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ELECTROMAGNETIC SCATTERING OF NEUTRONS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

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* * * * *

The Ohio State University 1968

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CHAPTER I

BACKGROUND

The effect of the electromagnetic interaction between a neutron and a scattering nucleus can be important in interpretation of scattering data. In this study, the influence of the interaction between the magnetic moment of the neutron and the coulomb field of the target is evaluated. This electromagnetic term is relatively long range (having a $1/r^3$ dependence) and is of spin-orbit form. This latter fact indicates a probable influence on polarization phenomena as well as on the differential scattering cross-section and, in the case of neutron-proton scattering, an influence on many of the measurable double and triple scattering parameters.

The influence of the magnetic moment-coulomb field interaction on particle scattering was first brought to general attention by Mott\(^{(1)}\) when he pointed out the polarizing effects resulting from electron scattering by nuclei. The importance of this force in neutron-nuclei interactions was first pointed out by Schwinger\(^{(2)}\), who calculated in the Born approximation the influence of this interaction on the elastic


\[^{(2)}\text{J. Schwinger, Phys. Rev. 73, 407 (1948).}\]
scattering cross-section and on the neutron polarization. His results indicated a rather pronounced effect for small angle scattering. This magnetic moment-coulomb field interaction is referred to in the literature variously as Mott-Schwinger Scattering, Schwinger Scattering, and simply electromagnetic scattering. This work will use the first cited term and abbreviate it as M-S scattering. Eriksson (3) has pointed out the importance of this electromagnetic scattering for protons. He calculated the effect on polarization for 130 Mev Protons scattered from aluminum, carbon, and iron by a high energy limiting form of the WKB approximation, and concluded the effect was noticeable up to scattering angles of 10°. Heckrotte (4) performed a similar calculation for proton-carbon scattering at 300 Mev and obtained results similar to those of Eriksson. Sample (5) has treated the effect in the neutron scattering problem by handling the electromagnetic interaction as a perturbation and using wave functions obtained from hard-sphere scattering as the zero-order approximation. He gets results entirely consistent with those of other investigators cited. Monahan and Elwyn (6) have found an influence on polarization of neutrons with energies less than 1 Mev


that extends to angles as large as 24°. They have used an approach based on a generalized Born approximation (which allows for the simultaneous evaluation of nuclear and electromagnetic effects) and found the inclusion of electromagnetic scattering to be necessary in interpreting the experimental polarization data of Elwyn\(^7\) et. al. Redmond\(^8\) pointed out that Monahan and Elwyn's first order correction to the Born approximation can be an over correction and proposed a more accurate approximation using a technique whereby a phase shift function is calculated at successive distances from the origin in a step-wise manner.

The modification of the nucleon-nucleon scattering problem by inclusion of the M-S force has received some attention. Garren\(^9\) included the M-S effect in an approximate relativistic calculation of high energy p-p scattering by use of the Born approximation. Breit and co-workers have considered the problem in connection with high energy p-p scattering\(^{10}(11)(12)\) as well as in a more general


\(^{(8)}\) R. F. Redmond, Phys, Rev. 140, B1267 (1965).


context\textsuperscript{(13)(14)}. In the more recent works\textsuperscript{(13)(14)}, Breit pointed out that wave distortion by the nuclear interaction invalidates the Born approximation treatments for low $\ell$ value partial waves, and then calculated some consequences of the force for high $\ell$ waves only (Reference (13)). In addition, the n-p problem was discussed and differences between n-p and p-p scattering introduced by the M-S interaction were pointed out. It was found that for p-p scattering at 147 Mev, the polarization, $P(\theta)$, the correlation coefficient $C_{KP}(\theta)$, and the triple scattering parameter $A(\theta)$ are appreciably affected at small scattering angle although it is doubtful that measurements of $C_{KP}(\theta)$ and $A(\theta)$ can be made with sufficient accuracy to detect this influence. In the case of $P(\theta)$, however, a noticeable improvement in agreement with experimental data was achieved in the range of 8° to 20°. The basic approximation used in all these treatments was the plane-wave Born approximation.

The fact that M-S scattering can significantly influence experimental results has been firmly established in the references cited. In addition, the polarization produced by M-S scattering has been experimentally demonstrated by Voss and Wilson\textsuperscript{(15)} in the


\textsuperscript{(14)} G. Breit, Revs. Mod. Phys. 34, 766 (1962).

\textsuperscript{(15)} R. G. P. Voss and R. Wilson, Phil. Mag. 1, 175 (1956).
scattering of 100 Mev neutrons from uranium. In all cases, the methods which have been used to calculate the effects of the M-S interaction are approximate ones. In most cases the Born approximation has been used. In the analysis of nucleon-nucleon scattering data the effects of M-S scattering have been largely ignored although it is not entirely clear under what conditions they may be influential. It would, therefore, seem very desirable to develop more accurate techniques for treatment of the M-S force and to systematically evaluate the domains of importance relative to this interaction.

The remainder of this paper is divided into three chapters, the next chapter deals with the theory of neutron scattering by spinless targets (neutron-nuclei scattering) and with the theory of the neutron-proton scattering problem. In both instances, a formalism for dealing with M-S scattering is developed. In the second chapter it is also shown that it is feasible to develop and use M-S wave functions in optical model calculations in somewhat the same spirit as coulomb wave functions are used. In the third chapter the results of the formalism are applied to models to determine the regions of importance of the M-S interactions in a systematic manner. In addition some miscellaneous effects are evaluated; namely, the effect on the p wave level widths, the influence on the deuteron binding energy, and the applicability of the general formalisms to the proton-nucleus and proton-proton scattering problems.

The fourth chapter is one of summary and conclusions.

The treatment is nonrelativistic in all cases.
CHAPTER II

THEORY OF MOTT-SCHWINGER SCATTERING

General Formalism In The Neutron-Nucleus Interaction

Preliminaries. The M-S force arises from the interaction of the magnetic moment of one particle with the coulomb field of the other. In the particular case being considered, the anomalous magnetic moment of the neutron and the coulomb field of the nucleus are involved. The general form of the M-S potential is,

$$ V_{MS}(r) = \text{constant} \frac{\overrightarrow{\ell} \cdot \overrightarrow{s}}{r^3}, $$

where $\overrightarrow{\ell}$ and $\overrightarrow{s}$ represent the neutron orbital and spin angular momenta respectively. A derivation of the potential is given in Appendix A.

It would, of course, be possible to solve the Schrödinger Equation numerically with such a potential included. This procedure, however, would result in considerable numerical work primarily because of the long range nature of the force relative to nuclear forces. The potential, $V_{MS}$, extends to distances comparable to electron orbits and, therefore, requires the range of numerical integration of the radial equation for a given partial wave to be extended by orders of magnitude. In addition, the number of partial waves which are necessary to obtain
a satisfactory solution is greatly increased over the pure nuclear force problem.

It appears that a superior method exists for the evaluation of this interaction. The basis for this method appears in a paper by Calogero\(^{(1)}\) although he credits the equations to the earlier fundamental work of Faxen and Holtsmark\(^{(2)}\). For the sake of completeness, that part of Calogero's paper which is pertinent to this study will be reviewed in what follows.

**Review of Calogero's Results.** That part of Calogero's paper essential to this work is easily developed.

Consider the equation for the radial wave function in the form,

\[
\varphi''_l(\rho) + \left[1 - U(\rho) - \frac{L(L+1)}{\rho^2}\right] \varphi_l(\rho) = 0 , \tag{1}
\]

where \(\rho = \sqrt{\frac{2mE}{\hbar^2}}r = kr\), and \(U(\rho) = \frac{2m}{\hbar^2} \frac{V(\rho)}{k^2}\). A comparison of the equivalent integral equation for \(\varphi_l\),

\[
\varphi_l(\rho) = j_l(\rho) - \int_0^\rho \left[ j_l(\rho)n_l(\rho') - n_l(\rho)j_l(\rho') \right] U(\rho') \varphi_l(\rho') \rho'^2 d\rho' , \tag{2}
\]

---


(where \( j_\ell(\rho) \) and \( n_\ell(\rho) \) are the spherical Bessel and Neuman functions\(^{(3)}\)) with a solution of the form,

\[
\varphi_\ell(\rho) = C_\ell(\rho) j_\ell(\rho) - S_\ell(\rho) n_\ell(\rho) ,
\]

leads to the following pair of coupled integral equations for \( C_\ell(\rho) \) and \( S_\ell(\rho) \),

\[
C_\ell(\rho) = 1 - \int_0^\rho n_\ell(\rho') U(\rho') \left[ C_\ell(\rho') j_\ell(\rho') - S_\ell(\rho') n_\ell(\rho') \right] \rho'^2 d\rho' ,
\]

and

\[
S_\ell(\rho) = -\int_0^\rho j_\ell(\rho') U(\rho') \left[ C_\ell(\rho') j_\ell(\rho') - S_\ell(\rho') n_\ell(\rho') \right] \rho'^2 d\rho' .
\]

After differentiation these equations yield two coupled differential equations,

\[
C_\ell'(\rho) = -U(\rho) \rho^2 n_\ell(\rho) \left[ C_\ell(\rho) j_\ell(\rho) - S_\ell(\rho) n_\ell(\rho) \right] ,
\]

and

\[
S_\ell'(\rho) = -U(\rho) \rho^2 j_\ell(\rho) \left[ C_\ell(\rho) j_\ell(\rho) - S_\ell(\rho) n_\ell(\rho) \right] .
\]

\(^{(3)}\) Handbook of Mathematical Functions (Edited by M. Abramowitz and I. A. Stegun) National Bureau of Standards 437 (1965).
In Calogero's paper, little more is done with these equations. Instead a nonlinear equation for \( \tan \delta_{\ell}(r) \) is derived and its consequences are pursued in detail.

\[
\frac{d}{dp} \tan \delta_{\ell}(p) = -U(p)\rho^2 \left[ j_{\ell}(p) - \tan \delta_{\ell}(p) n_{\ell}(p) \right]^2.
\] (8)

For the present purpose, Equations (6) and (7) appear to be more useful. First, they are linear, and therefore are more easily dealt with in terms of both analytical approximations and numerical integration procedures. Further, \( C_{\ell}(p) \) and \( S_{\ell}(p) \) are better behaved functions than \( \tan \delta_{\ell}(p) \) since they are bounded and more slowly varying. In addition, radial wave functions are obtained through use of (6) and (7), and the effect of the M-S potential can be included in optical model calculations in a manner very similar to that used to account for the coulomb interaction.

In the next section, specialization to the M-S potential is made and the method of solution is developed.

**Method of Solution.** For completeness the Schrödinger Equation for the radial wave function is given with the M-S term as the only potential energy term in the Hamiltonian.

\[
\frac{d^2R_{\ell j}}{dp^2} + \frac{2}{\rho} \frac{dR_{\ell j}}{dp} + \left[ 1 - \frac{L(L+1)}{\rho^2} - \frac{\lambda L_{\ell j}}{\rho^3} \right] R_{\ell j} = 0 ,
\] (9)
where

\[ \lambda_{tj} = \frac{Ze^2|\mu_n|k}{mc^2} \beta_{tj} \]

and

\[ Z = \text{atomic number of target nucleus} \]
\[ e = \text{electron charge} \]
\[ \mu_n = \text{neutron magnetic moment} \]
\[ m = \text{neutron mass} \]
\[ c = \text{velocity of light} \]
\[ k = \text{neutron wave number} \]
\[ \beta_{tj} = t \text{ for } j = t + 1/2 \]
\[ = -(t+1) \text{ for } J = t - 1/2. \]

The two values of \( \beta_{tj} \) are the eigen values of the \( \vec{t} \cdot \vec{s} \) operator (see Appendix A).

The resultant equations for \( C_{tj}(\rho) \) and \( S_{tj}(\rho) \) are,

\[ C'_{tj}(\rho) = -\frac{\lambda_{tj}}{\rho} \left[ C_{tj}(\rho) n_{t}(\rho) j_{t}(\rho) - S_{tj}(\rho) n_{t}^2(\rho) \right], \quad (10) \]

and

\[ S'_{tj}(\rho) = -\frac{\lambda_{tj}}{\rho} \left[ C_{tj}(\rho) j_{t}^2(\rho) - S_{tj}(\rho) n_{t}(\rho) j_{t}(\rho) \right]. \quad (11) \]
At this point it will be shown that solutions to (10) and (11) are expressible in terms of solutions to the general Volterra Equation of the second kind and convenient properties of the solutions will be obtained from this result.

Define the functions \( f_{t_j}(\rho) \) and \( g_{t_j}(\rho) \) by,

\[
S_{t_j}(\rho) = t_{t_j} f_{t_j}(\rho) e^{-\lambda_{t_j} \int_{\rho}^{\infty} s_{t_j}(\rho') \frac{n_{t_j}(\rho')}{\rho'} d\rho'},
\]

and

\[
C_{t_j} = g_{t_j}(\rho) e^{-\lambda_{t_j} \int_{\rho}^{\infty} s_{t_j}(\rho') \frac{n_{t_j}(\rho')}{\rho'} d\rho'},
\]

where

\[
t_{t_j} = \tan \delta_{t_j} = \lim_{\rho \to \infty} \frac{S_{t_j}(\rho)}{C_{t_j}(\rho)},
\]

and

\[
\lim_{\rho \to \infty} f_{t_j}(\rho) = 1; \quad \lim_{\rho \to \infty} g_{t_j}(\rho) = 1.
\]

Substitution of these forms into (10) and (11) gives,
\[ f'_{Lj}(\rho) = -\frac{\lambda_{Lj} j^2_{L}(\rho)}{\rho} g_{Lj}(\rho) e^{-2 \lambda_{Lj} \int_{\rho}^{\infty} \frac{j_{L}(\rho') n_{L}(\rho')}{\rho'} d\rho'} \quad \text{, (15)} \]

\[ g'_{Lj}(\rho) = \lambda_{Lj} j^2_{L}(\rho) f_{Lj}(\rho) e^{-2 \lambda_{Lj} \int_{\rho}^{\infty} \frac{j_{L}(\rho') n_{L}(\rho')}{\rho'} d\rho'} \quad \text{. (16)} \]

Integration of (15) with the condition \( \lim_{\rho \to \infty} f_{Lj}(\rho) = 1 \), gives,

\[ f_{Lj}(\rho) = 1 + \frac{\lambda_{Lj} j^2_{L}(\rho)}{\rho} \int_{\rho}^{\infty} g_{Lj}(\rho') e^{-2 \lambda_{Lj} \int_{\rho'}^{\infty} \frac{j_{L}(\rho'') n_{L}(\rho'')}{\rho''} d\rho''} d\rho' \quad \text{. (17)} \]

Similarly, integration of (16) combined with \( f_{Lj}(\rho) \) from (17) gives,

\[ g_{Lj}(\rho) = 1 - \lambda_{Lj} j^2_{L}(\rho) \int_{\rho}^{\infty} e^{-2 \lambda_{Lj} \int_{\rho'}^{\infty} \frac{j_{L}(\rho'') n_{L}(\rho'')}{\rho''} d\rho''} \left[ 1 + \frac{\lambda_{Lj} j^2_{L}(\rho'')}{\rho''} g_{Lj}(\rho'') e^{2 \lambda_{Lj} \int_{\rho''}^{\infty} \frac{j_{L}(\rho'''') n_{L}(\rho'''')}{\rho''''} d\rho''''} \right] d\rho' \quad \text{. (18)} \]
or, after some manipulation,

\[ g_{\ell j}(\rho) = 1 - \lambda_{\ell j} t_{\ell j} \int_{\rho}^{\infty} \frac{n_{\ell'}^{2}(\rho'')}{\rho''} e^{-2\lambda_{\ell j} b_{\ell}(\rho')} d\rho' \]

\[ - \lambda_{\ell j} \int_{\rho}^{\infty} \frac{j_{\ell'}^{2}(\rho'')}{\rho''} s_{\ell j}(\rho'') e^{-2\lambda_{\ell j} b_{\ell}(\rho')} d\rho'' \int_{\rho}^{\infty} \frac{n_{\ell'}^{2}(\rho'')}{\rho''} e^{-2\lambda_{\ell j} b_{\ell}(\rho')} d\rho'', \] (19)

where

\[ b_{\ell}(\rho) = \int_{\rho}^{\infty} \frac{j_{\ell'}(\rho') n_{\ell}(\rho')}{\rho'} d\rho' \]

and the change in the limits and the order of integration is explained in Appendix B.

Equation (19) can be written in the form,

\[ g_{\ell j}(\rho) = 1 - \lambda_{\ell j} t_{\ell j} \int_{\rho}^{\infty} \frac{n_{\ell'}^{2}(\rho'')}{\rho''} e^{-2\lambda_{\ell j} b_{\ell}(\rho')} d\rho' \]

\[ - \lambda_{\ell j} \int_{\rho}^{\infty} H_{\ell j}(\rho, \rho') g_{\ell j}(\rho') d\rho', \] (20)

where
A change of variable, \( \rho' = 1/\omega \), in the two integrals of Equation (20) is necessary to obtain the desired form,

\[
H_{t_j}(\rho, \rho') = \frac{j_{\ell}^{2}(\rho')}{\rho'} e^{2\lambda_{t_j} b_{t}(\rho')} \int_{\rho}^{\rho'} \frac{n_{t}^{2}(\rho'')}{\rho''} e^{-2\lambda_{t_j} b_{t}(\rho'')} d\rho'' . \tag{21}
\]

The form of Equation (22) is that of a Volterra Equation of the second kind. It is known that for \( H_{t_j}(\omega \rho) \) bounded in the region of interest and for the first integral also bounded, a solution in the form of a Liouville-Neuman series exists and converges for all values of \( \lambda_{t_j}^{2} \). The necessary conditions of boundedness are met if

\[
H_{t_j}(\rho, \omega) = \frac{j_{\ell}^{2}(\omega)}{\omega} e^{2\lambda_{t_j} b_{t}(\omega)} \int_{\rho}^{1/\omega} \frac{n_{t}^{2}(\rho'')}{\rho''} e^{-2\lambda_{t_j} b_{t}(\rho'')} d\rho'' . \tag{23}
\]

\( H_{t_j}(\rho, \omega) \) and \( g_{t_j}(\rho) \) are the kernel and the solution of Equation (22), respectively.

\( H_{t_j}(\rho, \omega) \)

\( g_{t_j}(\rho) \)

\( \lambda_{t_j} \)

\( \rho \)

\( \omega \)

\( j_{\ell} \)

\( n_{t} \)

\( b_{t} \)

\( \lambda_{t_j} \)

\( \rho' \)

\( \rho'' \)

\( d\rho'' \)

\( \int \)

\( \frac{1}{\omega} \)

\( \omega \)

\( \lambda_{t_j} b_{t} \)

\( \lambda_{t_j}^{2} \)

\( \rho' \)

\( \rho'' \)

\( d\rho'' \)

\( \int \)

\( \frac{1}{\omega} \)

\( \omega \)

\( \lambda_{t_j} b_{t} \)

\( \lambda_{t_j}^{2} \)

\( \rho' \)

\( \rho'' \)

\( d\rho'' \)

\( \int \)

\( \frac{1}{\omega} \)

\( \omega \)

\( \lambda_{t_j} b_{t} \)

\( \lambda_{t_j}^{2} \)

\( \rho' \)

\( \rho'' \)

\( d\rho'' \)

\( \int \)

\( \frac{1}{\omega} \)

\( \omega \)

\( \lambda_{t_j} b_{t} \)

\( \lambda_{t_j}^{2} \)

\( \rho' \)

\( \rho'' \)

\( d\rho'' \)

\( \int \)

\( \frac{1}{\omega} \)

\( \omega \)

\( \lambda_{t_j} b_{t} \)

\( \lambda_{t_j}^{2} \)

\( \rho' \)

\( \rho'' \)

\( d\rho'' \)

\( \int \)

\( \frac{1}{\omega} \)

\( \omega \)

\( \lambda_{t_j} b_{t} \)

\( \lambda_{t_j}^{2} \)

\( \rho' \)

\( \rho'' \)

\( d\rho'' \)

\( \int \)

\( \frac{1}{\omega} \)

\( \omega \)

\( \lambda_{t_j} b_{t} \)

\( \lambda_{t_j}^{2} \)

\( \rho' \)

\( \rho'' \)

\( d\rho'' \)

\( \int \)

\( \frac{1}{\omega} \)

\( \omega \)

\( \lambda_{t_j} b_{t} \)

\( \lambda_{t_j}^{2} \)

\( \rho' \)

\( \rho'' \)

\( d\rho'' \)

\( \int \)

\( \frac{1}{\omega} \)

\( \omega \)

\( \lambda_{t_j} b_{t} \)

\( \lambda_{t_j}^{2} \)

\( \rho' \)

\( \rho'' \)

\( d\rho'' \)

\( \int \)

\( \frac{1}{\omega} \)

\( \omega \)

\( \lambda_{t_j} b_{t} \)

\( \lambda_{t_j}^{2} \)

\( \rho' \)

\( \rho'' \)

\( d\rho'' \)

\( \int \)

\( \frac{1}{\omega} \)

\( \omega \)

\( \lambda_{t_j} b_{t} \)

\( \lambda_{t_j}^{2} \)

\( \rho' \)

\( \rho'' \)

\( d\rho'' \)

\( \int \)

\( \frac{1}{\omega} \)

\( \omega \)

\( \lambda_{t_j} b_{t} \)

\( \lambda_{t_j}^{2} \)

\( \rho' \)

\( \rho'' \)

\( d\rho'' \)

\( \int \)
the point \( \rho = 0 \) is excluded from the region of integration in the integral comprising the coefficient of \( \lambda_{lj} t_{lj} \), and if, in addition, \( \omega \) itself is bounded in the expression for \( H_{lj}(\rho, \omega) \). This latter condition results because \( b_{L}(x) \) and \( c_{L}(x) \) behave as \(-1/x\) and \( 1/x^{2L+2} \) as is shown later (Equations 41 and 42). For all other values of \( \rho \) and \( \omega \) the functions are well-behaved.

The solution to Equation (22) is then given by (4),

\[
g_{lj}(\rho) = \sum_{n=0}^{\infty} (\lambda^{2})^{n} g_{ljn}(\rho),
\]

where

\[
g_{lj0}(\rho) = 1 - \lambda_{lj} t_{lj} \int_{0}^{1/\rho} \frac{2(n+1)}{\omega} e^{-2\lambda_{lj} b_{L}(\omega)} d\omega,
\]

and

\[
g_{ljn}(\rho) = -\int_{0}^{1/\rho} H_{lj}(\rho, \omega) g_{ljn-1}(\omega) d\omega.
\]

For these results, it is not difficult to see that the general form for \( g_{lj}(\rho) \) can be expressed as,

\[
g_{lj}(\rho) = t_{lj} G_{lj}^{(1)}(\rho) + G_{lj}^{(2)}(\rho),
\]

where \( G_{lj}^{(1)} \) and \( G_{lj}^{(2)} \) are independent of \( t_{lj} \).
It can be seen from Equation (17) that $f_{\ell j}(\rho)$ is also well behaved away from the origin, and that the dependence of $f_{\ell j}(\rho)$ on $t_{\ell j}$ can be expressed as,

$$f_{\ell j}(\rho) = \frac{1}{t_{\ell j}} F_{\ell j}^{(1)}(\rho) + F_{\ell j}^{(2)}(\rho).$$  \hspace{1cm} (28)

In principle, Equations (24), (25), (26), and (17) could be used to determine the radial wave function with the M-S term present. However, the numerical work involved in this method is great (assuming several terms are required in the series for sufficient accuracy) and the method is inferior to a numerical integration of the differential equation form. It might appear that a good deal of effort has gone into the development of a formulation that is not useful. It is important to note, however, that the establishment of the forms (27) and (28) will prove essential in obtaining the differential equation formulation.

It is worth noting that although boundary values used in the present formulation are the asymptotic values, it is possible to formulate the problem in terms of quantities prescribed at some suitable point near the origin. This cut-off radius would, of course, be an additional parameter, and for that reason the formulation given here was deemed the more useful.

By using Equations (27) and (28) in Equations (15) and (16) and equating coefficients of like powers of $t_{\ell j}$, the following set of equations result,
The appropriate asymptotic values are given through Equations (27) and (28), and the expression immediately following Equation (14). They are,

\[
\begin{align*}
F_{l,j}^{(1)} (\rho) &\sim 0, \quad G_{l,j}^{(1)} (\rho) \sim 0 \\
F_{l,j}^{(2)} (\rho) &\sim 1, \quad G_{l,j}^{(2)} (\rho) \sim 1
\end{align*}
\]  

Now, one can presumably solve the set of Equations (29) through (33) at least numerically and then if the phase shift
function is known at the nuclear radius, \( \rho_c \), \( (\tan \delta_{t_j}(\rho_c)) \equiv \)

\[
\frac{S_{t_j}(\rho_c)}{C_{t_j}(\rho_c)}
\]

Equations (12) and (13) give,

\[
t_{t_j} = \frac{F_{t_j}^{(1)}(\rho_c) e^{-2\lambda_{t_j} b_{t_j}(\rho_c)}}{\tan \delta_{t_j}(\rho_c) G_{t_j}(\rho_c) - F_{t_j}^{(2)}(\rho_c) e^{-2\lambda_{t_j} b_{t_j}(\rho_c)}} - \tan \delta_{t_j}(\rho_c) C_{t_j}(\rho_c).
\]

(34)

In the next section, approximate analytical solutions to Equations (29) through (33) are developed and the resulting formula for Equation (34) is compared to various approximations in the literature. A discussion of the numerical integration of the equations is postponed until later.

**Approximate Solutions.** In order to establish the basis for the approximate solutions, it is necessary to consider the behavior of certain integrals which are involved. Explicit forms for these integrals are given by Redmond(5) and are given here for convenience. To facilitate later reference, the symbols \( a_{t_j}(\rho) \), and \( c_{t_j}(\rho) \) are introduced here.

\[
a_{t_j}(\rho) = \int_\rho^\infty \frac{j_{t_j}^2(\rho')}{\rho'} \, d\rho' = \\
\frac{1}{2 \lambda (\lambda + 1)} \left\{ 1 - \rho^2 \left[ (1 - \lambda^2) j_{t_j}^2(\rho) + j_{t_j-1}^2(\rho) - 2 \lambda \rho^{-1} j_{t_j}(\rho) j_{t_j-1}(\rho) \right] \right\}.
\]

---

$$b_\ell (\rho) = \int_\rho^\infty \frac{j_\ell (\rho') n_\ell (\rho')}{\rho'} \, d\rho' =$$

$$\frac{-\rho^2}{2\ell (\ell + 1)} \left\{ (1 - \rho^{-2}) j_\ell (\rho) n_\ell (\rho) + j_{\ell -1}(\rho) n_{\ell -1}(\rho) \right.$$ 

$$- \left[ j_\ell (\rho) n_{\ell -1}(\rho) + j_{\ell -1}(\rho) n_\ell (\rho) \right] \rho^{-1} \right\} \cdot \quad (36)$$

$$c_\ell (\rho) = \int_\rho^\infty \frac{n_\ell^2 (\rho')}{\rho'} \, d\rho' =$$

$$\frac{1}{2\ell (\ell + 1)} \left\{ 1 - \rho^2 \left[ (1 - \rho^{-2}) n_\ell^2 (\rho) + n_{\ell -1}^2 (\rho) - 2 \rho^{-1} n_\ell (\rho) n_{\ell -1}(\rho) \right] \right\} . \quad (37)$$

By using the small \( \rho \) \( (i.e. \rho \ll \ell) \) limiting forms for the functions \( j_\ell (\rho) \) and \( n_\ell (\rho) \), \( (3) \)

$$\lim_{\rho \to \infty} j_\ell (\rho) \to \frac{\rho^\ell}{1.3.5.7-\cdots-(2\ell+1)} \quad (38)$$

and

$$\lim_{\rho \to \infty} n_\ell (\rho) \to -\frac{1.3.5.7-\cdots-(2\ell-1)}{\rho^{\ell+1}} \quad , \quad (39)$$

it is relatively easy to establish the behavior of the three functions for small \( \rho \) values.
The function \( a_\ell(p) \) is seen to be constant (for a fixed \( \ell \)) at small \( p \), whereas \( c_\ell(p) \) changes rapidly and is singular at the origin.

The necessary condition that the exponential factors in Equations (29)-(32) be replaced by unity is \( 2\lambda_{ij}b_\ell(p) \ll 1 \). Using the definition of \( \lambda_{ij} \) following Equation (9) with the small \( p \) approximation to \( b_\ell(p) \) leads to a statement of the necessary condition as, \( r \gg 3 \times 10^{-14} \text{ cm} \) where \( r \) is the radial distance from the center of the nucleus. As this quantity is smaller than a nucleon radius, one can conclude that the exponential factor can be replaced by unity if \( k \) is not too large, since it is necessary that \( kr \ll \ell \).

Now if one assumes in Equation (29), that \( G_{ij}^{(2)} \) is well represented by its asymptotic value of unity (numerical results verify this assumption for \( p \) not too small), then

\[
F_{ij}^{(1)}(p) = \lambda_{ij}a_\ell(p). \quad (43)
\]
Also, using this result in Equation (30) and noting (Equation (35)) that over most of the range where $a_\ell(\rho)$ is appreciably different from zero, $a_\ell(\rho) \approx a_\ell(0)$, one gets,

$$G_{\ell j}^{(2)}(\rho) = 1 - \lambda_{\ell j}^2 a_\ell(0) c_\ell(\rho). \quad (44)$$

In applying the same kind of approximations to Equations (31) and (32), one finds,

$$F_{\ell j}^{(2)}(\rho) = 1, \quad (45)$$

and

$$G_{\ell j}^{(1)}(\rho) = -\lambda_{\ell j} c_\ell(\rho). \quad (46)$$

The range of validity of these approximations can be determined by examining the result of substituting Equations (43) through (46) into the differential Equations (29) through (32).

In order for Equations (43) and (44) to be good approximations, it can be seen that $|\lambda_{\ell j}^2 a_\ell(0) c_\ell(\rho)| \ll 1$ should be true. More explicitly, using Equation (40) and (42) in this inequality, one gets for a necessary condition,

$$\rho^{2\ell+2} \gg \frac{\lambda_{\ell j}^2 [(2\ell-1)!!]^2}{4\ell(\ell+1)^2}. \quad (47)$$
For Equations (45) and (46) to be valid, it is necessary that
\[ F_{l,j}^{(2)} \] be small, or
\[ \left| \lambda_{l,j}^2 \frac{j_{l}(\rho)}{\rho} c_{l}(\rho) \right| \ll 1, \]
which can be expressed through use of Equation (42) as,
\[ \rho^3 \gg \frac{\lambda_{l,j}^2}{(2l+1)^2(2l+1)}. \] (48)

It is interesting to use these approximate results in Equation (34), and to compare the resulting form with other approximations.
The result of Equations (43) through (46) substituted in Equation (34) gives, after a little rearranging,
\[ t_{l,j} = \frac{\tan \delta_{l,j}(\rho_c) - \lambda_{l,j} a_{l}(\rho_c) e^{-2\lambda_{l,j} b_{l}(\rho_c)} - \lambda_{l,j}^2 \tan \delta_{l,j}(\rho_c) a_{l}(\rho_c) c_{l}(\rho_c)}{e^{-2\lambda_{l,j} b_{l}(\rho_c)} + \tan \delta_{l,j}(\rho_c) \lambda_{l,j} c_{l}(\rho_c)} \]. (49)

For comparison, the plane-wave Born approximation gives,
\[ t_{l,j} = \tan \delta_{l,j}(\rho_c) - \lambda_{l,j} a_{l}(\rho_c), \] (50)
and Monahan and Elwyn\(^{(6)}\) derive,

The Monahan and Elwyn result is very much like Equation (49) and in fact, they are identical through terms of the order of $\lambda_{l_j}$ if $\lambda_{l_j} b_{l} (\rho_{c})$ can be well approximated by zero. It has already been established that this is a good approximation.

The same approximations give, for the radial wave function outside the nuclear radius,

$$
R_{l}(\rho) = 1 - \lambda_{l_j}^{2} a_{l}(\rho) c_{l}(\rho)
$$

$$
- \lambda_{l_j} c_{l}(\rho) \left[ \frac{\tan \delta_{l_j}(\rho_{c}) - \lambda_{l_j} a_{l}(\rho_{c}) - \lambda_{l_j}^{2} a_{l}(\rho_{c}) c_{l}(\rho_{c}) \tan \delta_{l_j}(\rho_{c})}{1 + \lambda_{l_j} \tan \delta_{l_j}(\rho_{c}) c_{l}(\rho_{c})} \right] n_{l}(\rho).
$$

(52)

**Numerical Solutions.** Equations (29) through (32) with the boundary conditions, Equation (33), can, of course, be solved exactly numerically. Inspection shows the two pairs of coupled equations to be identical. That is Equations (29) and (30) are identical to Equations (31) and (32), so the same program will solve each set and
it is only necessary to apply the appropriate boundary conditions. For convenience in applying the boundary conditions, a change of variable was made as, \( z = 1/\rho \). Thus, the asymptotic values of the functions were applied at \( z = 0 \) and the numerical integrations performed outward to a desired value of \( z \) on the CDC-6400 digital computer. The method used was the well-known Runge-Kutta Method as applied to simultaneous equations.\(^7\) The inherent error in this method is of the order of \((\Delta z)^5\), or for the cases done here, \(10^{-10}\) or less since the largest value of \( \Delta z \) ever used was \(10^{-2}\).

It is perhaps appropriate to point out here that this procedure requires much less numerical work than would direct integration of the Schrödinger Equation. Further, it provides a better physical basis for interpreting various approximate methods.

It is clear that the general technique described here could be applied to the scattering of protons by nuclei. To make this application it is only necessary to replace the spherical Bessel functions, \( j_{\ell}(\rho) \) and \( n_{\ell}(\rho) \) by the appropriate coulomb wave functions, usually designated as \( F_{\ell}(\rho) \) and \( G_{\ell}(\rho) \). The approximate solutions are not as useful in this case since the analogues to Equations (35)-(37) are not known. It is perhaps also true that the effects are not as interesting in the proton case due to the large small-angle cross-

section that occurs for coulomb scattering which would tend to obscure the M-S influence on polarization and differential cross section, both of which are most influenced at small scattering angles.

The Optical Model. One of the objectives of this study was to assess the effects of the M-S force on neutron scattering. The most straightforward way to do this seemed to be to compare the results of optical model calculations with and without the M-S term in the potential. To this end, the optical model program, CLOUDY, written and used by Cassola and Koshel\(^{(8)}\) in optical model studies, was obtained and modified to include the M-S term when desired. The form of the nuclear potential used for this problem was,

\[
V(r) = - V_c f(r) - i W_c g(r) + V_{so} \left( \frac{1}{r} \frac{df}{dr} \right) \sigma \cdot \hat{L} \left( \frac{\hbar}{m_c} \right)^2 ,
\]

where \(f(r)\) is the Woods-Saxon form,

\[
f(r) = \left[ 1 + \exp \left( \frac{r-R}{a_s} \right) \right]^{-1} ,
\]

and \(g(r)\) is a surface absorption term of the form,

\[
g(r) = 4 \exp \left( \frac{r-R}{a_D} \right) \left[ 1 + \exp \left( \frac{r-R}{a_D} \right) \right]^{-2} .
\]

No attempt was made to find optimum optical model parameters based on experimental data, but rather, the parameters chosen were based on the nonlocal parameters of Perey and Buck\(^9\) as evaluated by Elwyn et al.\(^{10}\) The basic parameters used in the calculations were,

\[
\begin{align*}
V_c &= 47.20 - 0.27E\;\text{Mev}, \\
W_c &= 9.6\;\text{Mev}, \\
V_{so} &= 7.2\;\text{Mev}, \\
\alpha_D &= 0.66\;\text{fm}^{-1}, \\
\alpha_s &= 0.47\;\text{fm}^{-1}, \text{ and} \\
R &= 1.27\;\text{A}^{1/3}\;\text{fm}.
\end{align*}
\]

For neutron energies less than about 5 Mev, the effects of compound-elastic scattering must be included. In this process the neutron is absorbed by the nucleus to form a compound state, and emitted without change in energy in times too short to distinguish this process from shape-elastic scattering. In the optical model, this contribution is included in the reaction cross-section. Compound-elastically scattered neutrons are unpolarized and have a different angular distribution than those scattered via the shape-elastic process, and hence exert an influence on the experimentally determined polarizations and cross-sections. To account for this effect


In the present calculations, a simple model used by Percy and Buck\(^\text{(9)}\) was adopted. In this model, the total compound-elastic cross-section, \(\sigma_{\text{CE}}\), was taken to be the optical model reaction cross section, \(\sigma_r\), less the experimentally determined nonelastic cross section, \(\sigma_N\). The process was assumed to be isotropic, and the polarization is then

\[
\frac{d\sigma_s(\theta)}{d\Omega} \text{diluted by the ratio } \frac{d\sigma_s(\theta)}{d\Omega} \frac{\sigma_{\text{CE}}}{4\pi}, \text{ where } \frac{d\sigma_s(\theta)}{d\Omega} \text{ is the shape elastic differential cross section.}
\]

The optical-model calculations determine the phase shifts, \(\delta_{Lj}(\rho_c)\) to be used in Equation (34). That is, by using the optical model program to solve the Schrodinger Equation numerically out to \(\rho_c\) with the complex potential as described and including the M-S potential, the phase shifts \(\delta_{Lj}(\rho_c)\) can be found. This, of course, is the phase shift that would result if the potential included in its determination were the entire potential. In the present application, it is the argument of the phase shift function, \(\tan \delta_{Lj}(\rho)\), at \(\rho_c\). The actual tangent of the phase shift, then, as given by Equation (34), is the large \(r\) limit of the phase shift function. The cut-off value, \(\rho_c\), must only be large enough to include all effects of the nuclear potential.

Since the interaction is assumed to be independent of the spin of the target nucleus, it is permissible to treat the scattering as being off a target of zero spin. The treatment of scattering of spin 1/2 particles by spin zero targets is well known and the results are
readily available in the literature. (11)(12)(13) If the scattering is analyzed in terms of partial waves, the pertinent results for scattering of an unpolarized beam can be expressed as,

\[
\sigma_s = \frac{n}{k^2} \sum_{\ell=0}^{\infty} \left\{ (\ell+1)\left[1 - \eta^+_{\ell} |^2 + \ell(1 - |\eta^-_{\ell}|^2) \right] \right\}, \quad (56) \\
\sigma_r = \frac{n}{k^2} \sum_{\ell=0}^{\infty} \left\{ (\ell+1)(1 - |\eta^+_{\ell}|^2) + \ell(1 - |\eta^-_{\ell}|^2) \right\}, \quad (57)
\]

\[
\frac{d\sigma_s(\theta)}{d\Omega} = |g(\theta)|^2 + |h(\theta)|^2, \quad (58)
\]

\[
\bar{P}_{S.E.}(\theta) = \frac{2 \text{Re} \ g^*(\theta) \ h(\theta)}{d\sigma_s(\theta)} \quad \frac{\vec{n}}{d\Omega}
\]

\[
(\vec{n} = \vec{k}_i \times \vec{k}_f)
\]

where

\[
g(\theta) = \frac{1}{2ik} \sum_{\ell=0}^{\infty} \left\{ (\ell+1)(\eta^+_{\ell} - 1) + \ell(\eta^-_{\ell} - 1) \right\} P_{\ell}(\cos \theta), \quad (60)
\]


\[
\begin{align*}
\hat{h}(\theta) &= + \frac{1}{2k} \sum_{l=0}^{\infty} \left( \eta^+_l - \eta^-_l \right) P^1_L(\cos \theta), \\
\end{align*}
\]

(61)

where

\[
\begin{align*}
\eta^+_l &= \eta^0_l, \ l+1/2 = e^{2i \delta, l+1/2}, \quad \text{and} \\
\eta^-_l &= \eta^0_l, \ l-1/2 = e^{2i \delta, l-1/2}.
\end{align*}
\]

The observable quantities defined on the left-hand side of the above equations are,

- \(\sigma_s\) - total scattering cross section
- \(\sigma_r\) - reaction cross section
- \(\frac{d\sigma_s(\theta)}{d\Omega}\) - differential scattering cross section

and \(P(\theta)\) is the neutron polarization after scattering. Further, \(\text{Re}\) denotes the real part of the quantity involved, and \(P^1_L(\cos \theta)\) and \(P^1_L(\cos \theta)\) represent Legendre polynomials and associated Legendre polynomials respectively.

Now it happens that in Equation (61), the series converges slowly when M-S scattering is included. However, for sufficiently large \(l\), one expects the Born approximation to give accurate results, hence, one can include the effect of all \(l\) above some appropriate cut-off value, \(l_c\). This type of correction was carried out by Sample\(^{14}\) and is derived in Appendix C. The resulting equation for \(\hat{h}(\theta)\) is,

\[ h(\theta) = \frac{i}{2k} \sum_{\ell=0}^{\ell_c} \left\{ \left[ \eta_{\ell}^+ - \eta_{\ell}^- \right] + \left( \frac{2\ell+1}{\ell(\ell+1)} \right) i\gamma k \right\} p_{\ell}^1(\cos \theta) - \frac{i\gamma}{2} \cot \frac{\theta}{2}. \]  \hspace{1cm} (62)

In this expression, \( \gamma \) is a constant related to the M-S interaction and is,

\[ \gamma = |\mu_n| Z \frac{e^2}{mc^2}. \]  \hspace{1cm} (63)

In all other sums, the convergence is rapid, and one may achieve sufficient accuracy by truncating the series with a reasonable value of \( \ell_c \).

In addition to the two extremes of no M-S scattering and numerical treatment including M-S scattering, it is also of interest to include the results of the Born approximation for comparison. The Born approximation is derived in Appendix C (See also Wolfenstein\(^{(15)}\) and Reference (12)). In the Born approximation,

\[ t_{\ell j} = \tan \delta_{\ell j}(\rho_c) - \lambda_{\ell j} a_{\ell}(\rho_c), \]  \hspace{1cm} (50)

and the M-S contribution to \( h(\theta) \) is,

\[ h_{\text{MS}}(\theta) = -\frac{i\gamma}{2} \cot \frac{\theta}{2}. \]  \hspace{1cm} (64)

\(^{(15)}\) L. Wolfenstein, Annual Reviews of Nuc. Sci. 6, 43 (1956).
In the application here, $t_{lj}$ was found from Equation (30) and the calculations were carried out as in the exact case.

In applying the optical model, the $\rho_c$ and $\ell_c$ values suggested by Hodgson (16) are, $\ell_c = 2.4 \rho_N$ and $\rho_c = \rho_N + 7 \rho_s$, where $\rho_N$ corresponds to the nuclear radius, $R$. In addition, it was determined that ~100 mesh points within the nuclear radius gave good results (i.e. changing from 100 to 200 mesh points made only negligible changes in the results). Actually, in most cases $\ell_c$, $\rho_c$, and the number of mesh points were chosen substantially greater than these values given above. It was also found that moderate variation of the charge radius about the nuclear radius had no appreciable influence on the results so the charge radius was taken to be the nuclear radius in all cases.

General Formalism In The Neutron-Proton Interaction

Preliminaries. The treatment of the M-S interaction in the nucleon-nucleon scattering problem is a somewhat different matter than that of the nucleon-nucleus problem. The essential difference results because the influence of particle spin occurs in a more involved way. For p-p scattering, since the magnetic moment of each particle is acted on by the coulomb field of the other, the M-S interaction is dependent on $\ell \cdot s$ where $s$ is the total spin of the system, and for n-p scattering, the analogous term is $\ell \cdot s_n$ which does not

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commute with total spin. Of course, the nuclear force part of the Hamiltonian also contains a dependence on total spin, (the tensor force; a spin-spin force etc.) and that fact alone introduces new scattering amplitudes into the problem.

The question of interest in nucleon-nucleon scattering concerns any possible influence on the interpretation of experimental data which might incur if M-S scattering were included in a correct manner. The current method of using nucleon-nucleon scattering data (cross-section, polarization, depolarization, etc.) is to attempt to find that set of phase shifts which best represents the entirety of experimental results. The best result here is defined in terms of chi-squared minimization. Inclusion of the M-S term in the p-p scattering problem would bring about no changes in the general formalism. That is, the forms for the scattering amplitudes would not be changed, and of course, no change in the chi-squared minimized phase shifts would result, except possibly for the relatively minor influence of large $l$ partial waves where the usual practice is to include their contribution through analytical treatment of the one-pion exchange potential. Breit has included the large $l$ partial wave effect of the M-S force in the p-p scattering problem and found it to be small,

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but not negligible. Although the charge-independent nuclear Hamiltonian introduces no spin-mixing, on the other hand, for n-p scattering, inclusion of the M-S force mixes the singlet and triplet spin states (consequently the two isotopic spin states are also mixed) and does change the interpretation of the phase shifts since new scattering amplitudes are introduced. The usual practice is to regard this coupling between spin states to be negligibly weak and, in fact to neglect the M-S interaction entirely.\(^{(18)(19)}\) There are good reasons for neglecting this effect in the phase shift analyses. First, in contrast with p-p scattering, there is insufficient n-p scattering data to allow one to effect a reliable determination of phase shifts.\(^{(20)}\) Further, essentially all the data is at scattering angles somewhat larger than one expects the M-S influence to be evident. Since the nuclear spin-orbit force is of relatively short range so it is less important at lower energies,\(^{(21)}\) one might expect spin-orbit force dependent effects to be influenced by the relatively weak M-S force in this lower energy range.


The consideration that inclusion of the M-S force introduces no changes in the p-p scattering formalism coupled with the considerations that the large forward angle p-p cross section tends to obscure the M-S force influence, and that the use of coulomb wave functions introduces additional numerical complexities, led to a decision to investigate the influence of the M-S force on n-p scattering.

In what follows, in this chapter, the n-p scattering formalism is developed including the M-S force, and a distorted wave approximation is used to find the appropriate phase shifts. In Chapter III, the effect of M-S scattering on measurable scattering parameters is calculated and compared to experimental results. In addition, the effect of the M-S force on the bound-state problem is discussed, and an estimate of the influence on the deuteron binding energy is made.

**The Coupled Radial Equations.** Consider a beam of neutrons with given spin, s, and given total angular momentum projection, m. Assume the momentum vector of the beam to have a direction such that it lies along the quantization axis. It is convenient in this case to express the wave function in terms of the so-called spin-angle functions, $\gamma^m_{j s}$, as,

$$\gamma^m = \frac{1}{kr} \sum A_{js} j_{js} \gamma^m_{j s} F_{j s}(kr), \quad (65)$$

where the $A_{js}$ are normalization constants, $F_{j s}(kr)$ represents the radial dependence, and
\[ \gamma_{J\ell S}^m = \sum_{m_S} \langle \ell, s, m-m_s, s, m_s / Jm \rangle \gamma_{\ell, \ell, \ell}^{m-m_s} (\theta, \varphi) \chi_s^{m_s}. \] 

(66)

In Equation (66), the quantity, \( \langle \ell, s, m-m_s, s, m_s / Jm \rangle \), represents a Clebsch-Gordan coefficient as defined by Condon and Shortley (22), the \( \gamma_{\ell, \ell, \ell}^{m-m_s} (\theta, \varphi) \) represent the well-known spherical harmonics (22), and \( \chi_s^{m_s} \) represents the spin wave function with total spin, \( s \), and \( z \)-projection, \( m_s \). Since \( \ell \cdot s \) does not commute with \( s^2 \), it follows that an eigenfunction of \( s^2 \) (e.g., \( \gamma_{J\ell S}^m \)) is not an eigenfunction of \( \ell \cdot s \). It is, therefore, clear that \( \ell \cdot s \) will mix singlet and triplet spin states and from parity conservation considerations, these will be states where \( J=\ell \). In order to illustrate this mixing explicitly and to derive the appropriate radial wave equations, it is necessary to evaluate the expression \( \langle \ell \cdot s \rangle \gamma_{J\ell S}^m \) for the four possible states. This tedious but straightforward procedure is outlined in Appendix D. The results are listed below.

\[ \langle \ell \cdot s \rangle \gamma_{J, \ell, \ell}^m = \frac{l}{2} \langle \ell \rangle \gamma_{J, \ell, \ell}^m \]

(67)

\[ \langle \ell \cdot s \rangle \gamma_{J, \ell, \ell}^m = -\frac{(\ell+1)}{2} \gamma_{J, \ell, \ell}^m \]

\[ (\mathbf{\ell} \cdot \mathbf{s}) \psi^m_{\ell \ell_0} = -\frac{1}{2} \psi^m_{\ell \ell_1} + \frac{1}{2} [\ell (\ell + 1) ]^{1/2} \psi^m_{\ell \ell_0} \]

\[ (\mathbf{\ell} \cdot \mathbf{s}) \psi^m_{\ell \ell_1} = \frac{1}{2} [\ell (\ell + 1) ]^{1/2} \psi^m_{\ell \ell_1} \]

Note, that if \( \ell = 0 \), one does not appear to get the expected result in the third expression. The reason is, of course, that there is no state where \( J = \ell = 0 \) and \( s = 1 \). That is, \( \psi^0_{001} = 0 \).

Outside the nuclear force range the Schrödinger Equation with the M-S term included (Equation (9)) can be written,

\[ \frac{1}{\rho} \frac{d}{d\rho} \left( \rho^2 \frac{d\psi^m_{\ell}}{d\rho} \right) + \left[ 1 - \frac{\ell (\ell + 1)}{\rho^2} \right] \psi^m_{\ell} - \frac{\gamma}{\rho^3} (\mathbf{\ell} \cdot \mathbf{s}) \psi^m_{\ell} = 0 , \quad (68) \]

where

\[ \gamma = \frac{e^2}{mc^2} \rho |u_n| ; \quad \psi^m = \sum_{\ell} \psi^m_{\ell} ; \]

and

\[ \psi^m_{\ell} = \psi^m_{\ell \ell_0} R_{\ell_0} + \psi^m_{\ell \ell_1} R_{\ell_1} + \psi^m_{\ell+1 \ell_1} R_{\ell+1} + \psi^m_{\ell-1 \ell_1} R_{\ell-1} \]

In this expression, \( R_{\ell_0} = \frac{1}{kr} A_{\ell \ell_0} F_{\ell \ell_0} \) etc.

When this form for \( \psi^m_{\ell} \) is substituted in Equation (68), with the results from Equation (67), the four radial equations are obtained by multiplying that result on the left, in turn by \( \psi^{m*}_{\ell \ell_0} \), \( \psi^{m*}_{\ell \ell_1} \).
$\gamma_{m^*}^{m^*}$, $\gamma_{l+1,l,1}$, and $\gamma_{l-1,l,1}$ and integrating over $4\pi$ solid angle. The resulting radial equations are.

$$
\frac{1}{\rho^2} \frac{d}{d\rho} \left( \rho^2 \frac{d R_{L0}}{d\rho} \right) + \left[ 1 - \frac{\ell(\ell+1)}{\rho^2} \right] R_{L0} - \frac{\gamma}{2\rho^3} \left[ \ell(\ell+1) \right]^{1/2} R_{L1} = 0 \quad (69)
$$

$$
\frac{1}{\rho^2} \frac{d}{d\rho} \left( \rho^2 \frac{d R_{L1}}{d\rho} \right) + \left[ 1 - \frac{\ell(\ell+1)}{\rho^2} \right] R_{L1} + \frac{\gamma}{2\rho^3} R_{L1} - \frac{\gamma}{2\rho^3} \left[ \ell(\ell+1) \right]^{1/2} R_{L0} = 0 \quad (70)
$$

$$
\frac{1}{\rho^2} \frac{d}{d\rho} \left( \rho^2 \frac{d R_{L+1,1}}{d\rho} \right) + \left[ 1 - \frac{\ell(\ell+1)}{\rho^2} \right] R_{L+1,1} - \frac{\gamma}{2\rho^3} \ell R_{L+1,1} = 0 \quad (71)
$$

$$
\frac{1}{\rho^2} \frac{d}{d\rho} \left( \rho^2 \frac{d R_{L-1,1}}{d\rho} \right) + \left[ 1 - \frac{\ell(\ell+1)}{\rho^2} \right] R_{L-1,1} + \frac{\gamma}{2\rho^3} (\ell+1) R_{L-1,1} = 0 \quad (72)
$$

It should be noted that within the range of nuclear forces the tensor force couples the two states $l=J\pm 1$. This coupling must be accounted for in expressing scattering amplitudes in terms of phase shifts. The effects of this latter coupling on pertinent matrix elements is well known, and is readily available in the literature (see, for example, Reference 13). However, as this $l=J\pm 1$ coupling is entirely independent of the spin-state mixing, it can be omitted in the development here, and still easily be included in whatever final results require it.
Derivation of the M Matrix. Since coupled wave functions such as $R_{L0}$ and $R_{L1}$ imply an amplitude loss or decrease of a given state between entrance and exit channels, the phase shifts associated with $R_{L0}$ and $R_{L1}$ of Equations (69) and (70) are complex. It is customary, and in many ways more convenient to deal with real phase shifts. This can be accomplished if, in the asymptotic region, one takes proper linear combinations of the coupled functions as the wave functions.

In addition to the two real phase shifts, a third real quantity, called a mixing parameter is also introduced. These three new parameters are related to the complex phase shifts of $R_{L0}$ and $R_{L1}$ through equivalence of the asymptotic forms of the radial wave functions or equivalence of the corresponding S-matrix element forms. There are two general methods customarily used for defining the real or eigen phase shifts. The phase shifts are known as the Blatt and Biedenharn (23) and the Stapp (24) (or sometimes "bar") phase shifts, referring to those who introduced the techniques. Both definitions are discussed by Mott and Massey (13). The Blatt and Biedenharn type of phase shifts will be used here. This choice is arbitrary, and given one set of parameters, one can always find the other.

Taking a linear combination of states, one may express the asymptotic form of the wave function as,


where \( \hat{\psi}^m_{t+1,t1} = \gamma^m_{t+1,t1} \),

\[ \hat{\psi}^m_{t-1,t1} = \gamma^m_{t-1,t1} \]

\[ \hat{\psi}^{m+} = \gamma^m_{I1} + \tan \epsilon_t \gamma^m_{I0} \], and

\[ \hat{\psi}^{m-} = \gamma^m_{I1} - \cot \epsilon_t \gamma^m_{I0} \]

The relationship between the last two forms results from the orthogonality condition. In this notation \( t \) ranges over the indices 1, +, and −, as indicated, and \( \epsilon_t \) represents the mixing parameter mentioned above. The + and − subscripts correspond to the signs on the right-hand side of the appropriate expressions.

If the incident direction is selected along the quantization (z) axis, so the projection of orbital angular momentum, \( m_s = 0 \) for the incident beam, then since \( m \) is a good quantum number, the final value of \( m \) will be the projection of spin, \( m_s \), of the incident beam. The asymptotic form of the wave function can then also be expressed as,

\[ \psi^m_s \sim e^{i\rho} \cos \theta \chi_{s'}^{m'} + \frac{e^{i\rho}}{r} \sum_{m_s} N_{m_s}s^{m_s} \chi_{s}^{m_s} \], \hspace{1cm} (74)
with spin, \( s' \), and projection, \( m'_s \). The quantities \( M_{s m s' m'} \) represent the amplitudes for scattering from the initial state to the final state with spin, \( s \), and projection, \( m_s \), and constitute the elements of the \( M \) matrix. As there are four possible values of the pair \( (s, m_s) \) the \( M \) matrix is square and of order four.

In order to relate the two asymptotic expressions (73) and (74) such that the \( M \) matrix elements can be determined, it is necessary to use the following expansions,

\[
e^{i\rho \cos \theta} = \sum_{\ell} i^{\ell} (2\ell+1) j_{\ell}(\rho) P_{\ell}(\cos \theta), \quad \text{and}
\]

\[
P_{\ell}(\cos \theta) \chi_{s'}^{m'} = \left( \frac{4\pi}{2\ell+1} \right)^{1/2} \sum_{J=|\ell-s|}^{J=\ell+s} (s'0 m'_{s}/Jm') j_{J}\chi_{J\ell s}^{m}. \quad \text{and}
\]

Substituting these expressions into Equation (74), equating the result to Equation (73), and using the asymptotic form for \( j_{\ell}(\rho) \), namely,

\[
j_{\ell}(\rho) \sim \frac{1}{2i\rho} \left[ e^{i(\rho - \frac{\ell\pi}{2})} - e^{-i(\rho - \frac{\ell\pi}{2})} \right],
\]

together with the asymptotic form for \( F(\rho) \),
\[ F_{J\ell t}(\rho) \sim \frac{1}{2i} \left[ e^{i(\rho - \frac{\ell \pi}{2} + \delta_{J\ell t})} - e^{-i(\rho - \frac{\ell \pi}{2} + \delta_{J\ell t})} \right], \]

where \( \delta_{J\ell t} \) is the appropriate phase shift, yields for a given value of \( \ell \),

\[ \frac{(4\pi)^{1/2}i\ell(2\ell+1)^{1/2}}{2i\rho} \left[ e^{i(\rho - \frac{\ell \pi}{2})} - e^{-i(\rho - \frac{\ell \pi}{2})} \right] \sum_{J} (\ell s'0m'/Jm')_{J\ell s'}^{m'} \]

\[ + \frac{e^{i\rho}}{r} \sum_{m_s} M_{sm_s s'm'} \chi_{s'}^{m'} \]

\[ = \frac{1}{2i\rho} \sum_{J} A_{J\ell t}^{m_s} \left[ e^{i(\rho - \frac{\ell \pi}{2} + \delta_{J\ell t})} - e^{-i(\rho - \frac{\ell \pi}{2} + \delta_{J\ell t})} \right]. \quad (75) \]

Note that since \( j_{\ell}(\rho) \) and \( F_{J\ell t}(\rho) \) are solutions to the same differential equation, their asymptotic forms differ only by a phase shift.

Equating the coefficient of \( e^{-i\rho} \) to zero, on both sides of Equation (75) gives,

\[ A_{\ell+1,t}^{m_s} = e^{i\delta_{\ell+1,t} \ell} c_{\ell}(\ell s'0m'/\ell+1 m') \]

\[ A_{\ell-1,t}^{m_s} = e^{i\delta_{\ell-1,t} \ell} c_{\ell}(\ell s'0m'/\ell-1 m') \quad (76) \]
\[
\begin{align*}
\left[ A_{ll+} e^{-i\delta_{ll+}} + A_{ll-} e^{-i\delta_{ll-}} \right] & \begin{pmatrix} m^s_s \\ m^s_{s'} \end{pmatrix} = \cot \varepsilon \begin{pmatrix} m^s_s \\ m^s_{s'} \end{pmatrix}, \\
\left[ A_{ll+} e^{-i\delta_{ll+}} \tan \varepsilon + A_{ll-} e^{-i\delta_{ll-}} \right] & \begin{pmatrix} m^s_s \\ m^s_{s'} \end{pmatrix} = \cot \varepsilon \begin{pmatrix} m^s_s \\ m^s_{s'} \end{pmatrix}, \\
\end{align*}
\]

where, \( C_\ell = \sqrt{4\pi(2\ell+1)} \).

Equations (76) indicate the \( A_{ll} \) are, not unexpectedly, dependent on \( s' \). In order that the notation reflect this dependence, the coefficients \( A_{ll} \) are hereafter designated as \( A_{ll} \).

Upon performing the necessary algebra, in Equations (76), one obtains the following equations for the eight \( A \) coefficients (for a given \( \ell \)); Equations (77) for \( s' = 1 \) and Equations (78) for \( s' = 0 \).

\[
\begin{align*}
A_{\ell+1, \ell, 1} &= e^{i\delta_{\ell+1, \ell, 1}} \tan \varepsilon + \cot \varepsilon \begin{pmatrix} m^s_s \\ m^s_{s'} \end{pmatrix}, \\
A_{\ell+1, \ell, 1} &= e^{i\delta_{\ell+1, \ell, 1}} \tan \varepsilon + \cot \varepsilon \begin{pmatrix} m^s_s \\ m^s_{s'} \end{pmatrix}, \\
A_{\ell-1, \ell, 1} &= e^{i\delta_{\ell-1, \ell, 1}} \tan \varepsilon + \cot \varepsilon \begin{pmatrix} m^s_s \\ m^s_{s'} \end{pmatrix}, \\
A_{\ell-1, \ell, 1} &= e^{i\delta_{\ell-1, \ell, 1}} \tan \varepsilon + \cot \varepsilon \begin{pmatrix} m^s_s \\ m^s_{s'} \end{pmatrix}, \\
A_{\ell+1} &= \frac{C_\ell (\ell 10 m_s^s / \ell 1 m_s^s)}{\tan \varepsilon + \cot \varepsilon} e^{i\delta_{\ell+1}}, \\
A_{\ell-1} &= \frac{C_\ell (\ell 10 m_s^s / \ell 1 m_s^s)}{\tan \varepsilon + \cot \varepsilon} e^{i\delta_{\ell-1}}.
\end{align*}
\]
and

\[ A^m_{l+1, l, t0} = 0 \text{ since } (\ell 00m^l_s/l+1 m^l_s) = 0 \]

\[ A^m_s = 0 \]

\[ A^m_{l-0} = 0 \]

\[ A^m_{l+0} = \frac{C^l (\ell 00m^l_s/l m^l_s)}{\tan \epsilon^l + \cot \epsilon^l} e^{i \delta^l m_l} \]

\[ A^m_{l-0} = - \frac{C^l (\ell 00m^l_s/l m^l_s)}{\tan \epsilon^l + \cot \epsilon^l} e^{i \delta^l m_l} \]

(78)

For an incoming singlet partial wave \((s' = 0)\), substitution of Equation (78) into Equation (75) gives,

\[ \sum_{m^s} M^s_{m^s 00} \chi^s_s = \frac{C^l}{k} \left\{ \frac{e^{2i \delta^l m_l + 2i \delta^l m_l -} - e^{2i \delta^l m_l}}{\tan \epsilon^l + \cot \epsilon^l} \chi^0_{m^l m^l} \right. \]

\[ + \left[ \frac{\tan \epsilon^l e^{2i \delta^l m_l + 2i \delta^l m_l -} + \cot \epsilon^l e^{2i \delta^l m_l -}}{\tan \epsilon^l + \cot \epsilon^l} - 1 \right] \chi^0_{m^l 0} \} \]

(79)

and for an incoming triplet wave \((s' = 1)\), Equation (77) substituted in Equation (75) gives,

\[ \sum_{m^s} M^s_{m^s 1m'} \chi^s_s = \frac{C^l}{k} \{ (\ell 10m^l_s/l+1 m^l_s) \left[ e^{2i \delta^l m_l + 1, l + 1} - 1 \right] \chi^m_{m^l m^l} \} \]
where \( C^l = \frac{\sqrt{4\pi(2l+1)}}{2l} \).

Substituting the explicit form for the \( \frac{\partial}{\partial \mathcal{J}_s} \) (Equation (66)), into Equation (80) yields the various elements of the \( M \) matrix. These elements are listed in Table 1. In obtaining these expressions, the relationships between spherical harmonics and Legendre polynomials\(^\text{(22)}\), has been employed. Namely,

\[
Y^m_l = \frac{1}{\sqrt{2\pi}} P^m_l(\theta) \, e^{im\phi},
\]

where

\[
P^m_l(\theta) = (-1)^m \left[ \frac{2l+1}{2} \frac{(l-m)!}{(l+m)!} \right]^{1/2} \sin^m \theta \frac{d}{d\cos \theta} \frac{d^m}{d(\cos \theta)^m} P_l(\cos \theta) \quad (m > 0)
\]

and

\[
P^{-m}_l = (-1)^m P^m_l.
\]

The quantities, \( Q \), which appear, are defined as,
\[
Q_{\pm 1} = \frac{e^{2i\delta_{\ell\pm 1,\ell,1}} - 1}{2i}
\]

(81)

\[
Q_{\ell 0} = \frac{1}{2i} \left[ \tan \epsilon_{\ell} e^{2i\delta_{\ell+}} + \cot \epsilon_{\ell} e^{2i\delta_{\ell-}} \right] - 1
\]

(82)

\[
Q_{\ell 1} = \frac{1}{2i} \left[ \cot \epsilon_{\ell} e^{2i\delta_{\ell+}} + \tan \epsilon_{\ell} e^{2i\delta_{\ell-}} \right] - 1
\]

(83)

\[
Q_{\ell} = \frac{1}{2i} \left[ e^{2i\delta_{\ell+}} - e^{2i\delta_{\ell-}} \right]
\]

(84)

It is important to note that if one had included the tensor force effects of mixing \( \ell = J \pm 1 \) states the phase shifts, \( \delta_{\ell\pm 1,\ell,1} \), would have been dependent on \( m \), the \( z \) projection of the total angular momentum (see Reference 13). This does not imply that \( m \) is not invariant, but occurs because of the nature of the coupling. The \( m \) dependence can be removed by introduction of \( \ell \) mixing eigen phase shifts and mixing parameters. It is, perhaps, also worth noting that this \( m \) dependence does not occur for the case under consideration where \( J = \ell \) singlet and triplet states are coupled. This fact can perhaps be seen most easily through the properties of the \( S \) matrix discussed later. It is not difficult to show that in this case, \( m \) dependent phase shifts lead to an unsymmetric \( S \) matrix, which is in violation of time reversal invariance.
Forms for the scattering amplitudes that apply when the
coupling between \( \ell = J + 1 \) and \( \ell = J - 1 \) partial waves is included
are listed in Appendix F if they differ from the Table 1 entry. The
bar phase shifts are used for these \( \ell \) coupled states, because this is
the customary way of reporting them.

**Forms For Scattering Amplitudes.** It should be emphasized that
the matrix elements as defined above refer to a particular representa-
tion of the spin wave function. The representation chosen here is
that most frequently seen, and is given by,

\[
\psi_{\text{spin}} = \begin{pmatrix}
0 \\
\chi_1 \\
1 \\
\chi_1 \\
0 \\
\chi_1 \\
1 \\
\chi_1
\end{pmatrix}
\]

(85)

It happens that if one sets \( \epsilon_\ell = 0 \) or \( \epsilon_\ell = \frac{\pi}{2} \), the matrix
elements reduce to forms derived by others\(^{(13)(25)}\) for the case of
no spin mixing and no \( \ell \)-wave mixing.

In most applications the choice of \( \varphi \) is arbitrary since the
scattering process is azimuthally symmetric, and it is convenient to
set \( \varphi = 0 \), corresponding to selecting the \( y \) direction along \( \vec{k}_{\text{in}} \times \vec{k}_{\text{out}} \).

TABLE 1

Scattering Amplitude Forms

\[ M_{0000} = \frac{1}{k} \sum_{l} (2l+1) Q_{l0} P_{l}^{e}(\cos \theta) \]

\[ M_{0010} = 0 \]

\[ M_{1000} = 0 \]

\[ M_{0011} = \frac{1}{k} \sum_{l} \left( \frac{2l+1}{[2l(l+1)]^{1/2}} Q_{l} \sin \theta e^{i\phi} P_{l}^{r}(\cos \theta) \right) \]

\[ M_{001-1} = \frac{1}{k} \sum_{l} \left( \frac{2l+1}{[2l(l+1)]^{1/2}} Q_{l} \sin \theta e^{-i\phi} P_{l}^{r}(\cos \theta) \right) \]

\[ M_{1100} = - M_{001-1} \]

\[ M_{1-100} = - M_{0011} \]

\[ M_{1111} = \frac{1}{2k} \sum_{l} \left[ (\ell+2) Q_{\ell+1} + (\ell-1) Q_{\ell-1} + (2\ell+1) Q_{\ell-1} \right] P_{l}^{e}(\cos \theta) \]

\[ M_{1011} = - \frac{1}{\sqrt{2k}} \sum_{l} \left[ - \left( \frac{2l+3}{l(l+1)} Q_{l+1} + \frac{l-1}{l} Q_{l-1} + \frac{2l+1}{l(l+1)} Q_{l-1} \right) \right] P_{l}^{r}(\cos \theta) \times \sin \theta e^{i\phi} \]
\( M_{1111} = \frac{1}{2k} \sum_{l} \left[ \frac{1}{l+1} Q_{l+1} + \frac{1}{l} Q_{l-1} - \frac{2l+1}{l(l+1)} Q_{l} \right] P_{l}^{m}(\cos \theta) \sin^{2} \theta e^{i\varphi} \)

\( M_{1110} = -\frac{1}{\sqrt{2k}} \sum_{l} (Q_{l+1} - Q_{l-1}) P_{l}^{m}(\cos \theta) \sin \theta e^{-i\varphi} \)

\( M_{111-1} = M_{1-111} e^{-4i\varphi} \)

\( M_{1010} = \frac{1}{k} \sum_{l} \left[ (l+1) Q_{l+1} + lQ_{l-1} \right] P_{l}(\cos \theta) \)

\( M_{1-110} = -M_{1110} e^{2i\varphi} \)

\( M_{101-1} = -M_{1011} e^{-2i\varphi} \)

\( M_{1-11-1} = M_{1111} \)

Elements of the S matrix are defined in terms of the amplitudes \( M_{sm's'm_s'} \) in a most general way by Blatt and Biedenharn in their fundamental review paper. Their Equation (3.14) as applied to the case of interest here is,

\[
M_{sm's'm_s'} = \frac{\pi i}{k} \sum_{J,m,l,v} \sqrt{2l+1} \left( \langle l's'm_s'/JM \rangle \langle l's'm_s'/JM \rangle (\delta_{s's_s'} - S_{lss'}^{J}) \right) Y_{l}^{m_s'} \quad (86)
\]
The elements of the S matrix, $S_{\alpha \beta}$, are defined in the usual manner i.e. the ratio of the amplitude of the radial outgoing wave in channel $\beta$ to the amplitude of the radial incoming wave in channel $\alpha$. Using Equation (86) together with Table 1, one finds for $J = l$ waves,

$$
\begin{align*}
S_{l01} &= S_{l10}^l = 2i Q_l \\
S_{l00}' &= 2i Q_{l0} + 1 \\
S_{l11} &= 2i Q_{l1} + 1
\end{align*}
$$

(87)

It is easily seen that the two by two $S_{lss}^l$ matrix has the expected properties of symmetry and unitarity.

In the next section, a generalized form for the M matrix with singlet-triplet mixing is developed, and further the matrix is characterized in terms of some useful parameters.

**General Considerations and the M Matrix.** It is clear that any four by four matrix can be expressed as a linear combination of, at most, sixteen base matrices. For base matrices, it is convenient to use the unit matrix, and matrices formed from direct products of the Pauli spin matrices with each other and with the unit $(2 \times 2)$ matrix in the spin-spaces of the two particles. It will happen that invariance principles will eliminate certain of these forms, and one is eventually left with only six base matrices. That is, the requirement that the scattering matrix be invariant under rotation, time inversion, and
space reflection allows only six terms to be retained in the general representation. All this is well-known, and was first developed by Wolfenstein and Ashkin (26). It happens that if one assumes charge independence of the interaction, then only five base matrices are allowed. Since the nuclear forces are believed to be charge independent, the usual treatments of n-p scattering do not include the consequences of charge dependent interactions. It is also true, that the general treatments in the literature are somewhat obscure in regard to detail. It, therefore, seems desirable to present the general development in some detail.

Although one can choose the cartesian components of the Pauli spin matrices to form the base matrices, it is perhaps more customary to use the irreducible tensor component forms. The use of irreducible tensors offers no particular advantage in the present case, but provides a treatment which is easily extended to higher spins.

The irreducible tensor forms for the matrices operating in the spin space of a spin-1/2 particle are:

\[ \sigma_{00} = I_{00} \text{ (identity matrix)}, \]
\[ \sigma_{11} = -\frac{1}{\sqrt{2}} (\sigma_x + i\sigma_y), \]
\[ \sigma_{10} = \sigma_z, \]

and 

\[ \sigma_{1-1} = \frac{1}{\sqrt{2}} \left( \sigma_x - i \sigma_y \right), \]

where \( \sigma_x, \sigma_y, \sigma_z \) are the usual Pauli matrices.

In order to distinguish between spin spaces, a superscript, 1, will be used for the neutron, and 2 for the proton.

The coordinate system usually chosen is defined by the unit vectors \( \hat{\mathbf{A}}, \hat{\mathbf{P}}, \) and \( \hat{\mathbf{K}} \) where,

\[ \hat{\mathbf{A}} = \frac{\mathbf{k}_{\text{in}} \times \mathbf{k}_{\text{out}}}{|\mathbf{k}_{\text{in}} \times \mathbf{k}_{\text{out}}|}, \quad \hat{\mathbf{P}} = \frac{\mathbf{k}_{\text{in}} + \mathbf{k}_{\text{out}}}{|\mathbf{k}_{\text{in}} + \mathbf{k}_{\text{out}}|}, \quad \text{and} \quad \hat{\mathbf{K}} = \frac{\mathbf{k}_{\text{out}} - \mathbf{k}_{\text{in}}}{|\mathbf{k}_{\text{out}} - \mathbf{k}_{\text{in}}|}. \]

In the above definitions, \( \mathbf{k}_{\text{in}} \) and \( \mathbf{k}_{\text{out}} \) represent the wave propagation vectors before and after the scattering process. From these unit vectors, spherical tensors are formed as;

\[ n_{11} = -\frac{1}{\sqrt{2}} (n_x + i n_y), \]

\[ n_{10} = n_z, \]

and

\[ n_{1-1} = \frac{1}{\sqrt{2}} (n_x - i n_y), \]

with analogous definitions being made from the cartesian components of \( \hat{\mathbf{P}} \) and \( \hat{\mathbf{K}} \).

Now, in order to arrive at the general form of the M matrix, consider first, how various quantities behave under space reflection and time inversion. This is illustrated in Table 2.
<table>
<thead>
<tr>
<th></th>
<th>Space Reflection</th>
<th>Time Inversion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_{\ell m} \rightarrow \sigma_{\ell m}$</td>
<td>$\sigma_{\ell m} \rightarrow -\sigma_{\ell m}$</td>
<td></td>
</tr>
<tr>
<td>$\sigma_{\ell m} \rightarrow \sigma_{\ell m}$</td>
<td>$\sigma_{\ell m} \rightarrow -\sigma_{\ell m}$</td>
<td></td>
</tr>
<tr>
<td>$\overline{k}<em>{\text{in}} \rightarrow -\overline{k}</em>{\text{in}}$</td>
<td>$\overline{k}<em>{\text{in}} \rightarrow -\overline{k}</em>{\text{out}}$</td>
<td></td>
</tr>
<tr>
<td>$\overline{k}<em>{\text{out}} \rightarrow -\overline{k}</em>{\text{out}}$</td>
<td>$\overline{k}<em>{\text{out}} \rightarrow -\overline{k}</em>{\text{in}}$</td>
<td></td>
</tr>
<tr>
<td>$n_{\ell m} \rightarrow n_{\ell m}$</td>
<td>$n_{\ell m} \rightarrow -n_{\ell m}$</td>
<td></td>
</tr>
<tr>
<td>$P_{\ell m} \rightarrow -P_{\ell m}$</td>
<td>$P_{\ell m} \rightarrow -P_{\ell m}$</td>
<td></td>
</tr>
<tr>
<td>$K_{\ell m} \rightarrow -K_{\ell m}$</td>
<td>$K_{\ell m} \rightarrow -K_{\ell m}$</td>
<td></td>
</tr>
</tbody>
</table>

In Table 2 the subscript, $m$, refers to any component $(1,0,-1)$ of the quantity involved.
Using the properties listed in Table 2, the behavior of the scalar products between the spin matrices and the unit vectors is deducible, as listed in Table 3.

**TABLE 3**

**SIGNS OF SCALAR PRODUCTS UNDER TRANSFORMATION**

<table>
<thead>
<tr>
<th>Scalar Product</th>
<th>Space Reflection</th>
<th>Time Inversion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_{1m}^{(1)} \cdot n_{1m}$</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>$\sigma_{1m}^{(1)} \cdot P_{1m}$</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>$\sigma_{1m}^{(1)} \cdot K_{1m}$</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$I_{00}$</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>$\sigma_{1m}^{(2)} \cdot n_{1m}$</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>$\sigma_{1m}^{(2)} \cdot P_{1m}$</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>$\sigma_{1m}^{(2)} \cdot K_{1m}$</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>
In Table 3 the summation convention for repeated indices is used, e.g. \( \sigma^{(1)}_{1m} \cdot n_{1m} = \sigma^{(1)}_{11} n_{11} + \sigma^{(1)}_{10} n_{10} + \sigma^{(1)}_{1-1} n_{1-1} \) etc.

The \( M \) matrix is to be represented by a linear combination of products of matrices operating in the two-spin spaces. Thus, taking products between \( \sigma^{(1)} \) and \( \sigma^{(2)} \) terms with the same space-time signature, \( M \) is expressible as;

\[
M = I_{00} [A + D \sigma^{(2)}_{1m} \cdot n_{1m}] + \sigma^{(1)}_{1m} \cdot n_{1m} [C + B \sigma^{(2)}_{1m} \cdot n_{1m}]
\]

\[
+ E(\sigma^{(1)}_{1m} \cdot P_{1m})(\sigma^{(2)}_{1m} \cdot P_{1m}) + F(\sigma^{(1)}_{1m} \cdot K_{1m})(\sigma^{(2)}_{1m} \cdot K_{1m})
\]

or

\[
M = A + D \sigma_{\vec{r}}^{(2)} + C \sigma_{\vec{r}}^{(1)} + B \sigma_{\vec{r}}^{(1)} \sigma_{\vec{r}}^{(2)} + E \sigma_{\vec{p}}^{(1)} \sigma_{\vec{p}}^{(2)} + F \sigma_{\vec{K}}^{(1)} \sigma_{\vec{K}}^{(2)} \quad (88)
\]

Thus, the six parameter result is obtained. In the above expression, \( A, B, C, D, E, \) and \( F \) are, in general, complex functions of \( \varphi \) and \( \theta \). The \( M \) matrix is given explicitly in terms of these parameters if one regards the \( \sigma \)'s above as four by four matrices obtained through the direct product operation with the unit matrix. In applying this procedure, a certain representation of the spin wave function is implied, and this representation is different than that used in the derivation of Table 1. The two representations are,
For Table 1,  
\[ \psi_s = \begin{pmatrix}
\frac{1}{\sqrt{2}} (\alpha_1 \beta_2 - \alpha_2 \beta_1) \\
\alpha_1 \\
\frac{1}{\sqrt{2}} (\alpha_1 \beta_2 + \alpha_2 \beta_1) \\
\beta_1 \\
\end{pmatrix} \]  

(89)

and for Equation (88) using the direct product operation,  
\[ \psi_s = \begin{pmatrix}
\alpha_1 \\
\frac{1}{\sqrt{2}} (\alpha_1 \beta_2 + \alpha_2 \beta_1) \\
\beta_1 \\
\end{pmatrix} \]  

(90)

In this notation \( \alpha \) and \( \beta \) refer to spin up and spin down respectively, and the subscript, 1 or 2, refers to the appropriate spin space or particle (neutron or proton respectively).

Thus in order to arrive at the appropriate forms for the \( \sigma \) products one must effect a unitary transformation on the matrices that were obtained by taking direct products. It is also possible to find the desired result directly by inspection. This is done by using the
known result of operating on the components of Equation (89) with the two-by-two Pauli matrices, and then by inspection, one can determine the four-by-four matrix that gives the same result. For example,

\[ \sigma_x^{(1)} \frac{1}{\sqrt{2}} (\alpha_1 \beta_2 - \alpha_2 \beta_1) = \frac{1}{\sqrt{2}} (\beta_1 \beta_2 - \alpha_2 \alpha_1) = \frac{1}{\sqrt{2}} (x_1^{-1} - x_1) , \]

\[ \sigma_x^{(1)} \alpha_1 \alpha_2 = \beta_1 \alpha_2 = \frac{1}{\sqrt{2}} (x_1^0 - x_0^0) , \]

\[ \sigma_x^{(1)} \frac{1}{\sqrt{2}} (\alpha_1 \beta_2 + \alpha_2 \beta_1) = \frac{1}{\sqrt{2}} (\beta_1 \beta_2 + \alpha_2 \alpha_1) = \frac{1}{\sqrt{2}} (x_1^{-1} + x_1^1) , \]

\[ \sigma_x^{(1)} \beta_1 \beta_2 = \alpha_1 \beta_2 = \frac{1}{\sqrt{2}} (x_1^0 + x_0^0) . \]

Thus, the result of the four-by-four matrix representing \( \sigma_x^{(1)} \) operating on the form Equation (89) is,

\[
\begin{pmatrix}
0 \\
\chi_0 \\
1 \\
\chi_1 \\
0 \\
\chi_1 \\
-1 \\
\chi_1 \\
\end{pmatrix}
= \frac{1}{\sqrt{2}}
\begin{pmatrix}
-1 & 1 \\
\chi_1 & -\chi_1 \\
0 & -1 \\
\chi_1 & -\chi_0 \\
0 & -1 \\
\chi_1 & -\chi_1 \\
0 & 1 \\
\chi_1 & \chi_0 \\
\end{pmatrix}.
\]

From this relation, one can easily see that;
\[ \sigma_x^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \]

In addition to determination of the various \( \sigma \) components of interest, it is necessary to find the components of \( \hat{n}, \hat{p}, \) and \( \hat{K} \). As usual, one chooses \( \vec{k}_{\text{in}} \) in the \( z \) direction, then, \( \vec{k}_{\text{in}} = e_z \), and
\[ \vec{k}_{\text{out}} = \sin \theta \cos \phi \ e_x + \sin \theta \sin \phi \ e_y + \cos \theta \ e_z, \]
and from this taken with the definitions of \( \hat{n}, \hat{p}, \) and \( \hat{K} \) (Page 51) one finds the results in Table 4.

Using the results of Table 4, and following the procedure illustrated to determine the four-by-four Pauli matrices, \( \sigma_x^{(1)} \), one can determine the four-by-four forms of the matrices of Equation (88).

The appropriate Pauli matrices are listed below in Table 5.

The procedure for obtaining, for example, \( \sigma_{lm}^{(1)} \cdot n_{lm} \) is,
\[ \sigma_{lm}^{(1)} \cdot n_{lm} = \sigma_{11}^{(1)} n_{11}^{*} + \sigma_{10}^{(1)} n_{10}^{*} + \sigma_{1-1}^{(1)} n_{1-1}^{*}, \quad (92) \]

where the \( \sigma_{lj}^{(1)} \) quantities have been previously defined. By carrying out the operations indicated in Equation (92) and similar operations
for the other matrices in Equation (88), one obtains the results listed in Table 6.

### Table 4

**UNIT VECTOR COMPONENTS**

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_{11} = -\frac{i}{\sqrt{2}}e^{i\varphi}$</td>
<td>$p_{11} = -\frac{\sin \theta}{2\sqrt{2}}e^{i\varphi}$</td>
<td>$k_{11} = -\frac{\cos \theta}{\sqrt{2}}e^{i\varphi}$</td>
<td></td>
</tr>
<tr>
<td>$n_{10} = 0$</td>
<td>$p_{10} = \cos \frac{\theta}{2}$</td>
<td>$k_{10} = -\sin \frac{\theta}{2}$</td>
<td></td>
</tr>
<tr>
<td>$n_{1-1} = -\frac{i}{\sqrt{2}}e^{-i\varphi}$</td>
<td>$p_{1-1} = -\frac{\sin \theta}{2\sqrt{2}}e^{-i\varphi}$</td>
<td>$k_{1-1} = -\frac{\cos \theta}{\sqrt{2}}e^{-i\varphi}$</td>
<td></td>
</tr>
</tbody>
</table>

### Table 5

**FOUR-BY-FOUR PAULI MATRICES**

\[
\sigma_x^{(1)} = \frac{1}{\sqrt{2}}
\begin{pmatrix}
0 & -1 & 0 & 1 \\
-1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0
\end{pmatrix}
\]
\( \sigma_y^{(1)} = \frac{i}{\sqrt{2}} \cdot \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix} \)

\( \sigma_z^{(1)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \)

\( \sigma_x^{(2)} = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix} \)

\( \sigma_y^{(2)} = \frac{i}{\sqrt{2}} \cdot \begin{pmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix} \)
TABLE 5 (Continued)

\[ \sigma_z^{(2)} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]

TABLE 6

BASE MATRICES FOR M MATRIX REPRESENTATION

\[ \sigma_{lm}^{(1)} \cdot n_{lm} = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -i \varphi & 0 & e^{i \varphi} \\ -e^{-i \varphi} & 0 & e^{i \varphi} & 0 \\ 0 & -e^{-i \varphi} & 0 & e^{i \varphi} \\ -e^{-i \varphi} & 0 & -e^{-i \varphi} & 0 \end{pmatrix} \]
\[ \sigma_{1m}^{(2)} \cdot n_{1m} = \frac{1}{\sqrt{2}} \]

\[
\begin{pmatrix}
0 & -e^{-i\varphi} & 0 & -e^{i\varphi} \\
e^{i\varphi} & 0 & e^{i\varphi} & 0 \\
0 & -e^{-i\varphi} & 0 & e^{i\varphi} \\
e^{-i\varphi} & 0 & -e^{-i\varphi} & 0
\end{pmatrix}
\]

\[
(1)\quad (2) =
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & -e^{2i\varphi} \\
0 & 0 & 1 & 0 \\
0 & -e^{-2i\varphi} & 0 & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos 2\varphi \quad \frac{2 \sin \theta \cos \theta}{\sqrt{2}} e^{i\varphi} & \sin 2\varphi \quad e^{2i\varphi} \\
0 & \frac{2}{\sqrt{2}} \sin \theta \cos \theta e^{-i\varphi} & \sin 2\varphi - \cos \frac{3}{2} \quad -\frac{2}{\sqrt{2}} \sin \theta \cos \theta e^{i\varphi} \\
0 & \sin 2\varphi \quad e^{-2i\varphi} & -\frac{2}{\sqrt{2}} \sin \theta \cos \theta e^{-i\varphi} & \cos 2\varphi \quad e^{2i\varphi}
\end{pmatrix}
\]
It is, of course, possible to express the quantities $A$, $B$, $C$, $D$, $E$, and $F$ in terms of the matrix elements previously derived and given in Table 1. In order to save writing, the matrix element, $M_{ss's's'}$, will be redesignated as indicated below in the expression for $M$.

$$M = \begin{pmatrix} M_{ss} & M_c & 0 & M_c \\ -M_c & M_{11} & M_{10} & M_{1-1} \\ 0 & M_{01} & M_{00} & -M_{01} \\ -M_c & M_{1-1} & -M_{10} & M_{11} \end{pmatrix}$$  \text{(93)}
It will be useful to summarize the trace properties of various products of Pauli matrices.

\[
\begin{align*}
\text{Tr } \sigma_i^{(1)} \sigma_j^{(2)} &= 0 \\
\text{Tr } \sigma_i^{(1)} \sigma_j^{(1)} &= 4 \delta_{ij} \\
\text{Tr } \sigma_i^{(1)} \sigma_j^{(1)*} &= -4 \\
\text{Tr } \sigma_i^{(2)} \sigma_j^{(2)} &= 4 \delta_{ij}
\end{align*}
\]

These results (with the normalization factor, 4, replaced by 2), are easily obtained from the two by two Pauli matrices.

Further, since the Pauli matrices are traceless and the unit matrix has trace, 4, from Equation (88), the trace of \( M \) is equal to 4A. These considerations lead to the following expressions from which the parameters A through F can be related to the \( M_{ij} \).

\[
\begin{align*}
\text{Tr } M &= 4A \\
\text{Tr } (\sigma_i^{(2)} M) &= 4D \\
\text{Tr } (\sigma_i^{(1)} M) &= 4C \\
\text{Tr } (\sigma_i^{(1)} \sigma_j^{(2)} M) &= 4B \\
\text{Tr } (\sigma_i^{(1)} \sigma_j^{(1)} M) &= 4E \\
\text{Tr } (\sigma_i^{(2)} \sigma_j^{(2)} M) &= 4F
\end{align*}
\]
An alternative, equivalent form of Equation (88) is frequently seen in the literature. \(^{(27)}\)

\[ M = a + b \left[ \sigma_{\alpha}^{(1)} - \sigma_{\alpha}^{(2)} \right] + c \left[ \sigma_{\beta}^{(1)} + \sigma_{\beta}^{(2)} \right] + m \sigma_{\alpha}^{(1)} \sigma_{\alpha}^{(2)} \]

\[ + g \left[ \sigma_{\alpha}^{(1)} \sigma_{\alpha}^{(2)} + \sigma_{\beta}^{(1)} \sigma_{\beta}^{(2)} \right] + h \left[ \sigma_{\alpha}^{(1)} \sigma_{\beta}^{(2)} - \sigma_{\beta}^{(1)} \sigma_{\alpha}^{(2)} \right] \] \hspace{1cm} (95)

In this expression, if one postulates charge independence, then particles (1) and (2) are to be regarded as identical and \( M \) should be symmetric upon particle exchange. Thus, \( b \) would vanish and only five parameters would be necessary. In Equation (88), this same condition (charge independence) simply means that \( C = D \).

It is a simple matter to relate the quantities \( A \) through \( F \) of Equation (88) to those \( a \) through \( h \) of Equation (95). These relationships together with those given by Equation (94) are summarized in Table 7. The set \( a \) through \( h \) are sometimes referred to as the Wolfenstein parameters, presumably because of the early use of this formulation by Wolfenstein \(^{(28)}\).


These relationships listed in Table 7, are identical with those of Reference (27) except for the introduction of the parameter, \( b \), which does not appear in Reference (27). The final entry in the Table, i.e. the expression for \( h \), does not result from the procedures described, but is a consequence of time-reversal invariance. The procedure is to note that if one could neglect time-reversal invariance, one could introduce into Equation (95) two new parameters, \( l \) and \( j \), which would be coefficients of the terms \( \sigma_{\text{P}}^{(1)} \sigma_{\text{K}}^{(2)} \) and \( \sigma_{\text{K}}^{(1)} \sigma_{\text{P}}^{(2)} \). If these parameters are included and then subsequently the requirement made that the quantities \( l \) and \( j \) vanish in order to preserve time-reversal invariance, the result is,

\[
\sqrt{2} [M_{10} + M_{01}] \cos \theta = \sin \theta [M_{11} - M_{1-1} - M_{00}]. \tag{96}
\]

This result is identical with that obtained when \( M_c \) or \( b \) is taken to be zero.\(^{(27)}\) Thus, Equation (96) combined with the result for \( h \) as obtained by the procedures indicated, leads to the final entry in Table 7.

The Scattering Parameters. This section will be concerned with the derivation of quantities usually observed in scattering experiments in terms of the Wolfenstein parameters. The procedure for this derivation is outlined in Reference (27), and results are given there for the case of a charge independent interaction (no singlet-triplet coupling).
The spin-density matrix (or density matrix) formalism is a convenient framework for this development. The basics of this formalism are discussed in the literature, and will not be developed here.

**TABLE 7**

**RELATIONSHIP BETWEEN WOLFENSTEIN PARAMETERS AND SCATTERING AMPLITUDES**

<table>
<thead>
<tr>
<th>Wolfenstein Parameter (Eq. 100)</th>
<th>Equivalent Form (Eq. 93)</th>
<th>Equivalent Form (Table 3)(Eq. 98)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>A</td>
<td>$\frac{1}{4}[2M_{11} + M_{00} + M_{ss}]$</td>
</tr>
<tr>
<td>b</td>
<td>$\frac{C - D}{2}$</td>
<td>$\frac{i}{\sqrt{2}} M_c$</td>
</tr>
<tr>
<td>c</td>
<td>$\frac{C + D}{2}$</td>
<td>$\frac{i}{2\sqrt{2}} [M_{01} - M_{10}]$</td>
</tr>
<tr>
<td>m</td>
<td>B</td>
<td>$\frac{1}{4}[- 2M_{1-1} + M_{00} - M_{ss}]$</td>
</tr>
<tr>
<td>g</td>
<td>$\frac{E + F}{2}$</td>
<td>$\frac{1}{4}[2M_{11} + M_{1-1} - M_{ss}]$</td>
</tr>
<tr>
<td>h</td>
<td>$\frac{E - F}{2}$</td>
<td>$\frac{\sqrt{2}}{4 \sin \theta} [M_{01} + M_{10}]$</td>
</tr>
</tbody>
</table>
The density matrix is defined as,

\[ \rho = \psi \psi^+ \]  

(97)

where \( \psi \) refers to the spin part of the wave function only.

If \( \rho_i \) represents the initial (before scattering) density matrix and \( \rho_f \) indicates the final (after scattering) matrix, then,

\[ \rho_f = M \rho_i M^+ \]  

(98)

The utility of \( \rho \) lies in its relationship to the cross section and to the polarization (spin expectation value) of a beam of particles which undergo a scattering. Namely,

\[ \frac{d\sigma}{d\Omega} = \frac{\text{Tr} \rho_f}{\text{Tr} \rho_i} \]  

(99)

and

\[ P = < \vec{\sigma} > = \frac{\text{Tr}(\rho_f \vec{\sigma})}{\text{Tr} \rho_f} \]  

(100)

where \( \text{Tr} (A) \) means to evaluate the trace of the matrix involved, and \( < \vec{\sigma} > \) refers to the averaged spin vector (polarization) of the beam after scattering.

In the subsequent treatment, all quantities of interest such as \( \rho, P, \frac{d\sigma}{d\Omega} \), etc. refer to the incident neutron beam.

For an unpolarized beam, the density matrix normalized to unit incident flux is from Equation (97),
\[
\rho_f = \frac{1}{4} I ,
\]  
(101)

where \(I\) is the 4 x 4 identity matrix.

Equations (98) and (99) then easily give the well known result, 4 (d\sigma/d\Omega)_u = \text{Tr}(M M^+), which together with Equation (95) gives

\[
4 (\frac{d\sigma}{d\Omega})_u = |a|^2 + |m|^2 + 2(|b|^2 + |c|^2 + |g|^2 + |h|^2) .
\]  
(102)

Here the subscript, \(u\), indicates the cross section is for an unpolarized incident beam.

If one makes the usual choice of incident wave vector, \(\bar{k}_{in}\) along the \(z\) axis, and sets \(\phi = 0\), then the vector \(\bar{k}_{in} \times \bar{k}_{out}\) which is normal to the scattering plane is in the \(y\) direction, and the neutron polarization is \(<\sigma^{(1)}_y>\), or

\[
4 (\frac{d\sigma}{d\Omega})_u <\sigma^{(1)}_y> = \text{Tr}(M M^+ \sigma^{(1)}_y) .
\]  
(103)

Again, using Equation (95) along with the relationships for evaluating the traces of products of matrices in the two spin spaces Equation (103) can be shown to give,

\[
\frac{d\sigma}{d\Omega}_u <\sigma^{(1)}_y> = -2 \text{Re}[c^*(a+m)] + 2 \text{Re}[b^*(a-m)] .
\]  
(104)

The two quantities evaluated \((\frac{d\sigma}{d\Omega}_u, \text{ and } P_f)\) can be determined by, at most, two scatterings of the beam. Of the remaining possible
observables at a given scattering angle (in principle there are 256 in all)\(^{(27)}\), five triple scattering parameters have been measured. These will be discussed next. In deriving these quantities, an incident polarized beam is assumed. Wolfenstein\(^{(28)}\) shows the most general expression for the polarization after a second scattering can be written as,

\[
\frac{d\sigma_2}{d\Omega} \overrightarrow{P}_2 = \frac{d\sigma}{d\Omega} u \left\{ \hat{n}_2 [P_2 + DP_1 (\hat{n}_1 \cdot \hat{n}_2) + \hat{s}_2 [AP_1 (\hat{n}_1 \cdot \hat{k}_1) + RP_1 (\hat{n}_1 \cdot \hat{n}_2 \times \hat{k}_1)]
\right.
\]

\[
\left. + \hat{k}_2 [A'P_1 (\hat{n}_1 \cdot \hat{k}_1) + R'P_1 (\hat{n}_1 \cdot \hat{n}_2 \times \hat{k}_1)] \right\}, \quad (105)
\]

where \(\hat{k}_1\) and \(\hat{k}_2\) are unit vectors in the laboratory directions of the incident and scattered beams respectively, and

\[
\hat{n}_2 = \frac{\hat{k}_1 \times \hat{k}_2}{|\hat{k}_1 \times \hat{k}_2|}, \quad \text{and} \quad \hat{s}_2 = \hat{n}_2 \times \hat{k}_2.
\]

Further, \(P_1\hat{n}_1\) represents the polarization of the incident beam, \(P_2\hat{n}_2\) is the polarization which would have occurred as a result of the scattering, had the incident beam been unpolarized, \(\overrightarrow{P}_2'\) is the actual polarization after scattering, \(\frac{d\sigma_2}{d\Omega}\) is the cross section for the polarized beam, and \(D, A, A', R,\) and \(R'\) are the triple scattering parameters.

The density matrix for a beam with polarization, \(\overrightarrow{P}\), can be shown to be\(^{(11)}\)

\[
\rho = \frac{1}{4} (1 + \overrightarrow{P} \cdot \overrightarrow{P}) \quad . \quad (106)
\]

This is shown in Appendix F.
To find $D$, the depolarization parameter, we assume that $P_1 = 1$, and that the scattering plane is such that $\hat{n}_2$ is parallel to $\hat{n}_1$. Then, from Equation (105)

$$
\frac{d\sigma_2}{d\Omega} \overrightarrow{P}_2 = \frac{d\sigma_2}{d\Omega} \left[ P_2 + D \right] \hat{n}_2.
$$

(107)

Now, it is shown in References (11) and (14) that, as a consequence of time reversal invariance,

$$
\text{Tr}(M \sigma_1 M^+) = \text{Tr}(M^+ \sigma_1^+).
$$

(108)

Using Equation (108) along with Equations (99)-(101), one can easily show,

$$
\hat{n}_2 \cdot \overrightarrow{P}_2 = \frac{P_2 + D}{1 + P_2}.
$$

(109)

Further, Equations (98)-(100) and Equation (106) yield,

$$
\hat{n}_2 \cdot \overrightarrow{P}_2 = \frac{\text{Tr} \left\{ \frac{1}{4} \left[ M(1 + \sigma_{n_2}) M^+ \right] \sigma_{n_2}^+ \right\}}{\left( 1 + P_2 \right) \frac{d\sigma}{d\Omega} \left[ P_2 + D \right]}.
$$

(110)

Upon equating Equations (109) and (110), and solving for $D$, one finds

$$
4 \frac{d\sigma}{d\Omega} D = \text{Tr}[M \sigma_{n_2} M^+ \sigma_{n_2}].
$$

(111)
Use of Equation (95) for $M$ gives the explicit result,

$$\frac{d\sigma}{d\Omega}(1-D) = 4|g|^2 + 4|h|^2.$$ (112)

To determine $R$, one again sets $P_1 = 1$, and now selects the scattering plane such that $\hat{n}_1 = \hat{n}_2 \times \hat{k}_1$. In this case,

$$\frac{d\sigma_2}{d\Omega} = \text{Tr}(\rho_f) = \text{Tr}\left(\frac{1}{4} [M(1+\sigma_{\hat{n}_1}) M^+]\right) = \frac{1}{4} \text{Tr}(M M^+) + \frac{1}{4} \text{Tr}(M \sigma_{\hat{n}_1} M^+) .$$

The first factor is just $\frac{d\sigma}{d\Omega} u$, and the second is $\frac{d\sigma}{d\Omega} u P_{\hat{n}_1}$, where $P_{\hat{n}_1}$ is the polarization component along $\hat{n}_1$ resulting from scattering an unpolarized incident beam. Since the polarization vector is in the direction $\hat{n}_2$ in this case, and $\hat{n}_1 \perp \hat{n}_2$, it follows that $P_{\hat{n}_1} = 0$. Thus, Equation (105) gives:

$$\overrightarrow{P}_2 = \hat{n}_2 \ P_2 + \hat{s}_2 \ R + \hat{k}_2 \ R' ,$$ (113)

or,

$$R = \hat{s}_2 \cdot \overrightarrow{P}_2.$$ (114)

From Equations (98)-(100) one may write,

$$\hat{s}_2 \cdot \overrightarrow{P}_2 = \frac{\text{Tr}(\rho_f \sigma_{\hat{s}_2})}{\text{Tr} \rho_f} ,$$

or
\[
R = \frac{\text{Tr}[M \frac{1}{4}(1 + \sigma_{\hat{a}_1}) M^+ \sigma_{\hat{s}_2}]}{\text{Tr}[M \frac{1}{4}(1 + \sigma_{\hat{a}_1}) M^+]} , \quad \text{or}
\]

\[4 \frac{d\sigma}{d\Omega} R = \text{Tr}[M \sigma_{\hat{a}_1} M^+ \sigma_{\hat{s}_2}] . \quad (115)\]

In order to use Equation (115), it is necessary to relate the coordinate systems \(\hat{a}_2, \hat{b}_2, \hat{s}_2; \hat{a}_2, \hat{p}, \hat{k};\) and \(\hat{a}_1, \hat{a}_2, \hat{k}_1.\) The necessary relationships are;

\[
\hat{a}_1 = \hat{k} \cos \frac{\theta}{2} + \hat{p} \sin \frac{\theta}{2} ,
\]

\[
\hat{s}_2 = \hat{k} = -\hat{k}_1 \sin \frac{\theta}{2} + \hat{a}_1 \cos \frac{\theta}{2} , \quad (116)
\]

and

\[
\hat{k}_2 = \hat{p} .
\]

In Equation (116), \(\theta\) is the scattering angle in the center of mass system, and it is important to note that \(\hat{k}_1 \cdot \hat{k}_2 = \cos \theta_L,\) where \(\theta_L\) is the scattering angle in the laboratory coordinate system. Equations (95) and (116) applied to Equation (115) give:

\[
R \frac{d\sigma}{d\Omega} u = [|a|^2 - |m|^2 - 4 \text{Re } c b^* - 4 \text{Re } g h^*] \cos \frac{\theta}{2}
\]

\[ - 2 \text{Re } i[a^*(c+b) - m^*(c-b)] \sin \frac{\theta}{2} . \quad (117)\]

The quantity \(\text{Re}\) means "take the real part".
It is clear that $R'$ is given by a similar procedure, starting with Equation (113).

In order to find $A$ and $A'$ it is necessary to rotate $\hat{n}_1$ by a magnetic field such that it lies along $k_1$. Then, one again sets $P_1 = 1$, and it is easy to see that $\frac{d\sigma_2}{d\Omega} = \frac{d\sigma}{d\Omega}$. With these results, Equation (105) becomes,

$$\bar{P}_2 = \hat{n}_2 P_2 + \hat{s}_2 A + \hat{k}_2 A'$$

(118)

Now, the same procedures used to find $R$ can be used to find $A$ and $A'$.

The two remaining observables that have been measured in scattering experiments are two of the so-called correlation coefficients. These are defined as,

$$4 \frac{d\sigma}{d\Omega} C_{nn} = Tr(M M^+ \sigma_{\hat{n}_2}^{(1)} \sigma_{\hat{n}_2}^{(2)})$$

(119)

and

$$4 \frac{d\sigma}{d\Omega} C_{KP} = Tr(M M^+ \sigma_{\hat{R}}^{(1)} \sigma_{\hat{P}}^{(2)})$$

(120)

and, of course, are a measure of the correlation between neutron and proton polarizations after scattering. Table 8 summarizes the results.
### TABLE 8

**OBSERVABLES IN TERMS OF WOLFENSTEIN PARAMETERS**

<table>
<thead>
<tr>
<th>Observable</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{d\sigma}{d\Omega_u}$</td>
<td>$</td>
</tr>
<tr>
<td>$\frac{d\sigma}{d\Omega_u}$</td>
<td>$\bar{P}$</td>
</tr>
<tr>
<td>$\frac{d\sigma}{d\Omega_u}$</td>
<td>$(1-D)$</td>
</tr>
<tr>
<td>$\frac{d\sigma}{d\Omega_u}$</td>
<td>$R$</td>
</tr>
<tr>
<td>$\frac{d\sigma}{d\Omega_u}$</td>
<td></td>
</tr>
<tr>
<td>$\frac{d\sigma}{d\Omega_u}$</td>
<td>$R'$</td>
</tr>
<tr>
<td>$\frac{d\sigma}{d\Omega_u}$</td>
<td></td>
</tr>
<tr>
<td>$\frac{d\sigma}{d\Omega_u}$</td>
<td>$A$</td>
</tr>
<tr>
<td>$\frac{d\sigma}{d\Omega_u}$</td>
<td></td>
</tr>
<tr>
<td>$\frac{d\sigma}{d\Omega_u}$</td>
<td>$A'$</td>
</tr>
<tr>
<td>$\frac{d\sigma}{d\Omega_u}$</td>
<td></td>
</tr>
<tr>
<td>$\frac{d\sigma}{d\Omega_u}$</td>
<td>$C_{nn}$</td>
</tr>
<tr>
<td>$\frac{d\sigma}{d\Omega_u}$</td>
<td>$C_{KP}$</td>
</tr>
</tbody>
</table>
Phase Shifts by Distorted Wave Calculations. In order to evaluate the effect of the M-S force on the observables it is necessary to find the effect on the phase shifts. To accomplish this it is necessary to know the phase shifts due to the nuclear interaction alone. It will be assumed that those phase shifts determined from experimental data are the nuclear phase shifts. This is probably a good approximation since the effects of the M-S force are expected to be evident only at small angles, and most of the experimental data that is used is from relatively large angle scatterings.

The equations pertinent to determination of the phase shifts are Equations (69)-(72). In Equations (71) and (72), the radial functions do not correspond to pure $l$ states due to the tensor force as has been discussed previously. The phase shifts associated with these mixed wave functions are real provided the mixing is done properly, and the M-S term will modify them, just as in the case of nuclear scattering, through application of Equation (34) or its approximate form, Equation (48). These mixed state phase shifts are not unique since one has some choice in the way the mixing is described. The coupling scheme in general use is that of Stapp, and the phase shifts are referred to as Stapp, or sometimes bar phase shifts. This is presented in a number of places and will not be repeated here (see, for example, Reference 13).

Equations (69) and (70) are coupled, through the M-S force outside the nuclear radius, and hence modification of the phase shifts
associated with the \( j = \ell \) states is more difficult. For convenience Equations (69) and (70) are repeated.

\[
\frac{1}{\rho^2} \frac{d}{d\rho} \left( \rho^2 \frac{dR_{\ell_0}}{d\rho} \right) + \left[ 1 - \frac{\ell (\ell + 1)}{\rho^2} \right] R_{\ell_0} - \frac{\gamma}{2\rho^3} [\ell (\ell + 1)]^{1/2} R_{\ell_1} = 0 \tag{69}
\]

\[
\frac{1}{\rho^2} \frac{d}{d\rho} \left( \rho^2 \frac{dR_{\ell_1}}{d\rho} \right) + \left[ 1 - \frac{\rho (\ell + 1)}{\rho^2} \right] R_{\ell_1} + \frac{\gamma}{2\rho^3} R_{\ell_1} - \frac{\gamma}{2\rho^3} [\ell (\ell + 1)]^{1/2} R_{\ell_0} = 0 \tag{70}
\]

If the scattering matrix (for \( j = \ell \)) is expressed in terms of the phase shifts associated with the pure state wave functions, it will contain nonzero off-diagonal elements. Further, the phase shifts associated with \( R_{\ell_0} \) and \( R_{\ell_1} \) are complex. It is the complex phase shifts that are of interest to the calculations to be done here. To estimate these, it will be assumed that, first a pure singlet wave is being scattered, and then in turn, a pure triplet wave. This allows one to make simplifying approximations in Equations (69) and (70), which give approximate solutions for the off-diagonal elements of the scattering matrix and to the real parts of the phase shifts. The unitarity requirement on the scattering matrix then gives the imaginary part of the phase shifts.

First consider an incoming singlet wave. Since the coupling is very weak, it will be true that \( R_{\ell_0} \gg R_{\ell_1} \), and one can approximate Equations (69) and (70) as,
\frac{1}{\rho^2} \frac{d}{d\rho} \left( \rho^2 \frac{dR_{\ell\ell_0}}{d\rho} \right) + \left[ 1 - \frac{\ell(\ell+1)}{\rho^2} \right] R_{\ell_0} = 0, \quad (121)

and

\frac{1}{\rho^2} \frac{d}{d\rho} \left( \rho^2 \frac{dR_{\ell_1}}{d\rho} \right) + \left[ 1 - \frac{\ell(\ell+1)}{\rho^2} \right] R_{\ell_0} - \frac{\nu}{2\rho^3} \left[ \ell(\ell+1) \right]^{1/2} R_{\ell_0} = 0. \quad (122)

On the other hand, for an incoming triplet wave \( R_{\ell_1} \gg R_{\ell_0} \) and one has,

\frac{1}{\rho^2} \frac{d}{d\rho} \left( \rho^2 \frac{dR_{\ell_1}}{d\rho} \right) + \left[ 1 - \frac{\ell(\ell+1)}{\rho^2} \right] R_{\ell_0} - \frac{\nu}{2\rho^3} \left[ \ell(\ell+1) \right]^{1/2} R_{\ell_1} = 0, \quad (123)

and

\frac{1}{\rho^2} \frac{d}{d\rho} \left( \rho^2 \frac{dR_{\ell_1}}{d\rho} \right) + \left[ 1 - \frac{\ell(\ell+1)}{\rho^2} \right] R_{\ell_1} + \frac{\nu R_{\ell_1}}{2\rho^3} = 0. \quad (124)

These results indicate that the incident singlet wave is not phase-shifted (Equation 121) by the M-S force in this approximation, and that the incident triplet wave is shifted (Equation 124) through the \( \frac{\nu}{2\rho^3} \) term. Thus, the real parts of the phase shifts are approximated by the nuclear phase shift in the singlet case, and the nuclear phase shift modified in the usual manner by the \( \frac{\nu}{2\rho^3} \) term, in the triplet case.

Now in the case of Equations (121) and (122), for an incoming wave of unit amplitude, the asymptotic solutions are of the
form, \( R_{\ell 0} \sim e^{-i \rho} - S_{\infty} e^{i \rho} \) and \( R_{\ell 1} \sim S_{10} e^{i \rho} \). Further, these solutions should have matching logarithmic derivatives at the nuclear radius or at a cut-off radius corresponding to the range of the nuclear force.

The solution for \( R_{\ell 0} \) presents no difficulty and is of the form,

\[
R_{\ell 0} = A_{\ell} \left[ \cos \delta_{\ell 0} \ j_{\ell}(\rho) - \sin \delta_{\ell 0} \ n_{\ell}(\rho) \right]. \tag{125}
\]

The solution for \( R_{\ell 1} \) is obtained by writing a particular solution of Equation (122) by use of the Green's function

\[
G(\rho, \rho') = i \ j_{\ell}(\rho) \ h^{(1)}_{\ell}(\rho')
\]

where \( \rho_0 \) and \( \rho_1 \) indicate the lesser and greater of \( \rho \) and \( \rho' \). One must add the homogenous solution, \( h^{(1)}_{\ell}(\rho) \), to get the complete solution. (In the above expressions, \( h^{(1)}_{\ell} = j_{\ell}(\rho) + i n_{\ell}(\rho) \).) The solution of Equation (122) can now be written as,

\[
R_{\ell 1} = A_{\ell} h^{(1)}_{\ell}(\rho) - \frac{i \nu}{2} [\ell(\ell+1)]^{1/2} \left\{ h^{(1)}_{\ell}(\rho) \int_{\rho_0}^{\rho} \frac{j_{\ell}(\rho') \ R_{\ell 0}(\rho')}{\rho'} \ d\rho' \right\} + j_{\ell}(\rho) \int_{\rho_0}^{\rho} \frac{h^{(1)}_{\ell}(\rho') \ R_{\ell 0}(\rho')}{\rho'} \ d\rho'. \tag{126}
\]

In this expression \( A_{\ell} \) is to be determined by the condition at \( \rho_c \)

\( (= k r_c) \), where \( r_c \) is the nuclear force range.
To determine \( \alpha_\ell \), one first differentiates Equation (126),

\[
R_{\ell 1}'(\rho) = \alpha_\ell h^{(1)'}(\rho) - \frac{i\gamma[\ell(\ell+1)]^{1/2}}{2} j_\ell(\rho) \int_0^\infty \frac{h^{(1)'}(\rho') R_{\ell 0}(\rho')}{\rho'} \, d\rho',
\]

(127)

and then equates logarithmic derivatives at \( \rho = \rho_c \) to get,

\[
\frac{R_{\ell 1}'(\rho_c)}{R_{\ell 1}(\rho_c)} = \frac{\cos \delta_{\ell 1} j_\ell(\rho_c) - \sin \delta_{\ell 1} n_\ell(\rho_c)}{\cos \delta_{\ell 1} j_\ell(\rho_c) - \sin \delta_{\ell 1} n_\ell(\rho_c)}. \tag{128}
\]

Substitution of Equations (126) and (127) into Equation (128) gives,

\[
\alpha_\ell = \frac{-i K_\ell \tan \delta_{\ell 1} \int_0^\infty \frac{h^{(1)'}(\rho') R_{\ell 0}(\rho')}{\rho'} \, d\rho'}{\tan \delta_{\ell 1} + i}, \tag{129}
\]

where \( K_\ell = -\frac{\gamma}{2}[\ell(\ell+1)]^{1/2} \).

At this point it is worth noting that if in Equations (123) and (124) one assumes a form for \( R_{\ell 1} \) as,

\[
R_{\ell 1} \rightarrow [\cos \delta_{\ell 1} j_\ell(\rho) - \sin \delta_{\ell 1} n_\ell(\rho)],
\]

which is certainly valid, then one can follow through completely analogous steps as in the present case, with the result that the solutions for \( R_{\ell 0} \) and \( R_{\ell 1} \) are interchanged along with their respective phase shifts.

The asymptotic form for \( R_{\ell 1} \), from Equation (126) is,

\[
R_{\ell 1} \sim \left[ \alpha_\ell + i K_\ell \int_0^\infty \frac{j_\ell(\rho') R_{\ell 0}(\rho')}{\rho'} \, d\rho' \right] \frac{1}{\rho} e^{i[\rho - (\ell+1)\pi]} \left( \frac{\ell+1}{2} \right). \tag{130}
\]
The value of $A_\ell$ in Equation (125) can be found just as the analogous constants given in Equation (77) were determined, and one finds $A_\ell = i^{\ell+1} \sqrt{4\pi(2\ell+1)} e^{i\delta_{lo}}$. Using this value in Equation (125) and substituting $R_{lo}$ from Equation (125) and $A_\ell$ from Equation (129) in Equation (130), one finds an expression for $R_{ll}$ that is symmetric in $\delta_{ll}$ and $\delta_{lo}$.

$$ R_{ll} = \frac{i^{(\ell+1)} \sqrt{4\pi(2\ell+1)} e^{i(p - \frac{\delta_{ll} + \delta_{lo}}{2})}}{\rho [\sin\delta_{ll}\cos\delta_{lo} + \cos\delta_{ll}\sin\delta_{lo} + i(\cos\delta_{lo}\cos\delta_{ll} - \sin\delta_{lo}\sin\delta_{ll})]} \times [a_\ell(\rho_c)\cos\delta_{lo}\cos\delta_{ll} - b_\ell(\rho_c)(\sin\delta_{lo}\cos\delta_{ll} + \sin\delta_{ll}\cos\delta_{lo}) + c_\ell(\rho_c)\sin\delta_{ll}\sin\delta_{lo}] .$$  \hspace{1cm} (131)

In this expression, $a_\ell(\rho_c)$, $b_\ell(\rho_c)$ and $c_\ell(\rho_c)$ are given by Equations (35)-(37).

The fact that $R_{ll}$ is symmetric in $\delta_{ll}$ and $\delta_{lo}$ together with the fact that for the triplet incident wave $R_{lo}$ and $R_{ll}$ merely interchange forms together with an interchange in phase shifts means that the scattering matrix is symmetric as it should be.

Now, the form of the $j = \ell$ scattering matrix in this approximation is,
where $S_{10} \equiv S_{01}$, $\delta_{L0}$ is the nuclear singlet phase shift, and $\delta_{L1}$ is the nuclear triplet phase shift modified by an M-S contribution. It can be seen that the imaginary part of the phase shift must be the same in both diagonal elements to preserve unitarity. Further, the value of $\beta_L$ can be determined through the unitarity condition.

Consider the asymptotic solutions to Equations (121) and (122),

$$R_{L0} \sim \frac{1}{2i\rho} \frac{\sqrt{4\pi(2\ell+1)}}{i} \left[ e^{2i(\delta_{L0} + i\beta_L)} - e^{i(\rho - \frac{\ell\pi}{2})} - e^{-i(\rho - \frac{\ell\pi}{2})} \right], \quad (133)$$

and

$$R_{L1} \sim \frac{2i(\ell+1)\sqrt{4\pi(2\ell+1)}}{2i\rho} e^{i\delta_{L0}K_L} \left[ \int_{\rho_c}^{\infty} \frac{j_{L}(\rho')\psi_{L0}(\rho')}{\rho'} d\rho' - \tan \delta_{L1} \int_{\rho_c}^{\infty} \frac{n_{L}(\rho')\psi_{L0}(\rho')}{\rho'} d\rho' \right] e^{i(\rho - \frac{\ell\pi}{2})}, \quad (134)$$
where \( \varphi_L = \left[ \cos \delta_L \ j_L(\rho) - \sin \delta_L \ n_L(\rho) \right] \). Now the unitarity condition requires that \( |S_{\alpha \alpha}|^2 + |S_{10}|^2 = 1 \), or, from Equations (133) and (134),

\[
e^{-\bar{\beta}_L} = 1 - 4(K_L)^2 \left\{ \frac{\int_{\rho}^{\infty} j_L(\rho') j_L(\rho') d\rho' - \tan \delta_{L1} \int_{\rho}^{\infty} n_L(\rho') j_L(\rho') d\rho'}{\tan^2 \delta_{L1} + 1} \right\}^2,
\]

(135)

or for \( \beta_L \) small, approximately,

\[
\beta_L = (K_L)^2 \left\{ \frac{\int_{\rho}^{\infty} j_L(\rho') \varphi_L(\rho') d\rho' - \tan \delta_{L1} \int_{\rho}^{\infty} n_L(\rho') \varphi_L(\rho') d\rho'}{\tan^2 \delta_{L1} + 1} \right\}.
\]

(136)

The connection between the complex phase shifts determined here, and the real phase shifts and mixing parameters which were previously introduced, (i.e. the quantities \( \delta_{L+} \), \( \delta_{L-} \), \( \cot \epsilon_L \) which were used to characterize the spin mixing scattering problem), are readily given through comparison of the scattering matrix elements expressed in the two forms (see Equation (87)).
\[
2i(\delta_{\ell 0} + i\beta_{\ell}) = \tan \epsilon_{\ell} e^{2i\delta_{\ell+}} + \cot \epsilon_{\ell} e^{2i\delta_{\ell-}} \tag{137}
\]

\[
2i(\delta_{\ell 1} + i\beta_{\ell}) = \cot \epsilon_{\ell} e^{2i\delta_{\ell+}} + \tan \epsilon_{\ell} e^{2i\delta_{\ell-}} \tag{138}
\]

These two complex equations lead to four expressions, of which only three are independent, the remaining equation being given by the first three and the unitarity condition. These three independent equations are sufficient to determine \(\delta_{\ell+}\), \(\delta_{\ell-}\), and \(\epsilon_{\ell}\) in terms of \(\delta_{\ell 0}\), \(\delta_{\ell 1}\), and \(\beta_{\ell}\).

By carrying out the appropriate algebra in Equations (137) and (138) one obtains the desired relations.

First, define

\[
\alpha_1 = e^{-2\beta_{\ell}} [\cos 2\delta_{\ell 0} + \cos 2\delta_{\ell 1}] \\
\alpha_2 = e^{-2\beta_{\ell}} [\sin 2\delta_{\ell 0} + \sin 2\delta_{\ell 1}] \tag{139}
\]

Then,

\[
\delta_{\ell+} = \frac{1}{2} \left\{ \cos^{-1} \left[ \frac{\alpha_1}{\sqrt{\alpha_1^2 + \alpha_2^2}} \right] + \cos^{-1} \left[ \frac{1}{2}(\alpha_1^2 + \alpha_2^2)^{1/2} \right] \right\}, \tag{140}
\]
\[ \delta_{\ell^-} = \frac{1}{2} \left\{ \cos^{-1} \frac{\alpha_1}{\sqrt{\alpha_1^2 + \alpha_2^2}} - \cos^{-1} \left[ \frac{1}{2}(\alpha_1^2 + \alpha_2^2)^{1/2} \right] \right\}, \quad (141) \]

and

\[ \epsilon_{\ell} = \tan^{-1} \left[ \left( \frac{e^{-2\beta_{\ell}} \cos 2\delta_{\ell_0} - \cos 2\delta_{\ell} \cos 2\delta_{\ell_0}}{\cos 2\delta_{\ell} - e^{-2\beta_{\ell}} \cos 2\delta_{\ell_0}} \right)^{1/2} \right]. \quad (142) \]

Thus, this distorted wave calculation together with the experimentally determined singlet and triplet phase shifts provide the means for estimating the influence of the M-S force on n-p scattering. Phase shifts determined in this manner include the effects of wave distortion, even for small \( \ell \) in contrast to those determined by the plane-wave Born approximation as discussed by Breit(18).
CHAPTER III

RESULTS OF THE CALCULATIONS

Neutron-Nucleus Scattering

Results of the Calculations. Typical results of the optical model calculations are presented in Figures 1 through 29. Calculations were done for Al, Mn, Nb, and Bi at energies of 0.5, 1, 7, 14.5, and 24 Mev. Those results which are repetitious, or which do not show any features of particular interest have been omitted. The differential cross-sections are compared with experimental data from BNL-400\(^1\) in some cases, and with the optical model results of Agee and Rosen\(^2\) in other cases, as noted on the curves. In general the agreement is good, the only exception being Al at 1 Mev, (not shown) and Bi at 0.5 (Figure 18) and 1 Mev (not shown). These cases of relatively poor agreement are probably attributable to the approximate method of treating compound elastic scattering in the present calculations. The overall, generally good agreement with experiment serves to establish


a degree of confidence in the choice of optical model parameters. In
general, the inclusion of the M-S term has very little influence on
\( \frac{d\sigma(\theta)}{d\Omega} \) except at very small angles, \( \sim 2^\circ \) or less. This effect was
expected, and is greatest at low energies and for high Z targets, and
was given in all cases considered, to an accuracy of about 1\% or
better by the Born approximation for the phase shifts. In fact, a
somewhat unexpected result of the study was that use of the Born
approximation for the phase shifts (Equation 50) was adequate in all
cases for both \( \frac{d\sigma(\theta)}{d\Omega} \), and \( P(\theta) \). Polarizations calculated with the
M-S effect included exactly and through use of Equation (50) agreed
to within 0.5\% in all cases. This result can be explained qualita-
tively on the basis that one expects the Born approximation to be
valid when \( E \gg V \), and for the values of \( \rho_c \) which were used, this
condition was always satisfied. Further insight into the validity of
the Born approximation is given from the approximate solutions for the
phase shifts as given in Equation (49). The Born approximation is
seen to result from Equation (49) if \( c_L(\rho_c) \) is small, and this is the
case for \( \rho_c \) large. That is, for \( \rho_c \) large enough Equations (43)
through (46) become,

\[
F^{(1)}_{\ell j}(\rho) = \lambda_{\ell j} a_{\ell}(\rho), \tag{143}
\]

\[
G^{(2)}_{\ell j}(\rho) = 1, \tag{144}
\]

\[
F^{(2)}_{\ell j}(\rho) = 1, \tag{145}
\]
and

\[ G_{lj}^{(1)}(p) = 0 \]

(146)

and these results substituted in Equation (34) give the Born approximation.

The calculations for P(\theta) gave results that were not entirely expected. The small-angle influence of the M-S term previously discussed is clearly present in all cases. This effect, like the small-angle effect on \( \frac{d\sigma(\theta)}{d\Omega} \), is generally greatest at low energies and high Z. The calculation for Nb at 0.5 Mev (Figure 11) substantiates the influence of the M-S force as calculated in References (3) and (4), although there is a general lack of agreement with the experimental results of Reference (4) at 56° and 86°. This "small-angle" effect is seen to exert a significant influence on P(\theta) for Bi at 0.5 Mev at angles as large as 40°-50° (Figure 17). In all cases, at the low energies investigated, this influence was evident out to 20°, and as just noted, to 50° in the extreme case, however, for the highest energy (24 Mev) this influence was restricted to the 10°-15° range. For all isotopes studied, this small-angle effect was the only one observed at low energies. However, for the higher Z nuclei and at the higher energies, the polarization is seen to be influenced by


the M-S term in a somewhat different manner. Note, for example, Mn at 7 Mev in the 50°-60° range (Figure 7), Nb at 7 Mev and 14.5 Mev in the 30°-60° range (Figures 13 and 15), and Bi at the higher energies in the 25°-40° range. Indeed, Bi at 7 Mev shows an effect at 130°-140° (Figure 20). An examination of the curves indicates that these effects on polarization occur at angles which correspond to regions near minima in the cross-section curve.

The general results of the polarization can be explained qualitatively. At very small angles, the cot $\frac{\theta}{2}$ term (see Equation (62)) dominates the quantity, $h(\theta)$, when the M-S term is included in the potential. As $\theta$ increases, $|h(\theta)|$ tends to become small relative to $|g(\theta)|$ and effects on polarization are not seen, unless $|g(\theta)|$ is near a minimum. That is, at most angles (not too small) $|g|^2 \gg |h|^2$ and $\sigma \sim |g|^2$. However, at or near cross-section minima $|g|^2$ and $|h|^2$ can be of the same order. Since, also, the influence of the M-S term is usually stronger on $h(\theta)$ than on $g(\theta)$, the M-S effect on polarization can be expected to be nonnegligible when $|h| \sim |g|$. At even larger angles, the cot $\frac{\theta}{2}$ contribution to $h(\theta)$ becomes very small and even though $\frac{d\sigma}{d\Omega}$ is generally small at these large angles, $g(\theta)$ and $h(\theta)$ are changed so little by the M-S term that little effect on the polarization is noted. (For example: Bi at 7 Mev and 60°,(Figure 19).

It should be pointed out that although the M-S term might be expected to be more effective at higher energies, the nuclear spin-orbit term is also more influential at the higher energies and can be strong enough to obscure the effect of M-S scattering on polarization. This
is evidently the case for the lower Z nuclei since no "large angle"
effects are noted for Al, for Mn the "large angle" influence is small
at 14.5 (not shown) and 24 Mev, (Figure 9) and for Nb it is small at
24 Mev (not shown).

Since the strength of the nuclear spin-orbit term is not well-
known, it was varied to study the influence it might have on the cal-
culated polarization curves. At low energies, moderate variations of
V_s were found to exert virtually no influence on the polarization
curves. For example, for Nb at 0.5 Mev, the curves calculated for V_s =
7.2 Mev and V_s = 10 Mev were indistinguishable. This confirms similar
findings noted in Reference (4). In order to evaluate this effect at
higher energies, Bi polarizations at 7 Mev and 24 Mev were examined
for V_s values of 5 Mev and 10 Mev. The 24 Mev results are typical
and are shown here. In general the polarizations were larger for larger
V_s magnitudes, but the changes in polarizations due to the M-S term
were not substantially effected (see Figures 24 through 26).

Figures (28) and (29) show small angle polarizations for Bi
at 7 and 24 Mev. The finer angular spacing taken for these calculations
shows the polarizations reach nearly 100% at some angle (less than 1°)
in both cases. This means, of course, that the wider angular spacing
used in the other calculations may not show sufficient detail to give
the maximum polarization in all cases.

Influence of M-S Force on p Wave Resonances. The influence of
M-S scattering on p wave resonance level widths, and of course by impli-
cation, the p wave strength function and transmission coefficient can be
FIGURE 1. POLARIZATION OF 0.5 MEV NEUTRONS SCATTERED BY Al
FIGURE 2. DIFFERENTIAL SCATTERING CROSS SECTION FOR 0.5 MEV NEUTRONS SCATTERED BY Al
FIGURE 3. POLARIZATION OF 14.5 MEV NEUTRONS SCATTERED BY AI
FIGURE 4. DIFFERENTIAL CROSS SECTION FOR 14.5 MEV NEUTRONS SCATTERED BY Al
FIGURE 5. POLARIZATION OF 1 MEV NEUTRONS SCATTERED BY Mn

Calculated

+ No MS
o With MS

$P(\theta)$ percent

$\theta_{CM},$ degrees

$\theta = 01^\circ$
FIGURE 6. DIFFERENTIAL SCATTERING CROSS SECTION FOR 1 MEV NEUTRONS SCATTERED BY Mn
FIGURE 7. POLARIZATION OF 7 MEV NEUTRONS SCATTERED BY Mn
FIGURE 8. DIFFERENTIAL SCATTERING CROSS SECTION FOR 7 MEV NEUTRONS SCATTERED BY Mn
FIGURE 9. POLARIZATION OF 24 MEV NEUTRONS SCATTERED BY Mn
FIGURE 10. DIFFERENTIAL CROSS SECTION FOR 24 MEV NEUTRONS SCATTERED BY Mn
FIGURE 11. POLARIZATION OF 0.5 MEV NEUTRONS SCATTERED BY Nb
FIGURE 12. DIFFERENTIAL SCATTERING CROSS SECTION FOR 0.5 MEV NEUTRONS SCATTERED BY Nb
FIGURE 13. POLARIZATION OF 7 MEV NEUTRONS SCATTERED BY Nb
FIGURE 14. DIFFERENTIAL SCATTERING CROSS SECTION FOR NEUTRONS SCATTERED BY Nb AT 7 MEV
FIGURE 15. POLARIZATION OF 14.5 MEV NEUTRONS SCATTERED BY Nb
FIGURE 16. DIFFERENTIAL CROSS SECTION FOR 14.5 MEV NEUTRONS SCATTERED BY Nb
Figure 17. Polarization of 0.5 MeV neutrons scattered by Bi.
FIGURE 18. DIFFERENTIAL SCATTERING CROSS SECTION FOR 0.5 MEV NEUTRONS SCATTERED BY Bi
FIGURE 19. POLARIZATION OF 7 MEV NEUTRONS SCATTERED BY Bi
FIGURE 20. LARGE ANGLE POLARIZATION OF 7 MEV NEUTRONS SCATTERED BY Bi
FIGURE 21. DIFFERENTIAL SCATTERING CROSS SECTION FOR 7 MEV NEUTRONS SCATTERED BY Bi
FIGURE 22. POLARIZATION OF 14.5 MEV NEUTRONS SCATTERED BY Bi
FIGURE 23. DIFFERENTIAL CROSS SECTION FOR 14.5 MEV NEUTRONS SCATTERED BY Bi
FIGURE 24. POLARIZATION OF 24 MEV NEUTRONS SCATTERED BY Bi ($V_s = 7.2$ MEV)
FIGURE 25. POLARIZATION OF 24 MEV NEUTRONS SCATTERED BY Bi ($V_s = 5$ MEV)
FIGURE 26. POLARIZATION OF 24 MEV NEUTRONS SCATTERED BY Bi (Vs = 10 Mev)
FIGURE 27. DIFFERENTIAL CROSS SECTION FOR 24 MEV NEUTRONS SCATTERED BY Bi
FIGURE 28. SMALL ANGLE POLARIZATION OF 7 MEV NEUTRONS SCATTERED BY Bi (WITH M-S)
FIGURE 29. SMALL ANGLE POLARIZATION OF 24 MEV NEUTRONS SCATTERED BY Bi (WITH M-S)
estimated through use of the wave functions of Equation (3). In investigating this effect the nomenclature of Blatt and Weisskopf was adopted and the pertinent results of their treatment extended to include the $j$ dependence introduced through the spin-orbit interaction.

The resonance cross-section is given by,

$$
\sigma_{r,\ell j} = (\ell + 1) \pi \kappa^2 \frac{(-4) S_\ell^+ \text{Im} f_\ell^+}{(\text{Ref}_\ell^+ - \Delta_\ell^+)^2 + (\text{Im} f_\ell^+ - S_\ell^+)^2} + \ln \kappa^2 \frac{(-4) S_\ell^- \text{Im} f_\ell^-}{(\text{Ref}_\ell^- - \Delta_\ell^-)^2 + (\text{Im} f_\ell^- - S_\ell^-)^2},
$$

(147)

where $f_\ell^\pm = R \left[ \frac{1}{U_\ell^\pm} \frac{d}{dr} \right]_{r=R}$, $U_\ell^\pm$ is the radial wave function comprised of a linear combination of incoming and outgoing waves, and

$$
\Delta_\ell^\pm + i S_\ell^\pm = R \left[ \frac{1}{\varphi_\ell^\pm} \frac{d}{dr} \right]_{r=R}
$$

where $\varphi_\ell^\pm$ is a solution of the radial Schrodinger Equation corresponding to an outgoing spherical wave. That is, $\varphi_\ell^\pm = G_\ell^\pm + i F_\ell^\pm$ where these two functions are linearly independent.

solutions of the radial Schrödinger Equation outside the nuclear radius
and \( U^\pm_L = F^\pm_L - t^\pm_L G^\pm_L; \) \( t^\pm_L = \) tangent of the phase shift, \( \delta^\pm_L \).

In the above designations, the + and - correspond to the cases
\( j = \ell + 1/2 \) and \( j = \ell - 1/2 \) respectively.

It is further shown in Reference (5) that,

\[
\Delta^\pm_L = R \left[ \frac{d}{dr} \frac{d G^\pm_L}{dr} - \frac{d}{dr} \frac{d F^\pm_L}{dr} \right]_{r=R}
\]

(148)

and

\[
S^\pm_L = R \left[ \frac{d}{dr} \frac{d F^\pm_L}{dr} + \frac{d}{dr} \frac{d G^\pm_L}{dr} \right]_{r=R}
\]

(149)

By use of the Wronskian relation between \( G^\pm_L \) and \( F^\pm_L \), Equation (149) becomes,

\[
S^\pm_L = \frac{kr}{(G^\pm_L)^2 + (F^\pm_L)^2} = kR V^\pm_L;
\]

(150)

and \( V^\pm_L \) is called the "penetration factor".

Now the \( \ell \)th wave cross section for formation of the compound
nucleus (through channel \( \alpha \)) is expressed as,

\[
\sigma_{c\ell}(\alpha) = (\ell+1) \pi \frac{\lambda^2}{\ell} T^+_\ell(\alpha) + \ell \pi \frac{\lambda^2}{\ell} T^-\ell(\alpha),
\]

(151)
where \( T_\ell^\pm(\alpha) \approx \frac{4k}{K} V_\ell^\pm \) for \( k \ll K \), \( \left( K = \frac{2m(E+V)}{\hbar^2} \right) \), and \( V \) is the depth of the nuclear potential well. Further, the channel width, \( \Gamma_\alpha^\pm \), is defined approximately by,

\[
\Gamma_\alpha^\pm \approx \Gamma_\ell^\pm(\alpha) \frac{D}{2\pi} = \frac{4k}{K} \frac{V_\ell^\pm}{2\pi} \frac{D}{2\pi}, \quad (D = \text{level spacing}) \text{, and}
\]

it is customary to write this expression as, \( \Gamma_\alpha^\pm = (2k R V_\ell^\pm) (\gamma_\alpha) \)

where \( \gamma_\alpha \) (called the reduced width) is dependent on conditions inside the nucleus.

In the absence of M-S scattering, \( V_\ell^+ = V_\ell^- \) and, \( F_\ell(R) = kR j_\ell(kR) \) and \( G_\ell(R) = kR n_\ell(kR) \). On the other hand, when M-S scattering is included, the solution \( U_\ell^\pm \) can be written,

\[
U_\ell^\pm = kR \left[ \sum t_{\ell j} C_{\ell j}^{(1)}(kR) + C_{\ell j}^{(2)}(kR) \right] e^{\lambda_{\ell j} b_{\ell}(kR)} j_{\ell}(kR)
- \left[ F_{\ell j}^{(1)}(kR) + t_{\ell j} F_{\ell j}^{(2)}(kR) \right] e^{\lambda_{\ell j} b_{\ell}(kR)} n_{\ell}(kR) \right].
\]

(152)

However, these two terms are not linearly independent solutions as can be seen in the way the auxiliary \( F \) and \( G \) functions are coupled through Equations (29) through (32). It is a simple matter to rearrange terms such that "proper" groupings occur as,
\[
U^\pm_L = kR \left\{ \left[ G^{(2)}_{Lj} e^{\lambda Lj} b_L j_L - F^{(1)}_{Lj} e^{-\lambda Lj} b_L n_L \right] - t_{Lj} \left[ G^{(1)}_{Lj} e^{\lambda Lj} b_L j_L + F^{(2)}_{Lj} e^{-\lambda Lj} b_L n_L \right] \right\} . \tag{153}
\]

The bracketed terms in the above expression are, then, the \( F_L \) and \( G_L \) used in the definition of \( V^\pm_L \).

On physical grounds, one might expect the penetration to be affected, (especially at low energy), by the electromagnetic interaction. It should be reduced for the repulsive potential, \((j = \ell + 1/2)\) and increased for the attractive one.

This effect was evaluated for the case \( E = 10^{-4} \) Mev, \( Z = 41 \), and \( kR = 1.205 \times 10^{-2} \). (According to Reference (6) \( _{41}\text{Nb}^{93} \) has a p-wave resonance near 100 ev.) The necessary functions were determined numerically and are listed in Table 9.

The resulting values for the penetration factors are, \( V^+_L = 1.43 \times 10^{-4} \), \( V^-_L = 1.48 \times 10^{-4} \) while for no M-S term, one finds \( V^+_L = 1.455 \times 10^{-4} \). The correct weighting (see Equations (147) and (151)) is (for p-waves) given by \( \overline{V}_L = (2V^+_L + V^-_L)/3 = 1.447 \times 10^{-4} \). Thus, the effect of the M-S term on the p-wave penetration factor is seen to be to decrease it by a very small amount, of the order of 1/2%.

### TABLE 9

**AUXILIARY FUNCTIONS EVALUATED FOR**

\( Z = 41, \ E = 10^{-4} \) Mev, and \( \rho = 1.205 \times 10^{-2} \)

<table>
<thead>
<tr>
<th>Function</th>
<th>( t = j+1/2 )</th>
<th>( t = j-1/2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G_{\ell j}^{(2)}(\rho) )</td>
<td>0.788</td>
<td>0.181</td>
</tr>
<tr>
<td>( F_{\ell j}^{(1)}(\rho) )</td>
<td>( 0.664 \times 10^{-4} )</td>
<td>( -0.132 \times 10^{-3} )</td>
</tr>
<tr>
<td>( G_{\ell j}^{(1)}(\rho) )</td>
<td>( -0.319 \times 10^{-4} )</td>
<td>( 0.616 \times 10^{-4} )</td>
</tr>
<tr>
<td>( F_{\ell j}^{(2)}(\rho) )</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>( e^{\lambda_{\ell j} b_{\ell}(\rho)} )</td>
<td>0.9927</td>
<td>1.0146</td>
</tr>
<tr>
<td>( e^{-\lambda_{\ell j} b_{\ell}(\rho)} )</td>
<td>1.0073</td>
<td>0.9854</td>
</tr>
</tbody>
</table>
Neutron-Proton Scattering

The Scattering Calculations. For convenience and completeness a brief summary of the procedure followed in performing the n-p scattering calculations is given here.

First, phase shifts based on fitting experimental data were taken from Reference (7) and assumed to be due to the nuclear interaction only. For the phase shifts corresponding to \( l = J \pm 1 \) partial waves, the M-S effect on the phase shifts was estimated through use of Equations (49). For \( l = J \) partial waves an imaginary part, \( \delta \), was determined from Equation (135) and combined with real parts, \( \delta^0 \), the nuclear phase shift associated with the singlet \( l = J \) wave, and \( \delta \) the nuclear phase shift as modified by Equation (49), that corresponds to the triplet \( l = J \) partial wave.

The appropriate amplitude forms (Table 2) were evaluated with \( \varphi = 0 \) (and therefore \( \sigma_n = \sigma_y \)). For those cases where the Born approximation to the M-S term gives a nonzero scattering amplitude, the high \( l \) wave contributions to the matrix element were included in the same manner as they were included in the \( h(\theta) \) term as given in Equation (62). That is, the series was truncated at the appropriate \( l \), and a correction, \( \Delta M \), was added to these matrix elements where it applied. The correction is,

\[ (7) \text{R. A. Arndt and M. H. MacGregor, Phys. Rev. 141, 873 (1966).} \]
\[ \Delta M = \pm \frac{\gamma}{2\sqrt{2}} \left[ \cot \frac{\theta}{2} - \sum_{\ell=0}^{\ell_c} \frac{2\ell+1}{\ell(\ell+1)} P^1_{\ell} (\cos \theta) \right], \]  
\hspace{1cm} (154)

where the positive sign goes with those elements below the principal diagonal, the negative sign with those above. It happens that only those matrix elements which contain the \( P^1_{\ell} (\cos \theta) \) factors in the sums have nonzero Born amplitudes. Hence, it can be seen that the only Wolfensteiu parameters (Table 7) that are altered by the M-S term as calculated by the Born approximation are \( b \) and \( c \). More explicitly, the corrected matrix elements can be written (in terms of those as given by Equation (90)) as, \( M_{c} = \Delta M, M_{01} = \Delta M, M_{10} = \Delta M \), where it is understood that the sums involved are truncated at \( \ell = \ell_c \).

As in the case of scattering by nuclei, the purpose of this study was not to obtain the best fit to experimental data, but to find the general effects of the M-S force. Customarily, in determining phase shifts through analysis of experimental data, the large \( \ell \) wave contributions have been included by assuming the nuclear potential acting on these partial waves is the one-pion exchange potential (OPEP).\(^8\) The effects of the OPEP on the amplitude matrix elements are normally included in the Born approximation in much the same spirit as large \( \ell \) wave contributions to the M-S scattering have been included here. This one-pion exchange contribution (OPEC) has not

been included in these calculations. Since it has a rather small effect on the observables\(^{(9)}\), and would not contribute anything to the evaluation of the M-S effects, it seemed to be an unnecessary complication. The end result of including the OPEC for large \(l\) waves would have been small changes in the calculated observables. These changes would be the same independent of whether the M-S force is included or not.

Nuclear phase shifts from Reference (7) at energies of 25, 50, 95, 142, and 210 Mev were used to evaluate the effect of the M-S force on the nine scattering parameters of Table 8. The calculations were performed on the Battelle CDC-6400 digital computer. Typical results are shown in Figures 30 through 41, together with the experimental data used for the nuclear phase shift determinations reported in Reference (7) and taken from References (9) and (10). In all cases, significant M-S effects are confined to small angles and hence do not exert any influence on the phase shift determinations. The relatively good agreement of the calculated results with the experimental data would seem to justify omission of the OPEC in this application. Results at other energies and for other scattering


parameters follow the pattern of those shown. It should be noted that the cutoff radius for the proton was taken at 1.7 f and that a 30 percent variation around this value was found to have an insignificant influence on the results.

The measure of the smallness of the effect on phase shifts etc. is illustrated in Table 10 where the eigen phase shifts, mixing parameters and the imaginary part of the complex phase shifts are listed for the 142 Mev case.

The results show that in general the M-S influence on the scattering parameters is confined to small scattering angles. In fact, for all parameters except the polarization and the correlation coefficient, $C^{(\theta)}_{KP}$, any significant influence is restricted to $\sim 2^\circ$ or less. For $P(\theta)$ and $C^{(\theta)}_{KP}$, the M-S effects are present at larger angles. The $C^{(\theta)}_{KP}$ influence at 25 Mev extends out to the 30°-40° range and the polarization effects out to $\sim 10^\circ$. Unfortunately, $C^{(\theta)}_{KP}$ at 25 Mev is small, being of the order of $10^{-3}$ at 30°, and it has not yet been measured at any energy in the n-p case. At higher energies (where $C^{(\theta)}_{KP}$ becomes larger) the influence is restricted to smaller angles as can be seen by examining the polarization curves at various energies. It was also found that the small angle effect on polarization becomes small at the higher energies.

The fact that $P(\theta)$ and $C^{(\theta)}_{KP}$ are the most strongly influenced parameters can be explained by the fact that they depend most strongly on the Wolfenstein parameters, b and c, and that these are the only Wolfenstein parameters that are influenced by the large $\ell$ wave
TABLE 10

EIGEN PHASE SHIFTS AND MIXING PARAMETERS FOR 142 MEV

(All Angles in Radians)

<table>
<thead>
<tr>
<th>$\ell$</th>
<th>$\delta_{\ell+}$</th>
<th>$\delta_{\ell-}$</th>
<th>$\epsilon_\ell$</th>
<th>$\beta_\ell$</th>
</tr>
</thead>
<tbody>
<tr>
<td>NoM-S 0</td>
<td>0.28693</td>
<td>0</td>
<td>$\pi/2$</td>
<td>0</td>
</tr>
<tr>
<td>M-S 0</td>
<td>0.28693</td>
<td>0</td>
<td>$\pi/2$</td>
<td>0</td>
</tr>
<tr>
<td>NoM-S 1</td>
<td>-0.29653</td>
<td>-0.27192</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>M-S 1</td>
<td>-0.29651</td>
<td>-0.27192</td>
<td>0.00117</td>
<td>$8.2 \times 10^{-10}$</td>
</tr>
<tr>
<td>NoM-S 2</td>
<td>0.41521</td>
<td>0.08552</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>M-S 2</td>
<td>0.41524</td>
<td>0.08552</td>
<td>0.00015</td>
<td>$2.27 \times 10^{-9}$</td>
</tr>
<tr>
<td>NoM-S 3</td>
<td>-0.03770</td>
<td>-0.01606</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>M-S 3</td>
<td>-0.03766</td>
<td>-0.01606</td>
<td>0.00602</td>
<td>$1.691 \times 10^{-8}$</td>
</tr>
<tr>
<td>NoM-S 4</td>
<td>0.07295</td>
<td>0.01099</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>M-S 4</td>
<td>0.07299</td>
<td>0.01100</td>
<td>0.00285</td>
<td>$3.117 \times 10^{-8}$</td>
</tr>
<tr>
<td>NoM-S 5</td>
<td>-0.01099</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>M-S 5</td>
<td>-0.01097</td>
<td>$2.5 \times 10^{-6}$</td>
<td>0.01510</td>
<td>$2.745 \times 10^{-8}$</td>
</tr>
</tbody>
</table>
correction term, $\Delta M$. These considerations would indicate that two of the four correlation coefficients pertinent to scattering of polarized beams\textsuperscript{(11)} might be rather strongly modified by the M-S force. These coefficients are defined in a manner similar to $C_{nn}$ and $C_{KP}$ but have never been measured and so they were not included in the calculations. Using the same nomenclature used in the derivation of the nine scattering parameters of Table 8, these coefficients are defined as,

\[
4 \left( \frac{d\sigma}{d\Omega} \right) u C_{nnn} = \text{Tr} \left[ M \sigma_{\hat{r}_2}^{(1)} M^+ \sigma_{\hat{r}_2}^{(1)} \sigma_{\hat{r}_2}^{(2)} \right] 
\]

\[
= -2 \text{Im} [(c-b)(a^*+m^*)] 
\]

\[
4 \left( \frac{d\sigma}{d\Omega} \right) u C_{nKP} = \text{Tr} \left[ M \sigma_{\hat{r}_2}^{(1)} M^+ \sigma_{\hat{r}_2}^{(1)} \sigma_{\hat{p}_2}^{(2)} \right] 
\]

\[
= -2 \text{Im} [(a-m)g^* + (a+m)h^*] 
\]

\[
4 \left( \frac{d\sigma}{d\Omega} \right) u C_{K2} \times \hat{k}_1, K, n = \text{Tr} \left[ M \sigma_{\hat{r}_2}^{(1)} \times \hat{k}_1 M^+ \sigma_{\hat{r}_2}^{(1)} \sigma_{\hat{r}_2}^{(2)} \right] 
\]

\[
= -2 \text{Re} [a(c^*-b^*) + m(c^*+b^*)] \cos \frac{\theta}{2} - 2 \text{Im} [ma^*+2bc^*] \sin \frac{\theta}{2} 
\]

\[
4 \left( \frac{d\sigma}{d\Omega} \right) u C_{K2} \times \hat{k}_1, nP = \text{Tr} \left[ M \sigma_{\hat{r}_2}^{1} \times \hat{k}_1 M^+ \sigma_{\hat{r}_2}^{(1)} \sigma_{\hat{p}_2}^{(2)} \right] 
\]

\[
\text{(11) L. Wolfenstei, An. Rev. of Nucl. Sci. 6, 43 (1956).}
\[ = 2 \text{Im} \left[ g^* (a+m) + h^* (a-m) \right] \cos \frac{\theta}{2} - 4 \text{Re} \left[ cg^* bh^* \right] \sin \frac{\theta}{2} \]  

These forms indicate a relatively strong M-S influence on \( C_{nnn} \) and \( C_{n2} \times k_1 \) and a weak influence on the remaining two coefficients.

The M-S Term and the Deuteron. For the sake of completeness, the influence of the M-S term on the binding energy of the deuteron has been estimated.

First, recall that the ground state of the deuteron is the only bound state, that is a triplet-even state, and that it is a mixture of S and D states (see Reference (12), Chapter 5). The wave function for the deuteron can be expressed as,

\[ r \psi = f_s(r) \, \phi_{101} + f_D(r) \, \phi_{121} , \]  

where \( \frac{f_s(r)}{r} \) and \( \frac{f_D(r)}{r} \) represent the radial wave functions for the singlet and triplet states respectively, and \( \phi_{1m} \) is the spin-angle function defined by Equation (66).

For the present purpose it is pertinent to note from Equation (67) that,

\[(12)\text{M. A. Preston, Physics of the Nucleus, Addison-Wesley, Reading, Mass. (1962).}\]
FIGURE 30. N-P DIFFERENTIAL SCATTERING CROSS SECTION AT 25 MEV
FIGURE 31. NEUTRON POLARIZATION IN N-P SCATTERING AT 25 MEV
FIGURE 32. SMALL ANGLE POLARIZATION FOR N-P SCATTERING AT 25 MEV
FIGURE 33. N-P DIFFERENTIAL SCATTERING CROSS SECTION AT 50 MEV
FIGURE 34. NEUTRON POLARIZATION FOR N-P SCATTERING AT 50 MEV
FIGURE 35. N-P DIFFERENTIAL SCATTERING CROSS SECTION AT 142 MEV
FIGURE 36. NEUTRON POLARIZATION IN N-P SCATTERING AT 142 MEV
FIGURE 37. NEUTRON DEPOLARIZATION FOR N-P SCATTERING AT 142 MEV
FIGURE 38. $R(\theta)$ FOR N-P SCATTERING AT 142 MEV
FIGURE 39. $A(\theta)$ IN N-P SCATTERING AT 142 MEV
FIGURE 40. CORRELATION COEFFICIENT, $C_{kp}$, FOR N-P SCATTERING AT 142 MEV
FIGURE 41. SMALL ANGLE SCATTERING PARAMETERS FOR N-P SCATTERING AT 142 MEV
and that all other matrix elements between the pure state spin-angle functions vanish.

Now, the M-S potential will be defined as constant inside the charge radius of the proton, and as varying as \( \frac{1}{r^3} \) in the exterior region. That is,

\[
V_{MS} = \left( \frac{he}{Mc} \right)^2 |\mu_n| \frac{\bar{l} \cdot \bar{s}_n}{r_c^3} \quad \text{for } r \leq r_c
\]

\[
= \left( \frac{he}{Mc} \right)^2 |\mu_n| \frac{\bar{l} \cdot \bar{s}_n}{r^3} \quad \text{for } r > r_c
\]

(161)

where the symbols have the meanings given after Equation (10), Chapter II, and \( r_c \) denotes the charge radius of the proton.

If the M-S term is treated as a perturbation, the change in binding energy is (in first order),

\[
\Delta E = \frac{\int \phi^* V_{MS} \psi r^2 \, dr}{\int \phi^* \psi r^2 \, dr} .
\]

(162)

Using Equations (159), (160), and (161), in Equation (162), one gets,
\[ \Delta E = -\frac{3}{2} \left( \frac{\hbar c}{M_C} \right)^2 \mu_n \frac{\int_0^r \frac{f_D^2(r)}{r^3} \, dr + \int_r^\infty \frac{f_D^2(r)}{r^3} \, dr}{\int_0^\infty (f_s^2(r) + f_D^2(r)) \, dr}. \] (163)

This expression was evaluated by using \( f_s(r) \) and \( f_D(r) \) as computed by Gartenhaus. (13) A value of \( 1.72 \times 10^{-13} \) cm. was used for \( r_c \) (see Reference (12), Page 130).

The resulting energy shift obtained from Equation (163) is, \( \Delta E = -1.64 \times 10^{-3} \) Mev. It was found that \( \Delta E \) was relatively insensitive to moderate variations in \( r_c \).

CHAPTER IV

SUMMARY AND CONCLUSIONS

The Neutron-Nucleus Problem

We conclude with a brief summary and a few statements relative to the conclusions which can be inferred.

Optical model calculations were performed to determine the effect of M-S scattering on the differential cross-section and on the polarizations resulting when neutrons are scattered by nuclei. The results were:

1. The differential cross-section is influenced only at angles of $\sim 2^\circ$ or less.

2. The polarization can be influenced at angles much greater than had been previously considered, primarily at angles which lie near minima in the cross-section curve, and at the higher energies considered.

3. For the energies and nuclei considered, the M-S influence tends to be large for large $Z$ and small $E$, although the "large-angle" effect is not present for $E < 1$ Mev.
(4) Use of the Born approximation to compute that part of the phase shift due to the M-S potential was found to be entirely adequate for all cases considered.

(5) The effect of the M-S potential on the p-wave penetration factor was found to be negligibly small.

One concludes from this that the polarization influence of M-S scattering extends to large angle, for \( Z \) not too small, and therefore, any optical model studies which would attempt to fit polarization data or make some evaluation of the nuclear spin-orbit term must include the M-S potential. Moderate variations of the nuclear spin-orbit strength around the Perey and Buck value of 7.2 Mev do not significantly influence this conclusion.

The Neutron-Proton Problem

To summarize, in Chapter II a formalism was developed to include the M-S force in the neutron-proton scattering problem. This procedure is of general interest because the M-S force results in singlet-triplet spin state mixing. Forms for the Wolfenstein parameters and the nine most common scattering parameters were derived with this state mixing included, and the resulting scattering amplitude matrix was also determined. The effect of the M-S term on phase shifts was evaluated through the use of a distorted wave calculation for low \( l \) waves, and the effect of large \( l \) waves was included in the Born approximation. Calculations for energies from 25-210 Mev were performed to find the influence of the M-S force on the scattering
parameters where it was assumed that phase shifts determined by best
Chi-squared fits to scattering data for the nucleon-nucleon problem
were due to the nuclear interaction only. In addition, the effect of
the M-S force on the binding energy of the deuteron was estimated.

From this study it can be concluded that:

(1) All nine scattering parameters are markedly
influenced by the M-S interaction, but this
influence is confined to small scattering angles
(less than 5°), except polarization and the
correlation coefficient, $C_{KP}$, at the lower
energies, and possibly the correlation coeffi-
cients, $C_{nnn}$, and $C_{n_2 \times k_1 K, n}$.

(2) The estimated effect on the deuteron binding
energy was found to be small as expected
($\sim -1.6 \times 10^{-3}$ Mev).
APPENDIX A

The Mott-Schwinger Potential

The M-S force arises from the motion of the magnetic dipole moment of the neutron in the electric field of the nucleus. It is more convenient to regard the force resulting from the motion of the nuclear charge in the magnetic field of the neutron. The force is given by, $F = q/c \vec{v} \times \vec{B}$, where $q$ is the total nuclear charge, $c$ is the velocity of light, $\vec{v}$ is the relative velocity vector, and $\vec{B}$ is the magnetic field resulting from $\mu_n$, the magnetic moment of the neutron. Now,

$$\vec{B} = -\nabla \left( \mu_n \cdot \frac{\vec{r}}{r^3} \right), \quad (164)$$

and

$$\mu_n = \frac{e\hbar}{2mc} \mu_n \sigma, \quad (165)$$

where $\frac{e\hbar}{2mc}$ is the nuclear magneton, $\mu_n$, is the neutron magnetic moment in nuclear magnetons, and $\hbar \sigma = 2\vec{s}$, where $\vec{s}$ is the neutron spin vector. Using $q = Ze$, and Equations (164), and (165) in the force equation along with the vector identity
\[ \nabla (\sigma \cdot \frac{r}{r^3}) = \sigma \times (\nabla \times \frac{r}{r^3}) + (\sigma \cdot \nabla) \frac{r}{r^3} = (\sigma \cdot \nabla) \frac{r}{r^3}, \]

results in,

\[ \vec{F} = -\frac{Ze^2}{mc^2} \mu_n \vec{v} \times \left( \vec{\sigma} \cdot \frac{\vec{e}_r}{r^3} \right), \quad (166) \]

where \( \vec{e}_r \) is the unit vector in the direction \( \vec{r} \).

From the relationship between force and potential, \( \vec{F} = -\nabla \vec{V} \), one finds,

\[ \vec{V} = -\frac{Ze^2}{2mc^2} \mu_n \frac{(\vec{v} \times \vec{e}_r) \cdot \vec{\sigma}}{r^2}. \]

This can be put in the \( \vec{l} \cdot \vec{\sigma} \) form by using, \( \hbar \vec{l} = -m_r \vec{v} \times \vec{r} \), where \( m_r \) is the reduced mass. This gives,

\[ \vec{V} = \frac{Ze^2}{2m_r c^2} \mu_n \frac{\vec{l} \cdot \vec{\sigma}}{r^3}. \]

The value of \( \mu_n \) used here is \( 1.91314 \).

Now, consider \( \vec{l} \cdot \vec{s} \).

Note, that since \( \vec{j} = \vec{l} + \vec{s} \),

\[ j(j+1) = (\vec{l}+\vec{s}) \cdot (\vec{l}+\vec{s}) = \vec{l}(\vec{l}+1) + 2\vec{l} \cdot \vec{s} + s(s+1) \]

or

\[ \vec{l} \cdot \vec{s} = \frac{j(j+1) - \vec{l}(\vec{l}+1) - s(s+1)}{2}. \]
For $J = \ell + 1/2$, $s = 1/2$,

$$\overline{\mathbf{l}} \cdot \overline{s} = \ell/2,$$

and for $j = \ell - 1/2$,

$$\overline{\mathbf{l}} \cdot \overline{s} = -\frac{(\ell+1)}{2}.$$
APPENDIX B

Change of Order of Integration in Equation 18

By writing \( F(\rho', \rho'') = F_1(\rho') F_2(\rho'') \), one can express the integral in question as,

\[
\int_\rho^{\infty} \int_{\rho'}^{\infty} F(\rho', \rho'') \, d\rho'' \, d\rho'.
\]

Now, the change in the limits and order of integration requires,

\[
\int_\rho^{\infty} \int_{\rho'}^{\infty} F(\rho', \rho'') \, d\rho'' = \int_{\rho'}^{\infty} \int_\rho^{\infty} F(\rho', \rho'') \, d\rho' \, d\rho''.
\]

That these two integrals are equivalent can be seen from the diagram below:

![Diagram showing region of integration](image-url)
(1) In the integral on the left-hand side of Equation (172) 
\( \rho' \) varies from \( \rho \) to \( \infty \) and \( \rho'' \) from \( \rho' \) to \( \infty \) for every \( \rho' \), while

(2) In the integral on the right-hand side, \( \rho'' \) varies from \( \rho \) to \( \infty \) and \( \rho' \) from \( \rho \) to \( \rho'' \) for every \( \rho'' \).

Thus, the function is integrated over the same region in both cases.
APPENDIX C

The Born Approximation

The Born approximation for the phase shift due to the M-S term is obtained following the treatment of Schiff\(^{(1)}\).

The radial form of Schrödinger Equation of interest is;

\[
\frac{1}{\rho} \frac{d}{d\rho} \left( \rho^2 \frac{dR_{\ell_j}}{d\rho} \right) + \left[ 1 - \frac{\ell(\ell+1)}{\rho^2} \right] R_{\ell_j} = \frac{\lambda_{\ell_j}}{\rho^3} R_{\ell_j} . \tag{173}
\]

The radial function, \(R_{\ell_j}\), is taken as \(R_{\ell_j} = f_{\ell_j} + v_{\ell_j}\), where

\[
\frac{1}{\rho} \frac{d}{d\rho} \left( \rho^2 \frac{df_{\ell_j}}{d\rho} \right) + \left[ 1 - \frac{\ell(\ell+1)}{\rho^2} \right] f_{\ell_j} = 0 , \quad \text{and} \quad f_{\ell_j} \gg v_{\ell_j} .
\]

This allows Equation (173) to be approximated as,

\[
\frac{1}{\rho} \frac{d}{d\rho} \left( \rho^2 \frac{dv_{\ell_j}}{d\rho} \right) + \left[ 1 - \frac{\ell(\ell+1)}{\rho^2} \right] v_{\ell_j} = \frac{\lambda_{\ell_j}}{\rho^3} f_{\ell_j} . \tag{174}
\]

The solution of Equation (174) is given in terms of the appropriate Green's function, \(G(\rho, \rho')\),

\[ v_{\ell j} = \lambda_{\ell j} \int_0^\infty \frac{G(p, p') \, f_{\ell j}(p')}{p^{\prime 3}} \, \rho'^2 \, dp' , \quad (175) \]

where \( G(p, p') = j_{\ell}(p_<) \, n_{\ell}(p_>) \) where \( p_<, p_> \) are the lesser, greater, of \( p \) and \( p' \), and \( j_{\ell} \) and \( n_{\ell} \) are the solutions for \( f_{\ell j} \) regular and irregular at the origin. Now, asymptotically \( \rho > \rho' \), so one has

\[ R_{\ell j}(\rho) = j_{\ell}(\rho) + n_{\ell}(\rho) \lambda_{\ell j} \int_0^\infty \frac{j_{\ell}^2(\rho') \, dp'}{\rho'} , \]

and hence, one has for the tangent of the phase shift due to M-S scattering only,

\[ \tan \delta_{\ell j}^{MS} = - \lambda_{\ell j} \int_0^\infty \frac{j_{\ell}^2(\rho') \, dp'}{\rho'} . \quad (176) \]

The Born approximation for the scattering amplitude is derived from the expression,

\[ f(\theta, \varphi) = \frac{1}{4\pi} \int e^{-i \frac{\vec{k} \cdot \vec{r}}{\hbar}} U(\vec{r}) \, e^{i \frac{\vec{k}_0 \cdot \vec{r}'}{\hbar}} \, d\vec{r}' , \quad (177) \]

where \( U = - \frac{2m}{\hbar^2} V, \, \vec{k}_0 \) is the incident wave vector, and \( \vec{k} \) the scattered wave vector.

The M-S potential is most conveniently expressed here as

\[ U_{MS} = - \frac{\gamma \vec{g} \cdot (\vec{F} \times \vec{p})}{\hbar r^3} \text{ (see Appendix A).} \]
Thus,

\[ f(\theta, \varphi) = -\frac{\gamma \sigma \cdot \vec{k}'}{4\pi \hbar} \int \frac{e^{-i \vec{k} \cdot \vec{r}'}}{r'} \frac{e^{-i(k'_0 \cdot \vec{r}')}}{r'}^2 \sin \theta \, d\theta \, d\varphi \, dr' \]

\[ = \frac{\gamma \sigma \cdot \vec{k}'_0 \times \vec{k}}{4\pi} \int \frac{e^{-i \vec{r}' \cdot (\vec{k} - \vec{k}'_0)}}{|\vec{r}'|} \sin \theta \, d\theta \, d\varphi \, dr' . \tag{178} \]

Now, \( \vec{r}' = \vec{r}' \cos \theta e_K + \vec{r}' \sin \theta e_\perp \), where \( e_K \) is a unit vector in the direction \( \vec{K} = (\vec{k} - \vec{k}'_0) \), and \( K = 2k \sin \omega / 2 \) where \( \omega \) is the scattering angle and \( |k| = |k'_0| \). In the above expression for \( \vec{r}' \), \( e_\perp \) is a unit vector normal to \( e_K \).

Thus, \( \frac{\vec{r}'}{|\vec{r}'|} = \frac{\vec{k} - \vec{k}'_0}{K} \), and one can write

\[ f(\theta, \varphi) = \frac{\gamma \sigma \cdot \vec{k}'_0 \times \vec{k}}{4\pi K} \int (\vec{k} - \vec{k}'_0) \cos \theta \, e^{-i \vec{r}' \cdot \vec{K}} \sin \theta \, d\theta \, d\varphi \, dr' \]

because the \( \varphi \) integration causes the term involving \( e_\perp \) to vanish.

The cross product in the above expression can be written \( \vec{k}'_0 \times (\vec{k} - \vec{k}'_0) \) = \( k^2 \sin \omega \vec{n} \), where \( \vec{n} \) is the unit normal to the scattering plane.

This substitution reduces the amplitude expression to,

\[ f(\theta, \varphi) = \frac{\gamma \sigma \cdot \vec{n} \sin \omega k^2}{K} \int \cos \theta \, e^{-i \vec{k} \cdot \vec{r}'} \sin \theta \, d\theta \, d\varphi \, dr' \tag{179} \]

The integration can be performed and gives a factor,

\[ \frac{4 \pi i}{K} \], thus,
\[
 f(\theta, \phi) = \frac{1}{4} \frac{v \sin \omega}{\omega} \cdot \frac{1}{2} = \frac{1}{2} \cot \frac{\theta}{2} - \frac{1}{n} . \tag{180}
\]

The Born approximation for the M-S contribution to the phase shift is given by (see Equations (50) and (42)),

\[
 \delta_{lj} = \lambda_{lj} \int_{p'} \frac{j_2^2(p')}{p'} \ dp' = \lambda_{lj} \int_{\omega} \frac{j_2^2(p')}{p'} \ dp' = \lambda_{lj} \frac{1}{2l(l+1)} , \tag{181}
\]

where \( \lambda_{lj} = k \gamma \beta_{lj} \). Then one gets

\[
 \delta_{l, l+1/2} = \frac{kv}{2(l+1)} , \quad \text{and} \quad \delta_{l, l-1/2} = -\frac{kv}{2l} .
\]

Writing \( h(\theta) = \sum_{l=1}^{\infty} (2l+1) S_l P_l^1(\cos \theta) \), where

\[
 S_l = \frac{1}{2k} \frac{e^{2i\delta_{l, l+1/2}} - e^{-2i\delta_{l, l-1/2}}}{2l+1} , \quad \text{and using the above expressions for the phase shifts,}
\]

\[
 h(\theta) = \frac{1}{2} \sum_{l=1}^{\infty} \frac{2l+1}{l(l+1)} P_l^1(\cos \theta) . \tag{182}
\]

Thus, the large \( l \) contribution to \( \Pi(\theta) \) can be included by adding the Born approximation expression, Equation (180), to the partial series (up to \( l=l_c \)) which uses exact phase shifts, and then subtracting the Born approximation expression for those terms up to \( l=l_c \) which have been included twice.
\[ h(\theta) = \sum_{\ell=1}^{\ell_c} (2\ell+1) \left[ S_\ell + \frac{i\nu}{2} \frac{1}{\ell(\ell+1)} \right] P_\ell^{1}(\cos \theta) - \frac{i\nu}{2} \cot \frac{\theta}{2}, \quad (183) \]

(see Equation (62)).
APPENDIX D

Calculation of \((\mathbf{\ell} \cdot \mathbf{s})\) \(\mathcal{M}^{J\Sigma L}\)

The operator can be expressed in component form as;

\[
\mathbf{\ell} \cdot \mathbf{s} = M_x \frac{\sigma_{xn}}{2} + M_y \frac{\sigma_{yn}}{2} + M_z \frac{\sigma_{zn}}{2} .
\]  

(184)

The following results are useful in the evaluation and are readily obtained through the raising and lowering operators \(M^\pm = M_x \pm iM_y\),

\[
M_x Y^m_l = \frac{1}{2} \left\{ \left[ (l+m)(l-m+1) \right]^{1/2} \gamma^{m-1}_l + \left[ (l-m)(l+m+1) \right]^{1/2} \gamma^{m+1}_l \right\},
\]

\[
M_y Y^m_l = \frac{1}{2} \left\{ \left[ (l+m)(l-m+1) \right]^{1/2} \gamma^{m-1}_l - \left[ (l-m)(l+m+1) \right]^{1/2} \gamma^{m+1}_l \right\},
\]

(185)

and also \(M_z Y^m_l = m Y^m_l\).

Further, the results of the operation of the Pauli-spin matrices on the spin functions are needed.

\[
\begin{align*}
\sigma_{xn} \chi^1_1 &= \frac{1}{\sqrt{2}} \left[ \chi^0_1 - \chi^0_0 \right] \\
\sigma_{yn} \chi^1_1 &= \frac{1}{\sqrt{2}} \left[ \chi^0_1 - \chi^0_0 \right] \\
\sigma_{zn} \chi^1_1 &= \chi^1_1
\end{align*}
\]

(186)
\[
\begin{align*}
\sigma_{xn} x_1^o &= \frac{1}{\sqrt{2}} [x_1^1 + x_1^{-1}] \\
\sigma_{yn} x_1^o &= \frac{1}{\sqrt{2}} [x_1^{-1} - x_1^1] \\
\sigma_{zn} x_1^o &= x_1^o \\
\sigma_{xn} x_1^{-1} &= \frac{1}{\sqrt{2}} [x_1^o + x_1^o] \\
\sigma_{yn} x_1^{-1} &= -\frac{1}{\sqrt{2}} [x_1^o + x_1^o] \\
\sigma_{zn} x_1^{-1} &= -x_1^{-1}
\end{align*}
\]

(187)

\[
\begin{align*}
\sigma_{xn} x_0^o &= \frac{1}{\sqrt{2}} [x_1^{-1} - x_1^1] \\
\sigma_{yn} x_0^o &= \frac{1}{\sqrt{2}} [x_1^{-1} + x_1^1] \\
\sigma_{zn} x_0^o &= x_0^o
\end{align*}
\]

(188)

(189)

In the above,

\[
\begin{align*}
\chi_1^1 &= \alpha_n \alpha_p \\
\chi_1^o &= \frac{1}{\sqrt{2}} [\alpha_n \beta_p + \alpha_p \beta_n] \\
\chi_1^{-1} &= \beta_n \beta_p
\end{align*}
\]

and

\[
\begin{align*}
\chi_0^o &= \frac{1}{\sqrt{2}} [\alpha_n \beta_p - \alpha_p \beta_n]
\end{align*}
\]
where the subscripts refer to neutron and proton, and \( \alpha \) and \( \beta \) correspond to spin up and down respectively.

Using the results given above, with the definition of the spin-angle functions,

\[
\mathcal{g}_J^{m} = \sum_{m_s} (\ell s m m_s m_s/Jm) Y_\ell^{m-s} \chi_s^{m_s},
\]

it is tedious, but straightforward to obtain the results,

\[
(\ell \cdot s_n) \mathcal{g}_J^{m} \propto \mathcal{g}_{\ell+1,1,1} = \frac{1}{2} \mathcal{g}_{\ell+1,1,1},
\]

\[
(\ell \cdot s_n) \mathcal{g}_J^{m} \propto \mathcal{g}_{\ell-1,1,1} = -\frac{(\ell+1)}{2} \mathcal{g}_{\ell-1,1,1},
\]

\[
(\ell \cdot s_n) \mathcal{g}_J^{m} \propto \mathcal{g}_{\ell,1,1} = -\frac{1}{2} \mathcal{g}_{\ell,1,1} + \frac{1}{2} \ell(\ell+1)^{1/2} \mathcal{g}_{\ell,0,1},
\]

and

\[
(\ell \cdot s_n) \mathcal{g}_J^{m} \propto \mathcal{g}_{\ell,0,1} = \frac{1}{2} \ell(\ell+1)^{1/2} \mathcal{g}_{\ell,1,1}.
\]
APPENDIX E

Modified Scattering Amplitudes Including Tensor Coupling

In all cases, the phase shifts applicable to the $\ell = J+1$ states are the bar phase shifts. The following definitions replace those of Table 1.

\begin{align*}
M_{1111} &= \frac{1}{2k} \sum_{\ell} \left\{ (2\ell+1)Q_{\ell 1} + (\ell-1)\alpha_{\ell,\ell-1} + (\ell+2)\alpha_{\ell,\ell+1} - [\ell(\ell-1)]^{1/2} \alpha_{\ell-1} \right. \\
&\left. - [(\ell+1)(\ell+2)]^{1/2} \alpha_{\ell+1} \right\} P_{\ell}(\cos \theta), \quad (191)
\end{align*}

\begin{align*}
M_{1011} &= \frac{1}{2k} \sum_{\ell} \left\{ \frac{2\ell+1}{\ell(\ell+1)} Q_{\ell 1} + \left(\frac{\ell-1}{\ell}\right) \alpha_{\ell,\ell-1} - \left(\frac{\ell+2}{\ell+1}\right) \alpha_{\ell,\ell+1} - \left(\frac{\ell}{\ell+1}\right)^{1/2} \alpha_{\ell+1} \right. \\
&\left. + \left(\frac{\ell+2}{\ell+1}\right)^{1/2} \alpha_{\ell+1} \right\} P_{\ell}^{1}(\cos \theta) \sin \theta \ e^{i\phi}, \quad (192)
\end{align*}

\begin{align*}
M_{1-111} &= \frac{1}{2k} \sum_{\ell} \left\{ - \frac{(2\ell+1)}{\ell(\ell+1)} Q_{\ell 1} + \frac{1}{\ell} \alpha_{\ell,\ell-1} + \frac{1}{\ell+1} \alpha_{\ell,\ell+1} - [\ell(\ell-1)]^{-1/2} \alpha_{\ell-1} \right. \\
&\left. - [(\ell+1)(\ell+2)]^{1/2} \alpha_{\ell+1} \right\} P_{\ell}^{1}(\cos \theta) \sin^{2} \theta \ e^{2i\phi}, \quad (193)
\end{align*}


161
\[ M_{1110} = \frac{1}{2k} \sum_{l} \left\{ -\alpha_{l,l-1} + \alpha_{l,l+1} - \left( \frac{l-1}{l} \right) \alpha_{l-1} + \left( \frac{l+2}{l+1} \right)^{1/2} \alpha_{l+1} \right\} \]

\[ P_{l}(\cos \theta) \sin \theta e^{-i\phi} , \]  

(194)

\[ M_{1010} = \frac{1}{k} \sum_{l} \left\{ l\alpha_{l,l-1} + (l+1)\alpha_{l,l+1} + [l(l-1)]^{1/2} \alpha_{l-1} + [(l+1)(l+2)]^{1/2} \alpha_{l+1} \right\} P_{l}(\cos \theta) . \]  

(195)

In the above expressions, the \( \alpha \)'s are defined as,

\[ \alpha_{l,l-1} = \cos \frac{2}{\epsilon_{l-1}} \frac{2i\delta_{l,l-1}}{2i} , \]  

(196)

\[ \alpha_{l,l+1} = \cos \frac{2}{\epsilon_{l+1}} \frac{2i\delta_{l,l+1}}{2i} , \]  

(197)

\[ \alpha_{l-1} = \sin \frac{\epsilon_{l-1}}{2} \frac{i(\delta_{l,l-1} + \delta_{l-2,l-1})}{2} , \]  

(198)

\[ \alpha_{l+1} = \sin \frac{\epsilon_{l+1}}{2} \frac{i(\delta_{l+2,l+1} + \delta_{l,l+1})}{2} , \]  

(199)

and \( \delta_{l,j} \) and \( \epsilon_{j} \) are the bar phase shifts and mixing parameter respectively.
Illustration That $\rho^{(1)} = \frac{1}{4}(1 + \vec{P} \cdot \vec{\sigma})$ is the Density Matrix for a Polarized Beam

Note that for scattering of spin 0 targets, the density matrix is a 2 x 2 and can be represented as a linear combination of the unit matrix and the Pauli spin matrices. Further, rotational invariance will mean:

$$\rho^{(1)} = c \cdot \mathbb{1} + d \cdot \vec{\sigma}.$$  \hspace{1cm} (200)

The total flux (or beam intensity) is given by $\text{Tr} \rho$ (this follows from $\rho = \psi \psi^\dagger$).

Hence, $\text{Tr}(\rho) = 2c$ or $c = 1/2$.

Further, the polarization component, $P_m$, is given by:

$$P_m = \frac{\text{Tr}(\rho \sigma_m^{(1)})}{\text{Tr}\rho} = \text{Tr}\left[ \frac{\frac{1}{2} \sigma_m + (d \cdot \vec{\sigma}) \sigma_m^{(1)}}{1} \right]$$

or,

$$d_m = \frac{P_m}{2}.$$

\hspace{1cm} (201)
Thus, $p^{(1)} = \frac{1}{2}(1 + \vec{P} \cdot \vec{c})(1)$. Now, since $p$ is related to the spin space of the incident particle only (in this case, the neutron), it follows that the $4 \times 4$ form of $p$ is just the $4 \times 4$ form of $p$ as given. That is $p \rightarrow (1_{4x4} + \vec{P}_{4x4} \cdot \vec{c}_{4x4})$. In order to preserve the normalization to unit flux, and to be consistent with the definition of $p$ for an unpolarized beam, it is necessary to renormalize and one gets,

$$p = \frac{1}{4}(1 + \vec{P} \cdot \vec{c}) \quad (202)$$

Goldberger and Watson\(^{(1)}\) give a general form for spin $1/2$ particles scattered off spin $s_2$ particles of

$$p_1 = \frac{1}{2(2s_2+1)} \left[ 1 + \vec{P}_1 \cdot \vec{c} \right] \quad (203)$$

---