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By


*****

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CHAPTER I
INTRODUCTION

One of the dramatic antenna developments of the past decade has been the spiral antenna. This antenna now is used at frequencies from the beginning of the HF band (3 MHz) up to X-band (10,000 MHz) and even beyond. Variations of the spiral are used in almost every type of system employing antennas, particularly when circular polarization and broad bandwidth are desired. For example, the spiral antenna is finding wide applications in communications and electromagnetic warfare.

Although a good deal of study has been devoted to the spiral antenna since it was first conceived[1], its full potential has not been realized. Most of the emphasis has been placed on its broadband radiation and impedance characteristics, and bandwidths well over 10:1 are now readily achieved. Relatively little effort, however, has been devoted to the beam shaping and beam scanning possibilities of the spiral. It is the purpose of this study to develop a method of maximizing the directivity of a multielement, planar, equiangular spiral antenna in a particular direction in the hemisphere above the antenna for a given number of modes on the spiral. From a practical point of view this means that the spiral can provide beam scanning
through 360 degrees in azimuth and through a substantial range in elevation. Elevation scanning, however, is restricted by the fact that the radiation pattern goes to zero at the plane of the spiral (assuming the ideal case of infinite ground plane). The scanning can be obtained electronically by appropriate phase and amplitude control of the spiral elements.

The first spiral antenna, which took the form of an Archimedes spiral, was proposed by E.M. Turner in 1953[1]. Later, V.H. Rumsey advanced the theory that an equiangular spiral antenna of infinite length would have an infinite bandwidth[2]. A thorough investigation of planar, equiangular spiral antennas of finite size was accomplished by J.D. Dyson[3]. As a result of Dyson's work the first practical antenna was developed which could be made to approximate the characteristics of the infinite structure. The earliest equiangular spiral antennas were two element planar structures.

During the past decade a great deal of effort has been devoted to improving the performance of equiangular spiral antennas. One result of this effort is the equiangular spiral antenna with more than two elements[4]. With many elements, many different excitations may be used and an excitation may be determined that will optimize, in some sense, the performance of the antenna.
The object of this investigation is to demonstrate analytically the results of simultaneously exciting two or more higher order modes on a planar, equiangular spiral antenna when the excitation is such as to produce maximum directivity in a given direction. To accomplish this it is necessary to determine the excitation that will yield maximum directivity for a given number of modes on the antenna.

In Chapter II the excitation of a multielement, equiangular spiral antenna is shown to be expressible as the weighted sum of certain characteristic excitation vectors. Assumptions are discussed which make it possible to establish a one-to-one relationship between a characteristic excitation vector and a radiation mode on the spiral. The assumptions are justified in terms of experimental evidence obtained by other investigators.

The existence of spiral modes and their characteristic excitation has been experimentally demonstrated by Dyson and Mayes[4] on a four arm conical spiral, by Sivan-Sussman[5] on four and six arm planar spirals and by Ransom[6] on four arm planar spirals.

The excitation for maximum directivity is determined in Chapter III by finding the required excitation vector coefficient or weighting of the characteristic excitation vectors of each of the excited modes.
The directivity function of a multimode equiangular spiral is expressed as a ratio of quadratic forms which yields directly the excitation vector coefficients for maximum directivity in a specified direction and the value of the directivity in that direction. Both the excitation vector coefficients and maximum directivity appear in terms of the radiation fields of the spiral modes.

In the absence of expressions for the fields of an actual spiral antenna, it is possible to obtain explicit values for the excitation vector coefficients and the directivity by employing the fields of models of the spiral antenna. This is discussed in Chapter IV. The first model is an extension of the radiating ring model which is commonly used to describe the radiating mechanism and bandwidth characteristics of a spiral antenna[7, 8, 9]. The second model is the anisotropic sheet model proposed and analyzed by Cheo, Rumsey and Welch[10].

The results obtained using the fields of each of these models are presented in Chapter V. The results show that a significant improvement in the directivity of a multielement spiral may be achieved by properly exciting two or more modes on the spiral.

A general summary of the results of the investigation is given in Chapter VI and a possible DF application of the multimode spiral is briefly discussed.
CHAPTER II
THEORY OF A MULTIMODE SPIRAL

Introduction

In this chapter the radiation field of a multielement spiral antenna is shown to be expressible as a sum of the radiation fields of characteristic modes of the spiral antenna. These modes are defined with respect to certain excitation current vectors which form a basis for all possible excitation vectors subject to Kirchoff's current law.

The "radiation band" or "active region" theory of radiation from a spiral structure is described in terms of the characteristic modes of the spiral.

Finally, the assumptions required to establish a one-to-one relationship between a characteristic excitation and an active region are discussed.

The Multielement Spiral Antenna

A multielement, planar, spiral antenna may be generated by the family of curves

\[ \rho = \rho_0 e^{-a(\phi - \xi)} \]
In Eq. (1), \( \rho \) and \( \phi \) are the usual polar coordinates, \( a \) is the expansion factor which determines the rate of growth of the spiral, \( \rho_0 \) is the distance from the origin to the initial point of the curve and \( \delta_\ell \) is the angular width of an element. In practice \( \rho_m \) determines the extent of the feed region. The minimum radius of the spiral is given by

\[
\rho_m = \rho_0 e^{-2a\tau \pi}
\]

where \( \tau \) is the number of turns of the spiral around the origin. The expansion factor is related to the angle between a radius vector and a tangent to the curve by

\[
a = \cot \alpha
\]

In this investigation the structure of interest is an \( N \)-element, planar, self-complimentary, equiangular spiral. To generate this structure the parameter \( \delta_\ell \) in Eq. (1) is assigned values given by

\[
\delta_\ell = \frac{\ell \pi}{N} \quad (\ell = 0, 1, 2 \ldots 2N-1)
\]

and the initial point of the \( \ell \)-th curve is taken as the point at which

\[
\phi - \delta_\ell = 0
\]

A sketch of a four element spiral is shown in Fig. 1.
Fig. 1--A four-arm planar equiangular spiral with parameters shown.
The $N$ elements of a multielement spiral terminate in the feed region making $N$ terminals available for excitation.

**Excitation of a Multielement Spiral**

The theory of excitation of a four element conical spiral has been given by Dyson and Mayes [4]. The application of their results to an $N$ element planar spiral is included here for completeness.

The structure generated by Eq. (1) possesses $N$-fold rotational symmetry. That is, an axial rotation through $2\pi/N$ radians transforms the structure into itself. A scheme for the excitation of an $N$-element spiral antenna is shown in Fig. 2. If the current at the $k$-th terminal is denoted by $i_k$ the excitation can be described by a current vector

\[
\bar{I} = (i_N, i_1, i_2, \ldots, i_{N-1})
\]

The index on the terminal currents is defined modulo $N$.

A permutation of the excitation by one element produces a field which differs from that of $\bar{I}$ by a rotation in space of $2\pi/N$ radians.

The permuted excitation vector is written as

\[
\bar{I}_P = (i_1, i_2, i_3, \ldots, i_N)
\]

and is given by

\[
\bar{I}_P = P \bar{I}
\]
Fig. 2—Schematic representation of excitation of an N-arm spiral.
where $P$ is the $N \times N$ permutation matrix

$$P = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \end{bmatrix}$$

and the vectors $\overline{I}$ and $\overline{I}_P$ are now written as column vectors.

As shown in Appendix I the eigenvalues of $P$ are $e^{j \frac{2 \pi \sigma}{N}}$ with the corresponding eigenvectors given by

$$\overline{I}_{\sigma} = N^{-\frac{1}{2}} \left( 1, e^{\frac{j 2 \pi \sigma}{N}}, e^{\frac{j 4 \pi \sigma}{N}}, \ldots, e^{\frac{j 2 \pi \sigma (N-1)}{N}} \right)$$

where $\sigma$ may assume the value of each of the integers from 1 to $N$.

It is further shown in Appendix I that the $N$ vectors $\overline{I}_{\sigma}$ span the space of $N$ fold rotationally symmetric excitation vectors. However, under the condition that the $N$ spiral arms are the only terminals available, Kirchoff's current law requires that

$$\sum_{n=1}^{N} i_n = 0$$

Each of the eigenvectors $\overline{I}_{\sigma}$ except $\overline{I}_N$ satisfies this condition.

The remaining $N-1$ eigenvectors form an orthonormal basis for all possible excitations of an $N$ element, self-complimentary spiral antenna. Therefore, any excitation of an $N$ element, self-complimentary
spiral that satisfies the constraint of Eq. (11) may be written as a linear combination of the eigenvectors $\overline{I}_\sigma (\sigma = 1, 2, \ldots N-1)$.

The eigenvectors $\overline{I}_\sigma$ lead, in a natural way, to a definition of spiral antenna modes and an approximate description of the fields of a spiral antenna in terms of these modes.

**Spiral Antenna Modes**

Consider an $N$ element spiral antenna excited by one of its excitation basis vectors $\overline{I}_k$ where

$$
\overline{I}_k = N^{-\frac{1}{2}} \left( 1, e^{j \frac{2\pi k}{N}}, \ldots, e^{j \frac{2\pi k}{N}(N-1)} \right).
$$

Since the field of the antenna must satisfy Maxwell's equations in spherical coordinates the azimuthal variation of the field can be written in the form $e^{j n \phi}$ with $n$ an integer. In addition, the field must be single valued and only those solutions of Maxwell's equations for which $n = N\mu + k$ with ($\mu = 0, \pm 1, \pm 2 \ldots$) will satisfy the relationship between a rotation and a change in phase established by the $N$ fold rotational symmetry of the excitation vectors.

A solution for the electric field of the antenna with excitation $\overline{I}_k$ may be written in the form

$$
\overline{E}_k = \sum_{\mu} a_\mu \overline{\psi}_{N\mu + k}
$$
where \( \tilde{\psi}_{N\mu} \) is a vector function given by

\begin{equation}
\tilde{\psi}_n = \tilde{F}_n(r, \theta) e^{jn\phi}
\end{equation}

with the function \( F_n(r, \theta) \) a solution to the separated homogeneous wave equation in the variables \( r \) and \( \theta \).

Using the fields \( \tilde{E}_k \) of the excitation basis vectors \( \tilde{I}_k \) the fields of an arbitrary excitation vector \( \tilde{I} \) may be written as

\begin{equation}
\tilde{E} = \sum_{k=1}^{N-1} (\tilde{I}, \tilde{I}_k) \tilde{E}_k
\end{equation}

The product \( (\tilde{I}, \tilde{I}_k) \) in Eq. (15) is defined as

\begin{equation}
(\tilde{I}, \tilde{I}_k) = [I]^{\dagger} [I_k]
\end{equation}

where \([I_k]\) and \([I]\) are \( N \times 1 \) column matrices and \([I]^{\dagger}\) is the conjugate transpose of \([I]\).

Equation (15) is useful only if the coefficients \( a_{\mu} \) in the expansion of \( \tilde{E}_k \) are known. The problem of determining the exact value of these coefficients is as yet unsolved. However, by formulating the radiation band theory of the spiral antenna in terms of the characteristic excitations and comparing the results with experimental data we will show that the coefficients \( a_{\mu} \) with \( \mu \neq 0 \) can be approximated by \( a_{\mu} = 0 \).
Choosing the N-th terminal as phase reference and the characteristic excitations $i_k$, the phase of the excitation at the input to the m-th element is given by $e^{j \frac{2km}{N}}$. At a point on the m-th element a distance $S$ along the element the phase of the excitation is given by $e^{j \left[ \frac{2km}{N} - \psi(S) \right]}$.

The exact form of the function $\psi(S)$ depends upon the geometry and environment of the spiral. However, a solution to the boundary value problem associated with a planar sheath equiangular spiral antenna\[11\] indicates that to a good approximation the function $\psi(S)$ may be written as

\[(17) \quad \psi(S) = \beta_S S\]

where

\[(18) \quad \beta_S = \frac{2\pi}{\lambda_S}\]

and $\lambda_S$ is the wavelength along the arm.

At this point it is worthwhile to note that the wavelength $\lambda_S$ on a spiral arm is not the free space wavelength $\lambda_0$. It will be shown in Chapter V that $\lambda_S$ is less than $\lambda_0$; a result that was demonstrated experimentally by Dyson[3] and by Ransom[6]. Also, $\lambda_S$ approaches $\lambda_0$ only on tightly wound spirals.
Consider the 4 element spiral shown in Fig. 3. The spiral curves represent the center lines of the spiral elements. The phase of the excitation at the terminals is shown as that for $I_3$. The elements are divided into equal segments $\Delta S$. Assuming that the frequency of the excitation is that for which $\beta S \Delta S = \frac{\pi}{2}$, the phase with respect to the reference terminal changes by $90^\circ$ between each tic mark. Moving outward along any two adjacent elements a region is reached at which radially adjacent points are in phase. Circles which connect the in phase points on the elements define a region in which the phase along a radius is essentially constant. These regions are shown by the circular bands of Fig. 3. Figures 4 and 5 show a region of in phase excitation for $I_2$ and $I_3$ respectively. Simple constructions show that the mean circumference of the circular bands is given by

\[ C = (4\mu + k)\lambda_S \]

with ($\mu = 0, 1, 2, \ldots$).

Figure 3 shows two circular bands. The inner band corresponds to $\mu = 0$ in Eq. (19) while for the outer band $\mu = 1$. Figures 4 and 5 show circular bands only for $\mu = 0$.

The phase variation around the bands may be described as follows. Note that on any element of the 4 element spiral there are exactly $n$ intervals of length $\Delta S$ within the band whose mean circumference is
n \lambda_s. Thus the total phase shift along the portion of the element within
the band is $\frac{n\pi}{2}$. Since there are four elements the total phase shift
around the band is $2n\pi$. Thus the point to point variation in phase
around the band must be given by $n\phi$. The radial phase variation may
be inferred from Fig. 4 by noting the phase variation from element
to element while moving outward along a fixed radius. Between a
feed point and the inner edge of the circular band the phase variation
indicates a negative phase velocity or inward traveling wave. Across
the circular band there is no phase variation thus indicating an infinite
phase velocity. Beyond the outer edge of the circular band the element-
to-element phase variation indicates a positive phase velocity or out-
ward traveling wave. Cheo[10], et.al. observe the same radial phase
characteristics by calculating the phase of the excitation currents on an
anisotropic plane sheet model of the spiral antenna.

The principal contribution to the radiation field of the spiral
antenna is from the currents within the circular bands[6, 12]. Radiation
from currents outside these bands will tend to cancel and their contri-
bution to the radiation field is negligible. Experiment has shown that
if the maximum radius of a spiral antenna is reduced the affect on the
radiation pattern is very small until the reduced radius approaches
that of the outer edge of the inner most "radiation band". In other
words the contribution to the radiation field from the radiation bands
Fig. 3--First and fifth mode radiation bands on a four-arm spiral.
Fig. 4—Second mode radiation band on a four-arm spiral.
Fig. 5--Third mode radiation band on a four-arm spiral.
whose mean circumference is greater than $k \lambda_s$ may be neglected.

This result allows the coefficient $a_\mu$ in Eq. (13) to be taken as zero when $\mu > 0$. For values of $\mu$ less than zero the azimuth phase variation of the field associated with the coefficient $a_\mu$ is given by $e^{-j(N|\mu|-k)\phi}$. Since this variation is opposite to the preferred direction established by the winding sense of the spiral elements the contribution of these terms to the radiation field can be expected to be negligible. This fact has been demonstrated experimentally\cite{5}. Thus for $\mu < 0$ the coefficients $a_\mu$ also may be taken as zero.

Recalling that the characteristic excitation vector $\overline{I}_k$ produces a radiation field of the form given by Eq. (13) and utilizing the results of the foregoing discussion we have

\begin{equation}
\overline{E}_k = \overline{F}_k(r, \theta) e^{jk\phi}.
\end{equation}

Equation (20) will serve as the definition of the \textit{k}-th spiral antenna mode. That is, by definition the characteristic excitation $\overline{I}_k$ radiates the \textit{k}-th spiral mode. This definition is made regardless of the form of $\overline{F}_k(r, \theta)$. The term "multimode" spiral will be used to describe any spiral of three or more elements excited by two or more of its characteristic excitations.

Substituting Eq. (20) into Eq. (15) gives the field of an arbitrary excitation vector $\overline{I}$. That is
Thus, subject to the condition of conservation of current, the radiation field of an arbitrarily excited N-arm spiral can be written as the weighted sum of N-1 spiral antenna modes. This completes the analysis in terms of the spiral modes. However, in this investigation the problem of interest is a synthesis problem. The problem is that of determining the excitation vector $\mathbf{i}$ of Eq. (21) which will maximize the directivity in a given direction subject to a constraint on the number of modes excited on the spiral.

To put Eq. (21) in a slightly more amenable form we write the conjugate transpose of $\mathbf{i}$ as

$$[\mathbf{i}]^\dagger = \sum_{n=1}^{N-1} A_n [i_n]^\dagger .$$

Substituting Eq. (22) into Eq. (21) and recalling that the characteristic excitation vectors are mutually orthogonal, Eq. (21) becomes

$$\overline{E}(r, \theta, \phi) = \sum_n A_n \overline{F}_n (r, \theta) e^{jn\phi} .$$

The definition given by Eq. (20) and the formulation of Eq. (23) were obtained by using experimental evidence and physical reasoning.
to substantiate the assumption that in the ideal case the characteristic excitation $I_k$ produces a radiation field whose azimuth variation is given by $e^{jk\phi}$. As a practical matter the ideal case can be very closely approximated by careful fabrication of the antenna and feed network and by the proper choice of the number of arms ($N$).

This last point may be illustrated by recalling that because of symmetry requirements the only allowable solutions to Maxwell's equations for the characteristic excitation $I_k$ are those with the azimuth variation given by $e^{j(N\mu+k)\phi}$ with $\mu$ equal to $(0, \pm 1, \pm 2 \ldots)$. Experimental evidence supports the conclusion that even in the practical case radiation fields with the azimuth variation for which $\mu < 0$ are not present\textsuperscript{[5]}. Thus the azimuth variation of a practical spiral antenna excited by $I_k$ may be characterized by the integers $k, N+k, 2N+k$ and so on. To facilitate the discussion we define the modes for which $\mu \neq 0$ as harmonic modes ordered according to $\mu$. Hence, the $N+k$-th mode is the first harmonic of the $k$-th mode and the $2N+k$-th mode is the second harmonic mode. Since the characteristic excitation vectors are defined modulo $N$, it is apparent that all the excitations $I_{\mu N+k}$ are identical. Suppose that a planar spiral antenna is excited by its characteristic excitation vector $I_k$. In terms of the radiation band theory, if the maximum radius of the spiral is large enough the energy which is not radiated in the first radiation band (mean
circumference = \( \lambda_s \) will be partially radiated from the following bands (mean circumference = \((N+1)\lambda_s\), \((2N+1)\lambda_s\)...) and will result in radiation in harmonic modes. Experience has shown that harmonic mode radiation is generally insignificant with regard to its effect on the radiation pattern of the antenna. However, in applications requiring good axial ratios over an appreciable part of the pattern or uniform azimuth patterns, as in direction finding applications, harmonic mode radiation, particularly the first harmonic, can become significant.

One technique for suppressing harmonic mode radiation is to reduce the outer circumference of the spiral, thereby eliminating the harmonic radiation bands. A load ring, that is, a band of lossy material, may then be placed around the outer circumference to absorb the energy which would otherwise be reflected back toward the feed region\[^{12}\]. This technique limits the bandwidth of the spiral. A second technique for suppressing harmonic mode radiation consists of choosing \( N \) so that \((N+k)\lambda_s\), where \( k \) is the highest mode to be excited, is greater than the desired outer circumference of the spiral. A load ring is also employed in this configuration. This technique does not limit the bandwidth but if \( N \) is too large fabrication of the spiral can be difficult. Nonetheless the above discussion indicates that a practical spiral antenna can be made to closely approximate...
the ideal case. That is, the excitation $I_k$ produces a radiation field whose azimuth variation is given by $e^{jk\phi}$.

In the following chapter an expression is derived for determining the coefficients $A_n$ in Eq. (23) and hence the excitation vector $\bar{I}$ which maximizes the directivity of an $N$-element spiral antenna.
CHAPTER III
MAXIMIZATION OF THE DIRECTIVITY FUNCTION
OF A MULTIMODE SPIRAL

Introduction

In this chapter the directivity function for a multimode spiral antenna is obtained in terms of the characteristic modes defined in Chapter II. It is shown that the directivity function reduces to a ratio of quadratic forms with one nonzero eigenvalue. This eigenvalue is the maximum obtainable directivity in a given direction. Its associated eigenvector is the excitation vector required to produce this maximum directivity.

The Directivity Function of a Multimode Spiral

The directivity of an antenna is defined as the ratio of the radiation intensity in a given direction to the average radiation intensity of the antenna[13]. The directivity function from which the value of the directivity is determined is given by

\[
D(\theta_0, \phi_0) = \frac{4\pi U(\theta_0, \phi_0)}{\int_0^{2\pi} \int_0^\pi U(\theta, \phi) \sin \theta \, d\theta \, d\phi}
\]
where $U(\theta, \phi)$ is the radiation intensity. The radiation intensity of an antenna is given by[14]

$$U(\theta, \phi) = \frac{r^2}{Z_0} \left[ \overline{E}^* \cdot \overline{E} \right]$$

(25)

where $\overline{E}$ is the electric field in the radiation zone. Substitution of Eq. (25) into Eq. (24) gives

$$D = \frac{4\pi \left[ \overline{E}^* \cdot \overline{E} \right]}{\int_0^{2\pi} \int_0^{\pi} \left[ \overline{E}^* \cdot \overline{E} \right] \sin \theta \, d\theta \, d\phi}$$

(26)

According to the analysis of Chapter II the electric field of a multimode spiral antenna may be written as

$$\overline{E} = \sum A_n \overline{E}_n$$

(27)

From Eq. (20), $\overline{E}_n$ is given by

$$\overline{E}_n = \overline{F}_n(r, \theta) e^{in\phi}$$

(28)

In the radiation zone the radial component of $\overline{E}_n(r, \theta, \phi)$ is negligible and the vector function $\overline{F}_n(r, \theta)$ may be written in terms of its far field components as

$$\overline{F}_n(r, \theta) = \hat{x} F_{n\theta}(r, \theta) + \hat{\phi} F_{n\phi}(r, \theta)$$

(29)
Furthermore, the radial dependence of $F_n(\theta, \phi)$ and $F_n(\theta, \phi)$ is given by $e^{-jkr}$ and Eq. (28) may therefore be written as

\begin{equation}
E_n = \left[ \hat{f}_n(\theta) + \hat{g}_n(\theta) \right] \frac{e^{-jkr}}{r} e^{j\phi}
\end{equation}

where

\begin{align}
\hat{f}_n(\theta) &= F_n(\theta, \phi) r e^{jkr} \\
\hat{g}_n(\theta) &= F_n(\theta, \phi) r e^{jkr}
\end{align}

Substituting Eq. (30) into Eq. (27) and Eq. (27) into Eq. (26) gives an expression for the directivity function of a multimode spiral antenna in terms of its characteristic modes. The directivity function is

\begin{equation}
D(\theta, \phi) = \frac{4\pi}{\sum \sum \int_0^{2\pi} \int_0^{\pi} A_m^* \left[ f_m^*(\theta) f_n(\theta) + g_m^*(\theta) g_n(\theta) \right] A_n e^{j(n-m)\phi} \sin \theta d\theta d\phi}
\end{equation}

The integration over $\phi$ is easily carried out and the directivity function simplifies to

\begin{equation}
D = \frac{2}{\sum A_m^* \left[ \int_0^{2\pi} \left( |f_n|^2 + |g_n|^2 \right) \sin \theta d\theta \right] A_n}
\end{equation}
where in order to simplify notation the dependence of $D$ on $\theta_o$ is implied. For the special case of a circularly polarized field, which will be shown to be the case of interest, $f_n = g_n$ and Eq. (34) reduces to

$$D = \frac{2 \sum \sum A_m^* f_m^* f_n g_n e^{i(n-m)\phi_o}}{\sum A_n^* \left[ \int_0^\pi |f_n|^2 \sin \theta \, d\theta \right] A_n} \quad (35)$$

For convenience it is desirable to establish the following substitution and ordering convention. Let an $N$ element spiral be excited by $M$ characteristic modes ($M < N$). Let $n_1$ equal the smallest value of $n$ in Eq. (35). Let $n_2$ equal the next highest value of $n$ and on until all values of $n$ are exhausted. Now the functions $f_n e^{j n \phi_o}$ are ordered according to increasing values of $n$ and we make the substitution

$$\gamma_i = f_{n_i} e^{j n_i \phi_o} \quad (36)$$

under which Eq. (35) becomes

$$D = \frac{2 \sum_{m=1}^M \sum_{n=1}^M A_m^* \gamma_m^* \gamma_n g_n A_n}{\sum_{n=1}^M A_n^* \left[ \int_0^\pi |\gamma_n|^2 \sin \theta \, d\theta \right] A_n} \quad (37)$$
The right hand side of Eq. (37) may be written as a ratio of quadratic forms\cite{15, 16}. That is

\[
D = \frac{\sum_{m=1}^{M} \sum_{n=1}^{M} A_m^* \Gamma_{mn} A_n}{\sum_{n=1}^{M} A_n^* G_n A_n}
\]

where

\[
\Gamma_{mn} = \gamma_m^* \gamma_n
\]

and

\[
G_n = \frac{1}{2} \int_{0}^{\pi} |\gamma_n|^2 \sin \theta d\theta.
\]

The coefficients \(A_n\) which maximize the directivity may be readily determined by employing a theorem concerning the properties of a ratio of quadratic forms\cite{15, 16}.

**Maximization of the Directivity Function**

The maximization of the directivity function of a spiral antenna is but one example of the usefulness of the properties of a ratio of hermitian quadratic forms. Several examples of other applications
have been enumerated in the literature\[16, 17, 18\] and it therefore
seems worthwhile to consider these properties in some detail.

Examination of Eq. (38) shows that it may be written in matrix
form as

\[
D(A) = [A]^+ [\Gamma] [A] [A]^+ [G] [A]
\]

in which \([A]\) is an \(M \times 1\) column matrix whose elements are the \(A_n\)'s,
\([\Gamma]\) is an \(M \times M\) Hermitian matrix with elements \(\Gamma_{mn}\), \([G]\) is an
\(M \times M\) diagonal matrix with elements \(G_n\) and \([A]^+\) is the conjugate
transpose of \([A]\).

The applicable properties of the ratio of Hermitian quadratic
forms in Eq. (41) are contained in the following theorem\[15, 16\].

\([G]\) is positive definite then:

1. The roots of the characteristic equation

\[(42) \quad \text{determinant} ([\Gamma] - \lambda [G]) = 0\]

are real.

2. The maximum and minimum values of \(\lambda\) are the bounds of

\(D(A)\).

That is,
3. The right equality in Eq. (43) holds only when $[A]$ satisfies

$$[\Gamma] [A] = \lambda_{\text{MAX}} [G] [A]$$

while the left equality holds only when $[A]$ satisfies

$$[\Gamma] [A] = \lambda_{\text{MIN}} [G] [A] .$$

Thus, if $\lambda_M$ is the maximum value of the roots of Eq. (42) it is the maximum value of the directivity function for a given set of the elements $\Gamma_{mn}$. In addition, the elements of the eigenvector $[A]_M$ of Eq. (44) are the coefficients in the excitation synthesis equation, Eq. (45), which will maximize the directivity.

It remains then to find the maximum eigenvalue $\lambda_M$ and its associated eigenvector(s) $[A]_M$. To find $\lambda_M$ we note that the matrix $[\Gamma]$ may be written as

$$[\Gamma] = [\gamma] [\gamma]^\dagger$$

in which $[\gamma]$ is an $M \times 1$ column matrix with elements $\gamma_n$ and $[\gamma]^\dagger$ is the conjugate transpose of $[\gamma]$. Solving Eq. (42) for $\lambda = \lambda_M$ is equivalent to solving the matrix equation

$$\lambda \leq D(A) \leq \lambda_{\text{MAX}}$$
in which \([U]\) is the identity matrix. It is easy to show that \([G]\) has an inverse and that multiplying Eq. (47) from the left by \([\gamma]^\dagger [G]^{-1}\) gives

\[
(48) \quad ([\gamma]^\dagger [G]^{-1}[\gamma] - \lambda_M) [\gamma]^\dagger = 0.
\]

Equations (48) shows that Eq. (41) has not more than one nonzero eigenvalue, it being given by

\[
(49) \quad \lambda_M = [\gamma]^\dagger [G]^{-1} [\gamma].
\]

Since the eigenvalues of Eq. (41) are known to be either zero or \(\lambda_M\) and because of the particular form of the matrix \([\Gamma]\) a special technique may be employed to find the eigenvectors \([A]\) [16].

In what is to follow we make use of three theorems from the theory of linear vector spaces. Stated without proof they are:

**Theorem 1.** Eigenvectors associated with different eigenvalues are orthogonal;

**Theorem 2.** With each eigenvalue of multiplicity \(S\) there is associated a set of \(S\) linearly independent eigenvectors;
Theorem 3. In a linear vector space of $M$ dimensions any $M + 1$ vectors are linearly dependent.

Let $[A_0]_i$ represent the eigenvectors associated with the eigenvalue $\lambda = 0$ and let $[A]_i$ represent the eigenvectors associated with the eigenvalue $\lambda = \lambda_M$. Define the weighted inner product by

$$([A_0]_i, [A_0]_j) = [A_0]_i^\dagger [G] [A_0]_j$$

then from Theorem 1 we have

$$([A_0]_i, [A_0]_j) = [A_0]_i^\dagger [G] [A_0]_j = 0$$

The eigenvectors $[A_0]_i$ are solutions to the equation

$$[\gamma] [\gamma]^\dagger [A_0]_i = 0$$

It is a simple task to find a set of $M - 1$ linear independent vectors that satisfy Eq. (52). One such set is given by

$$[A_0]_1 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \\ \cdots \cdots \\ 0 \\ \cdots \cdots \\ 0 \\ \cdots \cdots \\ 0 \\ \cdots \cdots \\ 0 \\ \cdots \cdots \\ 0 \\ \cdots \cdots \\ 0 \\ \cdots \cdots \\ 0 \\ \cdots \cdots \\ 0 \\ \cdots \cdots \\ 0 \end{bmatrix}$$

$$[A_0]_2 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \\ \cdots \cdots \\ 0 \\ \cdots \cdots \\ 0 \\ \cdots \cdots \\ 0 \\ \cdots \cdots \\ 0 \\ \cdots \cdots \\ 0 \\ \cdots \cdots \\ 0 \\ \cdots \cdots \\ 0 \\ \cdots \cdots \\ 0 \end{bmatrix}$$

$$\cdots$$

$$[A_0]_{M-1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \end{bmatrix}$$

$$[A_0]_1 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \end{bmatrix}$$

$$[A_0]_2 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \end{bmatrix}$$

$$\cdots$$

$$[A_0]_{M-1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \end{bmatrix}$$

$$[A_0]_1 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \end{bmatrix}$$

$$[A_0]_2 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \end{bmatrix}$$

$$\cdots$$

$$[A_0]_{M-1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \end{bmatrix}$$
Therefore, by Theorem 2 the eigenvalue $\lambda = 0$ is of multiplicity $M - 1$ and by Theorem 1 and Theorem 3 there can be no more than one eigenvector $[A]_M$ associated with the eigenvalue $\lambda_M$. The eigenvector $[A]_M$ is determined by forming the inner product, according to Eq. (51), of $[A]_M$ with each of the eigenvectors $[A_0]_i$ given in Eq. (53). Suppose we let

$$[A]_M = \begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\vdots \\
\alpha_M
\end{bmatrix}$$

then forming the inner product of $[A]_M$ with each of the $[A_0]_i$ results in the system of $M - 1$ equations

$$\frac{\alpha_1 G_1}{\gamma_1} = \frac{\alpha_2 G_2}{\gamma_2}, \quad \frac{\alpha_1 G_1}{\gamma_1} = \frac{\alpha_2 G_2}{\gamma_2}, \quad \ldots, \quad \frac{\alpha_1 G_1}{\gamma_1} = \frac{\alpha_M G_M}{\gamma_M}. \tag{55}$$

Each of the equation in this system is satisfied if we choose

$$\alpha_n = \frac{\gamma_n}{G_n}. \tag{56}$$

Then by substitution Eq. (54) becomes
which by inspection may be written more concisely as

\[
\begin{bmatrix}
\frac{Y_1}{G_1} \\
\frac{Y_2}{G_2} \\
\vdots \\
\frac{Y_M}{G_M}
\end{bmatrix}
\]

which by inspection may be written more concisely as

\[
[A]_{\text{MAX}} = [G]^{-1} [Y].
\]

Performing the matrix multiplication indicated in Eq. (49) and Eq. (58) gives the maximized directivity as

\[
D = \sum \frac{|Y_n|^2}{\frac{1}{2} \int_0^{\pi} |Y_n|^2 \sin \theta \, d\theta}
\]

and the coefficient \( A_n \) as

\[
A_n = \frac{Y_n}{\frac{1}{2} \int_0^{\pi} |Y_n|^2 \sin \theta \, d\theta}
\]

The radiation field is therefore given by

\[
\bar{E} = p \sum_n \frac{f_n(\theta_0) f_n(\Theta) e^{jn(\Phi + \phi_0)}}{\frac{1}{2} \int_0^{\pi} |f_n(\Theta)|^2 \sin \theta \, d\theta}
\]
where \( \bar{p} \) is a polarization vector. The conjugate transpose of the excitation vector associated with the maximized directivity is given by

\[
\mathbf{I}_n^+ = \frac{1}{2} \int_0^{\pi} |\gamma_n|^2 \sin \theta d\theta
\]

where

\[
\mathbf{I}_n^+ = N^{-1/2} \begin{pmatrix}
1, e^{-j \frac{2\pi}{N}}, e^{-j \frac{4\pi}{N}}, \ldots, e^{-j \frac{2\pi(N-1)}{N}}
\end{pmatrix}
\]

Equation (59) and Eq. (62) specify the maximized directivity function and the associated excitation vector relative to a given number of modes on the spiral and to a given direction \( \theta_o, \phi_o \). The form of \( \gamma_n \), which is given in Eq. (36), shows that the directional dependence of Eq. (59) arises from the heretofore unconstrained choice of \( \theta_o \); the angle at which the directivity is to be maximized. It will be shown that there is one choice of \( \theta_o \) which will result in the largest possible value of the directivity. This particular value of the directivity is defined to be the optimum directivity. Although the excitation required to achieve optimum directivity and the resulting radiation field cannot be determined without knowing the functions \( f_n(\theta) \), the existence of a constraining relationship between the excitation for optimum directivity and the resulting radiation field can be demonstrated.
Optimum Directivity Constraint

The expression derived in the previous section are written in terms of the related functions \( \gamma_n(\theta) \) and \( f_n(\theta) \). For purposes of this section we make an additional change and write \( f_n(\theta) \) in complex form letting

\[
f_n(\theta) = \alpha_n(\theta) e^{j[nx(\theta) + \xi(\theta)]}
\]

where \( \alpha_n(\theta) \), \( x(\theta) \) and \( \xi(\theta) \) are real functions.

Suppose that \( \theta_d \) is the value of \( \theta \) that optimizes the directivity. Then by Eqs. (36), (59) and (54) \( \theta_d \) is a solution to the equation

\[
\sum \frac{\alpha_n(\theta)}{G_n} \alpha_n'(\theta) = 0
\]

where the prime denotes differentiation with respect to \( \theta \).

Substituting Eq. (64) into Eq. (61) and multiplying the latter by its complex conjugate gives the power pattern of the spiral with excitation for optimum directivity. The relative power pattern is given by

\[
|E|^2 = \sum \sum \frac{\alpha_m(\theta_d) \alpha_n(\theta_d)}{G_m G_n} \alpha_m(\theta) \alpha_n(\theta) \left[ \cos \{(n-m)(\chi(\theta_d) + \chi(\theta) + \phi + \phi_d)\} \right].
\]

Expressions for determining the direction of the power pattern maximum are obtained by setting the partial derivatives of \(|E|^2\) with
respect to \( \phi \) and \( \theta \) equal to zero. For the partial derivative with respect to \( \phi \) this gives

\[
\sum \sum n \frac{\alpha_m(\theta_d) \alpha_n(\theta_d)}{G_m \ G_n} \ \alpha_m(\theta) \ \alpha_n(\theta) \ [ \\
\sin \{(n-m) (X(0) + X(\theta) + \phi + \phi_o)\} = 0 .
\]

For the partial derivative with respect to \( \theta \) this gives

\[
\sum \sum \frac{\alpha_m(\theta_d) \alpha_n(\theta_d)}{G_m \ G_n} \ \{ \alpha_n(\theta) \ \alpha_m'(\theta) + \alpha'_n(\theta) \ \alpha_m(\theta) \} \ [ \\
\cos \{(n-m) (X(\theta_d) + X(\theta) + \phi + \phi_o)\} \\
- 2X'(\theta) \sum \sum n \frac{\alpha_m(\theta_d) \ \alpha_n(\theta)}{G_m \ G_n} \ [ \\
\sin \{(n-m) (X(\theta_d) + X(\theta) + \phi + \phi_o)\} = 0 .
\]

Equation (67) is satisfied for all values of \( \theta \) when

\[
\phi + \phi_o + X(\theta_d) + X(\theta) = 0 .
\]

Substituting Eq. (69) into Eq. (68) reduces the latter to

\[
\sum \sum \frac{\alpha_m(\theta_d) \alpha_n(\theta_d)}{G_m \ G_n} \ \{ \alpha_n(\theta) \ \alpha_m'(\theta) + \alpha'_n(\theta) \ \alpha_m(\theta) \} = 0
\]

which after some minor manipulation and rearrangement becomes
Comparing the second factor in Eq. (71) to Eq. (65) shows that θ_d is a solution to Eq. (71). Hence the direction of optimum directivity is constrained to be the direction of maximum radiation which depends only on the number of modes excited and the choice of φ_o. However, the maximized directivity function is independent of φ_o therefore, the choice of φ_o influences the azimuth location of the radiation maximum but not its magnitude. Thus if the relative phase of the excitation coefficients is changed the pattern will scan in azimuth. To scan the pattern by an amount ψ the phase of each mode is changed by an amount equal to (M-2)ψ, M being the order of the mode. Scanning in azimuth also occurs when the frequency is changed because on the spiral antenna a change in frequency is equivalent to a rotation of the antenna structure. Furthermore, the relationship between the amount of scan and the change in frequency is nonlinear [19].

In order to obtain explicit values for the optimum directivity of a spiral and for the associated excitation vector the radiation fields of the spiral modes must be determined. Because this problem is intractable for an actual spiral, models are utilized. In the following chapter two models are considered. They are the radiating ring model and the anisotropic sheet model of the spiral antenna.
CHAPTER IV
IDEAL MODELS OF A PLANAR, EQUIANGULAR, SPIRAL ANTENNA

Introduction

The characteristic spiral modes defined in Chapter II and employed in Chapter III arise from an idealization of the radiation field of a spiral antenna. Thus, nothing is lost in utilizing an ideal model of a spiral to obtain expressions for the radiation fields provided, of course, the fields so obtained approximate those obtained experimentally. Therefore, in this chapter we are concerned with obtaining expressions for the radiation fields of a spiral antenna model in order to be able to calculate specific values of the maximum directivity and its associated excitation vector.

The first model to be considered results from imposing certain experimentally determined conditions on the far fields of the radiation band model described in Chapter II. The application of these conditions results in an assumed equivalent current distribution whose radiation fields are readily obtained.

The second model considered is the anisotropic sheet model proposed and analysed by Cheo et. al [10]. The radiation fields of
this model, which they derived, are given along with a brief description of their analysis.

The anisotropic sheet model and models which are similar to the particular radiating ring model developed herein are the models that are commonly used by many investigators. Many of the references cited contain applications of one or the other of these models. Because of their common usage, it will be of interest to make a comparison of the optimum directivities and excitation current vectors of the two models for various mode configurations.

The Radiating Ring Model

The radiating ring model of a spiral antenna is derived from the radiation band theory described in Chapter II. It was argued in Chapter II that the radiation fields of the characteristic modes of a spiral antenna arise from currents within circular bands on the antenna and it was shown in Chapter II that the mean circumference of these bands is given by

\[ C_n = n \lambda_s \]  

in which \( \lambda_s \) is the wavelength on the spiral and \( n \) characterizes the azimuthal phase variation of the current. Neglecting the currents which do not contribute to the radiation field, the current distribution
on the antenna takes the form of a planar array of \( M \) concentric bands of current; \( M \) being the number of modes excited on the antenna. The radiating ring model is developed by replacing the bands of finite width with rings of infinitesimal width such that the radius of each ring is equal to the mean radius of the band it replaces and the azimuthal variation of the current in both cases is the same. According to the discussion of the excitation current vectors given in Chapter II, the current on the \( n \)-th ring can be written

\[
(73) \quad \overline{I}_n(\rho, \phi) = \overline{I}_n(\rho) e^{j n \phi}
\]

and for the ring model

\[
(74) \quad \overline{I}_n(\rho) = \overline{I}_n \delta(\rho - \rho_n)
\]

with

\[
(75) \quad 2\pi \rho_n = n\lambda_s.
\]

An array of concentric rings can be considered to be a degenerate spiral obtained by setting \( a \) equal to zero in the equation

\[
(76) \quad \rho = \rho_n e^{a\phi}.
\]
In cylindrical coordinates the current \( I_n \) is given by

\[
(77) \quad I_n = \hat{\rho} I_n \cos \alpha + \phi' I_n \sin \alpha
\]

and recalling that

\[
(78) \quad \alpha = \cot \alpha
\]

the current on the \( n \)-th ring is given by

\[
(79) \quad I_n = \phi' I_n
\]

Hence, the current density for the radiating ring model is given by

\[
(80) \quad J(\rho, \phi') = \phi' \sum_n I_n \frac{\delta(\rho - \rho_n)}{\rho} e^{jn\phi'}
\]

A schematic representation of the radiating ring model is shown in Fig. 6.

There are two conditions which must be imposed on the radiation field of the model before the model is complete. These conditions are established by considering the characteristics of the many measured radiation patterns of equiangular, self-complimentary spiral antennas which are displayed in the literature. Examination of a large number of experimentally obtained equiangular, spiral antenna radiation patterns indicates that such an antenna is capable of radiating a circularly polarized field over much of its pattern. For example in Reference 6 axial
ratios of less than 3 dB were obtained over 80% of the angular region of the radiation pattern. Thus, the first condition to be imposed on the radiation fields of an ideal model is that the field must be circularly polarized over the entire pattern. The second condition is related to the first condition by the fact that in almost every case examined the normal component of the electric field at the plane of the spiral \((z=0)\) is zero. Hence, if the radiation field is to be circularly polarized
everywhere it must be zero in the plane of the antenna. It is shown in Appendix III that these conditions will be satisfied by the radiation field of the model if the current distribution on the model is replaced by an identical distribution of circularly polarized multipole sources.

The electric field in the far field of an elementary, circularly polarized, first-order multipole source is derived in Appendix III and is given by

$$E_{\text{S}} = \left[ \hat{\theta} \mp j \hat{\phi} \right] E_s$$

with the scalar $E_s$ given by

$$E_s = \frac{k^2 Z_o I \pm \Delta z}{4\pi} \left[ \cos \theta \sin(\phi - \phi') \pm j \cos(\phi - \phi') \right] \frac{e^{-jkr}}{r} \cos \theta.$$  

For a current density $\overline{J}(\rho, \phi')$ the multipole moment $I \Delta z$ contained in a surface element $\rho d\rho d\phi'$ is $\overline{J}(\rho, \phi') \rho d\rho d\phi'$. Thus, for an azimuthally directed distribution of circularly polarized multipoles the scalar $E_s$ is given by

$$E_s = \frac{k^2 Z_o}{4\pi} \int_s \overline{J}(\rho, \phi') \left[ \cos \theta \sin(\phi - \phi') \right.$$

$$\left. \pm j \cos(\phi - \phi') \right] \frac{e^{-jkr}}{r} \cos \theta \rho d\rho d\phi'.$$

The radiation field of the ring model may be obtained directly from Eq. (83).
The Radiation Field of the Ring Model

The radiation integral of the ring model is obtained by substituting Eq. (80) into Eq. (83) and applying the usual far field approximations. Thus,

\[
E_s = \frac{k^2 Z_0 \cos \theta e^{-jkr}}{4\pi r} \sum_{n} I_n \left\{ \int_{0}^{2\pi} \int_{0}^{\infty} \delta(\rho - \rho_n) \left[ \cos \theta \sin(\phi - \phi') \right. \right.
\]
\[
\left. \pm j \cos(\phi - \phi') \right] e^{jm\phi'} e^{ik\rho \sin \theta \cos(\phi - \phi')} d\rho d\phi' \right\}.
\]

The integration over \( \rho \) is easily carried out. The integration over \( \phi' \) is also easily performed if we make use of the transformation given by

\[
e^{ik\rho \sin \theta \cos(\phi - \phi')} = \sum_{m=-\infty}^{\infty} j^m J_m(k\rho \sin \theta) e^{im(\phi - \phi')}
\]

and by substitution and the rearranging of terms obtain

\[
E_s = \frac{k^2 Z_0 \cos \theta e^{-jkr}}{4\pi r} \sum_{n} I_n \left\{ \sum_{m=-\infty}^{\infty} j^m J_m(k\rho \sin \theta) e^{im\phi} \right\}
\]
\[
\times \int_{0}^{2\pi} \left\{ \cos \theta \sin(\phi - \phi') \pm j \cos(\phi - \phi') \right\} e^{j(n-m)\phi'} d\phi'.
\]

It is a simple matter to show that

\[
\int_{0}^{2\pi} \cos(\phi - \phi') e^{j(n-m)\phi'} d\phi' = \pi e^{j\phi} \delta_{m,n+1}
\]

and that
\[ (88) \quad \int_0^{2\pi} \sin(\phi - \phi') e^{j(n-m)\phi'} d\phi' = \pm j \pi e^{\mp jn} \delta_{m,n+1} \]

where \( \delta_{m,n+1} \) is the Kronecker delta. With the help of Eq. (87) and Eq. (88), \( E_S \) is rewritten as

\[ (89) \quad E_S = \pm \frac{k^2 Z_0 \cos \theta e^{-jkr}}{4r} \sum_n m_n l_n e^{jnp} \left[ J_{n-1}(k\rho_n \sin \theta) - J_{n+1}(k\rho_n \sin \theta) \right] \pm \left[ J_{n+1}(k\rho_n \sin \theta) + J_{n-1}(k\rho_n \sin \theta) \right] \cos \theta \].

Further simplification is possible by utilizing the Bessel function identities

\[ (90) \quad \frac{1}{2} \left[ J_{n-1}(k\rho_n \sin \theta) + J_{n+1}(k\rho_n \sin \theta) \right] = \frac{n}{k\rho_n \sin \theta} J_n(k\rho_n \sin \theta) \]

and

\[ (91) \quad \frac{1}{2} \left[ J_{n-1}(k\rho_n \sin \theta) - J_{n+1}(k\rho_n \sin \theta) \right] = J_n'(k\rho_n \sin \theta) \]

and the fact that

\[ (92) \quad k\rho_n = \frac{n\lambda_s}{\lambda_0} \]

Thus, the radiation field of the radiating ring model of a multimode spiral is given by
It is encouraging to note that the field on the axis ($\theta = 0$) of the ring radiator model is zero for all values of $n$ except $n = 1$. For $n = 1$ the field is maximum on the axis. This is in complete agreement with the experimental results described in many of the references cited.

In Chapter III it was shown that the radiation field of a multi-mode spiral could be written as a sum of the radiation fields of the individual modes. That is, we wrote

$$\vec{E} = \sum_n A_n \overline{E}_n$$

with

$$\overline{E}_n = \left[ \hat{\theta} f_n(\theta) + \hat{\phi} g_n(\theta) \right] e^{-jkr} e^{jn\phi}$$

To obtain the functions $f_n(\theta)$ and $g_n(\theta)$ in terms of the radiation field of the ring radiator model, we simply compare Eq. (96) and Eq. (95) with Eq. (93). Then by letting
(97) \[ A_n = I_n \]

the comparison yields

(98) \[ f_n(\theta) = \pm j \frac{n k^2 Z_o}{2} \left\{ J_n\left(\frac{n \lambda_S}{\lambda_o} \sin \theta\right) + \frac{\lambda_o \cot \theta}{\lambda_S} J_n\left(\frac{n \lambda_S}{\lambda_o} \sin \theta\right) \right\} \cos \theta \]

and

(99) \[ g_n(\theta) = \mp j f_n(\theta) \].

Equation (98) will be used to calculate the maximum directivity and its associated eigenvector for the ring model. The results are presented in the following chapter. For purpose of comparison, the maximum directivity and its associated excitation vector will be calculated using the fields of a second ideal model. The second model is the Anisotropic Sheet model of an equiangular spiral antenna.

The Anisotropic Sheet Model

An analysis of the anisotropic sheet model of the planar, equiangular spiral antenna is given by Cheo, Rumsey and Welch [10]. A supplement to their original work is contained in Reference 20. For completeness, and for the convenience of the reader the technique is outlined here.

The model can be described by considering the self-complimentary case with many spiral elements covering the plane \( z = 0 \).
When the number of elements is taken to be infinite, the antenna assumes the form of an anisotropic sheet which is perfectly conducting in the direction of the elements and perfectly transparent in the orthogonal direction in the plane of the sheet. A solution of Maxwell's equations is obtained by imposing the boundary conditions at the surface and in the feed region and by imposing the radiation condition on a general solution of the scalar wave equation.

The boundary conditions at the surface are satisfied by requiring that \( E \) parallel to the elements vanish while \( E \) tangential to the surface and normal to the element is continuous and by choosing circularly polarized fields such that \( E = j Z_o \mathbf{H} \) above the surface \( (z > 0) \) and \( E = - j Z_o \mathbf{H} \) below the surface \( (z < 0) \). Again, the fields and the source of the fields which is located at the center of the antenna are assumed to have the same azimuthal dependence given by \( e^{j
\phi} \). The electric field satisfies

\[
E = \pm k \nabla \times (\hat{z} U) + \nabla \times \nabla \times (\hat{z} U)
\]

with the minus sign chosen for \( z > 0 \). The function \( U \) satisfies the scalar wave equation

\[
\nabla^2 U + k^2 U = 0
\]

and is expressed as a general solution of Eq. (101) by
\[
U = e^{j\phi} \int_0^{\pi} g(\alpha) J_n(\alpha \rho) e^{\pm jz\sqrt{k^2 - \alpha^2}} \alpha \, d\alpha
\]

again the minus sign is chosen for \( z > 0 \).

From the spiral geometry the condition that \( E \) parallel to the elements be zero gives

\[
E_\rho = E_\phi \quad \text{at} \quad Z = 0.
\]

Substituting from Eq. (100) with the negative sign gives

\[
\left( \frac{\partial^2 U}{\partial z \partial \rho} - \frac{k}{\rho} \frac{\partial U}{\partial \phi} \right) = \frac{1}{\rho} \left( \frac{\partial^2 U}{\partial \phi \partial z} + k \frac{\partial U}{\partial \rho} \right) \bigg|_{z=0}.
\]

Substituting Eq. (101) into Eq. (104) and evaluating the resultant integral gives an ordinary differential equation for \( g(\alpha) \) whose solution is

\[
g = \frac{I_n}{X} \left[ \frac{1 - (1 - x)^{\frac{1}{2}}}{1 + (1 - x)^{\frac{1}{2}}} \right]^{\frac{n}{2}} \frac{1}{\left[ 1 - j \alpha(1 - x)^{\frac{1}{2}} \right]} \left[ 1 - j \frac{n}{a} \right]
\]

in which

\[
x = \frac{\alpha^2}{k^2}
\]

and \( I_n \) is a constant dependent on the source strength. By putting Eq. (105) back into Eq. (104) and evaluating the integral in the
radiation zone the radiation field of the anisotropic sheet model is found to be given by

\[ E = I_n E_n e^{j\psi_n} e^{jn\phi} \]  

with

\[ E_n = \frac{\cot \theta \left[ \tan \frac{1}{2} \theta \right]^n}{\left[ 1 + a^2 \cos^2 \theta \right]^{1/2}} \frac{n \tan^{-1}(a \cos \theta)}{e^{a}} \]  

and the function \( f_n(\theta) \) is given by

\[ f_n(\theta) = E_n e^{j \tan^{-1}(a \cos \theta) + \frac{n}{2a} \ln (1 + a^2 \cos^2 \theta)} \]  

Numerical values for the maximum directivity, its associated excitation vector and the resultant field pattern for both models are obtained in the following chapter.
CHAPTER V
COMPUTED DIRECTIVITY, EXCITATION
AND PATTERNS OF THE SPIRAL MODELS

Introduction

This chapter is devoted to the presentation of numerical results obtained by evaluating the expressions for the maximized directivity function, excitation coefficients and radiation patterns derived in Chapter III. The modal fields of both the ring radiator model and the anisotropic sheet model are used and the results compared.

Before presenting the explicit results it will be beneficial to discuss the mode configurations to be considered and the notation to be used.

It was noted in Chapter IV that the field on the axis of the ring radiator model is zero for all values of n except n = 1. Examination of Eq. (108) shows that this is also true in the case of the anisotropic sheet model. Furthermore the radiation pattern of the lowest order mode (n = 1) is maximum on the axis and for the optimum directivity to occur in the axial direction only the lowest order mode is excited.

In this investigation we are concerned with configurations involving the second and higher order modes. Specifically, the configurations considered are labeled by M and consist of the second through M-th order modes. For example, the configuration with M = 4 consists of
the second, third and fourth order modes. Five multimode configurations are considered; those with M equal to three through seven. The second order mode alone (M = 2) is included for reference. In addition to the number of modes excited, the equations to be evaluated depend upon a parameter which must be specified. For the ring radiator model the parameter is the ratio of the wavelengths \( \lambda_s/\lambda_0 \). The specified values of \( \lambda_s/\lambda_0 \) are 1.0, 0.8, 0.6, 0.4, and 0.2. The parameter for the anisotropic sheet model is the tightness constant \( a \). The specified values of "a" are 0.05, 0.15, 0.30, 0.50, 0.75, and 1.05. These values of the parameters cover the range from what would be called tightly wound spirals to loosely wound spirals.

A total of 66 cases are considered and the results are presented in tabular and graphical form. However, because of the vast amount of data involved only representative plots are shown. The task of performing the necessary calculations would have been formidable and time consuming were it not for the digital computer. The relevant equations were programmed on an IBM 7094 and all 66 cases including the integrations were executed in less than 30 minutes. Flow charts for the programs are given in Appendix III.

In what is to follow the optimum directivity and direction of pattern maximum are determined first after which the excitation coefficients and principal plane radiation patterns are calculated.
Optimum Directivity

The optimum directivity \( (D_0) \) may be determined by solving Eq. (65) to obtain the direction of maximum radiation \( (\theta_d) \) and using this value of \( \theta \) in Eq. (59). However for the purpose of comparing the behavior of the maximized directivity function for different mode configurations it is more revealing to plot the maximized directivity function and obtain \( D_0 \) and \( \theta_d \) graphically.

Substituting Eq. (98) into Eq. (59) gives the maximized directivity function of the ring radiator model which is

\[
D(\theta_o) = \cos^2 \theta_o \sum_{n=2}^{M} \frac{[P_n(\theta_o)]^2}{\frac{1}{2} \int_0^{\pi} [P_n(\theta)]^2 \cos \theta \sin \theta \, d\theta}
\]

where

\[
P_n(\theta) = J_n \left( \frac{n \lambda_S}{\lambda_o} \sin \theta \right) + \frac{\lambda_o \cot \theta}{\lambda_S} J_n \left( \frac{n \lambda_S}{\lambda_o} \sin \theta \right)
\]

Similarly, substitution of Eq. (110) into Eq. (59) gives the maximized directivity function of the anisotropic sheet model which is

\[
D(\theta_o) = \frac{\cot^2 \theta_o}{1 + a^2 \cos^2 \theta_o} \sum_{n=2}^{M} \frac{[Q_n(\theta_o)]^2}{\frac{1}{2} \int_0^{\pi} [Q_n(\theta)]^2 \frac{\cot \theta \sin \theta}{1 + a^2 \cos^2 \theta} \, d\theta}
\]

where
Values of $D_0$ and $\theta_d$ obtained for the various mode configurations and winding parameters of the ring radiator model and the anisotropic sheet model are given in Table I and Table II, respectively. Plots of $D_0$ vs. $\theta_d$ which are representative of the behavior of the maximum directivity function with respect to $\theta_d$ and $M$ are shown in Figs. 7 and 9 for the ring radiator model and in Figs. 8 and 10 for the anisotropic sheet model. The value of $\lambda_S/\lambda_O = 0.8$ used in computing the data for Fig. 7 and the value of $a = 0.05$ used in computing the data for Fig. 8 correspond to models of tightly wound spirals. The value of $\lambda_S/\lambda_O = 0.20$ used in computing the data for Fig. 9 and the value of $a = 0.75$ used in computing the data for Fig. 10 correspond to models of very loosely wound spirals. The optimum directivity calculations for all the mode configurations and winding parameters considered are summarized in Fig. 11 for the ring radiator model and in Fig. 12 for the anisotropic sheet model. The optimum directivity of both tightly wound models ($\lambda_S/\lambda_O = 0.8$, $a = 0.05$) and loosely wound models ($\lambda_S/\lambda_O = 0.2$, $a = 0.75$) are plotted as a function of the mode configurations $M$. The lines connecting the plotted points serve to clearly indicate the trend of the optimum directivity.
| M = | 2 | 3 | 4 |
|-----|-----------------|
| $\lambda_s/\lambda_o$ | $D_0$ | $\theta_d$ | $D_0$ | $\theta_d$ | $D_0$ | $\theta_d$ |
| 1.0 | 2.4 | 34.2 | 4.7 | 40.0 | 7.0 | 43.0 |
| 0.8 | 2.3 | 37.5 | 4.4 | 41.7 | 6.3 | 45.6 |
| 0.6 | 2.2 | 38.2 | 4.2 | 42.4 | 6.1 | 47.1 |
| 0.4 | 2.1 | 38.8 | 4.1 | 44.3 | 5.9 | 49.6 |
| 0.2 | 2.1 | 40.0 | 4.0 | 45.0 | 5.9 | 50.0 |

| M = | 5 | 6 | 7 |
|-----|-----------------|
| $\lambda_s/\lambda_o$ | $D_0$ | $\theta_d$ | $D_0$ | $\theta_d$ | $D_0$ | $\theta_d$ |
| 1.0 | 9.4 | 46.1 | 11.2 | 47.5 | 13.4 | 48.8 |
| 0.8 | 8.3 | 48.8 | 10.4 | 51.4 | 12.3 | 53.4 |
| 0.6 | 8.0 | 50.8 | 10.0 | 54.0 | 11.9 | 56.2 |
| 0.4 | 7.8 | 52.7 | 9.8 | 55.5 | 11.7 | 58.3 |
| 0.2 | 7.8 | 53.5 | 9.8 | 57.0 | 11.7 | 59.5 |
### TABLE II
VALUES OF $D_0$ AND $\theta_d$ FOR THE ANISOTROPIC SHEET MODEL

<table>
<thead>
<tr>
<th>$M$ =</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>$D_0$</td>
<td>$\theta_d$</td>
<td>$D_0$</td>
</tr>
<tr>
<td>0.05</td>
<td>2.2</td>
<td>37.4</td>
<td>4.1</td>
</tr>
<tr>
<td>0.15</td>
<td>2.2</td>
<td>37.5</td>
<td>4.1</td>
</tr>
<tr>
<td>0.30</td>
<td>2.1</td>
<td>38.5</td>
<td>4.1</td>
</tr>
<tr>
<td>0.50</td>
<td>2.0</td>
<td>40.4</td>
<td>4.0</td>
</tr>
<tr>
<td>0.75</td>
<td>1.9</td>
<td>43.6</td>
<td>3.8</td>
</tr>
<tr>
<td>1.05</td>
<td>1.8</td>
<td>49.0</td>
<td>3.8</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$M$ =</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>$D_0$</td>
<td>$\theta_d$</td>
<td>$D_0$</td>
</tr>
<tr>
<td>0.05</td>
<td>8.1</td>
<td>49.3</td>
<td>10.1</td>
</tr>
<tr>
<td>0.15</td>
<td>8.1</td>
<td>49.6</td>
<td>10.1</td>
</tr>
<tr>
<td>0.30</td>
<td>8.0</td>
<td>50.5</td>
<td>10.1</td>
</tr>
<tr>
<td>0.50</td>
<td>8.0</td>
<td>52.2</td>
<td>10.0</td>
</tr>
<tr>
<td>0.75</td>
<td>7.9</td>
<td>55.4</td>
<td>10.0</td>
</tr>
<tr>
<td>1.05</td>
<td>7.9</td>
<td>58.6</td>
<td>10.0</td>
</tr>
</tbody>
</table>
Fig. 7—Maximum directivity vs. $\theta_0$ for six mode configurations on the ring radiator model with $\lambda_s/\lambda_o = 0.80$. 
Fig. 8—Maximum directivity vs. $\theta_o$ for six mode configurations on the anisotropic sheet model with $a = 0.05$. 
Fig. 9--Maximum directivity vs. $\theta_0$ for six mode configurations on the ring radiator model with $\lambda_s/\lambda_0 = 0.20$. 
Fig. 10--Maximum directivity vs. $\theta_0$ for six mode configurations on the anisotropic sheet model with $a = 0.75$. 
Fig. 11--Optimum directivity vs. mode configuration on the ring radiator model with $\lambda_s/\lambda_o = 0.8$ and $\lambda_s/\lambda_o = 0.2$. 
Fig. 12--Optimum directivity vs. mode configurations on the anisotropic sheet model with $a = 0.05$ and $a = 0.75$. 
Extensive calculations using intermediate values of the winding parameters indicate that if the given parameters represent bounds on the winding parameters the resultant directivities represent bounds on the optimum directivities. For example, with $M = 4$ and $a$ between 0.05 and 0.75 the optimum directivity will be between 6.1 and 5.8.

It is most interesting to note that for both models and for the configurations considered the optimum directivity is given to within 10\% by

\begin{equation}
D_0 = 2N
\end{equation}

where $N$ is the number of modes excited. We will postpone further discussion of this result until the conclusion of this chapter.

The direction ($\theta_d$) of the optimum directivity and hence the direction of the pattern maximum as a function of $M$ is shown in Fig. 13 for the ring radiator mode and in Fig. 14 for the anisotropic sheet model. Again, the plotted points represent bounds on the values of $\theta_d$.

With the values of $\theta_d$ given in Table I and Table II we can compute the excitation vector coefficients required to optimize the directivity of the models.
Fig. 13--Direction of optimum directivity vs. mode configurations on the ring radiator model with $\lambda_s/\lambda_o = 0.8$ and $\lambda_s/\lambda_o = 0.2$.

Fig. 14--Direction of optimum directivity vs. mode configuration on the anisotropic sheet model with $a = 0.05$ and $a = 0.75$. 
Excitation Vector Coefficients

Recalling the discussion of Chapter II concerning the characteristic excitations of an equiangular spiral antenna and the results of Chapter III, specifically Eq. (62), the excitation vector for optimum directivity is written as

\[
[I]^\dagger = \sum_{n=2}^{M} \frac{f_n(\theta_d) e^{jn\phi_o}}{ \frac{1}{2} \int_0^\pi |f_n(\theta)|^2 \sin \theta d\theta } [I_n]^\dagger
\]

where \([I_n]^\dagger\) is the conjugate transpose of the characteristic excitation of the \(n\)-th mode and is written as

\[
[I_n]^\dagger = N^{-\frac{1}{2}} \left[ 1, e^{-j \frac{2n\pi}{N}}, e^{-j \frac{4n\pi}{N}}, \ldots, e^{-j \frac{2n\pi}{N}(N-1)} \right]
\]

where \(N\) is the number of spiral elements. From Eq. (116) the excitation vector coefficients are seen to be given by

\[
A_n = \frac{f_n(\theta_d) e^{jn\phi_o}}{ \frac{1}{2} \int_0^\pi |f_n(\theta)|^2 \sin \theta d\theta }
\]

The functions \(f_n(\theta)\) have been determined for the models considered and are given by Eq. (98) for the fing radiator model and by Eq. (108) and Eq. (110) for the anisotropic sheet model.

Using the appropriate functions the excitation vector coefficients for the ring radiator model are given by
where \( P_n(\theta) \) is given by Eq. (112). For the anisotropic sheet model the excitation vector coefficients are

\[
A_n = \frac{P_n(\theta_d) \cos \theta_d e^{jn\left(\phi_0 - \frac{\pi}{2}\right)}}{\frac{k^2Z_0}{4} \int_0^\pi |P_n(\theta)|^2 \cos^2 \theta \sin \theta \, d\theta}
\]

(119)

\[
A_n = \left(\frac{\cot \theta_d}{[1 + a \cos \theta_d]^{\frac{1}{2}}}\right) \frac{Q_n(\theta_d) e^{\frac{j}{2} \tan^{-1}(a \cos \theta) + \frac{\pi}{2} \ln(1 + a^2 \cos^2 \theta_d) + \phi_0}}{\frac{1}{2} \int_0^\pi \frac{[Q_n(\theta)]^2 \cot^2 \theta \sin \theta}{1 + a^2 \cos^2 \theta} \, d\theta}
\]

(120)

where

\[
\beta = \frac{1}{2a} \ln(1 + a^2 \cos^2 \theta_d) + \phi_0
\]

(121)

and \( Q_n(\theta) \) is given by Eq. (114).

In terms of the value of the optimum directivity the choice of \( \phi_0 \) is arbitrary and will affect only the azimuth location of the pattern maximum. To simplify the computation \( \phi_0 \) is chosen to be equal to \( \pi/2 \) for the ring radiator model but for the anisotropic sheet model \( \phi_0 \) is chosen so that \( \beta \), given by Eq. (121), is equal to zero.

To effect a comparison between the various mode configurations, the excitation vector coefficients are normalized such that

\[
[1]^\dagger [1] = 1
\]

(122)
This is easily accomplished and the normalized excitation vector coefficients become

\[
A_n = \frac{A_n}{\left\{ \sum_{n=2}^{M} |A_n|^2 \right\}^{\frac{1}{2}}}
\]

The normalized excitation coefficients for the various mode configurations are given in Table III for the ring radiator model with \( \lambda_s/\lambda_o = 0.8 \) and \( 0.2 \) and in Table IV for the anisotropic sheet model with \( a = 0.05 \) and \( 0.75 \). Values of the normalized excitation vector coefficients for other values of the parameters \( \lambda_s/\lambda_o \) and "a" are given in Appendix IV.

A comparison of the tabulated excitation vector coefficients for both models indicates an expected similarity in that for all the mode configurations and values of the tightness parameters considered the excitation vector coefficients increase monotonically with the order of the excited mode. Of more practical importance, however, is the difference in the magnitude of the coefficients associated with the two models. This difference may be illustrated by comparing the curves of Fig. 15 with those of Fig. 16. Figure 15 shows the logarithm of the ratio of the coefficient of the highest order mode to the coefficient of the second order mode as a function of \( \lambda_s/\lambda_o \) with
<table>
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<th>$M = \frac{\lambda_s}{\lambda_c}$</th>
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<td>$A_2$</td>
<td>0.0751</td>
<td>0.1125</td>
<td>0.1715</td>
<td>0.2970</td>
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<td>$A_3$</td>
<td>0.1570</td>
<td>0.2250</td>
<td>0.3365</td>
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<td>0.8525</td>
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<td>$A_4$</td>
<td>0.2610</td>
<td>0.3670</td>
<td>0.5343</td>
<td>0.7910</td>
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<tr>
<td>$A_5$</td>
<td>0.3865</td>
<td>0.5345</td>
<td>0.7525</td>
<td>----</td>
<td>----</td>
</tr>
<tr>
<td>$A_6$</td>
<td>0.5310</td>
<td>0.7175</td>
<td>----</td>
<td>----</td>
<td>----</td>
</tr>
<tr>
<td>$A_7$</td>
<td>0.6875</td>
<td>----</td>
<td>----</td>
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</table>

$\lambda_s/\lambda_c = 0.20$

<table>
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<th>6</th>
<th>5</th>
<th>4</th>
<th>3</th>
</tr>
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<tbody>
<tr>
<td>$A_2$</td>
<td>0.0002</td>
<td>0.0010</td>
<td>0.0052</td>
<td>0.0279</td>
<td>0.1580</td>
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<tr>
<td>$A_3$</td>
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<td>0.0072</td>
<td>0.0364</td>
<td>0.1860</td>
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<tr>
<td>$A_4$</td>
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<td>0.0414</td>
<td>0.2020</td>
<td>0.9822</td>
<td>----</td>
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<td>0.1935</td>
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<tr>
<td>$A_6$</td>
<td>0.2175</td>
<td>0.9810</td>
<td>----</td>
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</tr>
<tr>
<td>$A_7$</td>
<td>0.9785</td>
<td>----</td>
<td>----</td>
<td>----</td>
<td>----</td>
</tr>
<tr>
<td>$A_m$</td>
<td>$A_{7}$</td>
<td>$A_{6}$</td>
<td>$A_{5}$</td>
<td>$A_{4}$</td>
<td>$A_{3}$</td>
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<td>--------</td>
<td>--------</td>
<td>--------</td>
<td>--------</td>
<td>--------</td>
</tr>
<tr>
<td>$M=0.05$</td>
<td>0.2328</td>
<td>0.2804</td>
<td>0.3500</td>
<td>0.4555</td>
<td>0.6338</td>
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<tr>
<td>$M=0.75$</td>
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<td>0.2712</td>
<td>0.3383</td>
<td>0.4405</td>
<td>0.6210</td>
</tr>
</tbody>
</table>

TABLE IV
NORMALIZED EXCITATION VECTOR COEFFICIENTS
FOR THE ANISOTROPIC SHEET MODEL

$a = 0.05$

$a = 0.75$
Fig. 15--Ratio of the maximum to minimum excitation vector coefficients vs. $\lambda_s/\lambda_o$. 

\[ \frac{\Sigma M - 1}{\Sigma^2} \]
Fig. 16--Ratio of the maximum to minimum excitation vector coefficients vs. "a".
the mode configurations on the ring radiator model as a parameter.

Figure 16 shows the ratio of the coefficient of the highest order mode to the coefficient of the second order mode as a function of the tightness parameter "a" with the mode configurations on the anisotropic sheet model as a parameter. Both the similarity and the large differences between models in the ratio of the coefficients can be explained in terms of the effective radiation resistance of the spiral modes. This will be discussed further in the final chapter. The remainder of this chapter is devoted to the presentation of typical radiation patterns of the models.

Radiation Patterns

The expressions for the radiation patterns of the models of a multimode spiral have been evaluated for the various mode configurations previously described. The elevation patterns ($\phi$ = constant) were computed with the excitation required to produce a pattern maximum in the plane $\phi = 0$. The azimuth patterns ($\theta$ = constant) are computed with $\theta = \theta_d$. Furthermore, all the radiation patterns shown are normalized to the maximum value of the pattern of the particular configuration and parameter shown.

Elevation patterns of the ring radiator model with $M$ equal to 3, 5 and 7 and with $\lambda_s/\lambda_o$ equal to 0.8 are shown in Fig. 17, those
Fig. 17--Elevation patterns of the ring radiator model with $\lambda_s/\lambda_o = 0.8$ and $M = 3, 5$ and 7.

with $M$ equal to 4 and 6 and $\lambda_s/\lambda_o$ equal to 0.8 are shown in Fig. 18. Elevation patterns of the anisotropic sheet model with "a" equal to 0.05 and $M$ equal to 3, 5, and 7 are shown in Fig. 19 and with $M$ equal to 4 and 6 in Fig. 20. These patterns are representative of the elevation patterns of the models for all the tightness parameters considered. It is interesting to note the difference in lobe structure in the direction $\phi$ equal to $\pi$ when the number of modes excited is
A configuration with M even (number of modes is odd) always produces a single lobe with a maximum that is larger than either of the two lobes produced by the configuration with one less mode. Thus if the back lobe to main lobe ratio is to be minimized a multimode spiral should be excited in pairs of one even and one odd ordered mode. The beamwidth of the elevation patterns of both models decreases by approximately $6^\circ$ and the number of modes
Fig. 19--Elevation patterns of the anisotropic sheet model with $a = 0.05$ and $M = 3, 5$ and 7.

excited is increased from two to six. This fact will gain significance when the azimuth patterns are considered. The change in the direction of pattern maximum with a change in mode configuration has been discussed and is shown in Fig. 13 and Fig. 14 for both models. The similarity of the elevation radiation patterns of the models with the same mode configuration but different tightness parameters is illustrated by Figs. 21 and 22. Figure 21 shows the elevation radiation
Fig. 20—Elevation patterns of the anisotropic sheet model with $a = 0.05$ and $M = 4$ and 6.

Patterns of the ring radiator model with $M$ equal to 7 and $\lambda_s/\lambda_0$ equal to 0.8 and 0.2, while Fig. 22 shows the elevation pattern of the anisotropic sheet model with $M$ equal to 7 and "$a" equals 0.05 and 0.75. With the exception of the direction of pattern maximum ($\theta_d$) there is very little difference in the elevation patterns of the spiral models with different tightness parameters. Although the mode configuration with $M$ equal to 7 is the only one shown, no significant
Fig. 21—Elevation patterns of the ring radiator model with $M = 7$ and $\lambda_s/\lambda_0 = 0.8$ and 0.2.

- differences, other than the difference in the value of $\theta_d$, were found in the elevation patterns of the other configurations considered. The shift in $\theta_d$ occurs because in terms of the free space wavelength $\lambda_0$ the radial spacing between modes decreases as the spiral becomes less lightly wound. A careful comparison of the elevation patterns reveals only insignificant differences between the elevation patterns of the two models with the same mode configuration and similar tightness parameters.
Fig. 22—Elevation patterns of the anisotropic sheet model with $M = 7$ and $a = 0.05$ and $0.75$.

Azimuth radiation patterns which are typical of both models for all the values of the tightness parameters considered are shown in Fig. 23. The similarity of the azimuth patterns of both models for the various mode configurations and tightness parameters is illustrated by Fig. 24. Figure 24 shows the azimuth radiation patterns of both models with $M$ equal to 7 and 3 and for the tightness parameters $\lambda_s/\lambda_o$ equal to 0.8 and 0.2 and "$a$" equal to 0.05 and 0.75. For the
Fig. 23--Azimuth patterns of the ring radiator model with $\lambda_s/\lambda_0 = 0.8$ and $M = 3, 4, 5, 6$ and 7.

Fig. 24--Comparison of the azimuth patterns of the ring radiator model and the anisotropic sheet model for two values of $a$ and $\lambda_s/\lambda_0$ with $M = 3$ and 7.
configuration with \( M \) equal to 3 the azimuth patterns for \( \lambda_s/\lambda_0 \) equal to 0.8 and "\( a \)" equal to 0.05 are indistinguishable. The only appreciable difference in the azimuth patterns of a given mode configuration is in the vicinity of the minima.

From a comparison of the change in elevation and azimuth patterns with a change in mode configurations it is apparent that the increase in directivity which accompanies an increase in the number of modes excited on the spiral is predominantly associated with a reduction in beamwidth in the azimuth plane.

Comparisons show that the computed optimum directivities and radiation patterns of the two models are in good agreement. This result is not too surprising if we recall that with excitation for maximum directivity, the directivity in a given direction is the sum of the directivities of the individual modes in that direction. Thus, if the modes have approximately equal directivity and radiate relatively broad patterns the optimum directivity of a spiral antenna may be approximated by a constant times the number of modes excited. The constant is, of course, the maximum directivity of one of the modes. Such an approximation was proposed earlier in this chapter on the basis of computed results.

Although the shapes of the mode radiation patterns are similar they differ in the direction at which maximum radiation occurs in the
radiation intensity in a given direction, and in the total power radiated for a unit input current which is proportional to the radiation resistance. Because of the first two differences the excitation mode currents are proportional to the field strength of that mode in the desired direction. The difference in radiation resistance accounts for the appearance of the integrated radiation intensity in the numerator of the expression for the excitation current coefficients. With the excitation required, the expression for the radiation field appears the same as that of a uniformly illuminated broadside linear array. The optimum directivity of such an array is given by the product of the directivity of an element and the number of elements. Furthermore the radiation pattern of such an array is relatively insensitive to small changes in the element pattern. This explains the similarity in the radiation patterns between different models and between the same model with different tightness parameters.

The large differences in the calculated current ratios particularly between the loosely wound spiral models is attributable to the fact that the excitation vector coefficients are inversely proportional to the radiation resistance of the excited mode. Since the width of the radiating current bands on the ring radiator model was assumed to be infinitesimal the radiation resistance decreases very rapidly as the radius of the ring is reduced. The rate of decrease increases
as the order of the mode is increased. Thus the current required to radiate a given amount of power is much greater for a loosely wound ring radiator model than for a tightly wound model. The current ratios on the anisotropic sheet model are relatively insensitive to changes in the tightness parameter and, as indicated by the slopes of the curves in Fig. 16, to the order of the excited mode. By comparison then, the behavior of the excitation coefficients on the anisotropic sheet model may be associated with the finite width of the radiation bands. That is, increasing the tightness parameter or the order of mode increases the width of the radiation bands. The change in width of the radiation bands has been observed in experiment[6].

In light of the above discussion there is little to recommend the radiating ring model over the anisotropic sheet model of the spiral antenna. However, the ring radiator, or current band model, does offer a clear picture of the radiating mechanism and broad band nature of the spiral antenna.

The similarities of the calculated optimum directivity and radiation patterns of the two models can be used to establish a relationship between the wavelength on the spiral and the tightness parameter. This was done by comparing the many curves and patterns calculated for both models. A curve relating the wavelength
to the tightness parameter is shown in Fig. 25. It should be emphasized that the curve was obtained in a non-rigorous manner and that the experimental point shown is actually an average value[6]. Nonetheless, the fact that the wavelength on a spiral is less than the free space wavelength is clearly established.

![Graph](image)

*Fig. 25--The wavelength ratio $\lambda_0/\lambda_0$ vs. the tightness parameter $a$.*

Determining the amplitude and phase of the excitation of a multimode spiral is easily accomplished. The required excitation is determined by summing the properly weighted characteristic excitation vectors of the spiral. The necessary feed network may then be designed and implemented and used to excite the spiral through an N wire transmission line. A possible configuration for the excitation
of the second and third order modes on a four-arm spiral is shown in Fig. 26. For higher order modes on multiarm spirals the feed networks would of course be somewhat more complex, particularly when the characteristic excitation vectors are unequally weighted as in the case of excitation for optimum directivity.

Fig. 26 -- Possible feed network for exciting the second and third mode on a four-arm spiral.
CHAPTER VI
SUMMARY AND CONCLUSION

Introduction

This chapter is devoted to summarizing the results of this investigation and to a discussion of the conclusions which are drawn from these results. In addition, a possible application for a multimode spiral antenna is suggested.

Summary

The formulation and solution of the problem considered in this investigation are dependent upon the definition of the spiral antenna modes given in Chapter II. The definition was obtained by using experimental evidence and physical reasoning to substantiate the assumption that the characteristic excitation $I_k$ produces a radiation field whose azimuth variation is given by $e^{jk\phi}$.

In Chapter III it was shown that when the radiation field of a multimode spiral antenna is written in terms of the fields of the spiral antenna modes, the directivity function is expressible as a ratio of hermitian quadratic forms. The properties of a ratio of hermitian quadratic forms facilitate the determination of expressions for the maximum directivity and the excitation vector coefficients required to produce it.
For a given number of modes the maximum directivity represents the maximum achievable directivity in a given elevation direction $\theta_o$. The fact that the value of the maximum directivity depends on the choice of $\theta_o$ indicates that a particular choice of $\theta_o$ will yield a value for the directivity which is greater than for any other value of $\theta_o$. This value of the directivity is defined as the optimum directivity. When the spiral antenna is excited to produce optimum directivity the direction of optimum directivity and the direction of pattern maximum coincide. This is not the case when the directivity in some arbitrary direction is maximized. The value of optimum directivity is independent of the azimuth coordinate whereas the relative phase of the excitation vector coefficients depend on the choice of azimuth direction $\phi_o$. Thus if the relative phase of the excitation coefficients is changed the pattern will scan in azimuth.

In order to obtain and compare explicit values for the optimum directivity of a spiral antenna and for the associated excitation vector coefficients the radiation fields of two idealized models of spiral antennas are employed. The radiating ring model is an extension of similar models employed by other investigators. This model is based on an approximation of the current distribution on a spiral antenna. The anisotropic sheet model proposed and analyzed by Cheo, et. al., is based on an approximation of the physical structure of a spiral antenna.
Comparisons indicate that the computed optimum directivities and radiation patterns of the two models are in good agreement. The ratio of the computed excitation vector coefficients do not compare favorably. The unfavorable comparison is attributed to neglecting the finite width of the radiation bands in the formulation of the ring radiator model.

Conclusions

An N element equiangular spiral antenna may be excited by one or more of its N-1 characteristic excitations. The results of this investigation clearly show that, for modes of higher order than the first, a significant increase in the directivity of a spiral antenna can be obtained by the controlled excitation of more than one mode. If M is the number of modes excited, the increase in directivity over single mode excitation is approximately the factor M. Since this result is based on the analysis of idealized models it should be considered an upper limit or design goal in the implementation of a practical spiral antenna.

In addition to being able to maximize the directivity in a particular direction by the use of higher order modes, the beam can be pointed closer to the horizon than heretofore possible if modes of higher order than the second are used. Furthermore, the beam can
be electronically scanned through 360 degrees in azimuth and through a substantial range (although less than 90 degrees) in elevation by controlled excitation of the spiral elements.

The words "controlled excitation" imply the proper weighting of the excitation current vectors of the various modes employed. As shown in Chapter V and indicated by the entries in Table III and Table IV, and the tables in Appendix IV, the excitation vector coefficients for optimum directivity depend on the parameters of the spiral and the number of modes excited. Thus, without some estimate as to the required excitation vector coefficients, achieving optimum directivity in practice would probably require a large number of trial and error experiments. The computed excitation vector coefficients given in the tables fill the need for a first approximation to the required excitation vector coefficients for optimum directivity. On the other hand, since the radiation patterns of the mode configurations considered are rather broad in both azimuth and elevation, minor errors in the excitation vector coefficients will not greatly affect the directivity of the spiral and the computed values might produce results which are close enough for practical purposes. It is the author's opinion that the best estimates of the required excitation vector coefficients for achieving the optimum directivity of a practical spiral
antenna are those computed for the anisotropic sheet model. This opinion is based entirely on the fact that the radiating ring model neglects the width of the radiation bands.

In conclusion the following discussion suggests a possible application of a multimode spiral antenna in a passive DF system.

Kaiser, et. al. [21] have proposed and implemented a passive spiral DF system utilizing the first and second modes on a spiral antenna. The outputs corresponding to the individual modes are fed into a phase comparator. Since the phase difference between the first and second modes is equal to \( \phi \) plus a constant (which can be made equal to zero) the output of the phase comparator is a measure of the azimuth direction of the incoming signal. Elevation information is obtained by amplitude comparison.

A similar system could be implemented utilizing the second and third order modes (or any two consecutively ordered modes) since the phase difference between these modes is also equal to \( \phi \). This system would be capable of obtaining accurate azimuth and elevation DF information in directions nearer the plane of the antenna than would the first and second mode system.

Mode weighting for optimum directivity can be employed to produce nearly equal azimuth patterns for both modes and to improve the amplitude comparison by optimizing the slope at the crossover point.
APPENDIX I
EIGENVALUES AND EIGENVECTORS OF THE
PERMUTATION OPERATOR P

The permutation operator $P$ in an $N$ dimensional space may be
written as an $N \times N$ matrix given by

$$
P = \begin{bmatrix}
0 & 1 & 0 & 0 & \ldots & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 1 \\
1 & 0 & 0 & 0 & \ldots & 0
\end{bmatrix}
$$

(124)

If $I_{\sigma}$ is an eigenvector of $P$ and $\lambda_{\sigma}$ is the corresponding eigenvalue,
$\lambda_{\sigma}$ must satisfy the equation

$$
\text{determinant } |P - \lambda_{\sigma}| = 0 .
$$

(125)

Expanding the determinant of the matrix $|P - \lambda_{\sigma}|$ yields the eigen-
value equation

$$
\lambda_{\sigma}^N = 1
$$

(126)

the solutions of which are the $N$-th roots of unity; that is

$$
\lambda_{\sigma} = e^{\frac{2\pi\sigma}{N}} \quad (\sigma = 1, 2, 3 \ldots N).
$$

(127)
To determine the eigenvectors corresponding to the eigenvalue $\lambda_\sigma$

let

$$I_\sigma = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ \vdots \\ a_N \end{bmatrix}$$

then forming

$$PI_\sigma = \lambda_\sigma I_\sigma$$

we get the set of $N$ equations

$$a_2 = \lambda_\sigma a_1, \quad a_3 = \lambda_\sigma a_2, \quad \ldots \ldots \quad a_N = \lambda_\sigma a_{N-1}$$

One set of eigenvectors may be obtained by choosing $a_1$ to be unity.

This choice gives for the normalized eigenvectors

$$I_\sigma = \begin{bmatrix} 1 \\ j \frac{2\pi}{N} \sigma \\ e^{\frac{4\pi}{N} \sigma} \\ j \frac{2\pi}{N} (N-2) \sigma \\ e^{\frac{2\pi}{N} (N-1) \sigma} \\ \vdots \end{bmatrix}$$
Forming the inner product of the vector \( \mathbf{I}_\omega \) with the vector \( \mathbf{I}_\sigma \) gives

\[(132) \quad [\mathbf{I}_\omega]^* [\mathbf{I}_\sigma] = \sum_{k=1}^{N} \frac{j2\pi(k\sigma-\omega)}{N} = \delta_{\sigma,\omega+N}\]

where \([\mathbf{I}_\omega]^*\) is the complex conjugate transpose of \([\mathbf{I}_\omega]\) and \(\delta_{\sigma,\omega}\) is the Kroenecker delta. Thus, the eigenvectors are orthogonal to one another and are therefore a basis in the \(N\) space.
APPENDIX II

RADIATION FIELD OF A CIRCULARLY POLARIZED MULTipoLE SOURCE

Rumsey[22] has shown that the elementary source of fields which are circularly polarized at every point in the radiation zone consists of coincident infinitesimal electric and magnetic dipoles with dipole moments related by

\[ M_l = \pm j Z_o I I \]

The choice of sign determines the sense of the polarization.

Consider the arrangement of such sources shown in Fig. 27. The vector potentials associated with the electric and magnetic current elements are

\[ A_l = \frac{k M_l}{j 4\pi} h_o^{(2)}(kr) \]

and

\[ F_l = \frac{k M_l}{j 4\pi} h_o^{(2)}(kr) \]

respectively, where \( h_o^{(2)}(kr) \) is the zero-order spherical Hankel function and
Fig. 27--Coordinate system for circularly polarized multipole source.

\[ r = \left[ x^2 + y^2 + \left( z - \frac{\Delta z}{2} \right)^2 \right]^{\frac{1}{2}}. \]

The vector potentials of source 2 are given by Eq. (134) and Eq. (135) with \( \frac{\Delta z}{2} \) replaced by \( -\frac{\Delta z}{2} \) in Eq. (136). The magnetic vector potential of the pair of electric current elements may be written

\[ A_X = A^r_X(x, y, z - \frac{\Delta z}{2}) - A^r_X(x, y, z + \frac{\Delta z}{2}) \].
If the spacing $\Delta z$ is allowed to approach zero while the dipole moment is increased such that the product $\Delta z I \ell$ remains finite the vector potential becomes

\[
A_x \xrightarrow{\Delta z \to 0} - \Delta z \frac{\partial A_x}{\partial z}
\]

(138)

\[
= -j \frac{k^2 I \ell \Delta z}{4\pi} h_1^{(2)}(kr) \cos \theta
\]

(139)

where

\[
r = \left[ x^2 + y^2 + z^2 \right]^{\frac{1}{2}}
\]

(140)

Similarly, for the pair of magnetic current elements we have

\[
F_x = -j \frac{k^2 M \ell \Delta z}{4\pi} h_1^{(2)}(kr) \cos \theta
\]

(141)

It is readily apparent that the form of Eq. (135) and Eq. (141) is independent of the orientation of the dipole moments. Hence for arbitrarily directed current elements we have

\[
\overline{A} = \frac{k^2 I \ell \Delta z}{4\pi j} h_1^{(2)}(kr) \cos \theta
\]

(142)

\[
\overline{F} = \frac{k^2 M \ell \Delta z}{4\pi j} h_1^{(2)}(kr) \cos \theta
\]

(143)
Substitution of Eq. (133) into Eq. (143) gives

\[ F = \pm \frac{k^2 Z_0 \int \Delta z}{4\pi} h_1^{(2)}(kr) \cos \theta. \]  

To obtain the vector potentials in the radiation zone we make use of the large argument form of the spherical Hankel function given by

\[ h_1^{(2)}(kr) \xrightarrow{r \to \infty} - \frac{e^{-jkr}}{kr} \]

and by substitution the vector potentials become

\[ A = j \frac{k \int \Delta z}{4\pi} \frac{e^{-jkr}}{r} \cos \theta \]

\[ F = \pm j Z_0 \overline{A}. \]

Equation (146) and Eq. (147) give the vector potentials of a circularly polarized first-order multipole. In the radiation zone the components of the field are proportional to the components of the vector potentials and because of the \( \cos \theta \) term they will be zero in the plane \( z = 0 \). The radiation field of a circularly polarized first-order multipole source may be determined directly from \( \overline{A} \) when \( \overline{I} \) is known or assumed.
Assuming that $\overline{I}$ is given by

$$\overline{I} = - \hat{\phi} I \sin \phi' + \hat{\theta} I \cos \phi'$$

the spherical-coordinate components of the vector potentials in the radiation zone are

$$A_\theta = j A \cos \theta \sin (\phi - \phi')$$
$$A_\phi = j A \cos (\phi - \phi')$$
$$F_\theta = \pm j Z_o A_\theta$$
$$F_\phi = \pm j Z_o A_\phi$$

where

$$A = \frac{k I I A z}{4 \pi r} e^{-jk r \cos \theta} .$$

The electric field of the electric current source is given by

$$\overline{E}^e = - j \omega \mu \overline{A}$$

while the electric field of the magnetic current source is given by

$$\overline{E}^m = - j \omega \epsilon Z_o (\overline{F} \times \hat{r}) .$$
By direct substitution the electric field of a circularly polarized multipole which is azimuthally directed is found to be given by

(156) \[ \overline{E} = (\theta + j \phi) E_s \]

where

(157) \[ E_s = \frac{k^2 Z_0 I \Delta z}{4\pi} \left[ \cos \theta \sin(\phi - \phi') \pm j \cos(\phi - \phi') \right] e^{-jkr} \frac{\cos \theta}{r} \]

Thus the field is circularly polarized at every point in the radiation zone and is zero in the plane \( z = 0 \).
APPENDIX III
FLOW CHARTS OF THE PERTINENT
COMPUTER PROGRAMS

The computer programs used to evaluate the expressions
obtained in this investigation are straightforward and utilize standard
programming techniques. For this reason, only a block diagram and
two flow charts are used to present a general description of the
pertinent programs.

Figure 28 shows the block diagrams of the subroutines used to
calculate the fields of the Ring Radiator model and the Anisotropic
Sheet Model described in the text. Figure 29 shows the flow chart of
the program for computing the matrix elements $G_N$, and the directivity
as a function of $\theta$. Figure 30 shows the flow chart of the program for
computing the excitation vector coefficients and the elevation and
azimuth radiation patterns. Use of the subroutines allows the same
programs to be used without alterations for both models.

Simpson's rule employing eighty-eight intervals is used to
evaluate the integral in the expression for $G_N$. Since the integrand is
an even function the limits on the integral are taken as $0$ and $\pi/2$.

A list of the symbols used in the flow charts is included as part
of this appendix.
\( T = \frac{\lambda_s}{\lambda_o} \) or "A" in subroutine EFLD.

\( S = \theta \) in subroutine EFLD.

\( N \) = Order of modes.

\( M \) = Number of modes excited.

\( \theta_d \) = Direction at which optimum directivity occurs.

\( A_k \) = Excitation vector coefficient of \( K \)-th mode.

\( A_k^* \) = Normalized excitation vector coefficient of \( K \)-th mode.

\( E_{NK} \) = Electric field of \( K \)-th mode at \( \theta = \theta_d \) with \( \phi = \phi_o \).

\( E_N \) = Electric field of \( K \)-th mode as a function of \( \theta \) with \( \phi = \phi_o \).

\( E_M \) = Total electric field as a function of \( \theta \) with \( \phi = \phi_o \).

\( E_{PM} \) = Total electric field as a function of \( \theta \) with \( \phi = \phi_o - \pi \).

\( E_{NM} \) = Total electric field at \( \theta = \theta_d \) and \( \phi = \phi_o \).

\(|E_M|\) = Normalized value of the magnitude of \( E_M \).

\(|E_{PM}|\) = Normalized value of the magnitude of \( E_{PM} \).

\( F_{NM} \) = Real part of the electric field as a function of \( \phi \) with \( \theta = \theta_d \).

\( F_{PM} \) = Imaginary part of the electric field as a function of \( \phi \) with \( \theta = \theta_d \).

\(|F_M|\) = Normalized magnitude of the electric field as a function of \( \phi \) with \( \theta = \theta_d \).

\( G_N \) = Elements of the matrix \([G]\) given by Equation (40).
Fig. 28--Block diagrams outlining the subroutines for computing the radiation field of the two models.
Fig. 29--Flow chart of the computer program used to compute $G_N$ and $D(\theta)$.
Fig. 30--Flow chart of the computer program used to compute 
\( A_k \) and the elevation and azimuth radiation patterns.
APPENDIX IV
TABLES OF EXCITATION VECTOR COEFFICIENTS

This appendix contains tables of excitation vector coefficients for the ring radiator model and for the anisotropic sheet model of the spiral antenna. The tables are labeled according to the value of $\lambda_s/\lambda_o$ or the tightness parameter "a". Each table gives the normalized excitation vector coefficient $A_m$ for the various mode configurations described in Chapter V.

$\lambda_s/\lambda_o = 1.00$

<table>
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<tr>
<th>$A_m$</th>
<th>$A_2$</th>
<th>$A_3$</th>
<th>$A_4$</th>
<th>$A_5$</th>
<th>$A_6$</th>
<th>$A_7$</th>
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<td>0.2264</td>
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<td>0.4286</td>
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<td>0.7330</td>
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\[ \lambda_s / \lambda_o = 0.60 \]

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\[ \lambda_s / \lambda_o = 0.40 \]

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\[ a = 0.30 \]

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REFERENCES


