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The Ohio State University, Ph.D., 1968
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THE VECTOR SPACE AS A UNIFYING CONCEPT
IN SCHOOL MATHEMATICS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By
Timothy A. Riggle, B.A., M.Ed.

The Ohio State University
1968

Approved by
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CHAPTER I

PLAN OF THE STUDY

Introduction

Many mathematics educators place great emphasis upon building a mathematics curriculum around unifying concepts. These concepts unify the curriculum by providing relationships among topics. As a result each new topic is more comprehensible because of its association with previously studied material. In this way mathematics becomes a study of structures rather than a study of unrelated problem solving tools.

Often these unifying concepts are mathematical structures which will be studied in an abstract form by the students when they are mathematically mature. When this is the case, the curriculum may spiral around the concepts, repeatedly returning to them at increasing levels of sophistication. Thus, while the concept is giving unity to the curriculum, its abstract form is becoming recognizable to the student. He is able to move from a concrete experience to a complex and rigorous abstraction through a series of well defined steps.

This writer believes that the mathematical structure known as a vector space can be an invaluable unifying concept for a mathematics program. The history of the development of the vector space clearly
shows its diverse applicability to problems of both geometric and algebraic nature. Many topics presented in elementary mathematics courses could be elucidated by approaching them from a vector space standpoint. The presence of sets of objects with an addition and a scalar multiplication abound in mathematics. By the introduction of concrete examples of a vector space in the early grades and the return to further examples as the students mature, the curriculum can gain new unity, and an understanding of the purely abstract vector space may be attainable at an earlier stage than was before possible.

Procedure for the Investigation

It shall be the purpose of this study to show explicitly how the concept of vector space can become a unifying thread for mathematics programs from elementary school to pre-calculus college level mathematics. Chapter II of this dissertation contains opportunities for studying vectors at many levels, and demonstrates how emphasis upon the vector space structure where it appears could add organization to the mathematics program. The writer does not display all such opportunities and limits himself to a sufficient number of examples to show how the presentation should proceed from concrete to abstract. At the same time he designates a variety of topics to which the basics of a vector space apply. Only elementary properties of vector spaces are included. That is, the concern is the development of the concept of an additive group with scalar multiplication where the scalars are
elements of a field. No attempt is made to deal with vector functions, subspaces, multiplication of vectors, and similar advanced topics.

Although the writer feels that the value of his study lies in establishing the vector space structure as a unifying thread, he sees some merit in a demonstration of how the vector space concept might be taught at a specific level. In Chapter III, a detailed plan for incorporating the unifying thread into a general education mathematics course for college freshmen is presented. The plan describes how such students may study the set of real functions as a vector space. Such an approach, in the opinion of the researcher, will give to the students a powerful tool as well as a greater understanding of the vector space concept.

Chapter IV reports the observations that were made by the researcher when he used the plan of Chapter III to teach a mathematics class of college freshmen about the vector space of real functions. In Chapter V, the writer summarizes the entire study, draws conclusions, and makes observations about its significance.

Definition of Terms

Before entering into the discussion of the vector space as a unifying concept in school mathematics, it is necessary that agreement upon the basic meaning of the vector space concept be reached. In order that such agreement might be attained, the definition of a vector space is presented.
So that we might more fully comprehend the definition let us consider briefly the history of its development. Speaking of the French mathematician Joseph-Louis Lagrange, B. N. Delone states "In his 'Analytic Mechanics' published in 1788, Lagrange arithmetized forces, velocities, and accelerations in the same way as Descartes arithmetized points." (4; 213)\(^1\) This arithmetization consisted of decomposing a force into three components each of which acts parallel to the direction of the three axes of analytic geometry in space. Thus in the problems of mechanics, equations connecting forces can be written as three equations involving their components. Also a force, velocity, or acceleration may be represented by a directed segment in the 3-dimensional coordinate system of space. Figure 1 shows the representation of a force and of its components \(f_x\), \(f_y\), and \(f_z\).

---

\(^1\)The first set of digits represents the Bibliography entry; the second represents the page number.
These directed segments were to be given the name vectors. Delone tells us "It was only after a hundred years from the time of Lagrange that mathematicians and physicists, particularly under the developing theory of electricity, began on a wide scale to consider the general theory of such segments." (4; 214) Later mathematicians were to find that there was an "algebra of vectors." One such mathematician was Hermann Grassman, a German mathematician born in 1809, who developed theories on vector algebra and worked with a geometry of n-dimensions in which his theories found application. (14) Other mathematicians were to generalize this "algebra of vectors" still further. They found that there were sets of objects quite unlike directed segments or n-tuples of real numbers on which operations could be defined which satisfied the same "algebra of vectors." Indeed there was great conflict among mathematicians as to what set of objects would be most useful to applied mathematicians. E. T. Bell writes "By the second decade of the twentieth century there was a babel of conflicting vector algebras, each fluently spoken only by its inventor and his few chosen disciples." (2; 208) Bell also tells us of the ultimate victory of J. W. Gibbs, an American mathematician, over the German Hamilton. Hamilton had solved the problem of multiplication of vector elements by introducing the quaternions denoted by i, j, and k which are defined by $i^2 = j^2 = k^2 = ijk = -1$. Bell writes "The vector analysis of Gibbs gradually displaced quaternions as a practical applied algebra . . . ." (2; 209) The vectors of Gibbs were "a magnitude and direction taken together."
He states "the numerical description of a vector requires three numbers, but nothing prevents us from using a single letter for its symbolical designation." Gibbs' treatment of vector multiplication resembles the quaternion approach, but his vectors more closely resemble the n-dimensional geometry of Grassman and may be studied without any knowledge of quaternions.

What we must recognize is that the din arose over what particular algebra was the best interpretation of the general algebra. The vector space which will be defined is a general or abstract concept of which there are many interpretations.

It is the work of Gibbs, Vector Analysis, published by one of his students in 1901 that largely influenced the definition of a vector space as it is used in the writings of American mathematicians. The following definition is a refinement of the definition given by Dr. John Riner in his textbook, Basic Topics in Mathematics. This definition is chosen because of the frequent reference to this text in Chapter III.

Let $\mathbb{R}$ be the field of real numbers and let $W$ be a set of elements together with an operation of addition (+), and a scalar multiplication of elements of $W$ by elements of $\mathbb{R}$ yielding again elements of $W$ and which satisfy the following laws (VO)-(V8). Then $W$ is called a "real vector space" or a "vector space over $\mathbb{R}$."

Let $u$, $v$, and $w$ denote arbitrary elements of $W$ and $a$, $b$, $c$ be real numbers.
(V0) \( u + v \in W, \ a \cdot u \in W \)
(V1) \( u + v = v + u \)
(V2) \( (u + v) + w = u + (v + w) \)
(V3) \( u + \overline{v} = u \) for some fixed element \( \overline{v} \) in \( W \)
(V4) Given \( u \in W \) there is an element \( -u \) in \( W \) such that \( u + (-u) = \overline{0} \)
(V5) \( a \cdot (u + v) = a \cdot u + a \cdot v \)
(V6) \( (a + b) \cdot u = a \cdot u + b \cdot u \)
(V7) \( (ab) \cdot u = a \cdot (b \cdot u) \)
(V8) \( 1 \cdot u = u \) and \( 0 \cdot u = \overline{0} \)

The definition of a vector space over an arbitrary field \( F \) may be obtained by replacing \( \mathbb{R}^1 \) with \( F \) in the preceding discussion.

However, this study will deal only with the vector space over \( \mathbb{R}^1 \).
CHAPTER II
OPPORTUNITIES FOR STUDYING THE VECTOR SPACE
IN SCHOOL MATHEMATICS

General Remarks

Many textbooks currently in use contain sections dealing with the vector space concept. These sections usually introduce the concept by a treatment of directed line segments and then apply the concept to a topic such as complex numbers or geometric proofs. (6) This often makes the new topic easier to understand while introducing the student to the vector space concept. In spite of these advantages, this approach is often not feasible. A large amount of time must be spent teaching about a vector space and before the student has a chance to assimilate this new knowledge he is expected to apply it to a new topic. Unless the student is familiar with the beauty of vector methods it is of little or no value to show him that the set of complex numbers, for example, forms a vector space. Also quite often this introduction to the vector space concept is forgotten when, several years later, the student is faced with a course in Linear Algebra.

Perhaps increased emphasis on the vector space concept whenever it is applicable would help students to better understand a diversity of topics. If the vector space concept were to be
introduced to the children in the early grades in a concrete way, clearly defined steps could be taken to insure that the student was familiar with vector methods well in advance of the time when he is to apply them. Furthermore, the student's knowledge of vector spaces could continually be reinforced and extended until he is ready for an intensive study at the abstract level in a Linear Algebra course.

In this chapter the writer presents a demonstration of several topics found in the curriculum and the relationship of the vector space concept to them. It is hoped that this demonstration will establish the vector space as a unifying concept in school mathematics. Those responsible for curriculum planning will find here a variety of opportunities for teaching about vector spaces at various levels. In addition, the sequence of topics is meant to demonstrate how the approach moves from concrete to abstract as the mathematical maturity of the student increases. These opportunities are meant to be placed in a curriculum which spirals around the vector space concept. But the curriculum will no doubt spiral about other concepts such as set theory or number systems simultaneously.

The writer recognizes that a plan which provides certain experiences at designated places throughout the curriculum is not feasible. Piecemeal curriculum construction and the role of the teacher as classroom organizer make it impossible to guarantee that a certain child or class has previously had any specific experience. Therefore this writer is not presenting a curriculum
plan but rather a series of opportunities which, it is hoped, will stimulate mathematics educators to provide these and/or similar opportunities to their students.

Some of the opportunities are meant to extend knowledge of vector spaces; some are applications of what is known about vector spaces to new topics; some will add variety to what would otherwise be repetitious and boring drill work; and some will not deal with vector spaces directly but will be laying the groundwork for vector methods. Such concepts as coordinate systems and addition within a system coupled with multiplication from outside the system are necessary vector methods, and opportunities to learn about these concepts must be provided.

For those who seek to implement these opportunities at specific levels in the curriculum, two notes of caution are in order. Firstly, these opportunities, especially those below the high school level, are not intended as opportunities to introduce advanced terminology and theory of vector spaces. Theory and exact terminology will be provided only after the student is familiar with concrete examples of vector spaces. Secondly, it is dangerous to assume that the students have any background activity in common. The teacher must be ready to provide supplementary work to some or all of the students in order to establish common ground. The amount of necessary supplementary work should decrease as more teachers become aware of the unifying feature of the vector space concept. However, transient
and inattentive students are always present and will continue to make it necessary for the teacher to remain flexible when planning classroom work.

Opportunity One

Many educators feel that the first encounter that a student has with a new concept should be a concrete one. The Swiss educator Pestalozzi, for example, felt that whatever we see and visualize we remember more vividly than what we hear. (12; 267) Perhaps the best visualized vectors would be directed line segments. In grade one or perhaps in kindergarten it would be valuable to give the pupils a chance to add directed line segments. The directed line segments could be arrows and would perhaps be called trips. The pupil would learn about differences both in direction and magnitude. These trips are added by placing the tail of one arrow on the head of the other without disturbing the direction of either. Large arrows laid on the floor, or arrows on a flannelboard could be used.

Another way of providing this opportunity to the child is the use of a puzzle. This puzzle would consist of blocks with arrows of different magnitudes and directions painted on them together with a frame containing a beginning point and a goal. The child is instructed to place blocks in the frame so that the heads of each arrow meet with the tail of the next. This will require some choice since some arrows will end at corners of blocks while others will not. After a series of trips chosen to miss permanent obstacles
inserted on the frame itself, the goal would be attained. A drawing of such a puzzle may be seen in Figure 2. The solution shown is only one of several possibilities.

![Figure 2]

**Opportunity Two**

When the pupil is ready to learn how to add integers together the directed line segments can be used again. The following is one example of how to give the child insight concerning addition.

A scale of all positive integers is made, or perhaps a yardstick can be used. The child learns to add on the scale by placing arrows of desired magnitude next to one another. Thus in adding four and three, the result of the child's work is shown in Figure 3.

![Figure 3]
Further work can be designed to illustrate the commutative and associative laws of addition of numbers. Simultaneously the pupil will be learning about the commutative and associative laws for addition of vectors. Figure 4 shows the result of such an exercise.

![Figure 4](image)

**Figure 4**

**Opportunity Three**

As the student learns the compass directions, the concrete vector space composed of directed line segments can again be an aid. The child is given a grid on which to draw. The grid has the basic directions marked on it and may be drawn over a map. The child is given a series of trips which are to be completed one after another, until a destination is reached. In the following example eight directions are used as the child is asked to draw each trip on his grid.
1. one mile S  
2. two miles E  
3. one mile NE  
4. two miles SE  
5. one mile SW  
6. one mile S  
7. two miles W  
8. one mile SW  
9. two miles NW  
10. one mile NE  
11. What one trip would have reached the same destination?

Such an exercise again provides opportunity to learn about the commutative and associative laws of vector addition, but it also introduces the idea of inverse trips. The student learns that a trip two miles in a southerasterly direction is cancelled out by a trip of two miles in a northwesterly direction.
This experience also provides an opportunity for the student to learn about polar coordinates in an intuitive way. That is, the trip, two miles north can be given as the ordered pair (2, N). Such experiences will be meaningful as background material when the high school or college student is struggling with polar coordinates.

**Opportunity Four**

The use of ordered pairs and the coordinate plane is fundamental in working with vectors. It is this medium which is used to describe ordered pairs of real numbers, perhaps the most concrete and understandable vectors. For this reason, it is important that the pupils begin learning about the coordinate plane as soon as possible. When the children have learned to count, they are ready to learn about coordinates. A very popular game called "Battleship" is an excellent aid to learning how to associate pairs of integers with points in the plane. Two rectangular grids are prepared and labeled as in Figure 6. One grid is assigned to each team and they, without showing the opposing team, choose three successive dots as the position of their battleship. The dots in Figure 6 show the positions that two teams might choose for their ships. The object of the game is to sink the other team's battleship. The team taking the first turn fires three shots at the opposing team's ship by calling out pairs of numbers such as (5, 3), (2, 1) and (6, 6). A hit is made if the pair of numbers is one of those used in locating the ship. The other team will then tell them only if they have made a hit or not. It is
then the second team's turn to shoot. The game continues until a battleship is sunk; that is, when three successful shots have been made. For longer or shorter games, the number of integers used and the size of the battleships may be varied. For a more detailed account of how this game is played the reader is referred to the booklet, *Experiences in Mathematical Discovery: 1- Formulas, Graphs, and Patterns*, published by the National Council of Teachers of Mathematics. (13)

![Figure 6](image)

**Figure 6**

**Opportunity Five**

After the pupil has learned to add integers, the writer assumes along with other educators that drill work is desirable. However, for some pupils drill work is boring and they become inattentive and waste studying time. Perhaps the following experience would make drill in addition of integers more interesting while establishing a foundation for further work with vectors.

The pupil is given ordered pairs of integers and told to add the first integers from each pair and then the second integers. At
this point either a column type or a row type of notation would be acceptable.

\[(2,3)\hspace{1cm}\text{or}\hspace{1cm}(2,3) + (3,5) = (5,8)\]

\[+ (3,5)\]

\[(5,8)\]

If the pupils already know how to plot \((2,3)\) on a rectangular grid, the exercise can be made more meaningful and more interesting. Since the pupil associates \((2,3)\) with "2 over and 3 up" and \((3,5)\) with "3 over and 5 up", he is able to count 2 over and 3 up and then 3 over and 5 up and obtain the result of 5 over and 8 up. Thus the faltering student can check on the graph to see if the answer in his computation is correct.

Many teachers would not want to formalize at this point. When this is the case, this exercise would be provided only as a drill in integer addition and no attempt would be made to define addition of ordered pairs to obtain another ordered pair.

**Opportunity Six**

The number line was used earlier to introduce integer addition and can also be used to teach pupils how to subtract integers. The line is again marked in integer divisions and arrows are used to represent numbers. The arrows are placed on the scale and directions are given for subtracting one integer from a larger one. The two arrows are placed with their tails at zero. Next the arrow is drawn which must be added to the smaller arrow in order to obtain
the larger one. This arrow is moved to a number scale and measured.

As an example, consider how the subtraction of 3 from 6 might be performed. In Figure 7 arrows representing 6 and 3 are drawn with tails at 0. Then the arrow which must be added to 3 to obtain 6 is drawn. Finally, this arrow is placed with its tail at 0. The arrow is found to be the one which represents 3.

Figure 7

Thus, the operation of subtraction consists of finding what must be added to the second integer to obtain the first.

An alternative plan may be used by those who wish to treat the operation of subtraction as the addition of an inverse. In this case, the operation is simply addition, but the arrow representing the second number should be flipped around and pointing to the left before it is added to the other arrow.

This writer has chosen to treat subtraction as an operation in its own right. This is because he feels that the student will
be performing subtraction of directed segments in this way. The directed segment $\overrightarrow{AB}$ in Figure 8 could be subtracted from $\overrightarrow{AD}$ by simply finding the vector $\overrightarrow{BD}$ which completes the triangle.

![Figure 8](image)

No matter which approach to subtraction is used, valuable exercises can be given to the pupils. Exercises should be assigned to the students to instigate research. The pupils can do research concerning associativity and commutativity of subtraction since it is easy for them to perform subtraction in this graphical way. Some students may be simply told to check to see if subtraction is associative while for others it may be better to ask them if $(8 - 3) - 2 = 8 - (3 - 2)$ is a true statement.

**Opportunity Seven**

Many current mathematics programs do not emphasize multiplication from outside a given system. Most operations can be defined within a system and when understanding of the field axioms is sought, this is no doubt a desirable approach. However, this approach would seem to make the idea of scalar multiplication a little suspect to beginning students of the vector space. Presently considerable time is spent in developing field axioms in which properties of binary
operations on a given set are studied. This writer recognizes the field structure as an important unifying concept in school mathematics. However, vector methods such as scalar multiplication may be presented to the student concurrently with the development of the field of real numbers. Certainly real numbers form a vector space over the real numbers where scalar multiplication is the ordinary multiplication of real numbers. This feature can often be utilized to provide experiences in scalar multiplication while detracting nothing from the development of the field structure.

One example of such an experience is as follows: The pupils begin early in elementary school to work problems dealing with money. The pupil is taught that $3 \times \$12 = \$36$ since $\$12 + \$12 + \$12 = \$36$. Although it is apparent that the role of "3" is somewhat different from the role of "12," this fact isn't mentioned. This could provide a nice opportunity to talk about scalar multiplication. The role of "3" is that of a scalar, it comes from outside the set of numbers with "$\$" signs and to multiply by "3" we add $\$12 + \$12 + \$12$. This is precisely the role of scalar multiplication in vector spaces. We can add two numbers representing dollars and we can multiply numbers representing dollars by a real number and get numbers representing dollars. But we cannot multiply two numbers representing dollars nor can we add real numbers not representing dollars to those which do.

This approach, in addition to introducing scalar multiplication, may help the child to understand why he should add or multiply in a
particular problem. If the pupil were told "Johnny earned $3 a week delivering papers and he delivered papers for 12 weeks," the pupil would know that to find the total money earned he should use $3 as an addend 12 times which is equivalent to multiplying $3 by 12. He cannot add 3 and 12 since 12 doesn't represent a quantity of dollars.

Opportunity Eight

The School Mathematics Study Group has produced a text, Mathematics for the Elementary School - Grade 5, which contains a section called "Addition of Rational Numbers on the Number Line." The approach used clarifies the arithmetic process of addition and provides drill in addition of directed line segments. This technique is merely an extension of that presented in Opportunity Two.

The child has previously learned about ordering of rational numbers, least common denominators, and the measure of line segments. The line segment from 0 to 1 is called the unit segment and a line segment has measure 3 if it is congruent to the vector sum of 3 segments each of which is congruent to the unit segment. Similarly, a segment is of measure $\frac{3}{2}$ if it is composed of 3 segments each congruent to one half of the unit segment. Exercises given to the student are problems in adding rational numbers with like denominators. As an example consider the following problem:
The addition of $\frac{2}{3}$ and $\frac{5}{3}$ may be shown on a number line. First the number line must be scaled in thirds.

Next we draw a segment $\overline{XY}$ of measure $\frac{2}{3}$ and next to it a segment $\overline{YZ}$ of measure $\frac{5}{3}$.

Now we see that $\overline{XY}$ is the union of two segments each with measure $\frac{1}{3}$, so the measure of $\overline{XY}$ is $\frac{2}{3}$. $\overline{YZ}$ is the union of five congruent segments each with measure $\frac{1}{3}$, so the measure of $\overline{YZ}$ is $\frac{5}{3}$. $\overline{XZ}$ is the union of $(2 + 5)$ or 7 congruent segments each of measure $\frac{1}{3}$. Therefore, the measure of $\overline{XZ}$ is $\frac{7}{3}$. That is $\frac{2}{3} + \frac{5}{3} = \frac{2+5}{3} = \frac{7}{3}$. (18; 308)

This presentation, it can be seen, is rather precise in dealing with concepts such as measure and congruency. The curriculum is spiraling around these concepts and using these terms so the children will be familiar with them early in their mathematical training.

The child is learning about addition of directed segments or vectors but he is also becoming familiar with a type of scalar multiplication. In the above example, a line segment $\overline{XY}$ had measure because it was congruent to the vector sum of two segments of measure $\frac{1}{3}$. This is scalar multiplication and more will be said about it in Opportunity Nine.

**Opportunity Nine**

Since addition of rational numbers is a rather difficult calculation when first encountered, this writer along with other educators feels that drill is necessary. The following technique can be used to add variety to drill work while expanding the pupil's knowledge of rational numbers and vectors.
It is necessary to show the pupil how to picture a rational number graphically. As an example, let us graph $\frac{2}{3}$. This is done by counting over 3 and up 2 on a lattice work labelled with natural numbers. Once the point (3,2) is found a straight line is drawn from (0,0) passing through (3,2). This line represents the rational number whose name is $\frac{2}{3}$. Other names for $\frac{2}{3}$ may also be read from the graph. In Figure 9, since (6,4) is also on the line $\frac{4}{6}$ can be seen to be another name for $\frac{2}{3}$. An arrow can be drawn to designate whatever name is desired.

![Figure 9](image)

The operation of addition is performed in the following way. Consider $\frac{2}{3} + \frac{1}{2}$ and the pictures in Figure 9 of $\frac{2}{3}$ and in Figure 10 of $\frac{1}{2}$.

![Figure 10](image)
We can inspect the graphs and find two names, one for $\frac{2}{3}$ and one for $\frac{1}{2}$, which fall on the same vertical line and add the associated arrows on the same lattice. In this case we would choose the arrows at $(6,4)$ and $(6,3)$ and adding them we get the vertical arrow with its head at $(6,7)$ as shown in Figure 11.

Figure 11

This technique, though long in description, is quickly demonstrated, and the pupils can practice adding line segments and working with graphs while reviewing addition of rational numbers. Simultaneously, the pupil is learning more about rational numbers. He is learning that a rational number is really an equivalence class and may be named by any one of several fractions. The pupil can graphically display all fractions in one equivalence class.
The above technique has provided the opportunity to introduce a trick which will be used in high school mathematics classes as an aid in graphing functions. This trick will be discussed in detail in Chapter III, but the reader will note that in the above exercise we have found the graph of \( y = \frac{7}{6}x \) by adding the graphs of \( y = \frac{2}{3}x \) and \( y = \frac{1}{2}x \).

**Opportunity Ten**

The study of multiplication of rational numbers provides an excellent opportunity for teaching the child about scalar multiplication. The child is told that, prior to learning how to multiply rational numbers, he is going to learn how to multiply a rational number by a natural number. The child will agree that since \( \frac{1}{2} + \frac{1}{2} = 1 \), \( \frac{3}{4} + \frac{3}{4} = \frac{6}{4} \), and \( \frac{2}{5} + \frac{2}{5} = \frac{4}{5} \) it should be true that \( 2 \cdot \frac{1}{2} = 1 \), \( 2 \cdot \frac{3}{4} = \frac{6}{4} \), and \( 2 \cdot \frac{2}{5} = \frac{4}{5} \). From this the child can go on to learning about \( 9 \cdot \frac{3}{5} \) since he already knows about addition of rational numbers.

The child is able to learn a great deal about multiplication by experimenting. He knows that \( 12 = 9 + 3 \) so \( (9+3) \cdot \frac{3}{5} = 12 \cdot \frac{3}{5} \) but what about \( 9 \cdot \frac{3}{5} + 3 \cdot \frac{3}{5} \)? Also \( \frac{1}{2} + \frac{1}{2} = 1 \) so \( (\frac{1}{2} + \frac{1}{2}) \cdot \frac{3}{5} = \frac{3}{5} \) but perhaps \( (\frac{1}{2} + \frac{1}{2}) \cdot \frac{3}{5} \) should be the same as \( \frac{1}{2} \cdot \frac{3}{5} + \frac{1}{2} \cdot \frac{3}{5} \), whatever that is. After the child has had ample time to discover much about multiplication of rational numbers, the teacher may wish to move the development along by pointing out that \( \frac{12}{3} \) is really 4 so \( \frac{12}{3} \cdot \frac{2}{5} = 4 \cdot \frac{2}{5} = \frac{2}{5} + \frac{2}{5} + \frac{2}{5} + \frac{2}{5} = \frac{8}{5} \) and \( \frac{12}{3} \cdot \frac{2}{3} = \frac{24}{15} = \frac{8}{5} \). At this point the development may proceed by traditional methods.
This procedure has the advantage of providing the better student a chance to do research on his own, but in addition, it establishes scalar multiplication as a worthwhile activity. Most operations which children encounter in their study of arithmetic are binary operations on a set. They are repeatedly told to operate only with two elements from the same set. They have added two integers to obtain a third, multiplied two natural numbers, and added fractions with like denominators.

The scalar operation is of an entirely different nature. An element is operated on by an element from outside its set. The scalar operations do have closure; that is, the result of a scalar operation is another object from the original set. Scalar operations are worthwhile and lead to interesting results when accepted. However, they may be difficult to accept as valid operations when their nature is contrary to every other operation previously studied. The child needs some experience with this type of operation to relate with scalar multiplication in a vector space. It is important therefore that work of this type be provided to give the child experience with this uncommon type of operation.

Opportunity Eleven

This opportunity is an extension of Opportunity Three. In the earlier case, the child drew directed segments to represent trips in various directions of the compass. But when the child learns how to use a protractor and construct angles he is ready to draw trips in
many more directions. The child is asked to display the following successive trips on a sheet of paper using $\frac{1}{2}$ inch to represent a mile. He must be told that the angles are to be measured above the horizontal and in a counter-clockwise direction.

1. 3 miles at 30°
2. 2 miles at 45°
3. 1 mile at 60°
4. 3 miles at 210°
5. 1 mile at 240°
6. 1 mile at 110°
7. 1 mile at 90°

8. What one trip is equivalent to the above series of trips?

After the child has correctly drawn the trips, his paper should look like Figure 12.

![Figure 12]

The answer to question 8 is a trip of approximately 3-5/8 miles in a 70° direction. Answers will vary due to inaccuracies in measuring.
Many more exercises of the above type may be constructed. Exercises can be constructed to bring forth ideas about the commutative, associative, or inverse properties of vector addition. The reader will observe that in the above exercise the concept of inverse of a trip is in evidence.

The treatment described doesn't actually add more material to the mathematic class's already crowded schedule. These exercises will be used as drill in constructing and measuring angles. This drill is already present in most if not all curriculum plans. Hence, economy of time is provided as the student learns about vector addition simultaneously.

Opportunity Twelve

In the Scott, Foresman and Company series of texts for the Junior High School the idea of a rate pair is included. (21)(22) This inclusion is not unique and the idea may be found in other writings. (11)(20) The rate pair is much like a fraction but acts differently. As a fraction \( \frac{2}{5} \) may be read as "two fifths" but as a rate pair it is read "2 out of 5," "2 for 5," or "2 per 5." Because of the differences in use and meaning, this writer prefers to write \((2:5)\) to represent the rate pair. One basic difference is noted in the following problem: "Jimmy plays in two ball games and bats \( \frac{4}{5} \) times in both. If he got 2 hits in the first game and 1 in the second, what rate pair shows his average?"
Jimmy might solve this problem as follows. "My average was (2:4) in the first game and (1:4) in the second. Therefore, my average for both should be the sum of the two or (3:4)." If (2:4) and (1:4) were equivalent to $\frac{2}{4}$ and $\frac{1}{4}$ it is true that their sum would be $\frac{3}{4}$. But most students would not be happy with (3:4) as the answer. However if a sort of vector addition were employed and $(2:4) + (1:4) = (2+1:4+4)$ or (3:8), a more reasonable answer is obtained. On the other hand, if Jimmy batted (2:4) in three ball games his average for the three games would be $3 \cdot (2:4) = (3 \cdot 2:3 \cdot 4)$ or (6:12). Thus the set of rate pairs does have a scalar multiplication and an operation of addition which resemble the operations of a vector space.

Equality of these rate pairs is another practical idea. The rate pair $(a:b)$ is said to be equivalent to $(c:d)$ written $(a:b) \sim (c:d)$ if and only if $ad = bc$. It is seen that proportional equations become equations in rate pairs. If Johnny can buy 3 pencils for $15\$, he determines $x$, the cost of two pencils by equating $(3:15)$ with $(2:x)$.

Some caution must be observed in dealing with rate pairs. In solving some problems, fractions must be added as well as rate pairs and, though they look similar, the addition is different. Consider the following problem:

Joe can build a cabinet in four days and his father can build one in three days. If they work together, how long should it take them to build 15 cabinets?
The rate pair describing Joe's work is (¾ :1) or ¾ cabinet per day. His father's work is described by (½ :1). But their simultaneous rate is ((¾ + ½):1) which is not equivalent to (¾ :1) + (½ :1). We now know their combined rate is (7/12 :1) and we want to find x such that (7/12 :1) ~ (15: x) or 7/12·x = 15.

For further applications and to see how the concept of rate pairs is taught to junior high students, the interested reader is referred to Seeing Through Mathematics Book 1. (21)

Opportunity Thirteen

The authors of the Seeing Through Mathematics series of texts have developed a technique for introducing signed rational numbers. This technique is somewhat unique in that it uses such advanced ideas as equivalence classes, cross products and directed segments or vectors.

Assuming that the child is able to draw directed segments on a number line labeled with Ra, the set of unsigned rational numbers, this technique associates the directed segments with pairs of rational (x, y) where x and y are elements of Ra. The x represents the terminal point of the directed segment and y, the initial point. The measure of the directed segment is x - y and two directed segments are said to be equivalent if they have the same measure. That is (a, b) ~ (c, d) as elements of Ra × Ra if and only if a + c = b + d. Of course measure has meaning to the pupil only if a-b and c-d are elements of Ra and so the segments must be directed to the right at first.
A positive rational number is defined by the *Seeing Through Mathematics* authors to be:

An infinite set of equivalent ordered pairs of rational numbers of arithmetic, each of whose members determine a directed segment that is directed to the right. A positive rational number contains a basic pair whose second component is 0, along with all other ordered pairs, that are equivalent to the basic pair. (22; 98)

The set of positive rational numbers $R^p$ consists of elements like $\frac{4}{3}$ which is the standard name for the equivalence class containing $(\frac{4}{3}, 0)$.

In an analogous manner, a negative rational number is defined as

An infinite set of equivalent ordered pairs of rational numbers of arithmetic, each of whose members determines a directed segment that is directed to the left. (22; 103)

The set of negative rational numbers $R^-n$ contains elements like $\frac{-4}{6}$. The set of rational numbers $R$ is defined to be $R = R^p \cup R^-n \cup \{0\}$.

Once the set $R$ is defined, the addition of its elements can be demonstrated in terms of adding directed segments. Thus addition of signed rational numbers has definite meaning for the students and they are able to discover the rules governing addition of signed numbers.

The reader who is interested in implementing this opportunity in a classroom is referred to the full development as expounded in *Seeing Through Mathematics* Book 2. Although the presentation is rigorous, it is comprehensible to the good student and provides opportunity to introduce him to vector methods as well as other advanced ideas.
Opportunity Fourteen

If the student has a background of experience similar to that provided by Opportunities One through Thirteen, the teacher may wish to begin a formal study of directed line segments as a vector space. This includes listing the properties of this vector space along with verification of the properties.

To be completely rigorous it would be best to define a vector as an equivalence class of directed segments having the same magnitude and direction. However, since this is the first formalization of the properties the teacher may find it less confusing to talk about the set of directed segments whose initial point is the origin of the coordinate system. Thus a vector is to be a directed segment beginning at a point $O$ in the plane and ending at a point $P$. We shall designate this vector $\mathbf{OP}$ and it is shown with vectors $\mathbf{OQ}$ and $\mathbf{OR}$ in Figure 13.

![Figure 13](image)

The sum of two vectors is found by treating them as adjacent sides of a parallelogram. The sum of the two vectors is the diagonal
having its initial point at 0. Thus the sum of \( \vec{OP} \) and \( \vec{OR} \), denoted 
\( \vec{OP} + \vec{OR} \) is found in Figure 14.

![Figure 14](image)

At this point some teachers may wish to define a scalar 
multiplication using the set of real numbers for scalars. This 
writer, wishing to avoid the awkwardness of explaining how to find 
\( 3 \cdot \vec{AB} \), prefers to use the set of rational numbers as scalars. 
Multiplication of the above described vectors by any rational 
number can be performed with a high degree of accuracy. This 
multiplication will involve a Euclidean construction just as the 
addition of vectors involves the construction of a parallelogram. 
As an example let us multiply \( \vec{OP} \) in Figure 15 by \( \frac{3}{5} \). To do this 
we draw a second vector \( \vec{QR} \) and on it mark off 5 equal segments. We 
construct a line through the fifth division point \( Q \) and \( P \). Now 
draw a line parallel to \( \vec{OP} \) but passing through the third division 
point on \( \vec{OR} \). Label the point of intersection with \( \vec{OP} \), \( S \). Now \( \vec{OS} \) 
is \( \frac{3}{5} \cdot \vec{OP} \).
Once the child has learned the operations of addition and scalar multiplication, the properties of this vector space are listed. The children can then check these properties by working out examples of each. This experience is to be utilized to give the pupils experience with geometric constructions and a concrete example of a vector space for later reference.

**Opportunity Fifteen**

When the student has extended the coordinate system so that every point in the plane has a pair of real numbers associated with it, he is ready to study about another vector space. We shall let $R^2 = (a, b)$, $a$ and $b$ are real numbers. Under the proper definition of addition in $R^2$ and scalar multiplication of elements in $R^2$ becomes a real vector space. "Real" means that the field of scalars is the set of real numbers.
If \((a,b)\) and \((c,d)\) are elements of \(\mathbb{R}^2\) we define 
\((a,b) + (c,d) = (a+c,b+d)\).
If \(k\) is a real number we define 
\(k \cdot (a,b) = (ka,kb)\).

The properties of a real vector space listed on page 6 may be verified. Indeed it is expected that the teacher and/or the students would verify most if not all the properties of this \textit{real vector space}.

The graphical significance of this vector space should be explained and the students should graphically add and find scalar multiples of the vectors. This graphical significance results from the fact that any point \(P\) in the plane has a pair of real numbers associated with it. Thus any vector \(\overrightarrow{OP}\), as described in Opportunity Fourteen, has the coordinates of \(P\) associated with it. It is true that the graphical addition of vectors yields the same results as addition of the associated ordered pairs in \(\mathbb{R}^2\). This is illustrated in Figure 16, \(\overrightarrow{OP}\) is associated with \((1,2)\), \(\overrightarrow{OR}\) is associated with \((2,1)\) and \(\overrightarrow{OP} + \overrightarrow{OR}\) is associated with \((3,3) = (1,2) + (2,1)\).

Figure 16
Figure 16 also illustrates that if \( \vec{0} \) is associated with \( (2,3) \), then \( 3 \cdot \vec{0} \) is associated with \( 3 \cdot (2,3) = (6,9) \).

This opportunity is extremely important due to the simplicity and yet completeness of this vector space. Because of its simplicity it is frequently used to illustrate properties of vector space in linear algebra courses.

The isomorphism between \( \mathbb{R}^2 \) as a vector space and the set of directed segments \( \vec{0} \) as a vector space provides a beautiful example of the value of mathematical structures. Much can be learned about the geometric model by studying the same mathematical structure within an algebraic model. If the student is to appreciate abstract mathematics, it is widely felt that he must be provided with an experiential background through examples such as this.

For the teacher who seeks to implement this opportunity at the high school level, more needs to be said about scalar multiplication of directed line segments. The procedure is exactly as that proposed in Opportunity Fourteen. But there is a slight difficulty in multiplying by irrational numbers. It isn't of much value to demand that the students know how to multiply directed segments by irrational numbers, but the teacher may present an example such as the following to demonstrate how it could be done. The teacher may wish to point out that the difficulty lies in finding a line segment whose length is the irrational number.
Since it is possible to construct a line segment of length $\sqrt{2}$, multiplication of a directed segment $\overrightarrow{OP}$ by $\sqrt{2}$ can be performed. Suppose $\overrightarrow{OP}$ is in the first quadrant and the teacher has constructed a line of length $\sqrt{2}$ by using an isosceles right triangle. $(0, \sqrt{2})$ is found and a line is drawn through $P$ and $(0, 1)$. $\overrightarrow{OP}$ is then extended and a line is drawn through $(0, \sqrt{2})$ but parallel to the line connecting $(0, 1)$ and $P$. Where this line intersects the extension of $\overrightarrow{OP}$ is the head of $\sqrt{2} \cdot \overrightarrow{OP}$. This construction is illustrated in Figure 17.

For a full development of this opportunity, Chapter Four of John Riner's Basic Topics in Mathematics (15) may be read.

Opportunity Sixteen

Solving a system of equations for a simultaneous solution is an important part of every student's mathematical experience. It is also an occasion to study a useful application of the vector space concept.
The set of all polynomial equations forms a real vector space. Two equations are added by forming a new equation whose coefficients are the sum of the coefficients of the original equations. Thus the sum of \( x+2y+z+2 = 0 \) and \( x+y - 3 = 0 \) is \( (x+2y+z+2) + (x+y - 3) = 0 \) or \( 2x+3y+ -1 = 0 \). Note that it is easier to add them if they are put into a form with 0 on the right hand side first. If this is not done terms on the left hand side must be added to terms from the left hand side and terms from the right hand sides of the two equations must be added together. The real multiple of an equation has all of its coefficients multiplied by the real number. Thus \( 3 \cdot (x^2+x+2 = 0) \) is \( 3x^2+3x+6 = 0 \).

The equation \( k(ax+by+c) + l(dx+ey+f) = 0 \) is said to be dependent upon \( ax+by+c = 0 \) and \( dx+ey+f = 0 \). It is easily shown that \( (x_o,y_o) \) is a solution of \( ax+by+c = 0 \) and \( dx+ey+f = 0 \) only if it is a solution of \( k(ax+by+c) + l(dx+ey+f) = 0 \). Thus the problem of solving two equations in two variables is reduced to solving one equation which is dependent upon the two. By a judicious choice of \( k \) and \( l \), the problem of solving the dependent equation can be quite easy. As an example consider the following solution to the system \( x+2y+2 = 0 \) and \( 2x - 3y+2 = 0 \). Here we see that by choosing \( k = 3 \) and \( l = 2 \), we obtain a convenient dependent equation, \( 3(x+2y+2) + 2(2x - 3y+2) = 0 \). That is, \( 7x+0y+10 = 0 \) or \( x = \frac{10}{7} \); but \( - \frac{10}{7} + 2y+2 = 0 \) implies \( y = - \frac{2}{7} \) and a solution \( (-\frac{10}{7},-\frac{2}{7}) \) is obtained.
When the students are ready to solve three equations in three unknowns the same method will again aid them. This method of obtaining a solution is one reward of a vector treatment which the student can see. It should provide a new respect for vector spaces while introducing the student to the concept of linear dependence of vectors.

The reader may wish to see a fuller discussion of what has been presented here. Such a discussion may be found in High School Mathematics: Course One. (1; 543)

**Opportunity Seventeen**

Several of the newer mathematics programs have utilized vector methods in studying Euclidean geometry. Among the writers convinced of the value of such an approach are those in the University of Illinois Committee on School Mathematics and the School Mathematics Study Group. (16)(19) The SMSG team has chosen to call their vectors "rigid motions." But the UICSM writers, in particular Herbert Vaughan, have based their work on translations. The researcher will present a brief discussion of this approach here.

Steven Szabo of the University of Illinois suggests that the discussion of translations begin at the untuitive level. He feels that an intuitive idea of translation might be developed by sliding transparent plastic sheets. In this way a teacher should be able to make the following properties of translations plausible.
1. Given points P and Q, there is a translation that maps P onto Q.

2. A translation is determined by any point and its image.

3. The composition of a pair of translations is a translation.

4. Composition of translations is both commutative and associative.

5. The identity mapping on E (Euclidean 3-space) is a translation.

6. Each translation has an inverse, and this, also is a translation. (19; 218)

Properties 1 and 2 simply state what a translation is. Let us observe here that although P and Q would determine a translation, every point in the space would be moved by this translation. Consider Figure 18 showing the translation moves P onto Q but also moves T onto U and X onto Y.
Property 3 discusses a composition of translations. To compose two translations one is performed directly after the other. Figure 19 shows how the translation determined by P and Q followed by the translation determined by Q and R affects several points in the plane and produces the single translation determined by P and R.

![Figure 19](image)

This composition of translations is often called addition of translations and properties 4, 5, and 6 establish that it has the properties of vector addition.
Once these properties are clarified for the students, the teacher may wish to describe how translations may be multiplied by real numbers. This process is visualized much like multiplication of directed segments by real numbers. Basically, the product of a translation by a real number is a new translation in the same direction but associating points at a distance equal to the real number times the distance between associated points in the original translation. Figure 20 shows what happens to several points under a translation along with what happens under the product of the original translation and the real number 3.

Now the set of translations has a vector addition and scalar multiplication defined on it. It may be easily made plausible that the set forms a real vector space. Once this has been done the students may use these techniques to prove theorems from Euclidean geometry.
The reader who wishes to utilize this approach will find a more complete development in Steven Szabo's article "An Approach to Euclidean Geometry through Vectors," found in the March 1966 issue of The Mathematics Teacher. (19)

Opportunity Eighteen

The treatment of the set of complex numbers as a real vector space is by no means novel. Many textbooks, for both the high school and college student, treat complex numbers as such. (6)(15) The writer presents a short discussion here so that the reader may see how such a treatment fits into a curriculum which is spiralling about the vector space concept.

Many teachers define the set of complex numbers to be $C = \{a+bi : a$ and $b$ are real numbers$\}$. Here $i$ is defined by $i^2 = -1$ and one should note that $bi$ doesn't mean $i$ multiplied by $b$. Addition of two complex numbers $a+bi$ and $c+di$ where $a$, $b$, $c$, and $d$ are real numbers is defined by the equation $(a+bi)+(c+di) = (a+c)+(b+d)i$. Thus addition is a closed operation on $C$. A multiplication can be defined on $C$ but that comes later. What concerns us at this point is a scalar multiplication. If $r$ is a real number, we define $r \cdot (a+bi) = ra+rb$. Thus we have scalar multiplication as well as addition on $C$. The teacher may or may not wish to have the students verify all the properties of a real vector space.

The teacher may find it valuable to note the isomorphism between $\mathbb{R}^2 = \{(a,b) : a$ and $b$ are real numbers$\}$ and
\[ C = \{a+bi : a \text{ and } b \text{ are real numbers}\} \]. This isomorphism of vector spaces would associate the pair \((a, b)\) with the complex number \(a+bi\). Some teachers may want to share this isomorphism with their students.

The above-mentioned isomorphism establishes a one-to-one correspondence between elements of \(C\) and directed line segments on the Euclidean plane with initial points at the origin. The association pairs \(a+bi\) with the segment whose terminal point is \((a, b)\) as shown in Figure 21. Clearly \(3+2i\) is associated with \(\overrightarrow{OP}\).

![Figure 21](image)

Once this correspondence is established, the students may do research to see if addition is the same in the two systems. Perhaps the teacher would ask the students to add \(3+2i\) and \(2+3i\) and then add the associated directed line segments. Of course
(3+2i) + (2+3i) = 5+5i and Figure 22 shows a similar result is obtained when the directed segments are added.

Figure 22

Comparisons of the properties in the two systems may be made and further research involving scalar multiplication would be beneficial. By comparing complex numbers with a familiar system the student may be made more at ease and provided with the tools for doing mathematical research. This is part of the beauty of any mathematical structure. Once it is studied in one setting, the researcher has ideas to apply to other models which might possibly have the same structure. Perhaps the inclusion of such avenues for research in the curriculum would lead the way to preparing more pure mathematicians.
The study of the set of ordered triplets as a vector space could prove to be valuable in two ways. Perhaps the more important reason is the fact that these ordered triplets form coordinates for points in 3-space. Many applied mathematicians utilize three dimensional coordinates in their work. Functions defined on 3-space are often used extensively by engineers. Secondly, the study of ordered triplets can be used to demonstrate how mathematicians seek extensions or generalizations of what they have previously encountered. The simple extension of what is known about ordered pairs to ordered triplets and the resulting generalization to n-tuples (sequences of n numbers where n is a positive integer) is exemplary of how important mathematical truths are first realized.

The set of ordered triplets which is often called \( \mathbb{R}^3 \) is defined by the equation \( \mathbb{R}^3 = \{(x,y,z) : x, y, \text{ and } z \text{ are real}\} \). If \((x,y,z) \in \mathbb{R}^3 \) and \((u,v,w) \in \mathbb{R}^3 \), these elements are added \((x,y,z) + (u,v,w) = (x+u,y+v,z+w)\). Furthermore if \(a\) is any real number we can define a multiplication of \((x,y,z)\) by \(a\). The defining equation is \(a \cdot (x,y,z) = (ax,ay,az)\). Thus there is a binary operation of addition and a scalar multiplication defined on \( \mathbb{R}^3 \). If we define \((x,y,z) = (u,v,w)\) to mean \(x = u\), \(y = v\), and \(z = w\) the extension from \( \mathbb{R}^2 \) to \( \mathbb{R}^3 \) is complete.

The teacher can verify that \( \mathbb{R}^3 \) is a vector space or he may decide to require the students to do this. Once \( \mathbb{R}^3 \) is
established as a vector space, the teacher may wish to apply this knowledge to the study of geometry.

Just as $\mathbb{R}^2$ is used to study plane geometry, $\mathbb{R}^3$ can be used to study solid geometry. Three perpendicular axes are constructed. One axis is called the $x$-axis, a second the $y$-axis, and the third the $z$-axis. The intersection of the three axes is called the origin and each point can be associated with an ordered triplet. The first number, the $x$-coordinate, is the perpendicular distance from the point to the plane determined by the $y$ and $z$ axes. Similarly the $y$-coordinate is the perpendicular distance from the $xz$-plane and the third number or $z$-coordinate is the perpendicular distance from the point to the $xy$-plane. This is illustrated in Figure 23.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure23.png}
\caption{Figure 23}
\end{figure}
Now that each point in 3-space has an ordered triplet we can associate this ordered triplet with the directed line segment from the origin to the point. Such an association makes it possible to study solid geometry and prove theorems in solid geometry by analytical techniques using vectors. The reader who would like to learn more about this technique is referred to a booklet entitled *Vectors in Three Dimensional Geometry* published by the National Council of Teachers of Mathematics. (9)

**Opportunity Twenty**

Many traditional and recent textbooks utilize vector techniques to solve problems usually associated with Physics. These problems deal with those concepts from Physics in which both direction and magnitude are involved and hence may be pictured in a diagram by a directed line segment. Such concepts as force, speed, and light rays, may be so pictured. In this paper the writer will designate how such problems might be used in a trigonometry course.

The writer assumes that those students to whom this opportunity would be presented have learned about the three trigonometric functions, sine, cosine, and tangent. It is also assumed that they can find the values of the trigonometric functions of angles stated in degree and radian measure. No new vector space is to be presented here. The presentation is rather an application of the already familiar set of directed line segments.
The students should be informed as to what physical concepts will be represented by directed segments. This writer will confine his discussion to force and velocity. The teacher wishing to use this discussion in a classroom would need to spend time discussing vertical and horizontal components of directed segments in the plane. The following discussion could be used for this purpose.

A force of 30 lbs. exerted on an object at an angle of \(^{45}\)° with the horizontal has a vertical component of \(\sqrt{2}/2\cdot(30)\) lbs. and an equal horizontal component. This is true since the directed segment with length of 30 units and angle of \(^{45}\)° is the sum of a horizontal and a vertical vector both of length \((\sin^{45}\)°\) \(\cdot 30 = (\cos^{45}\)°\) \(\cdot 30\). These directed segments are shown in Figure 24.

![Figure 24](image-url)
Thus a vertical force of \( \sqrt{2} \cdot (30) \) lbs. and a horizontal force of \( \sqrt{2} \cdot (30) \) lbs. are exerted on the object simultaneously.

A few examples to show how to apply knowledge of vectors, trigonometry, and science in solving problems follow.

**Example 1.** What direction and speed should a plane fly if the pilot wants to fly to a destination due east at a speed of 345 m.p.h. when a 40 m.p.h. wind is blowing from the North?

Here we need to find what directed segment to add to a vertical segment of length 40 in order to obtain a horizontal segment of length 345. Figure 25 shows this in a diagram.

![Figure 25](image.png)

This figure shows that the directed segment needed forms an angle of \( \theta \) with the horizontal where \( \tan \theta = \frac{40}{345} \). By the Pythagorean Theorem, the length of this directed segment is found to be \( 345^2 + 40^2 \). Thus the direction is \( \arctan \frac{40}{345} \) and the speed is \( 345^2 + 40^2 \).
Example 2. A switch engine on one track pulls a freight car on a parallel track by a cable. If a force of 3200 pounds on the cable is needed to start the car when the cable is taut and forms angles of 30° with the tracks, what force would be needed to start the car when the engine is coupled to the car? (23; 171)

In this problem we need to find the horizontal component of a segment directed at an angle of 30° off the horizontal and of length 3200. If we diagram the problem as in Figure 26 we see that \( x \), the length of the horizontal component, satisfies the relation \( \frac{x}{3200} = \cos 30° \) or \( x = (\cos 30°) \times 3200 \).

![Figure 26](image)

Opportunity Twenty-one

Perhaps due to recent applications in the field of statistics, extensive study of matrices seems to be occurring more
frequently at the undergraduate and high school levels. The School Mathematics Study Group has produced a text, *Introduction to Matrix Algebra*, for use in the high school. (17) Other authors are following suit. The set of $m \times n$ real matrices, where $m$ and $n$ are fixed positive integers, is an example of a real vector space and thus relevant to our discussion.

Real matrices are rectangular arrays of real numbers. An $m \times n$ matrix is said to consist of $m$ rows and $n$ columns. An example is the $3 \times 4$ matrix $\begin{pmatrix} 5 & 9 & 6 & 1 \\ 2 & 1 & 3 & 4 \\ 8 & 7 & 0 & 6 \end{pmatrix}$. In applications these numbers are often probabilities or coefficients in a system of equations. In elementary work with matrices the study is often limited to a given set of small matrices. The discussion presented here will be limited to the set of $2 \times 2$ real matrices. This may be sufficient for some students while for others a teacher may wish to extend the discussion to the general case, $m \times n$ matrices.

Two $2 \times 2$ matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$ are added according to the equation. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$ An example of such addition is $\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 3 & 5 \end{pmatrix}$.

The properties of vector addition may be verified and it is found that $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is the additive identity while $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$ is the inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. 
Multiplication of matrices can be defined but it is complicated and many applications of the simpler vector space can be found. Therefore, scalar multiplication of 2 x 2 real matrices by real numbers is considered next.

To multiply the 2 x 2 matrix \((\begin{array}{cc} a & b \\ c & d \end{array})\) by a real number \(r\), we multiply each component of the matrix by \(r\). Thus \(r \cdot \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) = \left(\begin{array}{cc} ra & rb \\ rc & rd \end{array}\right)\). An example of scalar multiplication is \(3 \cdot \left(\begin{array}{cc} \pi & -1 \\ -2 & 7 \end{array}\right) = \left(\begin{array}{cc} 3\pi & -3 \\ -6 & 21 \end{array}\right)\). With this definition of scalar multiplication and addition as defined above, the set of 2 x 2 real matrices forms a vector space.

Once students have completed a study of the structure of this vector space, the teacher may wish to provide an intrinsic reward. Such a reward could take the form of an application like the following.

A certain insurance company offers two forms of family insurance. One form comes in units consisting of $5,000 life insurance on the husband, $2,000 life insurance on the wife, and $1,000 insurance on the life of any child. The cost of this insurance package is a $100 premium. The other plan provides $6,000 on the husband, $1,000 on the wife, and $1,000 on each child and has a $110 premium. All information needed about the two unit plans is provided by the matrices.

\[
\left(\begin{array}{cc} 5,000 & 2,000 \\ 1,000 & 100 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{cc} 6,000 & 1,000 \\ 1,000 & 110 \end{array}\right).
\]
The insurance agent can quickly find all information about various levels of family insurance by adding scalar multiples of the two matrices. Thus if a husband wanted $50,000 insurance on himself, the agent can quickly display the needed information.

\[
4 \cdot \begin{pmatrix} 5,000 & 2,000 \\ 1,000 & 100 \end{pmatrix} + 5 \cdot \begin{pmatrix} 6,000 & 1,000 \\ 1,000 & 110 \end{pmatrix} = \\
\begin{pmatrix} 20,000 & 8,000 \\ 4,000 & 800 \end{pmatrix} + \begin{pmatrix} 30,000 & 5,000 \\ 5,000 & 550 \end{pmatrix} = \\
\begin{pmatrix} 50,000 & 13,000 \\ 9,000 & 1,350 \end{pmatrix}.
\]

Here it is quickly read that a premium of $1,350 will provide $50,000 life insurance on the husband, $13,000 on the wife and $9,000 on each child. The agent might also discuss the plan described by \( 10 \cdot \begin{pmatrix} 5,000 & 2,000 \\ 1,000 & 100 \end{pmatrix} = \begin{pmatrix} 50,000 & 20,000 \\ 10,000 & 1,000 \end{pmatrix} \) and other combinations of the two unit plans which provide in the neighborhood of $50,000 insurance on the husband.

This completes the presentation of opportunities for introducing the vector space concept into the school mathematics curriculum. The opportunities have evolved from concrete experiences for children to complex mathematical situations for mature students. In the next chapter, the researcher will discuss how one such experience with the vector space of real functions might be presented to college students.
CHAPTER III
THE VECTOR SPACE OF REAL FUNCTIONS

In Chapter II, the vector space concept was examined as a unifying thread which might run through the school mathematics program. This is to be the major contribution of this study. It is the purpose of this chapter to show how the emphasis upon the vector space concept can be implemented at a specific level. The writer has chosen to illustrate how the vector space of real functions might be taught to a general education mathematics class composed mostly of college freshmen. However, the reader should not expect to find a scientific study of this classroom procedure. Such is not the purpose of this study. The researcher merely wishes to show that the unifying thread of Chapter II can be implemented.

At The Ohio State University, First Year College Mathematics I and II, sequence of two 5-quarter hour courses, is designed for liberal arts students who are majoring in areas which do not extensively use mathematics as a tool. Students must demonstrate a proficiency in basic algebraic computation before they are permitted to enroll in First Year College Mathematics I. In recent years Dr. John Riner's Basic Topics in Mathematics (15) has been used as a text for both of these courses. This book is not a survey text. The primary goal in the text is "imparting to the reader an awareness
of the nature of mathematics.... by giving the reader a chance to do a little mathematics." (15, Preface)

In First Year College Mathematics I, the student learns about the following: algebra of sets, functions, equivalence relations, algebra of real numbers, absolute value, graphs of equations and functions, graphs of straight lines, the vector space \( \mathbb{R}^2 = \{(a,b): a \text{ and } b \text{ are real numbers}\} \), and the vector space of the complex numbers.

This writer was assigned to teach First Year College Mathematics II at the Lima Campus of The Ohio State University during the Winter Quarter 1968. He considered the material of the course and the philosophy guiding its treatment to be consistent with the objective of implementing the vector space as a unifying concept. Therefore, this assignment provided him an opportunity to investigate, on a small scale, the value of teaching such students about the vector space of real functions. The series of lessons dealing with this concept followed a short study of trigonometry. The students had studied arc length, degrees and radians, trigonometric functions, periodicity, graphs of trigonometric functions, identities, inverse trigonometric functions, and solution of triangles.

In his preliminary planning, the investigator observed that many educators feel the student is motivated to study and thus learns more quickly if he feels a definite need for some new knowledge; also a student is more eager to study mathematics when he receives practical benefits as a direct result of his study. Jerome Bruner has written,
The best way to create interest in a subject is to render it worth knowing, which means to make the knowledge gained usable in one's thinking beyond the situation in which the learning has occurred. (3; 31)

John Dewey observed,

It thus becomes the office of the educator to select those things within the range of existing experience that have the promise and potentiality of presenting new problems which by stimulating new ways of observation and judgment will expand the area of further experience. He must constantly regard what is already won not as a fixed possession but as an agency and instrumentality for opening new fields which make new demands upon existing powers of observation and of intelligent use of memory. (5; 375)

For these reasons, this writer adopted the following three-step technique as an effective method for teaching mathematics. First the teacher confronts students with a sensible problem which establishes a need for a mathematical idea. Then, once the need is evident, a study of the structure that covers this idea is made. Finally, the student receives a "payoff" when he finds that the new mathematical structure is useful in solving new problems as well as the original.

The writer planned to use this three-step technique to teach First Year College Mathematics II students that the set of functions defined on real numbers forms a vector space. Since the students had not studied continuous functions, it was not planned to consider the fact that continuous real functions form a subspace. As in most mathematical writings for students of comparable mathematical maturity, the functions to be considered were, for the most part, continuous.
The writer hoped that the students would be able to apply the techniques to drawing smooth graphs without a full understanding of continuity. The series of lessons and method of presentation, as conceived prior to the first meeting of the class, were as follows.

Establishing a need. Since the student had had experience graphing linear and trigonometric functions, the graphing of a function such as \( y = \sin x + x \) seemed a sensible problem for him. It was a simple extension of the problems the student had already encountered. But the drawing of this graph is considerably more complicated. It is tedious finding the values of \( y \) which correspond to given values of \( x \). Even when a considerable number of ordered pairs are located on the graph, the extrapolated graph which best fits the points is by no means evident. Thus the student should soon feel a need for a new mathematical idea which would aid him in this graphing problem. The writer intended to let the students grapple with the problem from one class period to the next. He felt that many students would find the method which would allow them to graph the function. In any event the following notes were to be given to the class on the following day.

---

Often we will encounter functions which look very similar to functions with which we are familiar. But when we try to graph these functions, the procedures which we have been using do not help. However, there are effective aids to graphing many of these functions. Let us consider a few of these aids.
One function which often causes difficulty is \( f(x) = \sin x + x \).

However, this function is easily graphed if \( f_1(x) = \sin x \) and \( f_2(x) = x \) are graphed first. Consider Figure 27 where \( f_1(x) \) and \( f_2(x) \) are graphed on the same set of axes. Now for each point (actually as many as are convenient) on the \( x \)-axis draw two vertical vectors to the \( y \) values on each graph. Add these vertical vectors and obtain a resultant vector for each value of \( x \) as shown in the second picture of Figure 27. The graph of \( f(x) = \sin x + x \) is obtained by drawing a smooth curve through the ends of the resultant vectors.

Figure 27
Another function which we would like to graph is $f(x) = 2 \cos x$. A helpful technique to use for this problem is to first graph $f(x) = \cos x$. Again vertical vectors are drawn from several points on the $x$-axis to the graph. These vectors are then doubled. That is, they are added to themselves to obtain resultant vectors. The ends of the resultant vectors are then joined with a smooth curve to obtain the graph of $f(x) = 2 \cos x$. This technique is illustrated in Figure 28.
(3) An interesting graph is that of $f(x) = -x^2$. We can obtain this graph easily once we have graphed $p(x) = x^2$. Again we draw vertical vectors from many points on the x-axis to the graph. Then we draw the inverses to these vectors which would be vectors of the same length but in the opposite direction. This is illustrated in Figure 29, where the dashed vectors are the inverses of the original vectors. A smooth curve is drawn through the ends of the inverse vectors and the graph of $f(x) = -x^2$ is obtained.
(4) As a final example we will graph \( f(x) = \frac{1}{2}x^2 + 3x -1 \) on \([-3,3]\).

We first graph \( g(x) = \frac{1}{2}x^2 \), \( h(x) = 3x \) and \( k(x) = -1 \) all on the same set of axes. We draw three vertical vectors from each of several points on the \( x \)-axis to the graphs. Then we add the three vertical vectors from each point to obtain a resultant vector. A smooth curve is drawn through the ends of the resultant vectors and the graph of \( f(x) = \frac{1}{2}x^2 + 3x -1 \) on \([-3,3]\) is obtained. All four graphs are shown in Figure 30.
Exercises: Graph the following functions.

1. \( f(x) = 1 - \cos x \)
2. \( f(x) = -\tan x \)
3. \( f(x) = x^2 + \sin x \)
4. \( p(x) = x^2 + x + 1 \)
5. \( p(x) = x^2 + 2x - 1 \)
6. \( P(x) = 2x^2 - x + 1 \)

The above exercises were to be assigned to the class. They were to be completed by the next class session by using the techniques described in (1), (2), (3), and (4). In one way the need would have been satisfied. The students should have been able to graph more complicated functions such as \( f(x) = \sin x + x \). But another need would have been established. The students should have felt a need for a logical basis for what they were doing. They should have wanted to know just when functions can be added graphically, why the sum of \( y = \sin x \) and \( y = x \) isn't \( 2y = \sin x + x \), and how these techniques can be extended to other problems. Thus there would be, the researcher felt, a need to seek out a mathematical structure which covers these ideas and to explore this structure.

Discovery and study of the structure. Assuming the students would recognize the need for a study of the above techniques, the writer planned to lead them into the discovery that the set of real functions forms a vector space. Here the emphasis would not be on continuous functions but rather on the addition and scalar multiplication of functions. Laws governing the use of these operations were to be set forth. Since the students in First Year College Mathematics II would have studied only functions with a few discontinuities, it was
assumed they would automatically want to draw smooth graphs of functions. The following notes were to be given to the students and a lecture-discussion of the material would follow.

Some students will wonder why the techniques studied in (1), (2), (3), and (4) are valid. Certainly no logical basis has been laid. In an attempt to analyze what has been done, the most obvious feature is the presence of addition of functions. This addition is performed by adding the functional values for each value of \( x \) in the domain of the two functions. That is, when we add two functions such as \( f_1 = \{(x, f_1(x)) : x \in \mathbb{R}\} \) and \( f_2 = \{(x, f_2(x)) : x \in \mathbb{R}\} \), we get a new function \( f = \{(x, f_1(x) + f_2(x)) : x \in \mathbb{R}\} \). An example of this addition would be as follows. If \( f_1 = \{(1,3), (2,5), (3,7), (4,9)\} \) and \( f_2 = \{(1,1), (2,4), (3,9), (4,16)\} \), the \( f = f_1 + f_2 = \{(1,3+1), (2,5+4), (3,7+9), (4,9+16)\} \) or \( f = \{(1,4), (2,9), (3,16), (4,25)\} \).

Let us make a definition to describe this addition of functions.

**DEFINITION:** If \( f(x) \) and \( g(x) \) are defined on the same domain \( D \), \( F(x) = f(x) + g(x) \) is defined by the equation

\[
F = \{(x, f(x) + g(x)) : x \in D\}.
\]

A second technique is the multiplication of a function by a real number. This multiplication is performed by multiplying the functional value for each value of \( x \) in the domain by the real number. That is, if \( f = \{(1,2), (2,2), (3,3), (4,5)\} \) then \( 2 \cdot f = \{(1,4), (2,4), (3,6), (4,10)\} \). A definition to clearly state this property is needed.
DEFINITION: If \( f = \{(x, f(x)): x \in D\} \) and \( a \in \mathbb{R}^2 \) then 
\[ a \cdot f = \{(x, a \cdot f(x)): x \in D\}. \]

We already know an algebraic system in which addition of the elements and multiplication by real numbers are important. The system is a real vector space and it is true that the set of all functions defined on a given domain form a vector space.

Let \( F \) be the set of all functions defined on the domain \( \mathbb{R}^1 \).

We shall use \( f, g \) and \( h \) to denote elements of \( F \) and \( a, b \) and \( c \) to denote real numbers. Then the properties of \( F \), the real vector space, are as follows:

(V0) \( f + g \in F, \ a \cdot f \in F \)
(V1) \( f + g = g + f \)
(V2) \( (f + g) + h = f + (g + h) \)
(V3) \( f + 0 = f \) where \( 0 = \{(x,0): x \in \mathbb{R}^1\} \)
(V4) \( -f = \{(x,-f(x)): x \in \mathbb{R}^1\} \) has the property that \( -f + f = F + -f = 0 \).
(V5) \( a \cdot (f + g) = a \cdot f + a \cdot g \)
(V6) \( (a + b) \cdot f = a \cdot f + b \cdot f \)
(V7) \( (ab) \cdot f = a \cdot (b \cdot f) \)
(V8) \( 1 \cdot f = f, \ 0 \cdot f = 0 \)

These properties may be quickly verified as may be seen as we verify (V2).

In (V2) we note that \( f + g = \{(x, f(x) + g(x)): x \in \mathbb{R}^1\} \) so \((f + g) + h = \{(x, (f(x) + g(x)) + h(x)): x \in \mathbb{R}^1\} \) but \((f(x) + g(x)) + h(x) = f(x) + (g(x) + h(x)) \) by the associative property of addition of real numbers.
Thus \((f + g) + h = \{x, f(x) + (g(x)) + h(x): x \in \mathbb{R}^3\} = f + (g + h)\).

Therefore (V2) is verified.

Exercises:

1. Verify property (V3).
2. Verify property (V6).
3. If \(f_1 = \{(1,1), (2,4), (3,9), (4,16)\}\) and \(f_2 = \{(1,2), (2,3), (3,4), (4,5)\}\) then
   \[3 \cdot f_1 + 2 \cdot f_2 = \]
   \[-f_1 = \]
   \[3 \cdot (-f_1) = \]
   \[3 \cdot (f_1 + f_2) = \]

These exercises were to be assigned to the students to be completed for the next class session. Perhaps excessive formalism would cause many students to forget the sensible problem which first aroused their interest and as a result mathematics would lose its value for them. For this reason, only a few representative properties of the vector space were to be verified. The students were to be encouraged to work out examples of the other properties.

Application. It would save time if the student were to remember this example until he undertook the study of the abstract vector space. But perhaps students are not impressed with arguments for economy. Maybe they would try to remember this vector space after examination time if it appeared to have a practical value. There is an available practical application of the above vector space.
This application is a simple case of the "Lagrange Interpolation Formula." The work involves addition of functions and multiplication of functions by real numbers. Although the properties of a vector space are not mentioned, many computations could not be justified if the properties had not been demonstrated. As an example, the reader will note that it will be required to add three functions and thus the associative law of addition will be used automatically.

This writer felt that the application would be of interest to many students and demonstrate a practical use for what they had learned. Again the discussion is presented as it was to be given to the students.

A draftsman is making a sketch. He finds it is necessary that his drawing pass through (1,2), (3,4), and (6,6) on his coordinate paper. He must have an equation for each line he draws so that the information can be fed into a computer. Thus he must find the equation of a function whose graph contains (1,2), (3,4), and (6,6). Notice that the following function has values 1 at \( x = 1 \), 0 at \( x = 3 \), and 0 at \( x = 6 \).

\[
f_1(x) = \frac{(x - 3)(x - 6)}{(1 - 3)(1 - 6)} = \frac{(x - 3)(x - 6)}{10}
\]

\( 2 \cdot f_1 \) is such that \( 2 \cdot f_1(1) = 2 \), \( 2 \cdot f_1(3) = 0 \), and \( 2 \cdot f_1(6) = 0 \).

Proceeding in an analogous manner, we can define

\[
f_2(x) = \frac{(x - 1)(x - 6)}{(3 - 1)(3 - 6)} = \frac{(x - 1)(x - 6)}{6}
\]

so that \( 4 \cdot f_2(1) = 0 \), \( 4 \cdot f_2(3) = 4 \), \( 4 \cdot f_2(6) = 0 \).

And finally we define \( f_3(x) \) by

\[
f_3(x) = \frac{(x - 1)(x - 3)}{(6 - 1)(6 - 3)} = \frac{(x - 1)(x - 3)}{15}
\]

so that \( 6 \cdot f_3(1) = 0 \), \( 6 \cdot f_3(3) = 0 \), \( 6 \cdot f_3(6) = 6 \).
Thus it follows that:

\[(2 \cdot f_1 + 4 \cdot f_2 + 6 \cdot f_3)(1) = 2 \cdot f_1(1) + 4 \cdot f_2(1) + 6 \cdot f_3(1) = 2 + 0 + 0 = 2\]

\[(2 \cdot f_1 + 4 \cdot f_2 + 6 \cdot f_3)(3) = 2 \cdot f_1(3) + 4 \cdot f_2(3) + 6 \cdot f_3(3) = 0 + 4 + 0 = 4\]

\[(2 \cdot f_1 + 4 \cdot f_2 + 6 \cdot f_3)(6) = 2 \cdot f_1(6) + 4 \cdot f_2(6) + 6 \cdot f_3(6) = 0 + 0 + 6 = 6\]

That is, the function \( f = 2 \cdot f_1 + 4 \cdot f_2 + 6 \cdot f_3 \) has a graph which contains \((1,2), (3,4), \) and \((6,6)\). And we can state more precisely what \( f \) looks like.

\[f(x) = 2 \cdot \frac{(x - 3)(x - 6)}{10} + 4 \cdot \frac{(x - 1)(x - 6)}{6} + 6 \cdot \frac{(x - 1)(x - 3)}{15}\]

\[f(x) = \frac{1}{5}(x^2 - 9x + 18) - \frac{2}{3}(x^2 - 7x + 6) + \frac{2}{5}(x^2 - 4x + 3)\]

\[f(x) = \left(\frac{1}{5} - \frac{2}{3} + \frac{2}{5}\right)x^2 + \left(-\frac{9}{5} + \frac{14}{3} - \frac{8}{5}\right)x + \left(\frac{18}{5} - \frac{12}{3} + \frac{6}{5}\right)\]

\[f(x) = -\frac{1}{15}x^2 + 19\frac{1}{15}x + 4\frac{1}{5}\]

And as a check,

\[f(1) = -\frac{1}{15}(1) + 19\frac{1}{15}(1) + 4\frac{1}{5} = \frac{30}{15} = 2\]

\[f(3) = -\frac{1}{15}(3) + 19\frac{1}{15}(3) + 4\frac{1}{5} = \frac{60}{15} = 4\]

\[f(6) = -\frac{1}{15}(6) + 19\frac{1}{15}(6) + 4\frac{1}{5} = \frac{90}{15} = 6\]

Exercises:

1. Find the equation of a function \( f \) such that \( f(1) = 3, f(3) = 3, \) and \( f(9) = 1.\)

2. Suppose an object is travelling so that its distance from the starting point, \( s, \) is related to the time travelling, \( t, \) by a quadratic equation \( s = f(t) = a \cdot t^2 + b \cdot t + c.\)

If \( s = 5 \) when \( t = 1, \) \( s = -23 \) when \( t = 2, \) and \( s = -83 \) when \( t = 3, \) find

(i) \( f(t) \)

(ii) the distance from the starting point when \( t = 10\)

3. Find a function of the form \( f(x) = a \cdot x^2 + b \cdot x + c \) which approximates \( g(x) = \sin x \) on \( 0, \) using \( f(0) = 0, f(2) = 1, \) and \( f(\pi) = 0. \) Use \( f(x) \) to approximate \( \sin 1 \) and compare this with \( \sin 1 \) found in a table.
4. In more advanced mathematics, it is shown that
\[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \ldots \] where
\[ n! = n(n - 1)(n - 2)(n - 3) \ldots 3 \cdot 2 \cdot 1. \]
If \(-1 < x < 1\) then \( |\frac{x^5}{5!} - \frac{x^7}{7!} \ldots| < \frac{1}{120} \)
so \( \sin x = x - \frac{x^3}{6} \) very closely. Graph \( f(x) = \sin x \) on the interval \((-1, 1)\) using this approximation.

These exercises were to be discussed in class. It was hoped that they would give the student a preview of advanced mathematical topics such as infinite series and approximation. Thus the students were to see that they had studied an important mathematical tool and hopefully they would have gained some insight into the value of studying mathematical structures.

The Procedure for Reporting Results

The writer did not intend to conduct a controlled experiment in methods of classroom teaching. He merely wished to share his observations with others who may seek to implement the vector space concept in their own teaching. For this reason, he planned two subjective techniques of reporting his observations and asks the reader to accept them only as such.

The first technique he planned to use was a comparison of the results of two similarly constructed tests, one to be given prior to the study of the real vector space of functions and the other immediately following the study. The tests are included in the Appendixes of this paper. Each test has six items and the reader will note the similarity between respectively numbered items.
The properties (V0), ... (V8), taken from the students text, are to be given to the students on both tests. The purpose of this test is not to determine the extent of memorization of these properties.

The first test was to be administered immediately following the study of the vector space, $\mathbb{R}^2$, and the vector space of complex numbers in the First Year College Mathematics I course. Questions 1 and 2 which also appear on the second test were constructed to help the researcher determine to what extent the students had acquired the ability to see the presence of a vector space structure in a new setting. Question 3, also appearing on the second test, was designed to determine the students' ability to detect and identify scalar multiplication and vector addition on objects similar to those with which he has worked. In question 4, also common to the second test, the researcher hoped to elicit responses indicating to what extent the students recognized the abstract nature and general applicability of the vector space structure. Questions 5 and 6, dealing with linear dependence of vectors, were to be asked in order to give the students opportunities to demonstrate their ability to assimilate a new property of vectors and utilize it. These questions deal specifically with the vector spaces which had been recently studied.

The second test was to be given immediately following the study of the vector space of real functions. Questions 5 and 6 were changed for this test in order that the student would again be asked to apply linear dependence to a recently studied vector space. In
this case, the set of real functions should have been familiar as the
set of complex numbers should have been for the first test.

The writer recognized that the use of the same questions on
both tests might cause students to do better on the second test merely
from familiarity with the questions. He hoped to minimize this
influence by administering the second test at least a month after the
first and avoiding discussion of the questions on the first test. The
first test was not to be returned to the students.

The second subjective technique that the researcher planned to
use in reporting his observations was a daily record. This record was
to contain reports written by the researcher after each class session.
These reports were to be unstructured in order that he might express
himself on anything he considered pertinent.

The results of the teaching experience will be reported in
Chapter IV.
CHAPTER IV

THE TEACHING EXPERIENCE

On Wednesday, January 17, 1968 the researcher began to teach the series of lessons described in Chapter III to a First Year College Mathematics II class at the Lima Campus of The Ohio State University. This class consisted of 24 students who had successfully completed First Year College Mathematics I with this same teacher in the preceding quarter.

The study of the vector space of real functions continued until Friday, January 26 when an examination was given. This examination was discussed in Chapter III. The results of this examination will be compared with the results of an analogous examination given to the same students in First Year College Mathematics I. This comparison will be presented at the end of this chapter. Before this is done, the researcher wishes to present a discussion of how and when the series of lessons was presented. These remarks are not part of a scientific study but are presented merely to convey the manner in which the concept of a vector space can be taught along with other concepts and skills.

January 17 The first half of the class period was spent returning the students' tests over Trigonometry. After discussing the tests, twenty minutes were left to begin the discussion of the vector space of real functions. The students had been asked to graph \( f(x) = \)
x + sin x as a homework problem. The teacher discovered that one student had graphed \( f_1(x) = x \) and \( f_2(x) = \sin x \) and used these graphs to obtain the result. This student told the class how he had done the problem. After the student had graphed \( f_1(x) \) and \( f_2(x) \), he had read from the graph, as best as he could, the two values of \( y \) associated with a single value of \( x \). He then added the \( y \)-values arithmetically and graphed the point with the same \( x \)-coordinate and a \( y \)-coordinate whose value was the obtained arithmetic sum. The student had repeated this process for several \( x \) values and then connected the points with a smooth curve.

The researcher showed the students that it was unnecessary to approximate the \( y \)-coordinates and add arithmetically. He demonstrated the use of vector addition following the discussion in technique (1) of the lesson plan provided to the students. Using the one student’s discussion provided a natural introduction to the use of vector methods.

As the bell rang, the teacher asked the students to read about techniques (2), (3), and (4) from their lesson plan as homework. These techniques concerned graphing \( f(x) = 2 \cos x \), \( f(x) = -x^2 \), and \( f(x) = \sqrt[3]{x^2} + 3x - 1 \) by taking scalar multiples, inverses of vectors, and sums of vertical vectors. The students were also assigned \( f(x) = 1 - \cos x \), \( f(x) = -\tan x \) and \( g(x) = x^2 + \sin x \) to graph.

Later that day, the researcher studied the graphs of \( f(x) = x + \sin x \) that were handed in by 20 of the students. It was discovered that 12 had found correctly the value of \( x + \sin x \) for many values of \( x \) and then plotted the points \((x, x + \sin x)\). Several of the students
had plotted over 15 points in order to ascertain the shape of the curve. Only two students had graphed $f_1(x) = x$ and $f_2(x) = \sin x$ first. In each case arithmetic addition of $\sin x$ and $x$ was used. There were 6 homework papers which were not done correctly by any technique. The researcher noted that the breakdown of $f(x) = x + \sin x$ into two functions was not the natural approach for most of the students.

January 18 The presentation of the techniques described in paragraphs (2), (3) and (4) of the students' lesson sheet went smoothly. Students seemed to have no significant difficulty understanding the techniques. After discussing (2), the graphing of $f(x) = 2 \cos x$, a student asked about multiplication by a negative number which allowed for a natural extension to the graph of $f(x) = -x^2$. While discussing $f(x) = -x^2$ students' questions prompted the writer to extend the discussion to the graphing of $f(x) = -2x^2$ and $g(x) = -2 \cos x$. The teaching was oriented toward treating $f(x) = -2x^2$ as $f(x) = 2(-x^2)$. That is, the inverses of the vertical vectors from $f_1(x) = x^2$ were found and then multiplied by 2. This seemed to be more consistent with vector methods than the consideration of $-2(x^2)$. This multiplication would seem to involve a change in direction for the vector as well as a change in size.

While discussing the graph of $f(x) = \frac{3}{2}x^2 + 3x - 1$, the students questioning about the order in which the vertical vectors were added led to a discussion of the associative and commutative laws of vector addition as they applied to the graphing techniques. The students indicated that they understood that any pair of the three vertical vectors at a specific value of $x$ could be added first.
The class concluded with a discussion of the graph of \( f(x) = 1 - \cos x \) which was a problem assigned for homework. Here two approaches to the problem were discussed and the students seemed to understand the distinction. It was decided that to be consistent with the previous approach to graphing \( f(x) = 2(-x^2) \), this function should be treated as \( f(x) = 1 + (-\cos x) \). Hence the graphs of \( f_1(x) = 1 \) and \( f_2(x) = -\cos x \) were prepared first. Of course \( f_2(x) = -\cos x \) was graphed by the method discussed in paragraph (2).

The students listened intently to the discussion of the techniques of graphical addition and scalar multiplication of functions. Their questions indicated that they were able to follow the discussion with ease.

The homework assignment for the class was to graph the functions \( p(x) = x^2 + x + 1 \), \( p(x) = x^2 + 2x - 1 \) and \( p(x) = 2x^2 - x + 1 \).

When the researcher studied the homework that was collected that day, he found that 23 students had worked on the assignment. Of these students, 11 had graphed \( f(x) = 1 - \cos x \) by the correct vector methods, 19 had graphed \( f(x) = -\tan x \), and 16 had graphed \( g(x) = x^2 + \sin x \) correctly.

January 19 The class began with a discussion of the graph of \( p(x) = 2x^2 - x + 1 \) which had been assigned as homework. The students who had difficulty seemed to have it because they failed to use a large enough portion of paper. All students seemed to be aware of the proper vector methods of use.
The teacher discussed the value of the vector methods which had been studied. He emphasized that if the techniques were to be applied to new situations, an understanding of why such techniques were valid was a necessity. An understanding of the validity was to come from an analytical approach to what was being done.

The analytical discussion closely followed the written lesson plan. Exceptions to this plan were the use of the functions whose graphs were drawn in (1) and (2) as examples. For instance, when the addition of functions was discussed the teacher used \( f_1(x) = x, f_2(x) = \sin x \) and \( f(x) = f_1(x) + f_2(x) \) as an example. He pointed out that \((0,0) \in f_1, (\pi/2,1) \in f_1, (\pi/6,\pi/2) \in f_1 \) and \((\pi,-1) \in f_1 \), while \((0,0) \in f_2, (\pi/2,\pi/2) \in f_2, (\pi/6,\pi/6) \in f_2 \) and \((\pi,\pi) \in f_2 \). Therefore \((0,0) \in f, (\pi/2,1+\pi/2) \in f, (\pi/6,\pi/6) \in f \) and \((\pi,-1+\pi) \in f \).

When scalar multiplication had been defined. The teacher included the function previously discussed in (2) as an example. That is, after showing that \((0,1), (\pi/2,0), (\pi,-1), \) and \((\pi/4,\pi/2) \) were elements of \( f_1 \) where \( f_1(x) = \cos x \), it was demonstrated that \((0,2), (\pi/2,0), (\pi,-2), \) and \((\pi/4,\pi/2) \) were elements of \( f(x) = 2 \cdot f_1(x) \).

The students were asked to note that the analytic development was parallel to what had been done in the graphing situation. In the graphing situations, it was actually the \( y \)-values of points on the graphs that had been considered as vertical vectors.

The discussion, following the lesson plan, moved on to a consideration of \( F \), the set of all functions defined on \( \mathbb{R}^1 \). The students were told that \( F \) was a vector space, though quite unlike
the vector spaces previously studied. Property (VO) was discussed and only enough time to make an assignment remained. Problem 3 from the exercise set at the end of that section of the lesson plan was assigned. This problem involved adding functions and finding scalar multiples of functions.

The homework assignment for that day was graded and it was discovered that 23 of the 24 students enrolled in the class had handed in homework. Of the 23 papers handed in, 22 had \( p(x) = x^2 + x + 1 \) graphed correctly, and 19 had \( p(x) = x^2 + 2x - 1 \) as well as \( p(x) = 2x^2 - x + 1 \) graphed correctly. The researcher felt that the students had comprehended the graphical techniques quite well.

January 22 No questions were asked concerning the homework assignment that was collected. Thus the instructor assumed that the students were able to add functions with a finite domain together and also multiply them by real numbers.

The instructor discussed the properties (VO), (V1),...(V8) of the real vector space \( F \). This discussion included proofs of (V1), (V2), (V5), and (V8) along with specific examples to illustrate each of (VO),...(V8). Four or five students actively participated in the proofs. The students were not able to provide much help in proving (V1) and (V2) but after seeing these proofs they were able to prove (V5) and (V8) with little help from the instructor.

When the discussion of (VO),...(V8) was finished, a homework assignment was given. The students were asked to verify properties (V3) and (V6) and read the discussion in the remainder
of the study sheet. This reading assignment dealt with an example of the Lagrange Interpolation Formula as it applies to quadratics.

When the homework papers that had been collected were studied by the instructor, he found that 19 students had done all four exercises correctly while one person had made a few mistakes.

January 23 In answer to the request of several students, the instructor began the class session with a discussion of the proof of property (V6). It seems that several students had not yet discovered the power of translating the statements into set notation and then applying the laws of operations on real numbers to the second coordinates of ordered pairs in the set.

The proof of property (V6) as presented by the instructor follows.

(V6) states that \((a + b) \cdot f = a \cdot f + b \cdot f\) where the "+" on the left hand side of the equation is an operation on real numbers and the "+" on the right hand side represents addition of functions. Now if

\[ f = \{(x,f(x)) \mid x \in \mathbb{R}\} \]

\((a+b) \cdot f = \{(x,(a + b)f(x)) \mid x \in \mathbb{R}\}\)

but

\((a + b) \cdot f(x) = a \cdot f(x) + b \cdot f(x)\) by the distributive law for multiplication of real numbers over addition of real numbers. Therefore

\[ (a + b) \cdot f = \{(x,a \cdot f(x) + b \cdot f(x)) \mid x \in \mathbb{R}\} \]

which is by definition the sum \( \{ (x,a \cdot f(x)) \mid x \in \mathbb{R}\} + \{ (x,b \cdot f(x)) \mid x \in \mathbb{R}\} = a \cdot f + b \cdot f\). Thus the proof is complete.

Following this discussion the instructor demonstrated how to find a function whose graph contained \((1,2), (3,4)\) and \((6,6)\). This demonstration closely followed the discussion on the final pages of the students' study sheets. The instructor found that the students
had a great deal of difficulty understanding the procedure. This was indicated by the many questions which students asked.

Problems 1, 2, and 3 from the final exercise set were assigned as homework to be completed for January 24.

Nineteen homework papers were collected. Thirteen students had proved property (V3) correctly while twelve had proved property (V6) correctly. Every student who had attempted the assignment had proved at least one of the two properties.

January 24 This class session was spent in a discussion of the homework assignment. Several students wanted to know how each problem should have been done. After the instructor had completed the first problem, the students indicated that the further example had clarified the procedure for them. The instructor's solution of the problem did not differ significantly from that presented in the student lesson sheet.

The one change in presentation which seemed to help the students concerned the scalar multiplication. In finding a function, \( f \), such that \( f(1) = 3, f(3) = 3 \) and \( f(9) = 1 \), the first step was to define a function, \( f_1(x) = \frac{(x-3)(x-9)}{(1-3)(1-9)} \) so that \( f_1(3) = 0, f_1(9) = 0 \) and \( f_1(1) = 1 \). Then \( 3 \cdot f_1(1) = 3 \). One student said that it seemed a duplication of effort to define \( f_1 \) such that \( f_1(1) = 1 \) and then multiply it by 3. Why not define \( f_1(x) = \frac{3(x-3)(x-9)}{16} \) so that \( f_1(1) = 3 \)? Several students agreed that this was easier to understand. Therefore the instructor proceeded by discussing similar alternate definitions for \( f_2(x) \) and \( f_3(x) \).
The discussion of the three homework problems took the full class period. The students were told to review the entire series of lessons before the next class session and prepare questions about the topics which were still difficult to understand.

Nineteen homework papers were collected. Sixteen students had done the first exercise correctly, none had done the second exercise correctly and seven had completed the third exercise. The teacher felt that most students understood the procedure but did not wish to spend enough time to do all problems.

January 25 The instructor did not plan any lesson for this class session. The class time was devoted to answering specific questions of the students. Some students wanted to see proofs of properties of the vector space F. Two of the properties were proved. Some students wanted to see another example of how to define a quadratic function passing through three given points. Such an example was explained in detail.

One student asked if it was always necessary to prove (V0), ...(V8) in order to show that a set of objects with an addition and scalar multiplication was a vector space. The instructor took the opportunity to study the idea of a subspace at an intuitive level. He demonstrated that the set $G = \{ f \mid f(x) = mx + b, m \in \mathbb{R}^1, b \in \mathbb{R}^1 \}$ was a subset of F and addition and scalar multiplication in G were identical to the same operations in F. Thus it was only necessary to show G was closed under the operations, (V0), and the presence of inverses and an identity element, properties (V3) and (V4). The above discussion concluded the class session and hence the study of the vector space of real functions.
Comparison of Vector Space Examination Results

On Friday, January 26, 1968, the second form of the Vector Space Examination was given. The results of this examination will be compared with the results obtained from the first form of the examination. The first form was administered to the students on December 1, 1967, while they were enrolled in First Year College Mathematics I. At that time, the students had just completed a study of the vector spaces $\mathbb{K}^2$, the set of complex numbers, and the set of equivalence classes of directed line segments in the plane. Both forms of the Vector Space Examination are included in the Appendices of this paper.

The two forms of the examination were graded at the same time and they were interspersed first. Thus the instructor was unable to determine whether he was grading the first form or the second until he graded questions 5 and 6 which differed on the two forms. This procedure was used to insure a more uniform grading of the two test forms. In the following comparison of the results, the first form of the Vector Space Examination will be called Test 1 and the second form will be called Test 2.

The reader is asked to recall that the first two questions, which were common to both tests, were designed to determine the students' ability to see the presence of a vector space structure in a new setting. The students were asked to show that a set with an addition and scalar multiplication defined by a table had two of the properties of a vector space. Table 1 shows the scores
obtained by the students on these questions. A score of five was given to those students who completed each proof and partial credit was given for examples illustrating the property. The mean scores shown in the table indicate that the students achieved higher scores on all but question 2(b) on Test 2 after seeing the further example of a vector space, the set of real functions.

**TABLE 1**

RESULTS ON QUESTIONS 1 AND 2

<table>
<thead>
<tr>
<th>Scores for 1(a) on Test 1</th>
<th>Frequency</th>
<th>Frequency</th>
<th>Scores for 1(b) on Test 1</th>
<th>Frequency</th>
<th>Frequency</th>
</tr>
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<td>mean = 2.71</td>
<td></td>
<td></td>
<td>mean = 1.54</td>
<td>mean = 1.67</td>
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<table>
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<tr>
<th>Scores for 2(a) on Test 1</th>
<th>Frequency</th>
<th>Frequency</th>
<th>Scores for 2(b) on Test 1</th>
<th>Frequency</th>
<th>Frequency</th>
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<td></td>
<td>mean = 3.04</td>
<td>mean = 2.75</td>
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</table>

Table 2 shows a tabulation of the scores on questions 3, 4 and 5. Question 3 asked the students to devise matrix addition and the scalar product for 2 x 2 matrices. The students were asked to compute one
example of each. They were given 5 points each if they displayed
the correct technique irrespective of computational errors. The
mean score on question 3 is again higher on Test 2 than it was on
Test 1.

<table>
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<tr>
<th>Scores for 3(a)</th>
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<table>
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<th>Scores for 3(b)</th>
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mean=3.50 mean=3.92

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<th>Scores for 4</th>
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<th>Frequency on Test 2</th>
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mean=2.21 mean=2.08

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<th>Parts</th>
<th>Number of Correct Answers of 5</th>
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<td>(iii)</td>
<td>16</td>
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<td>(iv)</td>
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<tr>
<td>(v)</td>
<td>20</td>
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</table>

mean=3.88 mean=4.13

In question 4, the students were asked to explain what they felt
a real vector space is. The responses were graded on a scale from 0 to
5. A score of 5 was given if the student described it as an abstract
structure of which there are several examples. A score of 4 was given
to students who recognized that there were several sets of objects that
displayed an addition and scalar multiplication satisfying (V0),
...(V8) and hence were vector spaces. Students who said that it was
any set of objects satisfying (V0),...(V8) were given a score of 3.
If several examples of real vector spaces were displayed, a score of
2 was given. One example of a real vector space earned a score of 1.
Other responses were scored on the same scale to the extent that
they admitted the presence of an abstract structure. The mean scores
for question 4 indicate that students did better on Test 1.

Question 5 was designed to test the ability of students to
apply the concept of linear dependence to the vector spaces recently
studied. The number of correct responses to each of the five parts
of this question are recorded in Table 2. The number of correct
responses is clearly much higher for Test 1 than for Test 2. The
researcher feels, in retrospect, that the level of difficulty is
significantly higher for the questions on Test 2. Each part of
question 5 that was answered correctly received a score of 2 points.

Question 6 also dealt with the application of linear dependence
to recently studied real vector spaces. The questions for both Test
1 and Test 2 were found to be too difficult for the students. Only
one student received any credit on question 6. He received two points
credit on Test 2 since he remarked that if two functions were dependent
their graphs would cross the x-axis at the same point. This was the
only comparison of the two graphs of two linearly dependent functions
that the instructor deemed worthy of credit.

Table 3 shows the scores received by the twenty-four students on
both tests. The mean scores for both tests was 27.83.
TABLE 3
TOTAL SCORES

<table>
<thead>
<tr>
<th>Student</th>
<th>Total Scores on Test 1</th>
<th>Total Scores on Test 2</th>
<th>Total Scores for 1, 2, 3 and 4 Test 1</th>
<th>Total Scores for 1, 2, 3 and 4 Test 2</th>
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Since questions 1, 2, 3 and 4 were identical on both tests, the researcher felt it would be of value to compare the total scores obtained on these four questions. These scores are also shown in Table 3. The mean score on the first test was 19.63 and the mean score on the second test was 21.13. The item analysis that was made earlier suggested that the second mean would be higher. The writer decided that scores did not warrant an analysis of the difference of the means in order to determine the level of significance.
The researcher realizes that the size of the sample used in this study does not warrant far-reaching conclusions. With a word of caution to the reader that the following remarks are simply observations and not conclusions, the writer will explain what the test results indicate to him.

Before the first test was administered the students had studied several examples of real vector spaces. They had studied $\mathbb{R}^2 = \{(a, b) | a \in \mathbb{R}, b \in \mathbb{R}^2\}$, the set of complex numbers, and the set of equivalence classes of directed line segments in the plane. This was part of their work in First Year College Mathematics I. The next quarter in First Year College Mathematics II, the students completed a short study of Trigonometry and then studied the set of real functions from a vector space standpoint. They learned to graph complicated functions such as $f(x) = 2 \cos x + x^2$ and studied a simple case of the Lagrange Interpolation Formula. Then, a month after they had taken the first form of the Vector Space Examination, they took the second form. There was no significant difference between the two test results. This indicates to the researcher that the students were able to maintain a level of competency with vector spaces while learning new skills.

The study of the vector space structure was extensive in the First Year College Mathematics I course. Several examples were displayed. These examples were not studied for themselves, but only as examples of sets displaying the given structure. In First Year College Mathematics II, the study of the vector space of real functions was primarily centered on applicability. Yet when the students were
asked to respond to questions about the vector space concept, they answered as well as they had after the extensive study of that concept.

In conclusion, the writer believes that the use of the vector space structure in studying real functions gave a new unity to the material studied in First Year College Mathematics I and II while helping the students to sustain a working knowledge of the vector space concept.
CHAPTER V

SUMMARY

The writer feels that this study has produced significant results. These results seem to support the writer's thesis that the use of the vector space as a unifying concept in the mathematics curriculum would contribute to the effectiveness of the educational process. The increased effectiveness would result from the new relativeness of topics in the mathematics program and the added opportunity for the study of mathematical structures.

In Chapter II, topics common to many mathematics programs were examined. It was found that in varying degrees these topics possessed the structure known as a vector space. Some of the topics fully displayed the properties of a vector space. Others possessed a scalar multiplication or a vector addition. However, all were bound together by the unifying thread of the vector space concept. While the writer does not feel that all topics in the mathematics curriculum are related in this way, he feels that there are a sufficient number of such topics to warrant the use of the vector space as one of several unifying threads in a school mathematics program.

The researcher believes that many of the opportunities presented in Chapter II illustrate how the relatedness of topics adds to the effectiveness of teaching. Some opportunities were
used to make drill work more interesting. Drill in addition of integers, geometric constructions, the reading of protractors, and multiplication of rational numbers was presented in the vector space context to provide novelty within the repetition.

Many of the experiences in Chapter II provide intuitive work with advanced concepts. The researcher feels that this background of intuitive experience will facilitate the more rigorous study of the concepts at a later stage. The bulk of this intuitive background is preparatory for the analytical study of the abstract vector space. The writer feels that the increasing use of the vector space structure in pure and applied mathematics justifies his claim that this alone would be a significant contribution. However, the reader will remember the occurrence of isomorphisms, equivalence classes, and groups, at a concrete level, in Chapter II. The relationships existing between vector spaces and these other advanced topics automatically provides for their inclusion.

The researcher would like to clarify this concrete-abstract dichotomy with an example. He feels the emphasis of many current mathematics programs upon the field axioms has unintentionally placed the operation of scalar multiplication in a suspect position. He has attempted, in many of the opportunities in Chapter II, to present examples of multiplication of different kinds of objects by real numbers. Such experiences should, in the opinion of the writer, make the operation of scalar multiplication more palatable or at least more acceptable when it is encountered at the abstract level.
The increased emphasis upon mathematical structures which would be inherent in any mathematics program unified by the vector space concept, should also be of pedagogical benefit. The acknowledged presence of mathematical structures opens new avenues for student research, for any time that a given model displays some of the properties of a structure, the student can be encouraged to search for more. In Chapter II, the writer attempted to demonstrate various opportunities of this nature.

In addition to the aforementioned benefits to be accrued from an emphasis upon the vector space concept, the writer feels that there are certain time saving features. He is of the opinion that the student can learn about a new topic more quickly if he recognizes the presence of familiar properties. For example, if the student is comfortable with vector techniques, a graphical approach should provide new insight concerning the properties of addition of complex numbers.

The series of lessons dealing with the vector space of real functions provides an example of how two topics can be studied simultaneously. The results of the classroom procedure followed by the researcher seem to indicate the feasibility of such a plan. However, the writer recognizes that more extensive study would be necessary in order to establish a definite conclusion.

This study provides a longitudinal reference for curriculum workers and/or classroom teachers who seek to introduce the vector concept at a particular level. In addition to exhibiting specific opportunities the writer has sought to indicate the level of abstractness
appropriate for different grade levels. Therefore, this paper can be used as part of a framework to give continuity to the students' experiences or as a source of valuable supplementary activities.

The writer hopes to see the vector space structure established as a unifying concept in school mathematics. Perhaps this study will convince other mathematics educators that this is a worthwhile goal and they will see fit to contribute their efforts in order that it might be attained.
APPENDIX A

Vector Space Examination - Form 1

The examination questions begin on the next page. The following definition of a real vector space is to be used on this examination.

Definition: Let $W$ be any set on which an "addition" and "scalar multiplication" are defined. Then, if the following properties, $(V0) \ldots (V8)$ hold, $W$ is a vector space over the real numbers or a real vector space.

Let $u$, $v$, and $w$ denote arbitrary elements in $W$ and $a, b, c$ be real numbers.

$(V0)$ $u + v \in W, \ a \cdot u \in W$

$(V1)$ $u + v = v + u$

$(V2)$ $(u + v) + w = u + (v + w)$

$(V3)$ $u + \overline{0} = u$ for some fixed element $\overline{0}$ in $W$.

$(V4)$ Given $u \in W$ there is an element $-u$ in $W$ such that $u + -u = \overline{0}$

$(V5)$ $a \cdot (u + v) = a \cdot u + a \cdot v$

$(V6)$ $(a + b) \cdot u = a \cdot u + b \cdot u$

$(V7)$ $(ab) \cdot u = a \cdot (b \cdot u)$

$(V8)$ $1 \cdot u = u$ and $0 \cdot u = \overline{0}$
Questions

1. Let $W = \{A, B, O\}$ and $X = \{0, 1, 2\}$. Define addition on $W$ and multiplication of elements from $W$ by scalars from $X$ with the following tables.

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From the table we see that $A + A = B$ and $2 \cdot A = B$.

To show $W$ is a vector space over $X$ we must verify $(V_0), \ldots, (V_8)$. You need only verify $(V_4)$ and $(V_7)$. Use the space provided below to do this.

(a) Show the operation $+$ has property $(V_4)$.

(b) Show the operation $\cdot$ has property $(V_7)$. 
2. Let $W$ be the set of all quadratic functions on $\mathbb{R}$. Then $f \in W$ if and only if $f(x) = rx^2 + sx + t$, where $r, s$, and $t$ are real numbers and $r \neq 0$. For example $f(x) = 3x^2 + 2x + 1 \in W$. Let $+$ be defined on $W$ by

$$(rx^2 + sx + t) + (px^2 + mx + n) = (r+p)x^2 + (s+m)x + (t+n)$$

and let scalar multiplication be defined by

$$k(rx^2 + sx + t) = (kr)x^2 + (ks)x + (kt).$$

Let $u = rx^2 + sx + t$ and $v = px^2 + mx + n$ and show that (V5) and (V6) hold for this set.

(a) In the space below show that (V5) holds.

(b) In the space below show that (V6) holds.
3. Consider a real matrix to be a square arrangement of four real numbers, \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). An example of such a matrix is \( \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} \). The set of these square arrangements forms a real vector space. What matrix would you associate with the following scalar product and vector sum? Your answers will indicate what you think scalar multiplication and vector addition should be.

\[
3 \begin{pmatrix} 0 & 4 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} + \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} =
\]

4. Write a paragraph on what the phrase "real vector space" calls to your mind. Try to express only your own thoughts and do not quote phrases or definitions from the text or lectures.
5. Two vectors \( u \) and \( v \) are said to be linearly dependent if there are real numbers \( a \) and \( b \) not both zero such that
\[ a \, u + b \, v = 0. \]
Thus in the vector space \( \mathbb{R}^2 \) \((2,3)\) and \((8/3,4)\) are linearly dependent since
\[ 4 \,(2,3) + -3 \,(8/3,4) = (0,0) = 0. \]
Determine which of the following pairs of complex numbers are linearly dependent. If a pair is linearly dependent, show one pair of numbers that would work for \( a \) and \( b \).

(i) \( u = 3 + 2i \), \( v = 6 + 4i \)

(ii) \( u = 3 - 3i \), \( v = 5 + 5i \)

(iii) \( u = 2 + 3i \), \( v = 3 + 2i \)

(iv) \( u = 2i \), \( v = i \)

(v) \( u = 2i \), \( v = 3i \)

6. Let \( u = c + di \) and \( v = (c + di)^{-1} \) be two complex numbers. Are \( u \) and the conjugate of \( v \) linearly dependent for all values of \( c \) and \( d \)? Explain your answer.
APPENDIX B

Vector Space Examination - Form 2

The examination questions begin on the next page. The following definition of a real vector space is to be used on this examination.

Definition: Let \( W \) be any set on which an "addition" and "scalar multiplication" are defined. Then, if the following properties, (V0)...(V8) hold, \( W \) is a vector space over the real numbers or a real vector space.

Let \( u, v, \) and \( w \) denote arbitrary elements in \( W \) and \( a, b, c \) be real numbers.

1. (V0) \( u + v \in W, a \cdot u \in W \)
2. (V1) \( u + v = v + u \)
3. (V2) \( (u + v) + w = u + (v + w) \)
4. (V3) \( u + \bar{u} = 0 \) for some fixed element \( \bar{u} \) in \( W \).
5. (V4) Given \( u \in W \) there is an element \( -u \) in \( W \) such that \( u + (-u) = 0 \)
6. (V5) \( a \cdot (u + v) = a \cdot u + a \cdot v \)
7. (V6) \( (a + b) \cdot u = a \cdot u + b \cdot u \)
8. (V7) \( (ab) \cdot u = a \cdot (b \cdot u) \)
9. (V8) \( 1 \cdot u = u \) and \( 0 \cdot u = \bar{u} \)
Questions

1. Let $W = \{A, B, C\}$ and $X = \{0, 1, 2\}$. Define addition on $W$ and multiplication of elements from $W$ by scalars from $X$ with the following tables.

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From the table we see that $A + A = B$ and $2 \cdot A = B$.

To show $W$ is a vector space over $X$ we must verify $(V0), \ldots, (V8)$. You need only verify $(V4)$ and $(V7)$. Use the space provided below to do this.

(a) Show the operation $+$ has property $(V4)$.

(b) Show the operation $\cdot$ has property $(V7)$. 
2. Let \( W \) be the set of all quadratic functions on \( \mathbb{R}^n \).
Then \( f \in W \) if and only if \( f(x) = rx^2 + ax + t \), where
\( r, s, \) and \( t \) are real numbers and \( r \neq 0 \). For example
\( f(x) = 3x^2 + 2x + 1 \in W \). Let \( + \) be defined on \( W \) by
\[(rx^2 + ax + t) + (px^2 + mx + n) = (r+p)x^2 + (a+m)x + (t+n)\]
and let scalar multiplication be defined by
\[k \cdot (rx^2 + ax + t) = (kr)x^2 + (ks)x + (kt)\).
Let \( u = rx^2 + ax + t \) and \( v = px^2 + mx + n \) and
show that (V5) and (V6) hold for this set.

(a) In the space below show that (V5) holds.

(b) In the space below show that (V6) holds.
3. Consider a real matrix to be a square arrangement of four real numbers, \((a \ b)\). An example of such a matrix is \((2 \ _3)\). The set of these square arrangements forms a real vector space. What matrix would you associate with the following scalar product the vector sum. Your answers will indicate what you think scalar multiplication and vector addition should be.

\[
3 \cdot \begin{pmatrix} 0 & 4 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} + \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}
\]

4. Write a paragraph on that the phrase "real vector space" calls to your mind. Try to express only your own thoughts and do not quote phrases or definitions from the text or lectures.
5. Two vectors \( u \) and \( v \) are said to be linearly dependent if there are real numbers \( a \) and \( b \) not both zero such that \( a \cdot u + b \cdot v = 0 \). Thus in the vector space \( \mathbb{R}^2(2,3) \) and \( (8/3,4) \) are linearly dependent since
\[
4 \cdot (2,3) + -3 \cdot (8/3,4) = (0,0) = 0
\]
Determine which of the following pairs of functions are linearly dependent. If a pair is linearly dependent, show one pair of numbers that would work for \( a \) and \( b \).

(I) \( u = 3x^2 + 2x + 5 \), \( v = 6x^2 + 4x + 10 \)

(II) \( u = 5x^2 + 5x - 5 \), \( v = 3x^2 + 3x = 3 \)

(III) \( u = 3x + 2 \), \( v = 2x + 1 \)

(IV) \( u = 2x + 2 \), \( v = (x + 1)(x^2 - x + 1) \)

(V) \( u = \sin x \), \( v = \sin (x + \pi) \)

6. If two functions are linearly dependent, how would the graphs of the two functions compare?
BIBLIOGRAPHY


