CONSERVATION LAWS

FOR ELECTROMAGNETIC FIELDS

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By

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INTRODUCTION

In 1918, Emmy Noether (1) published two theorems on the invariance of the integral in variational problems in which she showed that invariance with respect to a group of transformations of the variables implies existence of certain identities. The first of these theorems states that if the invariance group is an \( r \)-parameter transformation group, then \( r \) linear combinations of the variational derivatives are divergences. In the second theorem it was shown that invariance with respect to an infinite group, in which the parameters are arbitrary functions, implies certain relations exist between the variational derivatives of the functional considered.

Precise statements of these theorems will be given with proofs in Chapter I. In following chapters these theorems will be extended to manifolds with affine connection, and then to variational principles for integrals which do not contain derivatives of the field variables.

Our concern here is with application of these theorems to invariance of electromagnetic fields. It has long been known that Maxwell's equations for electromagnetic fields can be produced from a variational principle. The precise form of this variational principle will be discussed in Chapters I and III. The corresponding Lagrangian is invariant under the ten parameter Lorentz group. Hence by application of the first Noether Theorem, we should find ten identities. This has been done, and the results are discussed in Chapter I. When Maxwell's equations are satisfied these
identities assume the form of conservation laws; i.e., each identity states that the divergence of some vector is zero. The ten conservation laws obtained are seen to represent conservation of energy and momentum for the field, and are physically meaningful.

In 1909, Bateman (2) showed that Maxwell's equations for electromagnetic fields in vacuum are invariant under the larger fifteen parameter conformal group, which contains the Lorentz group as a subgroup. Thus, the first Noether theorem if applied should produce fifteen conservation laws.

In 1921, E. Bessel-Hagen (3) carried out this application and produced the desired fifteen first integrals. Ten of these are the ten previously mentioned. The five remaining ones seem to have no known physical interpretation. It should be possible to show these five identities are not conservation laws, or else find a physical interpretation for them. The main objective of this paper is to discuss this problem.

It was believed that an explanation for the five surplus identities could be found by making a critical review of the application of Noether's theorem, expressing all equations in covariant form and giving close attention to the variational principle used. There was also some question about the meaning of conformal invariance in electrodynamics, so a careful study of this invariance had to be made. This dissertation consists of the critical review required, with some extensions and generalizations of the Noether theorems and the variational principle for electromagnetic fields.

In Chapter I a careful account of the Noether theorems and their application by Bessel-Hagen is given. At this point it can be shown that
four of the conservation laws can be derived from the other eleven with no further assumptions, so they are functionally dependent on these eleven conservation laws, although not linearly dependent. This sheds some light on the "mystery" of the surplus conservation laws.

In Chapter II the Noether theorems are stated and proved in covariant form. In Chapter III variational principles are discussed and a new one is introduced for electromagnetic fields. The Noether theorems are then modified to be applicable with this new variational principle. In Chapter IV a study of conformal transformations is made and it is shown that the electromagnetic field belongs to a larger class of tensor fields which are conformally invariant.

Conformal invariance in an arbitrary manifold is discussed and shown to follow from two assumptions:

i) The action integral is \( w = \int \sqrt{|g|} \left( -\frac{1}{2} F_{\mu \nu} F^{\mu \nu} \right) \);

ii) \( \delta g_{\alpha \beta} = 0 \); i.e., no local variation of the metric tensor is allowed.

In Chapter IV the modified form of the first Noether theorem is applied to conformally invariant tensor fields, and the resulting identities are given. If we now assume that we have a Minkowski space, the identities become the 15 identities produced by Bessel-Hagen. Fifteen identities arise because the conformal group in Minkowski space is a 15 parameter group. In other spaces the number of identities is different because the conformal group may depend upon a different number of parameters. It is also seen from this discussion that the identities arising from conformal invariance, particularly the Bessel-Hagen identities, apply to other fields besides the electromagnetic field.

Briefly, the results in this dissertation can be summarized as follows:
1) The earlier Noether Theorems have been stated and proven in covariant form;

ii) A new variational principle applicable to the electromagnetic field has been introduced;

iii) It has been shown the electromagnetic field belongs to a large class of conformally invariant fields; i.e., tensor fields $F$ corresponding to action integral,

$$w = \int_{\mathcal{M}} dx \sqrt{g} (-\frac{1}{4} F_{\alpha \beta} F^{\alpha \beta}),$$

under the new variational principle;

iv) Modified forms of the Noether theorems suitable for the new variational principle have been stated and proven;

v) Four of the fifteen conservation laws given by Bessel-Hagen are shown to be derivable from the remaining eleven, so only eleven are essentially independent in this sense;

vi) It is shown that some conservation laws produced recently by Lipkin (4) do not follow from the Noether theorems, and possibly should not be regarded as conservation laws, since they do not consist of first integrals of Maxwell's equations, and have no physical interpretation.
CHAPTER I

REVIEW OF THE NOETHER THEOREMS WITH APPLICATION TO ELECTRODYNAMICS

In this chapter an accurate account of the Noether Theorems will be given in modern form. The theorems will then be applied to invariance of the electromagnetic field and conservation laws produced. This is essentially the content of the paper by Bessel-Hagen, but some improvements in notation and organization of the arguments will be given here to produce a more readable account. Some criticism of the results will be given to indicate the purpose of the present work by the author.

1. Basic Concepts Concerning Variations

In the following discussion we shall use the following notation:

\[ x = (x^0, x^1, x^2, x^3) \] denotes a point in \( \mathbb{R}^4 \). Greek indices \( \alpha, \beta, \gamma, \ldots \) have the range 0,1,2,3. Latin indices \( i, j, k, \ldots \) have the range 1,2,\ldots,n, where n is considered as given. Functions \( u^i(x^0, x^1, x^2, x^3) \), \( (i = 1, \ldots, n) \) will be considered. Briefly, \( u(x) \) will denote the collection of \( n \) functions \( u^i(x^\alpha) \). The partial derivative of \( u^i \) with respect to \( x^\alpha \) will be denoted by \( \partial_\alpha u^i \), and \( \partial u \) will denote the collection of all partial derivatives \( \partial_\alpha u^i \).

An action integral,\n
\[ W[u] = \int_\Omega dx L(x,u,\partial u), \quad dx = dx^0 dx^1 dx^2 dx^3, \quad (1.1.1) \]

where \( \Omega \) is a region in \( \mathbb{R}^4 \), will be considered.
We shall assume a transformation,
\[ \bar{x}^{\alpha} = x^{\alpha} + \epsilon \xi'(x, u, \partial u) + o(\epsilon), \]
is given in which \( \epsilon \) is a parameter and \( o(\epsilon) \) is a term for which
\[ \lim_{\epsilon \to 0} \frac{o(\epsilon)}{\epsilon} = 0. \]
We shall let
\[ \Delta x^{\alpha} = \epsilon \xi'(x, u, \partial u) \]  
and call \( \Delta x^{\alpha} \) the variation of \( x^{\alpha} \). Transformations \( \Delta x^{\alpha} \) will be called
\textbf{infinitesimal transformations} and denoted by,
\[ \bar{x}^{\alpha} = x^{\alpha} + \Delta x^{\alpha}, \]
with the understanding that the term of \( o(\epsilon) \) has been dropped. Briefly, we may sometimes write \( \bar{x} = x + \Delta x \) to denote such transformations.

We shall also assume transformations,
\[ \bar{u}^1(\bar{x}) = u^1(x) + \epsilon w^1(x, u, \partial u) + o(\epsilon), \]
are given where \( \bar{x} \) is defined by \( \Delta x^{\alpha} \). We define
\[ \Delta u^1 = \epsilon w^1(x, u, \partial u) \]  
\textbf{to be the total variation of} \( u^1 \). Briefly, \( \Delta u^1 \) will be denoted by
the infinitesimal form,
\[ \bar{u}^1(\bar{x}) = u^1(x) + \Delta u^1. \]
Occasionally, \( \Delta u^1 \) will be abbreviated to
\[ \bar{u}(\bar{x}) = u(x) + \Delta u. \]

The quantity,
\[ \delta u^1 = \bar{u}^1(x) - u^1(x), \]
will be called the \textbf{local variation} of \( u^1 \).

\textbf{NOTE:} In the following calculations only first order terms in \( \Delta x, \Delta u, \)
or \( \delta u \) will be retained. The usual summation convention on indices
will be employed throughout all discussions; e.g., the sum \( \sum_{\alpha=0}^{3} A_{\alpha} A^{\alpha} \)
will be written as \( A_{\alpha} A^{\alpha} \).
Observe now that,
\[ \tilde{u}^1(\bar{x}) = \bar{u}^1(x + \Delta x) = \bar{u}^1(x) + \frac{\partial \bar{u}^1(x)}{\partial x^\alpha} \Delta x^\alpha. \]

Since \( \tilde{u}^1(x) = u^1(x) + \delta u^1 \), then
\[ \tilde{u}^1(\bar{x}) = u^1(x) + \delta u^1 + \partial_\alpha^1 u_1^\alpha \Delta x^\alpha + \frac{\partial \delta u^1}{\partial x^\alpha} \Delta x^\alpha. \]

Retaining only first order terms, as agreed,
\[ \tilde{u}^1(\bar{x}) = u^1(x) + \delta u^1 + \partial_\alpha^1 u_1^\alpha \Delta x^\alpha. \]

Then, \( \Delta u^1 = \delta u^1 + \partial_\alpha^1 u_1^\alpha \Delta x^\alpha \),

(1.1.7)

because \( \Delta u^1 = \tilde{u}^1(\bar{x}) - u^1(x) \).

Relationship (1.1.7) is fundamental in the following discussion.

Now we define the total variation for \( W[u] \), as given by (1.1.1), to be
\[ \Delta W = W[\bar{u}(\bar{x})] - W[u(x)]; \]

where \( \bar{x} \) and \( \bar{u}(\bar{x}) \) are prescribed by (1.1.3) and (1.1.5). We shall express \( \Delta W \) in terms of \( \Delta x^\alpha \) and \( \Delta u^1 \) by calculating,
\[ \Delta W = \int_{\bar{x}} \! dx \bar{x}(\bar{x}, \frac{\partial \bar{u}}{\partial x}) - \int_{x} \! dx \bar{x}(x, u, \frac{\partial u}{\partial x}). \]

(1.1.8)

Note first that
\[ d\bar{x} = \left| \frac{\partial \bar{x}^\alpha}{\partial x^\beta} \right| dx = \begin{vmatrix} \frac{\partial \bar{x}^0}{\partial x^0} & \frac{\partial \bar{x}^0}{\partial x^1} & \frac{\partial \bar{x}^0}{\partial x^2} & \frac{\partial \bar{x}^0}{\partial x^3} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\partial \bar{x}^3}{\partial x^0} & \cdots & \cdots & \frac{\partial \bar{x}^3}{\partial x^3} \end{vmatrix} dx \]

Since \( \bar{x}^\alpha = x^\alpha + \Delta x^\alpha \), then \( \frac{\partial \bar{x}^\alpha}{\partial x^\beta} = \delta_\beta^\alpha + \partial_\beta \Delta x^\alpha \).
Hence, 

\[
\left| \frac{\partial \mathbf{x}^\alpha}{\partial x^\beta} \right| = \begin{vmatrix}
1 + \frac{\partial \Delta x^0}{\partial x^0} & \frac{\partial \Delta x^0}{\partial x^1} & \frac{\partial \Delta x^0}{\partial x^2} & \frac{\partial \Delta x^0}{\partial x^3} \\
\vdots & \ddots & \ddots & \ddots \\
\frac{\partial \Delta x^3}{\partial x^0} & \cdots & \cdots & 1 + \frac{\partial \Delta x^3}{\partial x^3}
\end{vmatrix}
\]

Up to first order terms in \( \Delta x^\beta \) we have 

\[
\left| \frac{\partial \mathbf{x}^\alpha}{\partial x^\beta} \right| = 1 + \frac{\partial \Delta x^\alpha}{\partial x^\beta}.
\]

Hence, 

\[
\Delta x = (1 + \partial_\alpha \Delta x^\alpha)dx.
\] (1.1.9)

Now we must compute the total variation of the derivative \( \partial_\alpha u^1 \).

By definition,

\[
\Delta (\partial_\alpha u^1) = \frac{\partial \overline{u}^1(x)}{\partial x^\alpha} - \frac{\partial u^1(x)}{\partial x^\alpha}.
\] (1.1.10)

Then,

\[
\Delta (\partial_\alpha u^1) = \frac{\partial}{\partial x^\alpha}[\overline{u}^1(x) - u^1(x)] + \frac{\partial}{\partial x^\alpha}[u^1(x) - u^1(x)] + \left[\frac{\partial u^1(x)}{\partial x^\alpha} - \frac{\partial u^1(x)}{\partial x^\alpha}\right].
\]

Now we show the following:

i) \( \overline{u}^1(x) - u^1(x) = \Delta u^1 \) (to first order terms);

ii) \( \overline{\partial} (\Delta u^1) = \overline{\partial} (\Delta u^1) \)

iii) \( \overline{\partial} \frac{\partial u^1(x)}{\partial x^\alpha} - \partial \frac{\partial u^1(x)}{\partial x^\alpha} = -\frac{\partial u}{\partial x^\alpha} \cdot \frac{\partial \Delta x^\beta}{\partial x^\alpha} \)

From i), ii), and iii) we can conclude by substitution above

\[
\Delta (\partial_\alpha u^1) = \frac{\partial (\Delta u^1)}{\partial x^\alpha} + \frac{\partial}{\partial x^\alpha}(\partial_\beta \Delta u^1 \Delta x^\beta) - \frac{\partial \overline{u}^1}{\partial x^\beta} \frac{\partial \Delta x^\beta}{\partial x^\alpha}.
\]

\[
= \partial_\alpha (\Delta u^1) + \partial_\beta (\partial_\alpha u^1) \Delta x^\beta.
\]

Then since \( \delta (\partial_\alpha u^1) = \partial_\alpha (\Delta u^1) \), we obtain

\[
\Delta (\partial_\alpha u^1) = \delta (\partial_\alpha u^1) + \partial_\beta (\partial_\alpha u^1) \Delta x^\beta.
\] (1.1.11)
This is the result that we wish to establish.

Proof of i):

\[ u^1(\bar{x}) - u^1(\bar{x}) = \left[ u^1(x) + \frac{\partial u^1(x)}{\partial x^\beta} \Delta x^\beta \right] - \left[ u^1(x) + \frac{\partial u^1(x)}{\partial x^\beta} \Delta x^\beta \right] \]

\[ = \left[ u^1(x) - u^1(x) \right] + \frac{\partial}{\partial x^\beta} \left[ u^1(x) - u^1(x) \right] \Delta x^\beta \]

\[ = \delta u^1 + \gamma_\beta (\delta u^1) \Delta x^\beta. \]

Hence, to first order terms, \( u^1(\bar{x}) - u^1(\bar{x}) = \delta u^1 \).

Proof of ii):

\[ \frac{\partial (\delta u^1)}{\partial x^\alpha} = \frac{\partial (\delta u^1)}{\partial x^\beta} \frac{\partial x^\beta}{\partial x^\alpha} = \frac{\partial \delta u^1}{\partial x^\alpha} (\delta_\alpha + \gamma_\alpha \Delta x^\beta) \]

So,

\[ \frac{\partial (\delta u^1)}{\partial x^\alpha} = \frac{\partial (\delta u^1)}{\partial x^\beta} + \frac{\partial (\delta u^1)}{\partial x^\beta} \gamma_\alpha \Delta x^\beta. \]

Dropping the second term gives i).

Proof of iii):

\[ \frac{\partial u^1(\bar{x})}{\partial x^\alpha} = \frac{\partial u^1(\bar{x})}{\partial x^\beta} \frac{\partial x^\beta}{\partial x^\alpha} = \frac{\partial u^1(\bar{x})}{\partial x^\beta} \left[ \delta_\alpha + \gamma_\alpha \Delta x^\beta \right] \]

\[ = \frac{\partial u^1(\bar{x})}{\partial x^\alpha} + \frac{\partial u^1(\bar{x})}{\partial x^\beta} \Delta x^\beta. \]

Hence,

\[ \frac{\partial u^1(\bar{x})}{\partial x^\alpha} - \frac{\partial u^1(\bar{x})}{\partial x^\alpha} = -\frac{\partial u^1(\bar{x})}{\partial x^\beta} \Delta x^\beta. \quad (1.1.12) \]

So,

\[ \frac{\partial u^1(\bar{x})}{\partial x^\beta} = \frac{\partial u^1(\bar{x})}{\partial x^\beta} - \frac{\partial u^1(\bar{x})}{\partial x^\beta} \Delta x^\beta \]

follows from (1.1.12).

Substitution into the right side of (1.1.12) gives

\[ \frac{\partial u^1(\bar{x})}{\partial x^\alpha} - \frac{\partial u^1(\bar{x})}{\partial x^\alpha} = -\left[ \frac{\partial u^1(\bar{x})}{\partial x^\beta} - \frac{\partial u^1(\bar{x})}{\partial x^\beta} \right] \frac{\partial \Delta x^\beta}{\partial x^\alpha}. \]

So,

\[ \frac{\partial u^1(\bar{x})}{\partial x^\alpha} - \frac{\partial u^1(\bar{x})}{\partial x^\alpha} = -\frac{\partial u^1(\bar{x})}{\partial x^\beta} \Delta x^\beta \frac{\partial u^1(\bar{x})}{\partial x^\beta} \Delta x^\beta \]

\[ = -\gamma_\beta \left[ u^1(x) + \gamma_\alpha \Delta x^\lambda \right] \Delta x^\beta. \]

Therefore,

\[ \frac{\partial u^1(\bar{x})}{\partial x^\alpha} - \frac{\partial u^1(\bar{x})}{\partial x^\alpha} = -\frac{\partial u^1}{\partial x^\beta} \frac{\partial \Delta x^\beta}{\partial x^\alpha} \]

to first order terms.

Now i), ii), and iii) are proven, so the conclusion (1.1.11) is justified.
We may now proceed with the calculation of $\Delta W$ in (1.1.8). Because of (1.1.9),

$$\Delta W = \int \frac{1}{\partial} \left[(1 + \partial \Delta x^i) L(\bar{x}, \bar{u}, \frac{\partial \bar{u}}{\partial \bar{x}}) - L(x, u, \frac{\partial u}{\partial x})\right]. \quad (1.1.13)$$

Now, $L(\bar{x}, \bar{u}, \frac{\partial \bar{u}}{\partial \bar{x}}) = L(x, u, \frac{\partial u}{\partial x}) + \frac{\partial L}{\partial x^\lambda} \Delta x^\lambda + \frac{\partial L}{\partial u^i} \Delta u^i + \frac{\partial L}{\partial (\partial u^i)} \Delta (\partial u^i). \quad (1.1.14)$

Let

$$\Delta L = \frac{\partial L}{\partial x^\lambda} \Delta x^\lambda + \frac{\partial L}{\partial u^i} \Delta u^i + \frac{\partial L}{\partial (\partial u^i)} \Delta (\partial u^i). \quad (1.1.15)$$

Then,

$$\Delta W = \int \frac{1}{\partial} \left[(1 + \partial \Delta x^i) (L + \Delta L) - L\right].$$

Simplification gives

$$\Delta W = \int \frac{1}{\partial} \Delta x^i \left[(L + \Delta L) + L \partial \Delta x^i\right]. \quad (1.1.16)$$

From (1.1.15), (1.1.11), and (1.1.7),

$$\Delta L = \frac{\partial L}{\partial x^\lambda} \Delta x^\lambda + \frac{\partial L}{\partial u^i} (\Delta u^i + \partial _\rho u^i \Delta x^\rho) + \frac{\partial L}{\partial (\partial u^i)} \left[\delta (\partial u^i) + \partial _\lambda (\partial u^i) \Delta x^\lambda\right]. \quad (1.1.17)$$

Rearrangement of terms gives,

$$\Delta L = \left[\frac{\partial L}{\partial x^\lambda} + \frac{\partial L}{\partial u^i} \partial _\lambda u^i + \frac{\partial L}{\partial (\partial u^i)} \partial _\lambda (\partial u^i)\right] \Delta x^\lambda + \frac{\partial L}{\partial u^i} \Delta u^i + \frac{\partial L}{\partial (\partial u^i)} \delta (\partial u^i). \quad (1.1.17)$$

Within the first bracket, derivatives $\frac{\partial L}{\partial x^\lambda}$, $\frac{\partial L}{\partial u^i}$, and $\frac{\partial L}{\partial (\partial u^i)}$ are computed while treating $L$ as a function of independent variables $x^\lambda$, $u^i$, and $\partial _\lambda u^i$.

If we now take into account the fact that each $u^i$ is a function of variables $x^\lambda$, and each $\partial _\lambda u^i$ is a function of variables $x^\lambda$, then $L$ may be treated as a composition function of variables $x^\lambda$ only. At this point we shall agree to denote the composition derivative with respect to $x^\lambda$ by $\partial _\lambda L$. By use of the chain rule, we find $\partial _\lambda L$ and $\frac{\partial L}{\partial x^\lambda}$ are related by the equation

$$\partial _\lambda L = \frac{\partial L}{\partial x^\lambda} + \frac{\partial L}{\partial u^i} \partial _\lambda u^i + \frac{\partial L}{\partial (\partial u^i)} \partial _\lambda (\partial u^i). \quad (1.1.17)$$

This distinction between $\partial _\lambda L$ and $\frac{\partial L}{\partial x^\lambda}$ is essential in the following arguments. By use of this result, equation (1.1.17) becomes

$$\Delta L = \partial _\lambda L \Delta x^\lambda + \frac{\partial L}{\partial u^i} \Delta u^i + \frac{\partial L}{\partial (\partial u^i)} \delta (\partial u^i). \quad (1.1.17)$$
Then,
\[
\Delta L = \partial_\alpha L \Delta x^\alpha + \frac{\partial L}{\partial u^i} \delta u^i + \partial_\alpha \left( \frac{\partial L}{\partial (\partial u^i)} \delta u^i \right) - \partial_\alpha \left( \frac{\partial L}{\partial (\partial u^i)} \right) \delta u^i
\]
\[
= \partial_\alpha L \Delta x^\alpha + \partial_\alpha \left( \frac{\partial L}{\partial (\partial u^i)} \delta u^i \right) + \left[ \frac{\partial L}{\partial u^i} - \partial_\alpha \left( \frac{\partial L}{\partial (\partial u^i)} \right) \right] \delta u^i \quad (1.1.18)
\]
Let
\[
\frac{\delta L}{\delta u^i} = \frac{\partial L}{\partial u^i} - \partial_\alpha \left( \frac{\partial L}{\partial (\partial u^i)} \right) \quad (1.1.19)
\]
\(\delta L\) is called the variational derivative of \(L\).

Substitution of (1.1.18) into (1.1.16) gives
\[
\Delta W = \int d\alpha \left\{ \frac{\delta L}{\delta u^i} \delta u^i + \partial_\alpha \left( \frac{\partial L}{\partial (\partial u^i)} \delta u^i \right) + \partial_\alpha (L \Delta x^\alpha) \right\}.
\]

Therefore,
\[
\Delta W = \int d\alpha \frac{\delta L}{\delta u^i} \delta u^i + \int d\alpha \partial_\alpha \left\{ L \Delta x^\alpha + \frac{\partial L}{\partial (\partial u^i)} \delta u^i \right\} \quad (1.1.20)
\]
Equation (1.1.20) is the basic result needed for proof of the Noether theorems in the next section.

The following transformation of (1.1.20) is convenient. Since
\[
\delta u^i = \Delta u^i - \partial_\beta u^i \Delta x^\beta,
\]
then
\[
L \Delta x^\alpha + \frac{\partial L}{\partial (\partial u^i)} \delta u^i = L \Delta x^\alpha + \frac{\partial L}{\partial (\partial u^i)} \Delta u^i - \frac{\partial L}{\partial (\partial u^i)} \delta u^i \Delta x^\beta
\]
\[
= \partial_\beta \Delta x^\beta + \frac{\partial L}{\partial (\partial u^i)} \delta u^i,
\]
where
\[
\partial_\beta = L \delta_\beta - \frac{\partial L}{\partial (\partial u^i)} \partial_\beta u^i \quad (1.1.21)
\]
Hence, we obtain the alternate form
\[
\Delta W = \int d\alpha \frac{\delta L}{\delta u^i} (\Delta u^i - \partial_\beta u^i \Delta x^\beta) + \int d\alpha \left\{ \partial_\beta \Delta x^\beta + \frac{\partial L}{\partial (\partial u^i)} \Delta u^i \right\}. \quad (1.1.22)
\]
We now have \(\Delta W\) expressed directly in terms of the total variations \(\Delta x^\alpha\) and \(\Delta u^i\) as desired.
In equations (1.1.2) and (1.1.4) we specified 
\[ \Delta x^\alpha = \epsilon A^\alpha(x,u,\delta u) \quad \text{and} \quad \Delta u^i = \epsilon \omega^i(x,u,\delta u). \]
It should be clear, however, that no direct use of these assumptions was made in deriving (1.1.20) or (1.1.22). Therefore, these results still hold if we consider r-parameter transformations 
\[ \Delta x^\alpha = \epsilon A^\alpha(x,u,u) \quad \text{and} \quad \Delta u^i = \epsilon \omega^i_A(x,u,\delta u), \] (1.1.23)
depending on parameters \( \epsilon, \epsilon, \ldots, \epsilon \). In the following arguments the infinitesimal transformations will frequently have the form given in (1.1.23).

As a further generalization, suppose we have infinitesimal transformations of the form 
\[ \Delta x^\alpha = \Delta A^\alpha, \quad \Delta u^i = \Theta^i_A, \] (1.1.24)
where \( \Delta_A \) and \( \Theta^i_A \) denote linear operators, and functions \( \Delta \phi^A \) \( (A=1,2,\ldots,s) \) are arbitrary functions. Again, the results (1.1.20) and (1.1.22) hold true. We shall consider transformations of the form (1.1.24) in the following discussion.

2. The Noether Theorems

We shall assume an action integral 
\[ W = \int_{\Omega} \! dx L \]
is given as in the preceding section, and infinitesimal transformations 
\[ \bar{x}^\alpha = x^\alpha + \Delta x^\alpha \quad \text{and} \quad \bar{u}^i(x) = u^i(x) + \Delta u^i \]
are prescribed.

Indices \( A, B, C, \ldots \) have the range 1 to \( N \) in the following discussion.

Theorem 1. Suppose functions \( \xi^\alpha_A(x,u,\partial u/\partial x), \omega^i_A(x,u,\partial u/\partial x), \) and \( \gamma^\alpha_A(x,u,\partial u/\partial x) \)
exist such that for arbitrary parameters $\varepsilon^1, \ldots, \varepsilon^N$, we have
\[
\Delta W = \int_a^b dx \partial_\xi \Delta V^\alpha, \quad \text{for} \quad \Delta V^\alpha = \varepsilon^A \gamma^\alpha_A(x, u, du)
\]
when $\Delta x^\alpha = \varepsilon^A \xi^A_A(x, u, du)$ and $\Delta u^i = \varepsilon^A \omega^A_A(x, u, du)$. Then, $N$ relations exist of the form
\[
(\varepsilon^A \partial_\xi u^i - \omega^1_A) \frac{\delta L^\varepsilon}{\delta u^i} = \partial_\xi (T^\lambda_{\beta} \xi^\beta_A + \frac{\partial L}{\partial (\partial_\xi u^i)} \omega^1_A - \gamma^\alpha_A). \quad (1.2.1)
\]

**Proof.** Using the form of $\Delta x^\alpha, \Delta u^i$, and $\Delta W$ given, and equation (1.1.22),
\[
\int_a^b dx \partial_\xi (\varepsilon^A \gamma^\alpha_A) = \int_a^b dx (\varepsilon^A \omega^1_A - \partial_\xi u^i \varepsilon^A \xi^\beta_A) \frac{\delta L^\varepsilon}{\delta u^i} + \int_a^b dx \partial_\xi (T^\lambda_{\beta} \xi^\beta_A \varepsilon^A + \frac{\partial L}{\partial (\partial_\xi u^i)} \omega^1_A).
\]
Therefore,
\[
\int_a^b \left( \delta L \omega^1_A - \partial_\xi u^i \varepsilon^A \xi^\beta_A + \partial_\xi (T^\lambda_{\beta} \xi^\beta_A \varepsilon^A + \frac{\partial L}{\partial (\partial_\xi u^i)} \omega^1_A) - \partial_\xi \gamma^\alpha_A \right) \varepsilon^A = 0.
\]
Region $\Omega$ is arbitrary, and parameters $\varepsilon^A$ are arbitrary and independent, so we conclude
\[
(\omega^1_A - \xi^\beta_A \partial_\xi u^i) \frac{\delta L}{\delta u^i} + \partial_\xi (T^\lambda_{\beta} \xi^\beta_A + \frac{\partial L}{\partial (\partial_\xi u^i)} \omega^1_A - \gamma^\alpha_A) = 0.
\]
This is equivalent to (1.2.1).

When transformations $\vec{x}^\alpha = x^\alpha + \Delta x^\alpha$ and $\vec{u}^i(\vec{x}) = u^i(x) + \Delta u^i$ exist, such that $\Delta W$ is the integral of a divergence as in Theorem 1, we say $W$ is divergence invariant with respect to this class of transformations.

Theorem 1 then states that if $W$ is divergence invariant with respect to an $N$ parameter group of infinitesimal transformations, then $N$ linear combinations of the variational derivatives are divergences. The $N$-parameter group involved is sometimes called a symmetry group for the given action integral. Existence of an $N$-parameter symmetry group then implies existence of $N$ identities between variational derivatives and variables $u^i$.

**Theorem 2.** Suppose linear operators $\Lambda^\alpha_A, \Delta^\alpha_A$, and $\Theta^j_A$ exist, such that
\[
\Delta W = \int_a^b dx \partial_\alpha (\Lambda^\alpha_A \Delta^\alpha_A \Theta^j_A)
\]
for arbitrary functions $\Delta \phi^A$ which vanish on the boundary of $\Omega$, when

$$\Delta x^\alpha = \Delta_A^{\alpha} [\Delta \phi^A] \quad \text{and} \quad \Delta u^i = \partial_A^i [\Delta \phi^A].$$

Then, $N$ relations exist of the form

$$\tilde{\partial}_A^i \left[ \frac{\delta L}{\delta u^i} \right] - \Delta_A^{\alpha} \left[ \partial_{\alpha} u^k \frac{\delta L}{\delta u^k} \right] = 0 \quad (1.2.2)$$

Note: $\tilde{\partial}_A^i$ denotes the adjoint of operator $\partial_A^i$, and is defined by

$$\int_\Omega dx \partial_A^i [\phi] \psi = \int_\Omega dx \tilde{\partial}_A^i [\psi] \phi .$$

Similarly, $\tilde{\Delta}_A^{\alpha}$ is the adjoint of operator $\Delta_A^{\alpha}$.

Proof. Substitute $\Delta W, \Delta x^\alpha, \Delta u^i$ as above in (1.1.22). Then,

$$\int_\Omega dx \partial_A^i [\Delta A^{\alpha} [\Delta \phi^A]] = \int_\Omega dx \frac{\delta L}{\delta u^i} \left( \partial_A^i [\Delta \phi^A] - \partial_{\alpha} u^i \Delta_A^{\beta} [\Delta \phi^A] \right)$$

$$+ \int_\Omega dx \partial_{\lambda} \left( \partial_{\beta} \Delta_A^{\alpha} [\Delta \phi^A] + \frac{\partial L}{\partial (\partial_{\alpha} u^i)} \partial_A^i [\Delta \phi^A] \right) . \quad (1.2.3)$$

Now, each integral of a divergence expression over $\Omega$ is equivalent to an integral over the boundary of an expression which is identically zero on the boundary, because functions $\Delta \phi^A$ vanish on the boundary. Furthermore, operators $\Delta_A^{\alpha}, \partial_A^i, \Delta_A^{\alpha}$ are linear. Hence, each such integral in (1.2.3) must vanish. Then,

$$\int_\Omega dx \frac{\delta L}{\delta u^i} \partial_A^i [\Delta \phi^A] - \int_\Omega dx \frac{\delta L}{\delta u^i} \partial_{\alpha} u^i \Delta_A^{\beta} [\Delta \phi^A] = 0 .$$

So,

$$\int_\Omega dx \left\{ \partial_A^i \left[ \frac{\delta L}{\delta u^i} \right] - \tilde{\Delta}_A^{\alpha} \left[ \frac{\delta L}{\delta u^i} \partial_{\alpha} u^i \right] \right\} \Delta \phi^A = 0 .$$

Since the $\Delta \phi^A$ are arbitrary, we conclude that

$$\tilde{\partial}_A^i \left[ \frac{\delta L}{\delta u^i} \right] - \tilde{\Delta}_A^{\alpha} \left[ \frac{\delta L}{\delta u^i} \partial_{\alpha} u^i \right] = 0$$

as asserted.

3. Physical Significance of the Noether Theorems

If $\omega = \int_\Omega dx L(x, u, \frac{\partial u}{\partial x})$ is the action integral for a physical field with field variables $u^i$, then the field equations are

$$\frac{\delta L}{\delta u^i} = 0 . \quad (1.3.1)$$
If transformations exist as required in Theorem 1, then (1.2.1) states that when the field equations are satisfied we have $N$ identities, 

$$\partial_\alpha(T^\alpha_\beta \xi^\beta_A + \frac{\partial L}{\partial (\partial_\alpha u_A^i)} \omega^i_A - \gamma^i_A) = 0.$$ 

These identities have the form $\partial_\alpha P_A^\alpha = 0$ for vectors 

$$P_A^\alpha = T^\alpha_\beta \xi^\beta_A + \frac{\partial L}{\partial (\partial_\alpha u_A^i)} \omega^i_A - \gamma^i_A.$$ 

Physicists normally interpret statements of the form 

$$\partial_\alpha P_A^\alpha = 0$$

as conservation laws. If the coordinate system $x^0 = t$, $x^1 = x$, $x^2 = y$, and $x^3 = z$ is employed, then $P^0$ is interpreted as the spatial density of some quantity, and $\vec{P} = (P^1, P^2, P^3)$ is interpreted as the vector flux of that quantity in space. The reasoning is as follows. Equation (1.3.2) is explicitly

$$\frac{\partial P^0}{\partial t} + \frac{\partial P^1}{\partial x} + \frac{\partial P^2}{\partial y} + \frac{\partial P^3}{\partial z} = 0$$

in this case. Briefly, we have

$$\frac{\partial P^0}{\partial t} + \text{div} \vec{P} = 0.$$  

(1.3.3)

Integration of (1.3.3) over a 3 dimensional region $V$ in $R^3$ gives

$$\frac{\partial}{\partial t} \iiint_V P^0 \, dx \, dy \, dz + \iiint_V (\text{div} \vec{P}) \, dx \, dy \, dz = 0.$$ 

Thus,

$$\frac{\partial}{\partial t} \iiint_V P^0 \, dx \, dy \, dz + \iint_{\partial(V)} (\vec{P} \cdot \vec{n}) \, dS = 0.$$  

(1.3.4)

where $B(V)$ denotes the boundary of $V$. The second integral represents a flux through the boundary of $V$ and (1.3.4) states that this flux is the negative of the time rate of change of the quantity

$$\iiint_V P^0 \, dx \, dy \, dz.$$ 

If we extend $V$ to all of $R^3$ (assuming existence of the improper integrals) and assume $|\vec{P}|$ tends to zero "at infinity", then (1.3.4) becomes

$$\frac{\partial}{\partial t} \iiint_{R^3} P^0 \, dx \, dy \, dz = 0.$$
So the quantity

\[ \iiint p^0 \, dx \, dy \, dz \]

with spatial density \( p^0 \) is a constant in time; i.e., a conserved quantity.

Bessel-Hagen applied Noether Theorem 1 to the n-body problem to produce the ten well known first integrals from invariance with respect to the ten-parameter Galilean group of space rotations and translations, time translations, and transformations, of the form

\[ \dot{x}^k = x^k + \gamma^k t \quad ; \quad (k = 1, 2, 3). \]

In the same paper Bessel-Hagen applied the first Noether theorem to electromagnetic fields in vacuum to produce the identities which arise from invariance under the 15-parameter conformal group. This derivation is, however, not so simple as the previous one and presents several questions which should be answered. A review of this application will be given now with some modernization and reorganization of the arguments.

4. Application of Noether's Theorem 1

To The Electromagnetic Field

Indices \( i, j, k, \ldots \) in the following shall have the range 1, 2, 3, while Greek indices \( \alpha, \beta, \gamma, \ldots \) have the range 0, 1, 2, 3.

\[ \varepsilon_{k mn} = \varepsilon^{k mn} \]

is the usual permutation symbol which has the value +1 when \( kmn \) is an even permutation of 123, or the value -1 when \( kmn \) is an odd permutation. In all other cases the value of \( \varepsilon_{k mn} \) is zero.

A four-dimensional space is assumed, with points \( x = (x^0, x^1, x^2, x^3) = (it, x, y, z) \). A metric tensor \( g_{\alpha \beta} \) is assumed, with

\[ g_{\alpha \beta} = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ 1 & \text{if } \alpha = \beta \end{cases} \]
The electromagnetic field in vacuum consists of a pair of vector fields, \( \vec{E} \) and \( \vec{H} \), with field equations:

Ia). \( \text{div} \ \vec{E} = 0 \)  \hspace{1cm} Ib). \( \text{curl} \ \vec{H} = \frac{\partial \vec{E}}{\partial t} \) (Maxwell equations)

IIa). \( \text{div} \ \vec{H} = 0 \)  \hspace{1cm} IIb). \( \text{curl} \ \vec{E} = -\frac{\partial \vec{H}}{\partial t} \)

A four dimensional formulation of these equations can be produced as follows. Introduce a four-potential \( A_\alpha \) and define tensor

\[
F_{\alpha \beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha .
\]  \hspace{1cm} (1.4.1)

The vectors \( \vec{E} = (E_1, E_2, E_3) \) and \( \vec{H} = (H_1, H_2, H_3) \) are defined by

\[
E^k = E_k = F_{k0}, \quad \text{and} \quad H_k = H^k = \frac{1}{2} \varepsilon_{kmm} F^{mn} .
\]  \hspace{1cm} (1.4.2)

Then, equations Ia) and Ib) are equivalent to tensor equation

\[
\text{I.} \quad \partial_\alpha F^{\alpha \beta} = 0 .
\]

Equations IIa) and IIb) are equivalent to

\[
\text{II.} \quad \partial_\alpha F_{\beta \lambda} + \partial_\beta F_{\lambda \alpha} + \partial_\lambda F_{\alpha \beta} = 0 .
\]

Equation II is, however, an identity resulting from equation (1.4.1) which defines \( F_{\alpha \beta} \).

In this formulation potential \( A_\alpha \) represents the field. The field equations for \( A_\alpha \) are given by I which states

\[
\partial_\alpha (\partial_\alpha A_\beta) - \partial_\beta (\partial_\alpha A_\alpha) = 0 .
\]  \hspace{1cm} (1.4.3)

Tensor \( F_{\alpha \beta} \) is introduced as a convenient device for stating the field equation in the compact form, I, which is easily seen to be co-variant.

\( F_{\alpha \beta} \) also provides simple definitions (1.4.2) for \( \vec{E} \) and \( \vec{H} \) in this model.

Consider the integral

\[
W = \int L \left( -\frac{\partial F_{\mu \nu}}{\partial x^\nu} F^{\mu \nu} \right) = \int L(x, A_\alpha, \partial_\beta A_\alpha) .
\]

From (1.1.19) it is seen

\[
\frac{\delta L}{\delta A_\beta} = \partial_\alpha F^{\alpha \beta} ,
\]

so the integral given can be called the action integral for a field.
with field equations I. Hence, we consider \( W = -\frac{1}{2} \int \mathcal{L} \) to be an action integral for such fields.

It should be noted at this point that the action integral given here does not actually produce all of Maxwell's Equations (I and II). Equations II are produced by the auxiliary assumption (14.1). This is a defect of the proposed variational principle when application of the Noether theorems is considered.

Bateman\((2)\) showed in 1909 that the Maxwell Equations are invariant with respect to the transformations which transform the equation

\[
dx^2 + dy^2 + dz^2 - dt^2 = 0
\]

into

\[
d\tilde{x}_1^2 + d\tilde{y}_2^2 + d\tilde{z}_3^2 - d\tilde{t}_4^2 = 0.
\]

This group of transformations forms the conformal group. Bessel-Hagen gave a direct proof that the integral \( \int_{\mathcal{L}} dx (-\frac{1}{2} F_{\alpha \beta} F^{\alpha \beta}) \) is invariant with respect to the conformal group. Later in this paper a general definition (see 4.1.3) of conformal transformations will be given with a new proof of conformal invariance of the above action integral, so the Bessel-Hagen argument need not be reviewed here. It is well known \((5)\) that the conformal group can be represented in infinitesimal form as a 15 parameter group generated by the following transformations:

\[
\begin{align*}
\tilde{x}^\alpha &= x^\alpha + \tau^\alpha; \\
\tilde{x}^\alpha &= x^\alpha + g^{\alpha \lambda} \omega_{\lambda \beta} x^\beta, \quad \omega_{\lambda \beta} = -\omega_{\beta \lambda}; \\
\tilde{x}^\alpha &= x^\alpha + \gamma x^\alpha; \\
\tilde{x}^\alpha &= x^\alpha + (2x^\rho x^\alpha - x_\rho x^\beta g^{\alpha \beta}) \eta_\rho.
\end{align*}
\]

The 15 quantities \( \tau^\alpha, \omega_{\lambda \beta}, \gamma, \eta_\rho \) are independent parameters.

We take the above facts as background for application of Noether Theorem I.
In the following we let
\[ \epsilon^1 = \tau^0, \quad \epsilon^2 = \tau^1, \quad \epsilon^3 = \tau^2, \quad \epsilon^4 = \tau^3, \]
\[ \epsilon^5 = \omega_{01}, \quad \epsilon^6 = \omega_{02}, \quad \epsilon^7 = \omega_{03}, \quad \epsilon^8 = \omega_{12}, \quad \epsilon^9 = \omega_{13}, \quad \epsilon^{10} = \omega_{23}, \]
\[ \epsilon^{11} = \gamma, \quad \epsilon^{12} = \eta_0, \quad \epsilon^{13} = \eta_1, \quad \epsilon^{14} = \eta_2, \quad \text{and} \quad \epsilon^{15} = \eta_3. \]

Then, the previous group of transformations has the form
\[ \bar{x}^\alpha = x^\alpha + \zeta^\alpha_A \epsilon^A \quad (A=1,2,\ldots,15), \]
where the vectors \( \zeta^\alpha_A \) are as follows:
\[ \zeta^1 = \zeta^\alpha_1, \quad \zeta^2 = \zeta^\alpha_2 = \delta^\alpha_1, \quad \zeta^3 = \delta^\alpha_2, \quad \zeta^4 = \delta^\alpha_3; \]
\[ \zeta^5 = x^g \alpha^\alpha - x^g \alpha^\alpha, \quad \zeta^6 = x^g \alpha^\alpha - x^g \alpha^\alpha, \]
\[ \zeta^7 = x^g \alpha^\alpha - x^g \alpha^\alpha, \quad \zeta^8 = x^g \alpha^\alpha - x^g \alpha^\alpha, \]
\[ \zeta^9 = x^g \alpha^\alpha - x^g \alpha^\alpha, \quad \zeta^{10} = x^g \alpha^\alpha - x^g \alpha^\alpha; \]
\[ \zeta^{11} = x^\alpha; \]
\[ \zeta^{12} = 2x^\alpha - g^\alpha \chi^\alpha \chi^\alpha, \quad \zeta^{13} = 2x^\alpha - g^\alpha \chi^\alpha \chi^\alpha, \]
\[ \zeta^{14} = 2x^\alpha - g^\alpha \chi^\alpha \chi^\alpha, \quad \zeta^{15} = 2x^\alpha - g^\alpha \chi^\alpha \chi^\alpha. \]

Given transformation \( \bar{x}^\alpha = x^\alpha + \zeta^\alpha_A \epsilon^A \) it is assumed that vector \( A_\alpha(x) \) is transformed by
\[ \bar{A}_\beta(\bar{x}) = A_\alpha(x) \frac{\partial x^\alpha}{\partial \bar{x}^\beta}; \]
i.e., vector field \( A_\alpha \) is "dragged along" by the transformation of \( x^\alpha \).

This assumption is necessary in order to obtain the conformal invariance claimed earlier. In infinitesimal form (1.4.9) becomes
\[ \Delta A_\beta = \bar{A}_\beta(\bar{x}) - A_\beta(x) \]
\[ = A_\alpha(x) \left[ \frac{\partial x^\alpha}{\partial \bar{x}^\beta} \right] - A_\beta(x) \]
\[ = A_\alpha(x) \left[ \frac{\partial (\Delta x^\alpha)}{\partial \bar{x}^\beta} \right] - A_\beta(x) \]
\[ = - A_\alpha(x) \frac{\partial \Delta x^\alpha}{\partial \bar{x}^\beta}. \]
Also,
\[ \frac{\partial \Delta x^\alpha}{\partial x^\beta} = \frac{\partial \Delta x^\alpha}{\partial x^\rho} \frac{\partial x^\rho}{\partial x^\beta} \]
\[ = \frac{\partial \Delta x^\alpha}{\partial x^\beta} \left[ \delta^\rho_\beta + \partial_\beta \Delta x^\rho \right] \]
\[ = \frac{\partial \Delta x^\alpha}{\partial x^\beta} + \frac{\partial \Delta x^\rho}{\partial x^\beta} \frac{\partial \Delta x^\rho}{\partial x^\beta} . \]

Hence, to first order terms \( \frac{\partial \Delta x^\alpha}{\partial x^\beta} = \frac{\partial \Delta x^\alpha}{\partial x^\beta} = \partial_\beta \Delta x^\alpha \).

So,
\[ \Delta A^\beta = - A_\alpha (x) \partial_\beta \Delta x^\alpha . \] (1.4.10)

Substitution \( \Delta x^\alpha = \zeta^\alpha_A \epsilon^A \) into (1.4.10) gives \( \Delta A^\beta = - (A_\alpha \partial_\beta \zeta^\alpha_A) \epsilon^A . \)

Defining \( \omega_{\beta A} \) by \( \Delta A^\beta = \omega_{\beta A} \epsilon^A \) we see
\[ \omega_{\beta A} = - A_\alpha \partial_\beta \zeta^\alpha_A . \] (1.4.11)

Since \( \Delta W = 0 \) with transformations of \( x^\alpha \) and \( A^\beta \) defined by (1.4.4) and (1.4.9), we have \( \gamma^\alpha_A = 0 \) in the hypotheses of Noether Theorem 1. So application of the theorem leads to 15 identities (see 1.2.1) of the form
\[ \frac{\delta L}{\delta A^\beta} \left[ \partial_\alpha A^\beta \zeta^\alpha_A + A_\alpha \partial_\beta \zeta^\alpha_A \right] = \partial_\alpha \left( T_{\gamma \delta}^\alpha \zeta^\beta_A - \frac{\partial L}{\partial (\partial_\alpha A^\beta)} A_\lambda \partial_\delta \zeta^\lambda_A \right) . \] (1.4.12)

with vectors \( \zeta^\alpha_A \) given by (1.4.5) to (1.4.8). Since
\[ L = -\frac{1}{4} \sum_{\mu, \nu} \left( \partial_\mu A_\nu - \partial_\nu A_\mu \right)^2 , \]
then,
\[ \frac{\delta L}{\delta A^\beta} \left[ \partial_\alpha A^\beta \zeta^\alpha_A + A_\alpha \partial_\beta \zeta^\alpha_A \right] = \partial_\alpha \left( T_{\mu \nu}^{\alpha \beta} \zeta^\mu_A + F^{\alpha \beta} A_\lambda \partial_\nu \zeta^\lambda_A \right) . \]

Now, assuming field equations \( \frac{\delta L}{\delta A^\beta} = 0 \) hold we have 15 conservation laws:
\[ \partial_\alpha \left( T_{\gamma \delta}^\alpha \zeta^\beta_A + F^{\alpha \beta} A_\lambda \partial_\delta \zeta^\lambda_A \right) = 0 \] . (1.4.13)

If we should now substitute the forms for \( \zeta^\lambda_A \) as given earlier for the conformal group, the results would be rather complicated. The resulting identities would also involve the components \( A_\alpha \) of the four potential. This is objectionable on physical grounds because the four potential for a given electromagnetic field is not unique, and in fact is introduced as a mathematically convenient device with no direct physical interpretation. Bessel-Hagen modified the transformation (1.4.9)
by adding a gauge transformation and thereby arrived at simpler forms for the conservation laws in (1.4.13). It is of some interest to observe that the additional shift of the potential is not needed. The derivation of conservation laws is shortened somewhat by another approach which will be given here. We can show in this way that the resulting conservation laws are a consequence of conformal invariance alone, and do not in any way depend upon gauge invariance of the action integral, as might be supposed from the Bessel-Hagen argument.

Observe that in the definition (1.1.21) we have

$$ T^{\alpha}_{\beta} = L \delta^{\alpha}_{\beta} - \frac{\partial L}{\partial (\nabla u)} \gamma_{\beta} u \, . $$

We have in the present case,

$$ T^{\alpha}_{\beta} = L \delta^{\alpha}_{\beta} - \frac{\partial L}{\partial (\nabla A)} \partial_{\beta} A_{\lambda} = L \delta^{\alpha}_{\beta} + F^{\alpha \lambda} \partial_{\beta} A_{\lambda} \, . $$

Hence, (1.4.13) is

$$ \partial_{\alpha} [L \delta^{\alpha}_{\beta} + F^{\alpha \lambda} \partial_{\beta} A_{\lambda}] \xi^{\beta}_{A} + F^{\alpha \beta} \partial_{\beta} A_{\lambda} \xi^{\lambda}_{A} = 0 \, . $$

Since

$$ F^{\alpha \beta} A_{\lambda} \partial_{\beta} \xi^{\lambda}_{A} = F^{\alpha \beta} \left[ \partial_{\beta} (A_{\lambda} \xi^{\lambda}_{A}) - \partial_{\beta} A_{\lambda} \xi^{\lambda}_{A} \right] \, , $$

then

$$ \partial_{\alpha} [L \delta^{\alpha}_{\beta} + F^{\alpha \lambda} \partial_{\beta} A_{\lambda}] \xi^{\beta}_{A} + F^{\alpha \beta} \partial_{\beta} (A_{\lambda} \xi^{\lambda}_{A}) + F^{\alpha \beta} \partial_{\beta} A_{\lambda} \xi^{\lambda}_{A} = 0 \, . $$

Interchange summation indices $\beta, \lambda$ in the last term and rearrange terms.

Then,

$$ \partial_{\alpha} [L \delta^{\alpha}_{\beta} + F^{\alpha \lambda} (\partial_{\beta} A_{\lambda} - \partial_{\lambda} A_{\beta}) + F^{\alpha \beta} \partial_{\beta} (A_{\lambda} \xi^{\lambda}_{A})] = 0 \, . $$

Define

$$ S^{\alpha \beta}_{\gamma} = L \delta^{\alpha}_{\beta} + F^{\alpha \lambda} F_{\beta \lambda} \, . $$

Then,

$$ \partial_{\alpha} (S^{\alpha \beta}_{\gamma} \xi^{\beta}_{A}) + \partial_{\alpha} (F^{\alpha \beta} \partial_{\beta} (A_{\lambda} \xi^{\lambda}_{A})) = 0 \, . $$

We have assumed the field equations $\partial_{\alpha} F^{\alpha \beta} = 0$ hold, so in the second term we have $F^{\alpha \beta} \partial_{\beta} (A_{\lambda} \xi^{\lambda}_{A})$ which vanishes since $F^{\alpha \beta} = -F^{\beta \alpha}$.

In this way we arrive at the simplified form,

$$ \partial_{\alpha} (S^{\alpha \beta}_{\gamma} \xi^{\beta}_{A}) = 0 \, . $$

(1.4.16)
This is essentially the same result as obtained by Bessel-Hagen, but it is obtained by merely simplifying and rearranging terms of (1.4.13). It is of some interest to note how tensor $S_{\beta\alpha}$ arises as a natural consequence of the mathematical arguments. This tensor was introduced by Bessel-Hagen as the energy-momentum tensor for the field. We shall later show that this is in fact the physical significance of $S_{\beta\alpha}$, but we do not need to assume so at the present time.

Now, we substitute expressions (1.4.5), (1.4.6), (1.4.7), and (1.4.8) into (1.4.16) to obtain the explicit form of the conservation laws. A summary of the results is:

A = 1, 2, 3, 4 gives four identities, $\partial_\beta S_{\beta\alpha} = 0$, $(\alpha = 0, 1, 2, 3)$;

(A = 5) $\partial_\beta [S_{\beta\alpha}(x^6 g^{0\alpha} - x^6 g^{1\alpha})] = \partial_\beta (x^6 S_{\beta}^0 - x^6 S_{\beta}^1) = 0$;

(A = 6) $\partial_\beta [S_{\beta\alpha}(x^2 g^{0\alpha} - x^2 g^{2\alpha})] = \partial_\beta (x^2 S_{\beta}^0 - x^2 S_{\beta}^2) = 0$;

(A = 7) $\partial_\beta [S_{\beta\alpha}(x^3 g^{0\alpha} - x^3 g^{3\alpha})] = \partial_\beta (x^3 S_{\beta}^0 - x^3 S_{\beta}^3) = 0$;

(A = 8) $\partial_\beta [S_{\beta\alpha}(x^4 g^{1\alpha} - x^4 g^{2\alpha})] = \partial_\beta (x^4 S_{\beta}^1 - x^4 S_{\beta}^2) = 0$;

(A = 9) $\partial_\beta [S_{\beta\alpha}(x^4 g^{1\alpha} - x^4 g^{3\alpha})] = \partial_\beta (x^4 S_{\beta}^1 - x^4 S_{\beta}^3) = 0$;

(A = 10) $\partial_\beta [S_{\beta\alpha}(x^3 g^{2\alpha} - x^2 g^{3\alpha})] = \partial_\beta (x^3 S_{\beta}^2 - x^2 S_{\beta}^3) = 0$;

(A = 11) $\partial_\beta (x^\alpha S_{\beta\alpha}) = 0$;

(A = 12) $\partial_\beta [S_{\beta\alpha}(2x^\alpha x^\nu - g^{\alpha\nu} x_{\nu} x_{\nu})] = \partial_\beta [2x^\alpha x_{\beta\alpha} - (x_{\nu} x^\nu)S_{\beta\nu}] = 0$;

(A = 13) $\partial_\beta [S_{\beta\alpha}(x^1 x^\nu - g^{\alpha1} x_{\nu} x_{\nu})] = \partial_\beta [2x^1 x_{\beta\alpha} - (x_{\nu} x^\nu)S_{\beta1}] = 0$;

(A = 14) $\partial_\beta [S_{\beta\alpha}(2x^2 x^\nu - g^{\alpha2} x_{\nu} x_{\nu})] = \partial_\beta [2x^2 x_{\beta\alpha} - (x_{\nu} x^\nu)S_{\beta2}] = 0$;

(A = 15) $\partial_\beta [S_{\beta\alpha}(2x^3 x^\nu - g^{\alpha3} x_{\nu} x_{\nu})] = \partial_\beta [2x^3 x_{\beta\alpha} - (x_{\nu} x^\nu)S_{\beta3}] = 0$.

This listing is conveniently condensed as follows:

\begin{align*}
\partial_\beta S_{\beta\alpha} = 0, & \quad (\alpha = 0, 1, 2, 3); \quad (C1) \\
\partial_\beta (x^\alpha S_{\beta\alpha} - x^\alpha S_{\beta\alpha}) = 0, & \quad (\alpha, \lambda) \in \{ (1, 0), (2, 0), (3, 0), (2, 1), (3, 1), (3, 2) \}; \quad (C2) \\
\partial_\beta (x^\alpha S_{\beta\alpha}) = 0; & \quad (C3) \\
\partial_\beta (2x^\lambda x^\mu S_{\beta\alpha} - (x_{\nu} x^\nu)S_{\beta\alpha}) = 0, & \quad (\lambda = 0, 1, 2, 3). \quad (C4)
\end{align*}
Tensor $S^{\alpha\beta}$ can be shown to be the tensor defined by physicists to be the energy-momentum tensor for electromagnetic field $F_{\mu\nu}$. Then, (C1) represents conservation of energy and momentum. Equation (C2) represents conservation of angular momentum, and an analogue of the center of mass theorem in the n-body problem. Details for these physical interpretations are given by Bessel-Hagen.

The principal aim now is to point out that the statements (C1) and (C2) can be interpreted as conservation laws for some physical quantities. However, no physical interpretation is known for statements (C3) and (C4). Hence, from the viewpoint of the physicists, the application of the first Noether Theorem to electromagnetism has produced too many conservation laws.

5. Dependence of the Conservation Laws

A set of $r$ conservation laws, $\partial_\alpha P^\alpha_k = 0 \ (k = 1,\ldots,r)$, is said to be independent if vectors $P_1^\alpha,\ldots,P_r^\alpha$ are linearly independent. The 15 conservation laws derived in section 4 are independent in this sense. However, these conservation laws are related mathematically in another way, which may be responsible for the failure to find a physical interpretation for (C4). It can be shown that if we assume (C1),(C2), and (C3) to be true, then (C4) is a mathematical consequence of these assumptions. So (C4) has no new mathematical content. Many "conservation laws" can be produced from known conservation laws such as (C1) and (C2) without any new physical content. It can also be shown that the truth of (C1) does not imply the truth of (C2) or (C3). Statement (C3) cannot be derived from (C1) and (C2) alone. Proof of these claims will now be given.

First it should be noted that (C1) is in fact a mathematical
consequence of both definition (1.4.15) for $S_{\rho\sigma}$ and the Maxwell equations I and II. This is a straightforward calculation.

Carrying out the differentiations in (C2) we find,

$$\partial_\beta (x^\alpha S^\beta_\alpha - x^\lambda S^\beta_\lambda) = \delta_\beta^\alpha S^{\beta_\alpha} + x_\beta^\alpha S^\beta_\alpha - \delta_\beta^\lambda S^{\beta_\lambda} - x_\beta^\lambda S^\beta_\lambda = x_\beta^\alpha S^{\beta_\alpha} - x_\beta^\lambda S^\beta_\lambda + (S^{\alpha_\lambda} - S^{\lambda_\alpha}).$$

(1.5.1)

Then, (C2) follows from two facts, (Cl) and $S^{\alpha_\lambda} = S^{\lambda_\alpha}$.

It has not yet been shown that $S^{\alpha_\lambda} = S^{\lambda_\alpha}$, so we must prove this now. We have

$$S^{\beta_\lambda} = g^{\alpha_\lambda} S^{\beta_\alpha} = Lg^{\beta_\lambda} + g^{\alpha_\lambda} F^{\beta_\rho} F_{\alpha\rho}.$$ 

Thus,

$$S^{\beta_\lambda} = Lg^{\beta_\lambda} + g^{\alpha_\lambda} g_\alpha \gamma F^{\beta_\rho} F^{\tau_\sigma} = Lg^{\beta_\lambda} + S^{\lambda_\alpha} g_\alpha \gamma F^{\beta_\rho} F^{\tau_\sigma} = Lg^{\beta_\lambda} + g_\rho \sigma F^{\beta_\rho} F^{\lambda_\sigma}.$$ 

Then,

$$S^{\lambda_\beta} = Lg^{\lambda_\beta} + g_\rho \sigma F^{\lambda_\rho} F^{\beta_\sigma} = Lg^{\lambda_\beta} + g_\rho \sigma F^{\lambda_\rho} F^{\beta_\sigma}.$$ 

Since $g_{\rho\sigma} = g_{\sigma\rho}$, then $S^{\lambda_\beta} = S^{\beta_\lambda}$.

We have now shown (C2) follows from (Cl) and $S^{\lambda_\beta} = S^{\beta_\lambda}$.

In the case of (C3) we find,

$$\partial_\beta (x^\alpha S^\beta_\alpha) = x^\alpha \partial_\beta S^\beta_\alpha + \delta_\beta^\alpha S^\beta_\alpha = x^\alpha \partial_\beta S^\beta_\alpha + S^\alpha_\alpha.$$ 

But, $S^\alpha_\alpha = L \delta^\alpha_\alpha + F^{\alpha_\beta} F_{\alpha\beta} = (-L F^{\alpha_\beta} F_{\alpha\beta}) + F^{\alpha_\beta} F_{\alpha\beta} = 0$.

Then, (C3) is a consequence of (Cl) and $S^\alpha_\alpha = 0$.

Observe that (C2) and (C3) follow from (Cl), but in each case an additional fact about tensor $S^{\alpha_\beta}$ is required.

We shall now show that (C4) follows from (Cl), (C2), and (C3) with no other facts needed. Carrying out the differentiations in (C4) gives
\( \partial_{\beta}(2x^\lambda x^\gamma x^\beta_{\gamma} - x_{\gamma} x_{\gamma} x_{\beta}) = \\
= 2x^\lambda \partial_{\beta}(x^\gamma x^\beta_{\gamma}) + 2 \delta^\lambda_{\beta}(x^\gamma x^\gamma) - (x_{\gamma} x_{\gamma}) \partial_{\beta} S^{\beta \lambda} - S^{\beta \lambda} (x_{\gamma} x_{\gamma}) \\
= 2x^\lambda \partial_{\beta}(x^\gamma x^\beta_{\gamma}) + 2(x^\gamma x^\gamma) - (x_{\gamma} x_{\gamma}) \partial_{\beta} S^{\beta \lambda} - 2x_{\gamma} S^{\beta \lambda}.
\)

Since \( x^\gamma x^{\alpha}_{\gamma} = x_{\alpha} S^{\lambda}_{\alpha} = x_{\alpha} S^{\lambda}_{\alpha} \), then we have above
\[
= 2x^\lambda \partial_{\beta}(x^\gamma x^\beta_{\gamma}) - (x_{\gamma} x_{\gamma}) \partial_{\beta} S^{\beta \lambda} + 2x_{\gamma} (S^{\lambda}_{\beta} - S^{\beta \lambda}). \tag{1.5.2}
\]

If we assume (C1) and (C3) then the first two terms vanish. From (1.5.1) we see that the third term in (1.5.2) must vanish when (C1) and (C2) hold. Hence, as claimed (C4) is a purely mathematical consequence of (C1), (C2), and (C3).

The argument given here shows that (C1), (C2), and (C3) are logically independent in the sense that the truth of one or two of these will not be sufficient to guarantee any of the remaining ones are true. Contrary to that, the equation (C4) can be considered as logically superfluous, since (C4) can be deduced from (C1), (C2), and (C3).

Any search for a physical interpretation for (C4) seems pointless. (C3) may yet have a physical interpretation, however, so we now have eleven conservation laws remaining.

In the following chapters we shall proceed to generalize the results in this chapter to determine the role of our choice of coordinate system, choice of metric, and the significance of conformal invariance for electromagnetic fields. Furthermore, the variational principle used in this chapter is not satisfactory for application of the Noether theorems so an alternative approach will be given.
CHAPTER II

COVARIANT FORM OF THE NOETHER THEOREMS

In this chapter we begin the task of converting earlier results to covariant form, as the first stage toward generalizing the preceding stated results. The extent to which such generalization is possible will hopefully shed some light on the mathematical foundations leading to conformal invariance of the electromagnetic field and the consequent conservation laws. Since it is intended to obtain these conservation laws again by use of the Noether theorem, we must first revise this theorem so that it may be applied in a more general space.

1. Tensor Analysis on Manifolds

In the following discussion \( x = (x^0, x^1, x^2, x^3) \) shall denote a point in a four-dimensional manifold \( X \) with affine connection. Greek indices \( \alpha, \beta, \gamma, \ldots \) shall have the range 0,1,2,3. At each point \( x \in X \), we have a four dimensional linear space \( \mathcal{A}_4(x) \) with basis \( \mathbf{t}_\alpha(x) \). The basis systems at points \( (x^\alpha) \) and \( (x^\alpha + dx^\alpha) \) are related by

\[
\mathbf{t}_\lambda(x^\alpha + dx^\alpha) = \mathbf{t}_\lambda(x^\alpha) + \Gamma^\nu_{\lambda\rho}(x)dx^\rho \mathbf{t}_\nu(x) + O(dx^\alpha)\mathbf{t}_\lambda(x) ,
\]

where \( \lim_{\max|dx^\alpha| \to 0} O(dx^\alpha) = 0 \) and functions \( \Gamma^\nu_{\lambda\rho} \) are components of the affine connection on \( X \). Briefly, (2.1.1) will be denoted by

\[
\mathbf{t}_\lambda(x + dx) = \mathbf{t}_\lambda(x) + \Gamma^\nu_{\lambda\rho}(x)dx^\rho \mathbf{t}_\nu(x) ,
\]

where it is understood that only first order terms in \( dx^\rho \) are retained.
We assume further that \( X \) is a Riemannian Space. Hence, each \( \mathcal{A}_\lambda(x) \) is an inner-product space \( \mathcal{A}_\lambda \). The numbers

\[
  g_{\alpha\beta}(x) = (\vec{r}_\alpha(x), \vec{r}_\beta(x)),
\]

(2.1.2)

where \((\vec{r}_\alpha, \vec{r}_\beta)\) denotes the inner-product of \( \vec{r}_\alpha \) and \( \vec{r}_\beta \), are components of the metric tensor \( g_{\alpha\beta} \) on \( X \). In order to achieve compatibility of the metric tensor and the affine connection on \( X \) we assume

\[
  \partial_\lambda g_{\alpha\beta} = g_{\nu\beta} \Gamma^\nu_{\alpha\lambda} + g_{\alpha\nu} \Gamma^\nu_{\beta\lambda}.
\]

(2.1.3)

It is assumed that \( \Gamma^\nu_{\alpha\lambda} = \Gamma^\nu_{\lambda\alpha} \) (torsion-free connection).

It is assumed that \( g = |\det(g_{\alpha\beta})| \neq 0 \) on \( X \).

Point \( x \in X \) is regarded as having an invariant geometric meaning, and four-tuples \((x^0, x^1, x^2, x^3)\) are assigned to points \( x \) by some coordinate system. With respect to another coordinate system, point \( x \) has coordinates \((x'^\alpha)\). We assume some functions \( \phi^\alpha \) exist such that

\[
  x'^\alpha = \phi^\alpha(x^0, x^1, x^2, x^3), \quad \left| \frac{\partial x'^\alpha}{\partial x^\beta} \right| \neq 0
\]

in some neighborhood of \( x \). When such a coordinate transformation is introduced, it is assumed that new basis systems \( \vec{r}_\alpha(x) \) are selected at each \( x \in X \), so that

\[
  \vec{r}_\alpha(x) = \frac{\partial x'^\alpha}{\partial x^\beta}(x) \vec{r}_\beta(x).
\]

(2.1.4)

This assures that functions \( g_{\alpha\beta} \) are components of a second order covariant tensor on \( X \).

Suppose \( \vec{A} \) is a vector field on \( X \). Then, vector \( \vec{A}(x) \in \mathcal{A}_\lambda(x) \) is independent of the choice of coordinate system. We have

\[
  \vec{A}(x) = A^\alpha(x^\lambda) \vec{r}_\alpha(x)
\]

for some components \( A^\alpha \). In a new coordinate system we have new components \( A'^\alpha \) defined by

\[
  \vec{A}(x) = A'^\beta(x^\lambda) \vec{r}_\beta(x).
\]
Using (2.1.4) we find
\[ \Lambda'^\beta (x'^\lambda) \mathcal{I}_\beta (x) = \Lambda^\alpha (x^\lambda) \frac{\partial x'^\beta}{\partial x^\alpha} \mathcal{I}_\beta (x). \]
Hence,
\[ \Lambda'^\beta (x'^\lambda) = \Lambda^\alpha (x^\lambda) \frac{\partial x'^\beta}{\partial x^\alpha}, \]
which is the usual transformation rule given in tensor analysis for contravariant components of a vector. This transformation rule with the rule (2.1.4) simply expresses the invariant character of \( \Lambda(x) \), and gives the statement \( \Lambda'(x') = \Lambda(x) \mathcal{I}_\alpha (x) \) an invariant meaning under coordinate transformations.

2. Covariant Derivatives and Differentials

Assume \( \Lambda \) is a vector field on manifold \( X \). Let \( x + dx \) denote the point with coordinates \( x^\alpha + dx^\alpha \). Consider
\[ \Delta \Lambda (x) = \Lambda (x + dx) - \Lambda (x) \] (2.2.1)
Assume that linear spaces \( \mathcal{A}_k (x) \) are isomorphic copies of the same linear space \( \mathcal{A}_k \), except that a different basis system \( \mathcal{I}_\alpha (x) \) is selected for each point \( x \in X \). Hence, \( \Lambda (x + dx) \), \( \Lambda (x) \), and \( \Delta \Lambda (x) \) are vectors in the same linear space \( \mathcal{A}_k \). The object now is to find components of vector \( \Delta \Lambda (x) \) relative to base \( \mathcal{I}_\alpha (x) \). We have
\[ \Delta \Lambda (x) = \Lambda'^\lambda (x + dx) \mathcal{I}_\lambda (x + dx) - \Lambda^\alpha (x) \mathcal{I}_\alpha (x) \]
\[ = \left[ \frac{\partial A^\lambda}{\partial x^\alpha} dx^\lambda \right] \left[ \mathcal{I}_\lambda (x) + \Gamma^\gamma_{\lambda \rho} (x) dx^\gamma \mathcal{I}_\rho (x) \right] - \Lambda^\alpha (x) \mathcal{I}_\alpha (x) \]
\[ = \frac{\partial A^\lambda}{\partial x^\alpha} dx^\lambda \mathcal{I}_\lambda (x) + \Lambda^\alpha (x) \Gamma^\gamma_{\lambda \rho} (x) dx^\gamma \mathcal{I}_\rho (x) \]
if only first order terms in \( dx \) are retained. So,
\[ \Delta \Lambda (x) = (\frac{\partial A^\lambda}{\partial x^\alpha} + \Gamma^\lambda_{\alpha \rho} A^\rho) dx^\lambda \mathcal{I}_\lambda (x) = \nabla A^\lambda \mathcal{I}_\lambda (x), \]
where
\[ \nabla A^\lambda = (\frac{\partial A^\lambda}{\partial x^\alpha} + \Gamma^\lambda_{\alpha \rho} A^\rho) dx^\alpha. \] (2.2.2)
The components \( \nabla A^\lambda \) are components of the vector \( \Delta \Lambda (x) \), and usually we say \( \nabla A^\lambda \) is the covariant differential of \( A^\lambda (x) \). Coefficients \( \nabla^\epsilon A^\lambda = \frac{\partial A^\lambda}{\partial x^\alpha} + \Gamma^\lambda_{\alpha \rho} A^\rho \) are components of the covariant derivative of \( A^\lambda \).
It should be noted for future reference that the ordinary differential, 
\[ dA^\lambda = \partial_\alpha A^\lambda dx^\alpha, \]
represents the first order change in component functions \( A^\lambda \) for vector field \( \vec{A} \), while the covariant differential \( \nabla A^\lambda \) represents the components of the absolute vector change \( \Delta \vec{A} \) defined by (2.2.1).

We see from (2.2.2) that
\[ \nabla A^\lambda = dA^\lambda + \Gamma^\lambda_{\alpha\beta} A^\alpha dx^\beta. \quad (2.2.3) \]

3. Covariant Variations for Vector Fields

Let \( \vec{A} \) again denote a vector field on \( X \). Assume infinitesimal transformations,
\[ \bar{x}^\alpha = x^\alpha + \Delta x^\alpha, \quad (2.3.1) \]
\[ \bar{A}^\alpha(x) = A^\alpha(x) + \Delta (\vec{A}^\alpha), \quad (2.3.2) \]
are given. Formula (2.3.1) defines a point transformation in \( X \), so \( (\bar{x}^\alpha) \) denotes a new point \( \bar{x} \) in \( X \). Transformation (2.3.1) must not be interpreted as a coordinate transformation assigning new coordinates \( (\bar{x}^\alpha) \) to the same geometric point \( x \in X \). This distinction must be made to avoid confusion in the following. With this agreement, the term \( \Delta x^\alpha \) is a contravariant vector as the notation suggests.

Let \( \Delta \bar{x} = \Delta x^\alpha \bar{I}_\alpha(x) \). Formula (2.3.2) shall be interpreted as defining a new vector field \( \vec{B} \) on \( X \), where
\[ \vec{B}(\bar{x}) = \bar{A}^\alpha(\bar{x}) \bar{I}_\alpha(\bar{x}). \quad (2.3.3) \]

Then,
\[ \vec{B}(x) - \vec{A}(x) = \delta \vec{A}(x) \quad (2.3.4) \]
is a vector local variation. Since all vectors in (2.3.4) are at the same point \( x \) with base \( \bar{I}_\alpha(x) \), we may express (2.3.4) in terms of components relative to this base. Then
\[ \bar{A}^\alpha(x) - A^\alpha(x) = \delta A^\alpha(x), \quad (2.3.5) \]
if \( \delta A^\alpha \) is defined by \( \delta A^\alpha (x) = \delta A^\alpha (\vec{x}) \). For such local variations we have established that \( \delta A^\alpha \) may be interpreted either as the components of the variation \( \vec{\delta A} \), or as the variation of the components \( A^\alpha \) as expressed by (2.3.5).

Now, define the total variation of field \( \vec{A} \) to be

\[
\Delta \vec{A} = \vec{B}(x + \Delta x) - \vec{A}(x).
\]

We shall determine components of \( \Delta \vec{A}(x) \) relative to basis \( \vec{l}_\alpha (x) \). From definition (2.3.6)

\[
\Delta \vec{A} = \left[ \vec{B}(x + \Delta x) - \vec{B}(x) \right] + \left[ \vec{B}(x) - \vec{A}(x) \right].
\]

From (2.2.1) and (2.2.2) we see

\[
\vec{B}(x + \Delta x) - \vec{B}(x) = \nabla B \vec{l}_\lambda (x)
\]

\[
= \left( \partial_{\nu} B^\lambda + \Gamma^\lambda_{\nu\sigma} B^\sigma \right) \Delta x^\nu \vec{l}_\lambda (x).
\]

But,

\[
(\partial_{\nu} B^\lambda + \Gamma^\lambda_{\nu\sigma} B^\sigma) \Delta x^\nu = \left[ \partial_{\nu} (A^\lambda + \delta A^\lambda) + \Gamma^\lambda_{\nu\sigma} (A^\sigma + \delta A^\sigma) \right] \Delta x^\nu.
\]

Retention of only first order terms in \( \delta A^\alpha \) and \( \Delta x^\nu \) gives

\[
(\partial_{\nu} A^\lambda + \Gamma^\lambda_{\nu\sigma} A^\sigma) \Delta x^\nu = \nabla_{\nu} A^\lambda \Delta x^\nu.
\]

Also, \( \vec{B}(x) - \vec{A}(x) = \vec{\delta A}(x) = \delta A^\alpha \vec{l}_\alpha (x) \) in (2.3.7). So we find

\[
\Delta \vec{A} = (\nabla_{\nu} A^\lambda \Delta x^\nu + \delta A^\lambda) \vec{l}_\lambda (x) = (\Delta A^\lambda) \vec{l}_\lambda (x),
\]

where

\[
(\Delta A^\lambda) = \delta A^\lambda + \nabla_{\nu} A^\lambda \Delta x^\nu.
\]

The \( (\Delta A)^\alpha \) are components of the total variation \( \Delta \vec{A} \).

From (2.3.2) we get

\[
\Delta (A^\alpha) = \vec{A}^\alpha (\vec{x}) - A^\alpha (x)
\]

\[
= [A^\alpha (\vec{x}) - \vec{A}^\alpha (x)] + [A^\alpha (x) - \vec{A}^\alpha (x)].
\]

So,

\[
\Delta (A^\alpha) = \partial_{\nu} A^\alpha \Delta x^\nu + \delta A^\alpha.
\]

Hence, \( \Delta (A^\alpha) = \delta A^\alpha + \partial_{\nu} A^\alpha \Delta x^\nu \) is the total variation of the component \( A^\alpha \), while quantities

\[
(\Delta A)^\alpha = \delta A^\alpha + \nabla_{\nu} A^\alpha \Delta x^\nu
\]
are the components of the total variation $\Delta \bar{\Lambda}$. For total variations a distinction must be made between $\Delta (A^\alpha)$ and $(\Delta A)^\alpha$. This is quite important in the following work.

As a generalization of this fact, if a tensor field $\bar{\Lambda}_{\mu\nu}$ is prescribed analogous to $(2.3.2)$ by

$$\bar{\Lambda}_{\mu\nu}(x) = \Lambda_{\mu\nu}(x) + \Delta (\Lambda_{\mu\nu}),$$

we define

$$(\Delta A)_{\mu\nu} = \delta \Lambda_{\mu\nu} + \nabla_\lambda \Lambda_{\mu\nu} \Delta x^\lambda. \quad (2.3.11)$$

A similar definition can be made for covariant or contravariant tensors of any order.

4. Variation of an Action Integral

We now assume $\bar{\Lambda}$ is a vector field on $X$, and $L(x,\bar{\Lambda},D\bar{\Lambda})$ is a scalar valued function of $x$, $\Lambda(x)$, and the covariant derivative $D\Lambda$. $D\Lambda$ is a tensor with components

$$\nabla_\lambda \Lambda^\alpha = \partial_\lambda \Lambda^\alpha + \Gamma^\alpha_{\mu\beta} \Lambda^\beta$$

at $x \in X$.

An action integral

$$W[\bar{\Lambda}] = \int_\Omega dx \sqrt{g} L(x,\bar{\Lambda},D\bar{\Lambda}) \quad (2.4.1)$$

will be studied. Assume infinitesimal transformations

$$\bar{x}^\alpha = x^\alpha + \Delta x^\alpha \quad \text{and} \quad \bar{\Lambda}^\alpha(x) = \Lambda^\alpha(x) + \Delta (\Lambda^\alpha)$$

are given, as in $(2.3.1)$, $(2.3.2)$. Vectors $\bar{B}$, $\Delta \bar{\Lambda}$, $\delta \bar{\Lambda}$, are defined as in section 3. We define the total variation of $W$ to be

$$\Delta W = W[\bar{B}(\bar{x})] - W[\bar{\Lambda}(x)]. \quad (2.4.2)$$

Hence,

$$W = \int_\Omega dx \sqrt{g(x)} L(x,\bar{\Lambda},D\bar{\Lambda}) - \int_\Omega dx \sqrt{g(x)} L(x,\bar{\Lambda},D\bar{\Lambda}) \quad (2.4.3)$$

Calculation of $\Delta W$ in terms of $\Delta x^\alpha$ and $\delta \Lambda^\alpha$ can be performed by
paralleling the similar calculation in section 1 of Chapter I. The variation of \( g(x) = \left| \det(g_{\alpha\beta}(x)) \right| \) must be calculated first, however. Assume \( \det(g_{\alpha\beta}(x)) > 0 \) for convenience. Now,
\[
\sqrt{g(x)} = \sqrt{g(x)} + \frac{1}{2} \frac{\partial g}{\partial x^\lambda} \Delta x^\lambda .
\]
(2.4.4)

Since \( g = \det(g_{\alpha\beta}) \), then \( \frac{\partial g}{\partial g_{\alpha\beta}} = G^{\alpha\beta} \), where \( G^{\alpha\beta} \) is the cofactor of \( g_{\alpha\beta} \) in determinant \( g \). By definition of \( g^{\alpha\beta} \) we have \( G^{\alpha\beta} = g^{\alpha\beta} \). Hence,
\[
\frac{\partial g}{\partial g_{\alpha\beta}} = g^{\alpha\beta} .
\]
So,
\[
\frac{\partial g}{\partial x^\lambda} = \frac{\partial g}{\partial g_{\alpha\beta}} \partial_{\lambda} g_{\alpha\beta} = g^{\alpha\beta} \partial_{\lambda} g_{\alpha\beta} .
\]
Substitution in (2.4.4) gives
\[
\sqrt{g(x)} = \sqrt{g(x)} + \frac{1}{2} \frac{1}{\sqrt{g}} g^{\alpha\beta} \partial_{\lambda} g_{\alpha\beta} \Delta x^\lambda .
\]

Therefore,
\[
\sqrt{g(x)} = \sqrt{g(1 + \frac{1}{2} \frac{1}{\sqrt{g}} g^{\alpha\beta} \partial_{\lambda} g_{\alpha\beta} \Delta x^\lambda)} .
\]
(2.4.5)

By use of (2.1.3), we find
\[
g^{\alpha\beta} \partial_{\lambda} g_{\alpha\beta} = g^{\alpha\beta} g_{\nu\beta} \partial_{\lambda} g^{\nu\beta} + g^{\alpha\beta} g_{\alpha\nu} \partial_{\lambda} g^{\nu\beta} = 2 \partial_{\lambda} \delta_{\nu} ,
\]
(2.4.6)
because \( g^{\alpha\beta} g_{\nu\beta} = \delta_{\nu} \) and \( g^{\alpha\beta} g_{\alpha\nu} = \delta_{\nu} \).

Hence, from (2.4.5)
\[
\sqrt{g(x)} = \sqrt{g(1 + \frac{1}{2} \partial_{\lambda} \Delta x^\lambda)} .
\]
(2.4.7)

Use \( dx = dx(1 + \partial_{\alpha} \Delta x^\alpha) \) in (2.4.3) and denote the integrands by \( L = L(x, A, D) \) and \( L + \Delta L = L(x, B, D) \).

Then, we find
\[
\Delta W = \frac{1}{2} \int \left( dx(1 + \partial_{\alpha} \Delta x^\alpha)(1 + \frac{1}{2} \partial_{\lambda} \Delta x^\lambda) \sqrt{g(1 + \Delta L)} - \sqrt{g(L + \Delta L)} \right) .
\]
(2.4.8)
\[
= \frac{1}{2} \int \left( dx \sqrt{g} \left[ (1 + \partial_{\alpha} \Delta x^\alpha + \frac{1}{2} \partial_{\lambda} \Delta x^\lambda)(L + \Delta L) - L \right] \right) .
\]
(2.4.9)
The variatio of th e integrand $L$ is
\[ \Delta L = L(x, \vec{A}, \vec{B}) - L(x, \vec{A}, \vec{D}) \]
\[ = \frac{\partial L}{\partial x^\alpha} \Delta x^\alpha + \frac{\partial L}{\partial A^\alpha} \Delta A^\alpha + \frac{\partial L}{\partial (A^\alpha A^\beta)} \Delta A^\alpha \Delta A^\beta \]
\[ = \nabla_\lambda L \Delta x^\alpha + \frac{\partial L}{\partial A^\alpha} \Delta A^\alpha + \nabla_\lambda \left( \frac{\partial L}{\partial (A^\alpha A^\beta)} \right) \Delta A^\beta \].
\[ (2.4.10) \]

Here
\[ \nabla_\lambda L = \frac{\partial L}{\partial x^\lambda} + \frac{\partial L}{\partial A^\alpha} \nabla_\lambda A^\alpha + \frac{\partial L}{\partial (A^\alpha A^\beta)} \nabla_\lambda (A^\alpha A^\beta) \]
and
\[ \frac{\partial L}{\partial A^\alpha} - \nabla_\beta \left( \frac{\partial L}{\partial (A^\alpha A^\beta)} \right). \]
\[ (2.4.11) \]

Details are omitted above because at this point the argument is essentially the same as given in Chapter I while deriving (1.1.18). Substitution of (2.4.10) in (2.4.9) and rearranging terms gives
\[ \Delta W = \int dx^\beta \frac{\partial L}{\partial A^\alpha} \Delta A^\alpha + \int dx^\beta \{ L \Delta x^\alpha + \frac{\partial L}{\partial (A^\alpha A^\beta)} \Delta A^\beta \}. \]
\[ (2.4.12) \]

Equation (2.4.12) is the covariant form of (1.1.18) for a vector field $A^\alpha$.

In (2.4.12) we may replace $\Delta A^\alpha$ by $(\Delta A)^\alpha - \nabla_\lambda A^\alpha \Delta x^\lambda$, in the second term. Then,
\[ \Delta W = \int dx^\beta \frac{\partial L}{\partial A^\alpha} \Delta A^\alpha + \int dx^\beta \nabla_\alpha \left[ \Delta x^\alpha \frac{\partial L}{\partial (A^\alpha A^\beta)} \Delta x^\beta + \frac{\partial L}{\partial (A^\alpha A^\beta)} (\Delta A)^\beta \right] \]
\[ = \int dx^\beta \frac{\partial L}{\partial A^\alpha} \Delta A^\alpha + \int dx^\beta \nabla_\alpha \left[ (\Delta x)\Delta x^\alpha \frac{\partial L}{\partial (A^\alpha A^\beta)} \Delta x^\beta + \frac{\partial L}{\partial (A^\alpha A^\beta)} (\Delta A)^\beta \right] \]
\[ (2.4.13) \]
where
\[ T^\alpha_{\lambda} = L \frac{\partial L}{\partial (A^\alpha A^\beta)} \nabla_\lambda A^\beta. \]
\[ (2.4.14) \]

$T^\alpha_{\lambda}$ is sometimes called the canonical-energy momentum tensor for field $A^\alpha$.

5. Covariant Noether Theorems

Theorem 1. Suppose 3$r$ vectors $\vec{e}_1, \ldots, \vec{e}_r, \vec{\eta}_1, \ldots, \vec{\eta}_r, \vec{\gamma}_1, \ldots, \vec{\gamma}_r$ exist at each point $x \in \mathcal{X}$, such that for arbitrary real numbers $\epsilon^1, \ldots, \epsilon^r$, if $\vec{\Delta}x = \epsilon^k \vec{e}_k$ and $\vec{\Delta}A = \epsilon^k \vec{\eta}_k$, then $\Delta W = \int dx^\beta \text{div}(\epsilon^k \vec{\gamma}_k)$. 
Let \( \tilde{\xi}_k = \xi_k^{\lambda} \tilde{\gamma}_\lambda(x) \), \( \tilde{\eta}_k = \eta_k^{\alpha} \tilde{\gamma}_\alpha(x) \), and \( \tilde{\gamma}_k = \gamma_k^{\alpha} \tilde{\gamma}_\alpha(x) \). Then r identities exist of the form,

\[
\frac{\delta L}{\delta A^\beta} [\partial x^\alpha \xi_k^{\lambda} - \eta_k^{\beta}] = \nabla_\alpha \left[ T^\alpha_\beta \xi_k^{\lambda} + \frac{\partial L}{\partial (\nabla_\alpha A^\beta)} \eta_k^{\beta} - \gamma_k^{\alpha} \right] (k = 1, \ldots, r)
\]

**Proof.** The proof follows immediately from (2.4.13) using arbitrariness of region \( \mathcal{A} \) and parameters \( \xi_k^\lambda \). Details were given in Chapter I in first proof of the Noether theorem.

**Theorem 2.** Assume linear differential operators \( \Lambda_k^\alpha, \Delta_k^\alpha, \Theta_k^\alpha \) (k = 1, \ldots, r) exist, such that for arbitrary functions \( \Delta \phi^1(x^\alpha), \ldots, \Delta \phi^r(x^\alpha) \) we have \( \Delta \bar{w} = \int_{\mathcal{A}} \nabla_\alpha \left[ \Lambda_k^\alpha \left[ \Delta \phi^k \right] \right] \), when \( \Delta x^\alpha = \Delta_k^\alpha \left[ \Delta \phi^k \right] \) and \( (\Delta A)^\alpha = \Theta_k^\alpha \left[ \Delta \phi^k \right] \).

Then r identities exist of the form,

\[
\tilde{\Theta}_k^\lambda \left[ \frac{\delta L}{\delta A^\lambda} \right] - \tilde{\Delta}_k^\lambda \left[ \frac{\delta L}{\delta A^\lambda} \nabla_\alpha \right] = 0,
\]

where \( \tilde{\Theta}_k^\lambda \) denotes the adjoint of \( \Theta_k^\lambda \).

**Proof.** Essentially the same as proof of Noether Theorem 2 in Chapter I, except for use of (2.4.13) needed here.

At this point it would seem logical to consider conformal transformations in an arbitrary Riemanian manifold, and study conformal invariance of the electromagnetic field in arbitrary spaces. The author did in fact take that course during the investigation of this problem. While making this study, certain calculations suggested a basic change in the usual variational principle for electromagnetic fields could and should be introduced. By so doing it could be shown conformal invariance is a property of a much broader class of fields, including the electromagnetic field as a special case. It will be shown that the identities produced by Bessel-Hagen hold true for this generalization of electromagnetic fields.
In this chapter the setting has been established for the later chapters, and an attempt has been made to clarify some ideas, such as the distinction between $(\Delta A)_{\alpha}$ and $\Delta(A_{\alpha})$ in more general spaces. But now the Noether theorems will be temporarily shelved until after a discussion of variational principles is completed. Then, modified forms of the Noether theorems will be produced and applied to electromagnetic fields.
CHAPTER III

VARIATIONAL PRINCIPLES

In this chapter the usual variational principle for electromagnetic fields will be criticised, and a new principle introduced. We begin with a review of the form of classical variational principles for physical fields in order to determine the essential mathematical features of such principles.

1. Classical Variational Principles

\( A_\alpha \) denotes a vector field on manifold \( X \). Assume \( A_\alpha \) satisfies some field equations. The field equations are said to be derived from a variational principle if we can find an integral,

\[
W = \int_\Omega d\mathcal{V} \mathcal{L}(x,A_\alpha, \nabla_\alpha A_\beta),
\]  

(3.1.1)

such that the vanishing of the variation of \( W \) implies the field equations for \( A_\alpha \). Local variations \( \delta A_\alpha \) of the field are assumed, and the variation of \( W \) is

\[
\delta W = \int_\Omega d\mathcal{V} \mathcal{L}(x,A_\alpha + \delta A_\alpha, \nabla_\alpha (A_\beta + \delta A_\beta)) - \int_\Omega d\mathcal{V} \mathcal{L}(x,A_\alpha, \nabla_\alpha A_\beta)
\]

\[
= \int_\Omega d\mathcal{V} \left[ \frac{\partial \mathcal{L}}{\partial A_\alpha} \delta A_\alpha + \frac{\partial \mathcal{L}}{\partial (\nabla_\alpha A_\beta)} \nabla_\alpha (\delta A_\beta) \right]
\]

\[
= \int_\Omega d\mathcal{V} \left( \frac{\partial \mathcal{L}}{\partial A_\beta} \delta A_\beta + \int d\mathcal{V} \nabla_\alpha \left( \frac{\partial \mathcal{L}}{\partial (\nabla_\alpha A_\beta)} \delta A_\beta \right) \right),
\]

(3.1.2)

where

\[
\frac{\delta \mathcal{L}}{\delta A_\beta} = \frac{\partial \mathcal{L}}{\partial A_\beta} - \nabla_\alpha \left( \frac{\partial \mathcal{L}}{\partial (\nabla_\alpha A_\beta)} \right).
\]

(3.1.3)

Now we assume \( \delta W = 0 \) for all \( \delta A_\alpha \) such that \( \delta A_\alpha = 0 \) on the boundary.
of \( \Omega \). If we let \( B(\Omega) \) denote the boundary of \( \Omega \), then the second integral in (3.1.2) can be written
\[
\oint_{\partial(\Omega)} \frac{\partial L}{\partial (\nabla \alpha A_{\beta})} \delta A_{\beta}
\]
by use of the Gauss theorem, \( \int_{\Omega} \delta W = \oint_{\partial(\Omega)} \frac{\partial L}{\partial A_{\beta}} \delta A_{\beta} \). Since \( \delta A_{\beta} = 0 \) on \( B(\Omega) \), then the integral (3.1.4) vanishes. Hence, \( \delta W = \oint_{\partial(\Omega)} \frac{\partial L}{\partial A_{\beta}} \delta A_{\beta} \). \( \delta W = 0 \) for arbitrary \( \delta A_{\beta} \) now implies \( \frac{\partial L}{\partial A_{\beta}} = 0 \). If these differential equations are the field equations for \( A_{\alpha} \) we say they have been derived from a variational principle, and (3.1.1) is called an action integral for field \( A_{\beta} \).

2. Variational Principle for Maxwell's Equations

The electromagnetic field in vacuum is a second order, anti-symmetric tensor field \( F_{\alpha\beta} \) with field equations,
\[
\begin{align*}
I. & \quad \nabla_{\alpha} F^{\alpha\beta} = 0 \\
II. & \quad \epsilon^{\alpha\beta\lambda\nu} \nabla_{\beta} F_{\lambda\nu} = 0 .
\end{align*}
\]
In equation II, \( \epsilon^{\alpha\beta\lambda\nu} \) is the Ricci tensor,
\[
\epsilon^{\alpha\beta\lambda\nu} = \frac{1}{\sqrt{g}} \epsilon^{\alpha\beta\lambda\nu} .
\]
Symbols \( \epsilon^{\alpha\beta\lambda\nu} \) are permutation symbols defined by
\[
\epsilon^{\alpha\beta\lambda\nu} = \begin{cases} 
+1 & \text{if } \alpha\beta\lambda\nu \text{ is an even permutation of } 0,1,2,3 \\
-1 & \text{if } \alpha\beta\lambda\nu \text{ is an odd permutation of } 0,1,2,3 \\
0 & \text{in all other cases.}
\end{cases}
\]
It is assumed that
\[
F_{\alpha\beta} = \nabla_{\alpha} A_{\beta} - \nabla_{\beta} A_{\alpha} .
\]
Then II is automatically satisfied and I becomes a set of field equations for \( A_{\alpha} \). This reduces the electromagnetic field to a vector field \( A_{\alpha} \).

Now, assume an action integral,
\[
\mathcal{W} = \int_{\Omega} \delta W = \int_{\Omega} dx \sqrt{g} (-\frac{1}{4} F_{\mu\nu} F^{\mu\nu})
\]
The resulting field equations can be calculated explicitly from (3.1.3), but it is easier to calculate $\delta W$ directly from (3.2.2) as follows.

$$\delta W = -\frac{1}{2} \int \delta x \sqrt{g} \left( F_{\mu \nu} F^{\mu \nu} + (\delta F_{\mu \nu}) F^{\mu \nu} \right)$$

But, $\delta F_{\mu \nu} = \delta (\nabla_{\mu} A_{\nu} - \nabla_{\nu} A_{\mu}) = \nabla_{\mu}(\delta A_{\nu}) - \nabla_{\nu}(\delta A_{\mu})$

So, $\delta W = -\frac{1}{2} \int \delta x \sqrt{g} \left( F^{\mu \nu} \nabla_{\nu} \delta A_{\mu} - F^{\mu \nu} \nabla_{\nu} \delta A_{\mu} \right)$.

Using anti-symmetry of $F^{\mu \nu}$ and interchanging summing indices gives

$$\delta W = -\int \delta x \sqrt{g} F^{\mu \nu} \nabla_{\nu} \delta A_{\mu}$$

If $\delta W = 0$ for arbitrary $\delta A_{\mu}$ then $\nabla_{\mu} F^{\mu \nu} = 0$. This is the desired field equation I.

Because of the preceding result, it is usually stated that field equations I and II are derived from action integral (3.2.2) by a variational principle. In fact only I is derived in this way, while equation II is an identity resulting from the assumption,

$$F_{\mu \nu} = \nabla_{\mu} A_{\nu} - \nabla_{\nu} A_{\mu}.$$ 

This fact is particularly important when the first Noether theorem is applied, because then we must assume that all of the field equations are derived from the action integral.

3. A New Variational Principle

Classical variational principles, as outlined in section 1, were motivated by the idea of minimizing or maximizing the integral (3.1.1). When producing variational principles for physical fields, the idea of maximizing or minimizing the action integral does not play an essential role. What really matters is that the variation of the action integral
be stationary. It is also conventional to assume the integrand of an action integral must contain derivatives of the field variables. In the following we shall consider an integrand containing no derivatives, and introduce derivatives via the variations of the field. This can be done in a natural manner for tensor fields. We shall consider integrals of the form,

$$W = \int \sqrt{g} L(g_{\mu\nu}, A_\alpha, F_{\lambda\rho})$$

(3.3.1)

where $A_\alpha$ and $F_{\lambda\rho}$ are vector and tensor fields, respectively. It will be assumed that

$$\delta g_{\alpha\beta} = 0 .$$

However, $g_{\alpha\beta}(x)$ may vary from point to point in the manifold. Manifold metric properties are thereby left untouched.

Now, let a point transformation, $x'^\alpha = x^\alpha + \Delta x^\alpha$, be given. We shall use this transformation to generate the transformation of $A_\alpha$ and $F_{\alpha\beta}$ by assuming these fields are "dragged along", so that

$$\tilde{A}_\alpha(x') = A_\beta(x) \frac{\partial x'^\alpha}{\partial x^\beta}$$

and

$$\tilde{F}_{\alpha\beta}(x') = F_{\mu\nu}(x) \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta}$$

(3.3.2)

As in Chapter II, we again define

$$\Delta(A_\alpha) = \tilde{A}_\alpha(x) - A_\alpha(x)$$

(3.3.3)

$$\Delta A_\alpha = \delta A_\alpha + \nabla_\lambda A_\alpha \Delta x^\lambda$$

(3.3.4)

$$\Delta(F_{\alpha\beta}) = \tilde{F}_{\alpha\beta}(x) - F_{\alpha\beta}(x)$$

(3.3.5)

$$\Delta F_{\alpha\beta} = \delta F_{\alpha\beta} + \nabla_\lambda F_{\alpha\beta} \Delta x^\lambda$$

(3.3.6)

We must again distinguish between variation of components, $\Delta(A_\alpha)$, and components of variation ($\Delta A_\alpha$) in a covariant sense. Note that

$$\Delta g_{\alpha\beta} = \delta g_{\alpha\beta} + \nabla_\lambda g_{\alpha\beta} \Delta x^\lambda = 0 ,$$

because $\delta g_{\alpha\beta} = 0$ and $\nabla_\lambda g_{\alpha\beta} = 0$.

We shall now compute $\Delta A_{\alpha}$ and $\Delta F_{\alpha\beta}$, corresponding to
transformation (3.3.2). Because of (3.3.2),
\[
\tilde{A}_\alpha (x) = A_\mu (x) \frac{\partial x^\mu}{\partial \bar{x}^\alpha}
\]
\[= A_\mu (x) \left[ \delta^\mu_\alpha - \partial_\alpha \Delta x^\mu \right]
\[= A_\mu (x) - A_\mu (x) \partial_\alpha \Delta x^\mu.
\]
Hence,
\[
\Delta (A_\mu) = -A_\mu (x) \partial_\alpha \Delta x^\mu.
\]
Since,
\[
\Delta (A_\mu) = \delta \Delta + \partial_\lambda A_\lambda \Delta x^\lambda
\]
and,
\[
(\Delta A)_\alpha = \delta A_\alpha + \nabla_\lambda A_\alpha \Delta x^\lambda,
\]
then
\[
(\Delta A)_\alpha = \Delta (A_\alpha) - \partial_\lambda A_\lambda \Delta x^\lambda + \nabla_\lambda A_\lambda \Delta x^\lambda
\]
\[= -A_\lambda \partial_\alpha \Delta x^\lambda - \partial_\lambda A_\lambda \Delta x^\lambda + \nabla_\lambda A_\lambda \Delta x^\lambda
\]
\[= -A_\lambda \partial_\alpha \Delta x^\lambda - \Gamma^\mu_\alpha A_\mu \Delta x^\lambda.
\]
Thus,
\[
(\Delta A)_\alpha = -A_\mu (\partial_\alpha \Delta x^\mu + \Gamma^\mu_\alpha \Delta x^\lambda)
\]
\[= -A_\mu \nabla_\alpha \Delta x^\mu.
\]
(3.3.7)

Similarly,
\[
\bar{F}_{\alpha \beta} (x) = F_{\mu \nu} (x) \frac{\partial x^\mu}{\partial \bar{x}^\alpha} \frac{\partial x^\nu}{\partial \bar{x}^\beta}
\]
\[= F_{\mu \nu} (x) \left[ \delta^\mu_\alpha - \partial_\alpha \Delta x^\mu \right] \left[ \delta^\nu_\beta - \partial_\beta \Delta x^\nu \right]
\]
\[= F_{\alpha \beta} (x) - F_{\alpha \nu} (x) \partial_\beta \Delta x^\nu - F_{\mu \beta} (x) \partial_\alpha \Delta x^\mu.
\]
So,
\[
\Delta (F_{\alpha \beta}) = -F_{\alpha \nu} \partial_\beta \Delta x^\nu - F_{\nu \beta} \partial_\alpha \Delta x^\mu.
\]
Then,
\[
(\Delta F)_{\alpha \beta} = \delta F_{\alpha \beta} + \nabla_\lambda F_{\alpha \beta} \Delta x^\lambda
\]
\[= \Delta (F_{\alpha \beta}) - \partial_\lambda F_{\alpha \beta} \Delta x^\lambda + \nabla_\lambda F_{\alpha \beta} \Delta x^\lambda.
\]
So,
\[
(\Delta F)_{\alpha \beta} = -F_{\alpha \nu} \partial_\beta \Delta x^\nu - F_{\nu \beta} \partial_\alpha \Delta x^\mu + (\nabla_\lambda F_{\alpha \beta} - \partial_\lambda F_{\alpha \beta}) \Delta x^\lambda
\]
\[= -F_{\alpha \nu} \partial_\beta \Delta x^\nu - F_{\nu \beta} \partial_\alpha \Delta x^\mu - (\Gamma^\nu_\alpha F_{\nu \beta} + \Gamma^\mu_\beta F_{\nu \alpha}) \Delta x^\lambda
\]
\[= -F_{\alpha \nu} (\partial_\beta \Delta x^\nu + \Gamma^\nu_\alpha \Delta x^\lambda) - F_{\nu \beta} (\partial_\alpha \Delta x^\mu + \Gamma^\mu_\beta \Delta x^\lambda).
\]
Therefore,
\[
(\Delta F)_{\alpha \beta} = -F_{\alpha \nu} \nabla_\beta \Delta x^\nu - F_{\nu \beta} \nabla_\alpha \Delta x^\mu.
\]
(3.3.8)

Now we define the variation of \( W \) to be,
\[
W = \int d\bar{x} \sqrt{g(\bar{x})} L \left( g_{\alpha \beta}, A_\alpha + (\Delta A)_\alpha, F_{\mu \nu} + (\Delta F)_{\mu \nu} \right) - \int d\bar{x} \sqrt{g} L \left( g_{\alpha \beta}, A_\alpha, F_{\mu \nu} \right).
\]
(3.3.9)

It was shown in Chapter II, section 4, equation (2.4.7) that,
\[
\sqrt{g(\bar{x})} = \sqrt{g(1 + \Gamma^\nu_\alpha \Delta x^\nu)}.
\]
Since \( dx = dx(1 + \partial_\alpha \Delta x^\alpha) \), then
\[
d\bar{\nu}(x) = \nu_\alpha(1 + \alpha_\alpha \Delta x^\alpha)(1 + \beta_\alpha \Delta x^\alpha)
\]
\[= \nu_\alpha(1 + \alpha_\alpha \Delta x^\alpha) . \tag{3.3.10}\]

Hence,
\[
\Delta \nu = \int_\Omega d\bar{\nu}(x) \left\{ L + \frac{\partial L}{\partial A_\alpha} (\Delta A)_\alpha + \frac{\partial L}{\partial F_\alpha \beta} (\Delta F)_{\alpha \beta} \right\} - \int_\Omega d\bar{\nu}(g_{\alpha \beta}, A_\alpha, F_\alpha \beta)
\]
\[= \int_\Omega d\bar{\nu}(L \nu_\alpha \Delta x^\alpha + \frac{\partial L}{\partial A_\alpha} A_\alpha \nu_\beta \Delta x^\beta - \frac{\partial L}{\partial F_\alpha \beta} F_\alpha \nu_\beta \Delta x^\beta) . \tag{3.3.11}\]

Substitute (3.3.7) and (3.3.8) into (3.3.11). Then,
\[
\Delta \nu = \int_\Omega d\bar{\nu}(L \nu_\alpha \Delta x^\alpha - \frac{\partial L}{\partial A_\alpha} A_\alpha \nu_\beta \Delta x^\beta - \frac{\partial L}{\partial F_\alpha \beta} F_\alpha \nu_\beta \Delta x^\beta)
\]
\[= \int_\Omega d\bar{\nu}(S^\beta_\nu \Delta x^\beta) , \tag{3.3.12}\]
where
\[S^\beta_\nu = L S^\beta_\nu - \frac{\partial L}{\partial A_\alpha} A_\alpha - \frac{\partial L}{\partial F_\alpha \beta} F_\alpha - \frac{\partial L}{\partial F_\beta \alpha} . \tag{3.3.13}\]

Equation (3.3.12) is the basic working formula for the new variational principle. Integration by parts gives,
\[
\Delta \nu = -\int_\Omega d\bar{\nu}(\nu_\beta S^\beta_\nu) \Delta x^\beta + \int_\Omega d\bar{\nu}(S^\beta_\nu \Delta x^\beta) . \tag{3.3.14}\]

Now we assume a variational principle as follows. Given (3.3.1) we define \( \Delta \nu \) by (3.3.9), so that the explicit form is given by (3.3.14).

Then, we assume \( \Delta \nu \equiv 0 \) for all \( \Delta x^\alpha \) which vanish on the boundary of \( \Omega \).

This leads to field equations,
\[\nu_\beta S^\beta_\nu = 0 , \tag{3.3.15}\]
with \( S^\beta_\nu \) given by (3.3.13).

Equations (3.3.15) are differential equations as desired, and this seems to be a natural way to produce field equations from an integral whose integrand does not contain derivatives of the field variables.

In 1959, S. Drobot and A. Rybarski (6) employed a similar technique for a variational principal of hydromechanics. We shall show that this technique can be used for electromagnetic fields.
4. Application To The Electromagnetic Field

Assume in the preceding arguments we use
\[ W = \int \text{d}x \sqrt{g} \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right). \quad (F_{\alpha\beta} = -F_{\beta\alpha}) \]  

(3.4.1)

From (3.3.13) we find
\[ S^\alpha_\gamma = Lg_{\gamma}^\alpha - \frac{\partial}{\partial F_{\alpha\beta}} F_{\alpha\beta} - \frac{\partial L}{\partial F_{\gamma\alpha}}. \]  

(3.4.2)

It follows from \( L = \frac{1}{2} g^{\mu\alpha} g^\nu F_{\mu\nu} F_{\alpha\beta} \) that
\[ \frac{\partial L}{\partial F_{\alpha\beta}} = -\frac{1}{2} F_{\alpha\beta}. \]  

(3.4.3)

So,
\[ S^\alpha_\gamma = Lg_{\gamma}^\alpha + F_{\beta}^\alpha F_{\gamma\alpha} \]
\[ = -\frac{1}{4} F_{\lambda}^\mu F_{\nu}^\lambda S^\beta_\gamma + F_{\alpha\beta} F_{\alpha\gamma}. \]  

(3.4.4)

This is recognized to be the tensor defined by physicists as the energy-momentum tensor for electromagnetic field \( F_{\alpha\beta} \).

The field equations resulting from the new variational principle

\[ \nabla^\beta S^\alpha_\gamma = \nabla_\beta \left( -\frac{1}{4} S^\beta_\gamma F_{\lambda}^\mu F_{\nu}^\lambda + F_{\alpha\beta} F_{\alpha\gamma} \right) = 0. \]  

(3.4.5)

Equations (3.4.5) state that energy and momentum are conserved.

This new variational principle does not lead to Maxwell's Equations (see I, II, section 2, Chapter III) without further assumptions. But, it can be shown that if we assume \( F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha \), then equations I and II follow. Hence, this variational principle is more general than the earlier one in section 2 of the chapter. At this point we adopt (3.4.5) as field equations for a tensor field \( F_{\alpha\beta} \), and study invariance of this field with its resulting conservation laws. We are then studying a larger class of fields, which contains the electromagnetic field as a special case.

5. New Form for the Noether Theorems

Because of the change in variational principles, we must revise
the statement of the Noether theorems for action integrals,

$$ W = \int dx \sqrt{g} (g_{\alpha \beta}, \partial_{\alpha} F_{\mu \nu}) .$$

(3.5.1)

We shall assume that all variations are produced by "dragging along" fields $A_{\alpha}$ and $F_{\mu \nu}$, when point transformation $\bar{x}' = x' + \Delta x'$ is given. $\Delta W$ is defined by (3.3.9). With this background, the following theorems may be proven.

**Theorem 1 (Modified Noether).** Suppose that for $r$ vectors, 

$$ \gamma_k(x) = \xi_{\alpha}^k(x), $$

(k=1,...,r), there exist $r$ vectors, $\gamma_k(x) = \xi_{\alpha}^k(x)$, such that for arbitrary independent parameters $\in_1, \ldots, \in_r$, 

$$ \Delta x' = \xi_{\alpha}^k(x) \in_k \text{ implies } \Delta W = \int \sqrt{g} \nabla_{\beta} \left( \gamma_{\beta} \in_k \right).$$

Then $r$ identities exist of the form,

$$ \nabla_{\beta} \left( S_{\beta \gamma} \xi_{\gamma}^k - \gamma_{\beta} \right) = (\nabla_{\beta} S_{\beta \gamma}) \xi_{\gamma}^k. $$

(3.5.2)

**Proof.** By (3.3.14)

$$ \Delta W = \int \sqrt{g} \left\{ \nabla_{\beta} (S_{\beta \gamma} \Delta x') - (\nabla_{\beta} S_{\beta \gamma}) \Delta x' \right\}. $$

Hence,

$$ \int \sqrt{g} \left\{ \nabla_{\beta} \left( S_{\beta \gamma} \xi_{\gamma}^k \right) = \left\{ \nabla_{\beta} (S_{\beta \gamma} \xi_{\gamma}^k - (\nabla_{\beta} S_{\beta \gamma}) \xi_{\gamma}^k \right\} \in_k \right\}, $$

and

$$ \int \sqrt{g} \left\{ \nabla_{\beta} (S_{\beta \gamma} \xi_{\gamma}^k - (\nabla_{\beta} S_{\beta \gamma}) \xi_{\gamma}^k \right\} \in_k = 0 . $$

By arbitrariness of $\Lambda$ and $\in_k$ we conclude,

$$ \nabla_{\beta} (S_{\beta \gamma} \xi_{\gamma}^k - \gamma_{\beta} \in_k) = (\nabla_{\beta} S_{\beta \gamma}) \xi_{\gamma}^k. $$

As a consequence of this theorem when field equations $\nabla_{\beta} S_{\beta \gamma} = 0$ hold, and vectors $\xi_{\gamma}^k$, $\gamma_{\beta}$ exist as stipulated above, then identities

$$ \nabla_{\beta} (S_{\beta \gamma} \xi_{\gamma}^k - \gamma_{\beta} \in_k) = 0 $$

exist. These are assumed to represent conservation laws when interpreted physically.

**Theorem 2.** Assume linear operators $\Lambda_{\alpha}^\gamma$ and $\Delta_{\alpha}^\gamma$ exist, such that for
arbitrary functions, \( \Delta \phi^1(x'), \ldots, \Delta \phi^r(x') \), we have

\[
\Delta W = \int_\Omega dx \sqrt{g} \nabla_\alpha (\Lambda^\alpha_k[\Delta \phi^k]) \quad \text{when} \quad \Delta x^\alpha = \Delta^\alpha_k[\Delta \phi^k].
\]

Then \( r \) identities exist of the form

\[
\Lambda^\alpha_k[\nabla_\beta s^\beta_{\nu \alpha}] = 0. \tag{3.5.3}
\]

\textbf{Proof.} Substitution of the given forms for \( \Delta W \) and \( \Delta x^\alpha \) in (3.3.14) gives,

\[
\int_\Omega dx \sqrt{g} \nabla_\alpha (\Lambda^\alpha_k[\Delta \phi^k]) = \int_\Omega dx \sqrt{g} \left\{ \nabla_\beta (s^\beta_{\nu \alpha} \Lambda^\nu_k[\Delta \phi^k]) - [\nabla_\beta s^\beta_{\nu \alpha}] \Lambda^\nu_k[\Delta \phi^k] \right\}.
\]

The integral of each of the divergences is equivalent to an integral over the boundary of \( \Omega \) which vanishes, because \( \Delta \phi^k = 0 \) on the boundary, and operators \( \Lambda^\alpha_k \) and \( \Lambda^\nu_k \) are linear.

Hence,

\[
\int_\Omega dx \sqrt{g} (\nabla_\beta s^\beta_{\nu \alpha}) \Lambda^\nu_k[\Delta \phi^k] = 0.
\]

Then,

\[
\int_\Omega dx \sqrt{g} \Lambda^\nu_k[\nabla_\beta s^\beta_{\nu \alpha}] \Delta \phi^k = 0.
\]

Because functions \( \Delta \phi^k \) are arbitrary, we conclude

\[
\Lambda^\nu_k[\nabla_\beta s^\beta_{\nu \alpha}] = 0.
\]
CHAPTER IV

CONFORMAL INVARIANCE

In this chapter point transformations in a manifold will be considered, and a definition of conformal point transformations will be given. It will be shown that a very general class of tensor fields is conformally invariant, and this class includes the electromagnetic field in vacuum. The conservation laws which result from such invariance will be stated.

1. Conformal Transformations

A manifold $X$ with metric tensor $g_{\alpha\beta}$ is assumed. The manifold is assumed to be coordinatized, so that each point $x$ may be identified with an $n$-tuple $(x^1, \ldots, x^n)$ of real numbers $x$. (Greek indices $\alpha$, $\beta, \gamma, \ldots$ have the range 1 to $n$.) No coordinate transformations will be considered, so the $n$-tuple $(x^\alpha)$ assigned to point $x \in X$ remains fixed in the following. Transformations of the form

\[ \bar{x}^\alpha = x^\alpha + \Delta x^\alpha \]  \hspace{1cm} (4.1.1)

will be studied. Transformation (4.1.1) represents a point transformation of point $x$ into point $\bar{x}$, where $n$-tuple $(\bar{x}^\alpha)$ denotes the coordinates of the new point $x \in X$.

When transformation (4.1.1) is introduced it will always be assumed that

\[ \delta g_{\alpha\beta} = 0 ; \]

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that is, no local variation of the metric tensor will ever be used in the following discussions. The metric tensor components \( g_{\alpha \beta}(x) \) may vary with point \( x \) in the manifold, however, so it is not assumed that \( g_{\alpha \beta}(\bar{x}) = g_{\alpha \beta}(x) \). In general we shall have,
\[
g_{\alpha \beta}(\bar{x}) = g_{\alpha \beta}(x + \Delta x) = g_{\alpha \beta}(x) + \partial_{\lambda} g_{\alpha \beta}(x) \Delta x^{\lambda}. \tag{4.1.2}
\]

Now, let \( x + dx \) denote a point with coordinates \( (x^\alpha + dx^\alpha) \). Transformation (4.1.1) maps \( x + dx \rightarrow \bar{x} + d\bar{x} \), where \( \bar{x} + d\bar{x} \) is a point with some coordinates \( \bar{x}^\alpha + d\bar{x}^\alpha \).

**Definition 1.** The point transformation \( \bar{x}^\alpha = x^\alpha + \Delta x^\alpha \) is conformal if and only if for arbitrary \( dx^\alpha \) we have,
\[
g_{\alpha \beta}(\bar{x}) dx^\alpha dx^\beta = (1 + \Delta \phi(x)) g_{\alpha \beta}(x) dx^\alpha dx^\beta, \tag{4.1.3}
\]
and \( \Delta \phi \) is some infinitesimal scalar valued function with \( (1 + \Delta \phi(x)) > 0 \).

**Remark.** Since \( dx^{\alpha} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{\mu}} dx^{\mu} \), then (4.1.3) is equivalent to
\[
g_{\alpha \beta}(\bar{x}) \frac{\partial \bar{x}^{\alpha}}{\partial x^{\mu}} \frac{\partial \bar{x}^{\beta}}{\partial x^{\nu}} dx^{\mu} dx^{\nu} = (1 + \Delta \phi(x)) g_{\mu \nu}(x) dx^{\mu} dx^{\nu}.
\]
Hence, by arbitrariness of the \( dx^{\mu} \) we conclude \( \bar{x}^\alpha = x^\alpha + \Delta x^\alpha \) is conformal if and only if
\[
g_{\alpha \beta}(\bar{x}) \frac{\partial \bar{x}^{\alpha}}{\partial x^{\mu}} \frac{\partial \bar{x}^{\beta}}{\partial x^{\nu}} = (1 + \Delta \phi(x)) g_{\mu \nu}(x). \tag{4.1.4}
\]

Now we shall convert (4.1.4) to infinitesimal form, and derive a condition on \( \Delta x^\alpha \) which is necessary and sufficient for conformality of (4.1.1).

Into (4.1.4) we substitute
\[
g_{\alpha \beta}(\bar{x}) = g_{\alpha \beta}(x) + \partial_{\lambda} g_{\alpha \beta}(x) \Delta x^{\lambda} \quad \text{and} \quad \frac{\partial \bar{x}^{\alpha}}{\partial x^{\mu}} = \delta_{\alpha}^{\mu} + \partial_{\mu} \Delta x^{\alpha}.
\]
Then,
\[
(\delta_{\alpha}^{\lambda} + \partial_{\lambda} \Delta x^{\lambda})(\delta_{\mu}^{\nu} + \partial_{\mu} \Delta x^{\nu})(\delta_{\nu}^{\beta} + \partial_{\nu} \Delta x^{\beta}) = (1 + \Delta \phi) g_{\mu \nu}.
\]
So,
\[
g_{\mu \nu} + g_{\mu \beta} \partial_{\nu} \Delta x^{\beta} + g_{\nu \beta} \partial_{\mu} \Delta x^{\beta} + \partial_{\lambda} g_{\mu \nu} \Delta x^{\lambda} = g_{\mu \nu} + \Delta \phi g_{\mu \nu}.
\]
Hence,
\[
g_{\mu \beta} \partial_{\nu} \Delta x^{\beta} + g_{\nu \beta} \partial_{\mu} \Delta x^{\beta} + \partial_{\lambda} g_{\mu \nu} \Delta x^{\lambda} = \Delta \phi g_{\mu \nu}. \tag{4.1.5}
\]
It is preferable to have this condition expressed in terms of covariant derivatives $\nabla_\alpha \Delta x^\beta$. Since $\nabla_\alpha \Delta x^\beta = \partial_\alpha \Delta x^\beta + \Gamma^\beta_{\alpha\rho} \Delta x^\rho$, then

$$\partial_\alpha \Delta x^\beta = \nabla_\alpha \Delta x^\beta - \Gamma^\beta_{\alpha\rho} \Delta x^\rho.$$  

Hence, (4.1.5) becomes

$$g_{\mu\beta} \nabla_\nu \Delta x^\mu + g_{\alpha\nu} \nabla_\mu \Delta x^\nu - \Gamma^\beta_{\nu\rho} \Delta x^\rho g_{\mu\beta} - \Gamma^\alpha_{\mu\rho} \Delta x^\rho g_{\alpha\nu} + \partial_\rho g_{\mu\nu} \Delta x^\rho = \Delta \phi g_{\mu\nu}.$$  

Then,

$$\nabla_\nu \Delta x^\mu + \nabla_\mu \Delta x^\nu = \Delta \phi g_{\mu\nu}.$$  

But,

$$\partial_\rho g_{\mu\nu} - g_{\mu\rho} \Gamma^\rho_{\nu\mu} - g_{\nu\rho} \Gamma^\rho_{\mu\nu} = \nabla_\rho g_{\mu\nu} = 0.$$  

So (4.1.5) is equivalent to

$$\nabla_\nu \Delta x^\mu + \nabla_\mu \Delta x^\nu = \Delta \phi g_{\mu\nu}.$$  

Equation (4.1.6) gives a necessary and sufficient condition that (4.1.1) be conformal in an arbitrary Riemannian space $X$.

**Definition 2.** The point transformation $\bar{x}^\alpha = x^\alpha + \Delta x^\alpha$ is called a **motion** if and only if,

$$g_{\alpha\beta}(\bar{x}) d\bar{x}^\alpha d\bar{x}^\beta = g_{\beta\alpha}(x) dx^\alpha dx^\beta.$$  

Motions preserve distance locally, and form a special class of conformal transformations for which $\Delta \phi = 0$. Hence, by (4.1.6)

$$\bar{x}^\alpha = x^\alpha + \Delta x^\alpha$$  

is a motion if and only if

$$\nabla_\mu \Delta x^\nu + \nabla_\nu \Delta x^\mu = 0.$$  

(4.1.7)

A vector $\xi^\alpha$ is called a Killing vector if and only if $\nabla_\alpha \xi^\gamma + \nabla_\gamma \xi^\alpha = 0$.

Hence (4.1.7) states that $\bar{x}^\alpha = x^\alpha + \Delta x^\alpha$ is a motion if and only if $\Delta x^\alpha$ is a Killing vector.

2. **Conformal Invariance of Electromagnetic Fields**

An action integral,

$$W = \int_{\Omega} dx \sqrt{g} L = \int_{\Omega} dx \sqrt{g} (\frac{1}{2} F_{\mu\nu} F^{\mu\nu})$$  

is given. When point transformation (4.1.1) is given we shall assume the variation of tensor field $F_{\mu\nu}$ is induced by dragging along the field,
as described in Chapter III, section 3. This produces a variation \( \Delta W \) defined by (3.3.9). Variation \( \Delta W \) is given explicitly by (3.3.12), where we found
\[
\Delta W = \int \delta \sqrt{g} \, \gamma \, \nabla_{\beta} \Delta x^{\gamma} \tag{4.2.2}
\]
for tensor \( S^{\beta}_{\gamma} = L \delta^{\beta}_{\gamma} - \frac{\partial L}{\partial A^{\alpha}} - \frac{\partial L}{\partial F_{\alpha \beta}} - \frac{\partial L}{\partial F_{\beta \alpha}} \). \tag{4.2.3}

For the time being leave \( F_{\mu \nu} \) arbitrary; i.e., we do not need to assume \( F_{\mu \nu} = -F_{\nu \mu} \), or \( F_{\mu \nu} \) is the curl of a vector \( A_{\alpha} \).

As stated earlier in (3.4.3), for \( L \) as in (4.2.1) we have
\[
\frac{\partial L}{\partial F_{\alpha \beta}} = -F^{\alpha \beta}. \quad \text{Since} \quad \frac{\partial L}{\partial F_{\alpha \beta}} = 0 \, , \text{we have from (4.2.3)}
\]
\[
S^{\beta}_{\gamma} = L \delta^{\beta}_{\gamma} + \frac{1}{2} (F^{\alpha \beta} F_{\alpha \gamma} + F^{\beta \gamma} F_{\alpha \alpha}). \tag{4.2.4}
\]
Then, \( S^{\alpha \beta} = \delta^{\alpha \gamma} S^{\beta}_{\gamma} = L g^{\alpha \beta} + \frac{1}{2} g^{\alpha \gamma} (F^{\beta \gamma} F_{\alpha \nu} + F^{\gamma \nu} F_{\alpha \beta}) \). \tag{4.2.5}

**Remark 1.** \( S^{\alpha \beta} \) is symmetric; i.e., \( S^{\alpha \beta} = S^{\beta \alpha} \).

**Proof.** \( L g^{\alpha \beta} \) is symmetric.

Let \( K^{\alpha \beta} = g^{\alpha \gamma} S^{\beta \gamma} = g^{\alpha \gamma} L g^{\beta \delta} F_{\gamma \delta} \).

Thus, \( S^{\alpha \beta} = L g^{\alpha \beta} + \frac{1}{2} K^{\alpha \beta} \).

Then, \( K^{\alpha \beta} = g^{\alpha \gamma} g^{\beta \lambda} g_{\gamma \delta} F^{\lambda \sigma} F^{\sigma \delta} + g^{\alpha \gamma} g^{\beta \lambda} g_{\gamma \delta} F^{\lambda \delta} F^{\sigma \delta} \).

Using \( g^{\rho \gamma} g_{\nu \sigma} = \delta^{\rho \nu} \) and \( g^{\rho \gamma} g_{\nu \lambda} = \delta^{\rho \lambda} \) gives,
\[
K^{\alpha \beta} = \delta^{\rho \gamma} g_{\gamma \lambda} F^{\lambda \sigma} F^{\rho \sigma} + \delta^{\rho \lambda} g_{\rho \delta} F^{\beta \delta} F^{\lambda \delta}
\]
\[
= \delta_{\rho \lambda} F^{\beta \rho} F^{\lambda \sigma} + g_{\rho \delta} F^{\beta \delta} F^{\lambda \delta} \tag{4.2.6}
\]
Then, \( K^{\alpha \beta} = \delta_{\rho \lambda} F^{\beta \rho} F^{\lambda \sigma} + g_{\rho \delta} F^{\beta \delta} F^{\lambda \delta} \).

Interchanging summation indices \( \alpha \) and \( \lambda \) gives
\[
K^{\alpha \beta} = g^{\rho \lambda} F^{\alpha \beta} F^{\rho \delta} + g^{\rho \delta} F^{\alpha \beta} F^{\rho \lambda} \cdot \tag{4.2.6}
\]
Comparison with (4.2.6) shows \( K^{\alpha \beta} = K^{\beta \alpha} \) because \( g_{\alpha \lambda} = g_{\lambda \alpha} \).

**Remark 2.** \( S^{\alpha \beta} = 0 \) . (Assuming \( n = 4 \))

**Proof.** \( S^{\alpha \beta} = L \delta^{\alpha \beta} + \frac{1}{2} (F^{\alpha \beta} F_{\alpha \beta} + F^{\beta \alpha} F_{\beta \alpha}) \) by (4.2.4).

Since \( \delta^{\alpha \beta} = 4 \) and \( L = \frac{1}{4} F_{\alpha \beta} F^{\alpha \beta} \),
then \[ S^\beta_\beta = -F^\alpha_\alpha F^{\alpha}_\beta + F^\alpha_\beta F^{\alpha}_\beta = 0. \]

Now observe that (4.2.2) can be written
\[
\Delta W = \int \frac{dV}{\sqrt{g}} S^\alpha_\gamma S^\gamma_\beta \Delta x^\alpha = \frac{1}{2} \int \frac{dV}{\sqrt{g}} (S^\alpha_\gamma \Delta x^\gamma + S^\gamma_\beta \Delta x^\beta).
\]
Since \[ S^\alpha_\gamma = S^\gamma_\alpha, \]
then \[ \Delta W = \frac{1}{2} \int \frac{dV}{\sqrt{g}} S^\gamma_\beta (\nabla^\gamma \Delta x^\beta + \nabla^\beta \Delta x^\gamma). \quad (4.2.7) \]

**Theorem 1.** If \( \bar{x}^\gamma = x^\gamma + \Delta x^\gamma \) is a conformal point transformation, then \( \Delta W \) for action integral (4.2.1), as defined by (3.3.9), is zero.

**Proof.** If \( \bar{x}^\gamma = x^\gamma + \Delta x^\gamma \) is conformal, then by (4.1.6)
\[
\nabla^\gamma \Delta x^\gamma = \Delta \phi g^\gamma_\beta
\]
for some scalar \( \Delta \phi \). Substitution in (4.2.7) gives
\[
\Delta W = \frac{1}{2} \int \frac{dV}{\sqrt{g}} (S^\gamma_\beta g^\gamma_\beta) \Delta \phi.
\]
But, \[ S^\gamma_\beta g^\gamma_\beta = S^\gamma_\gamma, \]
and as shown above \( S^\gamma_\gamma = 0. \) Hence, \( \Delta W = 0. \) as claimed.

Because of the preceding theorem, we say action integral (4.2.1) is conformally invariant. It should be noted that conformal invariance has been proven for an arbitrary tensor field \( F^\gamma_\beta \); i.e., integral (4.2.1) is conformally invariant for every field \( F^\gamma_\beta \), and we particularly do not need assumptions such as \( F^\gamma_\beta = -F^\beta_\gamma \) or \( F^\gamma_\beta = \nabla^\gamma A_\beta - \nabla^\beta A_\gamma \). It should also be noted that no special metric tensor \( g^\gamma_\beta \) has been assumed, nor did we need any special kind of coordinate system. Hence, conformal invariance of (4.2.1) holds in every Riemannian space with \( n = 4. \)

It is seen that tensor \( S^\gamma_\beta \) in (4.2.2) determines the variation \( \Delta W \) when point transformation \( \bar{x}^\gamma = x^\gamma + \Delta x^\gamma \) is applied, and conformal invariance is a mathematical consequence of two facts:

1). \( S^\gamma_\beta = S^\gamma_\gamma \)

2). \( S^\gamma_\beta = 0 \). 

Both 1) and 2) are immediate consequences of the two assumptions,

\[ L = -\frac{1}{4} F^\gamma_\beta F^{\gamma}_\beta \]

and

\[ \delta g^\gamma_\beta = 0, \]
in the preceding calculations. If another choice for $L$ could be found such that $S^\beta_\nu$ defined by (4.2.3) were symmetric with $S^\beta_\nu = 0$, then the corresponding action integral would be conformally invariant. Apparently, (4.2.1) is the only action integral of the form (3.3.1) which is conformally invariant. It is easily verified that integrals of the form,

$$W = \int dx \sqrt{g}(A_\chi A^\lambda) ,$$  \hspace{1cm} (4.2.8)

are not conformally invariant. In fact we have $L = A^\lambda A^\alpha$, and from (4.2.3)

$$S^\beta_\nu = L \delta^\beta_\nu - \frac{\partial L}{\partial A_\beta} A_\nu = L \delta^\beta_\nu - 2A^\beta A_\nu .$$

Then, $S^\beta_\alpha = Lg^{\beta \alpha} - 2A^\beta A^\alpha$.

$S^\beta_\alpha = S^\alpha_\beta$ in this case, but

$$S^\beta_\alpha = L \delta^\beta_\alpha - 2A^\beta A^\alpha = L \delta^\beta_\alpha - 2A^\beta A^\alpha = 2A^\beta A^\alpha \neq 0 .$$

As shown in the preceding, $S^\beta_\beta = 0$ is needed for conformal invariance. However, $S^\alpha_\beta = S^\beta_\alpha$ implies that (4.2.8) is invariant under the group of motions.

3. Application of Noether Theorem

$\Delta W = 0$ for $W = \int dx \sqrt{g}(-\frac{1}{2}F_{\mu \nu}F^{\mu \nu})$, when $x^{\alpha} = x^{\alpha} + \Delta x^{\alpha}$ is a conformal transformation. Now, if an $r$-parameter group of conformal transformations,

$$\Delta x^{\alpha} = x^{\alpha} + \xi^{\alpha}(x) \in k , \hspace{1cm} (4.3.1)$$

exists, we have $r$ identities,

$$\nabla_\beta (S^\beta_\nu \xi^{\nu}_k) = (\nabla_\beta S^\beta_\nu) \xi^{\nu}_k . \hspace{1cm} (4.3.2)$$

By (4.2.4),

$$S^\beta_\nu = L \delta^\beta_\nu + \frac{1}{2}(F^{\beta \mu} F_{\mu \nu} + F_{\mu \nu} F^{\beta \mu}) .$$

Assuming field equations $\nabla_\beta S^\beta_\nu = 0$ hold, we obtain $r$ conservation laws,

$$\nabla_\beta (S^\beta_\nu \xi^{\nu}_k) = 0 . \hspace{1cm} (4.3.3)$$
The exact form of these conservation laws depends upon the nature of manifold $X$. It has been shown by Yano and Bochner (7) that in some manifolds no conformal transformations exist. If $X$ is Minkowski space, then we have the 15 parameter group of conformal transformations given in section 4 of Chapter I. Then we obtain from (4.3.3) the 15 conservation laws discussed in Chapter I.

Although the form of these conservation laws is the same as given earlier, there is something new which should be noted. In the present treatment the field equations

$$\nabla_\beta S^{\beta}_{\nu} = 0 \quad (4.3.4)$$

are assumed, and the 15 identities are statements about a field $F_{\alpha\beta}$ defined by these equations and (4.3.2). We have not assumed Maxwell's Equations as field equations. If Maxwell's equations are satisfied by $F_{\alpha\beta}$, and $F_{\alpha\beta} = -F_{\beta\alpha}$, then (4.3.4) is satisfied. So, the 15 identities do apply to the electromagnetic field. However, (4.3.4) may be true in some cases when Maxwell's equations do not hold. Hence, all conservation laws derived from (4.3.3) apply to a larger class of fields which includes the electromagnetic field, merely as a special case. We shall consider the nature of this inclusion in Chapter V.
CHAPTER V

MODELS FOR ELECTROMAGNETIC FIELDS

It is usually assumed when electromagnetic fields are mentioned that we are considering a field which satisfies Maxwell's equations. However, there exist several mathematical models for electromagnetic fields, so it is worthwhile to review and compare these models. From this discussion one can gain a better understanding of the previous results concerning fields which satisfy the field equations,

$$\nabla \alpha S^\alpha = 0,$$

proposed in (3.4.5) of Chapter III.

1. The $\vec{E}$, $\vec{H}$, Model

In most introductory discussions of electromagnetism the electromagnetic field is defined to be a combination of an electric field and a magnetic field, represented by vectors $\vec{E}$ and $\vec{H}$, respectively. These vectors satisfy Maxwell's equations throughout some region, subject to certain boundary conditions. Maxwell's equations are given in the form,

\begin{align*}
(Ia) \quad & \nabla \times \vec{H} = \frac{\partial \vec{E}}{\partial t} + \vec{J} \\
(IIa) \quad & \nabla \times \vec{E} = -\frac{\partial \vec{H}}{\partial t} \\
(Ib) \quad & \nabla \cdot \vec{E} = \rho \\
(IIb) \quad & \nabla \cdot \vec{H} = 0.
\end{align*}

(5.1.1)

In these equations $\rho$ represents a charge density in the given region, and $\vec{J}$ represents the electrical current present. We are concerned only with the case $\rho = 0$ and $\vec{J} = \vec{0}$, which we call the electromagnetic field in vacuum.
From (IIIb) we conclude \( \vec{H} = \nabla \times \vec{A} \) for some vector potential \( \vec{A} \). Then, because of (IIa) we argue,

\[
\nabla \times \vec{E} + \frac{\partial \vec{H}}{\partial t} = 0 \quad \text{and} \quad \vec{H} = \nabla \times \vec{A},
\]

so

\[
\nabla \times \vec{E} + \nabla \times \frac{\partial \vec{A}}{\partial t} = 0.
\]

Then,

\[
\nabla \times (\vec{E} + \frac{\partial \vec{A}}{\partial t}) = 0.
\]

It follows that \( \vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla \phi \)

for some scalar potential \( \phi \). Hence, we have \( \vec{E} \) and \( \vec{H} \) expressed in terms of scalar potential \( \phi \) and vector potential \( \vec{A} \) by,

\[
\vec{H} = \nabla \times \vec{A} \quad \text{and} \quad \vec{E} = -\nabla \phi - \frac{\partial \vec{A}}{\partial t}.
\]

(5.1.2)

Substitution of (5.1.2) into (Ia) and (Ib) gives equations which determine \( \phi \) and \( \vec{A} \). Assuming \( \vec{J} = 0 \), the result from (Ia) is

\[
\nabla \times (\nabla \times \vec{A}) = \frac{\partial}{\partial t} (-\nabla \phi - \frac{\partial \vec{A}}{\partial t}).
\]

Since \( \nabla \times (\nabla \times \vec{A}) = \nabla (\nabla \cdot \vec{A}) - \nabla ^2 \vec{A} \),

then \( \nabla (\nabla \cdot \vec{A}) - \nabla ^2 \vec{A} = -\frac{\partial \vec{A}}{\partial t} - \nabla \left( \frac{\partial \phi}{\partial t} \right) \).

So,

\[
\nabla ^2 \vec{A} - \frac{\partial ^2 \vec{A}}{\partial t ^2} = \nabla \left( \nabla \cdot \vec{A} + \frac{\partial \phi}{\partial t} \right).
\]

(5.1.3)

From (Ib) we obtain \( \nabla \cdot (-\nabla \phi - \frac{\partial \vec{A}}{\partial t}) = 0 \).

Hence,

\[
\nabla ^2 \phi = -\frac{\partial}{\partial t} (\nabla \cdot \vec{A})
\]

(5.1.4)

In order to simplify (5.1.3) and (5.1.4) the Lorentz Gauge condition,

\[
\nabla \cdot \vec{A} + \frac{\partial \phi}{\partial t} = 0,
\]

(5.1.5)

is sometimes imposed. Then (5.1.3) reduces to

\[
\nabla ^2 \vec{A} = \frac{\partial ^2 \vec{A}}{\partial t ^2}.
\]

(5.1.6)

Since in (5.1.4) we now have from (5.1.5) \( \nabla \cdot \vec{A} = -\frac{\partial \phi}{\partial t} \), then

\[
\nabla ^2 \phi = \frac{\partial ^2 \phi}{\partial t ^2}
\]

(5.1.7)
Hence, after imposing the Lorentz Gauge condition $\phi$ and the components of $\vec{A}$ must be solutions of the wave equation.

2. The Four-Vector Model

In order to produce a four-dimensional covariant model of the electromagnetic field, many writers define the field to be a vector field $A_\mu$ satisfying field equations,

$$g^{\mu\nu} \partial_\mu (\partial_\nu A_\lambda - \partial_\lambda A_\nu) = 0 . \tag{5.2.1}$$

A Minkowski space is assumed with $(x^0, x^1, x^2, x^3) = (t, x, y, z)$, and $g_{00} = -1, g_{11} = g_{22} = g_{33} = 1$. If $\alpha \neq \beta$, then $g_{\alpha\beta} = 0$.

Equations (5.2.1) simplify to

$$g^{\mu\nu} \partial_\mu \partial_\nu A^\lambda = \partial_\nu (\partial_\mu A^\mu) = 0 . \quad (\nu = 1, 2, 3) \tag{5.2.2}$$

$$g^{\mu\nu} \partial_\mu \partial_0 A_\lambda + \partial_0 (\partial_\nu A^\nu) = 0 . \tag{5.2.2'}$$

In order to justify these equations as field equations, we define $\phi$, $\vec{A}$, $\vec{E}$, and $\vec{H}$ as follows:

$$A_0 = -\phi, (A_1, A_2, A_3) = \vec{A}; \tag{5.2.3}$$

$$\vec{E} = -\nabla \phi - \frac{\partial \vec{A}}{\partial t}; \tag{5.2.4}$$

$$\vec{H} = \nabla \times \vec{A}. \tag{5.2.5}$$

Note: $A^0 = \phi$, $(A_1, A_2, A_3) = \vec{A}$ with the metric above.

These definitions are motivated by the previous discussion of potentials $\phi$ and $\vec{A}$. For this reason $A_\mu$ is frequently called the four-potential for the associated $\vec{E}$, $\vec{H}$ field.

Because of (5.2.4) and (5.2.5), Maxwell equations IIa and IIb are satisfied immediately. We shall show (5.2.1) implies Ia and Ib. Since $\nu = k$ in (5.2.2) we have

$$g^{\mu\nu} \partial_\mu \partial_\nu A^k - \partial_k (\partial_\mu A^\mu) = 0 .$$

Since $g^{\mu\nu} \partial_\mu \partial_\nu A^k = \nabla^2 A^k - \frac{\partial^2 A^k}{\partial t^2}$ and $\partial_\mu A^\mu = \nabla \cdot \vec{A} + \frac{\partial \phi}{\partial t}$,
then \[ \nabla^2 \vec{A} - \frac{\partial^2 \vec{A}}{\partial t^2} = \nabla \left( \nabla \cdot \vec{A} + \frac{\partial \phi}{\partial t} \right). \]

So, \[ \nabla \left( \nabla \cdot \vec{A} \right) - \nabla^2 \vec{A} = - \frac{\partial^2 \vec{A}}{\partial t^2} - \nabla \left( \frac{\partial \phi}{\partial t} \right). \]

But, \[ \nabla \left( \nabla \cdot \vec{A} \right) - \nabla^2 \vec{A} = \nabla \times (\nabla \times \vec{A}), \]

so \[ \nabla \times (\nabla \times \vec{A}) = \frac{\partial}{\partial t} \left( - \nabla \phi - \frac{\partial \vec{A}}{\partial t} \right). \]

Using (5.2.4) and (5.2.5) we find \[ \nabla \times \vec{A} = \frac{\partial E}{\partial t}. \]

Hence, (Ia) is established.

Equation (5.2.2)' states

\[ g^{\mu \nu} \partial_\mu \partial_\nu \phi + \frac{\partial}{\partial t} (\partial_\mu A^\mu) = 0. \]

This is equivalent to

\[ \nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2} = - \frac{\partial}{\partial t} \left( \nabla \cdot \vec{A} + \frac{\partial \phi}{\partial t} \right). \quad (5.2.6) \]

From (5.2.4), \[ \nabla \cdot E = - \nabla^2 \phi - \frac{\partial}{\partial t} \left( \nabla \cdot \vec{A} \right). \]

So, \[ \nabla^2 \phi = - \left( \nabla \cdot \vec{E} \right) - \frac{\partial}{\partial t} \left( \nabla \cdot \vec{A} \right). \]

Then, \[ \nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2} = - \left( \nabla \cdot \vec{E} \right) - \frac{\partial}{\partial t} \left( \nabla \cdot \vec{A} + \frac{\partial \phi}{\partial t} \right). \quad (5.2.7) \]

Comparison of (5.2.6) and (5.2.7) shows \[ \nabla \cdot \vec{E} = 0. \]

Hence, (Ib) is established.

Our purpose is to determine to what extent the various mathematical models for the electromagnetic field in vacuum are equivalent. The

four-vector model does produce Maxwell's equations, when vectors \( \vec{E} \) and \( \vec{H} \) are defined by (5.2.4) and (5.2.5). There is a modification of this model which also produces Maxwell's equations. The modification uses a different definition of vectors \( \vec{E} \) and \( \vec{H} \) and a tensor field \( F_{\alpha \beta} \).

Again, assume \( A_\alpha \) is a vector for which

\[ g^{\mu \nu} g^{\gamma \delta} \partial_\mu \left( \partial_\alpha A_\beta - \partial_\beta A_\alpha \right) = 0. \]

Define tensor \( F_{\alpha \beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \).
Then the preceding field equations for $A_\alpha$ can be written as
\[
\partial_\mu F^{\mu \nu} = 0. \tag{5.2.8}
\]

From $F_{\alpha \beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ we have identity
\[
\epsilon^{\alpha \beta \lambda \nu} \partial_\beta F_{\lambda \nu} = 0. \tag{5.2.9}
\]

Define vectors $\vec{E}$ and $\vec{H}$ by
\[
E_k = F_{k0} \quad \text{and} \quad H_k = *F_{0k}. \quad (k = 1, 2, 3) \tag{5.2.10}
\]

*\(F^{\alpha \beta}\) is the dual of \(F_{\alpha \beta}\) defined by
\[
*F^{\alpha \beta} = \frac{1}{2} \epsilon^{\alpha \beta \gamma \rho} F_{\gamma \rho}. \tag{5.2.11}
\]

From (5.2.10) \(F_{10} = E_1, F_{20} = E_2, \) and \(F_{30} = E_3.\)

Also, \(H_k = \frac{1}{2} \epsilon^{0 \kappa \lambda \rho} F_{\kappa \lambda \rho}.\)

Short calculations show \(H_1 = F_{23}, H_2 = -F_{13}, \) and \(H_3 = F_{12}.\)

We can now express \(F_{\alpha \beta}\) in terms of components of $\vec{E}$ and $\vec{H}$. We have
\[
(F_{\alpha \beta}) = \begin{vmatrix}
0 & -E_1 & -E_2 & -E_3 \\
E_1 & 0 & H_3 & -H_2 \\
E_2 & -H_3 & 0 & H_1 \\
E_3 & H_2 & -H_1 & 0
\end{vmatrix}. \tag{5.2.12}
\]

From \(F^{\alpha \beta} = g^{\alpha \mu} g^{\beta \nu} F_{\mu \nu},\) it follows that
\[
(F^{\alpha \beta}) = \begin{vmatrix}
0 & E_1 & E_2 & E_3 \\
-E_1 & 0 & H_3 & -H_2 \\
-E_2 & -H_3 & 0 & H_1 \\
-E_3 & H_2 & -H_1 & 0
\end{vmatrix}. \tag{5.2.13}
\]

From (5.2.8) and (5.2.13) it follows now that
\[
(Ib) \ \nabla \cdot \vec{E} = 0 \quad \text{and} \quad (Ia) \ \nabla \times \vec{H} = \frac{\partial \vec{E}}{\partial t}. \]

Identity (5.2.9) implies (IIa) and (IIb), as can be easily verified.
3. The Tensor Model

The previous discussion suggests that we could define an electromagnetic field to be a tensor, $F_{\alpha\beta}$, which is anti-symmetric, and satisfies

$$ (I)' \quad \partial_\alpha F^{\alpha\beta} = 0 \quad , $$

$$ (II)' \quad \epsilon^{\alpha\beta\gamma\delta} \partial_\gamma F_{\gamma\delta} = 0 \quad . $$

We could then define $\vec{E}$ and $\vec{H}$ by (5.2.10) as before, or by (5.2.13). Then, equations Ia, Ib, IIIa, IIIb would follow for vectors $\vec{E}$ and $\vec{H}$. Hence, we need not introduce vector potential $A_\alpha$ in this case. We shall adopt a generalization of this viewpoint and define an electromagnetic field to be an anti-symmetric tensor $F_{\alpha\beta}$ satisfying equations (I) and (II) above in the covariant form,

I. $\nabla_\alpha F^{\alpha\beta} = 0$

II. $\epsilon^{\alpha\beta\lambda\nu} \nabla_\lambda F_{\lambda\nu} = 0 \quad .$

It should be noted that the definition of vectors $\vec{E}$ and $\vec{H}$ via (5.2.13) is meaningful only in Minkowski space, while in more general spaces we deal only with anti-symmetric tensor $F_{\alpha\beta}$.

In Chapter III, section 4, field equations,

$$ \nabla_\rho S_{\mu\nu} = \nabla_\rho ( -\frac{1}{4} \delta^\Phi_{\mu\nu} F^{\lambda\rho} F_{\lambda\nu} + F^{\alpha\beta} F_{\alpha\nu} ) = 0 \quad , $$

were introduced. We shall now show how these equations are related to I and II. Equation (5.3.1) states

$$ -\frac{1}{4} \nabla_\nu ( F_{\lambda\rho} F_{\alpha\nu} ) + F^{\lambda\rho} \nabla_\rho F_{\lambda\nu} + F_{\alpha\nu} \nabla_\nu F^{\alpha\beta} = 0 \quad . $$

Then,

$$ -\frac{1}{4} F^{\lambda\rho} \nabla_\nu F_{\lambda\rho} + \frac{1}{2} ( F_{\lambda\rho} \nabla_\nu F_{\lambda\nu} + F_{\rho\lambda} \nabla_\lambda F_{\rho\nu} ) + F_{\alpha\nu} \nabla_\nu F^{\alpha\beta} = 0 \quad . $$

So,

$$ -\frac{1}{4} F^{\lambda\rho} ( \nabla_\nu F_{\lambda\rho} + \nabla_\rho F_{\lambda\nu} + \nabla_\lambda F_{\rho\nu} ) + F_{\alpha\nu} \nabla_\nu F^{\alpha\beta} = 0 \quad . $$

(5.3.2)

Let, $\epsilon^{\alpha\beta\lambda\nu} \nabla_\lambda F_{\alpha\nu} = V^\alpha \quad . $
Then equation II above is $V^\alpha = 0$. Furthermore, since $\epsilon^{\alpha \beta \lambda \nu} = \epsilon^{\alpha \lambda \nu \beta} = \epsilon^{\nu \beta \lambda \alpha}$, then adding $V^\alpha = \epsilon^{\alpha \beta \lambda \nu} \nabla_\beta F_{\lambda \nu}$, $V^\beta = \epsilon^{\alpha \lambda \nu \beta} \nabla_\nu F_{\lambda \beta}$, and $V^\nu = \epsilon^{\nu \beta \lambda \alpha} \nabla_\lambda F_{\beta \alpha}$ gives

$$3V^\alpha = \epsilon^{\alpha \beta \lambda \nu} (\nabla_\beta F_{\lambda \nu} + \nabla_\lambda F_{\nu \beta} + \nabla_\nu F_{\beta \lambda}).$$

So, $3\epsilon_{\alpha \rho \sigma \tau} V^\alpha = \epsilon_{\alpha \rho \sigma \tau} \epsilon^{\alpha \beta \lambda \nu} (\nabla_\beta F_{\lambda \nu} + \nabla_\lambda F_{\nu \beta} + \nabla_\nu F_{\beta \lambda})$

$$= \delta^{\beta \lambda \nu} \delta_{\rho \sigma \tau} C_{\beta \lambda \nu}.$$ Tensor $C_{\beta \lambda \nu} = \nabla_\lambda F_{\beta \nu} + \nabla_\nu F_{\beta \lambda}$ is totally anti-symmetric. It follows that $\delta^{\beta \lambda \nu} \delta_{\rho \sigma \tau} C_{\beta \lambda \nu} = 6 \epsilon_{\rho \sigma \tau}.$ Since $3\epsilon_{\alpha \rho \sigma \tau} V^\alpha = 6 \epsilon_{\rho \sigma \tau},$

then $C_{\rho \sigma \tau} = \frac{1}{2} \epsilon_{\alpha \rho \sigma \tau} V^\alpha.$

Substitution in (5.3.2) gives

$$-\frac{1}{2} F^{\lambda \rho} (\frac{1}{2} \epsilon_{\alpha \nu \lambda \rho} V^\alpha) + F_{\lambda \nu} F^{\lambda \rho} = 0.$$ Then $\nabla_\beta S^\beta_\nu = 0$ is equivalent to

$$-\frac{1}{2} F^{\lambda \rho} \epsilon_{\alpha \nu \lambda \rho} (\epsilon^{\alpha \beta \sigma \tau} \nabla_\beta F_{\sigma \tau}) + F_{\lambda \nu} (\nabla_\beta F^{\lambda \rho}) = 0.$$

(5.3.3)

If $\epsilon^{\alpha \beta \sigma \tau} \nabla_\beta F_{\sigma \tau} = 0$ and $\nabla_\beta F^{\alpha \beta} = 0$, then by (5.3.3) $\nabla_\beta S^\beta_\nu = 0$, so field equations (5.3.1) are satisfied when Maxwell's equations hold. However, the converse may not be true.

If we assume at this point $F_{\alpha \beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha$ for some vector potential $A_\alpha$, then $\epsilon^{\alpha \beta \sigma \tau} \nabla_\beta F_{\sigma \tau} = 0$ follows automatically. Hence, (5.3.3) becomes

$$F_{\alpha \nu} \nabla_\beta F^{\alpha \beta} = 0.$$ (5.3.4)

Let $\nabla_\beta F^{\alpha \beta} = \Gamma^\alpha$, so (5.3.4) has the form $F_{\alpha \nu} \Gamma^\alpha = 0$. In Minkowski spaces the determinant of $(F_{\alpha \beta}) = (\mathbf{e} \cdot \mathbf{h})^2$, which we shall assume is not zero. Then, system $F_{\alpha \nu} \Gamma^\alpha = 0$ has only the trivial solution $\Gamma^\alpha = 0.$
Hence, from (5.3.4) it follows that $\nabla_\beta F^{\alpha\beta} = 0$. We can recover field equations I and II by assuming,

$$F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha \quad \text{and} \quad \det(F_{\alpha\beta}) \neq 0.$$  \hfill (5.3.5)

Hence, we have shown some electromagnetic fields belong to the class of tensor fields $F_{\alpha\beta}$ which satisfy field equations (5.3.1).

Now one may ask whether (5.3.1) implies equations I and II, without additional assumptions such as (5.3.5). We have shown in (5.3.3) that (5.3.1) implies that a certain combination of I and II must vanish. Apparently, this is all that we can conclude. As evidence of this we now express (5.3.1) in terms of vectors $\vec{E}$ and $\vec{H}$ in a Minkowski space, and note that the results do not imply Maxwell's equations (5.1.1).

4. $\vec{E}$, $\vec{H}$ Form of the New Field Equations

The relation between vectors $\vec{E}$ and $\vec{H}$ and the tensors $F_{\alpha\beta}$ and $F^{\alpha\beta}$ is given in (5.2.12) and (5.2.13). Again, we assume the metric and coordinate system as chosen in section 2. Latin indices $i, j, k, \ldots$ range from 1 to 3. From (5.2.12) and (5.2.13) we see

$$E^k = F^k_{\ k} \quad \text{and} \quad F^{\alpha k} = -F^k_{\ \alpha} = E^k_{\ k}$$  \hfill (5.4.1)

Also,

$$H^k = H^k_{\ k} = 1/2 e_{kmn} F^{mn} \quad \text{and} \quad F^{mn} = F_{mn}.$$  \hfill (5.4.2)

Then, $\delta e_{ij} H^k_{\ k} = \frac{1}{2} \delta e_{ij} F^{mn} = \frac{1}{2} \delta^{ij} F_{mn} = F^{ij}$.  \hfill (5.4.3)

In (5.3.2) we have shown

$$\partial_\beta S^{\alpha\nu} = -\frac{1}{2} F^{\lambda\rho} C_{\nu\lambda\rho} + F_{\alpha\nu} \nabla_\rho F^{\rho\nu},$$  \hfill (5.4.4)

where $C_{\nu\lambda\rho} = \partial_\nu F^{\lambda\rho} + \partial_\rho F^{\nu\lambda} + \partial_\lambda F^{\nu\rho}$. We shall express $C_{\nu\lambda\rho}$ in terms of $E^k$ and $H^k$. Recall that $C_{\nu\lambda\rho}$ is totally anti-symmetric. If $\alpha \neq \beta = 0$, then $C_{\alpha k\beta} = 0$.

Also,

$$C_{\alpha k j} = \partial_\nu F_{\ k j} + \partial_\nu F_{\ j o} + \partial_\nu F_{\ \alpha k} = \partial_\nu (e_{ikj} H^k) + \partial_\nu \vec{E}_j - \partial_\nu \vec{E}_k.$$  \hfill (5.4.5)
From (5.4.4),
\[ \partial \beta S_{\alpha}^{\beta} = -A_{t}^{k}C_{\alpha k j} + F_{\alpha p} \partial \alpha_{t}^{p k}. \]
Using (5.4.5),
\[ \partial \beta S_{\alpha}^{\beta} = -A_{t}^{k}(e^{k j}H)q_{j}k + H_{t}^{i} + H_{k}^{j} - \partial \beta_{j}E_{k} - E_{p} (\partial \beta_{t}^{p o p} + \partial \beta_{t}^{p k p}). \]
So, \[ \partial \beta S_{\alpha}^{\beta} = -A_{t}^{k}(e^{k j}H)q_{j}k + H_{t}^{i} + H_{k}^{j} - E_{p} \partial \beta_{p}E_{j} - E_{p} \partial \beta_{p}E_{j} - E_{p} \partial \beta_{p}E_{j} - e^{k j}H_{t}^{i}q_{j}k + \partial \beta_{j}E_{k}. \]

\[ \partial \beta S_{\alpha}^{\beta} = -A_{t}^{k}(e^{k j}H)q_{j}k + H_{t}^{i} + H_{k}^{j} - \partial \beta_{j}E_{k} - E_{p} \partial \beta_{p}E_{j} - E_{p} \partial \beta_{p}E_{j} - e^{k j}H_{t}^{i}q_{j}k + \partial \beta_{j}E_{k}. \]

\[ \partial \beta S_{\alpha}^{\beta} = -(H \cdot \partial \beta_{t}^{H} + \partial \beta_{t}^{H}) - H \cdot \text{curl} \vec{E} + E \cdot \text{curl} \vec{H}. \]

\[ \partial \beta S_{\alpha}^{\beta} = -(H \cdot \partial \beta_{t}^{H} + \partial \beta_{t}^{H}) - H \cdot \text{curl} \vec{E} + E \cdot \text{curl} \vec{H}. \]

Hence, \[ \partial \beta S_{\alpha}^{\beta} = 0 \text{ iff } \vec{H} \cdot (\text{curl} \vec{E} + \partial \beta_{t}^{H}) - \vec{E} \cdot (\text{curl} \vec{H} - \partial \beta_{t}^{H}) = 0. \] (5.4.6)

Similarly, we may show \[ \partial \beta S_{\alpha}^{\beta} = 0 \text{ is equivalent to } \frac{\partial S_{\alpha}^{\beta}}{\partial t} + \partial \beta_{t}^{1k} = 0, \] (5.4.7)

where \[ S = \vec{E} \cdot \vec{H} \] and \[ P_{ik} = \frac{1}{2} (E^{2} + H^{2}) \delta_{ik} - (E_{k}E_{k} + H_{k}H_{k}). \]

Now, \[ \partial \beta_{t}^{1k} = \frac{1}{2} \partial \beta_{t}^{1k} (E^{2} + H^{2}) - E_{k} \partial \beta_{t}^{1k} E - E_{i} \partial \beta_{t}^{1k} E - H_{k} \partial \beta_{t}^{1k} H - H_{i} \partial \beta_{t}^{1k} H. \]

This is equivalent to
\[ \partial \beta_{t}^{1k} = E^{i} (\partial \beta_{k}^{1k} - \partial \beta_{k}^{1k}) + H^{i} (\partial \beta_{k}^{1k} - \partial \beta_{k}^{1k}) - E^{k} \partial \beta_{t}^{1k} E - H^{k} \partial \beta_{t}^{1k} H. \]

(5.4.8)

Now, \[ (\nabla \vec{E}) \vec{H} = (E_{y} \partial_{y} \vec{E} - \partial_{y} \vec{E}_{y}) + E_{z} (\partial_{z} \vec{E} - \partial_{z} \vec{E}_{z}), \]
\[ E_{x} (\partial_{x} \vec{E} - \partial_{y} \vec{E}_{x}) + E_{z} (\partial_{z} \vec{E} - \partial_{z} \vec{E}_{z}), \]
\[ E_{y} (\partial_{y} \vec{E}_{x} - \partial_{y} \vec{E}_{y} + E_{y} (\partial_{z} \vec{E} - \partial_{z} \vec{E}_{y} + E_{y} (\partial_{y} \vec{E} - \partial_{y} \vec{E}_{y} + E_{y} (\partial_{z} \vec{E} - \partial_{z} \vec{E}_{z})). \]

The \( k \)-th component has the form, \[ E^{i} (\partial \beta_{k}^{1k} - \partial \beta_{k}^{1k}). \]

Using this result in (5.4.8), we see \[ \partial \beta_{t}^{1k} \] is the \( k \)-th component of the vector
\[ \{- (\nabla \vec{E}) \vec{H} + (\nabla \vec{H}) \vec{E} + (\text{div} \vec{E}) \vec{H} + (\text{div} \vec{H}) \vec{E} \} . \]

Hence, from (5.4.7) we conclude
\[ \frac{\partial}{\partial t} (E \cdot \vec{H}) - \{- (\nabla \vec{E}) \vec{H} + (\nabla \vec{H}) \vec{E} + (\text{div} \vec{E}) \vec{H} + (\text{div} \vec{H}) \vec{E} \} = 0. \]

(5.4.9)

But, \[ \frac{\partial}{\partial t} (E \cdot \vec{H}) = \frac{\partial E_{x}}{\partial t} \frac{\partial \vec{H}_{x}}{\partial t} + \frac{\partial E_{y}}{\partial t} \frac{\partial \vec{H}_{y}}{\partial t} + \frac{\partial E_{z}}{\partial t} \frac{\partial \vec{H}_{z}}{\partial t} = - \frac{\partial E_{x}}{\partial t} \vec{H}_{x} + \frac{\partial E}{\partial t} \vec{H} . \]

So, in (5.4.9) we have
\[ - \frac{\partial E_{x}}{\partial t} + (\nabla \vec{E}) \vec{H} + \left\{ \frac{\partial \vec{E}_{x}}{\partial t} - (\nabla \vec{H}) \vec{H} - (\text{div} \vec{E}) \vec{H} - (\text{div} \vec{H}) \vec{E} \right\} = 0. \]

(5.4.10)
In summary, the tensor equation \( \partial_{\alpha} S^\alpha_{\beta} = 0 \) is equivalent to vector equations

\[
\begin{align*}
\text{i) } & \quad H \cdot (\text{curl } E + \frac{\partial H}{\partial t} - E \cdot (\text{curl } H - \frac{\partial E}{\partial t}) = 0 \quad \text{(from 5.4.6),} \\
\text{ii) } & \quad \left(\frac{\partial E}{\partial t} - \text{curl } H\right) \times H - \left(\frac{\partial H}{\partial t} + \text{curl } E\right) \times E - (\text{div } E)E - (\text{div } H)H = 0
\end{align*}
\]

(from 5.4.10).

Equations i) and ii) are not Maxwell's equations Ia, Ib, IIa, and IIb, but i) and ii) are satisfied when the Maxwell equations hold. Apparently, i) and ii) may hold true also in some cases for which Maxwell's equations do not hold.

It would be very satisfying at this point if some condition, similar to the Lorentz gauge condition, could be found which would cause i) and ii) above to reduce to Maxwell's equations. But, no such condition is known.

It has been established in the preceding arguments that the conservation laws (4.3.2), and their special case in Chapter I, hold also in some cases when Maxwell's equations are not satisfied. Hence, they are statements about a more general class of physical fields than the electromagnetic field.
CHAPTER VI

CONCLUDING REMARKS

In this paper the primary concern has been those conservation laws for the electromagnetic field in vacuum which result from conformal invariance. The results obtained by Bessel-Hagen have been extended and generalized. In this chapter a summary of these new results will be given with some new questions which arose during this study and should have interesting answers.

Since 1964 some papers (see (4) and (8)) have been published which present infinitely many conservation laws, apparently unrelated to the ones considered here. It is natural to expect that these new conservation laws are due to some invariance of the field. If so, it should be possible to derive these conservation laws by use of the Noether theorems. In this chapter some results will be given for this problem.

1. Conclusions Concerning the Bessel-Hagen Results

This study began with the problem of clarifying the results presented by Bessel-Hagen in (3). It has been shown in Chapter I that four of his surplus conservation laws can be derived from the remaining eleven. Hence these four contain nothing new. A new variational principle for electromagnetic fields does not produce all of the field equations if we regard the electromagnetic field as a tensor field $F_{\alpha\beta}$ with field
equations,

\[ \nabla_{\lambda} F^{\alpha\beta} = 0 \]

\[ \zeta^{\alpha\beta\lambda\nu} \nabla_{\lambda} F_{\nu} = 0. \]

With the new variational principle no question arises concerning proper application of the first Noether theorem. The Bessel-Hagen results have been extended to arbitrary Riemannian Manifolds. It has been shown that in covariant form the action integral is still conformally invariant. Noether's theorem now leads to covariant conservation laws,

\[ \nabla_{\lambda} (S_{\alpha}^{\beta} \xi_{k}^{\beta}) = 0, \]

(6.1.1)

when \( \xi_{k}^{\beta} = x^{\beta} + \xi_{k}^{\beta} x^{k} \) represents the conformal group in a given manifold. In Minkowski space the conformal group is a fifteen-parameter transformation group. Hence, we obtain 15 conservation laws from (6.1.1). It is known (see (7), p. 55) that in some manifolds no conformal transformations exist. In such a manifold (6.1.1) yields no conservation laws.

2. Conformal Fields

It has been shown that the fifteen conservation laws produced by Bessel-Hagen are in fact statements about a class of fields \( F \) satisfying field equations.

\[ \nabla_{\beta} S_{\alpha}^{\beta} = 0 \]

(6.2.1)

with \( S_{\alpha}^{\beta} = - \frac{1}{4} F_{\alpha}^{\lambda} F^{\rho\lambda} S_{\rho}^{\beta} + \frac{1}{4} (F^{\alpha\beta} F_{\alpha\nu} + F^{\beta\alpha} F_{\nu\alpha}) \).

(6.2.2)

It is not necessary to assume \( F_{\alpha\beta} = -F_{\beta\alpha} \), since conformal invariance of the field does not depend on this assumption. If we assume \( \text{det}(F_{\alpha\beta}) \neq 0 \) and \( F_{\alpha\beta} = \nabla_{\alpha} A_{\beta} - \nabla_{\beta} A_{\alpha} \), then (6.2.1) yields Maxwell's equations. It would be very desirable to find some alternate condition, possibly geometric, which would reduce (6.2.1) to Maxwell's equations. However,
it is enough for present purposes to know that the fields described by (6.2.1) and (6.2.2) include the electromagnetic field as a special case.

It would also be interesting to determine the general solution for system (6.2.1) and (6.2.2), and thereby describe clearly the possible forms for such an $F_{\alpha\beta}$. Perhaps every solution must be of the form

$$F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha;$$

i.e., an electromagnetic field. However, equations (5.4.6) and (5.4.10) derived in Chapter V suggest that this is not the case.

The fields $F_{\alpha\beta}$ satisfying equations (6.2.1) and (6.2.2) may be called conformal fields, since they correspond to a conformally invariant action integral. It has been shown here that the Bessel-Hagen identities are statements which are true for all conformal fields. Determination of the precise nature of such fields is a new problem to be investigated.

3. Lipkin's Tensor

In 1964 D. M. Lipkin (4) published a paper in which existence of ten new conservation laws for electromagnetic fields was demonstrated. Briefly his results were as follows.

Define a tensor by

$$Z^{\mu\nu\kappa} = \frac{1}{4} \left[ g^{\lambda\kappa} k^\rho \epsilon_{\nu\kappa\rho\beta} + g_{\mu\lambda} \gamma_\beta \epsilon_{\nu\kappa\rho\beta} + g_{\nu\lambda} \epsilon_{\mu\kappa\rho\beta} + g_{\nu\lambda} \epsilon_{\mu\kappa\beta\rho} \right] F_{\rho \sigma} \partial_\sigma F_{\alpha \beta}$$

$$- \frac{1}{2} \left[ g_{\tau \sigma} \gamma_\rho \epsilon^{\nu\kappa\lambda\alpha} + g_{\tau \sigma} \epsilon^{\mu\kappa\lambda} \gamma_\alpha + g_{\tau \sigma} \epsilon^{\mu\kappa\lambda} \gamma_\rho \right] F_{\rho \sigma} \partial_\sigma F_{\alpha \beta}.$$  

(6.3.1)

It can be shown that if $F_{\alpha\beta}$ satisfies Maxwell's equations then

$$\partial_k Z^{\mu\nu\kappa} = 0.$$  

(6.3.2)

Interchanging $\mu$ and $\nu$ in (6.1.1) we see that $Z^{\mu\nu\kappa} = Z^{\nu\mu\kappa}$. Hence, (6.3.2) has the appearance of ten conservation laws. Lipkin converted
(6.3.2) to equivalent statements about vectors \( \hat{E} \) and \( \hat{H} \) hoping to determine the physical significance of (6.1.2). However, the results seem unrelated to conservation of energy, momentum, or any previously known conserved quantities. Lipkin claims that \( Z^{\mu \nu \kappa} \) cannot be reduced to a linear combination of derivatives of the energy-momentum tensor \( S^{\alpha \beta} \), but no proof of this is given.

Later in 1964 T. A. Morgan published a paper (8), showing that a large class of new conservation laws can be produced. These are represented by vanishing divergences for tensors of the form,

\[
V_{\mu \nu \alpha_i \cdots \alpha_m} = (\partial_{\alpha_{m-1}} \cdots \partial_{\alpha_1}) F_{\mu \sigma} (\partial_{\beta_{p-n}} \cdots \partial_{\beta_1}) F_{\nu \sigma}.
\]

As usual, \( * F_{\mu \nu} = \frac{1}{2} \epsilon^{\mu \nu \rho \sigma} F_{\rho \sigma} \). The tensor

\[
V_{\mu \nu \alpha} = \partial_{\alpha} F_{\mu \sigma} * F_{\nu} - \partial_{\alpha} F_{\mu \sigma} * F_{\nu},
\]

is claimed to be "essentially" Lipkin's Tensor, and,

\[
\partial_{\kappa} V_{\mu \nu} = 0.
\]

is equivalent to Lipkin's ten conservation laws. It can be shown that \( Z^{\mu \nu \kappa} \) is in fact the sum of two terms of the form of \( V^{\mu \nu \kappa} \) and another term whose divergence is always zero even when Maxwell's equations do not hold. Details of the calculations are omitted here because they play no essential role in the following. The intention here is merely to summarize results.

Since the papers by Lipkin and Morgan appeared, several papers, (9), (10), have been published which extended and generalized their results. The arguments are primarily calculations too lengthy to repeat here.

Our interest in these results lies in the possibility that all of these conservation laws might be derivable by the Noether theorems.
This would provide some unification of the scattered results and clarify the situation. We shall now consider the case of Lipkin's conservation laws from this viewpoint.

Reference to (3.5.2) shows that if (6.3.2) follows from the first Noether theorem, we must have

\[ Z^{\mu \nu \kappa} = S^k_\lambda \xi^{\lambda \mu \nu} - \gamma^{\kappa \mu \nu} \]  
(6.3.4)

where the transformation

\[ \tilde{x}^\alpha = x^\alpha + \xi^{\alpha \mu \nu} \xi_{\mu \nu} \]  
(6.3.5)

produces variation,

\[ \Delta \tilde{W} = \int dx \sqrt{\gamma} \lambda_q (\gamma^{\alpha \mu \nu} \xi_{\mu \nu}) \]

Quantities \( \xi_{\mu \nu} \) are parameters for the transformation group (6.3.5). But \( Z^{\mu \nu \kappa} \) as given by (6.3.4) does not contain derivatives of \( F_{\alpha \beta} \) because \( S^k_\lambda \) does not contain derivatives of \( F_{\alpha \beta} \). It is assumed that \( \xi^{\lambda \mu \nu} \) and \( \gamma^{\kappa \mu \nu} \) contain no derivatives of \( F_{\alpha \beta} \). Then (6.3.2) cannot be derived by Noether Theorem 1.

The second Noether theorem produces relations between the field equations. These are identities whose truth does not depend upon assuming the field equations are satisfied. The result (6.3.2) produced by Lipkin is not true unless the Maxwell equations are satisfied. Hence, this identity does not have the form of an identity resulting from application of Noether Theorem 2. So, equation (6.3.2) is not derivable directly from the second Noether theorem. It is possible, however, that (6.3.2) may be an indirect result following as a consequence of another identity derivable by the second Noether theorem. If not, then the new conservation laws cannot be derived by use of the Noether theorems. No invariance group is known which could lead to the desired identities.

Tensor \( Z^{\mu \nu \kappa} \) contains derivatives of \( F_{\alpha \beta} \), so equation (6.3.2) contains second derivatives of \( F_{\alpha \beta} \). Then, the new "conserved quantities"
are not first integrals of the field equations. Probably many relations of the form,

\[ \partial_\alpha W^{\alpha \beta \lambda \cdot \cdot} = 0 \]  

(6.3.6)
can be produced by successive differentiations of the field equations or previously known conservation laws. It is doubtful that such divergence expressions have any physical significance. It was shown in section 3 of Chapter I that we have good reason for interpreting statements of the form,

\[ \partial_\alpha P^\alpha = 0 , \]

as conservation laws. The arguments given do not apply, however, when \( P^\alpha \) is replaced by a tensor of higher order. In some cases statements such as (6.3.6) do represent conservation of physical quantities, but in other cases this may not be so. Hence, it may be incorrect to assume that (6.3.2) represents conservation of new physical quantities. Further research is required to determine whether this is so.
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