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the Degree Doctor of Philosophy in the Graduate
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By

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* * * * * * *

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CHAPTER I

INTRODUCTION AND RELATED LITERATURE

For efficient learning, the learner should discover by himself as large a fraction of the material to be learnt as feasible under the given circumstances - Polya (84: 608).¹

Background of the Investigation: The Space Age is making new demands of our schools. Because of the scientific orientation of our society more students should learn more mathematics. In addition, the explosion of knowledge has resulted in the expansion of the school curriculum in most areas. The importance of efficient learning by the student seems to be increasing.

According to Polya, student discovery is the key to efficient learning of mathematics. If this is true, then the history of mathematics which records for us the mathematical discoveries (creations) of the past should provide information about the origins of new developments in mathematics. The existence of independent, yet parallel in time, discoveries indicates that new mathematics may be a product of the period as much as it is a product of the individual;

¹The first set of digits represents the number of the Bibliography entry; the second represents the page number.
that is, the era provides the conditions or circumstances which precede a particular development in mathematics. A pattern of discovery in mathematics might be:

1) **Beginning**: a fertile set of circumstances.

2) **Growth**: development of a mathematical structure which is born of the fertile set of circumstances.

3) **Maturity**: the structure's place in mathematics and the development of new mathematics.

This pattern of discovery-evolution in pure mathematics probably is not that of the typical student. A modification is provided by some individuals who suggest that an attempt to solve some problem has provided the stimulus for many developments in mathematics. Speaking of instruction in the classroom, the authors of *Goals for School Mathematics* say, "We hope that many problems can be found (we know a few) that read, 'Here is a situation—think about it—what can you say?'' (47: 11). Thus a pattern of concept-evolution which may conceivably be the optimal one for the typical student might be:

1) **Beginning**: a problem.

2) **Growth**: development of a mathematical structure (model) which can be used to solve the problem.

3) **Maturity**: use of the structure in the solution of new problems.

Indeed, the inclusion of problems in textbooks by many authors seems to indicate that they believe problems can
and should have a place in mathematics instruction.

Now, the use of problem solving probably dates back to early instruction in mathematics. But, what has been the place of problem solving in recent instructional programs? It would appear that the problem most frequently follows the discussion of a new concept and the solution of the problems which the learner is actually called upon to solve requires (with only a few exceptions) the application of this new concept. The problem, thus placed, not only provides no opportunity for student discovery of the concept, but it actually robs him of opportunities for 'productive thinking.' It merely provides the student with a situation in which he can use (with or without understanding) a previously introduced concept.

Tentative Statement of the Scope of the Problem:
Logically, the presentation of the problem can occur before a discussion of the related topic, or it can follow such a discussion, or problems may be used both before and after a discussion of the related topic. Most authors of textbooks have placed the problems after a discussion of the related topic. Since the problems presented prior to such discussion seemed to have a potential as a stimulant of student investigation and learning, this investigation was an exploration of this use of problems. The primary phase involved a selection of the topic, determination of the type of problem to be
used, and the creation (or selection and modification) of the problem. The secondary phase involved the collection of information related to the question, "How does the timing of the problem-confrontation affect the student's learning of mathematics?"

Review of the Literature: As can be evidenced by numerous publications, the amount of research concerned with problem solving and discovery has greatly increased in recent years. While most "problem solving" research treats problem solving as an activity which has a single goal—an answer, "discovery-oriented" research assumes that teacher-promoted activity should lead the student (largely through his own efforts) to an understanding of some mathematical concept. Bolding says, "Discovering a property as a result of well-planned exercises has a distinct advantage over being told." (14:105). This statement suggests that well-planned exercises are the key to the promotion of this activity which ends with discovery. Butler and Wren emphasize the active role of the teacher in their discussion of the heuristic and genetic methods of teaching. They say that the well-chosen questions of the teacher should lead the student "to discover facts and information, relationships, and principles for himself . . ." (20:162). They theorize that the heuristic method "makes the student an active participant in the learning process and provides a spur to
quicken his interest since it places him in the role of at least a quasi investigator rather than a mere passive recipient of information." (20: 162).

Another facet of discovery is proposed by Hendrix when she tells us that the promotion of successful learning by discovery has no parallel as a solution to the motivation problem since "the discovery process itself is so exhilarating (to both children and adults) that it becomes its own motive in academic work." (50: 292). It is interesting to note that she seems to make discovery a part of induction when she says, "It is the recognition of the nonverbal awareness stage in induction learning that converts the classroom experience into that of actual discovery, the kind of thing that promotes a taste for and a delight in research." (50: 298). A slightly different view is given by Jones when he reports, "In my opinion intuition, induction, and application all still have important roles in both mathematics and teaching. However, these need not be lost in new curricula, but should be used in their proper relationship to further the discovery of proofs and relationships and to increase appreciation of the nature of proof and structure." (59: 198).

The importance of student effort is emphasized by Fawcett when he says, "Mathematics is the product of the human mind, and the mathematics teacher who serves his
students best will get them involved in the process of creating mathematics, new to them." (39: 455). Polya believes that problems promote discovery. "A great discovery solves a great problem, but there is a grain of discovery in the solution of any problem. Your problem may be modest; but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery." (83: v). In another book he writes, "Solving problems is the specific achievement of intelligence, and intelligence is the specific gift of mankind: solving problems can be regarded as the most characteristically human activity." (85: v).

According to Bingham, "Problem solving is the process of overcoming difficulties encountered in the attainment of an objective." (12: 11). She believes that "through opportunities to solve problems the child discovers and cultivates his abilities." (12: 3). She notes that "problems spring from a situation which contains both familiar and novel or uncertain elements." (12: 8).

As was mentioned earlier, much research which is related to problem solving has been done recently. Most of it has been concerned with techniques of obtaining the answer to the problem. It attempts to answer this question, "How do we find the answer to a problem most efficiently?"
This research has not been concerned with the "problem solving" of Polya. He is interested in the answer also, but more important, he is interested in (1) the solution itself, (2) the answer, and (3) the implications of the problem-solution-answer set.

The concept of teaching for discovery—including the use of problems to promote student involvement—seems to have wide acceptance. Has this acceptance had any influence on the mathematics curriculum? Speaking of the first algebra course, Kinsella lists as one of the common features of several programs (UICSM, SMSG, and CEEB), "Provision for more active participation of the learner—more self teaching and discovery, less telling and explaining by the teacher." (64: 47). Note that self teaching and discovery share equal emphasis. Later he provides additional comments about discovery teaching. "It is true that discovery teaching requires more time. It certainly seems absurd to demand that the teacher devise for each concept and generalization a sequence of student experiences leading to a discovery, whether or not the student is required to express his 'find' in an oral or written statement." (64: 83).

Many of the other writers that we have considered, although they believe it a sound technique, also have a word of caution about discovery teaching. Butler and Wren say "the heuristic method . . . is undeniably slower than
the lecture method, especially in the earlier stages of the work, and it is much more difficult to use." (20: 163). Later they write, "It (the genetic method) is probably the most difficult of all methods of teaching . . ." (20: 163). Hendrix phrases her concern in these words, "But such teaching is an art with a difficult technique. Further it is dependent upon sound and clearly written textbook materials." (50: 293). In Goals for School Mathematics we read, "It is obvious, however, that the discovery method is slow." (47: 17). In short, these individuals are concerned about (1) the time involved in discovery, and (2) the difficulty inherent in the effective use of the method.

This author believes that these concerns are justified. He further believes that there exists another difficulty which is as fundamental as is either of these. In a very recent article, Snyder points to the heart of the matter.

". . . Fundamentally, it (discovery teaching) is a process by which students are led to 'see' some general principle without having been told exactly what it is they are to see. Since the discovery process usually begins with material with which the students have some familiarity and requires each student to participate in the development, what is 'seen' by each student will come from his own background and, to this extent, will be meaningful to him.

It may be, however, that each student will discover something a little bit different from what the others see. This should cause no great alarm: this is exactly what happens every day
in every classroom. It is a result of individual differences among students . . ." (96: 374).

If this is true, what is the implication for discovery teaching and the learning of mathematics? It certainly means that any discovery which takes place is going to be individual; that is, each discovery will be a product of the student's mathematical background and creative ability.

Three things, then, seem to stand as obstacles to successful discovery teaching in a classroom situation. They are (1) a time element, (2) teacher inability to promote discovery, and (3) student inability to make the discovery—because of deficient background and limited ability. However, the importance of active participation, short of discovery, is emphasized by Polya. He says, "A reader who spent serious effort on a problem may profit by it even though he does not succeed in solving it." (87: xi). He thus suggests that the answer need not be the goal of each problem presentation. If "profit" is gained prior to the formulation of the final answer, then it must be true that significant learning can occur prior to the writing of the final answer. Hence, the problem—whether solved completely or not—frequently can serve as an instructional device which promotes learning.

**Difficulty of Securing Definitive Answers:** Although the review of literature provided some theoretical guidance in these studies, its contribution to the design of the
classroom studies was minimal. Consequently, the first classroom trial was intended as a pilot study which would provide an experiential background for the second study in these areas: (1) general design and execution, (2) selection of topic and type of problem, and (3) the development of materials. However, the conditions of the second study (school, course, and students) were so different that it was more nearly a new exploration than it was a re-exploration. In these conditions, the problems—which were newly developed and untried—did not always promote an effective student search (the first difficulty). In fact, student success indicated that some of the problems should not have been presented and others should have been presented in modified form.

The trial of a new technique produced the second difficulty which manifested itself in several ways. Some factors in this area were:

1) The approach was completely new to the students in these courses. Even the student who accepts the new approach finds that the newness presents a challenge because he is unfamiliar with its requirements. For example, the students did not easily adapt to the "fit your answer to the set of circumstances" which was required.

2) There was an initial resistance to the use of the problem technique on the part of some students. Part of the school term passed while this resistance lessened.
3) The problem technique may call for extra effort on the part of the student or require time which the student does not have. For example, in the second study many students held non-academic jobs.

4) As the term-length studies ended, the students did not appear to be transferring the learning of the problem to the related mathematical topic.

Since these factors could be due to the student's unfamiliarity with the technique and its requirements, their influence would probably be lessened in a year-long study or a situation of continuing use such as in a sequence of courses.

Finally, since learning is individual, there probably is no best way to teach mathematics. The existence of a best way would seem to demand a homogeneity of students which does not exist. Occasionally, a class may be homogeneous enough that there is a best method for that group. However, even students with approximately equivalent backgrounds probably learn mathematics best in different ways.

Each study was conducted in an available course in an existing set of circumstances. The course, having no college-level mathematics course as a prerequisite, was potentially the first course in college mathematics for the student. Although many of the students had completed a course in basic algebra (one had completed a 15 quarter hour sequence in calculus-analytic geometry), the student's background and ability appeared to vary greatly in the
courses. Hence, a third—and possibly the greatest—difficulty in securing definitive answers was that presented by the heterogeneity of the students who participated in the exploration.

Restatement of the Problem: Polya and the authors of Goals for School Mathematics suggest that problems which are presented before the classroom discussion of the related topic could promote student learning and discovery. Although this is probably true, a review of the literature, including abstracts of recent doctoral dissertations, does not seem to indicate that such problem utilization, if existent, is widespread. Hence, this investigation was an exploration of problem presentation which preceded the classroom consideration of related material. Since this author believes (1) that limited non-routine discovery occurs in the typical lower division classroom, and (2) that learning can occur prior to discovery, the principal objective of this problem presentation was "productive thinking" or pre-classroom learning, not discovery. The exploration involved:

1) General planning of the investigation.
2) Selection of mathematical topics.
3) A determination of the type of problem which seemed likely to promote the learning appropriate to the various topics.
4) Selection of or creation of the problem or sequence of problems which would promote the "appropriate" learning.
Thus, the major goal of this investigation was twofold: (1) the exploration/development of an "uncharted" method of mathematics instruction, and (2) the creation of essential problem materials.

Certain questions, which are related to the classroom phases of this "developmental/creative" investigation, will influence the discussion of the results of these studies. They are:

A) Primary

1) Did the problem-solution situation promote some "productive thinking?"

2) Was student understanding of mathematical topics advanced?

3) Did the technique produce test-measurable differences?

B) Secondary

1) Did this technique of problem presentation stimulate student activity in the attempted solution of an unfamiliar problem?

2) Was discovery of mathematical concepts promoted by the problem-solution situation?

3) Did this teaching technique produce a desirable/undesirable student reaction?

Although "statistical proof" of the superiority of this technique is not the goal of this study, some attempt to measure the effectiveness of the technique was made in items A-2 and A-3 above. The principal objectives of the study remain (1) the exploration of a technique, (2) the
development of the phases of execution in the classroom, and (3) the creation or selection of the required supporting materials.

**Plan for Future Chapters:** Research procedures will be presented in the next chapter. It will include a consideration of the assumptions and their influences on the design of the studies. The creation of the problem materials for the studies will be examined. In order to illustrate more precisely the form of the exploratory studies, brief sections of it will be considered. These focus attention on some of the topics used, the type of productive thinking or pre-learning which was desired for a particular topic, and the problem which was designed to promote this learning. The conditions—school, course, students—of the studies will be stated briefly. Then, the results of the presentation of the problem materials will be discussed in two chapters. Chapter III will be devoted to the analysis of numerical data while Chapter IV will present (1) the investigator's subjective analysis of group behavior, and (2) the students' views of the problem presentation as expressed in written form at the end of the school term. A summary of this investigation of problem utilization, suggestions for further study, and concluding subjective remarks will be presented in the final chapter.
CHAPTER II

RESEARCH PROCEDURES AND BASIC ASSUMPTIONS

Perhaps he will not reach his goal, but the search itself may prove more important than the goal. - Jacobson (55: 101).

Introduction: In the last chapter discovery teaching was examined briefly. Although many authors advocate its use in the mathematics classroom, most of them do so with some reservation. These reservations are related to the difficulties encountered in the use of discovery teaching and can probably be categorized as follows:

1. The greater time required for discovery teaching.
2. The difficulties inherent in the effective use of discovery teaching.
   a) The difficulty involved in promoting some (any) discovery.
   b) The difficulty involved in promoting the desired discovery.

It has been pointed out that promoting the desired discovery may present the greater challenge since it is in this area that the student's background and creative abilities dictate what personal discovery he will make.

Then problem-solving research was reviewed. It was noted that the goal of most of this research is to find
the technique which will lead the student most frequently and efficiently to the **correct answer**. Nevertheless, a few writers, including Polya, see the implications of the whole (problem-solution-answer) as much more significant than a correct answer. The answer is, indeed, a small part of the "whole" and the individual who does not look beyond the answer frequently is overlooking important implications. For example, an individual—given the formula—could calculate the energy release of hydrogen fusion and yet neither understand hydrogen fusion nor see the implications of this energy release.

Publications in these two areas were investigated because a problem situation presents a challenge and can provide the stimulus for a new discovery. Since the challenge-discovery pair can exist in the problem, the problem is seen as a possible stimulus of student activity, activity which could result in learning and perhaps some discovery.

**Assumptions:** As has been noted some authors have suggested that problems be used as proposed in mathematics education. However, no precise design of the application of the technique was found by this investigator. Consequently, these studies were exploratory-developmental. The plan of investigation was quite flexible with direction for the studies provided by quite general assumptions. The principal assumptions were:
1) Learning which occurs in one set of circumstances can help make related learning more efficient in different circumstances.

2) A problem can stimulate student activity.

3) Nonroutine discoveries in mathematics by undergraduate students will be limited in number.

4) Learning can occur prior to discovery.

5) The search for the solution of a problem may be as important as a correct solution.

6) Problem-solving/creative thought is a time-consuming process and should not be limited to the classroom.

7) The typical student should be guided in his search for new mathematics which is consistent with his maturity and background.

8) Classroom consideration of the problem and its related topic should follow the student's out-of-class search for a problem's solution.

These assumptions, together with the conditions of the study, had a guiding role in the exploration-development of this technique of problem utilization.

Discussion of Assumptions-Procedures: An examination of the "problem procedure" best illustrates the place of the assumptions in these studies. At the same time it will provide details about using problems to initiate the study of certain topics in mathematics. There are three identifiable phases in the problem technique. The first phase is the instructor's creation or selection of the problem; the second is the presentation of the problem and the student's search for the solution of the problem; and the third phase is the
classroom discussion or consideration of the problem and its related topic.

The Creation of the Problem: One does not point to an airplane and say, "That is (is not) a cat." One might, however, say "This (dog) is not a cat," or "This (cat) is a cat." The learning which takes place in either part of the second example would exceed the learning which results from the first statement. The first statement would not produce much learning since the two objects, airplane and cat, are not related in a "natural" way. In like fashion the problem, if it is to have the desired effect, must be related to the mathematical topic. If the procedure is to be used effectively, the instructor must find or create such problems.

Ideally, the problem would place the student in a mathematically challenging situation with many avenues open to his travel. This would satisfy the needs of the mathematics researcher, but probably would not be the optimum design for the stimulation of effective and efficient learning by the members of an average class. In fact, time and the sequential nature of a mathematics course limit the student's freedom in the selection of his own paths to a knowledge of mathematics. Since the student's individual search for the solution of the problem is conducted outside the classroom, where he will be unable to ask questions, his
time probably will be used very inefficiently if his efforts are not given direction. First, the problem should point the student in a direction consistent with the course material. Second, the problem should provide the guidance necessary for an efficient investigation. However, guidance should not provide the student with a road which he must follow, but focus his attention on challenges consistent with his background and abilities. In short, this author believes that for efficient student investigation, the efforts of the student should be given direction. The less mature the student is mathematically, the greater will be his need for direction in the solution of the problem and the search for implications.

Since few problems were found which satisfied the requirements of these studies, it was necessary to create (compose) most of them. The wide range of student abilities and backgrounds made problem creation difficult. A special attempt was made to avoid problems which would require a mathematical maturity which the typical student did not possess. The goal was to tailor each problem to the mathematical topic and the abilities of the student. Some of the criteria used in problem design were:

1) The student should be able to read it; that is, only the familiar may be used in the circumstances of the problem.

2) The set of circumstances must be new to each student.
3) Normally the solution should not require special knowledge in unfamiliar areas of mathematics. (When such knowledge was needed, it was presented in class prior to the distribution of the problem.)

4) The familiar should be put together in such a way that challenging questions can be asked.

5) The questions which are asked should be meaningful to all if student activity is to be stimulated.

6) Directions and guidance should be kept at a minimum, thus allowing great range in the attempt to find an answer which fits the circumstances of the problem.

7) Phases of the problem should be quite elementary so that a majority of the students will make some advance toward the solution.

The Presentation/Student Search: After the problem had been prepared, it was normally presented to the students without any classroom introduction. Occasionally, however, the nature of the problem seemed to suggest that the typical student would not get started without some introduction. Then, the classroom distribution of the problem was accompanied by a brief introduction to the problem. On other occasions, the solution of the problem required some information which the student had not yet studied. In this case, the introduction of the required pre-problem background accompanied the distribution of the problem. The information provided was in a sense equivalent to the classroom proof of a lemma which is useful in the proof of an assigned theorem.
In these studies the student was given at least two days for his search. This time provided the student with an opportunity to analyze the problem, to become familiar with its requirements, and to formulate conclusions. It provided him with an opportunity to check probable solutions, to discard unsound conclusions, and to construct new solutions. It provided him with an opportunity to seek the implications of a satisfactory solution. Although the student was given these opportunities, one of the basic assumptions was that the typical student would not be completely successful. Indeed, the goal was student participation in the examination of a situation which could stimulate student learning.

Since it seems that productive thought and creativity should not be time-budgeted to a fifty-minute class period, one of the distinct advantages of the problem as a stimulant of these activities is that it can be given to the student as an out-of-class exercise. However, the out-of-class search also provided an opportunity for student cooperation. In such cooperation all (both) participants are actively involved or some are (one is) not actively involved. In some instances the participation of two or more individuals may result in valid conclusions which no one individual would have produced. Even the student who only understands the work of the cooperative effort could profit from the
problem. Since only the individual who merely reproduces the work of another gains nothing from the problem, little was done to discourage a joint search.

The Problem in the Classroom: The goal of the problem presentation was student involvement in the learning of mathematics which is new to him. After the student's out-of-class search for a solution and its implications, the problem and the related (new) topic were considered in the classroom. Some expected results of this technique of introduction/exploration of the new topic were:

1) The slow learner would be better prepared to follow the crucial steps in the advance through the phases of development and the analysis of the implications of the new topic.

2) The student would be prepared for a more active role in the classroom discussion-development of the new material.

3) The able student, as well as the less able, should arrive at a more thorough understanding of the mathematical concept because of his experiences with "pre-mathematics." (47: 18).

The problem, therefore, is a supplement to the textbook and normal classroom discussion and development of material. It provides the student with opportunities for creative thought. It points the way to limited "original" investigation at the lower-division level while the textbook--because of the sequential presentation of all significant material--affords the student with little opportunity
for original investigation. The authors of *Goals for School Mathematics* speak of the importance of student exploration. They say, "Even in the later grades new concepts should be introduced by asking the class to explore possibilities. This will help to develop intuition equally with logical processes and keep alive a willingness to enter a pre-mathematics approach to new areas of mathematics." (47: 18). Later they write, "More difficult discovery exercises can be used to introduce mathematical ideas which will soon thereafter be explicitly presented in the text." (47: 28).

**Examples of Problems Used in the Studies:** The topic, the course and the abilities of the students all make demands of the problem. Consequently, the problems represented variety in form and level of difficulty. The amount of student activity required as well as the degree of critical thought required were definitely a function of the topic-problem-student. Some of the topics selected for introduction by the presentation of a problem, a consideration of the type of learning desired, and a sketch of the problem will serve to illustrate the character of the problems, their goals, and the plan for the realization of these goals.

**Example One:** One problem in each study was designed to promote a student investigation which would mold student background for a more complete understanding of the properties potentially possessed by a binary operation. This
investigator had observed in previous classes that the general student reaction to this material was that the properties were "obvious"—that their consideration was little more than unnecessary. To make the discussion more meaningful and to help focus attention on the importance and character of these "postulates" (elementary treatment of addition and multiplication in the set of real numbers) an operation which did not possess these properties was created and submitted to the students for analysis. Since the student is normally interested in techniques of grade determination, and since a technique of grade determination provided a suitable set of circumstances for the creation of an operation which lacked the properties to be discussed, it was selected as the subject of the problem. The goal of the problem was, therefore, a sort of supplemental-negative learning, a learning which results from the examination of an operation which is binary in form, but which lacks the properties to be introduced. A form of the problem (see problem 1, appendix A) is now considered.

Mr. Charles W. Goode has devised a system of determining student's grades which he calls cumulative-weighted grading, CWG.

I. Cumulative because—
1. The student's grade is determined after each examination.
2. All grades except the last one determined are discarded.
3. The one recorded grade and the last examination are used to determine the new grade.
II. Weighted because the student's last examination grade is given more weight in the determination of the new grade than is the student's present (recorded) grade.

<table>
<thead>
<tr>
<th>Student</th>
<th>Exam 1 CWG</th>
<th>Exam 2 CWG</th>
<th>Exam 3 CWG</th>
</tr>
</thead>
<tbody>
<tr>
<td>John</td>
<td>C</td>
<td>B</td>
<td>A</td>
</tr>
<tr>
<td>Mary</td>
<td>C</td>
<td>A</td>
<td>B</td>
</tr>
</tbody>
</table>

These examples show how Mr. Goode puts extra weight on the last examination of the student. Although the last examination is given extra weight in the determination of the student's new CWG, Mr. Goode, however, never changes the student's CWG more than one letter grade as the result of a single examination (see Mary in above example, second examination). In this problem we will assume that all examination grades are C, B, or A.

1. Beginning with the second examination, Mr. Goode uses a small table (similar to a multiplication table) to determine the student's CWG. Consistent with the above description of the CWG system and the given examples, complete Mr. Goode's "grade" table (Table VII).

<table>
<thead>
<tr>
<th>LAST EXAM GRADE</th>
<th>CWG</th>
<th>PRESENT</th>
<th>CWG</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>C</td>
<td>B</td>
<td>A</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>C</td>
<td>B</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>C</td>
<td>B</td>
</tr>
<tr>
<td></td>
<td>A</td>
<td>B</td>
<td>A</td>
</tr>
</tbody>
</table>

IN TABLE: NEW CWG

2. Find the student's CWG after the second examination.
3. Find the student's CWG after the third examination.

<table>
<thead>
<tr>
<th>Exam 1</th>
<th>Exam 2</th>
<th>CWG</th>
</tr>
</thead>
<tbody>
<tr>
<td>Henry</td>
<td>B</td>
<td>A</td>
</tr>
<tr>
<td>Paul</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>Al</td>
<td>B</td>
<td>B</td>
</tr>
</tbody>
</table>

4. After examination 6, Mr. Goode was very busy so he did not compute the new CWG although he did record the grades for examination 6. In fact, when he gave examination 7, he had not calculated the CWG after examination 6. Mr. Goode brought his record book up-to-date after examination 7. Karen objected to the CWG which Mr. Goode recorded for her. She believes that her CWG is A although Mr. Goode has (incorrectly) given her B. Mr. Goode's entries in his record book are correct (CWG after exam 5, C; exam 6, B; exam 7, A).

a. Explain how Mr. Goode used this data to get CWG, B.
b. Explain how Karen used this data to get CWG, A.
c. Explain the technique which must be followed when calculating a CWG from a series of scores.

(The goal of this section was to prepare the student for the introduction of the associative property.

Correct sequence of the two operations. Operation not performed in the proper sequence.

```
   C     B      A   \ C     B      A
     |     |      | \       |      |
    B     A      | \     B     A
       |      \   \    |      \    |
      A \ Result \       \ Result
```
As the illustration indicates, it is sometimes critical that operations be performed in some particular order.)

5. There is something "unique" about having a present CWG, B. Explain.

[The student is encouraged to look critically at an element which has some of the properties of an identity——B CWG X=X where X=A, B, or C. However, X CWG B≠ X for X = A or X = C. Part six of this problem provides the student with an opportunity to see a "second face" of the element, B.]

6. Jack received a B on examination 5 which made his new CWG a B. What was his CWG after examination 4? Explain your answer (see your table and numbers 2 & 3).

[This investigation should prepare the student for a discussion of the "cancellation" properties of addition and multiplication. Note that the operation does not have the cancellation property: A CWG B = C CWG B, but A ≠ C. In fact, X CWG B = B for X = A, B, or C.]

7. Each of the tables introduced below possesses a property (pattern) which does not exist in your CWG table. Examine the tables to discover this property, discuss.

<table>
<thead>
<tr>
<th>X</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>10</td>
<td>12</td>
<td>14</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>6</td>
<td>9</td>
<td>12</td>
<td>15</td>
<td>18</td>
<td>21</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>8</td>
<td>12</td>
<td>16</td>
<td>20</td>
<td>24</td>
<td>28</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>10</td>
<td>15</td>
<td>20</td>
<td>25</td>
<td>30</td>
<td>35</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>12</td>
<td>18</td>
<td>24</td>
<td>30</td>
<td>36</td>
<td>42</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>14</td>
<td>21</td>
<td>28</td>
<td>35</td>
<td>42</td>
<td>49</td>
</tr>
<tr>
<td>+</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
</tbody>
</table>

{0,1,2,3,4}
The elements in a set below a table are those elements which are in the body of the table.

The other tables were all symmetric about the upper-left to lower-right diagonal while the CWG table lacks this symmetry, i.e., the CWG operation is not commutative as is seen in parts a and b of #2.

8. The first three tables presented in problem 7 do not have a certain property which is possessed by the other tables. Study the tables to determine what the property is. Explain.

[Tables IV-VII have as entries only the elements of the finite set on which the operation is performed (closure) while the other tables do not have this property.]

The introductory section of this problem including parts 1, 2, & 3 was designed to familiarize the student with the "CWG" binary operation. The remaining portions of the problem, as indicated above, sought to prepare the student
for the classroom introduction-discussion of the commutative, associative, identity, and cancellation properties of addition in the set of real numbers. The principal component of this preparation was the study of a binary operation which did not have these properties. The goal of the problem along with the follow-up classroom presentation, was to help the student realize the true character of certain assumptions (postulates) which he normally judges so obvious that they need not be considered.

Example Two: A sequence of four problems was used to prepare the student for certain definitions in set theory, to provide exercises to fix the concepts involved, to introduce elementary graphing, and to create a background for the discussion of relations and functions. The first three sought to make the learning an almost effortless result of the student exercises completed after an attentive reading of the statement of the problem. This sequence illustrates how a problem can be used, not only in more than one capacity, but also as a portion of a spiral development of some topic. The goal of the four problems was to further student understanding of certain definitions and to promote a growing involvement in a pictorial representation of elements of sets, i.e., the development of a 1-1 correspondence between elements of a set and some visual display. This
activity was promoted without classroom discussion of the problems in the sequence. Although the numbers used in the sequence were from the set of whole numbers, the work completed was closely related to and provided a background for the classroom introduction of ordered pairs in $\mathbb{R} \times \mathbb{R}$ and related graphing in $\mathbb{E}$ which is normally studied in an algebra course. The first problem (see problem 8, Appendix A) with one example illustrative of the work required follows:

The numeral 26 is a symbolic representation of the number twenty-six (or two tens and six units). The numeral 62 is a symbolic representation of the number sixty-two (or six tens and two units). Hence, the two different combinations of the symbols 2 and 6 identify two different number concepts—twenty-six (twenty and six) and sixty-two (sixty and two). We conclude that the symbols used in our numeration system do not possess the commutative property.

Thus, a two-digit place-value numeral (base ten) has a tens digit and a units digit. In the exercises which follow, we will "build" sets of two-digit numerals (we will call 05 a two-digit numeral--zero tens and five units, hence five). The construction procedure is as follows:

1. The symbol for the "tens" digit is an element from a set named $T$.

2. The symbol for the "units" digit is an element from a set named $U$.

3. We will call the constructed sets of two digit numerals $S$.

In the following exercises, complete as indicated in the instructions. In addition, circle the elements in the array of numbers which correspond to the elements of the set $S$. 
Examples were provided to aid the student in the correct interpretation of the statement of the problem. Following the construction procedure outlined and other phases of the instructions, one section typical of the problem is:

\[ T = \{0,1,2,4\} \]
\[ U = \{3,4,5\} \]
\[ S = \{03, 04, 05, 13, 14, 15, 23, 24, 25, 43, 44, 45\} \]

The problem used the student's knowledge of numbers and numerals to prepare him for the definition of ordered pairs. This definition frequently presents some difficulties for students who have been working with unordered sets for a period of time. At the same time, the requirement to form "all possible two-digit numbers" from the two existing sets as instructed led to the formation of sets which are closely related to the Cartesian product of two sets. The setting up of the one-to-one correspondence between the elements of the set and certain elements of the array of
numbers provided the first step toward the "matching" of the elements of some ordered pairs with the points of the plane.

The second problem was designed to use the learning of the previous problem to make the definition of ordered pairs more clearly understood by the students, to promote student responses which can be recognized as the Cartesian product of two sets and thereby prepare the student for an introduction of the definition, and to continue the spiral development of the background preceding the classroom introduction of graphing in the Euclidean plane. This problem (see Problem 9, Appendix A) follows:

In our last problem we investigated the formation of two-digit numbers and the sets of numbers thus created. We noted that 26 ≠ 62. A new type of set is now introduced, a set in which order of elements is significant. Symbolically we write \((a,b)\), an ordered set with "first" element \(a\) and "second" element \(b\). When the ordered set has two elements, it is called an ordered pair.

Definition:

\((a,b) = (c,d)\) if and only if \(a = c\) and \(b = d\).

Note that, although \(\{a,b\} = \{b,a\}\), \((a,b) \neq (b,a)\) when \(a \neq b\). Most of the following items require that sets of ordered pairs be built using the element of two given sets. The procedure for construction of the ordered pairs is as follows:

1. The first element of the ordered pair is an element from a set named \(F\).

2. The second element of the ordered pair is an element from a set named \(S\).
3. $P$ is the name given to the set of all possible ordered pairs constructed from the sets $F$ and $S$.

In the following exercises, find $P$ and complete the item as illustrated in the above examples. In addition, circle the elements in the array of ordered pairs which correspond to the elements of the set $P$.

Following the construction procedure outlined and the instruction to circle those elements in the array which appear in the set of ordered pairs, one section typical of this problem is:

$$F = \{0, 2, 4\}$$

$$S = \{1, 2, 3, 4\}$$

$$P = \{(0,1), (0,2), (0,3), (0,4), (1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4)\}$$

This problem and exercises, of course, provide the background for the definition of the Cartesian product of two sets since the definition is simply an identification of and a naming of a binary operation which the student has already performed. In addition, it points out to the student that certain ordered pairs are included in certain sets and others are not included. Finally, for each element in his set of ordered pairs, the student circles the
corresponding (identical) element in an array of ordered pairs—an array which is a subset of \( W \times W \). The structure of the elements and the relative location of these elements within the array parallel the structure of the elements and the location of points in the "integral" portion of the first quadrant of the Euclidean plane.

Consistent with the goals of this sequence of problems, the third problem introduced the Cartesian product or cross product of two sets. The Cartesian product is presented as a binary operation on sets. The principal portion of the problem (see Problem 10, Appendix A) is:

Previously we have considered two binary operations on sets: the union of two sets and the intersection of two sets. We represent these operations symbolically as \( A \cup B \) and \( A \cap B \) respectively. In our last problem we constructed sets of ordered pairs. The formal definition of a third binary operation on sets, called the cross product or Cartesian product, is now provided.

Definition: \( A \times B = \{(x,y) \mid x \in A \text{ and } y \in B\} \)

Since the student had, in the preceding problem, already found the cross product of sets, it was necessary only to complete a few exercises after the presentation of the mathematical definition so that the operation would be associated with the proper mathematical terminology and symbolism when needed in a later context. So—what is essentially a repeat of the last problem, but using the conventional symbolism of mathematics—a typical portion
of the problem follows:

\[ A = \{0,1,2\} \]
\[ B = \{0,2,4\} \]
\[ A \times B = \{(0,0),(0,2),(0,4),(1,0),(1,2),(1,4),(2,0),(2,2),(2,4)\} \]

The degree of creative thought required in the first three problems of this sequence was not great. The principal demand was an ability to read with understanding. The student's analytic efforts made possible directed student activity which led him to "the door" of certain definitions. Only then did the following problem provide the student with the definition and opportunities to exhibit his knowledge of the definitions. Activity which paralleled the work with definitions, i.e., the circling of elements in an array, also created a student readiness for the first real steps in elementary graphing which is introduced in the fourth problem of the sequence.

The fourth and final problem in this sequence sought to re-emphasize the learning of the previous problems. In contrast to the first three problems which had been quite theoretical in nature, this problem was given a "practical" nature through the use of real circumstances. In it an attempt was made to illustrate that ordered pairs and related set theory could be used in circumstances which do not have mathematical origins. Consistent with the goal that the student be introduced to elementary graphing, the
problem was designed to lead the student to a recognition of the character of the (accepted) correspondence between ordered pairs of numbers and certain points in the plane. The selection of points was presented as an outgrowth of a desire to represent information in a pictorial fashion. The more critical portions of this problem (see Problem 11, Appendix A) are:

Nine students are identified by number in the display below. It also contains eight quiz scores for each of the students—four from the first half semester and four from the second half semester.

<table>
<thead>
<tr>
<th>STUDENTS</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>T 1</td>
<td>30</td>
<td>36</td>
<td>27</td>
<td>36</td>
<td>21</td>
<td>21</td>
<td>36</td>
<td>22</td>
<td>42</td>
</tr>
<tr>
<td>E 2</td>
<td>18</td>
<td>42</td>
<td>30</td>
<td>42</td>
<td>33</td>
<td>30</td>
<td>30</td>
<td>27</td>
<td>39</td>
</tr>
<tr>
<td>S 3</td>
<td>30</td>
<td>45</td>
<td>21</td>
<td>33</td>
<td>42</td>
<td>30</td>
<td>39</td>
<td>30</td>
<td>27</td>
</tr>
<tr>
<td>T 4</td>
<td>33</td>
<td>33</td>
<td>21</td>
<td>39</td>
<td>39</td>
<td>33</td>
<td>36</td>
<td>24</td>
<td>36</td>
</tr>
<tr>
<td>M</td>
<td>4</td>
<td>1</td>
<td>5</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>N 5</td>
<td>33</td>
<td>42</td>
<td>30</td>
<td>30</td>
<td>42</td>
<td>39</td>
<td>45</td>
<td>30</td>
<td>36</td>
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<tr>
<td>U 6</td>
<td>36</td>
<td>39</td>
<td>24</td>
<td>21</td>
<td>27</td>
<td>36</td>
<td>33</td>
<td>33</td>
<td>21</td>
</tr>
<tr>
<td>M 7</td>
<td>30</td>
<td>42</td>
<td>21</td>
<td>24</td>
<td>39</td>
<td>30</td>
<td>27</td>
<td>15</td>
<td>15</td>
</tr>
<tr>
<td>B 8</td>
<td>24</td>
<td>42</td>
<td>33</td>
<td>18</td>
<td>24</td>
<td>24</td>
<td>30</td>
<td>27</td>
<td>30</td>
</tr>
<tr>
<td>E</td>
<td>4</td>
<td>1</td>
<td>4</td>
<td>5</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>R N</td>
<td>4</td>
<td>1</td>
<td>5</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>

1 In the original problem more students were included—only nine are used in this paper so that arrays and graphs can be represented more efficiently.

2 Since the technique of grade determination was a function of the student's choice, results varied. Answers provided are taken from a student's paper.
1. If the only possible grades are 1 (highest grade), 2, 3, 4, and 5, determine—
   a. A grade for the combination of the first four scores—M.
   b. A grade for the combination of the second four scores—N.
   c. A final grade—F.
   d. Insert these grades in appropriate spaces in above display.

2. Describe in detail the method you used to determine the grades M, N, and F.

3. If \( R = \{(x,y)|x \text{ is a student's number and } y \text{ is his M}\} \), then:
   \[ R = \{(1,4),(2,1),(3,5),(4,2),(5,3),(6,4),(7,3),(8,4),(9,2)\} \]

   If \( S = \{(x,y)|x \text{ is a student's number and } y \text{ is his N}\} \), then:
   \[ S = \{(1,4),(2,1),(3,4),(4,5),(5,3),(6,3),(7,2),(8,4),(9,3)\} \]

   If \( T = \{(x,y)|x \text{ is a student's number and } y \text{ is his F}\} \), then:
   \[ T = \{(1,4),(2,1),(3,5),(4,3),(5,3),(6,3),(7,2),(8,4),(9,3)\} \]

4. a. In figure 1-a circle those elements which are elements of either \( R \) or \( S \), or both \( R \) and \( S \) (see #3). Note that this gives us a view of the grades of the students for the two half-semesters.

   b. In figure 1-b circle those elements which are elements of the set \( T \) (see #3). Note that this gives us a view of the grades (relative standing) of each of the students at the end of the semester.

\[
\begin{array}{cccccccc}
(1,5) & (2,5) & (3,5) & (4,5) & (5,5) & (6,5) & (7,5) & (8,5) & (9,5) \\
(1,4) & (2,4) & (3,4) & (4,4) & (5,4) & (6,4) & (7,4) & (8,4) & (9,4) \\
(1,3) & (2,3) & (3,3) & (4,3) & (5,3) & (6,3) & (7,3) & (8,3) & (9,3) \\
(1,2) & (2,2) & (3,2) & (4,2) & (5,2) & (6,2) & (7,2) & (8,2) & (9,2) \\
(1,1) & (2,1) & (3,1) & (4,1) & (5,1) & (6,1) & (7,1) & (8,1) & (9,1) \\
\end{array}
\]

Figure 1-a

\[
\begin{array}{cccccccc}
(1,5) & (2,5) & (3,5) & (4,5) & (5,5) & (6,5) & (7,5) & (8,5) & (9,5) \\
(1,4) & (2,4) & (3,4) & (4,4) & (5,4) & (6,4) & (7,4) & (8,4) & (9,4) \\
(1,3) & (2,3) & (3,3) & (4,3) & (5,3) & (6,3) & (7,3) & (8,3) & (9,3) \\
(1,2) & (2,2) & (3,2) & (4,2) & (5,2) & (6,2) & (7,2) & (8,2) & (9,2) \\
(1,1) & (2,1) & (3,1) & (4,1) & (5,1) & (6,1) & (7,1) & (8,1) & (9,1) \\
\end{array}
\]

Figure 1-b
Figure 2 contains two sets of lines: one set parallel-horizontal, the other set parallel-vertical. The horizontal lines are labeled 1, 2, 3, 4, or 5 while the vertical lines are labeled 1, 2, 3, \cdots, 9.

We can obtain an equivalent "student-grade" view if we define a one-to-one correspondence between ordered pairs and points, points which are the intersection of two lines—one vertical and one horizontal. We will define this correspondence as follows: the point which corresponds to the ordered pair $(x,y)$ is the point which is the unique intersection of the $x$-th vertical line and the $y$-th horizontal line (see examples in figure 2).

\[(2,3) \quad (5,4) \quad (7,1) \quad (8,2)\]

Figure 2

5. a. In figure 3-a circle those points which corresponds to the elements (see #3) of either R or S, or both R and S.
   b. In figure 3-b circle those points which correspond to the elements of T.
6. There is an interesting (important) difference existing between the two parts—a and b—of your "grade display" (either figure 1 or figure 3). Examine carefully these figures; determine the nature of this difference. Give an explanation of the difference.

Three parts of this problem are critical in this sequence which preceded a classroom discussion of graphing. (1) A set of ordered pairs was created which served as the solutions set of some relation. (2) Ordered pairs in this set were "paired" with elements in an array of ordered pairs. (3) Finally, the elements of the set of ordered pairs were used to select certain elements of a lattice of points—these points determined by the intersections of two sets of parallel lines; one set parallel-horizontal and the other set parallel-vertical.

Part six of this problem sought to promote student thinking beyond "graphing". It sets the stage for a classroom consideration of the relation-function concepts of mathematics. These were introduced at this time since the graphical implication of the definitions are frequently quite easily understood by the students. An interesting episode, which is described on page 73, occurred when a student asked a question concerning this part of the problem.

Critical in this example is the trial of a sequence of problems. This permitted a segmented approach to the several related topics which were to be introduced. It allowed a more gentle approach to portions of the material instead
of a rather abrupt approach through a two-day out-of-class search. This technique appeared to have certain advantages. It permitted greater freedom in the design of less involved problems, gave the student more total time for his personal development of topics, permitted a spiral development of certain phases of the material, and provided several opportunities for a reinforcement of the material learned in the early problems of the sequence.

Example Three: Often students have a distorted view of the real nature of mathematics and, in particular, of proof in mathematics. They do not understand the role of undefined terms, definitions, and the postulates. Many believe that the theorems proved in mathematics are only those which experience has already demonstrated as "truths." The goal of at least one problem in each of the studies was to help the student develop a more accurate understanding of the nature of mathematics.

In order to de-emphasize the "true from reality" view of the student, a problem situation which had (for the student) no real world model seemed desirable. Such a situation would permit an abstract investigation which proceeds from the definitions, postulates, and the hypotheses to a demonstration of the validity of the suggested conclusion. A topic which seemed to present an acceptable set of circumstances for this student quest was the binary
operation of multiplication (addition was used in a follow-up problem) in the set of rational numbers—provided the elements in the set and the operation are presented in a form which is unfamiliar to the student. Consequently, the "ordered pair" form of rational numbers was used in the problem, that is, \((a,b)\) was identified with \(a/b\), where \(a\) and \(b\) are integers and \(b \neq 0\). A statement of the problem (see Problem 6, Appendix A) with some of the exercises follows:

Postulates:

1. There exists a set of elements, \(W = \{0, 1, 2, 3, \ldots\}\). We will call this set the set of whole numbers.

2. We will assume the equality and order axioms for elements in the set of whole numbers.

3. We will assume the following properties for elements in the set \(W\): closure for addition and multiplication, commutative laws of addition and multiplication, associative laws of addition and multiplication, distributive law, identity element for addition (0), and identity element for multiplication (1).

Definition: \(S = \{(a,b) \mid a \in W, b \in W, \ b \neq 0\}\)

Definitions: \((a,b) \neq (c,d) = (ac,bd)\) This symbolic statement is the definition of a binary operation for elements of the set \(S\).

Definition: \((a,b) = (c,d)\) if and only if \(ad = bc\) This symbolic statement defines equality of elements in the set \(S\).

1. Show that \((a,c) = (a,c)\).

2. Show that \((a,b) = (ha, hb)\).
3. Prove that \( \# \) is a commutative binary operation.
   \[(a,b) \# (c,d) = (c,d) \# (a,b)\]

4. Prove that \( \# \) is an associative binary operation.
   \[[(a,b) \# (c,d)] \# (e,f) = (a,b) \# [(c,d) \# (e,f)]\]

5. Show that \( \# \) has the closure property on the set \( S \).
   \[(a,b) \# (c,d) \in S\]

6. Prove that \( \# \) has the cancellation property.
   If \((a,c) \# (h,k) = (y,z) \# (h,k)\), then \((a,c) = (y,z)\)

7. Name an identity element for \( \# \) on the set \( S \).
   Show that it has the properties of an identity.
   \[(a,b) \# ( , ) = ( , ) \# (a,b) = (a,b)\]

Prior to the presentation of the problem the whole number system had been studied. The necessity for and the character of undefined terms had been discussed. The use of the ordered pair definition of rational numbers created a situation which, for most students, appeared to have existence only in the definitions and postulates provided. The student, therefore, found himself in "an abstract" environment which seemed to have limited connection with anything he had previously experienced. The exercises sought to promote student demonstration of the existence of certain properties (theorems) within the system. The student was given the opportunity to "create mathematics new to him"

The student learning desired in this example was twofold. (1) The primary purpose was to make the student aware of the nature of mathematics—that mathematical systems are created and need not be related to known physical or real-life situations. This is not to suggest that mathematical
systems are always created without some application in mind or that if they are created without a knowledge of their applications to real situations, that they might not later be found useful in some real-world application. (2) The secondary purpose was to promote student demonstration of the existence of certain properties of the binary operation of addition in the set of rational numbers.

**Example Four:** The linear function was also first introduced to the student through a problem-promoted investigation. In example two the character of the problem (see problem 11, Appendix A) resulted in the plotting of isolated points, the points corresponding to ordered pairs with natural number components. The problem of this example, related to irrigation which is prevalent in the area of the institution (second study), presented circumstances for a continuum of points--a segment of a straight line. A portion of this problem follows (see problem 13, Appendix A).

You have an eighteen acre field which is to be irrigated. You have two sources of water for the irrigation of the field. Source A provides sufficient water for the irrigation of one acre per hour while source C provides sufficient water for the irrigation of two acres per hour. We will assume that the area irrigated increases continuously as the water is provided continuously. Consequently, we assume that the area irrigated at the end of one hour is one half that irrigated at the end of two hours, etc. In like fashion, we will assume that at the end of one-half hour the area irrigated will be one-half that irrigated at the end of one hour.
4. Two acres\(^3\) of the field have already been irrigated when you start \((t = 0)\) your portion of the project. You use water from source C.

a) Complete the table which will illustrate the relationship existing between certain times and the area irrigated.

<table>
<thead>
<tr>
<th>Time</th>
<th>0</th>
<th>1/4</th>
<th>1/2</th>
<th>1</th>
<th>3/2</th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>9</th>
<th>11</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>Acres</td>
<td>2</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b) Translate the information of the table (above) into a set of ordered pairs where the first element of the ordered pair corresponds to the time and the second element of the ordered pair corresponds to the area irrigated.

__________________________


c) In a previous problem we found an algebraic expression \(F(h) = h + 1\) which was useful in determining the miles traveled during the various hours of a trip. Relative to this problem (see problem 12, Appendix A) this algebraic expression had no interpretation for fractional numbers. That is, although we were able to ask about the distance traveled during the \(h\)-th hour, \((h, \text{ a natural number: } 1, 2, 3, 4, 5, 6);\) this expression gave nonsense answers if we used this expression with \(h = 5/2\). Is the same true in this problem? \(\underline{\text{__________}}\) Is it meaningful to ask how much area is irrigated during the first one half hour, the first five and one half hours? \(\underline{\text{__________}}\)

Explain.

\(^3\)The section presented here is the fourth of four similar introductory sections. See Appendix for other sections. The results from these other sections are introduced in part 5.
d) Write an algebraic expression \( D(t) \), which can be used to find the area (number of acres) irrigated at the various measures of time (after \( t = 0 \)). Write this expression in simplest form. Explain why you believe that it is a correct form. [Note: your answer must satisfy these requirements, \( D(0) = 2, D(1/2) = 3 \), etc.]

\[
D(t) = [2t + 2]
\]

5. Write below the algebraic expressions of 1-d, 2-d, 3-d, and 4-d. DOUBLE CHECK, are they correct?

\[
\begin{align*}
A(t) &= [t] \\
B(t) &= [t + 5] \\
C(t) &= [2t + 5] \\
D(t) &= [2t + 2]
\end{align*}
\]

a) Represent each of them graphically on figure 1. Label (with the corresponding ordered pairs) those points which you believe have special significance. (You may want to use the tables which you completed in the various sections—or the related set of ordered pairs—when you are graphing these expressions).

*** The following equations are not related to the irrigation problem. They are questions about the algebraic expressions and their graphs.

b) The graphs of \( A(t) \) and \( B(t) \) are in some respects similar. Explain.

[The graphs are parallel line segments.]

c) The graphs of \( C(t) \) and \( D(t) \) are in some respects similar. Explain.

[The graphs are parallel line segments.]

d) The similarity noted above does not exist between \( A(t) \) and \( C(t) \), between \( A(t) \) and \( D(t) \). Explain.

[Although the graphs are line segments, the graphs of the pairs given are not parallel.]
e) Look at the related algebraic expressions—these similarities and differences can be related to certain properties of these algebraic expressions. Explain.

[If the graphs of the expressions are parallel, then the coefficients of the t-term—term of degree one—are numerically equal.]

f) The graphs of B(t) and C(t) are in some respects similar. Explain.

[Both have the same initial point, (0,5).]

g) The similarity noted in f does not exist in the pairs—A(t) and B(t), A(t) and C(t), A(t) and D(t), B(t) and D(t). Explain.

[The graphs of the pairs listed do not have a common origin.]

h) Look at the algebraic expressions—the similarity and differences f and g can be related to certain properties of these algebraic expressions. Explain.

[If the graphs of a pair of expressions have identical numerical constants (b in ax + b) then the graphs will have a common origin (later called the y-intercept), otherwise their origins will differ.]

In this problem the student first supplied the missing numbers in a table of representative values, then used these tabular values to create a related set of representative ordered pairs which satisfied the conditions of the problem.

In part 4c a special attempt was made to promote student recognition of the continuous character of the "idealized" irrigation process. This is to be contrasted with the results of the previous problem (see problem 12, Appendix A) in which the defined relationship was valid only for a
certain subset of $\mathbb{N} \times \mathbb{N}$, that is, the elements in the ordered pairs must be natural numbers. Then, paralleling a requirement of the preceding problem, the student was asked to create an algebraic expression which would have the ordered pairs in his set in its solution set. Each of the four introductory sections gave rise to an algebraic linear expression which were presented in bracket form in section five: $A(t) = t$, $B(t) = t + 5$, $C(t) = 2t + 5$, and $D(t) = 2t + 2$. Equipped with the desired algebraic expressions and representatives of the solution set, the student, in 5a, was asked to graph these relations (functions). The graph is now presented.

Now the final portions of this problem were designed to focus student attention on certain properties exhibited in the graphs and the quality of the algebraic expression which
produced this visual property. The specific properties in mind were parallelism (related to slope of the line) and the starting point (the y-intercept when the whole of the Euclidean plane is our frame of reference). Finally the student was asked to examine the algebraic expression in an attempt to form some general conclusions regarding the graphs of expressions of the form, \( H(x) = ax + b \).

It should be noted that the questions in this problem occurred in pairs (or triples); first the student was asked to identify some property and then he was asked to note that some other pair lacked this property. It was hoped that this type of questioning would force the student to focus his attention on the desired property. As a precise example of this type of questioning let us consider part 5b and 5d.

5b) The graphs of \( A(t) \) and \( B(t) \) are in some respect similar. Explain.

Many answers are of course available to the student, thus giving him freedom in his examination, selection and identification. Some of his answers could have been: (1) They are both straight line segments. (2) The first component of the starting point is zero. (3) The second component of the terminal point is "18". (4) The line segments do not intersect. (5) The line segments are parallel. Now, with all these possible answers, how does the student limit or select his answer? He has indeed had an opportunity to
Let his mind wander. Then his selection or answer is subjected to another condition.

5d) The similarity noted above does not exist between A(t) and C(t), between A(t) and D(t). Explain.

This condition forces the student to discard all the enumerated answers except the answer, "They are parallel line segments". In general this two-condition search did promote student recognition of the property desired, however, the student was not so successful in identifying that part of the algebraic expression, which produced the particular property.

Technique Used with Control Group: The instructional form used for the teaching of new material to the students in the control group can probably best be described as a "classroom development" approach. A typical class period will best illustrate the technique.4

Approximately twenty minutes at the beginning of the class period was devoted to previously studied material. This would include questions of a general nature as well as specific questions about the problems which had been assigned. Following the question period, the new material was considered. The goal of the technique was the

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4This same approach to instruction was used with the exploratory group when a topic or section was considered which had not been introduced through a problem promoted investigation.
development of the new topic(s) in a mathematically sound fashion through the cooperative efforts of students and instructor. When the student contributions were limited, the essentials of the new material and the supporting details were presented by the instructor. It was not an authoritative presentation, but rather a development (rigor consistent with course and student level) of the topic by the instructor through a series of mathematically justifiable steps. After the introduction of the new material had been completed, some related examples were solved in class so the student would have some experience with the application of the theoretical material. At the end of the period the students were given an assignment which was to be completed prior to the beginning of the next period. The"homework" provided (1) an opportunity to develop a more complete understanding of the theoretical material, (2) an opportunity to use the new material in a more practical situation, and (3) an opportunity to develop some skill in the use of the material when the topic warrants it.

The outline of the classroom development of a particular topic will provide some detail. As an example the addition/subtraction of fractions will be considered. First, the sum of two fractions with identical denominators would be developed using the definitions, basic assumptions,
and previously established theorems. Then, the definition of least common multiple would be considered along with techniques of finding the least common multiple. Next, the fundamental theorem of fractions— if \( a, b, c \in \mathbb{R} \) and \( b, c \neq 0 \), then \( \frac{a}{b} = \frac{ac}{bc} \)— would be used to find the sum of any two fractions. Then, the parallel existing between the least common multiple and the least common denominator would be explored. Finally, the difference of two fractions would be considered with the result developed and validated within the rational number system. The period would end with the assignment of exercises related, for the most part, to those topics which had been considered during the class period.

In short, the topic is developed by the instructor and/or students theoretically and quite completely in the classroom. Then a few examples are considered in order to illustrate the application of the new theory. Finally, the student is given an out-of-class assignment so that he may come to a more complete understanding of the theoretical phases and develop an ability to work with the new material in non-theoretical situations.

Tests on Growth of Control and Exploratory Groups: It seemed that, even in these exploratory/developmental phases of the problem technique, some attempt should be made to measure student growth. In order to obtain data for a measure of relative student growth, this investigator taught
a control group as well as the exploratory group as indicated in the preceding section. Identical one-hour tests were given to the two groups at the beginning of the school term (see Test 1, Appendix B) and identical two-hour tests (see Test 2, Appendix B) were given at the end of the school term in Study One. The same pattern of testing was used in the second study (see Tests 3 and 4, Appendix B). These tests and the statistical measure used are discussed in greater detail in those sections of the next chapter which are devoted to the statistical results.

The Exploratory Studies: Two exploratory studies were undertaken in an effort to gain information about the proposed use of problems. Seven months of limited preparation preceded the first study. As one phase of this preparation, the investigator taught the course which was to be used in the first exploratory study. This provided an opportunity to become familiar with the content of the course, the "philosophy" of the course, and the typical student. Then, following the first study, another seven-month period preceded the second exploratory study. Extensive research was undertaken during this time interval. The analysis of this research and the results of the first study influenced the execution of the second study which was conducted in a different course in a different institution. Once again, this instructor taught the "exploratory" course during the
term preceding the study so that he might become familiar with the course, the students, and the institution.

Study 1: The first exploratory study was undertaken at a large midwestern university. The mathematics course used in the study was one designed for and required of all students in the elementary education curriculum. The only mathematics course required of these students, its goal was to promote an understanding of the mathematics which is taught in the elementary grades. How the children in the grades should be taught arithmetic was the subject of another course. The course met three days per week for a period of ten weeks. Many of the students—when given the opportunity to make comments about the course—said that the number of class sessions should be doubled or even tripled and that additional material should be included in the course.

The students, mostly female, were drawn from a large area geographically. Knowing that they would soon be teaching related material, most of the students were quite interested in the topics considered. Few of the students, who must maintain a 2.25 grade point average in this curriculum, had completed more than two years of mathematics in high school. More than one-half the students had completed a college course in basic algebra, but only three in the exploratory group and five in the control group had completed some other mathematics course.
In order to determine which of the two sections would be used as the exploratory group, an "exploratory" card and a "control" card were prepared. At the time of the morning class a fellow instructor selected from the deck of two cards the "control" card. The early section was then designated the control group. Each of the groups had been created by random assignment of the students to the various sections of the course at a given time period. Subjectively, the two groups seemed to be about equally well prepared for the course. In addition, both groups appeared to possess an inner motivation to study mathematics.

Study 2: The second exploratory study was conducted at a two-year college in the Far West. For the most part the students were from the lower half of their high school graduating class. Selection of this college was probably prompted by (1) friends are attending, (2) the school is close to home, (3) the student cost is minimal, and (4) the two-year program is a step toward the four-year college. Although there were some sincere students and there were some sincere, capable students, this investigator does not believe that these formed a majority group. Because of the low cost of attendance, the low interest level of the student, and a liberal withdrawal policy, the drop rate at the institution was quite high. Although the student may have had good intentions, he just didn't get started. The
result was that frequently class enrollment decreased from 30 per cent to 50 per cent during the term (true in most areas of study). In the two groups used in the study, 40 per cent (to the nearest 5 per cent) of the students were no longer attending at the end of the term.

An Intermediate Algebra course was used in the second study. It is the second course in a sequence of algebra courses: Beginning Algebra, Intermediate Algebra, and College Algebra. The Intermediate Algebra course was recommended for those students who (1) scored within a certain range on a placement test, (2) had completed at least four semesters of high school mathematics, or (3) had completed Beginning Algebra with a grade of "C" or better. The class had four sessions per week for a period of sixteen weeks (3 semester hours of credit). The investigator taught a 9:30 section of this course as well as a section at 10:30. Each of these was the only section of the course offered at the given hour. Hence, the student's selection of time period was involved in the formation of the sections, but it would seem that this selection would have had little to do with the character of the students. In this study the "exploratory" section was determined by the flip of a coin. In contrast to the students in Study 1, the students in Study 2 seemed reluctant to associate success in the course with future success although it could be used to satisfy the institution's degree (Associate of Arts) requirement in mathematics.
Conclusion: An outline of the use of problems to initiate certain topics in mathematics has been presented in this chapter. The assumptions which provided direction for the exploratory-developmental phases of the study were enumerated as were the criteria which influenced problem creation or selection. Some examples were included (1) to give an indication of some of the topics selected for problem introduction, (2) to illustrate the varying forms of learning desired as a result of the problem-promoted activity, and (3) to introduce some of the problems and exercises and to explain their part in the stimulation of the desired learning. Then, instruction in the control group was discussed. Following the general discussion, an outline of a typical class period provided some detail. The last major section provided pertinent information about the setting—student, course, and institution—in which each of the studies was undertaken.
CHAPTER III

ANALYSIS OF OBJECTIVE DATA

We hope that many problems can be found that read: "Here is a situation--think about it--what can you say?"—Goals for School Mathematics (47: 11).

Introduction: The first two chapters have presented (1) the background of the problem and a survey of related literature, (2) the assumption which influenced the design of the exploratory studies, and (3) a description of the use of problems to initiate the study of certain topics in mathematics. Results of the two studies are discussed in this and the following chapter. The categories considered are related to these questions. First, what are the immediate effects of using problems (in the exploratory group) to introduce new topics; that is, what can be said about problem-related student performance? Second, what are the more remote—and yet, more significant—effects of this use of problems; that is, what can be said about the student's learning of mathematics? Third, what are the side effects of such use of problems; that is, what can be said about group reaction to this technique of instruction? The discussion of the first two categories,
supported by numerical data, is presented in this chapter while group reaction is considered in Chapter IV.

**Performance of Exploratory Group on Problem:** The goal of the problem presentation was the stimulation of a "topic-related" exploration prior to the classroom introduction of the topic. Three levels of student performance seem critical in such an exploration. They are performance on the problem as a whole, performance on productive thinking items, and performance on discovery items.

Four problems - Problem 1, Problem 4, Problem 5, and Problem 13, (see Appendix A) - were selected for the numerical phase of the analysis of student performance. Three factors guided the selection of these four problems. First, the four represent three of the types of problems used. Second, they represent the two studies (two from each study). Third, they represent initial student attempts (early in the school term) as well as terminal student attempts to solve problems (late in the school term). Although the tabular data introduced is limited to the enumerated problems, other problems are considered when appropriate.

**Student Activity Promoted:** Did this technique of problem presentation stimulate activity which is related to an exploration/solution of an unfamiliar problem? Did the student search for answers to the challenges presented? Table 1 contains a nine-interval breakdown of student scores on the selected problems. The distribution of
TABLE 1.—Distribution of student scores (total score) on four selected problems from study 1 and study 2

<table>
<thead>
<tr>
<th>Score</th>
<th>Frequency</th>
<th>Score</th>
<th>Frequency</th>
<th>Score</th>
<th>Frequency</th>
<th>Score</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>33-34</td>
<td>1</td>
<td>25-27</td>
<td>2</td>
<td>45-48</td>
<td>0</td>
<td>46-50</td>
<td>1</td>
</tr>
<tr>
<td>31-32</td>
<td>1</td>
<td>22-24</td>
<td>2</td>
<td>41-44</td>
<td>2</td>
<td>41-45</td>
<td>5</td>
</tr>
<tr>
<td>29-30</td>
<td>3</td>
<td>19-21</td>
<td>5</td>
<td>37-40</td>
<td>3</td>
<td>36-40</td>
<td>4</td>
</tr>
<tr>
<td>27-28</td>
<td>2</td>
<td>16-18</td>
<td>2</td>
<td>33-36</td>
<td>8</td>
<td>31-35</td>
<td>6</td>
</tr>
<tr>
<td>25-26</td>
<td>4</td>
<td>13-15</td>
<td>3</td>
<td>29-32</td>
<td>4</td>
<td>26-30</td>
<td>1</td>
</tr>
<tr>
<td>23-24</td>
<td>6</td>
<td>10-12</td>
<td>4</td>
<td>25-28</td>
<td>1</td>
<td>21-25</td>
<td>1</td>
</tr>
<tr>
<td>21-22</td>
<td>5</td>
<td>7-9</td>
<td>1</td>
<td>21-24</td>
<td>1</td>
<td>16-20</td>
<td>0</td>
</tr>
<tr>
<td>19-20</td>
<td>4</td>
<td>4-6</td>
<td>6</td>
<td>17-20</td>
<td>0</td>
<td>11-15</td>
<td>1</td>
</tr>
<tr>
<td>17-18</td>
<td>3</td>
<td>0-3</td>
<td>4</td>
<td>13-16</td>
<td>1</td>
<td>5-10</td>
<td>1</td>
</tr>
</tbody>
</table>
scores in the table illustrates that the problem has stimulated some student activity in the attempt to solve the problem. In order to get a more precise measure of this activity, the average scores (means converted to per cent of possible score) were calculated for each of these problems. These average scores were:

Problem 1: average score, 70
Problem 4: average score, 43
Problem 5: average score, 68
Problem 13: average score, 69

The average student scores also indicate that the students (1) did attempt to solve the problem and (2) did make considerable progress in meeting the overall demands of the problem.

The average student score on problem 4 was considerably lower than it was on the other three. This problem is representative of a special category. In each of the two studies two problems were designed to illustrate for the student the nature of the game of mathematics; that is, they were to introduce the student to deductive proof in mathematics. To this end the problem introduced undefined terms (familiar to the student) and provided the definitions and postulates. The student activity promoted by the problem was a search for the deductive proof of certain properties or theorems. However, the typical student, with
limited background and mathematical maturity, experienced marginal success in his search.

In contrast to the numerous demands of the problems which required deductive proof, some problems were limited to a preliminary statement followed by fairly routine questions. A careful analysis of the statement of these problems was required with activity geared to reinforce the significant conclusions prompted by the analysis. This type of problem corresponded to the greatest level of student success. For example, the mean score on Problem 8 (Appendix A) was 46 (approximately 95 per cent). Other problems contained the preliminary statement with questions related either to the initial circumstances or to some additional conditions which were presented after the initial statement. The level of overall success—see averages for problems 1, 5, and 13 introduced on page 60—with such problems was not as high.

One problem (see Problem 12, Appendix A) promoted almost none of the desired activity. An algebraic expression for the sum of "n" consecutive natural numbers had an important part in the answers to most of the items in this problem. Since the students did not know of the existence of the required expression, the formula was derived (genetically) in class prior to the distribution of the problem. Nevertheless, few students were able to provide the correct answers to the questions presented.
In conclusion, the amount of overall activity promoted seemed to be related to the type of problem and its difficulty. Although the amount of activity may have varied, it was evident that the unfamiliar problem did promote considerable student activity.

**Productive Thinking Stimulated:** Did the student exploration of the problem situation stimulate some productive thinking? Some indication of the use of the term "productive thinking" should precede our discussion of this phase of the exploratory studies. The expression "productive thinking" was adopted by this investigator from *Productive Thinking* (104: ) by Max Wertheimer. Since we cannot be sure that the present interpretation is that of Wertheimer, an attempt will be made to make our interpretation clear. Our attention will be directed to the word "productive" rather than the word "thinking." Webster, in his *New Collegiate Dictionary* (103: 673) used, among others, the words, "creative," "fertile," and "originative" in his definition of productive. These words seem to imply what the author has in mind when he says, productive thinking. It is thinking which produces a conclusion which is new and alive, capable of growth, with avenues which potentially lure and challenge the student to advance beyond the original conclusion to new limits. The conclusion is not obtained mechanically and frequently it is not the
end—new discoveries lie beyond the original conclusion, and often the implications of the original conclusion are quite as important as the conclusion itself. Two items from Problem 5 (see Appendix A) will serve as concrete illustrations.

Item 2: Find T2 for each of the following players (Use your table).

Item 6: Each of the tables provided possesses a property (a pattern) which your tables in Item 1 of the problem do not possess. Explain.

Now in Item 2 the student refers to a table and writes his answer or perhaps he refers to one of the examples which provides the desired answer. T2 = 100. This investigator does not refer to this activity as productive thinking. Items of this kind will be classified "mechanical" as we could say that the answer to "3 + 4 = ____" is mechanical although some thought is involved.

However, in Item 6, the student is to examine several tables (see figure 4) which have been put into two distinct groups.\(^5\) His task is to find how the two classes differ. The difference is that the tables in one class are symmetric about a diagonal while no such symmetry exists in the tables of the other class. This difference is not obvious; the

\(^5\)See Sections 1 and 7 of Problem 5, Appendix A, for additional tables.
answer cannot be determined mechanically. This item, then, possesses a capacity for promoting productive thinking. Note that the "symmetry" conclusion should pose new questions and open new avenues to exploration.

<table>
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<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Figure 4

Since we wish to determine if productive thinking has been stimulated, we will look only at items which are phrased in such a way that such thinking would be promoted. The mechanical items will be eliminated from this analysis just as we would not ask, "How fast did the miler run the first one-hundred yards" although we might choose to do this. To proceed with the analysis of the productive thinking of the student, this investigator created a second scheme of numerical rating for use with the items which potentially promote this activity. This scheme is as follows:
0 - no activity, or some activity but not meaningful
1 - some activity, meaningful
2 - activity, "productive thinking," is or approaches that expected by this investigator
3 - activity, conclusions very good, goes beyond that (productive thinking) expected by the instructor, or is judged "discovery"

In this analysis four problems (the four problems used in the previous section: 1, 4, 5, and 13 (Appendix A)) were used. Tables 2 and 3 provide information about the number of students obtaining the various numerical scores on the "productive" items of these problems.

In order to complete the analysis, certain numerical measures were derived from the tabular entries. Using the 0-1-2-3 item rating, the average numerical score on each item was calculated for each of the problems. The results were:

- Problem 1: average numerical score, 1.1
- Problem 4: average numerical score, 0.8
- Problem 5: average numerical score, 1.2
- Problem 13: average numerical score, 1.1

The average-per-item score on these four problems is seen to lie in the 1.0 ± 0.2 interval. The center of this interval is 1.0 which corresponds to "some activity, meaningful." Although this result is somewhat short of "productive thinking and response," these averages portray the student as an investigator who is seeking an answer.
TABLE 2.—Number of students obtaining the various numerical indicators of student activity on two selected problems in study 1

<table>
<thead>
<tr>
<th>Numerical Classification of Student Activity</th>
<th>Item Number</th>
</tr>
</thead>
<tbody>
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<td>8 11 4 25 6</td>
</tr>
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<td>3</td>
<td>0 0 1 0 2</td>
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</table>

*Items selected for inclusion in the table are those designed to promote activity which author would classify as productive thinking/discovery.
TABLE 3.—Number of students obtaining the various numerical indicators of student activity on two problems in study 2

<table>
<thead>
<tr>
<th>Numerical Classification of Student Activity</th>
<th>Item Number</th>
<th>Problem 5</th>
<th>Problem 13</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1b 4c 5 6 7</td>
<td>6b 6c 6d 6e 6f 6g 6h</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0 1 2 8 4</td>
<td>4 4 4 8 4 12 11</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>6 14 15 8 7</td>
<td>3 3 4 10 4 3 7</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>14 5 3 4 9</td>
<td>13 13 10 2 12 5 2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0 0 0 0 0</td>
<td>0 0 2 0 0 0 0</td>
<td></td>
</tr>
</tbody>
</table>

*Items selected for inclusion in the table are those designed to promote activity which author would classify as productive thinking/discovery.*
which fits the circumstances of the problem and item. This degree of success indicates that the student is doing more than writing some answer to fill a blank space. Indeed, this student investigation (involvement) was one of the activities which should follow the distribution of the problem.

Although the average-per-item score is thus seen to be 1.0 ± 0.2, the tabular entries show that in the rating of each item some answers disclose the existence (subjective evaluation) of productive thinking (rated 2 or 3). To obtain a single number which is representative of this group of answers, the number of answers thus classified (235) was divided by the total number (number of students times number of items) of answers (675). The result was 0.35. This means that 35 per cent—or slightly more than one-third—of the answers qualified at this level.

These numerical results indicate that productive thinking on some items has taken place. If this productive thinking can be associated with a personal development (learning) of some mathematics, then many of the students have been personally involved in the learning of mathematics. This is important since the student's ultimate success in mathematics is going to be determined in large measure by the magnitude of his own successful efforts.
A second consideration of Problem 4, "the proof problem," discussed in the last section provides a concluding thought for this section. On this problem overall student activity (43 per cent) was considered low. Since student success was normally high on mechanical items, the absence of mechanical items on Problem 4 is seen as the probable cause for the low overall score. Nevertheless, this problem appears to have promoted productive thinking which is nearly on a par with that promoted by the other problems. In fact, if fewer proof (demanding) items had been presented (ten were presented), it is possible that the productive thinking measure per item may have been higher on this problem than on the others examined. This suggests that one of the greatest deficiencies of a problem could be its overadequate demands. We should note that the creation of an entirely new situation (advantages discussed elsewhere) introduced many--often mechanical--challenges, perhaps so many that the productive activity of the student (the principal goal) may have been lessened.

**On Discovery of Mathematical Concepts:** Did the student exploration of the problem situation and his search for answers to the challenges presented promote the discovery of mathematical concepts? As has been stated elsewhere, the student discovery which may occur is an unexpected bonus of the presentation of the problems. Since "productive
"productive thinking" items were created, frequently the potential for promoting discovery was present (see the discussion of our use of "productive thinking" on page 62). It was felt, therefore, that some of the students may make "mathematical" discoveries. A concise example which might promote discovery follows.

Situation: Two algebraic expressions:

\[ f(x) = 2x + 1 \]
\[ g(x) = 2x + 4 \]

Item A: Graph these expressions on the same set of coordinate axis.
Item B: What can you say about the graphs?

Answer: They are parallel straight lines.

Now the alert student would not stop here. He would ask himself, "What lies behind this parallelism?" or "Is there an algebraic explanation for this parallelism?" In this example a potential for further examination lies beyond the requested student response. Now the typical item did not directly promote student investigation beyond the answer. The concluding examination was a cooperative effort in the classroom. It remains true, however, that the student could have begun the search which resulted in "discovery" if he had chosen and he could have reported his results.

With this background we return to an analysis of student discovery of mathematical concepts. In an earlier
section (Page 65) a numerical rating scheme was developed for the classification of student activity on items which potentially promote productive thinking. The "3" rating in this scheme was: "activity, conclusions very good, goes beyond that (productive thinking) expected by the instructor, or is judged "discovery." The early introduction of this numerical category, which exceeded the requirements of the previous section, makes the results already reported serve our needs in the present section. Table 2 and Table 3, therefore, provide information about student discovery of mathematical concepts on four selected problems. Most of the items included in the tables are items which present to the alert, capable student opportunities for discovery. The tabular entries indicate that approximately one per cent of the item-answers qualify at level 3. In short, the student answers do not suggest that much discovery has taken place.

Before terminating this section some space should be devoted to a consideration of special attempts to promote discovery. Most of the items were problem-situation oriented; that is, most of the questions which were asked were directly related to the circumstances of the one problem presented. However, in Problem 13 (Appendix 4) questions were asked which required the student to go beyond the situation created in the problem. We read in this
problem. "The following questions are not related to the irrigation problem. They are questions about the algebraic expressions and their graphs." That is, the students were asked to go beyond the physical problem— they were to look at the results from an algebraic or abstract point of view. The intended purpose of this type of question was to get the student to that level of learning which could be called discovery— discovery promoted because of a special consideration of some physical problem. Unfortunately, the students found this departure from the circumstances of the problem very difficult. Their attention was focused on the problem— their answers were given in terms of the problem situation.

An additional example of student difficulty in getting away from the problem was illustrated clearly in Problem 11 (Appendix A) which was given prior to the classroom discussion of relation-function concepts and graphing. The item read as follows: "There is an interesting (important) difference existing between Part a and Part b (Figures 1 and 3). Examine carefully these figures: determine the nature of this difference. Give an explanation of this difference." Note that the item refers specifically to the graphs, not to the circumstances of the problem. The difference was that Part a illustrated a one-to-several correspondence (relation, one or two circles on each
vertical line in the figure) while Part b illustrated in this problem a many-to-one correspondence (function, one circle on each vertical line in the figure). Only a few of the students successfully identified this difference. As had happened in the problem discussed in the last paragraph, most of the answers were given in terms of the physical problem presented; in other words, most of the answers were about the "grades of students". However, as chance would have it, a student from the control section was visiting class on the day that we were discussing this problem. This student had not advanced to the related textbook material and he had not seen the problem. Yet, he was able to verbalize this difference almost immediately when he had seen only a portion of the figures displayed on the board. Although limited discovery had taken place, almost everyone recognized and understood the distinction pointed out by the visitor. Once the difference was understood, it was a simple matter to give the proper names to the concepts. We take this opportunity to again point out that "learning with understanding" which follows the "pre-learning" promoted by the problems was the goal of the problem presentation.

In conclusion, it seems that limited student discovery occurred at least on the few items designed to promote it. It should be pointed out, however, that the
student may have made discoveries which he did not report because the phraseology of the item did not specifically request it.

**Growth of Control and Exploratory Groups:** The earlier sections of this chapter have been devoted to an analysis of student performance on some of the problems which were used to introduce new topics during the school term. These analyses were necessarily related to the exploratory group. The remaining sections of this chapter will seek an answer to the question, "Is the problem technique of instruction superior to some more widely used form of mathematics instruction?" This author believes that the technique of instruction described in Chapter Two—referred to as "a classroom development approach"—is one of the more widely used techniques of mathematics instruction. It was, therefore, used as the instructional technique with the control groups in both studies.

Student choice of hour and/or institutional placement created the sections used in the two studies prior to the "chance" designation of one group as the control group and the other as the exploratory group. Since these creation procedures would seem to be random, can we assume (for statistical purposes) that the two groups were equivalent? The answer to this question is not required since statistical methods which correct for initial differences which
may have existed are available. Such a measure is the
covariance method of the analysis of variance. McNemar
says, "The method [covariance] is applicable whenever it
seems desirable to correct a difference on a dependent
variable which for some reason could not be controlled
by matching or by random sampling procedures". (69: 363).
This method makes allowance for an uncontrolled variable
and includes a sampling error adjustment which is needed
in testing the statistical significance of the difference
between "corrected" means. (69: 363). The covariance
method of the analysis of variance was used for the
statistical results which are reported in these sections.

Since the analysis of covariance effects an adjustment
in final means for some initial uncontrolled variable, some
beginning measure is essential. A one-hour test (see Test 1
[study one] and Test 3 [study two], Appendix B) given at the
beginning of the school term served as the uncontrolled or
criterion (independent) variable and provided the initial
scores. Although the analysis of covariance does permit
the use of more than one independent variable, this one-hour
pretest was the only independent variable used in the sta-
tistical calculations which are herein reported. Tables 4
and 6 contain the student scores used in the statistical
calculations.
### TABLE 4.—Test scores for subjects in study one

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<tr>
<th>Control Group</th>
<th>Exploratory Group</th>
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<td>25</td>
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**On Routine Test Items:** One of the primary phases of the discussion of results should be related to the student's learning of the material typical of the course. "Did the
technique produce test-measurable differences?" The intent of this phase of the analysis of student learning can be made more precise by phrasing it as a null hypothesis.

The introduction of new topics through a problem promoted student exploration is not significantly superior to a "classroom development" approach when the goal of the instruction is the learning of the routine material of the course.

In the context of this hypothesis, routine is not to be identified with trivial, but rather with material which is relatively straight-forward or which follows some familiar pattern although the work involved may be quite difficult.

In Study One the total score on the final examination served as the dependent variable for the analysis of co-variance. The between-groups adjusted sum of squares was found to be 1568.27 (see Table 5), while the within-groups

<table>
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<th>Source of Variation</th>
<th>df</th>
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<th>Variance</th>
<th>F Ratio</th>
</tr>
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<td>Within</td>
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<td>35,145.17</td>
<td>595.68</td>
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<tr>
<td>Total</td>
<td>60</td>
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<td>F (0.05 level) = 4.00</td>
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</table>
adjusted sum of squares was 25,145.17. Using these adjusted sums of squares among (1568.27) and within (595.68) variances were determined. Finally an F of 2.63 was calculated. Since the F for 1 and 59 degrees of freedom is approximately 4.00 (0.05 level of significance), the statistical results do not suggest that there are significant group differences on the (dependent variable) final examination over and above those which would be expected because of differences which existed

TABLE 6.—Test scores for subjects in study two

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</table>
on the uncontrolled variable. However, the corrected means—control group, (116.24)/exploratory group, (126.35)—do provide some information about the "direction" of test results.

In Study Two, Part I of the final examination contained those items (routine) which are typical of (1) the material normally taught, (2) the material normally learned by students, and (3) the material which normally appears on examinations (see Test 4, Appendix B). As in study one the one-hour pretest was taken as covariate. Based on a between groups variance estimate of 208.02 and a within-groups variance estimate of 329.40, an F ratio of 0.63 (see Table 7) was computed. Since the F for 1 and 33 degrees of freedom

<table>
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<tr>
<th>Source of Variation</th>
<th>df</th>
<th>Adjusted Sum of Squares</th>
<th>Variance</th>
<th>F Ratio</th>
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<td>208.02</td>
<td>208.02</td>
<td>0.63</td>
</tr>
<tr>
<td>Within</td>
<td>33</td>
<td>10,870.22</td>
<td>329.40</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>34</td>
<td>11,078.24</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

is 4.14 (0.05 level of significance) it cannot be concluded that in Study Two there are significant group differences on Part I of the final examination over and above those which
would be expected because of initial differences. Never­
theless, the adjusted means—as provided by the covariance
 technique—were 64.72 for the control group and 69.62 for
the exploratory group.

In conclusion, the components in the test of the null
hypothesis of this section were the pretest, the appropriate
section (designed to measure student learning of the routine
material of the course) of the final examination, and the
analysis of covariance. The data obtained from the test
instruments did not reveal the existence of significant
(.05 level) group differences although the corrected mean
for the exploratory group exceeded the corrected mean for
the control group in both studies.

**On Student Understanding:** One of the primary goals
of problem presentation was to promote student exploration
and critical thought. Does such activity foster a more
significant growth of student understanding? Although it
is difficult—if it is at all possible—to measure under­
standing, some attempt to analyze the growth of student
understanding seemed consistent with the exploratory nature
of the investigation. Following the pattern of the last
section our objective can be made more exact through the
introduction of a null hypothesis.

The introduction of new topics through a problem­
promoted student exploration is not significantly
superior to a classroom development approach when
the goal of the instruction is the development of an in-depth understanding of the fundamental concepts of the course.

This necessarily imperfect analysis of the growth of student understanding was influenced by two premises. First, it is not possible to measure student understanding directly (one can measure the length of a table), therefore, some indirect technique must be used. Second, an attempt to obtain some numerical measure is probably more desirable than is a completely subjective evaluation.

The instrument designed to measure student understanding (Study Two only) appeared as Part II of the Final Examination. It consisted of two companion sections. The first section, containing twenty items, had the general form of a true-false and/or multiple choice test. The inclusion of variables in the statements resulted in the creation of items which were (1) false for all values of the variable, (2) true for certain replacements of the variable and false for others, or (3) true for all replacements. Three examples from the instrument, Test 5, Part II-A (Appendix B) will illustrate (1) the types of statements included, and (2) the use of each of the three answers.

1. If \( x = 4 \), then \( x^2 - 7x + 12 = 0 \). (Number 20)

   Answer, A. A = always true. The "if" condition requires that \( x=4 \), but if \( x = 4 \), the "then" is satisfied.
2. If $a \in \mathbb{R}$ (real numbers) then there exists $h \in \mathbb{R}$ such that $(a)(h) = 1$. (Number 21)

Answer, $S$. $S =$ sometimes true and sometimes false. The "if" condition is satisfied when $a = 3$, the "then" is satisfied by selection $h = 1/3$. $\text{TRUE}$ The "if" condition is satisfied when $a = 0$, the "then" cannot be satisfied. $\text{FALSE}$

3. If $A$, $B$, and $C$ are non-empty sets, then $A \times (B \times C) = (A \times B) \times C$. (Number 23)

Answer, $N$. $N =$ never true (always false). When the "if" condition is satisfied, the "then" cannot be satisfied.

Hopefully, the student's ability to select the proper answer--frequently requiring an ability to make subtle distinctions while considering "special" cases--would give a meaningful measure of the student's understanding of some of the fundamental ideas represented in the twenty statements of this section.

Since test questions of this type are not common, the students in both groups were told of their presence in the final examination and the special character of the answer which was required. In addition, examples were included in the classroom discussion. In spite of its unique character, the students seemed to have little difficulty understanding what was required in this twenty item section.

The companion section, Test 4, Part II-B (Appendix B) required a creative student response. Related to the first section, it frequently prompted a second critical analysis
of answers given in Part II-A. The student was instructed as follows:

The correct answer for at least five of the answers in Part II-A was S. Identify five of these statements, then for each of the five identified statements (a) give an example in which the statement is true and (b) give an example in which the statement is false.

This section called for student creation of examples (see 2 above) which satisfied certain specified conditions. Quite frequently special examples were required which in turn meant that often only a limited number (perhaps only one) of each type example existed. It was hoped that this creative section would also provide a measure of student understanding.

Once again the statistical measure used to test the null hypothesis was the analysis of covariance. The dependent variable was the specially designed Part II of the final examination and the one-hour pretest was again selected as covariate. The between-groups adjusted sum of squares was found to be 237.61 (see Table 8) while the within-groups adjusted sum of squares was 9,992.68. Using these adjusted sums of squares and the appropriate degrees of freedom, between-group (237.61) and within-group (302.81) group variances were determined. The ratio of these variances was an F of 0.78. Since the F for 1 and 33 degrees of freedom is approximately 4.14 (0.05 level of significance)
TABLE 8: Analysis of Covariance: Uncontrolled (Criterion) Variable: Term Pretest; Covariate: Final Examination, Part II.

<table>
<thead>
<tr>
<th>Source of Variation</th>
<th>df</th>
<th>Adjusted Sum of Squares</th>
<th>Variance</th>
<th>F Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Between</td>
<td>1</td>
<td>237.61</td>
<td>237.61</td>
<td>0.78</td>
</tr>
<tr>
<td>Within</td>
<td>33</td>
<td>9,992.69</td>
<td>302.81</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>34</td>
<td>10,230.30</td>
<td></td>
<td>F (0.05 level) = 4.14</td>
</tr>
</tbody>
</table>

The statistical results do not suggest that significant group differences exist on the section of the final which was designed to measure understanding. However, the statistically adjusted mean for the exploratory group (50.10) did—as in the previous section—exceed that of the control group (44.87).

Conclusion: This chapter contains an analysis of the numerical data. The first three sections were devoted to a measure of student performance on some of the problems which were used to initiate the study of certain topics during the school term. The significant conclusions were: (1) the problem did stimulate student activity in the attempted solution of the unfamiliar problem, (2) the attempted solution of the problem did promote productive thinking, and (3) evidence that discovery of mathematical concepts had occurred was almost nonexistent. Recall that
discovery was not one of the desired goals of the problems and, therefore, did not influence their design. In general, student response was approximately at the level expected.

The last two sections were devoted to a statistical analysis of the growth of the control and exploratory groups. The two phases of the attempt to measure growth were related to (1) the student's learning of the routine material of the course and (2) the student's development of an in-depth understanding of the fundamental concepts of the course. The data for the statistical calculations (analysis of covariance) were obtained from a term pretest and the final examination. Since no differences at the .05 level of significance were disclosed by the data obtained from these instruments, the results suggest that any real differences in growth which may have occurred were not measured by the test instruments used in these studies.
CHAPTER IV

SUBJECTIVE REACTIONS TO EXPLORATORY AND
CONTROL GROUP BEHAVIOR

It is an excellent idea for a problem to foreshadow future developments, to teach as well as to test. - Fine (40: 99).

Introduction: In the previous chapter it was noted that, although the problems did promote the desired student activity and productive thinking, data obtained from the term pretest and the final examination do not reveal differences in the growth between the control and exploratory groups. Whatever the tests measured, it appeared to be about equivalent for both groups. In contrast to the numerical measures and results considered in that chapter, this chapter is a subjective report of some of the investigator's observations and student's comments which are indicative of group behavior. It attempts to present a picture of the student reaction to the use of out-of-class problems.

General Observations, Both Studies: "What answer do you want?" Spell it out for us. In neither situation was the first problem greeted with great enthusiasm. The students felt that the first problem was not related to the
course since it did not look like the material in the
textbook—in fact, "We have not seen anything like this
previously." The problem was labeled extra work by
many. They were especially bewildered by the following:

1. Frequently the problem did not point out
   clearly the answers which were to be obtained.

2. Often the answer was not a number or an
   equation but a verbal answer to a question
   about a set of circumstances (This is
   mathematics?).

3. Unlike anything they had encountered in
   mathematics, the problem would require
   special effort since a ready-made answer
   was not available and they could not follow
   the pattern "illustrated in an example."

4. The work required was just too difficult.

As the student's familiarity with the problems and their
requirements increased, the original resistance (lack of
acceptance) decreased. The resistance-acceptance phases
of problem presentation can probably be summarized as
follows:

1. There would have been less resistance to the
   problem technique if the problem had been
   strictly algebra and if there had been a
   unique-precise answer to the questions pre-
   sented (A question which has a unique-precise
   answer would not have promoted the desired
   exploration/search.)

2. There was more general acceptance when the
   student began to realize that:

   a) There was an important connection between
      the problem and the mathematical material.

   b) The goal was not a unique-precise answer
      but one which was a reasonable explanation
consistent with the set of circumstances presented.

c) His answer did not necessarily look exactly like that of the instructor.

Quite early in the term most students seemed to accept the problem as they would accept a scheduled test. In fact, student comments indicate that--before the end of the school term--a majority welcomed the occasional problem as an interesting and challenging task which was more acceptable than "another textbook assignment."

**Student Comments, Study One:** "I am glad I took this course. It has been good for me." This was the report of the student in the course who had the greatest preparation in the field of mathematics--3 semesters of a calculus and analytic geometry sequence. The students in both groups (in general) found the course interesting, challenging, and at times difficult to understand. At least one in each group did not expect the material which was presented. One student in the control group felt more time should be devoted to methods of teaching the material while another in the same group was critical because the instructor used a second approach to the material when a student did not follow the development when first presented.

Many of the students in the experimental group reported that the problems were very challenging--frustrating--discouraging. Some welcomed the challenge and "tried to meet
the challenge." They found the problems frustrating but also found that they "were forced to go beyond mere mem-
orization and into true understanding." Although dis­
couraged at times in the course, "I can truthfully say that I have learned more about math, if not reasoning, thinking, and problem solving . . ." Another wrote, "I like your method of teaching--letting us discover things for ourselves, then relating this information to things in the book, . . . because we will retain these principles longer." One didn't "really understand the purpose of giving us problems before having the material." As a group, they felt that the problems (challenging, diffi-
cult, frustrating, discouraging) helped bring them to a more complete understanding of the material.

**Student Comments, Study II:** General comments about the course brought to light the following facts. Two in the exploratory group did not like the emphasis placed on the modern concepts including the basic laws. At least one in each group felt that more time should be spent on the "more difficult" material. One individual in the con­
trol group suggested that more time be used working prob­lems of the type to be assigned.

The exploratory group had many comments to make about the problem-solution situations which were obviously upper-
most in their thoughts. Three-fourths of the students
were pleased with the homework problems (at the end of the course). These students reported that the problems were challenging, stimulating, and difficult, but that they were good. Several saw them as an aid to an understanding of the material. Others were led to see mathematics in a new light—as a useful subject instead of just mathematics. For example, one student wrote, "I feel that these problems helped me to realize how much mathematics means to us even in our daily lives."

Within the group of students who expressed a favorable reaction to the problems were some who voiced some reservations. The most frequently registered complaint was that the proof problems, Problems 6 and 7 (Appendix A), were too difficult.

Not all the students were happy with the problems—even at the end of the course. Three stated their disapproval of the problem technique. One said that they were not necessary to a learning of intermediate algebra. The comment of a second was more direct: "Those problems didn't do anything for me except, that is, for a few headaches..." The third credited the problems with his loss of interest in the course. He said, "It was those problems you gave us early in the semester. They were either too involved or just too simple. They messed up the whole semester so I just lost interest in the whole
thing." One of the above added, "The class in general didn't appreciate being used as guinea pigs (He was the only person who made such a comment—no other individual in the group even hinted at this conclusion).

There were many more positive comments. For example, "The problems were helpful in many ways and caused some real thinking... They are a nice change from the regular textbook homework." Or again in the same vein, "The problems that we have had throughout the semester I thought were very good... I found that on some of them you really had to stop and think." The problem "keeps you thinking of algebra while it lets your mind run free with new ideas and concepts and how to solve it," wrote another. One of the better students in the group verbalizes the "getting hooked to the problem" result which was discussed earlier in the section on "Student Discovery." He says, "I found that at times I became so involved in the surface problem that I failed to see the underlying principles." Additional passages from comments by students in both exploratory groups are given in Appendix C.

**Student Reaction to the Use of Problems:** Although group attitude and morale vary in some degree from class to class, groups of students at a given level—say, intermediate algebra—frequently do exhibit similar characteristics. In Study Two an attempt to detect radical
differences in group attitude/morale/interest which could be categorized "student reaction to the new use of problems" was undertaken. The critical question was "Do important group differences exist?" rather than "What differences exist?". It seemed to this investigator that if a radical difference existed a relatively crude instrument could detect it. To this end a 35-statement opinion survey, Test 5 (Appendix B), was prepared and given to both groups in Study 2. The statements of the opinion survey were related to the teaching of mathematics, the learning of mathematics, and mathematics itself. The student was told to "select the phrase which best describes your feelings about the statement." The phrases were: A) I strongly agree; B) I agree; C) I neither agree nor disagree; D) I disagree; or E) I strongly disagree. It was hoped that class reaction to the various items and combinations of items would provide the data for the analysis of student reaction. An example will provide information about the theoretical value of the survey. Let us examine Item 14. "We have a very good textbook (or at least a good one)." Now if most of one group agrees with the statement while most of the other group disagrees with it, some difference is seen to exist. This difference could indicate that some experience in the class has turned one group against the text while the other remains pleased with
it. Perhaps, more subtly, some experience has caused one group of students to dislike algebra and the dislike for the text merely is an indication of the feeling toward the course.

After the survey had been completed, each answer was given a numerical value: I strongly agree, 5; I agree, 4; I neither agree nor disagree, 3; I disagree, 2; and I strongly disagree, 1. Using these numerical values, the mean was calculated (to the nearest tenth) for each item and each group. Table 9 lists these means as well as the difference between means for each item. Attention was then directed to those items which had a difference of means of the two groups of 0.5 or more. These items in descending order of the difference are listed in Table 10.

Although some differences are seen to exist in the reactions to the statements, these differences—all less than 1.0—do not indicate to this investigator that an important difference exists between the groups. Nevertheless, the responses on some of the items do provide opportunities for comment.

Standing at the top of the "difference" list was this item, "On the average I spent more than eight hours per week (outside of class) studying for this course." Group tabulation of the responses to this item revealed that three in the exploratory group strongly agree and
TABLE 9.—Group Means on Items of the Opinion Survey calculated after student responses had been given a numerical value, Test 5 (Appendix B)

<table>
<thead>
<tr>
<th>Item Number</th>
<th>Mean Exploratory Group</th>
<th>Mean Control Group</th>
<th>Difference of Means</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.4</td>
<td>3.0</td>
<td>+0.4</td>
</tr>
<tr>
<td>2</td>
<td>3.0</td>
<td>3.2</td>
<td>-0.2</td>
</tr>
<tr>
<td>3</td>
<td>3.6</td>
<td>3.2</td>
<td>+0.4</td>
</tr>
<tr>
<td>4</td>
<td>1.9</td>
<td>2.1</td>
<td>-0.2</td>
</tr>
<tr>
<td>5</td>
<td>1.8</td>
<td>1.9</td>
<td>-0.1</td>
</tr>
<tr>
<td>6</td>
<td>1.8</td>
<td>1.9</td>
<td>+0.1</td>
</tr>
<tr>
<td>7</td>
<td>2.2</td>
<td>3.0</td>
<td>-0.8</td>
</tr>
<tr>
<td>8</td>
<td>3.6</td>
<td>2.8</td>
<td>+0.8</td>
</tr>
<tr>
<td>9</td>
<td>3.3</td>
<td>3.0</td>
<td>+0.3</td>
</tr>
<tr>
<td>10</td>
<td>3.6</td>
<td>4.2</td>
<td>-0.6</td>
</tr>
<tr>
<td>11</td>
<td>1.6</td>
<td>1.5</td>
<td>+0.1</td>
</tr>
<tr>
<td>12</td>
<td>3.7</td>
<td>3.6</td>
<td>+0.1</td>
</tr>
<tr>
<td>13</td>
<td>4.1</td>
<td>3.5</td>
<td>+0.6</td>
</tr>
<tr>
<td>14</td>
<td>3.6</td>
<td>3.6</td>
<td>0.0</td>
</tr>
<tr>
<td>15</td>
<td>3.2</td>
<td>3.2</td>
<td>0.0</td>
</tr>
<tr>
<td>16</td>
<td>1.5</td>
<td>1.6</td>
<td>-0.1</td>
</tr>
<tr>
<td>17</td>
<td>2.8</td>
<td>2.6</td>
<td>+0.2</td>
</tr>
<tr>
<td>18</td>
<td>3.4</td>
<td>3.0</td>
<td>+0.4</td>
</tr>
<tr>
<td>19</td>
<td>1.7</td>
<td>1.6</td>
<td>+0.1</td>
</tr>
<tr>
<td>20</td>
<td>1.8</td>
<td>1.7</td>
<td>+0.1</td>
</tr>
<tr>
<td>21</td>
<td>4.2</td>
<td>3.7</td>
<td>+0.5</td>
</tr>
<tr>
<td>22</td>
<td>2.1</td>
<td>2.8</td>
<td>-0.7</td>
</tr>
<tr>
<td>23</td>
<td>4.0</td>
<td>3.8</td>
<td>+0.2</td>
</tr>
<tr>
<td>24</td>
<td>3.6</td>
<td>3.6</td>
<td>0.0</td>
</tr>
<tr>
<td>25</td>
<td>2.4</td>
<td>1.6</td>
<td>+0.8</td>
</tr>
<tr>
<td>26</td>
<td>2.0</td>
<td>1.6</td>
<td>+0.4</td>
</tr>
<tr>
<td>27</td>
<td>2.0</td>
<td>2.1</td>
<td>-0.1</td>
</tr>
<tr>
<td>28</td>
<td>3.0</td>
<td>2.1</td>
<td>+0.9</td>
</tr>
<tr>
<td>29</td>
<td>2.2</td>
<td>2.4</td>
<td>+0.2</td>
</tr>
<tr>
<td>30</td>
<td>2.2</td>
<td>2.4</td>
<td>-0.2</td>
</tr>
<tr>
<td>31</td>
<td>4.1</td>
<td>4.0</td>
<td>+0.1</td>
</tr>
<tr>
<td>32</td>
<td>3.4</td>
<td>3.6</td>
<td>-0.2</td>
</tr>
<tr>
<td>33</td>
<td>1.7</td>
<td>1.7</td>
<td>0.0</td>
</tr>
<tr>
<td>34</td>
<td>4.2</td>
<td>3.7</td>
<td>+0.5</td>
</tr>
<tr>
<td>35</td>
<td>4.0</td>
<td>3.8</td>
<td>+0.2</td>
</tr>
</tbody>
</table>
TABLE 10.—Items of the opinion survey on which the difference of group means was equal to or greater than 0.5

<table>
<thead>
<tr>
<th>Item Number / Item</th>
<th>Exploratory Group</th>
<th>Control Group</th>
<th>Difference of Means</th>
</tr>
</thead>
<tbody>
<tr>
<td>28. On the average I spent more than eight hours per week—(outside of class)</td>
<td>3.0</td>
<td>2.1</td>
<td>+0.9</td>
</tr>
<tr>
<td>studying for this course.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7. The book need not be supplemented by individual notes, just the class lecture.</td>
<td>2.2</td>
<td>3.0</td>
<td>-0.8</td>
</tr>
<tr>
<td>8. Non-textbook assignments should be given occasionally.</td>
<td>3.6</td>
<td>2.8</td>
<td>+0.8</td>
</tr>
<tr>
<td>15. Mathematics through algebra should be required of all students.</td>
<td>2.4</td>
<td>1.6</td>
<td>+0.8</td>
</tr>
<tr>
<td>22. We covered too much material in this course.</td>
<td>2.1</td>
<td>2.8</td>
<td>-0.7</td>
</tr>
<tr>
<td>11. Individuals tend to think that mathematics is difficult—this belief could result in a mental block.</td>
<td>3.6</td>
<td>4.2</td>
<td>-0.6</td>
</tr>
<tr>
<td>13. Review for an examination on the day before an examination is a good policy.</td>
<td>4.1</td>
<td>3.5</td>
<td>+0.6</td>
</tr>
<tr>
<td>21. Because of the scientific orientation of our society more people must learn mathematics.</td>
<td>4.2</td>
<td>3.7</td>
<td>+0.5</td>
</tr>
<tr>
<td>34. Practical problems can promote the development of new mathematics.</td>
<td>4.2</td>
<td>3.7</td>
<td>+0.5</td>
</tr>
</tbody>
</table>

CIn descending order of the "difference of means"
five agree with the statement, while none in the control group strongly agree and only two agree with the statement. This result suggested that the students in the exploratory group devoted more time to the study of mathematics. Kersch reported a similar finding in an experiment with discovery teaching. He stated, "The data supports the hypothesis that self-discovery motivates the student to practice more and thus to remember and transfer more than he might if taught directly." (63: 65).

The next item of special interest was Number 8: "Non-textbook assignments should be given occasionally." This was the only item on which the means of the two groups stood on different sides of 3.0 (I neither agree nor disagree). In this item the exploratory section tended to agree while the control section tended to disagree. The results suggested that the students in the exploratory group see value in the non-textbook assignments and that the students in the control group who knew of the special problems do not see value in these assignments. (One student in the control group told the investigator that it was unfair to the members of the exploratory group to give them these extra problems since they required time which could have been better spent on the study of textbook material.) Other comments of the students in the exploratory group tended to support the results obtained on this
particular item. One should not, however, overlook entirely the possibility that the students may have a tendency to agree with the techniques of their instructor.

Another combination of items in the above list is of interest. The answers to Items 21 and 34 suggest to this investigator that the students in the exploratory group may have a better understanding of the nature of mathematics, of the development of mathematics, and of the value of mathematics in today's society than does the control group. Once again, other comments by students in the exploratory group support this interpretation of their answers to these two items, while the students in the control group made no comments which are useful in the interpretation of their answers.

Although student selections were approximately equivalent, a difference of means greater than or equal to 0.5 was recorded on certain items. Some of these items indicated that students in the exploratory group 1) devoted more time to the study of mathematics, 2) saw positive value in the assignment of special problems, and 3) had a more detailed picture of mathematics and its place in our civilization. However, the answers on the opinion survey did not indicate that radical group differences in student attitude, morale, and interest exist.
Conclusion: The comments of some students indicated that they were not happy with the introduction of new topics through a problem-promoted exploration. However, three-fourths of the students in Study Two expressed an acceptance of the problem technique in their end-of-course comments. Students reported that the problems were challenging, stimulating, and difficult, but that they were good. Several saw them as an aid to an understanding of the material. Finally, as was evidenced by the opinion survey in Study Two, no radical group differences in attitude, morale or interest seemed to exist at the end of the term. Probably the most significant disclosure of the survey was that the typical student in the exploratory group devoted more time—on the average—to the study of mathematics than did his counterpart in the control group. This fact, together with the statistical results of the previous chapter, suggested the following conclusion. Since the data did not reveal the existence of significant group differences in either one term study and since the students in the exploratory group have indicated that they devoted more time to the study of mathematics, it is possible that using problems to initiate the study of certain topics in mathematics may have resulted in less efficient student learning.
CHAPTER V

SUMMARY AND CONCLUSIONS

--they should be extending the frontiers of their mathematical knowledge largely by their own efforts, instead of expecting everything to be handed to them on a plate. Smith (95: 89).

Introduction: The use of problems to initiate the study of certain topics in mathematics was explored in two classroom studies. The writings of Wertheimer, Katona, and Polya stimulated this investigator's first interest in the use of problems. Polya, who sees efficient learning related in a fundamental way to discovery, suggests that problems can stimulate discovery. Jacobson, in an essay—"The Role of Problems in the Development of Mathematical Activity"—which was published as these studies were being completed said, "Perhaps he will not reach his goal, but the search may prove more important than the goal." (55: 101). Indeed, the goal of the problem presentation was the stimulation of a student search or exploration promoted and guided by a problem which preceded the classroom discussion of certain topics. The principal phases of the investigation were: (1) the exploration/
development of this utilization of problems and (2) the creation of essential problem materials.

**Observations:** The problems sought to promote an out-of-class student search—a search or exploration related to a topic which was soon to be considered in the classroom. As evidenced by the numerical data of Chapter III, many of the problems did promote the desired search and, in addition, the results of special scoring techniques suggested that productive thinking did occur on approximately one-third of the answers to the items which provided the appropriate circumstances. Although many of the problems did provide opportunities for the discovery of mathematical concepts, the few items which were geared to promote it apparently failed to do so. Some students may have made discoveries which they did not report since the item did not elicit such a response. In short, student performance on the problems can be summarized as follows: (1) an out-of-class student search was promoted, (2) some productive thinking seemed to occur, and (3) there was little evidence that student discovery occurred.

Problem-related student performance on a majority of the problems approached that anticipated by this investigators. Did the problems and the related explorations affect the student's learning of mathematics? A term pre-test and the final examination provided data for a
statistical analysis of the learning of the students in the control and exploratory groups. An attempt to measure two phases of student learning—(1) the learning of the routine material of the course and (2) the development of an in-depth understanding of the fundamental concepts—was undertaken in Study Two while in Study One the single measure obtained was related to both these facets of learning. Statistical results did not suggest that significant (0.05) group differences as measured by the test instruments existed. This does not suggest conclusively that no real difference existed, but only that no differences were revealed by the instruments used.

Finally, student reaction to this use of problems was principally neutral or positive. Three-fourths of the students in Study Two accepted the problem assignments, found them challenging, and saw value in them. However, a few were not pleased with the problem assignments. A "student opinion" survey failed to suggest that a radical difference in attitude, interest, or morale existed between the control group and the exploratory group. The results, as interpreted by this investigator, did suggest that the students in the exploratory group (1) devoted more time to the study of mathematics, (2) saw value in the non-textbook (problem) assignments, and (3) may have had a better understanding of mathematics than did the students in the
control group. However, the additional study time by the students in the exploratory group certainly does suggest that in Study Two a positive correlation between this utilization of problems and efficient learning has not been established.

**Review of Difficulties:** A review of the literature, including abstracts of recent doctoral dissertations, did not indicate that this technique of problem utilization—if existent—is widespread. Since little information was available to assist in the design of the classroom studies, the studies were exploratory in nature. It seems quite likely that the full potential of the problem technique was not realized in these exploratory circumstances. The possibility, therefore, does exist that the control-exploratory results in these studies are as much a function of differences in the effectiveness of the treatment as it is a function of differences in treatment.

This technique of problem utilization was completely new to the students in both studies. A lack of acceptance of this use of problems was noted early in the school term. In addition, students may resist a new method of teaching that makes them work harder and longer than they expected. It does seem to this investigator that these handicaps could be lessened if the technique were used for two or more school terms—a condition which was not possible in either study.
Since the mathematics courses used in these studies had no college level prerequisite, the groups of students lacked homogeniety—a factor so important that it alone may have almost ruled out the possibility of obtaining results which suggest a difference which is statistically significant at the 0.05 level. It should be noted, also, that a technique which is superior with one group may be inferior with another group of students at the same level and, therefore, it will be difficult to establish in a small number of trials the real value of a new technique. In short, there probably is no "best" way to teach mathematics although there may be "better" ways for certain groups of students.

Finally, some form of evaluation is necessary to establish the existence of a difference between groups. But, the ability to detect a difference is a function of the instrument used. Test results which seem to suggest that no difference exists, really indicate only that the instrument did not measure a difference.

Problem Utilization, Conclusions and Suggestions:
In order to better prepare himself for the proper selection of topics, the creation of the problem material, this investigator taught the "exploratory courses" to equivalent students prior to each study in an attempt to discover the student and the material. Although the information thus
gained played a significant part in the selection of topics and the design of the problems, some of the problems were not properly geared to the needs of the particular classroom situation. However, useful information gained from problems used in these studies should influence the design of problem material in future studies.

The typical student usually exhibits a "so what's new" attitude when he is introduced to the postulates—such as, \( a + b = b + a \)—for the real number system (elementary level). A problem-introduced binary operation—not completely unlike addition—which did not possess properties such as commutativity, associativity, etc. did appear to help make more meaningful the classroom discussion of the postulates and may have helped the student understand the nature of the postulates. This type of problem seemed to have potential and should be used when appropriate in future studies.

One of the criteria of problem design was, "Normally the solution should not require special knowledge in unfamiliar areas of mathematics. (When such knowledge was needed, it was presented in class prior to the distribution of the problem.)" One problem was a notable attempt to present required mathematics concurrent with the distribution of the problem. Although it was presented genetically in the classroom, the student seemed unable to learn
the material thoroughly enough to permit its use in the problem exploration. In future work with the lower division student, it is recommended that problems which require concurrent introduction of new mathematics be avoided—or approached cautiously.

Some problems were designed to familiarize the students with most of the essential concepts of some new definition. In most instances, the students were led to use the new definitions prior to their introduction. This approach seemed to be quite successful and in some cases was so successful that only the mathematical name and symbolism had to be added in the classroom. This type of problem—quite short and not too difficult—was readily accepted by the students and did promote the desired learning. When applicable, such problems probably have a continuing position in this use of problems.

Since deductive proof has such a central position in mathematics, problems in each study were designed to emphasize the postulate-theorem relationship. In order to stress this relationship, postulates which seemed to have no connection with mathematics familiar to the student were presented in the problem. Then, statements which follow from them were to be proved by the students. Student success with these problems was very limited. If such problems are used in future studies, the number of
theorems to be proved probably should be greatly reduced from the 7-10 used in these studies. Such problems should be presented only when it has been determined that they have some chance of encountering a mature response from the students.

Some general conclusions—not related to a special type of problem—were promoted by these exploratory studies. (1) Long, involved problems should be avoided. In fact, one of the greatest deficiencies of several problems was their overadequate demands or overextended search. (2) Attempts should be made to keep the mechanics (mechanical challenges) of the problem to a minimum with major emphasis on the productive thinking. (3) Consistent with the above, problems from mathematics which do not require the introductory (mechanical) section should be tried. (4) More numerous and less demanding problems should be used, especially in the introductory phases of the problem technique. This would probably help promote the acceptance of this use of problems. (5) Finally, "sequences of problems" probably have special potential in promoting the student search and related productive thinking. The following was noted earlier in the text.

This permitted a segmented approach to the several related topics which were to be introduced. It allowed a more gentle approach to portions of the material instead of a rather abrupt approach through a two-day out-of-class search. This technique appeared to have certain advantages.
It permitted greater freedom in the design of less involved problems, gave the student more total time for his personal development of topics, permitted a spiral development of certain phases of the material, and provided several opportunities for a reinforcement of the material learned in the earlier problems of the sequence.

The sequence of problems approach may be particularly useful with lower division students.

Perhaps the instructor-investigator—in future studies—should seek assistance in (1) the selection of the mathematical topic, (2) the selection of the problem, and (3) the preparation of a draft of the problem which best "fits" the student. Perhaps a "brain-storming" session of three to six instructors and former students would result in the selection of topics whose study is most effectively initiated by the problem technique. The group could also select the problem situation and produce a rough draft of the circumstances which would promote the desired activity. This would later be refined and put into final form by the instructor. Although those in the mathematics department should not be excluded, it does seem that teachers in other departments would be a good source for problems and new ideas.

Value and Potential Weaknesses: This investigator does believe that the problems used prior to the classroom consideration of certain topics did (1) stimulate student activity in the attempted solution of the problem and (2)
the attempted solution of the problem did promote some productive thinking. This technique puts more of the burden of the learning of mathematics on the shoulders of the student. Although the students may resist a procedure which makes them work harder and longer than they expected to work, it does seem that such a procedure might have value in that the material learned in this way may be retained for a longer period of time. Also, the promotion of the explorations and the development of the student's ability to work individually could be the greatest advantage derived from this use of problems since the student's ultimate success in the study of mathematics is going to be related to the work that he does as an individual. Continued success in mathematics becomes more and more a function of the student's learning outside the classroom.

Unfortunately, weaknesses in the proposed use of problems are evident in these exploratory studies. The students may reject the technique if additional time must be devoted to the study of mathematics. The student might feel this was justified if quantity or quality of learning were significantly superior. However, it seems quite obvious that only properly designed problems will produce the desired results. How does one design the problem which will promote the desired activity after the topic has been selected? Although the design of the problems was foreseen as one of
the difficult tasks, it was an even greater task than had been anticipated. Not only was the creation of the problem found difficult, but also it was discovered that the exploratory problem did not always promote the desired productive thinking. This, then, is just a modified form of the difficulty existent in the promotion of discovery learning which this use of problems sought to circumvent. Perhaps productive thinking, which is seen as a more immediate goal preceding discovery, can be promoted more consistently. However, it has not been established that the problem promoted productive thinking does result in more efficient learning. Nevertheless, these studies have provided information about the proposed use of problems and will, hopefully, provide some direction for future studies. More restricted investigation should be undertaken as suggested in the following.

1. The use of this technique of instruction at an earlier age may promote greater ultimate success and longer-lasting results. For a parallel we can contrast the ease with which a child learns a language with the difficulties experienced by most adults.

2. It may be necessary to use the technique for a longer period of time. As the basketball player learns the correct moves with extensive participation and practice, the student may learn the correct moves in problem solving (asking himself the critical questions) only after extensive exposure to problem solving situations.

3. Greater success may be experienced with students who are above average while limited success may be the result with average and below average
students. We must not discard or classify as undesirable an instructional technique which appears to be useful only with superior students. It must now be obvious to most instructors that many students who are superior (in other areas) just never succeed in mathematics. We should be searching for methods which promote the success of these students.

4. Some classes of students may find problems from the physical and social sciences the stimulant which promotes the desired activity in the study of mathematics. Such problems would increase the interest of the student, would tie together his learning in several areas, and help to de-emphasize the division which exists in the various departments today. Not only would the student's learning of mathematics be promoted, but also his knowledge of mathematics would be made more useful to him.

Of course, if it is discovered that the problem technique does produce superior learning when used in some more rigidly controlled circumstances, then the design problem arises again. It probably could not be assumed that all teachers could create the required problems. In addition, it could not be assumed that the problems of one instructor, successful with his students, would be equally successful when a second instructor presents them to his students. In other words, the question arises, "Are the problems produced and the producing instructor separable?"

Conclusions: This author does believe that the use of problems to initiate the study of certain topics in mathematics did stimulate student interest, student readiness, and "productive thinking." Although these three evidences of
student activity fall short of student discovery (the key to efficient learning—Polya), the problems did promote the learning of some new mathematics outside the classroom.

The general organization of the study, the problems created, the use of existing groups of students, and the tests used to measure student performance all reflect the exploratory character of the studies and the requirement that they be conducted in available situations. Significant results have been analyzed and reported. The complex variables involved and the extremely broad character of these studies prompt caution in the formation of conclusions. Results have pointed to the need for additional exploratory studies before any general conclusions are formulated. Four words seem to summarize the really critical variables which must remain a part of all studies of the proposed use of problems: students, problems, instructors, and measurement. If, through valid measurement, the technique is found to be useful, but not always useful, then we must seek an answer to the question, "What problems, with which students, and by which instructor?"
APPENDIX A

PROBLEMS
Mr. Charles W. Goode has devised a system of determining student's grades which he calls cumulative-weighted grading, CWG.

I. Cumulative because--
1. The student's grade is determined after each examination.
2. All grades except the last one determined are discarded.
3. The one recorded grade and the last examination are used to determine the new recorded grade.

II. Weighted because the student's last examination grade is given more weight in the determination of the new grade than is the student's present (recorded) grade.

In this problem we will assume that all examination grades are A, B, or C. Some examples which illustrate the CWG system follow:

John: First Exam: C Recorded CWG: C
Second Exam: B Recorded CWG: B
Third Exam: A Recorded CWG: A

Mary: First Exam: C Recorded CWG: C
Second Exam: A Recorded CWG: B
Third Exam: C Recorded CWG: C

These examples show how Mr. Goode puts extra weight on the last examination of the student. Although the last examination is given extra weight in the determination of the student's new CWG, Mr. Goode never changes the student's CWG more than one letter grade as the result of a single examination (see Mary in the above example, second examination). Another example follows:

Tom: Present CWG, A; Last Exam, C; New CWG, B

1. Beginning with the second examination Mr. Goode uses a small table (similar to a multiplication table) to determine the student's new CWG. Consistent with the above description of the CWG system and the given examples, complete Mr. Goode's "grade" table.
2. Find the student's CWG after the second examination.

<table>
<thead>
<tr>
<th>First Exam</th>
<th>Second Exam</th>
<th>CWG</th>
</tr>
</thead>
<tbody>
<tr>
<td>Henry</td>
<td>B</td>
<td>A</td>
</tr>
<tr>
<td>Paul</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>Alfred</td>
<td>B</td>
<td>B</td>
</tr>
</tbody>
</table>

3. Find the student's CWG after the third examination.

<table>
<thead>
<tr>
<th>Exam 1</th>
<th>Exam 2</th>
<th>Exam 3</th>
<th>CWG</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mary</td>
<td>B</td>
<td>C</td>
<td>A</td>
</tr>
<tr>
<td>Sue</td>
<td>A</td>
<td>C</td>
<td>B</td>
</tr>
<tr>
<td>Jane</td>
<td>A</td>
<td>B</td>
<td>A</td>
</tr>
</tbody>
</table>

4. After examination 6, Mr. Goode was very busy so he did not compute the new CWG although he did record the grades for examination 6. In fact, when he gave examination 7, he had not calculated the CWG after examination 6. Mr. Goode brought his record book up-to-date after examination 7. Karen objected to the CWG which Mr. Goode recorded for her. She believes that her CWG is A although Mr. Goode has (incorrectly) given her B. Mr. Goode's entries in his record book are correct—CWG after exam 5, C; exam 6, B; exam 7, A.

a. Explain how Mr. Goode used this data to get the B.
b. Explain how Karen used this data to get an A.
c. Comment on the technique which must be followed when calculating a CWG from a series of scores.

5. There is something unique about having a present CWG, B. Explain.

6. Jack received a B on examination 5 which made his new CWG a B. What was his CWG after examination 4? Give an explanation (see problems 1 and 2).
7. Each of the tables introduced below possess a property (pattern) which does not exist in your CWG table. Examine the tables to discover this property, discuss.

<table>
<thead>
<tr>
<th>X</th>
<th>1 2 3 4 5 6 7</th>
<th>+</th>
<th>0 1 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 2 3 4 5 6 7</td>
<td>0</td>
<td>0 1 2</td>
</tr>
<tr>
<td>2</td>
<td>2 4 6 8 10 12 14</td>
<td>1</td>
<td>1 2 3</td>
</tr>
<tr>
<td>3</td>
<td>3 6 9 12 15 18 21</td>
<td>2</td>
<td>2 3 4</td>
</tr>
<tr>
<td>4</td>
<td>4 8 12 16 20 24 28</td>
<td>0,1,2,3,4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>5 10 15 20 25 30 35</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>6 12 18 24 30 36 42</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>7 14 21 28 35 42 49</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>X</th>
<th>0 1 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0 0 0</td>
</tr>
<tr>
<td>1</td>
<td>0 1 2</td>
</tr>
<tr>
<td>2</td>
<td>0 2 4</td>
</tr>
</tbody>
</table>

{0,1,2,4}

<table>
<thead>
<tr>
<th>X</th>
<th>0 1 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0 0 0</td>
</tr>
<tr>
<td>1</td>
<td>1 1 2</td>
</tr>
<tr>
<td>2</td>
<td>2 0 1</td>
</tr>
</tbody>
</table>

{0,1,2}

* | E A B U |
---|---------|
E | E A B U |
A | A A U U |
B | B U B U |
U | U U U U |

{E,A,B,U}

>>> The elements in a set below a table are those elements which are in the body of the table.

8. The first three tables presented in problem 7 do not have a certain property which is possessed by the other tables. Study the tables to determine what the property is. Explain.
Problem 2, Study 1

Miss Smith is very interested in the arithmetic progress of ten of her sixth grade pupils. Each of these students received a C or a D in his fifth grade arithmetic. These students are identified by letters in this problem. The numerical score each received on his last arithmetic test is given below.

H 72 __
J 87 __
K 97 __
M 79 __
N 81 __
P 83 __
R 76 __
S 68 __
T 93 __
W 78 __

1. Miss Smith assigns letter grades as follows: 90-100, A; 80-89, B; 70-79, C; and 60-69, D. Give each student his letter grade (space provided above).

2. In the above sketch, group those students who have the same letter grade.

3. List, using set notation, those students who have
   a) a grade of A, call the set A. A = ___________
   b) a grade of B, call the set B. B = ___________
   c) a grade of C, call the set C. C = ___________
   d) a grade of D, call the set D. D = ___________

4. List, using set notation, those students who have
   a) a grade of B or C  E = _________________
   b) a grade of A or D  F = _________________
   c) a grade of A or B or D  G = _________________
5. List, using set notation, those students who are in both set E and set G. What is another name for this set?

6. List, using set notation, those students who are in both set E and set F. What is another name for this set?

7. List as a set those students who are in both set F and set G. Call this set Z. What property does the set Z possess?

8. List all the students who are in set E or set G. Call this set U. Identify an interesting property of the set U.

9. The number of students who have a grade of C is ____. Arrange these students in a set according to their numerical scores, highest score first. Arrange the "A" students in a set according to their numerical scores, lowest score first.
   C(special) = _______________
   A(special) = _______________

10. List, using set notation, all the students who are in set F or set G. Call this set V. Give another name for the set V.

11. List as a set all the students who did not receive a grade of C. Call this set L. Identify two different properties possessed by the pair of sets, C and L.
Problem 3, Study 1

A set of students in this class are to solve a set of problems. All the problems in the set must be solved. These students, as a group, decide that teams should be created to solve the various problems. When the organizational work is completed, the following is noted.

P1. If M and N are students, there exists a team (maybe more than one) on which they serve together.

P2. If M and N are different students, there exists at most one team on which they serve together.

P3. Any two teams have at least one student in common.

P4. There exists at least one (problem to be solved) team.

P5. Every team has at least three students (members).

P6. Not all students serve on the same team.

P7. No team has more than three students (members).

Certain statements are deduced from the above set of statements. These other statements (T1, T2, T3, and T4) are given below. "Prove" (show how they follow from the above statements) them.

T1. Any two different students serve together on exactly one team.

T2. Any two different teams have exactly one student (member) in common.

T3. There exist three students who do not serve together on a team.

T4. Every team has exactly three students (members).

In addition, the number of students in the set (a subset of this class) and the number of teams (problems to be solved) are determined by the above statements. Find the number of students and the number of teams. Give a complete discussion of how you were able to find the number of students and the number of teams (You may use statements T1, T2, T3, and T4 which you have already proved).
Problem 4, Study 1

The set of whole numbers, \( W \), is the set \( \{0, 1, 2, \ldots \} \). The elements of this set are used to build a new set on which two new binary operations \(#\) and \(*\) are defined. The elements of this set are ordered pairs, \((a, b)\), where \( a \) and \( b \) are elements from the set of whole numbers and the \( b \neq 0 \).

We will assume that the elements in \( W \) have the commutative property for addition and multiplication, the associative property for addition and multiplication, and the distributive property.

Definition: \((a, b) \# (c, d) = (ad + bc, bd)\)

Definition: \((a, b) * (c, d) = (ac, bd)\)

Definition: \((a, b) = (c, d)\) if and only if \( ad = bc \)

1. Show that \((a, c) = (a, c)\).

2. Show that \((a, b) = (ka, kb)\).

3. Prove that \(#\) is a commutative binary operation.

4. Prove that \(#\) is an associative binary operation.

5. Prove that \(*\) is a commutative binary operation.

6. Prove that \(*\) is an associative binary operation.

7. Name an identity element for \(#\). Show that it has the properties of an identity.

8. Name an identity element for \(*\). Show that it has the properties of an identity.

9. Show that both operations have the closure property.

10. Show that both operations have the cancellation property.
Problem 5, Study 2

R. U. A. Winner: We are all participants in a game which is played with a spinning wheel. The numbers 10, 50, and 100 appear on this wheel. The player spins the wheel. When the wheel stops, an indicator is pointing toward one of the numbers. This number is used in the determination of the player's score.

Each player (in turn) spins the wheel three times. The three numbers "selected" by these three spins are used to determine the player's score. Although a temporary score is determined after the first spin and the second spin, the score determined after the third spin is the player's score for that round of play. The numbers selected by the indicator are used to determine the player's score as follows:

1. \( T_1 \), the temporary score after the first spin, is the first number selected, \( S_1 \).

2. \( T_2 \), the temporary score after the second spin, is determined in the following way.
   a. If the number selected \( S_2 \) is smaller than \( T_1 \), then \( T_2 \) is one score unit less than \( T_1 \).
   b. If the second number selected \( S_2 \) is the same as \( T_1 \), then \( T_2 \) is the same as \( T_1 \).
   c. If the number selected \( S_2 \) is larger than \( T_1 \), then \( T_2 \) is one score unit larger than \( T_1 \).

3. \( T_3 \), the player's final score for that round, is determined in the following way (similar to \( T_2 \)).
   a. If the third number selected \( S_3 \) is smaller than \( T_2 \), then \( T_3 \) is one score unit less than \( T_2 \).
b. If the third number selected (S3) is the same as T2, then T3 is the same as T2.

c. If the third number selected is larger than T2, then T3 is one score unit larger than T2.

In this game there are only three score units. They are 10, 50, 100. Throughout this problem S1 is the first number selected, T1 is the temporary score after the first spin, S2 is the second number selected, etc. Some examples of score determination follow.

Example: S1  S2  S3
50  100  10

T1  T2  T3
50  50  50

Examples S1  S2  S3
10  100  100

T1  T2  T3
50  50  50

1. Although T1 is determined by the first spin, TWO numbers play a part in the determination of T2 and T3. Since only two numbers are used in the determination of these scores (T2 and T3), we are able to develop tables—much like a multiplication table—which can be useful in this game.

a. Consistent with the above description of scoring, complete the tables for T2 and T3.

<table>
<thead>
<tr>
<th></th>
<th>10</th>
<th>50</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>T2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>10</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>10</th>
<th>50</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>T1</td>
<td>50</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>

b. Is it necessary that we complete two tables? Give an explanation supporting your answer.
2. Find T2 for each of the following players (Use your table).

<table>
<thead>
<tr>
<th></th>
<th>T1</th>
<th>S2</th>
<th>T2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Henry</td>
<td>50</td>
<td>100</td>
<td></td>
</tr>
<tr>
<td>Paul</td>
<td>100</td>
<td>50</td>
<td></td>
</tr>
<tr>
<td>Alfred</td>
<td>50</td>
<td>50</td>
<td></td>
</tr>
</tbody>
</table>

3. Find T3 for each of the following players (Use your tables).

<table>
<thead>
<tr>
<th></th>
<th>S1</th>
<th>T1</th>
<th>S2</th>
<th>T2</th>
<th>S3</th>
<th>T3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mary</td>
<td>50</td>
<td></td>
<td>10</td>
<td></td>
<td>100</td>
<td></td>
</tr>
<tr>
<td>Susan</td>
<td>100</td>
<td></td>
<td>10</td>
<td></td>
<td>50</td>
<td></td>
</tr>
<tr>
<td>Janet</td>
<td>100</td>
<td></td>
<td>50</td>
<td></td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>

4. During one round Fred had the following scores: S1, 10; S2, 50; S3, 100. Larry says that Fred's T3 is 100. Jack says that Fred's T3 is 50. Who is correct? Examine this situation very carefully. (Larry and Jack have the correct numbers listed in the proper order.)

a) Explain how Larry used these numbers to get his result.
b) Explain how Jack used these numbers to get his result.
c) Discuss an important principle which must be used when determining a T3 from a series of numbers.

5. There is something unique about a temporary score of 50. Examine your tables, then explain.

6. Norman's S2 was 50. This number together with his T1 resulted in a T2 of 50. What was his T1? Give an explanation (See your tables and problem 2).

7. Each of the tables below possesses a property (a pattern) which your tables in Problem 1 do not possess. Explain (Also see Henry and Paul, Item 2).

<table>
<thead>
<tr>
<th>+</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>x</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>6</td>
<td>9</td>
</tr>
</tbody>
</table>

{0,1,2,3,4} {1,2,3,4,6,9}
The elements in a set below a table are those elements which are in the body of the table.

8. The first two tables presented in problem 7 (above) do not possess a certain property which is possessed by the other tables. Study the tables to determine what the property is. Explain.
Problem 6: Study 2

Postulates:

1. There exists a set of elements, \( W = \{0, 1, 2, 3, \ldots\} \). We will call this set the set of whole numbers.

2. We will assume the equality and order axioms for elements in the set of whole numbers.

3. We will assume the following properties for elements in the set \( W \): closure for addition and multiplication, commutative laws of addition and multiplication, associative laws of addition and multiplication, distributive law, identity element for addition (0), and identity element for multiplication (1).

Definition: \( S = \{(a, b) | a \in W, b \in W, b \neq 0\} \)

Definition: \((a, b) \neq (c, d) = (ac, bd)\) This symbolic statement is the definition of a binary operation for elements of the set \( S \).

Definition: \((a, b) = (c, d)\) if and only if \( ad = bc \) This symbolic statement defines equality of elements in the set \( S \).

1. Show that \((a, c) = (a, c)\).
2. Show that \((a, b) = (ha, hb)\).
3. Prove that \# is a commutative binary operation.
\[ (a, b) \# (c, d) = (c, d) \# (a, b) \]
4. Prove that \# is an associative binary operation.
\[ [(a, b) \# (c, d)] \# (e, f) = (a, b) \# [(c, d) \# (e, f)] \]
5. Show that \# has the closure property on the set \( S \).
\[ (a, b) \# (c, d) \in S \]
6. Prove that \# has the cancellation property.
If \((a, c) \# (h, k) = (y, z) \# (h, k)\), then \((a, c) = (y, z)\)
7. Name an identity element for \# on the set \( S \). Show that it has the properties of an identity.
\[ (a, b) \# (1, 1) = (1, 1) \# (a, b) = (a, b) \]
Postulates:

1. There exists a set of elements, \( W = \{0, 1, 2, 3, \ldots \} \). We will call it the set of whole numbers.

2. We will assume the equality and order axioms for elements in the set of whole numbers.

3. We will assume the following properties for elements in the set \( W \): closure for addition and multiplication, commutative laws for addition and multiplication, associative laws for addition and multiplication, distributive law, identity element for addition (0), and identity element for multiplication (1).

Definition: \( S = \{(a, b) \mid a \in W, b \in W, b \neq 0\} \)

Definition: \((a, b) \# (c, d) = (ac, bd)\)

Definition: \((a, b) * (c, d) = (ad + bc, bd)\)

Definition: \((a, b) = (c, d) \text{ if and only if } ad = bc\).

1. Prove that * is a commutative binary operation.

2. Prove that * is an associative binary operation.

3. Prove that * has the cancellation property.

4. Name an identity element for * on the set \( S \). Show that it has the properties of an identity.

5. Prove that the operation \# is distributive with respect to the operation *.

\((a, b) \# [(c, d) * (e, f)] = (a, b) \# (c, d) * (a, b) \# (e, f)\)
Problem 8, Study 2

The numeral 26 is a symbolic representation of the number twenty-six (or two tens and six units). The numeral 62 is a symbolic representation of the number sixty-two (or six tens and two units). Hence, the two different combinations of the symbols 2 and 6 identify two different number concepts—twenty-six (twenty and six) and sixty-two (sixty and two). We conclude that the symbols used in our numeration system do not possess the commutative property.

Thus, a two-digit place-value numeral (base ten) has a tens digit and a units digit. In the exercises which follow, we will "build" sets of two-digit numerals (we will call 05 a two-digit numeral—zero tens and five units, hence five). The construction procedure is as follows:

1. The symbol for the "tens" digit is an element from a set named Ti.
2. The symbol for the "units" digit is an element from a set named Ui.
3. We will call the constructed sets of two-digit numerals Si.

Example: 

\[ T_e = \{1, 2\} \quad n(T_e) = 2 \]
\[ U_e = \{1, 2\} \quad n(U_e) = 2 \]
\[ S_e = \{11, 12, 21, 22\} \quad n(S_e) = 4 \]

Example: 

\[ T_f = \{0, 1, 2\} \quad n(T_f) = 3 \]
\[ U_f = \{6, 7\} \quad n(U_f) = 2 \]
\[ S_f = \{06, 07, 16, 17, 26, 27\} \quad n(S_f) = 6 \]

In the following exercises, fill the blanks as indicated in the examples above. In addition, circle the elements in the array of numbers which correspond to the elements of the set Si.

Although an identical array of numbers was included in each of the items of this problem, it is presented only in item one of this Appendix.
1. \( T_1 = \{1, 2\} \)  \quad n(T_1) = \\
\( U_1 = \{2, 3\} \)  \quad n(U_1) = \\
\( S_1 = \quad \)  \quad n(S_1) = \\
\begin{tabular}{cccccccc}
06 & 16 & 26 & 36 & 46 & 56 & 66 \\
05 & 15 & 25 & 35 & 45 & 55 & 65 \\
04 & 14 & 24 & 34 & 44 & 54 & 64 \\
03 & 13 & 23 & 33 & 43 & 53 & 63 \\
02 & 12 & 22 & 32 & 42 & 52 & 62 \\
01 & 11 & 21 & 31 & 41 & 51 & 61 \\
00 & 10 & 20 & 30 & 40 & 50 & 60 \\
\end{tabular}

2. \( T_2 = \{0, 1, 2, 4\} \)  \quad n(T_2) = \\
\( U_2 = \{3, 4, 5\} \)  \quad n(U_2) = \\
\( S_2 = \quad \)  \quad n(S_2) = \\

3. \( T_3 = \{0, 2, 4\} \)  \quad n(T_3) = \\
\( U_3 = \{1, 2, 3, 4\} \)  \quad n(U_3) = \\
\( S_3 = \quad \)  \quad n(S_3) = \\

4. \( T_4 = \{4, 6\} \)  \quad n(T_4) = \\
\( U_4 = \{1, 2, 3, 4, 5\} \)  \quad n(U_4) = \\
\( S_4 = \quad \)  \quad n(S_4) = \\

5. \( T_5 = \{2, 3\} \)  \quad n(T_5) = \\
\( U_5 = \{0, 1, 2, 3, 4, 5\} \)  \quad n(U_5) = \\
\( S_5 = \quad \)  \quad n(S_5) = \\

6. \( T_6 = \{0, 1, 2, 3, 4, 5, 6\} \)  \quad n(T_6) = \\
\( U_6 = \{3, 4, 5\} \)  \quad n(U_6) = \\
\( S_6 = \quad \)  \quad n(S_6) = \\

In the following exercises, find the required intersection—then circle the elements in the array of numbers which correspond to the elements of the intersection.

7. \( S_1 \cap S_2 = \quad \)  \quad 8. \( S_3 \cap S_4 = \quad \)

9. \( S_5 \cap S_6 = \quad \)
Problem 9, Study 2

In our last problem we investigated the formation of two-digit numbers and the sets of numbers thus created. We noted that 26 ≠ 62. A new type of set is now introduced, a set in which the order of elements is significant. Symbolically we write \((a,b)\), an ordered set with "first" element \(a\) and "second" element \(b\). When the ordered set has two elements, it is called an ordered pair.

Definition: \((a,b) = (c,d)\) if and only if \(a = c\) and \(b = d\).

Note that, although \({a,b}\) = \({b,a}\), \((a,b) \neq (b,a)\) when \(a \neq b\). Most of the following items require that sets of ordered pairs be built using the element of two given sets. The procedure for construction of the ordered pairs is as follows:

1. The first element of the ordered pair is an element from a set named \(F_1\).
2. The second element of the ordered pair is an element from a set named \(S_1\).
3. \(P_1\) is the name given to the set of all possible ordered pairs constructed from the sets \(F_1\) and \(S_1\).

Example:

\[
\begin{align*}
F_1 &= \{1,2\} & n(F_1) &= 2 \\
S_1 &= \{1,2\} & n(S_1) &= 2 \\
P_1 &= \{(1,1), (1,2), (2,1), (2,2)\} & n(P_1) &= 4 \\
\end{align*}
\]

Example:

\[
\begin{align*}
F_2 &= \{0,1,2\} & n(F_2) &= 3 \\
S_2 &= \{6,7\} & n(S_2) &= 2 \\
P_2 &= \{(0,6), (0,7), (1,6), (1,7), (2,6), (2,7)\} & n(P_2) &= 6 \\
\end{align*}
\]

In the following exercises, find \(P_1\) and complete the item as illustrated in the above examples. In addition, circle the elements in the array of ordered pairs which correspond to the elements of the set \(P_1\).

Although an identical array of numbers was included in each of the items of this problem, it is presented only in item one of this appendix.
1. \( F_1 = \{1, 2\} \)
   \( S_1 = \{2, 3\} \)
   \( P_1 = \)
   \[
   (0, 6) \quad (1, 6) \quad (2, 6) \quad (3, 6) \quad (4, 6) \quad (5, 6) \quad (6, 6) \\
   (0, 5) \quad (1, 5) \quad (2, 5) \quad (3, 5) \quad (4, 5) \quad (5, 5) \quad (6, 5) \\
   (0, 4) \quad (1, 4) \quad (2, 4) \quad (3, 4) \quad (4, 4) \quad (5, 4) \quad (6, 4) \\
   (0, 3) \quad (1, 3) \quad (2, 3) \quad (3, 3) \quad (4, 3) \quad (5, 3) \quad (6, 3) \\
   (0, 2) \quad (1, 2) \quad (2, 2) \quad (3, 2) \quad (4, 2) \quad (5, 2) \quad (6, 2) \\
   (0, 1) \quad (1, 1) \quad (2, 1) \quad (3, 1) \quad (4, 1) \quad (5, 1) \quad (6, 1) \\
   (0, 0) \quad (1, 0) \quad (2, 0) \quad (3, 0) \quad (4, 0) \quad (5, 0) \quad (6, 0)
   \]
   \( n(F_1) = \) \( n(S_1) = \) \( n(P_1) = \)

2. \( F_2 = \{0, 1, 2, 4\} \)
   \( S_2 = \{3, 4, 5\} \)
   \( P_2 = \)
   \( n(F_2) = \) \( n(S_2) = \) \( n(P_2) = \)

3. \( F_3 = \{0, 2, 4\} \)
   \( S_3 = \{1, 2, 3, 4\} \)
   \( P_3 = \)
   \( n(F_3) = \) \( n(S_3) = \) \( n(P_3) = \)

4. \( F_4 = \{4, 6\} \)
   \( S_4 = \{1, 2, 3, 4, 5\} \)
   \( P_4 = \)
   \( n(F_4) = \) \( n(S_4) = \) \( n(P_4) = \)

5. \( F_5 = \{2, 3\} \)
   \( S_5 = \{0, 1, 2, 3, 4, 5\} \)
   \( P_5 = \)
   \( n(F_5) = \) \( n(S_5) = \) \( n(P_5) = \)

6. \( F_6 = \{0, 1, 2, 3, 4, 5, 6\} \)
   \( S_6 = \{3, 4, 5\} \)
   \( P_6 = \)
   \( n(F_6) = \) \( n(S_6) = \) \( n(P_6) = \)

In the following exercises, find the required intersection—then circle the elements in the array of ordered pairs which correspond to the elements of the intersection.

7. \( P_1 \cap P_2 = \)
   \( P_5 \cap P_6 = \)

8. \( P_3 \cap P_4 = \)
Previously we have considered two binary operations on sets: the union of two sets and the intersection of two sets. We represent these operations symbolically as $A \cup B$ and $A \cap B$ respectively. In our last problem we constructed sets of ordered pairs. The formal definition of a third binary operation on sets, called the cross product or Cartesian product, is now provided.

**Definition:** $A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$

In the following eight items use the sets provided to find the cross product of the two sets. Also find the number of elements in the sets as indicated.

- $A = \{0, 1, 2\}$
- $B = \{0, 2, 4\}$
- $C = \{0, 1, 2\}$
- $D = \{1, 2, 3, 4\}$
- $\emptyset = \text{empty set}$

<table>
<thead>
<tr>
<th>Item</th>
<th>Sets</th>
<th>Cross Product</th>
<th>Number of Elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$A \times B$</td>
<td>${(0,0), (0,2), (0,4), (1,0), (1,2), (1,4), (2,0), (2,2), (2,4)}$</td>
<td>9</td>
</tr>
<tr>
<td>2.</td>
<td>$A \times C$</td>
<td>${(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2)}$</td>
<td>9</td>
</tr>
<tr>
<td>3.</td>
<td>$A \times D$</td>
<td>${(0,1), (0,2), (0,3), (0,4), (1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4)}$</td>
<td>12</td>
</tr>
<tr>
<td>4.</td>
<td>$B \times C$</td>
<td>${(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2)}$</td>
<td>9</td>
</tr>
<tr>
<td>5.</td>
<td>$B \times D$</td>
<td>${(0,1), (0,2), (0,3), (0,4), (1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4)}$</td>
<td>12</td>
</tr>
<tr>
<td>6.</td>
<td>$C \times D$</td>
<td>${(0,1), (0,2), (0,3), (0,4), (1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4)}$</td>
<td>12</td>
</tr>
<tr>
<td>7.</td>
<td>$D \times C$</td>
<td>${(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2)}$</td>
<td>9</td>
</tr>
<tr>
<td>8.</td>
<td>$A \times \emptyset$</td>
<td>$\emptyset$</td>
<td>0</td>
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</tbody>
</table>

9. The numerical value of $n(E \times F)$ is uniquely determined by the sets $E$ and $F$ where $E$ and $F$ are any two sets. What can you say about $n(E \times F)$? Explain.
Problem 11, Study 2

Nine students are identified by number in the display below. It also contains eight quiz scores for each of the students—four from the first half semester and four from the second half semester. [In the original problem more students were included—only nine are used in this paper so that arrays and graphs can be represented more efficiently.]

<table>
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<th>STUDENTS</th>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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</tbody>
</table>

1. If the only possible grades are 1 (highest grade), 2, 3, 4, and 5, determine...
   a. A grade for the combination of the first four scores, M.
   b. A grade for the combination of the second four scores, N.
   c. A final grade, F.
   d. Insert these grades in the appropriate spaces in the above display.

2. Describe in detail the method you used to determine the grades M, N, and F.

3. If \( R = \{(x,y) | x \text{ is a student's number and } y \text{ is his M grade}\} \), then \( R = \) ____________________________

If \( S = \{(x,y) | x \text{ is a student's number and } y \text{ is his N grade}\} \), then \( S = \) ____________________________
If \( T = \{(x,y) \mid x \text{ is a student's number and } y \text{ is his grade}\} \), then \( F = \ldots \)

4. In Figure 1-a circle those elements which are elements of either \( R \) or \( S \), or both \( R \) and \( S \) (see #3). Note that this gives us a view of the grades of the students for the two half semesters.

In Figure 1-b circle those elements which are elements of the set \( T \) (see #3). Note that this gives us a view of the grades (relative standing) of each of the students at the end of the semester.

\[
\begin{align*}
(1,5) & \quad (2,5) & \quad (3,5) & \quad (4,5) & \quad (5,5) & \quad (6,5) & \quad (7,5) & \quad (8,5) & \quad (9,5) \\
(1,4) & \quad (2,4) & \quad (3,4) & \quad (4,4) & \quad (5,4) & \quad (6,4) & \quad (7,4) & \quad (8,4) & \quad (9,4) \\
(1,3) & \quad (2,3) & \quad (3,3) & \quad (4,3) & \quad (5,3) & \quad (6,3) & \quad (7,3) & \quad (8,3) & \quad (9,3) \\
(1,2) & \quad (2,2) & \quad (3,2) & \quad (4,2) & \quad (5,2) & \quad (6,2) & \quad (7,2) & \quad (8,2) & \quad (9,2) \\
(1,1) & \quad (2,1) & \quad (3,1) & \quad (4,1) & \quad (5,1) & \quad (6,1) & \quad (7,1) & \quad (8,1) & \quad (9,1)
\end{align*}
\]

Figure 1-a

\[
\begin{align*}
(1,5) & \quad (2,5) & \quad (3,5) & \quad (4,5) & \quad (5,5) & \quad (6,5) & \quad (7,5) & \quad (8,5) & \quad (9,5) \\
(1,4) & \quad (2,4) & \quad (3,4) & \quad (4,4) & \quad (5,4) & \quad (6,4) & \quad (7,4) & \quad (8,4) & \quad (9,4) \\
(1,3) & \quad (2,3) & \quad (3,3) & \quad (4,3) & \quad (5,3) & \quad (6,3) & \quad (7,3) & \quad (8,3) & \quad (9,3) \\
(1,2) & \quad (2,2) & \quad (3,2) & \quad (4,2) & \quad (5,2) & \quad (6,2) & \quad (7,2) & \quad (8,2) & \quad (9,2) \\
(1,1) & \quad (2,1) & \quad (3,1) & \quad (4,1) & \quad (5,1) & \quad (6,1) & \quad (7,1) & \quad (8,1) & \quad (9,1)
\end{align*}
\]

Figure 1-b

5. The elements in the two parts of Figure 1 are the elements of the set \( A \times B \) where \( A = \{1,2,3,\ldots,9\} \) and \( B = \{1,2,3,4,5\} \). Note that you have circled some of the elements in each of the two parts of Figure 1. Use the terminology of set theory to describe the relationship which exists between the "circled" elements and the elements of \( A \times B \).

Figure 2 contains two sets of lines: one set parallel-horizontal, the other set parallel-vertical. The horizontal lines are labeled 1, 2, 3, 4, or 5 while the vertical lines are labeled 1, 2, 3, \ldots, 9.

We can obtain an equivalent "student-grade" view if we define a one-to-one correspondence between ordered pairs and points, points which are the intersection of lines—one vertical and one horizontal. We will define this correspondence as follows: the point which corresponds
to the ordered pair \((x, y)\) is the point which is the unique intersection of the \(x\)-th vertical line and the \(y\)-th horizontal line (see examples in Figure 2).

\((2,3) \quad (5,4) \quad (7,1) \quad (8,2)\)

6. a. In Figure 3-a circle those points which correspond to the elements (see #3) of either \(R\) or \(S\), or both \(R\) and \(S\).
   b. In Figure 3-b circle those points which (see #3) correspond to the elements of \(T\).

7. There is an interesting (important) difference existing between the two parts \([a\) and \(b]\) of your "grade display" (either Figure 1 or Figure 3). Examine carefully these figures; determine the nature of this difference. Give an explanation of the difference.
Problem 12, Study 2

During the spring recess you decide to pedal your bicycle up Problem Canyon from Here to There. Your plan of travel is as follows:

First hour: one mile north, then one mile east
Second hour: one mile north, then two miles east
Third hour: one mile north, then three miles east
Nth hour: one mile north, then N miles east

1. Trace your route on the figure.

2. Produce an algebraic expression, $F(h)$, which can be used to find the distance traveled in a given hour, $h$.

3. Produce an algebraic expression, $G(h)$, which can be used to find the distance measured from Here (measured along the road traveled) at the end of the various hours, $h$.

4. Produce an algebraic expression, $H(h)$, which can be used to find the distance from Here (as the crow flies) at the end of the various hours, $h$. 
Problem 13, Study 2

You have an eighteen acre field which is to be irrigated. You have two sources of water for the irrigation of the field. Source A provides sufficient water for the irrigation of one acre per hour while source C provides sufficient water for the irrigation of two acres per hour. We will assume that the area irrigated increases continuously as the water is provided continuously. Consequently, we assume that the area irrigated at the end of one hour is one half that irrigated at the end of two hours, etc. In like fashion, we will assume that at the end of one-half hour the area irrigated will be one-half that irrigated at the end of one hour.

1. At \( t = 0 \) you start the irrigation of the 18 acre field using water from source A.

a) Complete the table which will illustrate the relationship existing between certain times and the area irrigated.

<table>
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<tr>
<th>Time</th>
<th>Acres</th>
</tr>
</thead>
<tbody>
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<tr>
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<tr>
<td>11</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td></td>
</tr>
</tbody>
</table>

b) Translate the information of the table (above) into a set of ordered pairs where the first element of the ordered pair corresponds to the time and the second element of the ordered pair corresponds to the area irrigated.

<table>
<thead>
<tr>
<th>Time</th>
<th>Acres</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<tr>
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</tr>
<tr>
<td>9</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td></td>
</tr>
</tbody>
</table>

c) In a previous problem we found an algebraic expression \( F(h) = h + 1 \) which was useful in determining the miles traveled during the various hours of a trip. Relative to this problem (see problem 12, Appendix A) this algebraic expression had no interpretation for fractional numbers. That is, although we were able to ask about the distance traveled during the \( h \)-th hour, \( h \), a natural number: 1, 2, 3, 4, 5, 6; this expression gave nonsense answers if we used this expression with \( h = 5/2 \). Is the same true in this problem? Is it meaningful to ask how
much area is irrigated during the first one-half hour, the first five and one-half hours? Explain.

d) Write an algebraic expression, $A(t)$, which can be used to find the area (number of acres) irrigated at various measures of time (after $t = 0$). Write this expression in simplest form. Explain why you believe that it is a correct form. [Note: Your answer must satisfy these requirements: $A(0) = 0$, $A(1/2) = 1/2$, $A(7) = 7$, $A(13) = 13$, etc.]

$$A(t) =$$

2. Five (5) acres of the field have already been irrigated when you start ($t = 0$) your portion of the project. You use water from source A. We will again use assumption (area increases continuously) of problem 1.

a) Complete a table which will illustrate the relationship existing between your time at the project and the area of the field irrigated (as in 1-a).

b) Translate the information of the table into a set of ordered pairs (as in 1-b).

c) Answer the question of 1-c as it is related to #2.

d) Write an algebraic expression, $B(t)$, which can be used to find the area (number of acres) irrigated at the various measures of (your) time ($t = 0$, and after). Write this expression in simplest form. Explain why you believe that it is a correct form. [Note: $B(0) = 5$, $B(2) = 7$, $B(5) = 10$]

$$B(t) =$$

3. Five (5) acres of the field have already been irrigated when you start ($t = 0$) your portion of the project. You use water from source C. Complete this problem using the instructions of #1 as these instructions would relate to this problem.
a) Complete the table, etc.

b) Translate the information, etc.

c) Answer the question of 1-c as it is related to #3.

d) Write an algebraic expression, \( C(t) \), etc.

\[
C(t) = \text{[Note: } C(0) = 5, C(2) = 9, C(7) = 19]\]

4. Two (2) acres of the field have already been irrigated when you start \( t = 0 \) — your portion of the project. You use water from source C. Complete this problem using the instructions of #1 (as these instructions would relate to this problem).

a) Complete the table, etc.

b) Translate the information, etc.

c) Answer the question of 1-c as it is related to #4.

d) Write an algebraic expression, \( D(t) \), etc.

\[
D(t) = \]

5. Write below the algebraic expressions of 1-d, 2-d, 4-d. DOUBLE CHECK, are they correct?

\[
A(t) = \quad C(t) = \\
B(t) = \quad D(t) =
\]

Represent each of these graphically on Figure 1. Label (with the corresponding ordered pair) those points which you believe have special significance. (You may want to use the table which you completed in the various problems—or the related set of ordered pairs—when graphing these expressions.)
6. The following questions are not related to this irrigation problem. They are questions about the algebraic expressions and their graphs.

a) The graphs of $A(t)$ and $B(t)$ are in some respect similar. Explain.

b) The graphs of $C(t)$ and $D(t)$ are in some respect similar. Explain.

c) The similarity noted above does not exist between $A(t)$ and $C(t)$, between $A(t)$ and $D(t)$. Explain.

d) Look at the algebraic expressions—these similarities and differences can be related to certain properties of these algebraic expressions. Explain.

e) The graphs of $B(t)$ and $C(t)$ are in some respect similar. Explain.

f) The similarity noted in e does not exist in the pairs—$A(t)$ and $B(t)$, $A(t)$ and $C(t)$, $A(t)$ and $D(t)$, $B(t)$ and $D(t)$. Explain.
g) Look at the algebraic expressions—the similarity and differences (e and f) can be related to certain properties of these algebraic expressions. Explain.
APPENDIX B

TESTS
Test 1, Pre-test, Study 1

Answer T (true) if the statement is always true, otherwise answer F (false).

1. All prime numbers are odd.
2. The square of an odd number is odd.
3. Subtraction of natural numbers is commutative.
4. If \( A = \{ a \} \), then \( A \) has two proper subsets.
5. If \( A \cap B \) and \( B \subseteq C \), then \( A \cap C \).
6. For any natural numbers, \( (m)(0) = m \) and \( m + 0 = m \).
7. If \( A \supseteq B \) and \( B \supseteq A \), then \( A = B \).
8. The set \( \{ 1 \} \) is closed under multiplication.
9. If \( A = \{ 1, 2 \} \) and \( B = (1, 2) \), then \( A = B \).
10. If \( A = \{ 1 \} \) and \( B = \{ 5 \} \), then \( n(A) = n(B) \).

Complete, use table for 13-15. Note: 4(five) = 4(base five)

11. 21(five) = _____(ten)  
    \[
    \begin{array}{c|cccc}
    \# & 0 & 1 & 2 & 3 \\
    \hline
    0 & 0 & 1 & 2 & 3 \\
    1 & 1 & 2 & 3 & 4 \\
    2 & 2 & 3 & 4 & 0 \\
    3 & 3 & 4 & 0 & 1 \\
    4 & 4 & 0 & 1 & 2 \\
    \end{array}
    \]

12. 3(five) + 4(five) = _____(five)

13. 2 \# 3 = _____

14. The identity for \( \# \) is _____.

15. 4\# = 3.

Identify by letter the property used in each.
A. Commutative property of addition
B. Commutative property of multiplication
C. Associative property of addition
D. Associative property of multiplication
E. Distributive property

16. \( (ab + cd) + ef = ab + (cd + ef) \)
17. \( a + b + c = a + c + b \)
18. \( a(b + c) = ab + ac \)
19. \( ab + cd + ef = ab + dc + ef \)
20. \( ([a + b]c)d = [a + b] (cd) \)

Items selected are representative of the 50 item test.
Test 2, Study 2

Use the sets \( U \) (the universal set), \( A, B, C, \) and \( D \) to answer questions 1-6.

\( U = \{J, U, N, E, 3\} \quad A = \{J, U, N, E\} \quad B = \{N, E\} \quad C = \{3\} \quad D = \{N, 3\} \)

1. \( B' = \) ________________
2. \( n(A) = \) ________________
3. \( A \cup D = \) ________________
4. \( B \cap D = \) ________________
5. \( B \times D = \) ________________
6. The classification \( U \rightarrow A, B, C, D \) is exhaustive but not exclusive (Select a correct combination of two sets by circling their letters.).

7. \( D_{12} \cap M_3 = \) ________________
8. If \( R = \{a, b\} \), list all the subsets of the set \( R \).

____________________________

Name the basic property illustrated in each (9, 10).

9. \( ad + bc + ef = ad + bc + fe: \) ________________
10. \( ([x + y] + z) + u = [x + y] + (z + u): \) ________________

11. If \( x, y \in \mathbb{N} \) (the set of natural numbers), the truth set of the statement \( 0 \leq x + 2y < 3 \) is ________________
12. If \( S = \{1 + 3 + 5, 7 + 9\} \), then \( n(S) = \) ________________
13. \( 2(\text{base thirteen}) = \) ____ (base three)
14. \( 12_{10}(\text{base three}) = \) ____ (base three)
15. \( 566(\text{base ten}) = \) ____ (eleven)

Fill the blanks as follows, 16-30:

A, if the statement is true for all replacements of the variables.

S, if the statement is true for certain replacements of the variable(s) and false for others.

F, if the statement is false for all replacements of the variable.

Example: The operation of finding the greatest common divisor of a set of numbers is a binary operation. The answer is \( S \) since the statement is
true if there are two elements in the set, however, one might not consider it a binary operation if there are more than two elements in the set.

16. \((A \times B) \times C = A \times (B \times C)\) for non-empty sets \(A, B,\) and \(C\).

17. If \(A \subseteq B\) and \(B \subseteq C\), then \(A \subseteq C\).

18. \(A \times C = C \times A\).

19. If \(A \neq B\) and \(B \neq C\), then \(A \neq C\).

20. If \(p\) is a prime, then \(p\) is an odd element of \(\mathbb{N}\).

21. If \(R\) is an equivalence relation on \(S\), then \(a \, R \, a\) for \(a \in S\).

22. The statement \(2x = 2\) is a proposition.

23. \([a] \subseteq A\)

24. If \(A \subseteq U\) (the universal set), then \((A \cup U) = U\).

25. If a binary operation, \(\#\), defined on \(S\) has the cancellation property if \(a \# b = a \# c\) for \(a, b, c \in S\), then \(b = c\).

26. If \(A = \{a, b\}\), then \(a \subseteq A\).

27. If \(R\) and \(S\) are any two non-empty sets, then \(S \subseteq (R \times S)\).

28. If \(A\) is a finite set and \(B \subseteq A\), then \(n(A) \neq n(B)\).

29. If \(A\) and \(B\) are subsets of some set, \(U\), then \(A \supseteq (A \cap B)\).

30. If "\(i\)" is the identity for some binary operation, \(\#\), defined on \(S\), then \(a \# i = i \# a\) for \(a \in S\).

The correct answer for some (at least four) of the above statements is \(S\). Identify four of these statements by number, then (for each) give an example in which the statement is true and an example in which the statement is false: problems 31-34.

31.

32.

33.

34.

35. We have defined the sum of two natural numbers in terms of operations on sets.

a. Give this definition for the sum, \(a + b\), where \(a, b \in \mathbb{N}\).

b. Give a complete illustration of the use of this definition for the sum, \(5 + 6\).
36. Construct a multiplication table (Table 1) for modulo-six arithmetic.

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Table 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>0</td>
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<tr>
<td>---</td>
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</tr>
<tr>
<td>0</td>
<td>0</td>
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<tr>
<td>1</td>
<td>b</td>
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<td>2</td>
<td>c</td>
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<td>3</td>
<td>d</td>
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<td>4</td>
<td></td>
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<tr>
<td>5</td>
<td></td>
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</tbody>
</table>

* is the inverse operation

X is the inverse operation

Use the above tables to complete 37-44. If no correct answer exists, write none.

37. The identity of Table 1 is ___.
38. The identity of Table 2 is ___.
39. 2 X 3 = ___
40. 4 X 2 = ___
41. 1 X 5 = ___
42. d * c = ___
43. b * d = ___
44. a * b = ___

Our universal set is R where R = {a, b|a, b ∈ N}, that is; the elements of the set are ordered pairs.

Definition: (a, b) = (c, d) iff a = c and b = d. We shall define a binary operation, &, on a subset S of the set R. Quite naturally, the elements of S are also ordered pairs. S = {a, b|a ∈ N and 1 = n{∅}}. Our binary operation is defined as follows: (a, 1) & (b, 1) = (ab - 1, a + b) In your answers (45-50), you may assume that the operations of addition and multiplication of elements of N have the five basic properties.

45. Are the elements of S all elements of R? Prove.

Five words associated with binary operations are listed below. Would & be associated with these words? Prove.

46. Closure: 47. Commutative: 48. Associative:
Test 3, Pretest, Study 2

1. \((-7) (6) = \) 
2. \((\sqrt[63]{6}) / (0) = \) 
3. \(0 / (6) = \) 
4. \((x^4) (x^2) = \) 
5. \(x^2 a / (x^{a+1}) = \) 
6. \((x^2)^4 = \) 
7. \(1 \div 3 + 1 \div 4 = \) 
8. \(\frac{5}{7} - \frac{3}{14} = \) 
9. \(\frac{(x + 3) (x - 7)}{(x + 3)} = \) 
10. \((16)(2) = \) 
11. \(\frac{x^2 + 2x + 1}{x + 1} = \) 
12. \(\sqrt{8} + 3\sqrt{2} = \) 
13. \(11 - (-4) = \) 
14. \(\frac{3}{4} \div \frac{5}{4} = \) 
15. If \(x - 3 = 10\), \(x = \) 
16. \(\frac{3}{\sqrt[3]{a^3}} = \) 
17. If \(x + 2 = 2x + 5\), \(x = \) 
18. \(7x = 21\), \(x = \) 
19. \((x - 5)(x + 3) = 0\), \(x = \) 
20. \(3h^2 + 10h - 8 = 0\), \(h = \) 
21. Find the product: \((2x + y)(2x - y) = \) 
22. Factor: \(2x^2 - 9x - 5 = \) 
23. \(\frac{2}{x - 1} + \frac{3}{x + 1} = \) 
24. A rectangle has a perimeter of 35 inches. It is known that five times the length of one side diminished by two times the length of the other side is equal to the perimeter. What are the dimensions of the rectangle?

Items selected are representative of the 66 item test.
Test 4, Study 2

1. If \( A = \{1, 2, 3, 4, 5, 7\} \), \( B = \{2, 4, 6, 8\} \), and \( C = \{1, 3\} \):
   
a. \( A \cup B = \) ______________________

b. \( B \cap A = \) ______________________

c. \( B \times C = \) ______________________

2. Solve for \( t \): \( 2t(2t + 5) = 3(t + 5) + 4t^2 \)

3. Solve for \( x \): \( \sqrt{x + 10} = 6x - 10 \)

4. Solve for \( y \): \( y^2 + 4y = 7 \) (Use quadratic formula)

5. Solve the system:
   \[
   \begin{align*}
   x + 2y + z &= 14 \\
   -2x + 3y - 2z &= 0 \\
   5x + z &= 10
   \end{align*}
   \]

6. Solve for \( x \): \( (x + 2)(2 - x) \leq 0 \). Represent your solution set on a line graph. CAREFUL.

7. Write in the form \( a + bi \): \( (1 + 2i)/(3 - 4i) \)

8. Sketch the graph of \( y = x^2 - 4x + 3 \). Use domain \(-1 \leq x \leq 5\).

9. A sum of \$2700 is invested, part at 4 per cent and the remainder at 5 per cent. Find the yearly interest on both investments if the interest on each investment is the same.

10. If the initial velocity of a falling body is 45 ft/sec, then the distance that it will fall in a vacuum is \( s = 45t + 16t^2 \). How long will it take the body to fall 279 feet?

Read the statement carefully. If the statement is always true (or true for all replacement of the variable), answer A. If the statement is true for certain replacements of the variable(s) and false for others, answer S. If the statement is never true (false for all replacements), answer N.

11. If \( * \) is a mathematical operation and \( a, b \) are elements such that \( a * b \) has meaning, then \( a * b = b * a \).

12. If \( P(t) = 0.4(t) \) is an algebraic expression which gives the portion of this test completed as a function of time for some student, then the domain of \( t \) is \( \mathbb{R} \).

13. If \( f(t) = 2t + 19 \) and \( g(t) = 2t - 1/2 \), then the graphs of \( f(t) \) and \( g(t) \) are parallel.

14. \( \frac{(x - 3)(x - 1)}{x - 1} = x + 3 \)

15. If \( 2x + 3 = 11 \), then \( x = 4 \).
16. If \( A \cup B = B \), then \( A \subseteq B \).
17. If \( A \) and \( B \) are sets, then \( A \cup B = B \cup A \).
18. The graph of \( y = 2x^2 - 13x + 2 \) is a parabola which opens upward.
19. If \( a, b, c \in \mathbb{N} \) and \( a \leq b \), then \( (a \leq b) \leq (b \leq c) \).
20. If \( x = 4 \), then \( x^2 - 7x + 12 = 0 \).
21. If \( a \in \mathbb{R} \), then there exists \( h \in \mathbb{R} \) such that \( (a)(h) = 1 \).
22. The least common multiple of the set \( \{2, 6, 12\} \) is 2.
23. If \( A, B, \) and \( C \) are non-empty sets, then \( A \times (B \times C) = (A \times B) \times C \).
24. The second term (usual order) in the expansion of \( (2x + y)^5 \) is \( 80x^2y \).
25. If \( G \cup \emptyset = \emptyset \), then \( G = \emptyset \).
26. \( b^2 - 4ac \) is called the discriminant.
27. If \( a, b \in \mathbb{R} \), then \( \sqrt{a} \sqrt{b} = \sqrt{(a)(b)} \).
28. The graph of \( 3x^2 + 9y^2 = 9 \) is a circle.
29. \( n(A \cup B) = n(A) + n(B) \).
30. If \( A \) and \( B \) are non-empty sets, then \( n(A \times B) = n(A) \cdot n(B) \).

The correct answer for at least five of the above statements was \( \emptyset \). Identify five of these statements by number, then...

a) give an example in which the statement is true

b) give an example in which the statement is false.
Test 5, Opinion Survey, Study 2

Select the phrase which best describes your feelings about the statement.

A - I strongly agree.
B - I agree.
C - I neither agree nor disagree.
D - I disagree.
E - I strongly disagree.

1. Each assignment should be turned in.
2. Some individual help should be given during the class period.
3. The lecture method is the best approach to the teaching of algebra.
4. Our instructor spends too much time re-explaining basic material to the slow student.
5. We have had too many tests this semester.
6. Only those who have a "mathematical bent" can learn mathematics.
7. The book need not be supplemented by individual notes, just the class lecture.
8. Non-textbook assignments should be given occasionally.
9. After a homework exercise has been discussed in class, an additional assignment should be made in that exercise.
10. Individuals tend to think that mathematics is difficult—this belief could result in a mental block.
11. Mathematics is a subject only for "way out" people.
12. Total grade points seem to be the best way to determine the final grade.
13. Review for an examination on the day before an examination is a good policy.
14. We have a very good textbook (or at least a good one).
15. Mathematics through algebra should be required of all students.
16. My adviser forced me to take this class.
17. Problems should be assigned prior to the discussion of the related material.
18. Change in mathematics requires that a new text be adopted every two years.
19. There should be little discussion in a mathematics class—the instructor should use most of the time proving theorems.
20. I would have learned more mathematics if our instructor had taught us less (I would have been required to do more work.).
21. Because of the scientific orientation of our society more people must learn mathematics.
22. We covered too much material in this course.
23. Problems should be used to initiate the study of the various topics in mathematics.
24. The solution of the textbook problems requires a real understanding of the related material.
25. There should be a written homework assignment made on the day prior to an examination.
26. On an examination a problem with an incorrect answer should receive no credit.
27. Our instructor requires too much work of the student.
28. On the average I spent more than eight hours per week (outside of class) studying intermediate algebra.
29. Set theory in no way contributes to a more complete understanding of algebra.
30. The mathematical material that I studied this semester is not the material that I expected to study.
31. I believe that I now have a better understanding of "the nature of mathematics" than I had at the start of the course.
32. I believe that I am now prepared for college algebra.
33. I care less for mathematics now than I did at the beginning of this course.

34. Practical problems can promote the development of new mathematics.

35. Students should extend the frontiers of their mathematical knowledge largely by their own effort.
APPENDIX C

STUDENT COMMENTS
I like your teaching method—letting us discover things for ourselves, then relating this information to the things in the book. We will retain these principles longer.

Although the problems which required a search for meaning frustrated me, I realize now that they forced us to go beyond mere memorization to true understanding.

Although I have become discouraged at times in this course, I can truthfully say that I have learned more about math—if not reasoning, thinking, and problem solving.

I don't really understand the purpose of giving us problems before having the material, but I imagine they serve a purpose. The problems helped me discover what I didn't know so I was able to go over those things more carefully (maybe that was the purpose).

Some of the homework was very challenging and I really tried to meet the challenge. This was a good method of study and thinking.

These passages are the sections of student comments which appear to be related to "the use of problems." Although some statements have been rephrased, the author believes that the meaning of the text has not been altered.
The special homework makes you think of methods and ways for the solution of a problem.

I enjoyed the course because it was challenging although I don't know that I liked the extra take home problems so well.

I think that the student should have been given a better idea in the beginning of what was expected of him in relation to the problems.

I am glad that I took this course. It has been good for me.

Those problems you threw in early in the semester were either too involved or just too simple. They messed up the whole semester so I just lost interest in the whole thing.

Now their value [the problems] is more appreciated.

At first the problems were a pain in the neck ... I felt good when I could figure out one of those problems because they made you think. (I actually believe that the problems were worthwhile.)

Although a few of the work sheets did help, some that you gave at the start of the year mixed me up and I haven't come out of my daze yet.

The non-textbook assignments were helpful. If they did nothing else, they made a person think. I found that at times I became so involved in the surface problem that I failed to see the underlying principle.
I thought that the problems were good. I found that some of them really made you stop and think. They were a challenge, but they were not unsolvable. I liked them.

The problems were helpful in many ways and caused some thinking. But a few were very confusing to me at the time, especially the ones on proof. They are a nice change from the regular textbook homework.

I feel that these problems helped me to realize how much mathematics means to us even in our daily lives. The problems could have been explained more thoroughly by the instructor upon their completion.

The outside, seemingly unrelated, material was of some benefit, but I feel it would have been more useful if we could have had just a little more information on the problems. However, if this had been the case, I doubt that the problem would have had its intended value.

I think that the extra problems on the whole have helped—especially the last four or five.

Those problems didn't do anything for me—except for a few headaches. I still don't fully understand your purpose for giving us—or for that matter, giving anyone—those problems . . . . The class in general didn't appreciate the idea of being used as guinea pigs.

For me the problems which were assigned ranged from easy to very difficult. However, I was very happy when I
found that I could work most of the ones which incorporated more basic ideas.

I felt that some of the extra exercises we did were fun and I enjoyed doing them. Besides learning from them— they changed the routine assignments. Others I did not understand and felt they were completely useless.
BIBLIOGRAPHY


