad E_x = 0 \\
& \quad E_y = \Gamma (j \omega \mu) \psi_h \\
& \quad E_z = j \omega \mu \frac{\partial \psi_h}{\partial y} \\
& \quad H_x = k_o \psi_h + \frac{\partial^2 \psi_h}{\partial x^2} \\
& \quad H_y = \frac{\partial^2 \psi_h}{\partial y \partial x} \\
& \quad H_z = -\Gamma \frac{\partial \psi_h}{\partial x}
\end{align}

and for the spatially orthogonal mode

\begin{align}
\text{(68)} & \quad E_x = -\Gamma (j \omega \mu) \psi_h \\
& \quad E_y = 0 \\
& \quad E_z = -j \omega \mu \frac{\partial \psi_h}{\partial x} \\
& \quad H_x = \frac{\partial^2 \psi_h}{\partial x \partial y}
\end{align}
\[ H_y = k_0^2 \psi_h + \frac{\partial^2 \psi_h}{\partial y^2} \]

\[ H_z = -\Gamma \frac{\partial \psi_h}{\partial y} \]

(69) LSM modes

\[ E_x = k_0^2 \psi_e + \frac{\partial^2 \psi_e}{\partial x^2} \]

\[ E_y = \frac{\partial^2 \psi_e}{\partial y \partial x} \]

\[ E_z = -\Gamma \frac{\partial \psi_e}{\partial x} \]

\[ H_x = 0 \]

\[ H_y = -\Gamma j\omega \epsilon \psi_e \]

\[ H_z = -j\omega \epsilon \frac{\partial \psi_e}{\partial y} \]

and for the spatially orthogonal mode

(70)

\[ E_x = \frac{\partial^2 \psi_e}{\partial x \partial y} \]

\[ E_y = k_0^2 \psi_e + \frac{\partial^2 \psi_e}{\partial y^2} \]

\[ E_z = -\Gamma \frac{\partial \psi_e}{\partial y} \]

\[ H_x = \Gamma (j\omega \epsilon) \psi_e \]

\[ H_y = 0 \]

\[ H_z = j\omega \epsilon \frac{\partial \psi_e}{\partial x} \]
In this application it is desired to place propagating impedance surfaces on the planes \( y = 0 \) and \( y = H \) and conducting surfaces at \( x = 0 \) and \( x = W \). The potential functions \( \psi_e \) and \( \psi_h \), must be chosen in such a manner that the impedance boundary conditions of the surface will be satisfied as well as the boundary conditions imposed by the vertical walls. The propagating surfaces are of two types, those that present an isotropic surface impedance, such as a dielectric clad conducting plane, and those that present an anisotropic surface impedance, such as a corrugated surface. Examination of the field equations eliminates some of these mode types from consideration.

For the isotropic surface, the following conditions must be satisfied

\[
Z_1 = \pm \frac{E_x}{H_z} \left| \frac{E_x}{E_z} \right|_{y=0} = \pm \frac{E_z}{H_x} \left| \frac{E_z}{E_x} \right|_{y=H} = Z_2
\]

(71)

where the upper sign is associated with the surface \( y = 0 \) and the lower sign is associated with the surface \( y = H \). Two modes may be eliminated from consideration immediately; the LSE mode defined in Eq. (67) and the LSM mode defined in Eq. (69). The field components require a zero and an infinite impedance condition respectively.

The impedance boundary condition for the anisotropic surfaces utilized here is
\[ Z_2 = \frac{1}{H} \left. \frac{E_z}{H} \right|_{y=0}^{y=H} \]

which is used in conjunction with

\[ E_x \bigg|_{y=0}^{y=H} = 0 = Z_1. \]

The impedance condition of the anisotropic surface eliminates the LSM mode defined in Eq. (69) since this mode requires an infinite impedance. Two other modes, the LSE mode defined in Eq. (68) and the LSM mode defined in Eq. (70), may also be eliminated by considering the boundary conditions imposed by the impedance surface for the anisotropic case. By considering the field components in terms of the potential functions (in Eq. (70) define \( \psi = \frac{\partial \psi_e}{\partial y} \)), it is to be noted that both the value of a function and its derivative is specified along the boundaries formed by the propagating surfaces. This results in an overspecification of the boundary conditions. A discussion of boundary conditions is contained in Reference 54. The wave equation is classified in this reference as a hyperbolic equation and the specification of both the value and the derivative of a function along a boundary is the Cauchy boundary condition. This overspecification of boundary conditions in this problem restricts the value of the impedance. If suitable potential functions are chosen, the LSE mode defined in Eq. (68) requires a zero impedance condition to satisfy these conditions while the LSM mode defined in Eq. (70) requires an infinite impedance condition.
Thus, for the case of surfaces with finite surface impedance, two modes, the LSE mode defined in Eq. (68) and the LSM mode defined in Eq. (70), must be considered when the propagating surface presents an isotropic surface impedance; and one mode, the LSE mode defined in Eq. (67), must be considered when the propagating surface presents an anisotropic surface impedance.

As a specific example of a hybrid waveguide with propagating surfaces of the anisotropic type, consider the example of the corrugated surface as the propagating surface. This surface presents an anisotropic surface impedance of the type defined in Eq. (72). For this configuration the LSE mode defined in Eq. (67), as previously discussed may propagate. A suitable potential for this problem that satisfies the boundary conditions imposed by the vertical walls is

\[
\psi_h = \sin \frac{\pi x}{W} \left( e^{-P_0 y} \pm e^{-P_0 (H-y)} \right)
\]

Here the \(e^{j\omega t - k z}\) dependence is implied and it is assumed that the width of the waveguide is restricted such that the higher order modes of the form

\[
\psi_h = \sin \frac{n \pi x}{W} \left( e^{-P_0 y} \pm e^{-P_0 (H-y)} \right); n > 1
\]

are cutoff. The field equations are then given by
(76) \[ E_x = 0 \]

\[ E_y = \Gamma (j\omega \mu_0) \sin \frac{\pi x}{W} \left( e^{-\rho_0 y} \pm e^{-\rho_0 (H-y)} \right) \]

\[ E_z = -j\omega \mu_0 \rho_0 \sin \frac{\pi x}{W} \left( e^{-\rho_0 y} \pm e^{-\rho_0 (H-y)} \right) \]

\[ H_x = \left( k_0^2 - \frac{\pi}{W} \right)^2 \sin \frac{\pi x}{W} \left( e^{-\rho_0 y} \pm e^{-\rho_0 (H-y)} \right) \]

\[ H_y = -\frac{\pi}{W} \rho_0 \cos \frac{\pi x}{W} \left( e^{-\rho_0 y} \pm e^{-\rho_0 (H-y)} \right) \]

\[ H_z = -\Gamma \frac{\pi}{W} \cos \frac{\pi x}{W} \left( e^{-\rho_0 y} \pm e^{-\rho_0 (H-y)} \right) \]

\[ \rho_0^2 - \left( \frac{\pi}{\omega} \right)^2 + \Gamma^2 + k^2 = 0 \]

It is to be noted that when \( \rho_0 = 0 \) the solution reduces to the ordinary TE\(_{01} \) waveguide mode.

In order to relate the parameters of the propagating surface and the parameter \( \rho_0 \) it is necessary to examine the impedance boundary condition defined in Eq. (72). The corrugated surface as previously discussed in Eq. (7) presents an impedance

(77) \[ Z_{cor} = j \frac{b-a}{b} Z_{TE_{01}} \tan B_{TE_{01}} h \]

The modal structure of the waveguide presents an impedance
Matching these two impedances yields the following equation which relates the modal structure of the waveguide to the parameters of the propagating surface.

\[ Z_{\text{GUIDE}} = \frac{E_z}{Hx} \mid _{y=0}^{y=H} \]

\[ = \frac{j \rho_0}{\omega \varepsilon_0} \left\{ \frac{\rho_0 H}{\cosh \left[ \frac{\rho_0 H}{2} \right] \left[ 1 - \left( \frac{\lambda}{2W} \right)^2 \right]} \right\} + \text{sign} \]

\[ = \frac{j \rho_0}{\omega \varepsilon_0} \left\{ \frac{\rho_0 H}{\cosh \left[ \frac{\rho_0 H}{2} \right] \left[ 1 - \left( \frac{\lambda}{2W} \right)^2 \right]} \right\} - \text{sign} \]

The parameter \( \rho_0 \) and its relation to the impedance surface may be directly related to the propagation properties of the parallel impedance surfaces. The impedance configuration chosen here corresponds to the equal impedance case discussed in detail in Chapter III. It is to be noted that Eq. (79) has the same form as
Eq. (40) and thus the impedance loci in Fig. 8 may be used to obtain the eigenvalue $p_0$. The parameter $p_0$ may have real or imaginary values and satisfy Eq. (79). Real values correspond to surface wave modes and imaginary values correspond to bounce or higher order waveguide modes. In this section the surface is chosen to be propagating, i.e. the surface impedance is chosen to be inductive. Hence, both the surface wave and bounce modes must be considered.

In the preceding section the surface was cutoff so that only the higher order bounce modes could propagate. This analysis thus provides a method of analyzing the higher order modes of the preceding section.

The waves in this case are perpendicularly polarized with respect to the impedance surfaces and the impedance loci depicted in Fig. 8 is applicable. For a small value of inductive impedance appropriate for a low loss configuration, the antisymmetric mode (-sign in the potential function given in Eq. (74)) will be rejected, as was discussed for Mode C in the parallel plate case. For a tall waveguide properly excited most of the energy will be contained in the surface wave mode. In addition the bounce modes, as concluded by Wait [27], attenuate more rapidly than the surface wave mode. Thus only the symmetric surface wave mode (+ sign in the potential function given in Eq. (74)) will be considered. An example of the transverse decay
rate $P_0$ as a function of the corrugation height $h$ is given for a waveguide one wavelength in width ($W = 1$) and five wavelengths in height ($H = 5$) in Fig. 14.

The cutoff frequency for these hybrid modes may be obtained from the eigenvalue relation given in Eq. (76). The cutoff wavenumber, $k_c$, may be obtained by setting $\Gamma$ equal to zero. The value, $P_0$, may be obtained, for a specified value of impedance, from the impedance loci in Fig. 8 as previously discussed. Thus the cutoff wavenumber, $k_c$, is equal to $\left(\left(\frac{\pi}{W}\right)^2 - P_0^2\right)^{1/2}$. It is to be noted that $P_0$ is real for bound modes and imaginary for bounce modes and thus the sign of $P_0^2$ changes between the bound modes and the bounce modes. The cutoff condition for the bound mode is thus when the guide width, $W$, is somewhat less than one-half wavelength. For the bounce modes, the cutoff frequencies as a function of the waveguide height, are spaced approximately in half wavelength intervals as may be seen from Fig. 8.

The attenuation constant for this corrugated waveguide may be obtained by using the field components defined in Eq. (76). The attenuation constant is defined as

\begin{equation}
(80) \quad \alpha = \frac{P_L}{2P}
\end{equation}

where $P_L$ is the power loss in the walls and $P$ is the power flow in the waveguide. The power flow in the waveguide is given by
Fig. 14--Relation between the modal structure of one wavelength by five wavelengths waveguide and the propagating surface.
For this mode, \( E_x = 0 \) and the \( \hat{z} \) component of the Poynting vector is \( E_y \times H_x^* \). The power flow down the waveguide is given by

\[
P = \frac{1}{2} \text{Re} \iint \vec{E} \times \vec{H}^* \cdot \hat{z} \, dS
\]

waveguide cross section

where \( \beta \) is the propagation constant, i.e. \( \Gamma = j\beta \). The power loss on the vertical walls is identical for both walls and the total loss is given by

\[
P_{LV} = \frac{R_s}{2} \int_0^H \left\{ |H_y|^2 + |H_z|^2 \right\} \bigg|_{x=0} \, dy
\]

\[
P_{LV} = R_s \left( \frac{\pi}{W} \right)^2 \left( p_o + \beta^2 \left( \frac{1-e^{-2p_0H}}{p_o} \right) + 2He^{-p_0H} k_o^2 \left( 1 - \left( \frac{\lambda}{2W} \right)^2 \right) \right)
\]

The tangential components of the magnetic field on the horizontal walls are \( H_x \) and \( H_z \). The losses on the top and bottom walls are identical. The propagating surface has fields induced in the corrugations; the losses in the corrugations may be taken into account by the \( \delta_{SUR} \) factor defined in Eq. (12). The dissipation loss in the horizontal walls is given by
\[ P_{\text{LH}} = 2 \left[ \frac{R_s}{2} \int_0^W \left\{ |H_x|^2 + \delta_{\text{SUR}} |H_z|^2 \right\} \, dx \right] \]

\[ = R_s \frac{W}{2} \left( 1 + e^{-P_0 H} \right)^2 \left[ \kappa_o \left( 1 - \left( \frac{\lambda}{2W} \right)^2 \right)^2 + \delta_{\text{SUR}} \beta^2 \left( \frac{\pi}{W} \right)^2 \right] \]

The attenuation constant is equal to

\[ \alpha = \frac{P_{\text{LV}} + P_{\text{LH}}}{2 P} \]

\[ = \frac{2R_s \left( \frac{\pi}{W} \right)^2 (P_0^2 + \beta^2) \left( \frac{1 - e^{-2P_0 H}}{P_0} \right) + 2He^{-P_0 H} (\beta^2 - P_0^2)}{Z_0 \beta k_o^3 \left( 1 - \left( \frac{\lambda}{2W} \right)^2 \right)^2 \left( \frac{\pi}{W} \right)^2} \]

\[ + \frac{R_s WlHe^{-P_0 H}}{Z_0 \beta k_o^3 \left( 1 - \left( \frac{\lambda}{2W} \right)^2 \right)^2 \left( \frac{\pi}{W} \right)^2} \left[ \kappa_o \left( 1 - \left( \frac{\lambda}{2W} \right)^2 \right)^2 + \delta_{\text{SUR}} \beta^2 \left( \frac{\pi}{W} \right)^2 \right] \]

In order to obtain a low loss configuration, the wave should be loosely bound to the propagating surface; that is, \( P_0 \) should be small.

Consider the limit of the attenuation constant as \( P_0 \to 0 \). Note that for this limit
\[ \beta \rightarrow k_0 \left(1 - \left(\frac{\lambda}{2W}\right)^2\right)^{1/2} \]

\[ k_c \rightarrow \frac{\pi}{W} \]

\[ \lim_{P_0 \rightarrow 0} \frac{1 - c^{-2P_0H}}{P_0} = \lim_{P_0 \rightarrow 0} \frac{2He^{-2P_0H}}{1} = 2H \]

\[ \delta_{SUR} \rightarrow 1 \]

The attenuation constant thus approaches

\[ \lim_{P_0 \rightarrow 0} \alpha \rightarrow \frac{2R_s}{Z_0} \left\{ \left( k_c^2 \left( \beta \right)^2 + \frac{W}{2} \left( 2 \right)^2 \left( \beta^2 + k_c^2 \right) \right) \right\} \]

\[ = \frac{R_s}{Z_0 \sqrt{1 - \left( \frac{f_c}{f} \right)^2}} \left[ \frac{2}{W} \left( \frac{f_c^2}{f} + \frac{1}{H} \right) \right] \]

It is to be noted that this expression is identical to that of the rectangular waveguide given in Eq. (63). Thus, it is to be expected that for small values of \( P_0 \) the attenuation constant has the same character as the rectangular waveguide attenuation constant. The attenuation constant has been evaluated for a waveguide one wavelength by five wavelengths in internal cross section for two different choices of surface parameters in Table IV. These parameters were not selected to optimize the attenuation constant but illustrate
### TABLE IV. ATTENUATION CONSTANTS FOR HYBRID WAVEGUIDE

<table>
<thead>
<tr>
<th>Frequency Range</th>
<th>Internal Waveguide Dimensions (inches)</th>
<th>Operating Frequency GHz</th>
<th>$p_0 = 0.4$</th>
<th>$p_0 = 0.525$</th>
<th>Ordinary 2:1 Guide</th>
<th>Infinite Parallel Plate Waveguide</th>
</tr>
</thead>
<tbody>
<tr>
<td>L-Band</td>
<td>6.500 x 33.00</td>
<td>1.806</td>
<td>0.0619</td>
<td>0.101</td>
<td>0.175</td>
<td>0.0350</td>
</tr>
<tr>
<td>S-Band</td>
<td>2.840 x 14.20</td>
<td>4.156</td>
<td>0.214</td>
<td>0.346</td>
<td>0.605</td>
<td>0.121</td>
</tr>
<tr>
<td>G-Band</td>
<td>1.372 x 9.360</td>
<td>6.304</td>
<td>0.399</td>
<td>0.645</td>
<td>1.134</td>
<td>0.227</td>
</tr>
<tr>
<td>J-Band</td>
<td>1.372 x 6.860</td>
<td>8.602</td>
<td>0.635</td>
<td>1.028</td>
<td>1.800</td>
<td>0.360</td>
</tr>
<tr>
<td>H-Band</td>
<td>1.122 x 5.610</td>
<td>10.518</td>
<td>0.865</td>
<td>1.400</td>
<td>2.44</td>
<td>0.489</td>
</tr>
<tr>
<td>X-Band</td>
<td>0.900 x 4.500</td>
<td>13.114</td>
<td>1.203</td>
<td>1.948</td>
<td>3.39</td>
<td>0.677</td>
</tr>
<tr>
<td>KU-Band</td>
<td>0.622 x 3.110</td>
<td>18.992</td>
<td>2.091</td>
<td>3.389</td>
<td>5.90</td>
<td>1.18</td>
</tr>
<tr>
<td>K-Band</td>
<td>0.420 x 2.100</td>
<td>28.094</td>
<td>3.778</td>
<td>6.109</td>
<td>10.61</td>
<td>2.14</td>
</tr>
<tr>
<td>KA-Band</td>
<td>0.280 x 1.400</td>
<td>42.172</td>
<td>6.946</td>
<td>11.218</td>
<td>19.51</td>
<td>3.92</td>
</tr>
</tbody>
</table>

* Waveguides constructed from aluminum ($\sigma = 34.7 \times 10^6$ mhos/meter).

** K equals corrugation gap width divided by corrugation spacing.
the attenuation obtained by two examples of a practical choice of parameters. The ordinary rectangular waveguide and the infinite \( \text{TE}_{01} \) parallel plate waveguide attenuation constants are also given for comparison purposes.

The modal structure of a waveguide containing propagating surfaces that present an isotropic surface impedance will now be considered. The dielectric clad ground plane is perhaps the best known example of such a surface. This surface, used as a surface wave structure, supports both parallel and perpendicular polarized surface waves depending on the thickness and relative permittivity of the structure used. In the discussion of the modal structure of this waveguide, two mode types must be considered as previously discussed, the LSE mode defined in Eq. (68) and the LSM mode type defined in Eq. (70). That two mode types must be considered is not surprising in view of the fact that this impedance surface as discussed in Chapter II will support surface waves of both polarizations. For the LSE mode type a suitable potential is given by

\[
\psi_h = \cos \frac{\pi x}{W} \left( e^{-P_1 y} \pm e^{-P_1 (H-y)} \right)
\]

The field components for this mode are given by
\begin{align}
(87) & \quad E_x = -j\omega \mu \Gamma \cos \frac{\pi x}{W} \left( e^{-P_1y} \pm e^{-P_1(H-y)} \right) \\
& \quad E_y = 0 \\
& \quad E_z = j\omega \mu \frac{\pi}{W} \sin \frac{\pi x}{W} \left( e^{-P_1y} \pm e^{-P_1(H-y)} \right) \\
& \quad H_x = \frac{\pi}{W} \rho_1 \sin \frac{\pi x}{W} \left( e^{-P_1y} \mp e^{-P_1(H-y)} \right) \\
& \quad H_y = (k^2 + \rho^2) \cos \frac{\pi x}{W} \left( e^{-P_1y} \pm e^{-P_1(H-y)} \right) \\
& \quad H_z = \Gamma \rho_1 \cos \frac{\pi x}{W} \left( e^{-P_1y} \mp e^{-P_1(H-y)} \right) \\
& \quad \rho^2 - \frac{\pi^2}{W^2} + \Gamma^2 + k^2 = 0 \\

\text{For the LSM mode, a suitable potential is given by} \\
(88) & \quad \psi_e = \sin \frac{\pi x}{W} \left( e^{-P_2y} \pm e^{-P_2(H-y)} \right) \\
\text{The field components for this mode are given by} \\
(89) & \quad E_x = -\frac{\pi}{W} \rho_2 \cos \frac{\pi x}{W} \left( e^{-P_2y} \mp e^{-P_2(H-y)} \right) \\
& \quad E_y = (k^2 + \rho^2) \sin \frac{\pi x}{W} \left( e^{-P_2y} \pm e^{-P_2(H-y)} \right) \\
& \quad E_z = \Gamma \rho_2 \sin \frac{\pi x}{W} \left( e^{-P_2y} \mp e^{-P_2(H-y)} \right) \\
& \quad H_x = j\omega \epsilon \Gamma \sin \frac{\pi x}{W} \left( e^{-P_2y} \pm e^{-P_2(H-y)} \right) \\
& \quad H_y = 0 \\
& \quad H_z = j\omega \epsilon \frac{\pi}{W} \cos \frac{\pi x}{W} \left( e^{-P_2y} \pm e^{-P_2(H-y)} \right) 
\end{align}
Again it has been assumed that the width of the waveguide is restricted to permit only one bounce mode in the \( x \) direction.

The relation between the impedance surface parameters and the modal parameters \( p_1 \) and \( p_2 \) will be considered now. The dielectric clad ground plane discussed in Chapter II is a suitable surface that will provide an isotropic surface impedance. The impedance used in Chapter II and given in Table II is valid for surfaces that are infinitely wide. For a finite width surface, as in the case of the corrugated surface, the finite width must be taken into account and hence the appropriate impedance becomes

\[
P_2 - \left( \frac{\pi}{W} \right)^2 + \Gamma^2 + k^2 = 0
\]

(90)

\[
Z = j \frac{Z_0}{\sqrt{\mu}} \tan \left[ \omega \sqrt{\mu} \sqrt{\epsilon} \mu \right] \left( 1 - \left( \frac{\lambda}{2W} \right)^2 \right) \frac{1}{t}
\]

where \( Z_0 = \sqrt{\frac{\mu}{\epsilon}} \) and \( W \) is the width of the dielectric surface of thickness \( t \). The modal impedance as given in Eq. (71) for the LSE mode defined in Eq. (87) is

(91)

\[
Z = -j\omega\mu_0 \frac{\coth \frac{p_1H}{2}}{p_1} \pm \text{sign}
\]

\[
= -j\omega\mu_0 \frac{\tan h \frac{p_1H}{2}}{p_1} \mp \text{sign}
\]
and the modal impedance for the LSM mode defined in Eq. (89) is

\[(92)\quad Z = j \frac{P_2}{\omega_0} \tan h \frac{P_2 H}{2} + \text{sign} \]

\[= j \frac{P_2}{\omega_0} \coth \frac{P_2 H}{2} - \text{sign} \]

The parameters \(p_1\) and \(p_2\) may be obtained by equating the surface impedance in Eq. (90) to the modal impedances in Eqs. (91) and (92). The following equations are then obtained

\[(93)\quad \text{LSE mode} \]

\[\tan \left[k \left(1 - \left(\frac{\lambda}{2W}\right)^2\right)^{\frac{1}{2}} t\right] = - k \frac{\coth \frac{P_1 H}{2}}{p_1} + \text{sign} \]

\[= - k \frac{\tan h \frac{P_1 H}{2}}{p_1} - \text{sign} \]

\[\text{LSM mode} \]

\[\tan \left[k \left(1 - \left(\frac{\lambda}{2W}\right)^2\right)^{\frac{1}{2}} t\right] = k p_2 \tan h \frac{P_2 H}{2} + \text{sign} \]

\[= k p_2 \coth \frac{P_2 H}{2} - \text{sign} \]

where \(k = \omega \sqrt{\mu_0 \epsilon_0} \).

As was the case in both the parallel plate waveguide and the previously considered anisotropic waveguide, both real and imaginary values of \(p_1\) and \(p_2\) will satisfy the relations in Eq. (93). Again real values of \(p_1\) and \(p_2\) correspond to surface wave type modes while
imaginary values of $p_1$ and $p_2$ correspond to bounce modes. The impedance loci used for the parallel plate waveguide may be again utilized to determine the mode types of this configuration. The impedance loci for the LSE mode corresponds to Fig. 9 and the impedance loci for the LSM mode corresponds to Fig. 8.

In a low loss application it is desired to obtain a modal form similar to the ordinary $\text{TE}_{01}$ mode but in a guide with increased height. This dictates the use of a thin dielectric slab which would have an inductive reactance. This configuration would support a surface wave type LSM mode with the positive sign used in the potential function of Eq. (88). The surface wave mode with the negative sign in the potential function of Eq. (88) may be rejected by choosing a suitably small value of inductive reactance for the impedance surface. The surface wave modes for the LSE modes are rejected since an impedance surface with capacitive reactance is required as may be seen from Eq. (91). The higher order bounce modes may also be determined from these impedance loci.

It must be remembered that the eigenvalue relations given in Eq. (87) and (89) require the additional factor for the third dimension in determining the cutoff frequencies for these higher order modes. It is also to be noted that the impedance surface destroys the degeneracy that exists between the higher order TE and TM modes in an
ordinary waveguide since the values $p_1$ and $p_2$ and hence the phase velocities of the LSE and LSM modes are different. Thus coupled hybrid mode propagation and its associated problems as discussed in Reference 5 are no longer a problem.

D. Rectangular Waveguide with Propagating and Cutoff Walls

The rectangular waveguide with cutoff walls and one with propagating walls were considered in the preceding sections. For the waveguide with cutoff walls, the cutoff surfaces were used to reject cross polarized modal components. The waveguide with propagating surfaces was shown to have bound surface wave type modes which have the same modal character as the ordinary $\text{TE}_{01}$ mode for a rectangular waveguide with increased height. It was desired to combine these two waveguide types to take advantage of both of their modal properties.

A configuration attempted early in this study was a waveguide constructed from corrugated surfaces which possess an anisotropic surface impedance. In terms of the impedance boundary conditions defined in Eq. (60), the impedances are $Z_1$ and $Z_4$ equal to zero with $Z_2$ and $Z_3$ non-zero. These impedances do not satisfy the impedance compatibility condition defined in Eq. (61) and, indeed, an attempt to find a potential function or combinations of potential
functions which would satisfy this configuration proved fruitless. A configuration that does satisfy the impedance compatibility condition is one with two of the walls isotropic and the other two anisotropic.

The impedance boundary conditions as defined in Eq. (60) for this case are $Z_3$ equals $Z_4$ and non-zero, $Z_1$ equals zero, and $Z_2$ is non-zero. This configuration will now be discussed.

The modal configuration of the waveguide with isotropic and anisotropic surfaces may be obtained from the Hertzian potential formulation in Eq. (67). There are two forms of potential functions appropriate for this case, one symmetric in the $x$ direction and the other antisymmetric, which are given by

\begin{equation}
\psi = \begin{cases} 
\sin k_1 \left(x - \frac{W}{2}\right) \left(e^{-\rho_0 y} \pm e^{-\rho_0 (H-y)}\right) & \text{antisymmetric} \\
\cos k_1 \left(x - \frac{W}{2}\right) & \text{symmetric}
\end{cases}
\end{equation}

and

\[- k_1^2 + \rho_0^2 + \Gamma^2 + k^2 = 0 \]

The field components for this mode become

\begin{equation}
E_x = 0
\end{equation}

\begin{equation}
E_y = \Gamma (j\omega \mu) \begin{pmatrix}
\sin k_1 \left(x - \frac{W}{2}\right) \\
\cos k_1 \left(x - \frac{W}{2}\right)
\end{pmatrix} \left(e^{-\rho_0 y} \pm e^{-\rho_0 (H-y)}\right)
\end{equation}
\[ E_z = -P_o \Gamma (j\omega \mu) \begin{pmatrix} \sin k_1 (x-\frac{W}{2}) \\ \cos k_1 (x-\frac{W}{2}) \end{pmatrix} (e^{-P_o y} \pm e^{-P_o (H-y)}) \]

\[ H_x = (k_o^2 - k_1^2) \begin{pmatrix} \sin k_1 (x-\frac{W}{2}) \\ \cos k_1 (x-\frac{W}{2}) \end{pmatrix} (e^{-P_o y} \pm e^{-P_o (H-y)}) \]

\[ H_y = -P_o k_1 \begin{pmatrix} \cos k_1 (x-\frac{W}{2}) \\ -\sin k_1 (x-\frac{W}{2}) \end{pmatrix} (e^{-P_o y} \pm e^{-P_o (H-y)}) \]

\[ H_z = -\Gamma k_1 \begin{pmatrix} \cos k_1 (x-\frac{W}{2}) \\ -\sin k_1 (x-\frac{W}{2}) \end{pmatrix} (e^{-P_o y} \pm e^{-P_o (H-y)}) \]

Using these field components, the boundary conditions become

\[ Z_3 = Z_4 = \begin{pmatrix} -j\omega \mu \frac{\tan k_1 \frac{W}{2}}{k_1} \\ j\omega \mu \frac{\cot k_1 \frac{W}{2}}{k_1} \end{pmatrix} \text{ for the symmetric case} \]

(96)

\[ Z_2 = \begin{pmatrix} j \frac{P_o}{\omega \epsilon} \tan h \frac{P_o}{2} \\ j \frac{P_o}{\omega \epsilon} \cot h \frac{P_o}{2} \end{pmatrix} \text{ for } + \text{ sign in } \psi_2 \]

\[ \text{for } - \text{ sign in } \psi_2 \]
Since the waveguide is assumed lossless both $p_0$ and $k_1$ may assume either pure real or pure imaginary values. These impedances are of the same form as those plotted in Fig. 8 and 9 and thus the impedance loci may be again used to analyze this mode. The real values correspond, as in the parallel plate configuration, to modes bound to the surfaces and the imaginary values correspond to bounce modes. In order to simulate a TE$_{01}$ mode it is necessary to make the impedances $Z_2$, $Z_3$, and $Z_4$ all inductive.

Consider the dependence of the modal solution in the $x$ direction first. The wave comprising the modal solution is polarized parallel to the surfaces $x = 0$ and $x = W$ and as demonstrated by Eq. (96) the impedance boundary condition has the same form as the parallel plate case. In this case $Z_3$ and $Z_4$ are chosen inductive in order to eliminate bound modes on the vertical walls. As may be seen in Fig. 9 (note $H$ is to be replaced by $W$ in Fig. 9 for this geometry), the antisymmetric mode begins to propagate when the guide width is somewhat less than one-half wavelength and the symmetric mode begins to propagate when the guide width is somewhat less than one wavelength. The propagation is thus quite similar to an ordinary conducting waveguide.

The $y$ dependence of this modal solution may be similarly analyzed. The wave comprising this mode is perpendicularly polarized with respect to the surfaces $y = 0$ and $y = H$ and again, as
demonstrated by Eq. (96), the impedance boundary condition has
the same form as the parallel plate case. Thus the impedance loci
in Fig. 8 is appropriate. For these surfaces, it is desired to
weakly bind the wave to the horizontal surfaces in order to eliminate
the antisymmetric bound wave (the form referred to as mode C in
Fig 8). The weak binding also reduces the loss mechanism on the
horizontal walls.

The cutoff condition may be obtained from the eigenvalue relation
for this mode by setting $\Gamma$ equal to zero. The values of $k_1$ and $p_0$
may be obtained from the appropriate impedance loci for a specified
value of impedance and thus all parameters necessary to evaluate the
cutoff wavenumber are specified. For inductive impedance surfaces
with small values, the cutoff frequencies will be very similar to
those of an ordinary waveguide.

This waveguide appears to have limited application for a low loss
structure. The presence of the impedance surfaces in this configura-
tion does not allow for much oversizing in a dominant mode of
operation. In addition, the use of dielectrics as impedance surfaces
increases the loss mechanism over an ordinary conducting surface.
This waveguide configuration may find application in a radiating
system since the use of two impedance surfaces provides more control
over the phase velocity of the structure.
Another configuration that satisfied the impedance compatibility relation given in Eq. (61) is $Z_2$ and $Z_3$ equal to zero and $Z_1$ and $Z_4$ non-zero. This configuration may be realized in practice by using a corrugated surface with the corrugations running longitudinally along the waveguide axis. Such a waveguide, as will be described later, has been constructed.

The modal structure of this waveguide may be obtained by considering the current components this waveguide may support. For this configuration only longitudinally directed current components may be supported. Consider the ordinary rectangular waveguide modes. The current components may be obtained from the usual $\hat{n} \times \hat{H}$ relation using the magnetic field components of the ordinary waveguide evaluated at the waveguide boundaries. The TM type modes have only longitudinally directed current components while TE modes have both transverse and longitudinal current components. Since the modified waveguide will support only longitudinal current components the TE modes will be rejected and only the TM modes will propagate. Thus, this waveguide was named the "E-guide."

The field components for the modes within this waveguide are given by [55]
\begin{align}
E_x &= -a_{pq} \frac{p\pi}{W} \cos \frac{p\pi x}{W} \sin \frac{q\pi y}{H} \\
E_y &= -a_{pq} \frac{q\pi}{H} \sin \frac{p\pi x}{W} \cos \frac{q\pi y}{H} \\
E_z &= a_{pq} \frac{K_{pq}}{\Gamma_{pq}} \sin \frac{p\pi x}{W} \sin \frac{q\pi y}{H} \\
H_x &= a_{pq} \frac{j\omega \epsilon}{\Gamma_{pq}} \frac{q\pi}{H} \sin \frac{p\pi x}{W} \cos \frac{q\pi y}{H} \\
H_y &= -a_{pq} \frac{j\omega \epsilon}{\Gamma_{pq}} \frac{p\pi}{W} \cos \frac{p\pi x}{W} \sin \frac{q\pi y}{H} \\
H_z &= 0
\end{align}

where
\begin{align}
K_{pq}^2 &= \left( \frac{p\pi}{W} \right)^2 + \left( \frac{q\pi}{H} \right)^2 \\
\Gamma_{pq}^2 &= K_{pq}^2 - \omega^2 \mu \epsilon
\end{align}

and p and q have integer values. It is important to notice that H and W are measured between the corrugation teeth on opposite walls.

The impedance values $Z_1$ and $Z_4$ as defined in Eq. (60) must be considered. $H_z$ is zero as a consequence of the modal structure being TM but in addition the tangential electric field components as given in Eq. (9) are also zero when evaluated at the walls. Thus the modal field does not "sense" the impedances $Z_1$ and $Z_4$. 
The dominant field in this waveguide is the $TM_{11}$ mode. The ordinary waveguide is characterized by a dominant $TE_{01}$ mode and the waveguide configurations previously considered have all had a dominant mode similar to the ordinary $TE_{01}$ mode. One application that immediately suggests itself is using this guide as a TM radiating line source. The radiation from a slotted rectangular waveguide excited by a $TM_{11}$ mode has already been considered [56]. Use of the "E-guide" provides a structure that operates in a dominant $TM_{11}$ mode without mode conversion problems associated with the ordinary waveguide.

The attenuation constant for an ordinary waveguide operating in the $TM_{11}$ mode is given by [57]

$$
\alpha = \frac{2R_S}{HZ_0} \sqrt{\frac{1 - \left(\frac{f_c}{f}\right)^2}{\left(\frac{H}{W}\right)^3 + 1}} \left(\frac{H}{W} + 1\right)
$$

This attenuation constant assumes the waveguide walls are conducting surfaces whereas the surfaces are corrugated. The loss mechanism of the corrugated surface operated in this manner must be considered. The theoretical analysis predicts no fields induced within the corrugations and hence the loss mechanism consists only of dissipation on the top of the corrugation teeth. The loss mechanism is thus similar to that of the $TE$ polarized scattering loss and the attenuation constant is reduced by the ratio $a/b$ where $a$ and $b$ are defined in Fig. 3. This ratio is always less than unity.
The "E-guide" operates in a dominant mode until the dimensions of one of the walls exceeds one wavelength. For a low loss waveguide, a square configuration is desirable and thus the waveguide operates in a dominant mode until the cross section exceeds one square wavelength. The ordinary rectangular waveguide, by comparison, operates in a dominant TE₀₁ mode until the waveguide cross section exceeds one wavelength by one-half wavelength. Thus the "E-guide" operates in a dominant mode with a much larger waveguide cross section than a conventional rectangular waveguide. For this reason, it is expected that the attenuation constant will be less.

The attenuation constant of the normal TE₀₁ rectangular waveguide, \( \alpha_N \), given in Eq. (63) and that of the "E-guide", \( \alpha_E \), were compared at the point at which the next highest order mode begins to propagate in both guides and it was found that \( \alpha_N = 1.12 \frac{b}{a} \alpha_E \) where \( b \) and \( a \) are the corrugation dimensions (\( b > a \)), thus verifying the intuitive statement. In practice the teeth should be made as thin as practical to reduce the loss mechanism. As previously discussed, \( b/a \) should exceed 2 in order that the impedance analysis be valid. For the section of constructed waveguide to be discussed, \( b/a \) equals 3.

A six foot length of this waveguide has been constructed and the modal characteristics predicted by the above theory were verified. This waveguide is depicted in the photograph in Fig. 15. The cross
section of this waveguide, as measured between corrugation teeth on opposite walls, is 3 1/8 inches square. The calculated cutoff frequency for this waveguide operating in its dominant TM$_{11}$ mode is 2.68 GHz.

The theory developed above indicates the dominant mode of operation is TM$_{11}$ and that the ordinary TE$_{01}$ rectangular waveguide mode cannot be supported. This theoretical conclusion was demonstrated experimentally on this section of waveguide. The waveguide was excited by a pyramidal horn with a fairly long taper. Such a horn, exciting an ordinary waveguide, would strongly couple energy into the dominant TE$_{01}$ mode. Excitation of the corrugated waveguide failed to couple any detectable amounts of energy in a TE$_{01}$ mode component. The field detected by probe measurements consisted of TM$_{11}$ and TM$_{21}$ field components depending on the frequency and orientation of the horn.

The waveguide was then excited in its dominant TM$_{11}$ mode with the probe mounted in the end plate appearing in the photograph of Fig. 16. This probe was used to excite the "E-guide" over the frequency range from 2 to 4 GHz. Below 2.7 GHz corresponding to the TM$_{11}$ mode cutoff frequency, no energy could be detected at the output. Above 2.7 GHz probe measurements indicated a TM$_{11}$ mode was propagating. Above 2.7 GHz the waveguide and probe system could be well matched using a triple stub tuner while below 2.7 GHz
Fig. 15--Section of "E-Guide.

Fig. 16--TM_{11} probe mounted in end plate.
no match could be achieved indicating coupling into a large reactive load. Such an impedance is presented by coupling into a cutoff waveguide.

The guide wavelength for the TM_{11} mode was also measured. The measurement was accomplished in the following manner. The waveguide was matched radiating into free space using a triple stub tuner and then a short circuit was placed within the waveguide structure. A standing wave was then observed in a slotted line as the short circuit was physically moved in the waveguide. The pattern of the VSWR observed in the slotted line was sharp and had good null depth indicating the waveguide was operating in a single dominant mode. The distance the short was moved between nulls in the VSWR pattern is equal to one-half the guide wavelength. The measured and calculated guide wavelength is shown in Fig. 17. Good correlation between measured and calculated values are indicated. The measured results are within the expected limits. This verifies that the assumed modal solution and boundary conditions are correct.

An attempt was also made to excite the waveguide with the TE_{01} type probe depicted in the photograph of Fig. 18. Above 2.7 GHz the TM_{11} mode propagates. Below 2.7 GHz energy still propagated through the six foot section but at a level 20-30 dB below that of the
Fig. 17--Measured and calculated guide wavelength for "E-guide".
TM$_{11}$ mode in its propagating region. Field probing the energy failed to identify a mode type. The fields were probed using both small loops and a dipole. An attempt to measure the guide wavelength with the short circuit as before resulted in a random set of values and thus no guide wavelength could be ascertained. A TE$_{01}$ type mode filter consisting of a set of parallel vanes was constructed and used within the waveguide. The maximum output energy level as determined by probe measurements was reduced by a minimum of 11 dB and this filtered energy was still indistinguishable as a mode type. It was concluded that this energy propagating below 2.7 GHz was probe radiation and evanescent mode effects since no coherent mode structure forming a propagation mechanism could be identified.

A center conductor was added to the waveguide in order to operate the waveguide as a rectangular coax as is depicted in the photograph in Fig. 19. With the addition of a center conductor, the waveguide operates in a dominant TEM mode. The waveguide in Fig. 19 has been operated in the 1.8 to 4.0 GHz range and this mode was experimentally observed by field probing. The guide wavelength of this rectangular coax was measured and the results in Fig. 20 are again within experimental accuracy. An analysis contained in Reference 58 gives a theoretical method of calculating the capacitance, inductance, and characteristic impedance of such structures and very recently design curves have been published [59].
Fig. 18.--**$\text{TE}_01$** probe mounted in end plate.

Fig. 19.--"E-guide" operated as rectangular coax.
Fig. 20--Measured and calculated guide wavelengths for the "E-guide" operated as a rectangular coax.
It is also possible to propagate higher order modes in a rectangular coax. This was demonstrated by operating the rectangular coax in Fig. 19 at 16 Ghz. To the best of the author's knowledge a theoretical study of these higher order modes has not been undertaken. The mechanism for propagation of these higher order modes is to bounce plane wave components between the inner and outer conductors. Some idea of the modal structure of these higher order modes may be seen by distorting the TE and TM waveguide fields remembering that the tangential field components must approach zero at the centered conductor. The mode indices must thus be 22. This means that the inner and outer conductors must be separated by one-half wavelength. Ghose reaches an identical conclusion [60]. The rectangular coax thus may be made larger than a circular coax where the next higher order mode propagates when the mean circumference is one wavelength [61]. It also appears that a propagation mechanism for the higher order modes would not be possible if the rectangular coax were constructed from a configuration that does not satisfy the impedance compatibility relation in Eq. (61). This would allow the rectangular coax to be greatly oversized.

F. Rectangular Waveguide with Impedance Structures Within the Cross Section

Thus far, impedance surfaces have been used to form the boundaries of a rectangular waveguide while the interior cross section of the waveguide has been assumed to be homogeneous and isotropic.
Impedance surface structures also may be utilized within the waveguide cross section. An example of this type of impedance loading in the rectangular geometry is the configuration depicted in Fig. 21 investigated by Vartanian, Ayres, and Helgesson [62]. Their motivation to investigate this configuration was to obtain an increase in both the bandwidth and power handling capabilities of the waveguide. The dielectric slab positioned in the center of the waveguide increased the bandwidth as may be seen by again considering the effect on the elemental plane waves comprising the TE₀₁ mode. The

Fig. 21--Geometry of dielectric loaded waveguide.
higher order modes in the waveguide are incident on the slab more closely to the normal to the surface and are less affected by the refractive effect of the dielectric slab. In this respect the dielectric loaded waveguide functions similarly to the ridge waveguide. The power handling capability of the waveguide is enhanced because the dielectric material with higher breakdown strength is placed in the region of highest field strength.

This geometry will be considered as a means of further reducing the attenuation. As discussed in Reference 62 the wall losses are comparable or lower than those of an ordinary rectangular waveguide. The loss mechanism contributed by the dielectric may be lowered by utilizing an artificial dielectric or perhaps using a Yagi array of elements as a structure to support the energy in the center of the waveguide. The use of the Yagi array as discussed in Chapter II is particularly attractive since it provides a means of incorporating an inherently low loss open transmission line in a closed structure.

One interesting feature of the configuration in Fig. 21 is that, for a proper choice of parameters, the fields decay exponentially from the slab. This property will be further discussed analytically in the following material. If the fields decay exponentially from the centered impedance surface, the loss mechanism on the vertical walls may be substantially reduced. This configuration can be incorporated into the hybrid configuration previously discussed.
Propagating surfaces may be used on the horizontal walls to increase the waveguide height while retaining a single mode of operation. In addition for this case cutoff impedance surfaces may be utilized on the vertical walls to reject cross polarized mode components.

The modal structure of the dielectric slab loaded waveguide may be obtained from the hybrid mode formulation previously defined. Consider the geometry of Fig. 21 with propagating surfaces on the horizontal walls. Again the LSE mode defined in Eq. (67) is appropriate for those propagating surfaces that present an anisotropic surface impedance. A suitable potential function for this configuration is given by

\[
\psi_h = \sin k_1 x \left( e^{-p_y} y \pm e^{-p_y} \left( H - y \right) \right); \quad 0 < x < \frac{W - d}{2}
\]

\[
= A \cos k_2 \left( x - \frac{W}{2} \right) \left( e^{-l_3 y} \pm e^{-l_3 y} \right), \quad \frac{W - d}{2} < x < \frac{W + d}{2}
\]

\[
= \sin k_1 (W - x) \left( e^{-p_3 y} \pm e^{-p_3 (H - y)} \right), \quad \frac{W + d}{2} < x < W
\]

This potential function was chosen to satisfy the boundary conditions and the physical symmetry conditions of the problem. This formulation differs from that of Reference 62 in that the \( y \) dependence imposed by the propagating surface is included here. The fields for the formulation given in Reference 62 are assumed uniform in the \( y \)
direction and hence an ordinary TE type mode results. In addition
the potential function \( g(x) \) used in Reference 62 is related to \( \psi_h \) by
\[
g(x) = -\Gamma \frac{\partial \psi_h}{\partial x}.
\]

The field components for this waveguide mode are given by

\[
\begin{align*}
E_x &= 0 \\
E_y &= \Gamma (j\omega \mu_o) \sin k_1 x \left( e^{-\psi_o y} \pm e^{-\psi_o (H-y)} \right) \\
E_z &= (j\omega \mu_o) (-\psi_o) \sin k_1 x \left( e^{-\psi_o y} \mp e^{-\psi_o (H-y)} \right) \\
H_x &= (k_o^2 - k_1^2) \sin k_1 x \left( e^{-\psi_o y} \pm e^{-\psi_o (H-y)} \right) \\
H_y &= k_1 (-\psi_o) \cos k_1 x \left( e^{-\psi_o y} \mp e^{-\psi_o (H-y)} \right) \\
H_z &= -\Gamma k_1 \cos k_1 x \left( e^{-\psi_o y} \pm e^{-\psi_o (H-y)} \right)
\end{align*}
\]

\[
\begin{align*}
E_x &= 0 \\
E_y &= \Gamma (j\omega \mu_o) A \cos k_2 \left( x - \frac{W}{2} \right) \left( e^{-\psi_o y} \pm e^{-\psi_o (H-y)} \right) \\
E_z &= j\omega \mu_o (-\psi_o) A \cos k_2 \left( x - \frac{W}{2} \right) \left( e^{-\psi_o y} \mp e^{-\psi_o (h-y)} \right) \\
H_x &= (\epsilon_r k_o^2 - k_2^2) A \cos k_2 \left( x - \frac{W}{2} \right) \left( e^{-\psi_o y} \mp e^{-\psi_o (h-y)} \right) \\
H_y &= k_2 \epsilon_3 A \sin k_2 \left( x - \frac{W}{2} \right) \left( e^{-\psi_o y} \mp e^{-\psi_o (h-y)} \right) \\
H_z &= \Gamma k_2 A \sin k_2 \left( x - \frac{W}{2} \right) \left( e^{-\psi_o y} \pm e^{-\psi_o (H-y)} \right)
\end{align*}
\]
\begin{align*}
E_x &= 0 \\
E_y &= \Gamma (j\omega \mu_0) \sin k_1 (W-x) \left( e^{-\text{Po}_y} + e^{-\text{Po}(H-y)} \right) \\
E_z &= j\omega \mu_0 (\text{Po}_q) \sin k_1 (W-x) \left( e^{-\text{Po}_y} + e^{-\text{Po}(H-y)} \right) \\
H_x &= (k_0^2 - k_1^2) \sin k_1 (W-x) \left( e^{-\text{Po}_y} + e^{-\text{Po}(H-y)} \right) \\
H_y &= \text{Po}_0 k_1 \cos k_1 (W-x) \left( e^{-\text{Po}_y} + e^{-\text{Po}(H-y)} \right) \\
H_z &= \Gamma k_1 \cos k_1 (W-x) \left( e^{-\text{Po}_y} + e^{-\text{Po}(H-y)} \right) \\
\frac{\text{Po}_0^2 - k_1^2 + \Gamma^2 + k_0^2}{2} &= 0 \\
\frac{\text{Po}_0^2 - k_2^2 + \Gamma^2 + \epsilon k_0^2}{2} &= 0
\end{align*}

The constant \( \Lambda \) and a relation between \( k_1 \) and \( k_2 \) may be obtained by enforcing the boundary conditions on the tangential electric and magnetic field components along the dielectric interface giving

\begin{equation}
A = \frac{\sin \frac{k_1(W-d)}{2}}{\cos \frac{k_2d}{2}}
\end{equation}

\begin{align*}
k_1 \cot k_1 \left( \frac{W-d}{2} \right) &= k_2 \tan \frac{k_2d}{2}
\end{align*}

The parameter \( \text{Po}_0 \) may be evaluated by enforcing the impedance boundary conditions at \( y=0 \) and \( y=H \). The impedance boundary condition
requires a discontinuous change of impedance at the boundaries between the dielectric loaded and unloaded regions. Physically this means that the dielectric should extend into the corrugations. There are five unknowns in this mode formulation to be evaluated: $k_1$, $k_2$, $\Gamma$, $A$, and $p_0$. Five independent equations for their solution have been obtained; two eigenvalue equations given in Eq. (100), two equations were obtained by enforcing the boundary conditions along the dielectric slab interface as given in Eq. (101), and finally the impedance boundary condition may be evaluated for the fifth equation.

A similar modal solution may be made for the case when the propagating surface presents an isotropic surface impedance. The hybrid mode formulations given in Eqs. (68) and (70) will have to be considered. The appropriate potential function for the LSM mode is

$$
\psi_e = \sin k_1 x \left( e^{-p_0 y} \pm e^{-p_0 (H-y)} \right) ; \quad 0 < x < \frac{W-d}{2}
$$

$$
= A \cos k_2 \left( x - \frac{W}{2} \right) \left( e^{-p_0 y} \pm e^{-p_0 (H-y)} \right) ; \quad \frac{W-d}{2} < x < \frac{W+d}{2}
$$

$$
= \sin k_1 (W-x) \left( e^{-p_0 y} \pm e^{-p_0 (H-y)} \right) ; \quad \frac{W+d}{2} < x < W
$$

and for the LSE mode is
The modal structure may be obtained in the same manner as the anisotropic case.

Thus far, a formal solution has been obtained for modes of the configuration depicted in Fig. 21. The propagation characteristics of these modes will now be considered. In Reference 62 the case of a uniform field structure in the y direction of Fig. 20 corresponding to \( p_0 = 0 \) has been considered. The vertical dependence for this equal impedance configuration has already been discussed for the parallel plate waveguide as well as the waveguides considered in preceding sections. Real values of \( p_0 \) correspond to bound waves on the horizontal walls while imaginary values of \( p_0 \) correspond to bounce modes. For the propagation in the vertical direction the character of the horizontal wave numbers \( k_1 \) and \( k_2 \) must be considered. Consider the nature of the propagation as the frequency is increased from its cutoff value. Initially the dielectric slab functions as a refractive medium as mentioned previously as the plane wave component traverses the waveguide cross section. As the frequency increases, the phase change in the dielectric becomes

\[
(103) \quad \psi_h = \cos k_1 x \left( e^{p_0 y} + e^{-p_0 (H-y)} \right); \quad 0 < x < \frac{W-d}{2}
\]

\[
= A \sin k_2 \left( x - \frac{W}{2} \right) \left( e^{p_0 y} + e^{-p_0 (H-y)} \right); \quad \frac{W-d}{2} < x < \frac{W+d}{2}
\]

\[
= - \cos k_1 \left( W-x \right) \left( e^{p_0 y} + e^{-p_0 (H-y)} \right); \quad \frac{W+d}{2} < x < W
\]
equal to 180°. Above this frequency the wave number in the dielectric
is real while the wavenumber in the air filled region is imaginary
and thus the fields in the air filled region decay away from the
dielectric. Thus the dielectric functions as a surface wave structure
as discussed in Chapter II within a closed waveguide geometry.

In the region in which the dielectric functions as a surface wave
structure, the dominant mode fields are quite weak on the walls at
x=0 and x=W. In fact, the field structure in this region of operation
are not significantly affected if the walls are removed altogether, a
fact experimentally observed in Reference 62. This suggests that
control over the higher order modes, which are strongly dependent
on the boundaries of these walls, could be achieved by modifying
these walls. Cutoff impedance surfaces, such as corrugated surfaces,
could be employed on the vertical walls to reject cross polarized
mode components. The higher order bounce modes in the x direction
are significantly affected by the vertical walls. The cutoff corrugated
surface will not satisfy the impedance compatibility relation in Eq. (61)
for the higher order bounce modes. The dominant mode of this wave-
guide, when the dielectric functions as a surface wave structure, is
very weak at the vertical walls and the boundary conditions provided
by the walls, as observed above, have little effect on the field
structure. The higher order modes, on the other hand, have
significant field values at the vertical walls and thus these modes
could be rejected by the boundary conditions on these walls.

Propagation of backward waves on this structure has been con-
sidered by Clarricoats [63]. In dielectric loaded circular waveguide
he demonstrated backward wave propagation is possible. He also
considered backward wave propagation in dielectric loaded rectan-
gular waveguide. His analysis indicates backward wave propagation
in this geometry is not possible.

Another placement of the dielectric surface is across the wave-
guide cross section as shown in Fig. 22. This waveguide is shown
with impedance surfaces forming the horizontal boundaries. When
these terminating boundaries are not present, the waveguide is
called an H-guide. This waveguide, as proposed by Tischer [64],
has received attention as a low loss transmission line. It attempts
to simulate an infinitely tall $TE_{01}$ waveguide whose attenuation
constant has been previously given in Eq. (63). The loss mechanism
provided by the horizontal walls is not present and thus the attenu-
ation constant as a function of frequency should behave as $K f^{-1/2}$. A
discussion of the mode types, both the TE type, $\bar{E} = \hat{y} \hat{E}_y$, and two
hybrid modes types characterized by $H_x = 0$ and $E_x = 0$, respectively,
may be found in Reference 65. The experimental performance of
this waveguide in the millimeter wavelength region is contained
in Reference 66. In this configuration, foamed material having a relative dielectric constant of 1.03 was used and propagation in the 27 to 50 GHz frequency range was considered. While this dielectric constant used in the impedance surface structure seems rather tenuous, experimental measurements confirm that it functions as a surface wave structure.

Fig. 22--Horizontal dielectric slab loaded waveguide.

A major problem in the design of such waveguides is the problem of terminations for the horizontal surfaces. If left unterminated, the waveguide may radiate at discontinuities increasing the effective attenuation and the potential interference problems of an open line.
result. In Reference 66 a lossy surface was considered to contain the field. The use of a surface wave structure to terminate the fields rather than adding the additional loss mechanism of the lossy surface, particularly in light of the analysis contained in Chapter III, seems to be a natural extension. The modal structure of this configuration will now be developed.

In Chapter III the formal solution for propagation between parallel-plates whose impedances are not equal has been presented. The modal solution for this structure may be viewed as two parallel plate waveguides on top of one another and confined in the third dimension. The case of isotropic surface impedances used throughout the structure may be solved using the hybrid mode formulation given in Eqs. (68) and (70). As shown in Reference 66 the modes in the centered dielectric region may have both symmetric and anti-symmetric distributions similar to those whose impedances are given in Table II. These impedance values must be modified for the third dimension as in Eq. (90). A suitable potential function for the LSM mode is given by

\[
\psi_e = \sin \frac{\pi x}{W} \left( e^{-\rho_0 y} + R e^{-\rho_0 \left( \frac{H+T}{2} - y \right)} \right) \quad ; \quad 0 < y < \frac{H-T}{2}
\]

\[
= A \sin \frac{\pi x}{W} \begin{cases} 
\cos k_1 \left( \frac{y-H}{2} \right) & \text{symmetric} \\
\sin k_1 \left( \frac{y-H}{2} \right) & \text{anti-symmetric} 
\end{cases} \quad ; \quad \frac{H-T}{2} < y < \frac{H+T}{2}
\]
and for the LSE mode

\[
\psi_h = \cos \frac{\pi x}{W} \left( e^{-P_0 \left( y - \frac{H+T}{2} \right)} + R e^{-P_0 (H-y)} \right), \quad \frac{H+T}{2} < y < H
\]

and a similar set of eigenvalue relations result. The LSM modes have the electric field polarized in the y direction and LSE modes are polarized in the x direction. The modal constants may be obtained by enforcing the boundary conditions.

These potential functions were chosen so that only one bounce mode in the x direction is considered; higher order bounce modes have an x dependence of \( \sin \frac{n \pi x}{W} \) with \( n \) an integer value. The y dependence for these modes will now be discussed. In Chapter II the modes in a dielectric slab were considered and a graphical means
of determining the number of propagating surface wave modes was indicated. Similar techniques may be employed for this case. The desired mode of propagation for low loss applications is the dominant symmetric LSM mode. Other higher order modes may be rejected by a proper choice of parameters. As was the case for the dielectric slab, placing a metallic plate in the center of the dielectric slab eliminates the anti-symmetric LSM mode and the symmetric LSE mode. This technique as pointed out in Reference 66 may be used for additional modal control.

The parameters of the surfaces as y=0 and y=11 may be chosen in such a manner as to make the fields on these surfaces very weak for the desired LSM mode. The field magnitudes on these surfaces are determined analytically by evaluating the modal structure for a particular configuration. In Reference 66 a lossy surface was proposed which could be treated analytically by the above analysis; for this analysis a reactive surface is used for the termination and the additional loss mechanism provided by a lossy termination has been eliminated.

The final configuration to be considered may be viewed as a hybrid combination of the proceeding two cases. The dielectric is in the shape of a cross as is shown in Fig. 22. The motivation for investigating this configuration was to provide a structure that could be solved exactly which might model the propagation of a Yagi array in a closed waveguide. The parameters of this configuration may
be chosen so that most of the dominant mode energy propagates in the manner of a bound surface wave in the centermost dielectric section. As discussed in Section E of Chapter II the propagation of energy along a Yagi array is quite similar to the propagation in this centermost region. A tapered section of the dielectric cross waveguide might also be used to launch energy onto a Yagi array.

Fig. 23--Cross section of dielectric cross waveguide.
The use of a Yagi array is particularly attractive for its low loss properties. Shefer [8] has considered experimentally the loss mechanism of such an array and found that increasing the number of elements in the Yagi transmission line did not increase the insertion loss measurably. The losses in a practical dielectric are thus thought to be much higher. Shefer [8] also noted that bends with a fairly small radius did not result in a substantial radiation loss so long as the axis of the elements remained parallel. Thus H-plane bend in a closed waveguide with a centered Yagi array could be negotiated without significant mode conversion.

The potential function for the waveguide configuration in Fig. 23 will now be given. The LSM mode has the desired polarization and a suitable potential function in each of the regions identified in Fig. 23 is given by

\begin{align}
\Psi_e &= (106) \nonumber \\
\text{Region} &
\begin{align*}
1 & \sin k_1 x \left( e^{-p_0 \left(y-H+T\right)} + R_e \right) e^{-p_0(H-y)} \\
2 & A \cos k_2 \left(x - \frac{W}{2}\right) \left( e^{-p_0 \left(y-H+T\right)} + R_e \right) e^{-p_0(H-y)} \\
3 & \sin k_1 (W-x) \left( e^{-p_0 \left(y-H+T\right)} + R_e \right) e^{-p_0(H-y)} \\
4 & B \sin k_1 x \cos k_3 \left(y-H\right)
\end{align*}
\end{align}

The parameters of this modal structure may be evaluated by enforcing the boundary conditions. Nine constants must be determined: $k_1$, $k_2$, $k_3$, $A$, $B$, $C$, $p_0$, $R_e$, and $\Gamma$. Nine independent equations may be obtained; there are four eigenvalue equations, matching tangential field components at $x = \frac{W}{2}$ yields two equations, matching tangential field components at $y = \frac{H+T}{2}$ yields two equations, and finally the impedance boundary condition at $y=0$ and $y=H$ yields the ninth equation.

In order to bind the energy to the centermost region, $k_1$ and $k_2$ should be imaginary and $p_0$ should be real. The solution to these equations is a formidable task. Some idea of the approximate values for these parameters may be obtained by further study of the proceeding waveguide examples. In order to chose the parameters of the Yagi
array to be substituted for the dielectric cross, the phase velocities of
the Yagi array and the dielectric cross waveguide should be matched.
Design data on the Yagi array is contained in Reference 46.

For the mode described above the fields decay away from the
center region and, with a proper choice of parameters, may be made
quite weak at the waveguide boundaries. Thus, the desired mode will
not be strongly dependent on the boundaries. Undesired modal com-
ponents, however, may be strongly influenced by the boundaries.
This suggests that the outer walls surrounding the surface wave
structure might be constructed from an impedance configuration that
does not satisfy the impedance compatibility relation given in Eq. (61).
An example of such a configuration is a corrugated waveguide with
the corrugations on all four walls transverse to the waveguide axis.
It appears that the surface wave structure would not be strongly
influenced by the presence of such a containing structure while the
undesired modal components could be rejected. Thus, a means of
surrounding an open transmission line with a structure that will not
support ordinary waveguide modes has been found.
The propagation of electromagnetic energy in both parallel plate
and rectangular waveguides having impedance surfaces has been
investigated. The general properties of impedance surfaces were
considered and the properties of several particular impedance surfaces
were summarized. The propagation between parallel impedance
planes was analyzed and the case of equal impedance planes, hereto­
fore unconsidered, was developed. A compatibility relation for the
impedance boundary conditions in a rectangular geometry that yield
a separable form of solution was developed and several examples of
waveguides meeting this criteria were considered.

The motivation behind this research effort has been the develop­
ment of a low loss transmission line system. The presence of an
impedance surface places additional constraints on the modal configu­
ration of a waveguide. These additional constraints are employed to
reject certain modal components. A series of waveguides that
satisfy the impedance compatibility condition were considered. A
waveguide was investigated that had vertical walls designed to reject
cross polarized \( \text{TE}_{n0} \) mode types and thus allow the waveguide
height to increase while retaining a dominant mode of operation. The reduction in the attenuation constant possible as a function of height increase was determined.

Waveguides with propagating horizontal surfaces were also considered. This configuration was motivated by the fact that propagating surfaces provide a mechanism to launch energy into a bound surface wave mode and thus reduce higher order mode content. For loosely bound waves, the modal form is similar to the ordinary \( \text{TE}_{01} \) waveguide mode and it was shown that the attenuation constant for this configuration in its limiting form approaches that of the ordinary \( \text{TE}_{01} \) mode. Attenuation constant calculations in Table IV indicate that for a practical choice of parameters, the attenuation constant approaches that of an infinitely tall waveguide.

Another configuration developed was the "E-guide", a waveguide that will support only TM modes. This waveguide was also operated as a rectangular coax and a model was proposed whose container will support no waveguide modes. The propagating mode of the rectangular coax is TEM and the higher order modes should not propagate in this model. Thus, the waveguide cross section can be increased in size and therefore the attenuation constant will be reduced.
Finally, impedance surface structures were incorporated within the waveguide cross section and a configuration that will model the propagation of a Yagi array in a closed waveguide structure was considered. The use of a Yagi array is particularly attractive because of its inherent low loss mechanism and a method of enclosing such a structure has been found. Further development of this last approach appears particularly promising.

In general, the waveguides considered here are oversized and thus the power handling capability, both peak and average, will be enhanced over that of conventional waveguide. In addition, the bandwidth of these waveguides is increased by oversizing.

The use of impedance surfaces in a waveguide also suggests potential applications for radiating systems. The use of the "E-guide" as a radiating TM line source is one possibility that was mentioned. The use of the waveguide with impedance surfaces boundaries reduces the modal content possible. In an array of truncated waveguides, an important problem is that of treating higher order mode effects in the apertures and thus the waveguides with impedance surfaces having fewer modes that will satisfy the boundary conditions may result in a more tractable problem. Also utilization of a waveguide with increased height increases the directivity of the E plane element pattern and may therefore improve
overall array performance. Finally, use of a waveguide that encloses a surface wave structure provides a modal configuration inherently capable of feeding a surface wave antenna.

The use of impedance surfaces in the design of microwave components is also a fruitful area for further research. The analysis considered here is valid for an infinitely long transmission line. In a practical microwave system the effects of discontinuities must be considered. The modal structure of waveguides has been considered from the point of view of modal propagation by homogeneous and inhomogeneous plane wave components. The effect of discontinuities may be assessed by considering the scattering caused by these discontinuities. Kay[67] has considered the scattering by an abrupt change in the surface impedance by using a Weiner-Hopf technique. Barlow and Brown[68] have considered some approximate solutions to surface wave propagation on bends and corners. Chu and Kouyoumjian[69] have considered surface wave diffraction by a wedge. These references may be helpful in considering the effects of discontinuities in a waveguide structure.
In conclusion, the use of impedance surfaces in parallel plate and rectangular waveguides has been investigated. The ideas presented here may be applied to other geometries. The use of impedance surfaces suggests several areas of fruitful research in the design of microwave components and radiating devices.
APPENDIX I
COMPARISON OF RECTANGULAR AND CIRCULAR WAVEGUIDE ATTENUATION CONSTANTS

The purpose of this appendix is to compare the attenuation constants of the rectangular and circular waveguides in their dominant modes of operation. The attenuation constant for the TE$_{01}$ rectangular waveguide mode (see Eq. (63)) is

\[
\alpha_R = \frac{R_s}{Z_0} \left[ \frac{1}{1 + \left( \frac{f_{cR}}{f} \right)^2} \right] \left[ \frac{1}{H} + \frac{2}{W} \left( \frac{f_{cR}}{f} \right)^2 \right]
\]

and $f_{cR}$ is the rectangular waveguide cutoff frequency. The attenuation constant for the TE$_{11}$ circular waveguide mode is [70]

\[
\alpha_C = \frac{R_s}{cZ_0} \left[ \frac{1}{1 - \left( \frac{f_{cC}}{f} \right)^2} \right] \left[ \left( \frac{f_{cC}}{f} \right)^2 + 0.4185 \right]
\]

where $c$ is the radius of the waveguide and $f_{cC}$ is the circular waveguide cutoff frequency.

For dominant mode propagation these attenuation constants are minimum when the next highest mode just begins to propagate. Consider the ratio of these attenuation constants.
where the waveguides are assumed constructed from the same material.

Assume the height of the rectangular waveguide is small enough so that the next highest mode is \( \text{TE}_{02} \), thus \( f = 2f_{cR} \) at the frequency at which the next mode begins to propagate. The next highest mode in the circular guide that begins to propagate is the \( \text{TM}_{01} \) mode and a similar ratio of the cutoff frequencies using the roots of the Bessel functions yields \( f = 1.31f_{cC} \). It is now necessary to find \( c \) and \( w \) in terms of the cutoff frequencies. These relations yield

\[
(110) \quad f_{cR} = \frac{1}{2\sqrt{\varepsilon\mu}W} \quad ; \quad f_{cC} = \frac{1.841}{2\pi\sqrt{\varepsilon\mu}}
\]

Substituting these quantities the ratio of the attenuation constants become

\[
(111) \quad \frac{\alpha_R}{\alpha_C} = \frac{W}{H} \frac{1.841}{\pi} \frac{0.5000}{0.7655} \left( \frac{1 - \left( \frac{f_{cC}}{f} \right)^2}{1 - \left( \frac{f_{cR}}{f} \right)^2} \right)^{\frac{1}{2}} \left( \frac{1 + \frac{2H}{W} \left( \frac{f_{cR}}{f} \right)}{\left( \frac{f_{cC}}{f} \right)^2 + 0.4185} \right)
\]

where the height to width ratio is the only missing parameter. Using the ratio for ordinary rectangular waveguides \( H = \frac{W}{2} \) gives 0.700 for
\( \frac{\alpha_R}{\alpha_C} \). This means that the attenuation constant of the ordinary rectangular
is seventy per cent of that for the circular waveguide when both are
compared at the point when the next highest order mode begins to
propagate.
APPENDIX II
MODE ORTHOGONALITY PROOF

The orthogonality of the mode functions for the parallel plate waveguide is considered in this appendix. The orthogonality condition that these modes satisfy requires the evaluation of the integral

\[ I = \int_{0}^{H} \psi_1 \psi_2 \, dy \]

as explained in the text. The potential functions satisfy the following differential equation

\[ \frac{d^2 \psi}{dy^2} + (\Gamma^2 + k^2) \psi = 0 \]

Consider two solutions \( \psi_1 \) and \( \psi_2 \) of this differential equation. Multiplying each equation by the other solution gives

\[ \psi_2 \frac{d^2 \psi_1}{dy^2} + (\Gamma_1^2 + k^2) \psi_1 \psi_2 = 0 \]

\[ \psi_1 \frac{d^2 \psi_2}{dy^2} + (\Gamma_2^2 + k^2) \psi_1 \psi_2 = 0 \]

Subtracting yields

\[ \psi_2 \frac{d^2 \psi_1}{dy^2} - \psi_1 \frac{d^2 \psi_2}{dy^2} + (\Gamma_1^2 - \Gamma_2^2) \psi_1 \psi_2 = 0 \]
Integrating yields

\[ \int_{0}^{H} \psi_{1} \psi_{2} \, dy = \frac{1}{\Gamma_{z}^{2} - \Gamma_{1}^{2}} \int_{0}^{H} \left( \psi_{2} \frac{d^{2} \psi_{1}}{dy^{2}} - \psi_{1} \frac{d^{2} \psi_{2}}{dy^{2}} \right) \, dy \, . \]  

Each term of the second integral may be integrated by parts

\[ \int_{0}^{H} \psi_{2} \frac{d^{2} \psi_{1}}{dy^{2}} \, dy = \psi_{2} \frac{d\psi_{1}}{dy} \bigg|_{0}^{H} - \int_{0}^{H} \frac{d\psi_{1}}{dy} \frac{d\psi_{2}}{dy} \, dy \, , \]

\[ \int_{0}^{H} \psi_{1} \frac{d^{2} \psi_{2}}{dy^{2}} \, dy = \psi_{1} \frac{d\psi_{2}}{dy} \bigg|_{0}^{H} - \int_{0}^{H} \frac{d\psi_{1}}{dy} \frac{d\psi_{2}}{dy} \, dy \, . \]

Substitution yields

\[ \int_{0}^{H} \psi_{1} \psi_{2} \, dy = \frac{1}{\Gamma_{z}^{2} - \Gamma_{1}^{2}} \left[ \psi_{2} \frac{d\psi_{1}}{dy} \bigg|_{0}^{H} - \psi_{1} \frac{d\psi_{2}}{dy} \bigg|_{0}^{H} \right] \]

The boundary conditions for each mode may be written

\[ Z_{1} = \begin{cases} \frac{-E_{z}}{H_{x}} \bigg|_{y=0} = -j \frac{1}{\omega \epsilon} \frac{d\psi_{H}}{dy} \bigg|_{y=0} & \text{perpendicular polarized modes} \\ \frac{E_{z}}{H_{x}} \bigg|_{y=H} = j \frac{1}{\omega \epsilon} \frac{d\psi_{H}}{dy} \bigg|_{y=H} \end{cases} \]

\[ Z_{1} = \begin{cases} \frac{E_{x}}{H_{z}} \bigg|_{y=0} = j \omega \mu \psi_{E} \frac{d\psi_{E}}{dy} \bigg|_{y=0} \end{cases} \]

\[ Z_{2} = \begin{cases} \frac{-E_{x}}{H_{2}} \bigg|_{y=0} = -j \omega \mu \psi_{E} \frac{d\psi_{E}}{dy} \bigg|_{y=H} \end{cases} \] parallel polarized modes

\[ Z_{2} = \begin{cases} \frac{E_{x}}{H_{2}} \bigg|_{y=H} = j \omega \mu \psi_{E} \frac{d\psi_{E}}{dy} \bigg|_{y=H} \end{cases} \]
Using these boundary conditions, Eq. (118) becomes

\[ I = \frac{1}{\Gamma_2^2 - \Gamma_1^2} \left[ \frac{\psi_1 \psi_2 |_{y=H}}{j/\omega \mu} (Z_2 - Z_2^*) + \frac{\psi_1 \psi_2 |_{y=0}}{j/\omega \epsilon} (-Z_1 + Z_1^*) \right] = 0 \]

\[ = \frac{1}{\Gamma_2^2 - \Gamma_1^2} \left[ \frac{\psi_1 \psi_2 |_{y=H}}{1/j\omega \mu} \left( -\frac{1}{Z_2} + \frac{1}{Z_2^*} \right) - \frac{\psi_1 \psi_2 |_{y=0}}{1/j\omega \epsilon} \left( \frac{1}{Z_1} - \frac{1}{Z_1^*} \right) \right] = 0 \]

When \( \psi_1 \) equals \( \psi_2 \), the integral, evaluated in Eq. (50), equals

\[ I = \int_0^H \psi^2 \, dy = (1 + R^2) \frac{1 - e^{-2\rho_0 H}}{2\rho_0} + 2 \Re \left\{ H e^{-\rho_0 H} \right\} \]

where \( R \) denotes either \( R_H \) or \( R_E \) as used in Eqs. (27) and (28).

The more general form of orthogonality condition requires consideration of the following integral

\[ I' = \int_0^H \psi_1 \psi_2^* \, dy \]

A similar manipulation yields

\[ I' = \frac{1}{\Gamma_2^2 - \Gamma_1^2} \left[ \frac{\psi_2^* \psi_1 |_{y=H}}{j/\omega \mu} (Z_2 + Z_2^*) + \frac{\psi_2^* \psi_1 |_{y=0}}{j/\omega \epsilon} (-Z_1 - Z_1^*) \right] \]

\[ = \frac{1}{\Gamma_2^2 - \Gamma_1^2} \left[ \frac{\psi_2^* \psi_1 |_{y=H}}{1/j\omega \mu} \left( \frac{1}{Z_2} - \frac{1}{Z_2^*} \right) + \frac{\psi_2^* \psi_1 |_{y=0}}{1/j\omega \epsilon} \left( \frac{1}{Z_1} - \frac{1}{Z_1^*} \right) \right] \]

For the lossless case the impedances are pure reactive and the integrals equal zero. For the lossy case, these integrals are in general non-zero and the modes are no longer orthogonal.
APPENDIX III
IMPEDANCE COMPATIBILITY RELATION

In this appendix the impedance boundary conditions incorporated in a rectangular waveguide are examined for separable forms of modal solutions. The boundary conditions have been previously given in Eq. (60) and the geometry of the waveguide is depicted in Fig. 10. Assuming propagation in the positive z direction, the electric field in its most separable form may be written as

\[
E = [x f_1(x) f_2(y) + \hat{y} g_1(x) g_2(y) + \hat{z} h_1(x) h_2(y)] e^{-\Gamma z}
\]

The magnetic field, as obtained from Maxwell's equations, is

\[
H = \frac{j}{\omega \mu} [x (h_1(x) h_2(y) + \Gamma g_1(x) g_2(y)) + \hat{y} (-h_1(x) h_2(y) - \Gamma f_1(x) f_2(y)) + \hat{z} (g_1(x) g_2(y) - f_1(x) f_2(y))] e^{-\Gamma z}
\]

The wave equation requires

\[
\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (r^2 + k^2) \right] f_1(x) f_2(y) = 0
\]

In evaluating the boundary conditions, the field components must have the same functional form along the surface which they are evaluated. This requires
\( h_1(x) = K_1 g_1(x) \)
\( f_1(x) = K_2 g_1(x) \)
\( h_2(y) = K_3 f_2(y) \)
\( g_2(y) = K_4 f_2(y) \)

In addition assume

\( f_1' = - K_x^2 f_1 \)
\( f_2'' = - K_y^2 f_2 \)

It is also convenient to define

\( \delta = \frac{K_4}{K_2} \)
\( \gamma = \frac{\Gamma K_2}{K_1 K_3} \)

The field components may now be written in terms of \( f_1(x) f_2(y) \) and \( \delta \) and \( \gamma \).

\( E_x = f_1(x) f_2(y) \)
\( E_y = - \left( \frac{\delta}{K_x^2} \right) f_1(x) f_2(y) \)
\( E_z = - \left( \frac{\Gamma}{\gamma K_x^2} \right) f_1(x) f_2(y) \)
\( H_x = - \frac{j \Gamma}{\omega \mu \gamma K_x^2} \left( \frac{1}{\gamma} - \delta \right) f_1(x) f_2(y) \)
\[(130)\]

\[
H_y = -\frac{j\Gamma}{\omega\mu} \left( \frac{1}{\gamma} + 1 \right) f_1(x) f_2(y)
\]

\[
H_z = -\frac{j\Gamma}{\omega\mu} (5 - 1) f_1(x) f'_2(y)
\]

Notice that the magnetic field is also separable as these field components demonstrate.

The impedance conditions become

\[(131)\]

\[
Z_1 = \pm \frac{1}{j\omega\mu} \left( \frac{f_2(y)}{f'_2(y)} \right) \left|_{y=0}^{y=H} \right.
\]

\[
Z_2 = + \frac{1}{j\omega\mu} \left( \frac{f_2(y)}{f'_2(y)} \right) \left|_{y=0}^{y=H} \right.
\]

\[
Z_3 = \pm \frac{1}{j\omega\mu} \frac{f_1(x)}{f'_1(x)} \left|_{x=0}^{x=W} \right.
\]

\[
Z_4 = \pm \frac{1}{j\omega\mu} \frac{f_1(x)}{f'_1(x)} \left|_{x=0}^{x=W} \right.
\]

The wave Eq. (126) yields the following eigenvalue relation

\[(132)\]

\[-K_x^2 - K_y^2 + \Gamma^2 + k^2 = 0\]

Since the waveguide cross section is isotropic and source free,

\[\nabla \cdot \vec{E} = 0\]

which requires
Examination of the impedance boundary conditions given in 
Eq. (131) shows that the impedances are related as

\begin{align}
(134) \quad & Z_1 \left( \frac{1}{\delta} - 1 \right) = Z_2 \left( \frac{1}{\delta} + \gamma \right) \\
& Z_3 (1 + \gamma) = Z_4 \left( 1 - \frac{1}{\delta} \right)
\end{align}

These relations may be viewed as "compatibility relations" that must 
be satisfied in order that a separable form of solution be achieved.

Kramer's rule may be used to solve these two equations for \( \delta \) 
and \( \gamma \). This leads to the solution \( \delta = 1 \) and \( \gamma = -1 \). Using Eqs. (132) 
and (133) with these values requires \( k = 0 \) or, in other words, this 
solution is a zero frequency solution. This indicates that \( \gamma \) and \( \delta \) are 
not independent and that the determinant in Kramer's solution must 
be zero. Equating this determinant to zero yields the following 
relation

\begin{align}
(135) \quad & Z_1 Z_3 - Z_2 Z_3 + Z_2 Z_4 = 0
\end{align}

This equation summarizes the relation the impedances must satisfy 
in order that a separable form of solution exists.
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