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VITA

July 21, 1938  Born - Hamilton, Ohio
1960. . . .  B.S.Ed., Mathematics, Miami University, Oxford, Ohio
1960-1961  Graduate Assistant, School of Education, Miami University, Oxford, Ohio
1962-1964  Mathematics Teacher, Amanda Junior High School, Middletown, Ohio
1964-1965  National Science Foundation Academic Year Institute Participant, The Ohio State University, Columbus, Ohio
1965-1966  Teaching Associate, Dept. of Mathematics, The Ohio State University, Columbus, Ohio
1966-1968  Instructor, Dept. of Mathematics, The Ohio State University, Lima Campus, Lima, Ohio

FIELDS OF STUDY

Major Field:  Education

Mathematics Education.  Professor Harold C. Trimble

Minor Fields:

Mathematics.  Professor Leslie H. Miller
Elementary Education.  Professor Lowry W. Harding
Secondary Education.  Professor Jack R. Frymier
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CHAPTER I

INTRODUCTION

1.1 Background

The study of properties of an infinite set is hampered from the outset by the number of elements contained within the set. Inductive reasoning may lead to conjectures of properties of a system defined on the set, but a valid proof of these conjectures usually depends upon more subtle deductive procedures. Were these same properties also exhibited in a smaller finite set, an opportunity would be available to study the finite set in detail, thereby gaining a deeper insight into its basic structures which parallel those of the larger and more complex set. Such an analogy of basic properties exists between the set of rational numbers and certain finite modular number systems. The importance of finite number systems can be seen in the number of ways they are used to illustrate principles of higher algebra. Yet, these same modular systems may be comprehended by students in elementary schools.

In 1967 Yu-Mei Yu Lu completed a doctoral disserta-
tion [9] concerned principally with a finite twenty-five point geometry. This geometry exhibits in many ways the structure of Euclidean plane geometry, yet contains only twenty-five distinct points. Again, the finite system may be analyzed in great detail using inductive procedures to prove conjectures whose counterparts in the Euclidean plane can be proven only through a deductive process. Finite geometry is related to plane geometry in much the same manner as modular arithmetic is to the real number system. In fact, modular arithmetic may be used to describe many of the properties of this geometry.

While the first development of finite geometries is credited by T. J. Fletcher [6:34] to O. Veblen and W. H. Bussey [10], the description of twenty-five point geometry is most readily available to the layman through Lillian R. Leiber's book, The Education of T. C. Mits [8].

Twenty-five point geometry and its relation to certain topics in advanced Euclidean geometry was first introduced to Mrs. Lu and this writer in a class under Professor Leslie H. Miller in 1965. Working with Dr. Miller, Mrs. Lu investigated many aspects of this mathematical system and of

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1 A number in brackets indicates the number assigned the reference in the bibliography; a second number which appears in some of the brackets indicates the page within that reference.
related nine and forty-nine point geometries. One result of her investigation was the development of a definition for angular measurement which suggests the possibility of extending the finite geometric system to include a finite trigonometry, the topic of concern in this dissertation.

1.2 The Purpose of the Study

The purpose of this investigation is to extend a finite geometric system by defining a finite trigonometry whose properties are analogous to those of the trigonometry of the Euclidean plane. These properties include definitions of trigonometric functions arising from ratios of sides of a right triangle, the relations of elements of a given triangle through a Law of Sines and a Law of Cosines, and the relations of trigonometric functions expressed through identities. Hopefully, the development of finite trigonometry will prove as adaptable to learning situations as are finite number systems and finite geometry.

1.3 Design of the Study

As noted in Mrs. Lu's dissertation, relatively few articles are available which investigate the properties of twenty-five point geometry, and no articles or publications concerning a finite trigonometry based upon the twenty-five point geometry are known to exist. In fact, prior to the definition of angle measurement by Mrs. Lu,
articles by both Martyn H. Cundy [4] and Arthur F. Coxford, Jr. [3] state that there is no satisfactory definition of a general angle in the geometry.

The development of finite trigonometry in this study centers around two underlying concepts: analogy and generalization. The resulting finite trigonometry reveals a trigonometry whose properties closely parallel those of trigonometry of the Euclidean plane. Both have trigonometric definitions arising from right triangles, both contain a Law of Sines and a Law of Cosines, and both contain many of the same identities. The analogy in this investigation, however, goes beyond the properties of the two systems and extends into the actual development of the finite system.

The presentation of Euclidean trigonometry in a classroom situation quite often follows a historical development where the trigonometric functions are shown first as a means of solving problems concerning right triangles. A generalization of the functions then relates trigonometry to general triangles, and a second generalization relates the trigonometry to angles of all measures. The finite trigonometry in this study is first developed to parallel this historical approach. While Chapter II is not directly concerned with finite trigonometry, it presents a necessary background in finite geometry upon which the trigonometric system is built. Chapter III shows the first stage of
finite trigonometry. Here the trigonometry is presented only as it relates to right triangles and to its use in finding unknown elements of triangles in the twenty-five point geometry. Chapter IV discusses an attempt to extend the trigonometry of Chapter III by viewing the trigonometric functions in relation to all angles which are elements of finite triangles. Here the analogy to Euclidean trigonometry fails in an attempt to develop a Law of Cosines for the finite case. The writer therefore reconsiders in Chapter V certain concepts of the finite geometry in order that a closer analogy may result between Euclidean trigonometry and the finite trigonometry. With new definitions, notably in the measurement of angles and in distance numeration, Chapter V then presents a finite trigonometry which appears analogous to Euclidean trigonometry in almost every way.

Since still other teaching situations may present trigonometry through more recently developed approaches of analytic geometry or the study of functions related to the unit circle, Chapter VI discusses alternative approaches to finite trigonometry through coordinates assigned to the twenty-five points and relates the trigonometric functions to the coordinates of points on the unit circle. Also included in Chapter VI is one method of developing trigonometric identities for the finite system.
The role of generalization can be seen in the development of both Euclidean trigonometry and the finite system. Indeed, its role in the creation of mathematics cannot be over emphasized. Whereas Chapters II through VI were concerned with generalizations of the trigonometry within the twenty-five points, Chapter VII presents a generalization of the finite trigonometry to a larger set of finite points. The concepts of the twenty-five point trigonometry are shown to hold within a forty-nine point system which suggests that this trigonometry might also prove valid in other finite systems.

Chapter VIII discusses a classroom presentation of a portion of material contained in this study and Chapter IX suggests areas in which students and other investigators might create new mathematics and extend the finite systems.

1.4 Significance of the Study

The material in this study is directed to teachers of mathematics and as a consequence it is also designed as a reference for textbook authors and curriculum builders. It is the hope of the writer that these persons may adapt this material for use in teaching situations. These situations might well include classroom presentations, individual enrichment, or group enrichment through a mathematics club. Although a study of finite trigonometry may be most ideally suited as a topic for a mathematics club, and thus avoid
the concern of its placement in an already crowded curric­ulum, its merits may warrant its consideration in other learning situations.

In particular:

1. The finite system is relatively simple. Most junior high-aged students are capable of manipulating and understanding the finite trigonometry developed in Chapters III and IV, while all other aspects of the system are within the understanding of capable students at the secondary level.

2. The finite trigonometry presents a system of mathematics other than a study of the real numbers. Students at the secondary level often find it difficult to fully understand the nature of mathematics when their only view is through a study of real numbers. Less capable students (or for that matter, many capable ones) may never realize an opportunity to see mathematics from a broader viewpoint.

3. Finite trigonometry presents concepts which also underlie trigonometry of the Euclidean plane. Thus the finite system may be used as a pre-trigonometric motivation, or for those who have previously studied Euclidean trigonom­etry it may be used as an excellent means of reviewing trigonometric concepts.

4. The finite trigonometry sharpens the concept of proof. While inductive reasoning in Euclidean trigonometry may be used to justify conjectures concerning the system,
inductive reasoning cannot be used to present a valid proof. However, an inductive proof may be acceptable in the finite system since for each conjecture there exists only a finite number of possibilities which are necessary to examine. In addition, diagrams which sometimes cloud proofs in the Euclidean system are relegated to a position of lesser importance since no diagram can fully illustrate the figures of the finite system.

5. The area of finite trigonometry contains material in which the student himself may work and create new mathematics. This, perhaps, is the most important contribution of this study. Many aspects of this system are yet to be extended and the definitions which have been created, like all definitions, are subject to reconsideration. Again, the simplicity of the finite system allows for the opportunity of active student involvement.

Finally, it is hoped that the presentation of the finite trigonometry in this dissertation will lead its readers to still other possibilities within the finite system.
CHAPTER II

TWENTY-FIVE POINT GEOMETRY

2.1 Introduction

Historically, trigonometry of the Euclidean plane is a branch of geometry which developed from properties of right triangles. Since the finite trigonometry of this investigation has also risen from a study of geometry, Chapter II is included to make this dissertation more self-contained by providing for the reader an underlying foundation of twenty-five point geometry necessary for the development of a finite trigonometric system.

While the bibliography lists several references which discuss finite geometry, the most complete source of reference is the dissertation of Yu-Mei Yu Lu [9], "An Expository Presentation of Finite Geometries as a Resource for Teachers," mention of which will be found throughout this presentation. The definitions which follow appear almost verbatim from Mrs. Lu's dissertation. Mrs. Lu's work on the topic of this section reflects the article, "Geometric Diversions: A 25-Point Geometry," by Arthur Coxford, Jr. [3]. The definitions for multiplication and addition in the finite system appear in Mrs. Lu's study. The discussion
of possible sizes of triangles is the work of this investigator.

The Twenty-Five Point Geometry involves the first twenty-five letters of the English alphabet arranged in the three blocks as shown.

\[\begin{array}{ccc}
A & B & C & D & E & A & I & L & T & W & A & H & O & Q & X \\
F & G & H & I & J & S & V & E & H & K & N & P & W & E & G \\
K & L & M & N & O & G & O & R & U & D & V & D & F & M & T \\
P & Q & R & S & T & Y & C & F & N & Q & J & L & S & U & C \\
U & V & W & X & Y & M & P & X & B & J & R & Y & B & I & K \\
\end{array}\]

Block I \hspace{1cm} Block II \hspace{1cm} Block III

Fig. 1.—Twenty-Five Point Geometry

Definitions related to Figure 1:

**Point:** Each of the 25 letters is called a point.

**Line:** Each row or column in any block is called a line. (There are 30 lines each containing 5 points and each of the 25 points is on 6 lines.)

**Parallel lines:** Two lines are parallel if they have no point in common. (Lines are parallel only if they are rows or columns in the same block.)

**Perpendicular lines:** Two lines are perpendicular if one is a row and the other is a column in the same block.

**Row distance:** If two points are in a common row, the distance between the points is obtained by counting the least number of steps between the points where, when reach-
ing the end of a row, counting is continued by jumping to
the beginning of the same row. (For example, AB = CD = AE
and AC = AD = CE.)

**Column distance:** If two points are in a common column,
the distance between the points is obtained by counting the
least number of steps between the points. (Thus, AF = FK =
AU and AK = PF = AP.)

**Distance Notation:** In a row, distances of one and two
steps are designated by 1 and 2. In a column, distances of
one and two steps are designated by 1' and 2'. (The four
distances are distinct. AI = 1 and AM = 1', but AI ≠ AM.)

**Triangle:** A set of three points is called a triangle
if the points are not collinear.

**Circle:** A circle is the set of points a given distance
from a given point. (Each circle is composed of 6 points.
The circle with center R and radius 2' contains points
C, H, L, X, V, and N.)

**Midpoint:** A point is the midpoint of a given segment
if it is on the line through the segment and is equidistant
from the endpoints of the segment. (The midpoint of AB
is D.)

**Addition:** The addition of the numbers used to
represent distances is defined in Table 1. Addition between
row and column distances remains undefined.
TABLE 1

ADDITION TABLE

<table>
<thead>
<tr>
<th>+</th>
<th>1</th>
<th>2</th>
<th>0</th>
<th>1'</th>
<th>2'</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>2</td>
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<td>2'</td>
<td>2'</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2'</td>
<td>2'</td>
<td>2'</td>
<td>1'</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(Note that the inverse operation does not yield a unique result since both 1 + 2 = 2 and 1 + 1 = 2.)

Multiplication: Multiplication of numbers used to represent distances is defined in Table 2.

TABLE 2

MULTIPLICATION TABLE

<table>
<thead>
<tr>
<th>X</th>
<th>1</th>
<th>2</th>
<th>0</th>
<th>1'</th>
<th>2'</th>
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<tbody>
<tr>
<td>1</td>
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<td>2</td>
<td>0</td>
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<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
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<td>2'</td>
<td>1'</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

(It is of interest to note that multiplication and addition were originally defined so that the Theorem of Pythagoras...
would be valid in twenty-five point geometry.)

**Similar triangles:** Two triangles are similar if the distances between corresponding vertices are proportional.

For twenty-five point geometry one can verify many results which correspond to those in Euclidean geometry. For example, the theorem that the medians of any triangle are concurrent can be proved inductively by testing for all possible triangles, and the Euclidean postulate that two points determine exactly one line is a valid theorem in twenty-five point geometry since any two of the twenty-five points occur simultaneously in one and only one row or column of the three given blocks. For a further discussion of the properties of the twenty-five point geometry, the reader is referred to Mrs. Lu's dissertation.

2.2 Triangles in Twenty-Five Point Geometry

In Euclidean geometry it is possible to designate a specific triangle without reference to angle by denoting the three points which determine its vertices or by specifying the lengths of its three sides. Triangles in twenty-five point geometry can also be designated in the same manner. By determining the number of possible ways in which three non-collinear points may be selected from one of the three blocks, one can show there are exactly 2,000 different triangles within the system.
However, the fact that the number of different possible combinations of the four distances when selected three at a time is twenty indicates that there are at most, within the twenty-five point system, twenty different sizes of triangles. These twenty combinations are listed below.

1 1 1 1 2 1
2 2 2 2 2
1 1 1 1 1 1
1 1 1 1 1 1
1 1 2 1 2 2
2 2 2 2 2
1 2 2 1 2
2 2 2 2 2

Assuming that all twenty sizes of triangles do exist, one can readily demonstrate that there are eight cases in which the sum of the squares of two sides is equal to the square of the third side. These are triangles with sides 1, 1, 1'; 2, 1', 1; 1, 2', 2; 1, 2, 2'; 2', 2', 1; 2, 2, 2'; 1', 1', 2; and 2, 2', 1'. (The hypotenuse for each case is listed in the third position.) Yet, the definition of perpendicular lines within the rectangular array limits the legs of a right triangle to the four combinations of 1, 1'; 1, 2'; 2, 2', and 2, 1', and when one applies the Theorem of Pythagoras to these distances, the results indicate the existence of only four possible sizes of right triangles: 1, 1', 2'; 1, 2', 2; 2, 1', 1; and 2, 2', 1'. These observations lead to one of the following possible conclusions: there exist other lines which should be
defined as perpendicular which do not appear as such in the rectangular array; the converse of the Pythagorean Theorem is not valid; or triangles with sides of 1, 1, 1'; 1, 2', 2'; 2, 2, 2'; and 2, 1', 1', do not exist.

One method, which could well be adapted for use with high school students to determine the sizes of all triangles that do exist in twenty-five point geometry is shown below.

With point A as the center, each of the remaining twenty-four points is present once and only once on one of four concentric circles whose radii are 1, 2, 1', and 2'. Table 3 lists the points found on each circle and gives the distances from a selected point on each circle to the other points of that circle. These circles are similar in that

<table>
<thead>
<tr>
<th>TABLE 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>CIRCLES WITH A CENTER OF POINT A</td>
</tr>
</tbody>
</table>

| Radius: | Points on the circle: | Distance from one point to other points on the same circle: |
| --- |
| 1 | BX W E H I | BX = 1 BW = 2' BE = 2 |
|  |  | BH = 2' BI = 1 |
| 2 | DO L C Q T | DO = 2 DL = 1' DC = 1 |
|  |  | DQ = 1' DT = 2 |
| 1' | SF R M U N | SF = 1' ST = 1 SM = 2' |
|  |  | SU = 1 SN = 1' |
| 2' | JY P V G K | JY = 2' JP = 2 JV = 1' |
|  |  | JG = 2 JK = 2' |
distances between corresponding points are proportional.

In addition to serving as the center of the four concentric circles, the point A is also designated as one of the vertices of all triangles considered in the following cases.

**Case I. Triangles with sides of 1, 1, and 1.**

With point A as the fixed vertex, the other two vertices must lie on the circle of radius 1. One can readily find six such triangles whose sides are 1, 1, and 1, (Triangles BAX, SAW, WAE, EAH, HAI, and IAD). Since the point A could be replaced by any of the other twenty-four points with six triangles of side 1, 1, 1, for each fixed vertex, the total number of such triangles in twenty-five point geometry is \( \frac{25 \cdot 6}{3} = 50 \). Division by three is necessary since this is the number of times each triangle would be counted under this arrangement.

One soon discovers that similar triangles of sides 2, 2, 2; 2', 2', 2'; and 1', 1', 1' are contained in corresponding similar circles. Thus, each of these sizes is exhibited by fifty different triangles, accounting for a total of 200 different equilateral triangles.

**Case II. Triangles with sides of 1, 1, 2'.**

a) First assume point A is the fixed vertex at which meet the sides of length 1. In order to form triangles of side 1, 1, 2', the remaining two vertices must both lie on
the unit circle with a distance of 2' between them. There are six triangles, and only six, which satisfy this requirement.

b) Next assume point A is the fixed vertex joining the sides of length 1 and 2'. The point J represents a point a distance of 2' from point A. The distances from point J to each point on the unit circle are JB = 1, JW = 1', JH = 2, JX = 2, JE = 1', and JI = 1. (Note that the points JAB and JAI would form triangles of sides 1, 1, 2'.) Since there are six points on the circle of radius 2', and, as illustrated with the point J, with each of these points there exist two points on the unit circle which form triangles of sides 1, 1, 2', the total number of triangles of size 1, 1, 2', as derived through parts a) and b), is \( \frac{25 \cdot 6 + 25 \cdot 6 \cdot 2}{3} = 150 \). The preceding process repeated for the similar triangles 2, 2, 2'; 1', 1', 1; and 2', 2', 2 also results with 150 triangles for each case, a total of 600 isosceles triangles.

Case III. Triangles with sides of 1, 2, 1'.

a) First assume A is the fixed vertex which joins sides of length 1 and 2. The distances from point D, a point which is two units from point A, to each point on the circle one unit from A are DB = 2, DW = 2', DH = 2', DX = 1', DE = 1, and DI = 1', which indicate that points DAX and DAI form the desired triangles of sides 1, 2, 1'. Thus with each point on the circle of radius 2 there are two points on
the circle of radius 1 which satisfy the requirements for triangles of sides 1, 2, 1'.

b) Next assume point A is the fixed vertex joining sides of length 1' and 2. Again D is a point 2 units from point A and the distances from point D to each point on the circle of radius 1' from point A are DS = 2', DR = 2, DU = 1, DF = 1, DM = 2, and DN = 2'. Again it is noted that for each point on the circle of radius 2 there are two points on the circle of radius 1' which satisfy the requirements for a triangle with sides of 1, 2, 1'.

c) Finally, assume point A is the fixed vertex joining the sides of length 1 and 1'. The distances from point B, a point which is 1 unit from point A, to each point on the circle of radius 1' are BS = 1', BR = 2, BU = 2', BF = 2', BM = 2, and BN = 1'. Thus with each point on the circle of radius 1 there are two points on the circle of radius 1' which form triangles with sides 1, 2, 1'.

The total number of triangles with sides 1, 2, 1' can then be determined by assuming each of the twenty-five points may be the center of the concentric circles described (which is also the fixed point of each triangle) and allowing for repetition of triangles. Thus, there are $25 \cdot 6 \cdot 2 + 25 \cdot 6 \cdot 2 + 25 \cdot 6 \cdot 2 \over 3 = 300$ distinct triangles with sides of 1, 2, 1'. Similar procedures may be used to determine that for each set of similar triangles (triangles with sides
2, 1, 2'; 1', 2', 2; and 2', 1', 1) the number of distinct possibilities is also 300.

Thus far, twelve sizes of triangles have accounted for 2,000 different triangles, the total number of triangles which exist in twenty-five point geometry. One can therefore conclude that no triangles exist for the remaining eight possible combinations of lengths, and indeed, when one applies the preceding process to the remaining combinations of lengths, this conclusion is verified.

Case IV. There are no triangles formed by combination of lengths 1, 1, 1'; 2, 2, 2'; 1', 1', 2; or 2', 2', 1.

Proof: Assume a triangle with sides 1, 1, 1' exists and let point A be a fixed vertex joining the sides of length 1. The remaining vertices must lie on a circle of center A, radius 1. There are six points on this circle and the distances between any two of these points are 1, 2, or 2' (never 1'). Since the point A may be replaced by any of the remaining twenty-four points to yield the same result, the statement of Case IV must be valid.

Case V. There are no triangles formed by combination of lengths 1, 1, 2; 2, 2, 1; 1', 1', 2'; and 2', 2', 1', since all points related by these distances lie on the same line.

It is of interest to note that all combinations of lengths from Case IV form Pythagorean triples; that is, the sum of the square of two of the three lengths is equal to
the square of the third; yet, the converse of the Pythagorean Theorem holds in twenty-five point geometry since these combinations of length do not form sides of triangles. The combinations of length in Case V also hold an unusual property in that the sum of any two of the lengths equals the third.

Another consequence of the procedure used to classify triangles of twenty-five point geometry is the proof of a theorem which may be stated as: Two circles, related such that the center of the second is a point on the circumference of the first, intersect at most in two points (shown in Cases I, II, and III), or intersect at only one point (Case V), or do not intersect at all (Case IV). Its counterpart in Euclidean geometry is generally assumed true from diagrams.

It has been the purpose of this chapter to give a brief description of twenty-five point geometry. A discussion in greater detail of the triangles in this system has also been included since they are the source from which arises the finite trigonometry of the following chapters.
CHAPTER III

RIGHT-TRIANGLE TRIGONOMETRY

ON TWENTY-FIVE POINTS

3.1 Introduction

During the junior high school years the students are frequently brought into contact with problems whose solutions depend upon properties of triangles. Scale drawings, ratio, and proportion are used to solve problems involving indirect measurements, and the Theorem of Pythagoras is used as another tool, even though its proof is not formally presented until some time later. Solutions of right triangles by trigonometric means are quite commonly found in general mathematics courses and algebra courses for ninth year students and in some cases at an even earlier time. These topics could rightly be called topics of a pre-trigonometric nature.

The material in this chapter could well be adapted as another means of providing the students of this age level, or beyond, with an understanding of some of the concepts used in the study of trigonometry. It is concerned largely with numerical solutions to right triangles in a twenty-five point plane.
This approach is somewhat unique in that it is based upon a finite system of points which can be manipulated easily by students. The finite system is in contrast to the unlimited number of points of the Euclidean plane. The limited number of angles and limited number of sizes of triangles in the twenty-five point system means that solutions found by trigonometric formulas can be checked easily against the finite list of possible solutions. In addition, the students are also introduced to a new mathematical system which can be used as a setting to illustrate both common and unlike properties with the Euclidean geometry and trigonometry.

Since the trigonometry in this chapter is restricted to right triangles, only a short background study in twenty-five point geometry is necessary to acquire a knowledge of the basic definitions. However, a more thorough understanding of the twenty-five point geometry would definitely prove helpful in studying the trigonometric system.

3.2 The Right Triangles

The triangles of twenty-five point geometry which contain right angles are easily distinguished from other triangles by observing the positions, in the three blocks, of the three points designated as the vertices. If the three vertices are contained in exactly one row and one column of one of the three blocks, (not all in one row or one
column) the triangle must contain a right angle. This follows immediately from the definition of perpendicular lines. In addition, the previous classification of all sizes of triangles shows that any triangle whose lengths of sides form Pythagorean triples must also form a right triangle.

The classification of all triangles also demonstrates that there exist only four sizes of right triangles in this system, all of which are similar. A representation of these four right triangles is shown in Plate I. (Note that diagrams used to illustrate figures of finite geometry do not give a true representation since, in finite geometry, lines and other geometric figures consist of sets of isolated points.) Measurement of the acute angles which is shown in the figures of this plate were assigned according to the scheme discussed in the following section.

3.3 Measurement of Angles

Interior angles were defined by Yu-Mei Yu Lu in a manner such that the sum of the interior angles of a triangle totaled 180°. In addition, corresponding angles of similar triangles, that is, triangles in which the ratios of corresponding sides are equal, are also found to be equal.

Interior angles of a triangle are determined as follows:

1. The angle between two lines is 90° if one line is
PLATE I

REPRESENTATIVE ILLUSTRATIONS OF THE FOUR SIZES OF RIGHT TRIANGLES

Figure 2

Figure 3

Figure 4

Figure 5
a row and the other is a column in the same block.

(2) The angle between two lines is $60^\circ$ or $120^\circ$ if both lines are rows in different blocks or if both lines are columns in different blocks.

(3) The angle between two lines is $30^\circ$ or $150^\circ$ if the lines are rows and columns in different blocks.

(4) Let three lines intersect to form a triangle. The angles of the triangle are assigned numerical values so that the sum of the interior angles is $180^\circ$.

Employing this system of angular measurement defined by Mrs. Lu, one observes that the four sizes of right triangles contain only the angles of $90^\circ$, $60^\circ$, and $30^\circ$ in the positions as shown in Plate I. There are no acute angles other than $60^\circ$ and $30^\circ$.

3.4 Computations

Addition and multiplication of the numbers 1, 1', 2, and 2' have been defined in Chapter II. In addition to these operations, the definitions of the trigonometric functions as they relate to right triangles require that the operation of division (or some other means to determine the ratio between two numbers) be defined. Since the inverse of multiplication (see Table 2) does yield unique results, $\frac{a}{b}$ or $a \div b$, is defined as follows:

$$\frac{a}{b} = c, \text{ if and only if, } a = b \cdot c.$$
From the multiplication table, it is apparent that \( \frac{a}{b} \) is defined uniquely for all possible combinations of \( a \) and \( b \) where \( a \) and \( b \) are to be chosen from the four numbers assigned to distance.

3.5 The Trigonometric Functions

In keeping with the definitions of plane trigonometry, the sine of an angle in the twenty-five point system is defined as the ratio of the length of the side opposite the angle to the length of the hypotenuse.

Values are shown for the sine of 30° and sine of 60° as obtained from all four triangles shown in Plate I.

Figure:

\[
\begin{align*}
(2) \quad \sin 30° &= \frac{1'}{2} = 2 \quad \sin 60° = \frac{1'}{2} = 1' \\
(3) \quad \sin 30° &= \frac{2'}{1} = 2 \quad \sin 60° = \frac{2'}{1} = 1' \\
(4) \quad \sin 30° &= \frac{2}{1} = 2 \quad \sin 60° = \frac{1'}{1} = 1' \\
(5) \quad \sin 30° &= \frac{1}{2} = 2 \quad \sin 60° = \frac{2'}{2} = 1'
\end{align*}
\]

In all four cases the sine of 30° equals 2 and the sine of 60° equals 1'. Thus, between these angles and the sine is established a functional correspondence.

The cosine of an angle in the twenty-five point system is defined as the ratio of the length of the side adjacent to the angle to the length of the hypotenuse. It is
apparent from the four right triangles that in each case
\[ \cos 30^\circ = \sin 60^\circ = 1' \]
\[ \cos 60^\circ = \sin 30^\circ = 2. \]

Other trigonometric functions are defined in terms of the sine and cosine. Letting \( \theta \) represent any angle, the trigonometric functions of \( \theta \)—tangent, cosecant, secant, and cotangent—are defined:

\[ \tan \theta = \frac{\sin \theta}{\cos \theta} \]
\[ \csc \theta = \frac{1}{\sin \theta} \]
\[ \sec \theta = \frac{1}{\cos \theta} \]
\[ \cot \theta = \frac{\cos \theta}{\sin \theta} \]

From the preceding computations for sine and cosine, the values for all six trigonometric functions are derived for the angles of 30° and 60°.

**TABLE 4**

**TRIGONOMETRIC VALUES OF 30° AND 60°**

<table>
<thead>
<tr>
<th></th>
<th>sin</th>
<th>cos</th>
<th>tan</th>
<th>csc</th>
<th>sec</th>
<th>cot</th>
</tr>
</thead>
<tbody>
<tr>
<td>30°</td>
<td>2</td>
<td>1'</td>
<td>1'</td>
<td>2</td>
<td>2'</td>
<td>2'</td>
</tr>
<tr>
<td>60°</td>
<td>1'</td>
<td>2</td>
<td>2'</td>
<td>2'</td>
<td>2</td>
<td>1'</td>
</tr>
</tbody>
</table>
3.6 Solutions of Right Triangles

The three sides and the three angles of a triangle together comprise the six elements of a triangle. Typically, information is given regarding the length of one or more of the sides and/or measurement of one or more of the angles. The problem then consists of applying this information to determine one or more of the remaining elements of the triangle.

Inspection of the four possible right triangles provides the quickest means to solution of the missing elements in twenty-five point geometry if one of the known elements is a right angle. The interest in the trigonometric solution, however, lies in the challenge: Given three elements of the triangle; to derive a correct solution other than by inspection.

Example 3-1. It is known that the side opposite the $30^\circ$ angle of a right triangle is 2'. Determine the remaining elements.

The sine of $30^\circ = 2 = \frac{2'}{\text{hypotenuse}}$;

therefore, the hypotenuse = $\frac{2'}{2} = 1'$.

The tangent of $30^\circ = 1' = \frac{2'}{\text{side adj. } 30^\circ}$;

therefore, the side adjacent to the $30^\circ$ angle = $\frac{2'}{1'} = 2$.

Since the sum of the angles must total $180^\circ$ the remaining
angle must be 60°. A representation of this triangle is shown below. Inspection of the four right triangles also indicates this to be the only possible solution.

![Triangle Diagram]

Usually three elements must be known (one of which must be a side) in order to determine the remaining elements. In dealing only with right triangles, one of the known elements is always the 90° angle. However, by using the inspection method, only two elements are required to produce a unique solution if the two known elements are the 90° angle and the hypotenuse.

This chapter has dealt with the solutions for unknown elements of right triangles. In the following chapter an extension of the trigonometric concept is sought in order to provide solutions for unknown elements of any triangle.
4.1 Introduction

The search for generalizations of previous concepts is often an effective tool in the creation of new mathematics. Moreover, it may also lead to a fuller understanding of the original mathematical concept and its structures. A student who views trigonometry in the Euclidean plane with respect to only triangles containing right angles will never grasp the potential usefulness of the tool. But the generalization of the trigonometric functions to functions of all angles transforms the power of trigonometry far beyond the original purpose it served in solving problems concerning right triangles.

In this chapter an attempt is made to extend the twenty-five point trigonometry in a manner analogous to the extension of Euclidean trigonometry where trigonometric functions are first defined in terms of angles contained in right triangles and then extended to angles within general triangles. The generalization of this second concept, the
definition of trigonometric functions in relation to all
angles whether contained within a triangle or not, is
discussed in the chapter following this one.

Together, this and the preceding chapter define a
limited trigonometric system. Although the derivations
reproduced in this chapter require a more sophisticated
mathematical background than possessed by most junior high
school students, many students are capable of applying the
formulas obtained to specific triangles, especially in the
finite system where the number of sizes of triangles is
limited. Even this small extension will give students a
better insight into the usefulness of trigonometry.

Other students, with a background of high school
geometry should be able to handle all the material in this
chapter satisfactorily and should also find it helpful,
since the derivations of the Law of Sines and the Law of
Cosines, especially in the twenty-five point geometry,
emphasize the role of the betweenness concept which is
largely ignored in a general course in plane geometry. In
addition, the derivation of the Law of Cosines when applied
to the twenty-five point system illustrates the role that
uniqueness plays in defining an operation. It is the prob-
lem concerning uniqueness which motivates the redefinition
and restructuring of parts of the finite system in order
that it may be extended in a manner more closely analogous
to plane trigonometry.
4.2 Solutions of Oblique Triangles

By drawing appropriate auxiliary lines, any triangle in Euclidean geometry can be associated with one or more right triangles. The associated right triangles can then be used to determine the solutions for unknown elements in the original given triangle.

Solutions for all elements of a right triangle, which were discussed in the previous chapter, require a knowledge of two elements of the triangle, one of which must be a side, in addition to the known right angle. Solutions for unknown elements of oblique triangles also require a knowledge of three elements of the triangle, one of which must be a side. These solutions are divided into four general cases based upon the known elements:

(I) Two angles and a side given
(II) Two sides and an angle opposite one of these given
(III) Two sides and the included angle given
(IV) Three sides given.

The measurement of angles discussed in Chapter III assigns angles to the twelve sizes of triangles in twenty-five point geometry as shown in Table 5. There are eight sizes of oblique triangles.

4.3 The Law of Sines

The Law of Sines is used to find solutions to cases
I and II of Section 4.2. Case II is sometimes known as the ambiguous case since it is possible that two different solutions may result from the known elements.

One derivation for the Law of Sines in Euclidean trigonometry is now shown.

Consider first the triangle ABC as shown at the top of page 34 where both angles A and B are acute. The line CD is an auxiliary line perpendicular to the side AB.

**TABLE 5**

ANGLES ASSIGNED TO THE TWELVE SIZES OF TRIANGLES

<table>
<thead>
<tr>
<th>Sides of Triangles</th>
<th>Angles Opposite Corresponding Sides</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 1 1</td>
<td>60° 60° 60°</td>
</tr>
<tr>
<td>2 2 2</td>
<td>60° 60° 60°</td>
</tr>
<tr>
<td>1' 1' 1'</td>
<td>60° 60° 60°</td>
</tr>
<tr>
<td>2' 2' 2'</td>
<td>60° 60° 60°</td>
</tr>
<tr>
<td>1 1' 2'</td>
<td>60° 30° 90°</td>
</tr>
<tr>
<td>2 2' 1'</td>
<td>60° 30° 90°</td>
</tr>
<tr>
<td>1' 2 1</td>
<td>60° 30° 90°</td>
</tr>
<tr>
<td>2' 1 2</td>
<td>60° 30° 90°</td>
</tr>
<tr>
<td>1 1' 1'</td>
<td>120° 30° 30°</td>
</tr>
<tr>
<td>2 2' 2'</td>
<td>120° 30° 30°</td>
</tr>
<tr>
<td>1' 2 2</td>
<td>120° 30° 30°</td>
</tr>
<tr>
<td>2' 1 1</td>
<td>120° 30° 30°</td>
</tr>
</tbody>
</table>
From the definition of the sine:

(a) \( \sin A = \frac{CD}{AC} \) or \( CD = AC(\sin A) \)

(b) \( \sin B = \frac{CD}{BC} \) or \( CD = BC(\sin B) \).

Equating the results from (a) and (b),

(c) \( AC(\sin A) = BC(\sin B) \)

or (d) \( \frac{\sin A}{BC} = \frac{\sin B}{AC} \).

In the figure below, triangle ABC is shown with angle A as an obtuse angle. If \( \sin A = \sin(180^\circ - A) \), then state-
ments (a) and (b) are also valid for this triangle. Therefore, the sine of an angle is defined to be equal to the sine of its supplement. The resulting equality, statement (d), can then be generalized to the Law of Sines:

\[
\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c},
\]

where \(a\), \(b\), and \(c\), are the sides opposite the respective angles in any triangle.

In the triangle at the top of page 34 the auxiliary line cuts the side \(AB\) between points \(A\) and \(B\). It was then necessary to consider also the case where \(CD\) cuts \(AB\) outside the triangle. In the twenty-five point geometry this concept of betweenness has not been completely resolved. The problem can best be observed by choosing any two points on a line, such as the points \(A\) and \(B\) in block I. The midpoint of two given points is generally considered to be on the line segment between the two given points. By the definition of the midpoint, the midpoint of \(AB\) is \(D\); the midpoint of \(DB\) is \(C\); the midpoint of \(CB\) is \(E\); and the midpoint of \(EB\) is \(A\). Thus all points on the line segment \(AB\) appear to be between points \(A\) and \(B\), including the endpoint \(A\) itself. Thus, when the corresponding auxiliary line needed for the derivation of the Law of Sines is obtained in twenty-five point geometry, it is unknown whether this line cuts the interior of the triangle or lies entirely on the exterior.

Suppose, however, the preceding derivation is also
applicable to twenty-five point geometry with the trigonometric functions as they have been defined. An immediate result suggested by the application of the Law of Sines to any right triangle is that the sine of 90° be defined as 1. In particular, the Law of Sines applied to the right triangle of sides 1, 1', 2', results in,

\[
\frac{\sin 90^\circ}{2'} = \frac{\sin 60^\circ}{1};
\]

\[
\sin 90^\circ = 2'(\sin 60^\circ) = 2' \cdot 1' = 1.
\]

In addition to the four sizes of right triangles, the Law of Sines relates to the four sizes of equilateral triangles plus the four sizes of triangles containing angles of 30°, 30°, and 120°. These latter triangles are of particular interest since the sine of 120° has not previously been defined.

The extended definition yields,

\[
\sin 120^\circ = \sin(180^\circ - 120^\circ)
\]

\[
= \sin 60^\circ
\]

\[
= 1'.
\]

Solutions are shown for an example of Case I and an example of Case II, both of which may be solved by applying the Law of Sines.

Example 4-1. Case I. Two angles of a triangle are both 30° and the side opposite one of these is 1'. Find the remaining elements.

(a) Solution by inspection. The remaining angle must
be 120°. Therefore, by inspection of the four triangles containing 120°, 30°, 30°, the sides must be 1, 1', and 1', respectively.

(b) Solution by Law of Sines. Sides opposite equal angles of a triangle are equal. Therefore, two of the sides are of length 1'. The remaining angle must be 120°. Applying the Law of Sines,

$$\frac{\sin 30°}{1'} = \frac{\sin 120°}{x};$$

$$\frac{2}{1'} = \frac{1'}{x};$$

$$x = \frac{1' \cdot 1'}{2} = 1.$$

The sides are 1', 1', 1, with corresponding angles 30°, 30°, 120°.

Example 4-2. Case II. Two sides of a triangle are 1 and 1'. The angle opposite the side of length 1' is 30°. Find the remaining elements. Applying the Law of Sines,

$$\frac{\sin 30°}{1'} = \frac{\sin x}{1};$$

$$\sin x = \frac{2}{1'} = 1';$$

therefore, x is either 60° or 120°.

(a) If x = 60°, the remaining angle is 90°. From the definition of cosine, the hypotenuse, c, can be expressed by

$$c = \frac{1}{\cos 30°} = \frac{1}{\frac{1'}{2}} = 2'.$$
One solution, then, is a triangle of sides 1, 1', 2', with corresponding angles of 60°, 30°, and 90°.

(b) If $x = 120°$, the remaining angle is $30°$ thus forming an isosceles triangle of sides 1', 1', 1, with corresponding angles of 30°, 30°, 120°.

Inspection of the twelve sizes of triangles reveals that these are the only two possible solutions for the triangle of Example 4-2.

The Law of Sines may be proved valid in the twenty-five point trigonometry by its application to all triangles.

4.4 The Law of Cosines

The Law of Cosines is used in plane trigonometry to determine solutions for Cases III and IV described in Section 4.2. One means of its derivation in the Euclidean plane is now shown.

In the figure below triangle ABC is shown with angles A and B as acute. CD is an auxiliary line perpendicular to AB.
Let \( x = \text{length } AD \). Then \( c - x \) represents length DB.

By the Theorem of Pythagoras:

(a) \( (CD)^2 = b^2 - x^2 \)

and (b) \( (CD)^2 = a^2 - (c - x)^2 = a^2 - c^2 + 2cx - x^2 \).

Equating the results from (a) and (b) and replacing \( x \) by \( b(\cos A) \) gives

(c) \( a^2 = b^2 + c^2 - 2bc(\cos A) \).

The equation, (c), is called the Law of Cosines.

To extend this definition to include angles greater than 90° which are also elements of triangles, angle A is chosen as an obtuse angle as shown in triangle ABC below.

\[ CD \] is again drawn perpendicular to \( AB \). The letter \( x \) represents the length \( DA \). Again applying the Theorem of Pythagoras,

(d) \( (CD)^2 = b^2 - x^2 \),

and (e) \( (CD)^2 = a^2 - (c + x)^2 = a^2 - c^2 - 2cx - x^2 \).
Equating (d) and (e), and replacing \(x\) by 
\[ b \cdot \cos(180^\circ - A) \], results in the equation,
\[ (f) \quad a^2 = b^2 + c^2 + 2 \cdot b \cdot c \cdot \cos(180^\circ - A). \]

Thus, if equation (f) is to be equivalent to equation (c), the cosine of \((180^\circ - A)\) must be defined as the negative of the cosine of \(A\). In addition, if the cosine of \(90^\circ\) is defined as zero, then the Law of Cosines would appear as a generalization of the Theorem of Pythagoras.

This derivation of the Law of Cosines when applied to twenty-five point geometry results in several problems. One is the betweenness concept discussed in relation to the Law of Sines. Another concerns the operation of subtraction. With relation to the figure on page 38, the length \(AB\) was expressed by \(x\) and \(c - x\), that is, \(AB = x + (c - x)\). However, the inverse of addition of lengths in twenty-five point geometry does not yield unique results. Thus if \(AB\) were a length of 2, and \(x\) were 1, then \(c - x\) could be either 2 or 1, since both \(1 + 2 = 2\) and \(1 + 1 = 2\).

Secondly, to equate statement (f) with the Law of Cosines, the cosine of \((180^\circ - A)\) was defined as the negative of the cosine of \(A\). Negative numbers, in relation to the finite trigonometric system thus far defined, do not exist. Only those numbers used to designate lengths are used to define multiplication and addition.

These problems suggest that the Law of Cosines is not
valid in the twenty-five point system as defined, and this is indeed true. This is evident in the following example where an attempt is made to apply the Law of Cosines to a triangle in twenty-five point geometry.

Example 4-3. Two sides of a triangle are both of length 1 and the included angle is 60°. Determine the length of the third side.

Denoting the third side as \( a \) and applying the Law of Cosines yields,
\[
a^2 = l^2 + l^2 - 2 \cdot 1 \cdot 1 \cdot \cos 60°; \\
a^2 = 1 + 1 - 2 \cdot 1 \cdot 1 \cdot 2 = 2 - 1 = 1.
\]

If \( a^2 = 1 \), then \( a \) is either 1 or 2 since \( 1^2 = 1 \) and \( 2^2 = 1 \). If \( a^2 = 2 \), then \( a = 1' \) or \( a = 2' \) since both \((1')^2 \) and \((2')^2 = 2 \). Thus it appears that the third side could be any of the four lengths. However, solution by inspection of the twelve sizes of triangles reveals the only possible choice of the third side to be the length 1.

Two alternatives present themselves. Either to retain the geometric and trigonometric systems as defined or to alter previous definitions in order that the trigonometric functions may be extended. It seems reasonable to assume that any identities of plane trigonometry which involve negative numbers or subtraction could not even exist in twenty-five point trigonometry as presently defined.

The twenty-five point trigonometric system can be
characterized, to this point, by the definitions of the six trigonometric functions, the Law of Sines, the absence of a Law of Cosines, and the values of the trigonometric functions shown in Table 6.

**TABLE 6**

<table>
<thead>
<tr>
<th></th>
<th>sin</th>
<th>cos</th>
<th>tan</th>
<th>csc</th>
<th>sec</th>
<th>cot</th>
</tr>
</thead>
<tbody>
<tr>
<td>30°</td>
<td>2</td>
<td>1'</td>
<td>1'</td>
<td>2</td>
<td>2'</td>
<td>2'</td>
</tr>
<tr>
<td>60°</td>
<td>1'</td>
<td>2</td>
<td>2'</td>
<td>2</td>
<td>2</td>
<td>1'</td>
</tr>
<tr>
<td>90°</td>
<td>1</td>
<td>0</td>
<td>undef.</td>
<td>1</td>
<td>undef.</td>
<td>0</td>
</tr>
<tr>
<td>120°</td>
<td>1'</td>
<td>undef.</td>
<td>undef.</td>
<td>2'</td>
<td>undef.</td>
<td>undef.</td>
</tr>
</tbody>
</table>

However, complete retention of the finite trigonometric system as developed severely limits the close analogy that was sought between this system and the trigonometry of the Euclidean plane. In Chapter V, therefore, previous definitions of the finite system will be reconsidered in order to develop a finite trigonometry which more completely reflects properties of the Euclidean plane.
5.1 Introduction

In this chapter a trigonometry is developed that relates to all angles defined in the twenty-five point geometry. Although this trigonometry is a result of an attempt to generalize the trigonometry that has been described in the two previous chapters, the two trigonometries should be recognized as independent of one another. This is true since in this chapter part of the basic structure of the earlier trigonometry has been redefined. This change is most evident in the extended numeration system adopted and in a refinement in the definition of angle.

Much of the discussion in this chapter centers on the creation of a finite trigonometry with properties analogous to plane trigonometry, rather than describing the mechanics of a completed system. As such it in part presupposes a knowledge of Euclidean trigonometry. This portion of the chapter would be of interest to students who are concerned with developing a similar mathematical system of their own.
Since many of the definitions and decisions made are of an arbitrary nature, an alternate choice could lead to yet another finite trigonometry. For example, it may well be possible to define sides of triangles as directed segments, rather than as lengths, and retain angular measurement as defined by Yu-Mei Yu Lu. Furthermore, in a finite system such as this, angles might well be measured in some manner other than degrees of rotation.

Other portions of this chapter may well be of interest to those not concerned with the development of the trigonometry. The mechanics of the system could be taught easily to persons unfamiliar with Euclidean trigonometry; this could include students at the upper junior high level. Thus the finite system could be used as a motivational technique for plane trigonometry. This writer would be hesitant, however, to recommend that both the trigonometry developed in this chapter and that of the previous two chapters be presented to students of the junior high age.

The completed system should also interest students who have already completed a course in plane trigonometry. The analogies which appear between the finite and infinite trigonometries are of an interest in themselves and also serve as a means of reviewing the properties of plane trigonometry.

5.2 Directed Distances

Extension of finite trigonometry to all angles in the
finite system in a manner analogous to plane trigonometry suggests first that negative numbers must be introduced into the numeration system. The four distances in twenty-five point geometry were previously designated by 1, 1', 2, and 2'. With reference to geometric shapes these distances were also used to determine the lengths of sides of triangles, quadrilaterals, etc. As direction is often implied with reference to distance, it is often desirable to distinguish between distance and length.

Following the convention that a distance to the left or down from the reference point is usually designated negative, while a distance to the right or up is designated positive, the row distances in twenty-five point geometry are then represented by -2, -1, 0, 1, and 2, and column distances by -2', -1', 0, 1', or 2'.

Length is then defined to be the absolute value of distance. For example, where distance AE = -1, length AE = | -1 | = 1. Lengths are represented by only the numbers 0, 1, 1', 2, or 2'.

Addition and Multiplication are defined as shown in Tables 7 and 8.

Since a distance of +4 and a distance of -1 lead one to the same point, as do distances +3 and -2, etc., modular five numeration can be used to express distances.
For column distances a like situation exists. This leads to the definition:

\[ a' \equiv b', \text{ if and only if, } a \equiv b \pmod{5}. \]

Column distances may then be expressed by the numbers 0, 1', 2', 3', or 4', or their equivalents, 0, 1', 2', -2', -1'. The numbers -2, -1, -2', and -1' which appear in the multi-
TABLE 8
MULTIPLICATION TABLE FOR DIRECTED DISTANCES

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1'</th>
<th>2'</th>
<th>-2'</th>
<th>-1'</th>
</tr>
</thead>
<tbody>
<tr>
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<td>2'</td>
<td>-2'</td>
<td>-1'</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>-1</td>
<td>1</td>
<td>-2</td>
<td>0</td>
<td>2'</td>
<td>-1'</td>
<td>1'</td>
<td>-2'</td>
</tr>
<tr>
<td>-2</td>
<td>-2</td>
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<td>-1</td>
<td>2</td>
<td>0</td>
<td>-2'</td>
<td>1'</td>
<td>-1'</td>
<td>2'</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>-2</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>-1'</td>
<td>-2'</td>
<td>2'</td>
<td>1'</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1'</td>
<td>1'</td>
<td>2'</td>
<td>-2'</td>
<td>-1'</td>
<td>0</td>
<td>2</td>
<td>-1</td>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td>2'</td>
<td>2'</td>
<td>-1'</td>
<td>1'</td>
<td>-2'</td>
<td>0</td>
<td>-1</td>
<td>-2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>-2'</td>
<td>-2'</td>
<td>1'</td>
<td>-1'</td>
<td>2'</td>
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<td>1</td>
<td>2</td>
<td>-2</td>
<td>-1</td>
</tr>
<tr>
<td>-1'</td>
<td>-1'</td>
<td>-2'</td>
<td>2'</td>
<td>1'</td>
<td>0</td>
<td>-2</td>
<td>1</td>
<td>-1</td>
<td>2</td>
</tr>
</tbody>
</table>

It should be noted that although a base five modular numeration may be used with respect to twenty-five point geometry, there appears no need to go beyond the numbers
appearing in the multiplication and addition tables to describe this system. Since numerical description of this mathematical system, by definition, is confined to these closed tables, equality is used in this study rather than congruence.

Division of two numbers, \( a \div b \) or \( \frac{a}{b} \) (where numbers are limited to those used to express distance), is defined as the inverse of multiplication. Therefore \( \frac{a}{b} = c \), if and only if, \( b \cdot c = a \). This definition is again possible since the inverse of multiplication produces unique results for all divisors except zero. Division by zero is undefined. It is apparent from the table that all numbers can be written in the form \( \frac{a}{b} \) and, in this sense, all numbers in this system are rational.

Subtraction of two numbers, \( a - b \), is defined as the inverse of addition. Therefore, \( a - b = c \), if and only if, \( a = b + c \). Unlike the inverse of addition for addition defined only in terms of length, the inverse of addition of numbers assigned to directed segments gives a unique result.

The operation of square root can also be applied to this system, but as is true with the real numbers, it is apparent that more than one root exists for some numbers while the square root is undefined for others. To provide
a functional correspondence, the principal roots are defined as follows:

\[
\sqrt{0} = 0 \\
\sqrt{1} = 1 \\
\sqrt{2} = 1' \\
\sqrt{-2} = 2' \\
\sqrt{-1} = 2.
\]

This definition can be used to determine the length of a side when using the Theorem of Pythagoras or the Law of Cosines.

Although it is evident that negative numeration must be introduced into the trigonometry, its introduction creates other problems which must be resolved. For example, the common ratio between sides of similar triangles cannot now be expressed directly through numeration used for distance since the sides are given in terms of length. In particular, the ratios of the corresponding sides of similar triangles \(2, 2', 1', \) and \(1, 1', 2'\), become \(2, 2,\) and \(-2\) respectively when one computes the ratios using the distance tables. It is apparent, however, that the absolute values of these ratios are equal. Therefore, similar triangles are defined as triangles in which ratios of corresponding sides have the same absolute value.

Related to this is a problem of multiple values obtained from the definition of the trigonometric functions. If all previous concepts of triangles are retained, the
definition for the sine of an angle (see Section 3.5) results in more than one value for the sine of 30° when applied to the four sizes of right triangles. The right triangle 1, 1', 2' (see Figure 2) yields the value, \( \sin 30° = \frac{1'}{2} = -2 \), while the triangle 2, 2', 1' (see Figure 3) yields the value, \( \sin 30° = \frac{2'}{1} = 2 \). A refinement in the definition of angle measurement, discussed in the following section, is used to resolve this problem.

5.3 Measurement of Angles Redefined

Under the definition of angle measurement which has been proposed by Yu-Mei Yu Lu, when two lines intersect two supplementary angles are formed. Only when the angle is a part of some closed geometric figure can it be determined which of the two measures is to be discussed. It is thus desirable to reconsider the definition of angle measurement in order to extend the trigonometric functions. This section is one attempt to extend the concept of angle measurement in such a way that trigonometric values can also be applied outside the context of a triangle.

In plane Euclidean geometry there is some ambiguity in determining the rotation required to transform one line onto another. As an example, consider lines \( l_1 \) and \( l_2 \), as shown in the following illustration with angle AOB as 30°.
The rotation of line $l_1$ onto line $l_2$ about point 0 involves the transformation of point A onto point B or of point A onto point C. Counterclockwise rotations of either $30^\circ$ or $210^\circ$ will execute this transformation. Since both rotations transform line $l_1$ onto line $l_2$, these angles can be said to be congruent, modulo $180^\circ$.

The rotation could also have been performed in a counterclockwise direction which shall be denoted as negative. Again a rotation of $+30^\circ$ is congruent to a rotation of $-150^\circ$. 
-150°, modulo 180°. All rotations about point 0 which transform line \( l_1 \) onto line \( l_2 \) are congruent, modulo 180°.

Suppose, however, in a rotation about 0, lines are replaced by line segments or rays with a common endpoint, such as shown below.

It is desired to rotate line segment 0A onto line segment 0B such that points A and B lie the same direction from point 0. Again assigning positive values to counterclockwise rotation and negative to clockwise rotation, all rotations which transform the line segment 0A onto line segment 0B are congruent, modulo 360°.

In twenty-five point geometry, triangles are defined in terms of the points which form the vertices or the lengths between vertices. These lengths may also be interpreted as line segments. The preceding interpretation of angles formed by line segments could be applied to any angle in
twenty-five point geometry if the definition of angle were to draw upon some ordering of the points about the point used as the vertex. The method of construction of the original three blocks does contain such an ordering process.

The twenty-five points of block I may be arranged on a lattice in which the ratios of the horizontal units to the vertical units are $\sqrt{3}: 1$ (see Figure 7). When the points are rotated $60^\circ$ about point A, the points of block I are transformed onto the corresponding points of block II. A second rotation of $60^\circ$ produces block III. After six successive rotations, block I is again obtained.

To observe this transformation, it is only necessary to examine blocks I and II. In particular, the point B in block I is replaced by the point I in block II. The point I in block I is replaced by H in block II; the point H in block I is replaced by E, etc. In shortened notation: B → I → H → E → W → X → B, and the cycle then repeats itself.

Since distance is preserved in this transformation, these points all lie on a circle whose center is the point A and whose radius is one. (This can easily be verified by the previous definition of a circle.) Not only are the points on this circle obtained, but more important, a method is obtained for ordering these points.

This same transformation about point A also contains the following cycles: C → L → O → D → T → Q → C;
Fig. 6.—Angular Measurement Template
Fig. 7.—Repeated Array of Block I
$V \rightarrow P \rightarrow Y \rightarrow J \rightarrow K \rightarrow G \rightarrow V$; and $N \rightarrow U \rightarrow M \rightarrow R \rightarrow F \rightarrow S \rightarrow N$. These form the points on circles of center A and radii of 2, 1, and 2', respectively. An illustration of the rotation and ordering of the points about point A is shown in Figure 7.

Likewise, a 60° rotation of block I about point B provides an ordering of all points on the circles whose center is B; a rotation about point C provides an ordering of all points on the circles whose center is C, etc. Illustrations of the rotations and ordering of the points about points B and C are shown in Figure 8. Note that it is apparent from Figure 8 that all of these rotations are related by translations of the points in block I. Therefore, in order to determine easily this ordering of points about any one of the twenty-five points, a template based upon the rotation about point A can be constructed (see Figure 6). By shifting this template on the lattice, the ordering of all points about any desired point can quickly be obtained.

Note also that the first rotation of 60° turns any row in block I into a row in block II. Since all rows in any one block are defined as parallel, it appears that any row in block I would intersect any row in block II to form 60° and 120° angles. This same concept can be extended to include block III. Thus, an agreement can be established
between the preceding discussion and the definition of angular measurement proposed by Yu-Mei Yu Lu.

Since a column of block I is transformed into a column in block II by this 60° rotation, intersecting columns of blocks I and II form angles of 60° and 120° while a column from block I would intersect a row in block II to form angles of 30° and 150°. Again this rotation
could be extended to block III. Continuing in this manner, it is determined that all angles are multiples of 30°.

In Figure 7, point A is shown as the center of rotation. The points B and C were chosen as initial points of rotation since B and C are one and two units, respectively, to the right of point A in block I. The points U and P, which are distances of 1' and 2', respectively, above point A in block I, were chosen as initial points of the other two cycles, and were positioned with respect to B and C as shown since line UP is known to be perpendicular to line BC. All other points are positioned relative to the initial points by the successive rotations of 60°. Also shown in this figure are all six lines which go through the point A. The point A divides each of these lines such that on any line, there are only two points on either side of A.

The measure of angles in twenty-five point geometry between line segments with a common endpoint can thus be defined as the rotation about the common point required to transform the one segment onto the other, such that their endpoints lie on the same side of the vertex. All rotations which execute this transformation are equivalent, when considered modulo 360°. (As before, counter-clockwise rotations are denoted as positive.)

The angle BAH (where line segment BA is to be rotated onto line segment AH) requires a 120° rotation. The angle
HAB is measured by $240^\circ$. (This is also congruent to $-120^\circ$, modulo $360^\circ$.)

There are, however, two rotations which will transform any line through point A onto another line through point A. If one of these is a $30^\circ$ rotation, the other is $-150^\circ$ ($210^\circ$); if one is a $60^\circ$ rotation, the other is $-120^\circ$ ($240^\circ$), etc.

As an example, the line through the points A and B can be rotated onto the line through points A and H by either a $120^\circ$ rotation or a $300^\circ$ rotation.

It appears an arbitrary decision as to which of these interpretations shall be defined as the angle of rotation. Mrs. Lu chose to define angles in agreement with the preceding rotation of lines. In her geometrical system this appears completely adequate. However, an extension of finite trigonometry can be more readily accomplished from an approach to a definition of angle measurement through rotation of line segments. The definition of angle measurement based on rotation of line segments is therefore used throughout the remainder of this study.

5.4 The Measurement of Angles in Triangles

The triangle AUB is a right triangle with sides $AU = 1'$, $UB = 2'$ (UB is the hypotenuse.), and $AB = 1$.

When one measures the angles of rotation in this triangle the angles AUB, UBA, and BAU are found to be $60^\circ$, $30^\circ$, and $90^\circ$, respectively. However, when the angles of triangle
ABU are measured by the same process, one finds that the angles ABU, BUA, and UAB are 300°, 330°, and 270°, and since angles are congruent, modulo 360°, the angles may also be denoted as -60°, -30°, and -90°, respectively. Thus the angles of the one triangle are additive inverses of the corresponding angles of the second, and triangle ABU may then be denoted as the inverse of triangle AUB. Similarly, triangle ANB which contains angles of 120°, 30°, 30°, has as its inverse triangle ABN whose angles measure -120°, -30°, -30°.

It is desirable, however, to define measurement of angles within a triangle so that angles of congruent triangles be uniquely determined. Therefore, the following rule is established:

Within a given triangle a positive direction of rotation is chosen and the numerical values of each angle are assigned, modulo 360°, such that

1. The sum of the angles totals 180°,
2. At least two angles are positive,
3. No angle is greater than 180°.

A representation of the twelve sizes of triangles in twenty-five point geometry with their angular measurements is shown in Plates II, III, and IV.
PLATE II

A REPRESENTATION OF THE FOUR SIZES
OF SCALENE TRIANGLES

Figure 9

Figure 10

Figure 11

Figure 12
PLATE III

A REPRESENTATION OF THE FOUR SIZES OF EQUILATERAL TRIANGLES

Figure 13

Figure 14

Figure 15

Figure 16
PLATE IV

A REPRESENTATION OF THE FOUR SIZES
OF ISOSCELES TRIANGLES

Figure 17

Figure 18

Figure 19

Figure 20
5.5 The Trigonometric Functions of Angles Contained in Right Triangles.

As in Chapter III the trigonometric functions of an angle are again defined as functions of the sides of a right triangle. It is emphasized that the term right triangle is being applied only to a triangle which contains a $+90^\circ$ angle and that the hypotenuse is the side opposite the $+90^\circ$ angle. The trigonometric functions of an angle $\theta$, contained as an angle of a right triangle, are defined by:

$$\sin \theta = \frac{\text{side opp. } \theta}{\text{hypotenuse}}$$

$$\cos \theta = \frac{\text{side adj. } \theta}{\text{hypotenuse}}$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\csc \theta = \frac{1}{\sin \theta}$$

$$\sec \theta = \frac{1}{\cos \theta}$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta}.$$  

The triangles shown in Figures 9 and 12 yield the values:

$$\sin 30^\circ = -2 \quad \cos 30^\circ = -1'$$

$$\sin 60^\circ = -1' \quad \cos 60^\circ = -2$$

$$\sin -60^\circ = 1' \quad \cos -60^\circ = -2$$

$$\sin 150^\circ = -2 \quad \cos 150^\circ = 1'.$$

From the discussion on measurement of angles in a
triangle it is apparent that angles $-90^\circ$, $120^\circ$, and $150^\circ$ of the triangles with sides of 2, 2', 1', and 1', 2, 1, are in one sense equivalent to the angles $90^\circ$, $-120^\circ$, and $-150^\circ$, respectively. Had these angles been assigned to the two preceding sizes of triangles, the definition of the trigonometric functions would have also yielded the values:

$$\sin -120^\circ (240^\circ) = 1' \quad \cos -120^\circ = 2$$
$$\sin -150^\circ (210^\circ) = 2 \quad \cos -150^\circ = 1'.$$

These values are of interest since they are consistent with the values which are assigned to these angles later in this study.

5.6 Trigonometric Values of All Angles

The trigonometric functions in Euclidean trigonometry as presented in Chapters III and IV are first defined for angles contained as elements of right triangles. The derivation of the Law of Sines then suggests that for angles greater than $90^\circ$ the sine of the angle and its supplement should be equal, while the derivation of the Law of Cosines suggests that the cosine of an obtuse angle should be equal to the negative of the cosine of its supplement. Extending this method of assigning trigonometric values to the angles of the twenty-five point geometry, one first notes that the trigonometric functions have been defined for the angles of $30^\circ$, $60^\circ$, $150^\circ$, and $-60^\circ$. If the Law of Sines and the Law of Cosines are to be valid, then application of these laws to all triangles in the finite system will yield the trig-
onomatric values for any angle which appears as an element of one of the triangles. In addition to the angles of 30°, 60°, 150°, and -60° (300°), the angles contained as elements of at least one triangle are 90°, 120°, 240°, and 270°. The only angles of a circle which do not appear as part of some triangle are angles of 0°, 180°, 210°, and 330°. A method for obtaining the values of the sine for all angles which appear in at least one of the triangles is shown below.

From Figure 9: \[ \frac{\sin 90°}{2} = \frac{\sin 30°}{1}; \sin 90° = 1. \]

From Figure 17: \[ \frac{\sin 120°}{1} = \frac{\sin 30°}{1}; \sin 120° = -1'. \]

From Figure 20: \[ \frac{\sin 240°}{2} = \frac{\sin 150°}{1}; \sin 240° = -1'. \]

From Figure 10: \[ \frac{\sin 270°}{1} = \frac{\sin 120°}{2}; \sin 270° = -1'. \]

The Law of Cosines in the form, \[ \cos A = \frac{b^2 + c^2 - a^2}{2bc}, \]

is used to determine the cosines of the angles appearing as an element of at least one of the triangles.

From Figure 9: \[ \cos 90° = \frac{1'1^2 + 1'1^2 - 2'1^2}{2.1'.1} = 0. \]

From Figure 17: \[ \cos 120° = \frac{1'1^2 + 1'1^2 - 1^2}{2.1'.1} = 2. \]

From Figure 20: \[ \cos 240° = \frac{1^2 + 1^2 - 2'1^2}{2.1.1} = 2. \]

From Figure 10: \[ \cos 270° = \frac{2'1^2 + 2^2 - 1^2}{2.2'.2} = 0. \]

The trigonometric values thus far assigned are shown in Table 9.
### TABLE 9
TRIGONOMETRIC VALUES OF CERTAIN ANGLES

<table>
<thead>
<tr>
<th>Angle</th>
<th>Sine</th>
<th>Cosine</th>
</tr>
</thead>
<tbody>
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<td>0°</td>
<td>x</td>
<td>x</td>
</tr>
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<td>30°</td>
<td>-2</td>
<td>-1'</td>
</tr>
<tr>
<td>60°</td>
<td>-1'</td>
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<tr>
<td>90°</td>
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<tr>
<td>150°</td>
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<td>1'</td>
</tr>
<tr>
<td>180°</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>210°</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>240°</td>
<td>1'</td>
<td>2</td>
</tr>
<tr>
<td>270°</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>300°</td>
<td>1'</td>
<td>-2</td>
</tr>
<tr>
<td>330°</td>
<td>x</td>
<td>x</td>
</tr>
</tbody>
</table>

The values for the sine and cosine to be assigned to the angles 210° and 330° can be determined most easily by noting the symmetry that appears in Table 9 with respect to the sines and cosines of 90° and 270°. Thus the cosine of 210° is assigned the value 1'; cosine 330° the value of -1'; sine of 210° the value of 2; and the sine of 330° also the value of 2. A more difficult pattern to discern from the table is that the values for the sine and the cosine appear
in the same order. Beginning with the sine of $120^\circ$, the sine values are $-1'$, $-2$, $-1$' $-1$, $1'$, $2$, $2'$, $-2$, $-1'$, $1$. Beginning with the cosine of $30^\circ$ the cosine values are $-1'$, $-2$, $0$, $2$, $1'$, $-1'$, $2$, $0$, $-2$, $-1'$, $-2$. The sine and cosine values for the angles of $0^\circ$ and $180^\circ$ are assigned so that this pattern continues. Thus $\sin 0^\circ$ is assigned the value 0; $\cos 0^\circ$ is assigned the value 1; $\sin 180^\circ$ is assigned the value 0; and $\cos 180^\circ$ is assigned the value $-1$. The trigonometric values for all angles less than $360^\circ$ in the twenty-five point geometry are shown in Table 10.

TABLE 10
TRIGONOMETRIC VALUES OF ANGLES LESS THAN $360^\circ$

<table>
<thead>
<tr>
<th></th>
<th>sin</th>
<th>cos</th>
<th>tan</th>
<th>csc</th>
<th>sec</th>
<th>cot</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0^\circ$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>undef.</td>
<td>1</td>
<td>undef.</td>
</tr>
<tr>
<td>$30^\circ$</td>
<td>-2</td>
<td>-1'</td>
<td>1'</td>
<td>2</td>
<td>2'</td>
<td>-2'</td>
</tr>
<tr>
<td>$60^\circ$</td>
<td>-1'</td>
<td>-2</td>
<td>-2'</td>
<td>2'</td>
<td>2</td>
<td>1'</td>
</tr>
<tr>
<td>$90^\circ$</td>
<td>1</td>
<td>0</td>
<td>undef.</td>
<td>1</td>
<td>undef.</td>
<td>0</td>
</tr>
<tr>
<td>$120^\circ$</td>
<td>-1'</td>
<td>2</td>
<td>2'</td>
<td>2'</td>
<td>-2</td>
<td>-1'</td>
</tr>
<tr>
<td>$150^\circ$</td>
<td>-2</td>
<td>1'</td>
<td>-1'</td>
<td>2</td>
<td>-2'</td>
<td>2'</td>
</tr>
<tr>
<td>$180^\circ$</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>undef.</td>
<td>-1</td>
<td>undef.</td>
</tr>
<tr>
<td>$210^\circ$</td>
<td>2</td>
<td>1'</td>
<td>1'</td>
<td>-2</td>
<td>-2'</td>
<td>-2'</td>
</tr>
<tr>
<td>$240^\circ$</td>
<td>1'</td>
<td>2</td>
<td>-2'</td>
<td>-2'</td>
<td>-2</td>
<td>1'</td>
</tr>
<tr>
<td>$270^\circ$</td>
<td>-1</td>
<td>0</td>
<td>undef.</td>
<td>-1</td>
<td>undef.</td>
<td>0</td>
</tr>
<tr>
<td>$300^\circ$</td>
<td>1'</td>
<td>-2</td>
<td>2'</td>
<td>-2'</td>
<td>2</td>
<td>-1'</td>
</tr>
<tr>
<td>$330^\circ$</td>
<td>2</td>
<td>-1'</td>
<td>-1'</td>
<td>-2</td>
<td>2'</td>
<td>2'</td>
</tr>
</tbody>
</table>
5.7 Solutions for Elements of Triangles

The four categories of problems in which solutions of unknown elements of a triangle are classified are discussed in Chapter IV. Examples showing a trigonometric solution for a problem in each of these categories are given below.

Example 5-1. Two sides of a triangle are 1 and 1'. The angle opposite side 1' is given as 30°. Determine the length of the third side.

The Law of Sines is used to determine first the angle opposite side 1.

\[
\frac{\sin 30^\circ}{1'} = \frac{\sin x}{1};
\]

\[
\sin x = \frac{-2}{1'} = -1';
\]

angle \( x = 60^\circ \) or \( 120^\circ \).

If angle \( x = 60^\circ \), the remaining angle must be 90°; the remaining side is the hypotenuse. The hypotenuse equals \( \sqrt{1^2 + 1'^2} = 2' \).

If angle \( x = 120^\circ \), the remaining angle must be 30°; the triangle must be isosceles and the third side 1'.

Example 5-2. Two angles of a triangle are 30° and 120° with an included side of length 2. Determine the two remaining sides.

The third angle must equal 30°; therefore, the triangle is isosceles and the sides opposite the 30° angles are
both of length 2. The remaining side can be determined by application of the Law of Sines.

\[
\frac{\sin 120°}{x} = \frac{\sin 30°}{2}; \quad x = 1'.
\]

Example 5-3. The three sides of a triangle are 1, 1, and 2'. Determine the measure of the angles.

The Law of Cosines is used to determine the angle opposite side 2':

\[
\cos x = \frac{1^2 + 1^2 - 2^2}{2 \cdot 1 \cdot 1} = 2.
\]

Since some triangles contain negative angles, \( x \) might equal 120° or -120°. A check is needed to determine which solution is correct. If \( x = 120° \), then the two remaining angles must both be 30°. Placing these values in the Law of Sines one observes

\[
\frac{\sin 30°}{1} \neq \frac{\sin 120°}{2'}; \text{ therefore, } x \neq 120°.
\]

If \( x = -120° \) then the two remaining angles must both be 150° and,

\[
\frac{\sin 150°}{1} = \frac{\sin -120°}{2'},
\]

thereby establishing that the three angles are 150°, 150°, and -120°.

Example 5-4. Two sides of a triangle are 1 and 2 and the included angle is 120°. Determine the remaining side.

Application of the Law of Cosines yields,

\[
a^2 = 1^2 + 2^2 - 2 \cdot 1 \cdot 2 \cdot \cos 120°;
\]

\[
a^2 = 2; \text{ therefore, } a = 1'.
\]
The finite trigonometry developed in this chapter does contain many of the same properties of trigonometry of the Euclidean plane. These include basic definitions of the functions as ratios of sides of right triangles, the extension of the functions to angles contained as elements of any triangle, and a means to solution of unknown elements of triangles through the Law of Sines and the Law of Cosines.

Although trigonometry of the Euclidean plane is often introduced through the geometry of a right triangle, as was the finite trigonometry in this chapter, analytic geometry allows for equivalent definitions of the trigonometric functions through an algebraic interpretation of assigned coordinates. Euclidean trigonometry can thus be studied from both an algebraic and geometric viewpoint. It is the purpose of Chapter VI, then, to view finite trigonometry, in a like manner, through a finite rectangular coordinate system. The study of trigonometric identities in the finite case is also a concern of Chapter VI since these identities may be developed through the finite coordinate system.
CHAPTER VI

COORDINATE TRIGONOMETRY

6.1 Introduction

The development of the Cartesian coordinate system for the Euclidean plane provides one of the most powerful methods for studying the properties of geometric figures and serves as the joining link between the previously separate areas of geometry and algebra. Although trigonometry has historically been developed through the study of geometric figures, it too can be developed through analytic methods and as such also falls under the realm of algebra.

In this chapter a rectangular coordinate system is assigned to the twenty-five point geometry, and the trigonometric functions are defined with respect to properties of the coordinate system. A special unit circle is also constructed and its coordinates are discussed in relation to the trigonometric functions. The points on this unit circle are then used to develop selected identities for the finite system.

Although coordinate systems may be introduced quite
early in the school years, the applications of the distance formula and other more complex equations generally limit this material to students in the upper levels of secondary school and beyond. Likewise, its counterpart in the finite geometry and trigonometry is more appropriate to the high school or lower college level. The modular type coordinates and the assignment of the finite coordinate system about certain designated angles are two aspects of the finite trigonometry which limit it to this age level. However, students who are familiar with the rectangular coordinates of the Euclidean plane should find little difficulty with the finite coordinates, and finite trigonometry, through coordinates, could well be used to present the concepts of Euclidean trigonometry or to serve as an introduction to the same. Once again, an advantage of the finite system, especially with regard to the trigonometric identities of this chapter, is the availability of proof through the exhaustion of all cases.

6.2 The Coordinate System

Previous studies in twenty-five point geometry show two different systems of coordinates being applied to the twenty-five points. Fletcher [6] and Heidlage [7] both use a system whereby the column and row distances appear equal, while the coordinate system proposed by Mrs. Lu differentiates between column and row distances. The latter
system holds an advantage in that the equations of lines, circles, conics, etc., appear more closely analogous to their Euclidean counterparts.

The coordinate system shown in Figure 21 is essentially that presented by Mrs. Lu in her dissertation. It represents, however, only one of the three hundred arrangements in which this coordinate system can be assigned to the twenty-five points.

\[\begin{align*}
S(-2,2') & \quad T(-1,2') & \quad P(0,2') & \quad Q(1,2') & \quad R(2,2') & \quad S(-2,2') \\
X(-2,1') & \quad Y(-1,1') & \quad U(0,1') & \quad V(1,1') & \quad W(2,1') & \quad X(-2,1') \\
D(-2,0) & \quad E(-1,0) & \quad A(0,0) & \quad B(1,0) & \quad C(2,0) & \quad D(-2,0) \\
I(-2,-1') & \quad J(-1,-1') & \quad F(0,-1') & \quad G(1,-1') & \quad H(2,-1') & \quad I(-2,-1') \\
N(-2,-2') & \quad O(-1,-2') & \quad K(0,-2') & \quad L(1,-2') & \quad M(2,-2') & \quad N(-2,-2') \\
S(-2,2') & \quad T(-1,2') & \quad P(0,2') & \quad Q(1,2') & \quad R(2,2') & \quad S(-2,2')
\end{align*}\]

Fig. 21.—The Coordinate System

In Figure 21, point A is shown as the origin while the points on line AB form the X axis and the points on line AU form the Y axis. The abscissas are row distances measured from the Y axis, the ordinates are column distances measured from the X axis, and again the numbers \(-2, -1, -2', -1', \ldots\)
are chosen in preference to the numbers 3, 4, 3', and 4', their equivalents, modulo five. In addition, if the block is considered as an endless array, once a coordinate system is assigned, the coordinates of any single point remain constant no matter how often the point is repeated within the array; thus the need of modular numbers is eliminated.

Figure 21 also shows direction to the right and upward from the origin as positive. Three other choices in which direction might be assigned are to denote upward and to the left as positive, to denote to the left and downward as positive, or to denote downward and to the right as positive. Since, in addition to the four possible ways of assigning direction, the coordinate system may be assigned to any of the three blocks and any one of the twenty-five points within the three blocks may be used as the origin, there are three hundred ways of assigning this coordinate system to the twenty-five points.

An analogy is seen in plane analytic geometry where given any triangle in the Euclidean plane there are an infinite number of ways to assign rectangular coordinates to the vertices of the triangle. In the twenty-five point system, as well as the Euclidean plane, it is often desirable to assign the coordinates so that one of the vertices of the triangle appears as the origin and one of the sides lies on one of the axes. Examples of assigning coordinates
to various angles in twenty-five point geometry are shown later in this chapter.

6.3 The Trigonometric Functions

Suppose a coordinate system is assigned to the twenty-five points as in Figure 21 and then from any one of the twenty-five points the distance from the X axis to this point, the distance from the Y axis to this point, and the length from the origin to this point are determined. The resulting three numbers are always related such that the sum of the squares of the first two is equal to the square of the third. Thus each point determines a set of three numbers which, if considered as lengths, form the sides of a right triangle if the point is not on an axis. This formation of a right triangle, whose legs are directed distances, suggests that the definition of the trigonometric functions for angles contained as elements of a right triangle might be extended to determine the trigonometric values for any angle formed by the X axis and a line segment whose endpoints are the origin and one of the other twenty-five points.

In the definition of the trigonometric functions which follow, the point A represents the origin; the point Z represents any of the other twenty-four points, and the line segment AB represents the initial side of the angle which
lies on the X axis. Under these conditions, the

\[
\text{sine of angle } \angle BAZ = \frac{\text{the ordinate of } Z}{\text{the length of } AZ}, \text{ and}
\]

\[
\text{cosine of angle } \angle BAZ = \frac{\text{the abscissa of } Z}{\text{the length of } AZ}.
\]

The remaining trigonometric functions are defined in terms of the sine and cosine as in Chapter V.

The preceding definitions are used to determine the values of the sine and cosine of selected angles as shown below.

**Example 6-1.** The angle BAV represents a 30° rotation. The coordinates of V are (1,1'); the length AV is 2'; therefore, the sine of angle BAV is \(\frac{1'}{2'} = -2\), and the cosine of angle BAV is \(\frac{1}{2'} = -1'\).

**Example 6-2.** The angle BAP represents a 90° rotation. The coordinates of P are (0,2'); the length AP is 2'; therefore, the sine of angle BAP is \(\frac{2'}{2'} = 1\), and the cosine of angle BAP is \(\frac{0}{2'} = 0\).

**Example 6-3.** The angle CAQ represents a 300° rotation. The coordinates of Q are (1,2'); the length AQ is 2; therefore, the sine of angle CAQ is \(\frac{2'}{2} = 1\), and the cosine of angle CAQ is \(\frac{1}{2} = -2\).

In all cases illustrated above and in fact for all angles formed in twenty-five point geometry, the values obtained for the sine and cosine when one employs the coordinate definition are identical with those assigned.
through application of the Law of Sines and the Law of Cosines in Chapter V. Thus, the two definitions are equivalent.

6.4 Measurement of Angles

Thus far the coordinate system has been applied only where point A is the vertex and where the initial side of the angle selected was AB or AC. If, however, line segments AB or AC do not form the initial side of the angle being measured, then a shifting or reassignment of the coordinates is required. To reassign the coordinates the line containing the initial side of the angle is first located in one of the three blocks and this line becomes the X axis. The vertex is assigned as the origin and the two points of the initial side of the angle are assigned the coordinates (1,0) and (2,0), or (1',0) and (2',0). Note that if the initial side of the angle is contained in a line that forms a column of one of the three blocks, then the abscissas are column distances while the ordinates are row distances. Positive values on the Y axis are assigned to the first two points beyond the origin in a 90° counter-clockwise rotation from the initial side.

The following examples illustrate how the coordinates are assigned to determine, with the aid of the trigonometric values (Table 10), the measurement of any angle.
Example 6-4. Determine the measurement of angle IAH.

The point A in Block II is designated as the origin with coordinates of point I as (1,0). The length of AH is 1; the coordinates of H are (-2,-1'); therefore, the sine of angle IAH = \(-\frac{1'}{1} = -1'\), and the cosine of angle IAH = \(-\frac{2}{1} = -2\).

The trigonometric table is used to determine that angle IAH is a rotation of \(60^\circ\).

Example 6-5. Determine the measurement of angle QGC.

The coordinate system is assigned to Block I as shown below where the coordinates of G are (0,0), and the coordinates of Q are (2',0). Line QG determines the X axis.

The coordinates of C are (-1',1); the length of CG is 2'; therefore, the sine of angle QGC = \(\frac{1'}{2'} = -1'\), and the cosine of angle QGC = \(-\frac{1'}{2'} = 2\). Angle QGC must be a rotation of \(120^\circ\).
Example 6-6. Determine the measurement of angle TDE.

The coordinate system is assigned to Block III as shown below.

The point D is the origin; the point T has coordinates of (2,0); and the point E has coordinates of (-2,-1'). (The directions are assigned with positive, being to the left and downward.) The length of ED is 1; the sine of angle TDE = \(-\frac{1}{1'} = -1'\); the cosine of angle TDE = \(-\frac{2}{1} = -2\); therefore, angle TDE is a rotation of 60°.

6.5 The Trigonometric Point

The unit circle, that is, a circle whose center is at the origin and whose radius is one unit, occupies a unique role in Euclidean trigonometry since the coordinates of any point on the circle equal the values of the sine and cosine
of the angle formed by the positive X axis and the line joining the point with the origin.

In twenty-five point geometry, with one point designated as the origin, the remaining twenty-four points are found on four concentric circles. The unit circle, like all other circles, contains exactly six points. However, a point transformation of the circles can transform each point on one of the concentric circles to a corresponding point on the unit circle. This transformation can be performed by dividing the coordinates of each point by the radius of the circle upon which it lies. The transformation gives twelve different sets of coordinates which are positioned on the unit circle relative to the angle the line segment from the origin to the original point formed with the X axis. For example, with point A as the origin, the point I, whose coordinates are (-2,-1'), lies on the unit circle and determines the 60° angle of BAI. The point V, whose coordinates are (1,1'), lies on a circle of radius 2' and determines the 30° angle of BAV. Division of the coordinates of V by 2' results in the coordinates (-1',-2), which are then considered as a point on the unit circle between the points (1,0) and (-2,-1'). Six of the points are the ordinary points on the unit circle; the other six may be considered as non-ordinary (imaginary) points. Note that all twelve sets of coordinates satisfy the equation \( x^2 + y^2 = 1 \).
The resulting twelve sets of coordinates are defined as trigonometric points and are ordered on the unit circle as illustrated in Figure 22. The points are numbered counter-clockwise beginning with the point P(0) at the coordinates (1,0). Since it is also possible to designate the clockwise numeration as negative, the coordinates of P(-2) and P(10), etc., are equivalent.
In any circle of finite geometry with points ordered as in Figure 7, the distance formula can be used to confirm that in each case the length between adjacent points is equal to the radius of the circle. When the distance formula is applied to compute the distance between adjacent points on the twelve point unit circle, the result is always \( \sqrt{27} \), a number which is undefined in the finite system since no number when squared equals 2'. In determining the distance between the points \( P(7) \) and \( P(8) \), for example, one applies the distance formula to obtain

\[
\begin{align*}
\overline{P(7)P(8)} &= \sqrt{(2 - 1')^2 + (1' - 2)^2}, \\
&= \sqrt{(-1 - 2 \cdot 2 \cdot 1' + 2) + (2 - 2 \cdot 1' \cdot 2 - 1)}, \\
&= \sqrt{27}.
\end{align*}
\]

Note that the trigonometric points cannot actually be plotted on the finite coordinate system since there is a mixture of primed and unprimed numbers appearing as both abscissas and ordinates.

The interest in the twelve point unit circle, however, is that, consistent with the unit circle of plane trigonometry, the abscissa of each trigonometric point equals the cosine of the angle formed by the X axis and the line from the origin through the trigonometric point, and the ordinate of the trigonometric point equals the sine of the angle. Thus the sine and cosine of finite trigonometry may also
be defined in terms of the trigonometric points by
\[ \sin t = \text{ordinate of } P(t), \]
\[ \cos t = \text{abscissa of } P(t). \]
The formula, \( A = 30 \cdot t \), may be used here to convert to angular measurement when one is given the trigonometric points.

6.6 Identities

A study of the properties of the twelve point unit circle and of patterns within the table of trigonometric values (Table 10) provide means of introducing several important identities. With only twelve entries in each column of the trigonometric table, certain patterns are easily discernible. Of particular interest is the symmetry of the sine and cosine values about 180° which reveals that for all possible values for \( t \), \( \cos(-t) = \cos t \) and \( \sin(-t) = -\sin t \). The table also shows the values of the sine appearing in the same order as those of the cosine. This relationship may be expressed with the identity, \( \sin t = \cos(t - 90^\circ) \). However, since \( \cos t = \cos(-t) \), the preceding equation may also be written as \( \sin t = \cos(90^\circ - t) \), and moreover, by replacing \( t \) with \( (90^\circ - t) \) one may obtain the equivalent identity, \( \cos t = \sin(90^\circ - t) \).

On the unit circle it is apparent that the coordinates of any point, \( P(t) = (x, y) \), are related such that \( x^2 + y^2 = 1 \). Since, for points on the unit circle, the
sine of $t$ may be defined as the $y$ coordinate of $P(t)$, and the cosine as the $x$ coordinate, the previous equation may also be written as $\sin^2 t + \cos^2 t = 1$, an identity which can be readily verified by examining all twelve cases.

Another concern is the relationship between the coordinates of points such as $P(8 - 7)$ and the coordinates of $P(8)$ and $P(7)$. In more general terms the question concerns the relationship of the coordinates of $P(u + v)$ or $P(u - v)$ with the coordinates of $P(u)$ and $P(v)$. These points are positioned on a circle and the length of the chord connecting $P(u)$ and $P(v)$ is the same length as the chord connecting $P(u - v)$ and $P(0)$. The coordinates of these points, expressed in trigonometric form, may then be placed in the distance formula to determine $P(u)P(v)$ and $P(u - v)P(0)$.

$$P(u)P(v)^2 = (\cos v - \cos u)^2 + (\sin v - \sin u)^2,$$

$$= \cos^2 v - 2\cos v \cos u + \cos^2 u$$

$$+ \sin^2 v - 2\sin v \sin u + \sin^2 u,$$

$$= 2 - 2\cos v \cos u - 2\sin v \sin u.$$

$$P(u - v)P(0)^2 = (1 - \cos(u - v))^2 + (0 - \sin(u - v))^2$$

$$= 1 - 2\cos(u - v) + \cos^2(u - v) +$$

$$\sin^2(u - v).$$

Since the distances are equal, the right side of the
formulas may be equated and the terms arranged to give,

$$\cos(u - v) = \cos u \cos v + \sin u \sin v.$$  

A similar expression for $\cos(u + v)$ may be obtained by writing $\cos(u + v)$ as $\cos(u - (-v))$.

$$\cos(u + v) = \cos(u - (-v)) = \cos u \cos (-v) + \sin u \sin (-v),$$

and since $\cos(-v) = \cos v$ and $\sin(-v) = -\sin v$,

$$\cos(u + v) = \cos u \cos v - \sin u \sin v.$$  

An expression for $\sin(u + v)$ may be obtained by first equating $\sin(u + v)$ with $\cos(90^\circ - (u + v))$. Thus,

$$\sin(u + v) = \cos(90^\circ - (u + v)) = \cos((90^\circ - u) - v) = \cos(90^\circ - u)\cos v + \sin(90^\circ - u)\sin v = \sin u \cos v + \cos u \sin v.$$  

Also,

$$\sin(u - v) = \sin(u - (-v)) = \sin u \cos v - \cos u \sin v.$$  

It is not the purpose of this study to prove identities which may be found in a standard text of plane trigonometry, but to show that they may also be derived for the finite case. Many identities which are valid for Euclidean trigonometry can also be shown valid for the finite case. Verification of these identities may be obtained in the finite trigonometry by employing all possible combinations of angles for $u$ and $v$, a method which is impossible for the identities in Euclidean trigonometry.
Two identities of Euclidean trigonometry which are not applicable for all choices of angles in finite trigonometry are the identities used to determine the functions of half angles,

\[ \cos \frac{u}{2} = \pm \sqrt{\frac{1 + \cos u}{2}} , \text{ and} \]

\[ \sin \frac{u}{2} = \pm \sqrt{\frac{1 - \cos u}{2}} . \]

A substitution of 90° in these formulas gives both the sine and cosine of 45° as 2' (The positive value was arbitrarily assigned.), while other substitutions give,

\[ \sin 135° = 2' \quad \cos 135° = -2' \]
\[ \sin 225° = -2' \quad \cos 225° = -2' \]
\[ \sin 315° = -2' \quad \cos 315° = 2' \]

A check with the trigonometric table shows that, while the preceding angles are not defined in the finite system, the values of 2' and -2' are the only numbers of the finite numeration system which do not appear as values in the sine or cosine columns.

Other angles for which there are no half-angles defined include 30°, 150°, 210°, and 330°. Half-angles of these all lead to values which are undefined in the numeration system. For example, a substitution of 30° in the half-angle sine formula shows

\[ \sin \frac{30°}{2} = \pm \sqrt{\frac{1 + (-1')}{}{2}} \]
\[ = \pm \sqrt{-2 + 2'} . \]
Though it may be possible to assign trigonometric values to heretofore undefined angles, such an extension is beyond the scope of this study.

This chapter completes the study of the twenty-five point trigonometric system which was introduced in Chapter V. Equivalent definitions of the trigonometric functions were shown to arise from right triangles, from a finite coordinate system, and from a specially constructed unit circle. The trigonometry was then used to determine the measure of angles, to find solutions to unknown elements of finite triangles, and to develop trigonometric identities.

One may recall that the trigonometric system in this chapter and in Chapter IV arose from an attempt to generalize the trigonometry developed in Chapters III and IV. Chapter VII discusses yet another type of generalization of the finite trigonometry, that of developing a finite trigonometry for a larger finite set of numbers.
7.1 Introduction

The preceding chapters have been concerned with the development of trigonometry defined on twenty-five points. This chapter considers the possibility of extending the preceding concepts of finite trigonometry to other finite geometries and in particular to a finite geometry of forty-nine points.

The reader might well consider other extensions of the finite trigonometry into still other finite systems, of which there are an unlimited number. Students who find an interest in finite systems might wish to explore forty-nine point geometry and trigonometry or could easily adapt this finite trigonometry to the nine point geometry outlined in Chapter IX.

7.2 Forty-Nine Point Geometry

The forty-nine point geometry may be represented by forty-nine upper and lower case letters of the English alphabet arranged in four rectangular blocks as shown in Figure 23.
<table>
<thead>
<tr>
<th>Block I</th>
<th>Block II</th>
</tr>
</thead>
<tbody>
<tr>
<td>A t o j W R M</td>
<td>A J S c e n w</td>
</tr>
<tr>
<td>P D F r m h c</td>
<td>U W g p r C L</td>
</tr>
<tr>
<td>f a U I D w k</td>
<td>i k t E N P Y</td>
</tr>
<tr>
<td>u p d X S N B</td>
<td>v G I R b d m</td>
</tr>
<tr>
<td>L G s n i V Q</td>
<td>K T V f o x B</td>
</tr>
<tr>
<td>b O J E X l g</td>
<td>X h q s D M O</td>
</tr>
<tr>
<td>q e Y T H C v</td>
<td>l u F H Q a j</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Block III</th>
<th>Block IV</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Fig. 23.--Forty-Nine Point Geometry</td>
</tr>
</tbody>
</table>
The dissertation of Yu-Mei Yu Lu describes the patterns of the twenty-five point geometry which were carried into the arrangement of the forty-nine points. For simplification, the points here are represented by upper and lower case letters of the alphabet rather than by a combination of numerals and letters as in her presentation. Moreover, while the points of Blocks I and II are shown in the same positions as the points of Blocks I and II of her dissertation, the points of Block III of this presentation appear as Block IV in her work and the points shown here for Block IV correspond to an interchange of rows and columns of her Block III. These minor revisions were suggested by the rotation of Block I, discussed later in this chapter, to obtain the corresponding points of the remaining three blocks.

Unlike twenty-five point geometry, row and column distances within the same block are defined as equal, while distances in Blocks I and II are not equal to distances in Blocks III and IV. Directed row and column distances in Blocks I and II are denoted by the numbers -3, -2, -1, 0, 1, 2, and 3, while directed row and column distances in Blocks III and IV are denoted by the numbers -3', -2', -1', 0, 1', 2', and 3'. Addition and multiplication of these numbers are defined in Tables 11 and 12 respectively.

All other definitions in forty-nine point geometry
TABLE 11

ADDITION TABLE FOR THE FORTY-NINE POINT SYSTEM

<table>
<thead>
<tr>
<th>+</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>1'</th>
<th>2'</th>
<th>3'</th>
<th>-3'</th>
<th>-2'</th>
<th>-1'</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>-3</td>
<td>-2</td>
<td>-1</td>
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<td>1'</td>
<td>2'</td>
<td>3'</td>
<td>-3'</td>
<td>-2'</td>
<td>-1'</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>-3</td>
<td>-2</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>1'</td>
<td>2'</td>
<td>3'</td>
<td>-3'</td>
<td>-2'</td>
<td>-1'</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>-3</td>
<td>-2</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>1'</td>
<td>2'</td>
<td>3'</td>
<td>-3'</td>
<td>-2'</td>
<td>-1'</td>
<td></td>
</tr>
<tr>
<td>-3</td>
<td>-2</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>-3</td>
<td>1'</td>
<td>2'</td>
<td>3'</td>
<td>-3'</td>
<td>-2'</td>
<td>-1'</td>
<td></td>
</tr>
<tr>
<td>-2</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>-3</td>
<td>-2</td>
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<td>2'</td>
<td>3'</td>
<td>-3'</td>
<td>-2'</td>
<td>-1'</td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>-3</td>
<td>-2</td>
<td>-1</td>
<td>1'</td>
<td>2'</td>
<td>3'</td>
<td>-3'</td>
<td>-2'</td>
<td>-1'</td>
<td></td>
</tr>
</tbody>
</table>

remain equivalent to those of the twenty-five point system.

As consequences of the definitions for addition and multiplication, the Theorem of Pythagoras is valid in the forty-nine point system and all circles contain eight
points rather than six as was the case in twenty-five point geometry. Again, for a more complete discussion of the properties of this system the reader is referred to Mrs. Lu's dissertation.
7.3 Triangles in Forty-Nine Point Geometry

The number of possible ways of choosing three points, not all three of which are contained in a single row or column of any of the four blocks of Figure 23 is 16,464, which is to say that there exist 16,464 distinct triangles in forty-nine point geometry. These triangles may be classified according to the lengths of their sides by a process similar to that shown in Chapter II. Under this process one observes 392 triangles for each size of the similar triangles with sides 1, 1, 1'; 2, 2, 2'; 3, 3, 3'; 1', 1', 2; 2', 2', 3' and 3', 3', 1; 392 triangles for each size of similar triangles with sides 1, 1, 3; 2, 2, 1; 3, 3, 2; 1', 1', 3'; 2', 2', 1'; and 3', 3', 2'; 392 triangles for each size of similar triangles with sides 1, 1, 2'; 2, 2, 3'; 3, 3, 1'; 1', 1', 3; 2', 2', 1; and 3', 3', 2; 784 triangles for each size of similar triangles with sides 1, 2, 1'; 2, 3, 2'; 3, 1, 3'; 1', 2', 2; 2', 3', 3; and 3', 1', 1; and 784 triangles for each size of similar triangles with sides 1, 2, 2'; 2, 3, 3'; 3, 1, 1'; 1', 2', 3; 2', 3', 1; and 3', 1', 2.

7.4 Measurement of Angles

A definition of angle measurement in forty-nine point geometry, equivalent to the definition of angle measurement for twenty-five point geometry as developed in Chapter V, requires that the points appearing in Blocks II, III, and IV
must result from a rotation of Block I. To observe that such a rotation does exist, and to determine its nature, first observe the six cycles which occur in the transformation of points in Block I to points of Block II. The point B of Block I becomes point m of Block II; the point m of Block I becomes point r of Block II; that is,

(1) \[ B \rightarrow m \rightarrow r \rightarrow p \rightarrow G \rightarrow T \rightarrow H \rightarrow Q \rightarrow B \]
(2) \[ C \rightarrow a \rightarrow k \rightarrow Y \rightarrow F \rightarrow g \rightarrow O \rightarrow h \rightarrow C \]
(3) \[ D \rightarrow N \rightarrow d \rightarrow I \rightarrow E \rightarrow s \rightarrow V \rightarrow x \rightarrow D \]
(4) \[ t \rightarrow J \rightarrow q \rightarrow l \rightarrow M \rightarrow w \rightarrow P \rightarrow U \rightarrow t \]
(5) \[ o \rightarrow S \rightarrow b \rightarrow X \rightarrow R \rightarrow n \rightarrow f \rightarrow i \rightarrow o \]
(6) \[ J \rightarrow c \rightarrow L \rightarrow K \rightarrow W \rightarrow e \rightarrow u \rightarrow v \rightarrow J \]

As in twenty-five point geometry, these cycles form the six circles about point A whose radii are 1, 2, 3, 1', 2', and 3', respectively. If these points are equally spaced on the circles, then the arc between adjacent points must be 45°. Also observe that the cycles formed by the transformation of points in Block III to points in Block IV are identical to the preceding six cycles.

Figure 24 shows the points of Block I repeated as part of an endless array. The point A is shown as the center of a circle with a radius of thirteen units. The points which appear on this circle are those points which, in forty-nine point geometry, appear on the circle of radius 1, and in addition, except for repetition of every other letter, they
Fig. 24.—A Circle of Radius 1 in Forty-Nine Point Geometry

also appear in the same order as the points listed in the first cycle.

In particular, in Euclidean geometry a rotation of $\arctan \frac{5}{13}$ transforms point $B$ onto point $m$, and since in finite
geometry point m is considered as the same point no matter how often it occurs; a second rotation transforms point m onto point r, etc. In forty-nine point geometry this transformation is defined as a $45^\circ$ rotation since eight such transformations always yield the original point.

Limitation of space prevents showing that circles of radii 26 and 39 units about point A contain the same points and ordering as do the second and third cycles.

To obtain circles containing the ordering of cycles four, five, and six, first observe a secondary rectangular array of Block I as illustrated in Figure 26. This array, from which is obtained Block III of Figure 23, also forms an endless pattern and the distances between adjacent points of this array are designated as $l'$. With point A of this array designated as the center, a circle with a radius of 13' units constructed on this array will pass through, in order, all points listed in cycle four; a circle with a radius of 26' units will pass through, in order, all points listed in cycle five, and a circle with a radius of 39' units will pass through, in order, all points listed in cycle six. Again, in Euclidean geometry, a rotation of $\arctan \frac{5}{13}$ transforms one point onto the next point of the cycle.
Fig. 25.—Angular Measurement Template for Forty-nine Point Geometry
Fig. 26.—Secondary Array of Block I in Forty-nine Point Geometry
The positioning of points on concentric circles about point A is shown in Figure 27. Points B, C, and D, were chosen as initial points on the circles of unprimed radii since these points are contained in the same line, and points t, o, and j, were chosen as initial points on the circles of primed radii. The final positioning of points on circles of primed radii in relation to points on circles of unprimed radii was determined after considering the results of the identities for trigonometric functions of twice an angle. All angles in the illustration are multiples of $22\frac{1}{2}^\circ$.

As in twenty-five point geometry, a translation of these points will yield the respective positions of these points when rotated about any given point. Thus a template (Figure 25) can also be constructed for the points of forty-nine point geometry to facilitate angular measurement. (A distortion appears on the template shown in order to reduce it to a more compact size.)

7.5 Measurement of Angles in Triangles

The measurement of angles contained as elements of triangles of forty-nine point geometry is performed by the process outlined in Chapter V. Thus, the angles of a triangle are assigned, modulo $360^\circ$, such that

1. The sum of the angles totals $180^\circ$
2. At least two angles are positive
3. No angle is greater than $180^\circ$. 
Fig. 27.—Relative Positions of Points on Concentric Circles about Point A

The results are shown in Plates V, VI, VII, VIII, and IX. One should note that there are four different sizes of triangles which contain 90° angles—those of Figures 35, 38, 46, and 49.
PLATE V

TRIANGLES SIMILAR TO A TRIANGLE WITH SIDES 1, 1, 1'

Figure 28

Figure 29

Figure 30

Figure 31

Figure 32

Figure 33
PLATE VI

TRIANGLES SIMILAR TO A TRIANGLE WITH SIDES 1, 1, 3

Figure 34

Figure 35

Figure 36

Figure 37

Figure 38

Figure 39
PLATE VII
TRIANGLES SIMILAR TO A TRIANGLE
WITH SIDES 1, 1, 2

Figure 40

Figure 41

Figure 42

Figure 43

Figure 44

Figure 45
PLATE VIII

TRIANGLES SIMILAR TO A TRIANGLE
WITH SIDES 1, 2, 1'

Figure 46

Figure 47

Figure 48

Figure 49

Figure 50

Figure 51
PLATE IX

TRIANGLES SIMILAR TO A TRIANGLE
WITH SIDES 1, 2, 2'

Figure 52

Figure 53

Figure 54

Figure 55

Figure 56

Figure 57
7.6 The Trigonometric Functions

Of the methods discussed to obtain the trigonometric functions for twenty-five point trigonometry, the most direct appears to be the construction of the special unit circle. In forty-nine point trigonometry there are sixteen trigonometric points on this circle with coordinates as shown in Figure 58. Again the sine of \( t \) is defined as the \( Y \) coordinate of the trigonometric point \( t \) and the cosine of

Fig. 58.—The Trigonometric Points of the Forty-Nine Point System
t as the X coordinate of this point. Table 13 shows the trigonometric values which are obtained from the unit circle. Here the point t has been replaced by the angle formed with the X axis and the line from t to the origin.

**Table 13**

**Trigonometric Values for the Forty-Nine Point System**

<table>
<thead>
<tr>
<th>Angle</th>
<th>sin</th>
<th>cos</th>
<th>tan</th>
<th>csc</th>
<th>sec</th>
<th>cot</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>undef.</td>
<td>1</td>
<td>undef.</td>
</tr>
<tr>
<td>22°1/2</td>
<td>-3'</td>
<td>-1'</td>
<td>3</td>
<td>-1'</td>
<td>-3'</td>
<td>-2</td>
</tr>
<tr>
<td>45°</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>-3'</td>
<td>-3'</td>
<td>1</td>
</tr>
<tr>
<td>67°1/2</td>
<td>-1'</td>
<td>-3'</td>
<td>-2</td>
<td>-3'</td>
<td>-1'</td>
<td>3</td>
</tr>
<tr>
<td>90°</td>
<td>1</td>
<td>0</td>
<td>undef.</td>
<td>1</td>
<td>undef.</td>
<td>0</td>
</tr>
<tr>
<td>112°1/2</td>
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<td>3'</td>
<td>2</td>
<td>-3'</td>
<td>1'</td>
<td>-3</td>
</tr>
<tr>
<td>135°</td>
<td>2</td>
<td>-2</td>
<td>-1</td>
<td>-3</td>
<td>3</td>
<td>-1</td>
</tr>
<tr>
<td>157°1/2</td>
<td>-3'</td>
<td>1'</td>
<td>-3</td>
<td>-1'</td>
<td>3'</td>
<td>2</td>
</tr>
<tr>
<td>180°</td>
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<td>-1</td>
<td>0</td>
<td>undef.</td>
<td>-1</td>
<td>undef.</td>
</tr>
<tr>
<td>202°1/2</td>
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<td>1'</td>
<td>3</td>
<td>1'</td>
<td>3'</td>
<td>-2</td>
</tr>
<tr>
<td>225°</td>
<td>-2</td>
<td>-2</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>247°1/2</td>
<td>1'</td>
<td>3'</td>
<td>-2</td>
<td>3'</td>
<td>1'</td>
<td>3</td>
</tr>
<tr>
<td>270°</td>
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<td>0</td>
<td>undef.</td>
<td>-1</td>
<td>undef.</td>
<td>0</td>
</tr>
<tr>
<td>292°1/2</td>
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<td>3'</td>
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<td>-3</td>
</tr>
<tr>
<td>315°</td>
<td>-2</td>
<td>2</td>
<td>-1</td>
<td>3</td>
<td>-3</td>
<td>-1</td>
</tr>
<tr>
<td>337°1/2</td>
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<td>-1'</td>
<td>-3</td>
<td>1'</td>
<td>-3'</td>
<td>2</td>
</tr>
</tbody>
</table>
The triangles which contain $+90^\circ$ angles yield trigonometric values which are in agreement with those of Table 13. For example, one obtains from the properties of the triangle shown in Figure 38,

$$\sin 45^\circ = \frac{2}{1} = 2,$$
$$\cos 45^\circ = \frac{2}{1} = 2,$$

and from the properties of the triangle illustrated in Figure 46,

$$\sin 112\frac{1}{2}^\circ = \frac{2}{1} = -1,,$$
$$\cos 112\frac{1}{2}^\circ = \frac{1}{1} = 3.$$

7.7 Properties of the Trigonometric Functions

All properties which were shown to exist for the twenty-five point trigonometry can also be exhibited in the forty-nine point trigonometry. These include the Law of Sines, the Law of Cosines, and the trigonometric identities. Trigonometric solutions for unknown elements of triangles in forty-nine point trigonometry through the Law of Sines and the Law of Cosines are shown in the following examples.

Example 7-1. Two sides of a triangle are of lengths 1 and 2 with an included angle of $22\frac{1}{2}^\circ$. Determine the length of the third side.
Substituting the known values in the Law of Cosines, one obtains,

\[ a^2 = l^2 + 2\cdot 2' - 2\cdot 1\cdot 2'\cdot \cos 22\frac{1}{2}^0 \]

\[ = 1 + (-1) - (-3')\cdot (-1') = -1. \]

Therefore, side a must be of length 2'.

Example 7-2. Two sides of a triangle are of lengths 2' and 3 with the angle opposite side 2' given as 67\frac{1}{2}^0.

Determine the measurement of the remaining angles.

From the Law of Sines one obtains,

\[ \frac{\sin 67\frac{1}{2}^0}{2'} = \frac{\sin x}{3}, \]

\[ \sin x = 3\cdot \frac{-1}{2'} = -1'. \]

Angle x must therefore be 67\frac{1}{2}^0 with the third angle as 45^0.

Of the many identities which are valid in the forty-nine point trigonometry, of particular interest are those which give the trigonometric values for twice an angle—\( \sin 2u = 2\sin u \cos u \) and \( \cos 2u = \sin^2 u - \cos^2 u \) — since these were used to determine the relative positions of points on circles of primed radii with respect to points on circles of unprimed radii. An application of these shows that,

\[ \sin 45^0 = 2\sin 22\frac{1}{2}^0 \cos 22\frac{1}{2}^0 \]

\[ = 2\cdot (-3')\cdot (-1') = 2, \]

and \[ \cos 45^0 = \sin^2 22\frac{1}{2}^0 - \cos^2 22\frac{1}{2}^0 \]

\[ = (-3')^2 - (-1')^2 = 2. \]
Repeated applications of these identities with respect to all possible angles verifies their validity.

In all cases, substitution in the identities which give the functions for half an angle,

\[ \sin \frac{x}{2} = \pm \sqrt{\frac{1 - \cos x}{2}} \quad \text{and} \quad \cos \frac{x}{2} = \pm \sqrt{\frac{1 + \cos x}{2}}, \]

lead to undefined numbers for angles which are also undefined. For example,

\[ \sin \frac{221^\circ}{2} = \sqrt{\frac{1 - \cos 221^\circ}{2}} \]

\[ = \sqrt{-3 - 3}. \]

This chapter has shown that the concepts of the twenty-five point trigonometry can be applied to define a trigonometry of forty-nine points which also displays many properties of the trigonometry of the Euclidean plane. Although the writer has attempted to relate the construction of these finite trigonometries to teaching situations at various levels, there is a need for actual presentation to classes of mathematics, to mathematics clubs, and to individuals in order to determine the adaptability of this material. Chapter VIII describes one such classroom presentation of finite trigonometry conducted by the writer.
CHAPTER VIII

A CLASSROOM PRESENTATION

8.1 Introduction

The Ohio State University makes available as one of its course offerings a two quarter sequence course designed to present the "nature of mathematics." In this course students are introduced to such topics as functions, an axiomatic approach to algebra, coordinate systems, and matrix theory. Such a course lends itself well to a study of finite mathematics, and thus, through the cooperation of Mr. David Mader who taught the course First Year College Mathematics, the writer spent seven class periods during the Spring quarter of 1968 discussing with the members of this class certain concepts and properties of twenty-five point trigonometry.

There were fifteen students in this class, all freshmen and sophomores enrolled in University College at the Lima Campus of The Ohio State University. Of the fifteen students eight had never taken a course in trigonometry and three had never taken a course in either trigonometry or geometry.
A diary of this presentation is included as a resource for those readers who might wish to consider a similar adaption of finite trigonometry for other classroom situations.

8.2 The Classroom Presentation

Two days prior to this writer's meeting with the class Mr. Mader discussed with them some properties of modular arithmetic. The following day he distributed a handout containing the information shown on the following page and discussed with them some basic properties of twenty-five point geometry including the number of possible lines, the definitions of distance, and the definition of a midpoint. Emphasis was placed upon the idea that twenty-five point geometry was a modular system similar in that respect to modular arithmetic.

Thursday. The writer's first day with the class was spent discussing the definition of circle and illustrating how the three theorems (1) The three altitudes of a triangle are concurrent, (2) The perpendicular bisectors of the sides of a triangle are concurrent, and (3) The three medians of a triangle are concurrent, were related to twenty-five point geometry. Next the lengths of the sides of various triangles were determined, including those used to illustrate the previous theorems. The twelve different sizes of triangles were listed on the chalkboard and a homework assign-
Line: Each row or column in any block is called a line.

Parallel lines: Two lines are parallel if they have no point in common.

Perpendicular lines: Two lines are perpendicular if one is a row and the other is a column in the same block.

Row distance: If two points are in a common row, the distance between the points is the least number of steps between the points, where, when reaching the end of a row counting is continued by jumping to the beginning of the same row.

Column distance: If two points are in a common column, the distance between the points is the least number of steps between the points.

Distance Notation: Row distances are designated by 1 and 2; column distances by 1' and 2'.

Triangle: A set of three points is called a triangle if the points are not collinear.

Circle: A circle is the set of points a given distance from a given point.

Midpoint: A point is the midpoint of a given segment if it is on the line through the segment and is equidistant from the endpoints of the segment.

Theorems:

Through a given point not on a given line there is exactly one line parallel to the given line.

Through a given point there is exactly one line perpendicular to a given line.

The three altitudes of a triangle are concurrent.

The perpendicular bisectors of the sides of a triangle are concurrent.

The three medians of a triangle are concurrent.

The perpendicular bisector of a chord of a circle passes through the center of the circle.
ment was given consisting of two parts: (1) Pick three points which determine a triangle and use this triangle to illustrate the three theorems discussed in class, (2) Find an example of each of the twelve sizes of triangles.

Friday. The following day problems concerning the homework were first discussed. A few students were still not certain how to apply the definitions of medians and perpendicular bisectors of line segments in twenty-five point geometry. Most students, however, were able to find an example of each of the twelve sizes of triangles and examples were given of triangles which some students had difficulty in finding. The twelve sizes of triangles were also viewed as three sets of similar triangles which provided some motivation for a definition of multiplication which was to be discussed later. Measurement of the angles of triangles was the main objective for the class hour. Therefore, the generation of blocks II and III from a rotation of block I was discussed as a preliminary part of the definition of angle measurement. A repeated array of block I and a template (see Figures 7 and 6) were distributed to each student to aid in this discussion. The measurement of rotations in geometry of the Euclidean plane were also discussed to point out that rotations which differed by multiples of $360^{\circ}$ appear to be equivalent. The angles of a variety of triangles were measured and rules
established for assigning direction to the angles. The assignment was to measure the angles of the twelve triangles they had found during preparation of the assignment for the previous day.

Monday. The students did surprisingly well in measuring and assigning direction to the angles of the triangles. There were only a few cases in which students did not agree on the angles assigned to a particular size of triangle, but these were discussed and the problems resolved. As the students had been using the repeated array of block I for the measurement of angles, a natural extension was to assign coordinates to these points. The development of a distance formula led to the necessity of defining addition and multiplication. It was fairly easy to motivate the definition for multiplication of an unprimed number by an unprimed number, and of a primed number by an unprimed number by referring to principles of modular arithmetic. The establishment of a rule for multiplication of a primed number by a primed number was suggested by examination of the Theorem of Pythagoras with respect to the right triangle with sides of 1', 2, 1. Here it appears that $(1')^2$ should be defined as 2. Therefore, 1' was suggested to be equivalent to $\sqrt{2}$. By using $\sqrt{2}$ as equivalent to 1' and $2\sqrt{2}$ as equivalent to 2', the instructor and class then completed the multiplication table and established the distance formula. The assignment
was to use the distance formula to calculate the distances between the pairs of points, L and S, N and T, and U and Q.

**Tuesday.** Following a brief discussion of the homework, the major portion of the period was used to construct the special unit circle. Whereas previously the students could relate the development of the finite system to their previous studies, they had little motivation for construction of the unit circle other than the fact that the coordinates of the points of the circle were to have some interesting properties. (None of the students had ever seen the trigonometric functions defined in terms of the coordinates of the unit circle.) It was related to the students that the properties of these coordinates were so important that the coordinates were given special names of cosine and sine. The relationships of the coordinates led to the establishment of the identities \( \sin^2 t + \cos^2 t = 1 \), \( \cos t = \cos(-t) \), and \( \sin t = -\sin(-t) \). No assignment was given.

**Wednesday.** The unit circle and its coordinates were again placed on the chalkboard. Discussion and motivation were centered around the questions: (1) How are the coordinates of \( P(t) \) related to each other? (2) How are the coordinates of \( P(t) \) related to the coordinates of \( P(-t) \)? (3) How are the coordinates of \( P(2) \) related to the coordinates of \( P(5) \) and \( P(3) \)? How are the coordinates of \( P(4) \) related to the coordinates of \( P(4) \) and \( P(6) \)? How are
coordinates of \( P(u + v) \) or \( P(u - v) \) related to the coordinates of \( P(u) \) and \( P(v) \)? Answers to the first two questions were expressed in terms of both \( x \) and \( y \) coordinates and in trigonometric form. A considerable part of the period was spent developing the identity, \( \cos(u - v) = \cos u \cos v + \sin u \sin v \), as a partial answer to questions of part (3).

The assignment was to select several values of \( u \) and \( v \) and use these to test the validity of the identities of \( \cos(u + v) = \cos u \cos v - \sin u \sin v \) and \( \cos(u - v) = \cos u \cos v + \sin u \sin v \).

**Thursday.** Time did not permit the development of the identities for \( \sin(u + v) \) and \( \sin(u - v) \) so these formulas were given in class and several examples were used to illustrate them. In order to show that the trigonometric functions could be applied outside the context of the unit circle the Law of Sines and the Law of Cosines were presented, and three problems using these formulas were illustrated with triangles of finite geometry. For homework each person was assigned one trigonometric point; he was to express this point in six other ways as the sum or difference of two other numbers, and then to use these numbers in the trigonometric addition formulas to check their validity.

**Friday.** The first part of the class hour was spent in a short review. The writer also showed how the sine and cosine could be defined in terms of ratios of sides of right triangles to give equivalent results. In the remaining time
the students were given a short examination covering finite geometry and trigonometry. For the examination the students were also given a repeated array of block I and a template to measure angles. Of the fifteen students, one who was absent on Thursday and Friday did not take the examination.

A copy of the examination which required 30-40 minutes is shown on the following pages. The numbers in parentheses along the right hand margin indicate the number of people who did not receive full credit for their response. An analysis of the responses for problems 1c, 6b, and 8 showed that although most persons set up the problem correctly their work contained errors in computation. No doubt, had the students had more practice with computation in the system, or had they been given a multiplication table to use, a greater percentage would have answered these problems correctly. Difficulty with computation was also evident in the number of persons incorrectly solving problems 4a and 4b.

Problems 1a, 1b, 1c, 2, 3, 4, 6a, 6b, and 7 were assigned a credit of five points each, and problems 5 and 8 were assigned a credit of ten points each. Scores were given based on the number of missed credits subtracted from 100. The resulting scores were 95, 90, 90, 87, 84, 80, 79, 78, 75, 71, 70, 68, 64, and 58. The writer feels that he had covered a considerable amount of material in a short
Formulas:

\[
\cos(u + v) = \cos u \cos v - \sin u \sin v \\
\cos(u - v) = \cos u \cos v + \sin u \sin v \\
\sin(u + v) = \sin u \cos v + \cos u \sin v \\
\sin(u - v) = \sin u \cos v - \cos u \sin v \\
a^2 = b^2 + c^2 - 2bc \cos A \\
\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}
\]

1. Use point B of block I (or of the array) as the origin:
   a) Give the coordinates of point G. (2)
   b) Give the coordinates of point S. (4)
   c) Use the distance formula to find the distance between points G and S. (show work) (8)

2. Find the point at which the medians of triangle TRI meet. (2)

3. Measure the angles of triangle TRI. (0)
   \[
   \begin{align*}
   \text{angle T} &= \\
   \text{angle R} &= \\
   \text{angle I} &= \\
   \text{Total} &= 180^\circ
   \end{align*}
   \]

4. a) \((2')\times(-1') = \) (8)
   b) \(\sqrt{-1} = \) (8)
5. Fill in the coordinates of the remaining points:

\[ P(3) : (0, 1) \]
\[ P(4) : \]
\[ P(5) : \]
\[ P(6) : \]
\[ P(7) : \]
\[ P(8) : \]
\[ P(9) : (0, -1) \]
\[ P(10) : \]
\[ P(11) : \]

6. a) Using the coordinates of \( P(2) \) and \( P(3) \), find the \( \cos 5 \). (show work)

\[ \text{(7)} \]

b) Again, using the coordinates of \( P(2) \) and \( P(3) \), find \( \sin -1 \).

\[ \text{(8)} \]

7. \[ \sin \frac{x}{2} = \pm \sqrt{\frac{1 - \cos x}{2}} \] is a formula used to find the sine of an angle when the cosine of an angle twice as large is already known. For example, if one knows the cosine of 120°, one can use this formula to determine the sine of 60°. Give an example of an angle in twenty-five point trigonometry which would lead to an undefined situation when used in this formula.

\[ \text{(6)} \]

8. Two sides of a triangle are 1 and 2'. The included angle is 150°. Use the Law of Cosines to determine the length of the third side.

\[ \text{(8)} \]
time and was satisfied with the results of the examination.

Throughout the seven class periods the students' responses and comments indicated a real interest in the finite mathematical system. The writer is sure that the objectives of this course were well served by the study of twenty-five point geometry and trigonometry.
CHAPTER IX

CONCLUSION

The teaching of mathematics in the secondary schools is seemingly often directed toward a teaching of content or techniques of symbol manipulation. The nature of mathematics, its creation, and its development, while receiving renewed emphasis from time to time, still occupies a relatively minor position in mathematics curricula of the high school level. Although the major concern of this study was the development of a finite trigonometry, one of the objectives was to emphasize the roles of analogy and generalization in the creation of mathematics. The analogy of mathematical systems and the creation of mathematics through a search for generalizations are certainly part of the nature of mathematics. The finite trigonometry provides a vehicle for the presentation of this aspect of mathematics to high school students.

Another objective of this study was a presentation of material which capable high school students could themselves investigate to create mathematics. Finite trigonometry offers many opportunities for further investigation.
by high school students as well as for those at the graduate level. Immediately following are outlines of several areas related to this study which students or others might pursue.

(a) Chapter VII discusses the concepts of twenty-five point trigonometry on a set of forty-nine points. As mentioned in Chapter VII there are many other finite geometries; one of the simplest finite geometries is a nine point system. This geometry may be represented by nine letters arranged in two blocks as shown in Figure 59. (Other arrangements of block II are possible.)

```
A B C A F H
D E F E G C
G H I I B D
```

**Fig. 59. — Nine Point Geometry**

Row and column distances of block I are designated by 1 and row and column distances of block II are designated by 1'. It remains for the reader to apply the concepts of finite trigonometry to this system.

(b) Chapter VII shows the concepts of twenty-five point trigonometry extended to a forty-nine point system. If, as suggested, these same concepts also apply to a nine point system, it appears possible that they may also be
extended to an eighty-one point system, a one hundred twenty-one point system, or to a general case, \((2n - 1)^2\) points.

(c) An unusual consequence of the definition of angle in this study was the inclusion of negative angles as elements of triangles. From the writer's work in this area it appears possible to define an equivalent twenty-five point trigonometry where, by employing directed distances for sides of triangles, one may express all angles with a positive numeration while their sum totals \(180^\circ\). One of the reasons this trigonometry was discarded in favor of the presentation herein was that no direct means was found to assign direction to the sides of the triangles. (Another reason for the adoption presented in this study is the relatively widespread practice of defining sides of triangles in twenty-five point geometry by positive numeration while the definition of angle measurement for this system has only recently evolved.) No doubt, other trigonometries may be defined for this twenty-five point system if one considers other possible definitions of angle measurement.

(d) One may observe from the principal diagonal of the multiplication chart used in twenty-five point geometry that each of the unprimed numbers may be obtained from the square of at least one of the numbers used for distance numeration, yet no number when squared equals one of the primed numbers. Moreover, the product of two primed numbers always equals an
unprimed number. Thus the primed numbers in many respects appear to carry a role analogous to imaginary numbers of the complex number system. A more extensive development of the number system is also suggested by the identities for half-angle trigonometric functions (see Sec. 6.6) which require, for evaluation, definitions for numbers of the form \( \sqrt{a + b} \).

(e) An examination of textbooks which present a traditional approach to Euclidean plane trigonometry reveals rather large segments devoted to solutions of unknown elements of triangles through logarithms. Although a finite or modular system has little need for logarithmic solutions, an interesting relationship may be shown to exist when a logarithmic system is defined over a finite set of numbers.

In this dissertation the writer has investigated and developed a finite trigonometry, the study of which he believes would provide a meaningful experience for many students of mathematics. The writer has also outlined a brief presentation of finite trigonometry for a college level course, but much work still remains concerning the adaptability of the finite systems at all levels of instruction. It is hoped that some readers may wish to adapt this material to various age levels and investigate its feasibility, perhaps in their own classrooms. Still others may find through this presentation related areas of finite
mathematics which they will investigate. The writer would consider either of these outcomes as evidence of a fulfillment of the purpose of his investigation.


