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WEAK AND STRONG CONSTRUCTIONS
IN PROXIMITY SPACES

DISSERTATION.

Presented in Partial Fulfillment of the Requirements for
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BY

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* * * * *

The Ohio State University
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Studies in Algebraic Topology. Professor Paul V. Reichelderfer

Studies in Real Analysis. Professor Paul V. Reichelderfer

Studies in Algebra. Professors Alan Woods and Wolfgang Kappe
III.3. Weak and Strong Proximities of Proximal Convergence on Compacta

BIBLIOGRAPHY

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INTRODUCTION

The modern theory of proximity spaces began in 1951 with the publication of Efremovich’s *Infinitesimal Spaces* [9] (numbers in brackets refer to the bibliography) although as Thron [28] points out similar ideas were discussed earlier by other authors. We will use the definition of proximity spaces which has become standard (with one slight deviation, see 0.3.5) and can be found in Smirnov [25]. All topological notation and definitions not explicitly defined in this paper can be found in Kelley [16].

The theory of proximity spaces is an attempt to generalize the non-topological properties of metric spaces, for example completeness and total boundedness, to arbitrary completely regular topological spaces. The theory rests on the fact that, for metric spaces $(X,d)$ and $(X^*,d^*)$ a function $f:X \to X^*$ is uniformly continuous iff for $A,B \subseteq X$, $D(d)(A,B) = 0$ implies that $D(d^*)(f[A],f[B]) = 0$. Proximity spaces were defined and studied by V.A. Efremovich [9,10] and later studied by Ju. M. Smirnov [24,25,26,27].

It was not immediately obvious how one should define the concept of total boundedness for proximity spaces. In [10] Efremovich defined a proximity space to be totally
bounded iff it contains no subset which is proximally iso-
morphic to the discrete proximity on the integers. As
Smirnov points out in [26] this definition agrees with the
concept of total boundedness in metric spaces but does not
agree with the desirable conjecture: "A proximity space is
compact iff it is complete and totally bounded." since the
compactification of the discrete metric space on the
integers does not satisfy this requirement. (We note that
spaces which have no countably infinite proximally discrete
subspaces have been studied briefly by Katetov in [15].
They are precisely the proximally totally bounded proximity
spaces in which every subspace is also proximally totally
bounded.) In [26] Smirnov defines a proximity space to
be totally bounded if its completion is compact. This
brings up another problem encountered in the attempt to
generalize metric properties to proximity spaces.

In attempting to define a completion of proximity
spaces Smirnov sought to extend a property of complete
metric spaces; that they are closed in every metric space
of which they are a subset. This concept turns out to be
equivalent to compactness in proximity spaces and hence
not equivalent to completeness in metric spaces. A concept
of completion was defined by Smirnov and a proximity space
which is equal to its completion was called complete. This
definition turned out to be equivalent to the metric concept
of completeness.
In the process of discussing proximal extensions of proximity spaces, e.g., completions and compactifications, Smirnov introduced the concept of a $\delta$-cover [25,26]. Essentially, if we define a uniform space using covers as is done by Isbell [14], a $\delta$-cover of a proximity space is any cover which is contained in some uniformity which generates the proximity space. The obvious analog of entourage-like sets or $\delta$-entourages was first studied by Alfsen and Njastad in [4]. In this paper we will make a great deal of use of this concept. In 1.1 we show that a proximity space $(X, \delta)$ can be completely characterized by its class of $\delta$-entourages which we will call $\mathcal{P}(\delta)$.

An early result of the theory is the fact that there is a one-to-one correspondence between the category of proximity spaces and the category of totally bounded uniform spaces [1,11]. We then have associated with any proximity space $(X, \delta)$ two classes of entourages or $\delta$-entourages; $\mathcal{U}(\delta)$, the totally bounded uniformity associated with $(X, \delta)$, and $\mathcal{P}(\delta)$ the class of $\delta$-entourages for $(X, \delta)$.

One of the original disquieting results of proximity spaces is the fact that, in defining products of proximity spaces in the natural (see 1.8.2) way, one finds that the proximal product is not proximally isomorphic to the metric product in familiar cases. This seems to be true because the proximal product corresponds to the uniform product of
the totally bounded uniformities associated with the given proximity spaces. In [22] Poljakov expands on an idea originally due to Mrowka [20] and defines a weak and a strong product of proximity spaces corresponding to the two entourage classes mentioned earlier. Several results of these definitions are announced in [22].

It appears that, in attempting proximal constructions similar to those which have been made in topological spaces and uniform spaces, two constructions will generally be possible. The "weak" construction will correspond to the totally bounded uniform structure and the "strong" construction will correspond to the collection of $\delta$-entourages although generally not in the same manner. The terms weak and strong are appropriate since generally the former will be smaller than the latter in the usual ordering of proximity spaces (see I.2). The strong constructions are generally equivalent to metric constructions in the case of metrizable proximity spaces.

It is assumed that the reader has a working knowledge of topological spaces and uniform spaces. Although a knowledge of proximity spaces would be helpful, the relevant facts used are outlined in 0.3 and several references are given there.

In Chapter I we have outlined some of the topics of proximity spaces appearing in the literature and also filled in some basic gaps. Since the papers on proximity spaces
use diverse methods, the class $\mathcal{P}(\mathcal{S})$ mentioned above has been used as a unifying concept in defining or re-defining these topics. Although many of the results in Chapter I are known, much of I.2, I.3, some of I.6, and some of I.8 appear to be new. The author has attempted to indicate sources if a given theorem appears in the literature. Proofs are given to theorems which are only stated elsewhere, and some new proofs are given in terms of $\mathcal{P}(\mathcal{S})$ where necessary.

In Chapter II, the class $\mathcal{W}(\mathcal{S})$ and the class $\mathcal{P}(\mathcal{S})$ are used to introduce proximity relations on the space of all subsets of $X$ where $(X, \mathcal{S})$ is a proximity space. Although the definitions given have roots in the literature the results are new. In II.3 we apply some of the results of II.1 and II.2 to give some sufficient conditions for the proximal quotient of a proximally complete proximity space to be proximally complete. The two main theorems of section II.3, II.3.7 and II.3.15, are direct generalizations of a theorem due to Kelley [16].

In Chapter III strong and weak constructions of proximities for function spaces are studied. In III.2 a conjecture of Leader [18] concerning the equivalence of two types of convergence of functions into a proximity space is shown to be true. The results of this chapter seem to parallel the known results of similar constructions for uniform spaces.
0.1 Categories and Functors

0.1.1. Definition. A category is a triple $(\mathcal{C}, [\cdot, \cdot], \circ)$ with the following properties:

1. $\mathcal{C}$ is a collection whose elements are called objects.
2. For every $X$ and $Y$ in $\mathcal{C}$ there is a set $[X, Y]$ whose elements are called maps or morphisms with domain $X$ and image space $Y$.
3. If $f \in [X, Y]$ and $g \in [Y, Z]$ then there is an element $g \circ f \in [X, Z]$ called the composition of $f$ and $g$.
4. $h \circ (g \circ f) = (h \circ g) \circ f$ when both are defined.
5. If $X$ is an object then there is an element $i_X \in [X, X]$ such that $i_X \circ f = f$ and $g \circ i_X = g$ whenever these compositions are defined.

0.1.2. Definition. A covariant functor $\mathcal{F}$ from a category $(\mathcal{C}, [\cdot, \cdot], \circ)$ to a category $(\mathcal{C}', [\cdot, \cdot]', \circ')$ is a rule of correspondence which assigns to each object $X$ in $\mathcal{C}$ an object $\mathcal{F}(X)$ in $\mathcal{C}'$ and which assigns to each $f$ in $[X, Y]$ and element $\mathcal{F}(f)$ in $[\mathcal{F}(X), \mathcal{F}(Y)]'$ such that the following are satisfied:

1. $\mathcal{F}(i_X) = i' \mathcal{F}(X)$ for every $X$ in $\mathcal{C}$.
\[(2) \quad \mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)\] whenever \(g \circ f\) is defined.

We will use functional or operator notation to express the correspondence which is the functor, although a functor is not, strictly speaking, a function (see [12], p. 149).

0.1.3. **Definition.** If \((\mathcal{X}, [\ , \ ], \circ)\) and \((\mathcal{X}^*, [\ , \ ]^*, \circ^*)\) are categories, \(X \in \mathcal{X}\) implies that \(X \in \mathcal{X}^*\), \(X\) and \(Y\) in \(\mathcal{X}\) implies that \([X, Y] \subseteq [X, Y]^*\); and \(\circ = \circ^*\) when both are defined, then we will call \((\mathcal{X}, [\ , \ ], \circ)\) a subcategory of \((\mathcal{X}^*, [\ , \ ]^*, \circ^*)\). If \([X, Y] = [X, Y]^*\) then \((\mathcal{X}, [\ , \ ], \circ)\) is said to be a full subcategory of \((\mathcal{X}^*, [\ , \ ]^*, \circ^*)\).

0.2 **Some Specific Categories**

0.2.1. **Definition.** The category whose objects are sets, whose morphisms are single valued functions, and whose composition is functional composition will be denoted \((\mathcal{S}, [\ , \ ], \circ)\). Thus if \(X\) and \(Y\) are sets then \(X\) and \(Y\) are in \(\mathcal{S}\); \([X, Y]\) is the set of all single valued functions from \(X\) to \(Y\), and \(\circ\) is functional composition.

0.2.2. **Definition.** If \(D\) is a directed set and \(T\) is an element of \([D, X]\), then \(T\) is called a net ranging in \(X\). We will use the symbols \((T, D)\) to denote a net. A subnet of a net \((T, D)\) is a net \((T \circ j, E)\) where \(j : E \rightarrow D\) such that for each \(d \in D\) there is an \(e^* \in E\) such that \(j(e) \geq d\) whenever \(e \geq e^*\).
0.2.3. **Definition.** The category whose objects are topological spaces, whose morphisms are continuous functions, and whose composition is functional composition will be denoted by \((\mathcal{T}, t\, [\ ]\, , \circ)\). We will write \(t[X,Y]\) for \(t[(X,\mathcal{T}),(Y,\mathcal{T}')]\) when there is no danger of confusing the topologies involved.

0.2.4. **Definition.** The category whose objects are uniform spaces, whose morphisms are uniformly continuous functions, and whose composition is functional composition will be denoted by \((\mathcal{U}, u\, [\ ]\, , \circ)\). When it is necessary to mention the uniformities involved we will write \(u[(X,\mathcal{U}),(Y,\mathcal{U}')]\), but when the uniformities are clearly understood we will shorten this to \(u[X,Y]\). If \((X,\mathcal{U})\) is in \(\mathcal{U}\) then \(\mathcal{U}\) is the collection of entourages for the uniform space as defined in Kelley [16].

0.2.5. **Definition.** The category whose objects are totally bounded uniform spaces, whose morphisms are uniformly continuous functions, and whose composition is functional composition will be denoted by \((p\mathcal{U}, u\, [\ ]\, , \circ)\). We remark that this category is a full subcategory of the category of uniform spaces and thus we are justified in using the symbol \(u[\ ]\) for the morphisms of this category.

0.2.6. **Theorem.** There is a functor \(\tau^*\) from the category of uniform spaces to the category of topological spaces. For any object \((X,\mathcal{U})\) in \(\mathcal{U}\), \(\tau^*(X,\mathcal{U}) = (X, \sigma(\mathcal{U}))\) is the topological space in which \(0 \subseteq X\) is open iff for each
There is a $U \in \mathcal{U}$ such that $U[x] \subseteq X$. If $f$ is a uniformly continuous function, then $\mathcal{C}^*(f)(x) = f(x)$.

0.2.7. Definition. If $(X, \mathcal{U})$ and $(X^*, \mathcal{U}^*)$ are in $\mathcal{U}$ and $f \in [X, X^*]$ then $f$ will be called uniformly open iff $U \in \mathcal{U}$ implies there is a $U^* \in \mathcal{U}^*$ such that $U^*[f(x)] \subseteq f[U[x]]$ for all $x \in X$.

0.3 The Category of Proximity Spaces

0.3.1. Definition. A proximity space (p-space) is a pair $(X, \delta)$ where $X$ is a set and $\delta$ is a relation on the set of subsets of $X$ satisfying the following properties:

1. For $A, B \subseteq X$, $A \delta B$ iff $B \delta A$.
2. For $A, B, C \subseteq X$, $A \delta (B \cup C)$ iff $A \delta B$, $A \delta C$, or both.
3. $\{x\} \delta \{x\}$ for all $x \in X$.
4. $\emptyset \delta A$ for all $A \subseteq X$ ($\delta$ denotes the negation of $\delta$).
5. For $A, B \subseteq X$, $A \overline{\delta} B$ implies the existence of disjoint sets $C$ and $D$ such that $A \subseteq C$, $B \subseteq D$, and $A \overline{\delta} C$ and $B \overline{\delta} D$.

If $(X, \delta)$ is a p-space the relation $\delta$ is called a proximity relation (p-relation) on the set $X$.

0.3.2. Definition. If $(X, \delta)$ and $(X^*, \delta^*)$ are p-spaces and $f \in [X, X^*]$ then $f$ is said to be proximally continuous (p-continuous) iff $A, B \subseteq X$ and $A \delta B$ implies that $f[A] \delta^* f[B]$. If $f$ is one-to-one, onto, p-continuous, and $f^{-1}$ is also p-continuous, then $f$ is called a proximal isomorphism.
0.3.3. Remark. The notation \( f: (X, \mathcal{S}) \to (X^*, \mathcal{S}^*) \) is sometimes used in the literature to denote the fact that \( f \) is a proximally continuous function. Since maps other than \( p \)-continuous functions also depend on the proximities given (e.g., proximal quotient maps) we will use this notation only to indicate what proximities are involved, not to indicate proximal continuity. A similar remark holds for topological spaces and uniform spaces.

0.3.4. Theorem. The triple \( (P, p(\cdot, \cdot), \circ) \) where \( P \) is the class of all \( p \)-spaces, \( p[(X, \mathcal{S}), (X^*, \mathcal{S}^*)] \) is the set of all \( p \)-continuous functions from \( (X, \mathcal{S}) \) to \( (X^*, \mathcal{S}^*) \), and \( \circ \) is functional composition is a category, called the category of proximity spaces.

0.3.5. Remark. In much of the literature what we have called a \( p \)-space is called a generalized \( p \)-space. In this context a proximity space is then a generalized \( p \)-space which satisfies, in addition to (1) through (5) of 0.3.1,

(6) \( x \neq y \) implies that \( \{x\} \not\in \mathcal{S} \{y\} \).

We will call spaces which also satisfy (6) separated \( p \)-spaces.

0.3.6. Remark. The following paragraphs list briefly some definitions and properties which form the basis of proximity theory. For the interested reader details of proofs of these statements may be found in [11], [25], or in [28].
0.3.7. Definition. Suppose \((X, \delta)\) is a p-space.
For \(A, B \subseteq X\) we define \(A \subseteq B \iff A \subseteq \bar{B}\).

0.3.8. Theorem. If \(\subseteq\) is a relation on \(\mathcal{P}(X)\) and for
\(A, B \subseteq X\) we define \(A \subseteq B \iff A \subseteq \bar{B}\), then \((X, \delta)\) is a p-
space iff the following properties are satisfied.

1. \(X \subseteq X\).
2. For \(A, B \subseteq X\), \(A \subseteq B \implies A \subseteq B\).
3. For \(A, B, C, D \subseteq X\), \(A \subseteq B \subseteq C \subseteq D \implies A \subseteq D\).
4. For \(A \subseteq X\) and \(\{B_n\}\) a finite collection of subsets
   of \(X\), \(A \subseteq B_n\) for all \(n\) iff \(A \subseteq \bigcap B_n\).
5. For \(A, B \subseteq X\), \(A \subseteq B \implies \bar{B} \subseteq \bar{A}\).
6. For \(A, B \subseteq X\), \(A \subseteq B\) implies the existence of
   a \(C \subseteq X\) such that \(A \subseteq C \subseteq B\).

0.3.9. Theorem. There is a functor \(\Phi\) from the
category of uniform spaces to the category of proximity
spaces. For \((X, \mathcal{U})\) in \(\mathcal{U}\), the space \(\Phi(X, \mathcal{U}) = (X, \delta(\mathcal{U}))\)
is the set \(X\) with the relation \(\delta(\mathcal{U})\) defined by:
\(A \subseteq (\mathcal{U}) B \iff U[A] \cap U[B] \neq \emptyset\) for all \(U \in \mathcal{U}\). For \(f\) in
\(\mathcal{U}[X, X^*]\), \(\Phi(f)(x) = f(x)\).

0.3.10. Definitions. Suppose \((X, \delta)\) is a p-space.
1. \(\mathcal{U}(\delta) = \{U: (X, \mathcal{U}) \in \mathcal{U} \text{ and } \delta(\mathcal{U}) = \delta\}\). If
   \(\mathcal{U} \in \mathcal{U}(\delta)\) then we say \(\delta\) is generated by \(\mathcal{U}\) or \(\mathcal{U}\)
generates \(\delta\).
2. If \(I \neq \emptyset\) and \(\{A_1: i \in I\}\) is a cover of \(X\) then we
call \(\{A_1\}\) a proximal cover (p-cover) of \(X\) iff there
exists a collection \( \{B_i : i \in I\} \) such that

(a) \( \{B_i : i \in I\} \) is a cover of \( X \), and

(b) \( B_i \subseteq A_i \) for all \( i \in I \).

0.3.11. Theorems. Suppose \((X, \delta)\) is a \( p \)-space and \( \mathcal{U} \in \mathcal{P}(\delta) \).

(1) \( A \subseteq B \) iff \( U[A] \cap B \neq \emptyset \) for all \( U \in \mathcal{U} \).

(2) \( A \subseteq B \) iff \( U[A] \subseteq B \) for some \( U \in \mathcal{U} \).

0.3.12. Theorem. There is a functor \( \mu \) from the category of proximity spaces to the category of totally bounded uniform spaces. For \((X, \delta)\) in \( P \), the space \( \mu(X, \delta) = (X, \mathcal{U}(\delta)) \) is the set \( X \) and the uniform structure which has

\[ \bigcup \{ A_i \times A_i : 1 \leq i \leq n \}; \{ A_i \} \text{ is a finite } p-\text{cover of } X \right\} \]

as a base for its entourage filter. For \( f \in p[X;X^*] \),

\[ \mu(f)(x) = f(x) \].

0.3.13. Theorems. Suppose \((X, \delta)\) is a \( p \)-space.

(1) \( \mathcal{U}(\delta) \in \mathcal{P}(\delta) \). That is, \( \delta(\mathcal{U}(\delta)) = \delta \).

(2) \( \mathcal{U}(\delta) \) is the smallest element of \( \mathcal{P}(\delta) \). That is, \( \mathcal{U} \in \mathcal{P}(\delta) \) implies \( \mathcal{U}(\delta) \leq \mathcal{U} \).

(3) \( \mathcal{U}(\delta) \) is the only totally bounded uniformity in \( \mathcal{P}(\delta) \).

0.3.14. Theorem. There is a functor \( \mathcal{C} \) from the category of proximity spaces to the category of topological spaces. For \((X, \delta)\) in \( P \), \( \mathcal{C}(X, \delta) = (X, \mathcal{V}(\delta)) \) is the set \( X \) and the topology whose closure operator \( c \) is defined
by: \( cA = \{ x : \{ x \} \in A \} \). For \( f \in p[X,X^*] \), \( c(f)(x) = f(x) \).

0.3.15. Theorem. Suppose \((X,\mathcal{U})\) is in \( \mathcal{U} \), \((X,\mathcal{T})\) is in \( \mathcal{T} \), and \((X,\mathcal{S})\) is in \( \mathcal{P} \).

1. \( \mathcal{T}(\mathcal{U}) = \mathcal{T}(\mathcal{S}(\mathcal{U})) \).

2. There is a \( \rho \)-relation \( \mathcal{S}^* \) such that \( \mathcal{T}(\mathcal{S}^*) = \mathcal{T} \) iff \((X,\mathcal{T})\) is completely regular. There is a separated \( \rho \)-relation \( \mathcal{S}^* \) such that \( \mathcal{T}(\mathcal{S}^*) = \mathcal{T} \) iff \((X,\mathcal{T})\) is a Tychanoff space.

3. If \( c \) is the closure operator for \( \mathcal{T}(\mathcal{S}) \), then \( A \subseteq B \) iff \( cA \subseteq cB \).

4. For \( x \in X \), \( \mathcal{N}(x) = \{ N : N \) is a \( \mathcal{T}(\mathcal{S}) \) neighborhood of \( x \} \) is the set \( \{ B : \{ x \} \subseteq B \} \).

0.3.16. Definition. If \((X,\mathcal{S})\) is a \( \rho \)-space and \( A \subseteq X \), then for \( B, C \subseteq A \) define \( B \subseteq A \subseteq C \) iff \( B \subseteq C \). Then \((A,\mathcal{S}\mid A)\) is a \( \rho \)-space called the proximal subspace for \( A \). \( \mathcal{T}(\mathcal{S}) \cap A \) is \( \mathcal{T}(\mathcal{S}\mid A) \) and if \((X,\mathcal{U})\) is in \( \mathcal{U} \) then \((A,\mathcal{S}(\mathcal{U} \cap A \times A)) \) is \((A,\mathcal{S}(\mathcal{U})\mid A)\).

0.3.17. Definition. If \((X,\mathcal{S})\) and \((X,\mathcal{S}^*)\) are in \( \mathcal{S} \leq \mathcal{S}^* \) iff \( A \subseteq \mathcal{S} \subseteq B \) implies \( A \subseteq B \).

0.3.18. Theorem. If \((X,\mathcal{S})\) and \((X,\mathcal{S}^*)\) are in \( \mathcal{S} \leq \mathcal{S}^* \) implies that \( \mathcal{T}(\mathcal{S}) \subseteq \mathcal{T}(\mathcal{S}^*) \).

0.3.19. Definition. If \((X,d)\) is a pseudometric space, for \( A,B \subseteq X \) define \( D(d)(A,B) = \inf \{ d(a,b) : a \in A, b \in B \} \) if \( A \neq \emptyset \neq B \), and \( \infty \) if \( A \) or \( B \) is empty. Define \( A \subseteq \mathcal{S}(d) \subseteq B \) iff \( D(d)(A,B) = 0 \). Then \((X,\mathcal{S}(d))\) is a \( \rho \)-space. A \( \rho \)-space \((X,\mathcal{S})\) will be called pseudometrizable iff there is a
psuedometric d for X such that \( \delta(d) = \delta \). If \((X, \delta)\) is pseudometrizable and d is the pseudometric, then the pseudometric topology \( \tau_d \) is the topology \( \tau(\delta) \). If \( \mathcal{U}_d \) is the pseudometric uniform structure for d, it is not generally true that \( \mathcal{U}(\delta(d)) = \mathcal{U}_d \).

0.3.20. Definition. If \((X, \delta)\) is a p-space, define \( D(\delta) = \{d: d \text{ is a pseudometric on } X \text{ and } \delta(d) \leq \delta\} \). \( D(\delta) \) is called the gauge of \( \delta \).

0.4 Hyperspaces

0.4.0. Remark. Proofs of the following results and further details can be found in Caulfield's thesis [7].

0.4.1. Definition. If X is any set, then define \( \hat{X} = \{A: A \subseteq X\} \). If \( f \in [X, X^*] \), then define \( \hat{f} \in [\hat{X}, \hat{X^*}] \) such that \( \hat{f}(A) = \{f(a): a \in A\} \).

0.4.2. Definition. Suppose X is a set and \( S \subseteq X \times X \). Define \( H(S) = \{(A, B) \in \hat{X} \times \hat{X}: A \subseteq S[B] \text{ and } B \subseteq S[A]\} \).

0.4.3. Lemma. Suppose \( S \) and \( S^* \subseteq X \times X \). Then:

1. \( S \subseteq S^* \) implies \( H(S) \subseteq H(S^*) \) and conversely if \( S \) is symmetric.
2. \( H(S) = H(S)^{-1} \).
3. \( \Delta_X \subseteq S \) implies that \( \Delta_{\hat{X}} \subseteq H(S) \).
4. \( H(S) \circ H(S) \subseteq H(S \circ S) \).
5. \( H(S \cap S^*) \subseteq H(S) \cap H(S^*) \).
0.4.4. **Theorem.** Let \((X, \mathcal{U})\) be a uniform space and let \(\mathcal{Q}\) be any base for \(\mathcal{U}\). Then \(H(\mathcal{Q}) = \{H(B) : B \in \mathcal{Q}\}\) is a base for a uniformity for \(\hat{X}\). Any two such bases for \((X, \mathcal{U})\) generate the same uniformity for \(\hat{X}\). Denote this uniformity by \(\mathcal{U}\). \((\hat{X}, \mathcal{U})\) is called the uniform hyperspace generated by \((X, \mathcal{U})\).

0.4.5. **Theorem.** If \((X, \mathcal{U})\) is totally bounded then \((\hat{X}, \mathcal{U})\) is totally bounded. \((X, \mathcal{U})\) is compact iff \((\hat{X}, \mathcal{U})\) is compact.

### 0.5 Function Spaces

0.5.1. **Definitions.** Suppose \(X\) and \(X^*\) are sets. Suppose \(x \in X\), \(A \subseteq X\), \(B^* \subseteq X^*\), and \(V^* \subseteq X^* \times X^*\). Then define:

1. \((A, B^*) = \{f \in [X, X^*] : f[A] \subseteq B^*\}\).
2. \((x, B^*) = (\{x\}, B^*)\).
3. \(W(A, V^*) = \{(f, g) \in [X, X^*] \times [X, X^*] : (f(a), g(a)) \in V^*\text{ for all } a \in A\}\).
4. \(W(x, V^*) = W(\{x\}, V^*)\).
5. \(W(V^*) = W(X, V^*)\).

0.5.2. **Definitions.** Suppose \((X, \mathcal{T})\) and \((X^*, \mathcal{T}^*)\) are topological spaces; the following topologies are defined for \([X, X^*]\):

1. The topology of pointwise convergence has for a subbase \(\{(x, 0^*) : x \in X, 0^* \in \mathcal{T}^*\}\). This topology will be denoted by \(\mathcal{T}(p, \mathcal{T}^*)\) or simply \(\mathcal{T}(p)\) if it is clear what topology on \(X^*\) is involved.
(2) The compact-open topology has for a subbase 
\{(K,0^*): K \subseteq X, K \text{ compact}, 0^* \in \mathcal{U}^*\}. This topology will be denoted by \mathcal{U}(c,o,\mathcal{U}^*) or simply \mathcal{U}(c).

0.5.3. **Definitions.** Let \(X\) be a non-empty set and \((X^*, \mathcal{U}^*)\) be a uniform space. The following uniformities are defined for \([X,X^*]\).

(1) The uniformity of pointwise convergence has for a subbase \(\{W(x, U^*): x \in X, U^* \in \mathcal{U}^*\}\). This uniformity will be denoted by \mathcal{U}(p.o., \mathcal{U}^*) or simply \mathcal{U}(p.o.).

(2) The uniformity of uniform convergence has for a base \(\{W(U^*): U^* \in \mathcal{U}^*\}\). This uniformity will be denoted by \mathcal{U}(u.c., \mathcal{U}^*) or simply \mathcal{U}(u.c.).

(3) Suppose \((X, \mathcal{T})\) is a topological space. The uniformity of uniform convergence on compacta has for a subbase \(\{W(K, U^*): K \subseteq X, K \text{ compact}, U^* \in \mathcal{U}^*\}\). This uniformity will be denoted by \mathcal{U}(u.c., C, \mathcal{U}^*) or simply \mathcal{U}(u.c., C).

0.5.4. **Definition.** Suppose \(\mathcal{C} \subseteq [X,X^*], x \in X\). Then
\[\mathcal{C}[x] = \{f(x): f \in \mathcal{C}\}.\]
CHAPTER I

PROXIMITY SPACES

I.1 A Characterization of Proximity Spaces

I.1.1 Definition. Suppose $X$ is a set and $\{Q_n\}$ is a sequence of sets, $Q_n \subseteq X \times X$ for each $n$. We will call $\{Q_n\}$ a normal sequence iff:

1. $\Delta_X \subseteq Q_n$ for every $n$,
2. $Q_n = Q_n^{-1}$ for every $n$, and
3. $Q_n \circ Q_n \subseteq Q_{n-1}$ for $n \geq 2$.

I.1.2 Definition. A proximity structure is a non-empty collection $\mathcal{P} \subseteq 2^X \times X$ which satisfies the following:

1. $\Delta_X \subseteq P$ for all $P \in \mathcal{P}$.
2. If $P \in \mathcal{P}$ then there is a symmetric $P^* \in \mathcal{P}$ such that $P^* \circ P^* \subseteq P$.
3. If $P \in \mathcal{P}$ and $P \subseteq Q$, then $Q \in \mathcal{P}$.
4. If $P_1$ and $P_2 \in \mathcal{P}$ and $A \subseteq X$, there is a $P_3 \in \mathcal{P}$ such that $P_3[A] \subseteq P_1[A]$ for $i = 1, 2$.
5. If $V \subseteq X \times X$ and if there is a normal sequence $\{Q_n\}$ such that $Q_1 \circ Q_1 \subseteq V$ and for each $A \subseteq X$, and each $Q_n$ there is a $P \in \mathcal{P}$ such that $P[A] \subseteq Q_n[A]$, then $V \in \mathcal{P}$.
1.1.3. Remark. We note that, if $V$ and $\{Q_n\}$ are
as in condition (5) of 1.1.2 then each of the $Q_n$'s is
also in $\mathcal{P}$. The following paragraphs show that any set
endowed with a proximity structure (p-structure) also has
associated with it a proximity relation; that any prox-
imity relation induces a p-structure; and that the process
of switching back and forth between these structures is
commutative. This will establish the equivalence between
p-spaces and sets endowed with p-structures. We first
prove a lemma.

1.1.4. Lemma. If $(X, \delta)$ is a p-space and $\{Q_n\}$
is a normal sequence such that for $A \subseteq X$ and $n \in \mathbb{N},$
$A \subseteq Q_n[A]$, then there is a uniform structure $\mathcal{U}$ such
that $\delta(\mathcal{U}) = \delta$, and $\{Q_n\} \in \mathcal{U}$.

Proof. In [4] Alfsen and Njastad prove the following:
If $A \subseteq V[A]$ for all $A \subseteq X$, where $V \subseteq X \times X$, then for any $U$
in $\mathcal{U}(\mathcal{S})$, $A \subseteq (V \cap U)[A]$ for all $A \subseteq X$. We will use this
result to prove the lemma.

Define $\mathcal{U}$ to be $\mathcal{U}(\delta) \vee \mathcal{U}*$ where $\mathcal{U}*$ is the uniform
structure generated by $\{Q_n\}$. We claim that $\delta(\mathcal{U}) = \delta$.
Suppose $A \nsubseteq B$. Then there is a $U \in \mathcal{U}(\delta)$ such that
$U[A] \cap U[B] = \emptyset$. Since $\mathcal{U}(\delta) \subseteq \mathcal{U}$, we have that $A \nsubseteq (\mathcal{U}) B,$
and therefore $A \delta(\mathcal{U}) B$ implies $A \delta B$, or $\delta(\mathcal{U}) > \delta$.
On the other hand, suppose $A \subseteq B$ relative to $\delta(\mathcal{U})$. Then
by the definition of supremum of uniform spaces there is a
$U \in \mathcal{U}(\delta)$ and a $Q_n$ such that $A \subseteq (U \cap Q_n)[A] \subseteq B$. But
A \subseteq Q_n[A] and A \subseteq U[A], both with respect to S, imply that A \subseteq (U \cap Q_n)[A] by the above result of Alfsen and Njastad, and therefore A \subseteq B relative to S. Then, using the definition of A \subseteq B given in 0.3.7, we have that A \subseteq (U) B implies that A \subseteq B, or that A \subseteq B implies that A \subseteq (U) B. Thus S \Rightarrow S(U), completing the proof.

I.1.5. Theorem. For any p-space \((X, S)\) let \(\Phi(S) = \bigcup_{U: U \in \pi(S)}^\bigcup\) (see 0.3.10 (1)). Then \(\Phi(S)\) is a proximity structure.

Proof. Since \(P \in \Phi(S)\) implies the existence of a uniform structure \(U \in \pi(S)\) such that \(P \in U\), conditions (1), (2), and (3) of I.1.2 are automatically satisfied.

To show that condition (4) holds, let \(A \subseteq X\) and let \(P_1\) and \(P_2 \in \Phi(S)\). Then each \(P_i \subseteq U_i\), \(i = 1, 2\), for some \(U_i \in \pi(S)\). Since \(S(U_i) = S\) and \(A \subseteq P_i[A]\), we have, by 0.3.11, \(A \subseteq P_i[A]\) for \(i = 1, 2\). Consider \(U(S)\). Again by 0.3.11, \(A \subseteq P_i[A]\) implies the existence of entourages \(U_i \in \pi(S)\) such that \(U_i[A] \subseteq P_i[A]\) for \(i = 1, 2\). Let \(U\) be \(U_1 \cap U_2\). Then \(U[A] \subseteq P_i[A]\) for \(i = 1, 2\). Since \(U(S)\) is in \(\pi(S)\) and \(U \in \pi(S)\), \(U \in \Phi(S)\) and (4) is proved.

To show (5), let \(V\) and \(\{Q_n\}\) be such that \(Q_1 \circ Q_1 \subseteq V\), and for any \(A \subseteq X\) and any \(n \in N\), there is a \(P \in \Phi(S)\) such that \(P[A] \subseteq Q_n[A]\). Then \(P \in \Phi(S)\) implies that there is a uniform structure \(U\) such that \(P \in U\) and \(S(U) = S\). Thus \(A \subseteq X\) and \(n \in N\) implies that \(A \subseteq P[A] \subseteq Q_n[A]\) which implies that \(A \subseteq Q_n[A]\). By I.1.4, there is a uniform structure
such that $U^* \in \pi(\mathcal{S})$ and $\{Q_n\} \subseteq U^*$. Then $V \in U^*$ and $U^* \subseteq \mathcal{P}(\mathcal{S})$, thus proving (5).

1.1.6. Lemma. Suppose $\mathcal{P}$ is a proximity structure for $X$. Suppose $A, B \subseteq X$. Then there is a $P \in \mathcal{P}$ such that $P[A] \cap P[B] = \emptyset$ iff there is a $P^* \in \mathcal{P}$ such that $P^*[A] \cap B = \emptyset$.

Proof. If $P \in \mathcal{P}$ and $P[A] \cap P[B] = \emptyset$ then, since $B \subseteq P[B]$, $P[A] \cap B = \emptyset$. Suppose $P^* \in \mathcal{P}$ and $P^*[A] \cap B = \emptyset$. Let $P \in \mathcal{P}$ such that $P$ is symmetric and $P \circ P \in P^*$. Then we claim that $P[A] \cap P[B] = \emptyset$. For suppose not. Then there is an $a \in A$, $b \in B$, and an $x \in X$ such that $(a, x)$ and $(b, x)$ are in $P$. Then $(a, b) \in P \circ P \in P^*$ and hence $b \in P^*[A] \cap B$ which is a contradiction.

1.1.7. Theorem. If $\mathcal{P}$ is a proximity structure for $X$, define $A \delta(\mathcal{P}) B$ iff $P[A] \cap P[B] \neq \emptyset$ for all $P \in \mathcal{P}$. Then $\delta(\mathcal{P})$ is a proximity relation for $X$ and $\mathcal{P}(\delta(\mathcal{P}))$ is $\mathcal{P}$.

Proof. We first show that $\delta(\mathcal{P})$ is a proximity relation. For convenience let $\delta(\mathcal{P})$ be $\delta$. We show that conditions (1) through (5) of 0.3.1 are satisfied.

(1) From the definition of $\delta$, $A \delta B$ iff $B \delta A$ is obvious.

(2) Suppose $A \nolimits \delta (B \cup C)$. Then $P[A] \cap P[B \cup C] = \emptyset$ for some $P \in \mathcal{P}$. Since $P[B \cup C] = P[B] \cup P[C]$, this means that $P[A] \cap P[B] = \emptyset$ and $P[A] \cap P[C] = \emptyset$, which implies that $A \nolimits \nolimits \delta B$ and $A \nolimits \nolimits \delta C$. 


Suppose $A \not\subseteq B$ and $A \not\subseteq C$. Then there are sets $P_1$ and $P_2$ such that $P_1[A] \cap P_1[B] = \emptyset$ and $P_2[A] \cap P_2[C] = \emptyset$. Then $P_1[A] \cap B = \emptyset$ and $P_2[A] \cap C = \emptyset$. Using condition (4) of I.1.2 there is a $P^* \in \mathcal{P}$ such that $P^*[A] = P_1[A]$ for $i = 1, 2$. Then $P^*[A] \cap (B \cup C) = \emptyset$. By I.1.6 there is a $P \in \mathcal{P}$ such that $P[A] \cap P[B \cup C] = \emptyset$ and thus $A \not\subseteq (B \cup C)$.

(3) Since $\Delta_X \subseteq P$ for all $P \in \mathcal{P}$, $(x, x) \in P$ for all $x \in X$ and for all $P \in \mathcal{P}$. Thus $\{x\} \subseteq \{x\}$ for all $x \in X$.

(4) Since $P[\emptyset] = \emptyset$ for all $P \in \mathcal{P}$, $\emptyset \not\subseteq A$ for all $A \subseteq X$.

(5) Suppose $A \not\subseteq B$. Then there is a $P \in \mathcal{P}$ such that $P[A] \cap P[B] = \emptyset$. Let $C = P[A]$ and $D = P[B]$. Then $C$ and $D$ are disjoint, $A \subseteq C$, and $B \subseteq D$. We claim that $A \not\subseteq C$ and $B \not\subseteq C$. Using I.1.6, $P[A] \cap C = \emptyset$ implies the existence of a $P^* \in \mathcal{P}$ such that $P^*[A] \cap P^*[C, C] = \emptyset$ and hence $A \not\subseteq C$. Similarly $B \not\subseteq C$ and (5) is proved.

We next show that $\mathcal{P} \subseteq \mathcal{P}(\mathcal{S}(\mathcal{P}))$. Let $P \in \mathcal{P}$. Using condition (2) of I.1.2, define a sequence of sets $P_n \in \mathcal{P}$ where $P_1 \circ P_2 \subseteq P$, and $\{P_n\}$ is a normal sequence. We claim that for all $n \in \mathbb{N}$, and for all $A \subseteq X$, $A \subseteq P_n[A]$. For if not, then $P_{n+1}[A] \cap P_{n+1}[C(P_n[A])] \neq \emptyset$. Then there is an $a \in A$, $b \in P_{n+1}[A]$, and an $x \in X$ such that $(a, x)$ and $(b, x) \in P_{n+1}$. Then $(a, b) \in P_n$ and $b \in P_n[A]$. Since this is not possible, $A \subseteq P_n[A]$. Then by I.1.4, there is a uniform structure $\mathcal{U}^*$ such that $\{P_n\} \subseteq \mathcal{U}^*$ and $\mathcal{S}(\mathcal{U}^*) = \emptyset$. Then $P \in \mathcal{U}^*$, and $\mathcal{U}^* \subseteq \mathcal{P}(\mathcal{S}(\mathcal{P}))$, and thus $\mathcal{P} \subseteq \mathcal{P}(\mathcal{S}(\mathcal{P}))$. 
Conversely, let $U \in \mathcal{U}$ where $\mathcal{U} \in \pi(S)$. Let $\{U_1\}$ be such that each $U_1$ is symmetric, $U_1 \circ U_1 \subseteq U_{1-1}$ for $1 \geq 2$, and $U_1 \circ U_1 \subseteq U$. Then for each $i \in \mathbb{N}$ and each $A \subseteq X$, $A \subseteq U_1[A]$. Thus $A \in \mathcal{U}_1[A]$ which implies, by I.1.6, that there is a $P \in \mathcal{P}$ such that $P[A] \cap \mathcal{U}_1[A] = \emptyset$, or $P[A] \subseteq U_1[A]$. Then $\{U_1\}$ and $U$ satisfy the hypotheses of condition (5) of I.1.2, and therefore $U \in \mathcal{P}$.

I.1.8. Theorem. For any $p$-space $(X, \delta)$, $\delta$ is equal to $\delta(\mathcal{P}(S))$.

Proof. We first show that $\delta \leq \delta(\mathcal{P}(S))$ (see 0.3.17). Suppose $A \in \delta(\mathcal{P}(S)) B$. Then for all $P \in \mathcal{P}(S)$, $P[A] \cap P[B] \neq \emptyset$. Since $\mathcal{U}(S) \subseteq \mathcal{P}(S)$, the above statement is true for all $P = U \in \mathcal{U}(S)$, and thus $A \in \delta B$.

To see that $\delta(\mathcal{P}(S)) \leq \delta$, let $P \in \mathcal{P}(S)$. Let $U$ be such that $P \in \mathcal{U}$ and $\mathcal{U} \in \pi(S)$. Suppose $A \in \delta B$. Then $U[A] \cap U[B] \neq \emptyset$ for all $U \in \mathcal{U}$ in particular for $U = P$. Since this is true for all $P \in \mathcal{P}(S)$, we have that $A \in \delta B$ implies that $A \in \delta(\mathcal{P}(S)) B$.

I.1.9. Remark. A proximity space uniquely determines and is uniquely determined by its proximity structure. The concept of $p$-structure is not new. It is inherent in Smirnov's [27] maximal uniform pseudo-structures and in the maximal generalized uniform spaces of Alfsen and Njastad [4]. The conditions in I.1.2 in slightly different form can be found in [4]. Instead of considering the $p$-structure
as a maximal element of some other type of structure which
generates a given p-space, we can take this concept as a
definition of a proximity space. We will make use of this
concept throughout this paper.

1.1.10. Remark. The following theorems show some of
the similarities between p-structures and uniform structures.
We will call the elements of the class \( \mathcal{P}(\delta) \) \( \delta \)-entourages.

1.1.11. Theorem. Suppose \((X, \delta)\) and \((X^*, \delta^*)\) are
p-spaces with p-structures \( \mathcal{P}(\delta) \) and \( \mathcal{P}(\delta^*) \) respectively.
Suppose \( f \in [X, X^*] \). Then \( f \in p[X, X^*] \) iff for every \( P^* \in \mathcal{P}(\delta^*) \)
\( (xfx)^{-1}[P^*] \in \mathcal{P}(\delta) \).

Proof. Suppose \( f \in p[X, X^*] \). Then \( A \subseteq B \) implies that
\( f[A] \delta^* f[B] \). Let \( P^* \in \mathcal{P}(\delta^*) \) and let \( \{P^*_1\} \) be a normal
sequence in \( \mathcal{P}(\delta^*) \) such that \( P^*_1 \circ P^*_1 \subseteq P^* \). We claim that
\( \{(xfx)^{-1}[P^*_1]\} \) and \( (xfx)^{-1}[P^*] \) satisfy the hypotheses of
(5) of 1.1.2. Since \( \{P^*_1\} \) is a normal sequence, so is
\( \{(xfx)^{-1}[P^*_1]\} \). We claim that for \( A \subseteq X \) and \( i \in N \),
\( A \subseteq (xfx)^{-1}[P^*_1][A] \); and hence that there is a \( P \in \mathcal{P}(\delta) \)
such that \( P[A] \subseteq (xfx)^{-1}[P^*_1][A] \). Since \( P^*_1 \in \mathcal{P}(\delta^*) \),
\( f[A] \delta^* P^*_1[f[A]] \); that is, \( f[A] \delta^* P^*_1[f[A]] \). Thus
\( f^{-1}f[A] \delta \cap f^{-1}[P^*_1[f[A]]] \). Since \( A \subseteq f^{-1}f[A] \) and
\( f^{-1}[P^*_1[f[A]]] \subseteq (xfx)^{-1}[P^*_1][A] \), we have \( A \delta \cap ((xfx)^{-1}[P^*_1][A]) \)
or \( A \subseteq (xfx)^{-1}[P^*_1][A] \). Thus (5) of 1.1.2 implies that
\( (xfx)^{-1}[P^*] \in \mathcal{P}(\delta) \).
To show the converse, suppose \( P^* \in \mathcal{P}(\mathcal{S}^*) \) implies that \( (\mathcal{F}\mathcal{F})^{-1}[P^*] \in \mathcal{P}(\mathcal{S}) \). Let \( A \subseteq B \). Then for any \( P^* \in \mathcal{P}(\mathcal{S}^*) \) there is an \( x(P^*) \in X \) such that \( x(P^*) \in (\mathcal{F}\mathcal{F})^{-1}[P^*][A] \) and \( (\mathcal{F}\mathcal{F})^{-1}[P^*][B] \). Then \( f(x(P^*)) \in P^*[f[A]] \cap P^*[f[B]] \). Thus for \( P^* \in \mathcal{P}(\mathcal{S}^*) \), \( P^*[f[A]] \cap P^*[f[B]] \neq \emptyset \) and therefore \( f[A] \subseteq f[B] \). Therefore \( f \) is \( p \)-continuous.

**I.1.12. Lemma.** If \((X, \mathcal{S})\) is a \( p \)-space with \( p \)-structure \( \mathcal{P}(\mathcal{S}) \), then for any \( P \in \mathcal{P}(\mathcal{S}) \) there exists a \( P_c \in \mathcal{P}(\mathcal{S}) \) closed relative to \( \mathcal{S}(\mathcal{U}) \) and symmetric and a \( P_o \in \mathcal{P}(\mathcal{S}) \) open relative to \( \mathcal{S}(\mathcal{U}) \) and symmetric such that \( P \subseteq P_c \) and \( P \subseteq P_o \).

**Proof.** If \( P \in \mathcal{P}(\mathcal{S}) \) then there is a uniform structure \( \mathcal{U} \) such that \( P \subseteq \mathcal{U}, \mathcal{U} \subseteq \mathcal{P}(\mathcal{S}), \) and \( \mathcal{S}(\mathcal{U}) = \mathcal{S} \). Since \( \mathcal{S}(\mathcal{U}) = \mathcal{S}(\mathcal{S}(\mathcal{U})) = \mathcal{S}(\mathcal{S}) \), and since each \( U \in \mathcal{U} \) has the above properties, the lemma follows.

**I.1.13. Lemma.** If \((X, \mathcal{S})\) is a \( p \)-space with \( p \)-structure \( \mathcal{P}(\mathcal{S}) \) then for \( A, B \subseteq X \), \( A \subseteq B \) iff there is a \( P \) in \( \mathcal{P}(\mathcal{S}) \) such that \( P[A] = B \).

**Proof.** \( A \subseteq B \) implies that there is a \( U \in \mathcal{U}(\mathcal{S}) \) such that \( U[A] = B \). Since \( \mathcal{U}(\mathcal{S}) \subseteq \mathcal{P}(\mathcal{S}) \), one part of the lemma is proved.

Suppose \( P[A] = B \) for some \( P \in \mathcal{P}(\mathcal{S}) \). Since \( P \in \mathcal{P}(\mathcal{S}) \) there is a \( \mathcal{U} \in \mathcal{S}(\mathcal{U}) \) such that \( P \in \mathcal{U} \). Then \( A \subseteq B \) relative to \( \mathcal{S}(\mathcal{U}) \) by 0.3.11. Noting that \( \mathcal{U} \in \mathcal{S}(\mathcal{S}) \) iff \( \mathcal{S}(\mathcal{U}) = \mathcal{S} \) completes the proof.
1.1.14. Lemma. If \((X, \delta)\) is a p-space and \(A \subseteq X\), then \(\mathcal{P}(\delta) \cap (A \times A) \subseteq \mathcal{P}(\delta \upharpoonright A)\) (see 0.3.16).

Proof. Since \(\mathcal{P}(\delta) \cap (A \times A) = \bigcup_{U \in \mathcal{P}(\delta)} (A \times A) : U \subseteq \delta\) and \(\delta \cap (A \times A) = \delta \upharpoonright A\) by 0.3.16, the result follows.

1.1.15. Remark. It would sometimes be useful when dealing with subspaces to know more about the relationship between the p-structure of the whole space and the p-structure of the subspace than is given in 1.1.14. An interesting conjecture is: if \(A \subseteq X\), \((X, \delta)\) is a p-space, then \(\mathcal{P}(\delta \upharpoonright A) = \mathcal{P}(\delta) \cap (A \times A)\). That this is not even true for dense subsets \(A\) is shown in the following example.

1.1.16. Example. Let \(R\) be the real number system and \(d\) the Euclidean metric for \(R\). Let \(\delta(d)\) be the metric proximity for \(R\). Let \(\mathcal{U}(\delta(d))\) be as in 0.3.12. Let \((X^*, \mathcal{U}^*)\) be the uniform completion of \((R, \mathcal{U}(\delta(d)))\). Then \((X^*, \mathcal{U}^*)\) is compact. Let \(\delta^* = \delta(\mathcal{U}^*)\). We will show in I.5.7 that \(\mathcal{P}(\delta^*) = \mathcal{U}^*\). Hence \(\mathcal{P}(\delta^*) \cap (R \times R)\) is actually \(\mathcal{U}(\delta(d))\). But we will show in I.4.4 that \(\mathcal{P}(\delta(d)) = \mathcal{U}_d\) which is not \(\mathcal{U}(\delta(d))\). Noting that, for \(A, B \subseteq R\), \(A \subseteq (d) B\) iff \(A \delta^* B\) in \(X^*\) then shows that \(\delta(d) = \delta^*|R\) and that \(\mathcal{P}(\delta(d)) \neq \mathcal{P}(\delta^*) \cap (R \times R)\).

1.1.17. Lemma. Suppose \((X, \delta)\) is a p-space with p-structure \(\mathcal{P}(\delta)\).

(1) \(0 \in \mathcal{P}(\delta)\) iff for all \(x \in 0\) there is a \(P \in \mathcal{P}(\delta)\) such that \(P[x] \subseteq 0\).
(2) If $A \subseteq X$ is compact and $A \subseteq 0 \in \mathcal{O}(\delta)$ then there is a $P \subseteq \wp(\mathcal{S})$ such that $P[A] \subseteq 0$.

**Proof.** Each of the above results is true for each $\mathcal{U} \in \mathcal{U}(\mathcal{S})$. Since $\mathcal{O}(\mathcal{U}) = \mathcal{O}(\mathcal{S})$ for all $\mathcal{U} \in \mathcal{U}(\mathcal{S})$, the results follow.

### 1.2 Lattice of Proximities

**1.2.1. Remark.** In this section we discuss the partial order $\leq$ defined on the collection of all proximity relations for a given set (see 0.3.17). We will use the notation $\Pi(X)$ to denote the set of all proximity relations on the set $X$. Similarly $\mathcal{T}(X)$ and $\mathcal{U}(X)$ denote the set of all topologies for $X$ and uniformities for $X$ respectively.

**1.2.2. Theorem.** Suppose $\delta$ and $\delta^* \in \Pi(X)$. The following conditions are equivalent.

1. $\delta \leq \delta^*$.
2. For $A, B \subseteq X$, $A \delta^* B$ implies $A \delta B$.
3. For $A, B \subseteq X$, $A \not\subseteq B$ implies $A \not\subseteq B$.
4. For $A, B \subseteq X$, $A \subseteq B$ implies $A \subseteq B$.
5. $\mathcal{U}(\delta) \subseteq \mathcal{U}(\delta^*)$.
6. $\mathcal{P}(\delta) \subseteq \mathcal{P}(\delta^*)$.

**Proof.** (1) is equivalent to (2) by definition. We follow the pattern $(2) \rightarrow (3) \rightarrow (4) \rightarrow (5) \rightarrow (6) \rightarrow (2)$ in giving the proof.

$(2) \rightarrow (3)$ Suppose for $A, B \subseteq X$, $A \delta^* B$ implies $A \delta B$. Then by contraposition, $A \not\subseteq B$ implies $A \not\subseteq B$. 
(3)→(4) Suppose for $A, B \subseteq X$, $A \triangleleft B$ implies $A \triangleleft^* B$. Then $A \subseteq B$ iff $A \triangleleft CB$. But then $A \triangleleft^* CB$ which implies that $A \subseteq^* B$.

(4)→(5) From 0.3.12, $U \in \mathcal{U}(S)$ iff there is a finite cover $\{B_i\}$ of $X$ and sets $A_i$ such that $B_i \subseteq A_i$ and $\bigcup (A_i \times A_i)$ is contained in $U$. Since $B_i \subseteq A_i$ implies $B_i \subseteq^* A_i$, $\bigcup (A_i \times A_i) \subseteq U$ implies that $U \in \mathcal{U}(S^*)$. Thus $U \in \mathcal{U}(S^*)$ implies $U \in \mathcal{U}(S)$.

(5)→(6) Let $P \in \mathcal{P}(S)$. Then there is a normal sequence $\{P_n\}$ in $\mathcal{P}(S)$ such that $P_1 \circ P_1 \subseteq P$ and for $n \in N$, $A \subseteq X$, there is a $U \in \mathcal{U}(S)$ such that $U[A] \subseteq P_n[A]$. This follows from I.1.2 and the fact that, for $Q \in \mathcal{P}(S)$, $Q[A] \subseteq B$ iff $A \subseteq B$. iff there is a $U \in \mathcal{U}(S)$ such that $U[A] \subseteq B$. But $U \in \mathcal{U}(S)$ implies $U \in \mathcal{U}(S^*)$. Thus $A \subseteq^* P_n[A]$ for all $A \subseteq X$, $n \in N$. Thus, by I.1.13, for each $A$ and each $n$, there is a $P^*$ in $\mathcal{P}(S^*)$ such that $P^*[A] \subseteq P_n[A]$. Then by (5) of I.1.2 $P \in \mathcal{P}(S^*)$. Thus $P \in \mathcal{P}(S)$ implies $P$ is in $\mathcal{P}(S^*)$.

(6)→(2) Suppose $A \triangleleft^* B$. Then $P[A] \cap B \neq \emptyset$ for all $P \in \mathcal{P}(S^*)$ and thus for all $P \in \mathcal{P}(S) \subseteq \mathcal{P}(S^*)$, and therefore $A \triangleleft B$.

I.2.3. Theorem. If $\mathcal{U}$ and $\mathcal{U}^* \subseteq \mathcal{U}(X)$ and $\mathcal{U} \subseteq \mathcal{U}^*$, then $S(\mathcal{U}) \subseteq S(\mathcal{U}^*)$.

Proof. Suppose $A \triangleleft (\mathcal{U}^*) B$. Then $U[A] \cap U[B] \neq \emptyset$ for all $U \in \mathcal{U}^*$ and thus for all $U \in \mathcal{U}$. Therefore $A \triangleleft (\mathcal{U}) B$. \

\[ \]
I.2.4. Remark. Using (5) of I.2.2 and I.2.3 we could deduce many of the properties of the order \( \leq \) from the order \( \subseteq \) on \( \bigcup(X) \). There is however, some merit to a direct examination of \( \leq \) on \( \tau(X) \) and this will be our approach here.

I.2.5. Theorem. There is a smallest element \( \delta_m \) in \( \tau(X) \) called the trivial proximity. There is a largest element \( \delta_M \) in \( \tau(X) \) called the discrete proximity.

Proof. For \( A, B \) \( X \) define \( A \delta_m B \) iff \( A \) and \( B \) are both non-empty. We first show that \( \delta_m \) is a proximity relation using 0.3.1. Conditions (1) through (4) of 0.3.1 are trivially satisfied. To see that condition (5) is also satisfied, we first note that \( A \delta_m B \) iff one, or both, are empty. If only one, say \( A \), is empty we let \( C = X \) and \( D = \emptyset \); otherwise we choose \( C = D = \emptyset \) and \( C \) and \( D \) will satisfy condition (5).

That \( \delta_m \leq \delta \) for any \( \delta \in \tau(X) \) follows from the fact that, for any \( \delta \), \( A \delta B \) implies \( A \neq \emptyset \neq B \) by condition (4) of 0.3.1.

For the remainder of the theorem, let \( A \delta_M B \) iff \( A \cap B \neq \emptyset \). Again conditions (1) through (4) of 0.3.1 are trivially met by \( \delta_M \). If \( A \delta_M B \), we can set \( A = C \) and \( B = D \) and condition (5) will be met.

To see that \( \delta \leq \delta_M \) for all \( \delta \in \tau(X) \), we note that \( A \cap B \neq \emptyset \) implies that \( P[A] \cap P[B] \neq \emptyset \) for all \( P \in \mathcal{P}(\delta) \).
for all \( \delta \in \mathcal{P}(X) \), since \( \Delta_X \) is contained in every such \( \mathcal{P} \).

Thus \( A \mathcal{S} \mathcal{M} B \) implies \( A \mathcal{S} B \) for any \( \delta \in \mathcal{P}(X) \).

1.2.6. Remark. The above theorem does not appear explicitly in the literature but certainly is part of the folk-lore of the theory. Constructions similar to the following are used by several authors although none of them characterize the supremum of a non-empty family of proximities. Notationally, if \( \{A_i\} \) is a finite collection of subsets of \( X \) and \( A = \bigcup A_i \), then we will call \( \{A_i\} \) a finite cover from below of \( A \).

1.2.7. Theorem. Suppose \( \{\delta_\alpha : \alpha \in I\} \in \mathcal{P}(X) \) where \( I \neq \emptyset \). Define \( A \mathcal{S} \delta^* B \) iff for any \( \{A_i\} \) a finite cover from below of \( A \) and any \( \{B_j\} \) a finite cover from below of \( B \), there is an \( i^* \) and a \( j^* \) such that \( A_{i^*} \mathcal{S} \alpha B_{j^*} \) for all \( \alpha \in I \). Then \( \delta^* \) is a proximity relation for \( X \); \( \delta_\alpha \leq \delta^* \) for all \( \alpha \in I \); and \( \delta^* \) is the smallest proximity relation in \( \mathcal{P}(X) \) with these properties.

Proof. We first show that \( \delta^* \) is a proximity relation, again using 0.3.1. Clearly conditions (1), (3), and (4) are satisfies by \( \delta^* \).

(2) To see that condition (2) is satisfied, suppose \( A \mathcal{S}^* B \) and \( A \mathcal{S}^* C \). Then there are collections \( \{A_i\} \) and \( \{B_j\} \) finite covers from below for \( A \) and \( B \) respectively, \( \{A_k\} \) and \( \{C_m\} \) finite covers from below of \( A \) and \( C \) respectively, such that for any pair \( (i,j) \) there is an \( \alpha(i,j) \in I \) such that \( A_1 \mathcal{S}_{\alpha(i,j)} B_j \) and for any pair \( (k,m) \) there is an \( \alpha^*(k,m) \)
in I such that \( A_k^* \delta_{\alpha(k,m)} C_m \). We show that \( A \overset{\delta^*}{\rightarrow} (B \cup C) \).

Let \( D(1,k) = A_1 \cap A_k^* \) for each \( i \) and \( k \). Then \( \{D(1,k)\} \) is a finite cover from below of \( A \) and \( \{B_j^*\} \cup \{C_m^*\} \) is a finite cover from below of \( B \cup C \). Consider any \( (i,k) \) and any \( j \).

Since \( A_1 \delta_{\alpha(i,j)} B_j^* \), we have \( (A_1 \cap A_k^*) \delta_{\alpha(i,j)} B_j^* \).

Similarly, for \( (i,k) \) and \( m \) there is an \( \alpha \in I \) such that \( (A_1 \cap A_k^*) \delta_{\alpha} C_m^* \). Thus we have \( A \overset{\delta^*}{\rightarrow} B \) and \( A \overset{\delta^*}{\rightarrow} C \) implies \( A \overset{\delta^*}{\rightarrow} (B \cup C) \).

To show the converse, suppose \( A \overset{\delta^*}{\rightarrow} B \) (or \( A \overset{\delta^*}{\rightarrow} C \)).

Let \( \{A_i^*\} \) be a finite cover from below of \( A \) and \( \{D_j^*\} \) be a finite cover from below of \( B \cup C \). Then \( \{B \cap D_j^*\} \) is a finite cover from below of \( B \) and therefore there is an \( i^* \) and a \( j^* \) such that \( A_i^* \delta_{\alpha}(B \cap D_j^*) \) for all \( \alpha \in I \). Since \( B \cap D_j^* \subseteq D_j^* \), we have that \( A_i^* \delta_{\alpha} D_j^* \) for all \( \alpha \in I \). Thus \( A \overset{\delta^*}{\rightarrow} (B \cup C) \).

(5) We now show that (5) of 0.3.1 is satisfied.

Suppose \( A \overset{\delta^*}{\rightarrow} B \). Then there are \( \{A_i^*\} \) and \( \{B_j^*\} \) finite covers from below of \( A \) and \( B \) respectively such that for every pair \( (i,j) \) there is an \( \alpha(1,j) \in I \) such that \( A_1 \delta_{\alpha(1,j)} B_j^* \). The proof proceeds in two stages.

Fix \( i^* \). Then for each \( j \), there is an \( \alpha_j = \alpha(i^*,j) \) such that \( A_{i^*} \delta_{\alpha_j} B_j^* \). Then for each \( j \) there are sets \( C_j \) and \( D_j \) such that \( A_{i^*} \subseteq \alpha_j C_j \), \( B_j \subseteq \alpha_j D_j \) and \( C_j \cap D_j = \emptyset \).

Let \( C_1^* = \bigcap C_j \) and \( D_1^* = \bigcup D_j \). Then \( A_{i^*} \subseteq C_1^* \), \( B \subseteq D_1^* \) and \( C_1^* \cap D_1^* = \emptyset \). We claim that \( A_{i^*} \delta^* \subseteq C_1^* \), and that
B \delta* \subseteq D_1. Suppose A_1, \delta* \subseteq C_1. Since \cup C_j = \bigcup C_j, there is a j such that A_1_1, \delta_{\alpha_j} \subseteq C_j for all \alpha \in I. But A_1, \delta_{\alpha_j} \subseteq C_1. Hence A_1, \delta* \subseteq C_1. Suppose B \delta* \subseteq D_1. Then \subseteq D_1, \delta* \cup B_j, and there is a j such that \subseteq D_1, \delta_{\alpha_j} B_j for all \alpha \in I. Then \subseteq D_1, \delta_{\alpha_j} B_j and \subseteq D_1 \subseteq D_j implies \subseteq D_j \delta_{\alpha_j} B_j. But B_j \subseteq D_j and therefore A_1, \delta* B.

We then have, for each i, a C_i and D_i disjoint such that A_i \subseteq C_i, B \subseteq D_i and A_i \delta* \subseteq C_i, B \delta* \subseteq D_i. Let C = \bigcup C_i and D = \bigcap D_i. Then A \subseteq C, B \subseteq D, and B \cap D = \emptyset. If A \delta* \subseteq C then A_i \delta* \subseteq C for some i*. By condition (2) of 0.3.1 proved above, A_i \delta* \subseteq C_1. Since \subseteq C_i \subseteq C_1*. But since A_i \delta* \subseteq C_1 we must have A \delta* \subseteq C. If B \delta* \subseteq D then there would be an i* such that B \delta* \subseteq D_i* (\subseteq D = \bigcup \subseteq D_i), again by (2) of 0.3.1. But B \delta* \subseteq D_i*, and therefore B \delta* \subseteq D.

We next show that \delta* \Rightarrow \delta_{\alpha} for all \alpha \in I. Suppose A \delta* B. Since \{A_i\} and \{B_j\} are finite covers from below of A and B respectively, we have A \delta_{\alpha} B for all \alpha \in I. Thus \delta* \Rightarrow \delta_{\alpha} for all \alpha \in I.

Finally, to show \delta* is the smallest proximity relation with this property; suppose \delta \Rightarrow \delta_{\alpha} for all \alpha \in I. We show that \delta \Rightarrow \delta* *. Suppose A \delta B and \{A_i\} and \{B_j\} are finite covers from below of A and B respectively. Then by property (2) of 0.3.1, A \delta B implies that there is an i*
and a \( j^* \) such that \( A_1 \delta^* B_j^* \). But then \( A_1 \delta^* B_j^* \) for all \( \alpha \in I \). We can conclude from this that \( A \delta^* B \). Thus \( \delta > \delta^* \).

**1.2.8. Remark.** If \( \{ \delta_\alpha : \alpha \in I \} \subseteq \Pi(X) \), the \( \delta^* \) constructed in 1.2.7 will be denoted by \( \forall \{ \delta_\alpha : \alpha \in I \} \) or \( \sup \{ \delta_\alpha : \alpha \in I \} \), and called the supremum of the family \( \{ \delta_\alpha : \alpha \in I \} \). The following corollary will be useful later.

**1.2.9. Corollary.** Suppose \( I \neq \emptyset \) and suppose \( \{ \delta_\alpha : \alpha \in I \} \subseteq \Pi(X) \). Let \( \delta^* = \sup \{ \delta_\alpha : \alpha \in I \} \). Let \( \subseteq_\alpha \) be the \( p \)-neighborhood relation for \( \delta_\alpha \) and \( \subseteq^* \) the \( p \)-neighborhood relation for \( \delta^* \). Then \( A \subseteq^* B \) iff there is a collection \( \{ A_i \} \), a finite cover from below of \( A \), a finite collection \( \{ B_j^* \} \), \( B = \bigcap B_j^* \), such that for any pair \((i,j)\), \( A_i \subseteq_\alpha B_j^* \) for some \( \alpha \in I \).

**Proof.** Suppose \( A \subseteq^* B \). Then \( A \delta^* \subseteq B \) and there are \( \{ A_i \} \) and \( \{ C_j \} \) finite covers from below of \( A \) and \( \subseteq B \) respectively such that for any pair \((i,j)\), there is an \( \alpha \in I \) such that \( A_i \delta_\alpha C_j \). Let \( B_j = \subseteq C_j \). Then \( \subseteq B_j \) is equal to \( \subseteq C_j = B \). \( A_i \delta_\alpha C_j \) implies \( A_i \subseteq_\alpha \subseteq C_j \) and thus \( A_i \subseteq_\alpha B_j \).

Suppose there are collections \( \{ A_i \} \) and \( \{ B_j \} \). \( \{ A_i \} \) a finite cover from below of \( A \), \( \subseteq B_j = B \), such that for any pair \((i,j)\) there is an \( \alpha \in I \) such that \( A_i \subseteq_\alpha B_j \). Then \( \{ \subseteq B_j \} \) is a finite cover from below of \( \subseteq B \), and \( A_i \delta_\alpha \subseteq B_j \). Then by 1.2.7, \( A \delta^* \subseteq B \) or \( A \subseteq^* B \).
I.2.10. Theorem. For any non-empty set X, \((\mathcal{P}(X), \leq)\) is a complete lattice.

Proof. To show that \((\mathcal{P}(X), \leq)\) is a lattice, we first note that \(\leq\) is a partial order. Theorem I.2.7 shows that for \(\delta_1\) and \(\delta_2 \in \mathcal{P}(X)\) there is a \(\delta_1 \vee \delta_2 \in \mathcal{P}(X)\). For \(\delta_1\) and \(\delta_2 \in \mathcal{P}(X)\) we define \(\delta_1 \wedge \delta_2 = \sup\{\delta : \delta \leq \delta_i, i = 1,2\}\). Since \(\delta_m \leq \delta_1\) for \(i = 1,2\), the above family is not empty and its supremum exists by I.2.7; that \(\delta_1 \wedge \delta_2\) is actually the infimum for \(\delta_1\) follows from standard arguments.

To show that \((\mathcal{P}(X), \leq)\) is complete we note that for any non-empty family \(\{\delta_\alpha : \alpha \in I\} \subseteq \mathcal{P}(X)\), \(\sup\{\delta_\alpha : \alpha \in I\}\) exists by I.2.7. Defining \(\inf\{\delta_\alpha : \alpha \in I\} = \sup\{\delta : \delta \leq \delta_\alpha\}, \text{ for all } \alpha \in I\) and noting that \(\{\delta : \delta \leq \delta_\alpha\} \text{ for all } \alpha \in I\) is not empty completes the proof.

I.2.11. Theorem. Suppose \(I \neq \emptyset\), \(\{\delta_\alpha : \alpha \in I\} \subseteq \mathcal{P}(X)\). Let \(\mathcal{U}^* = \sup\{\mathcal{U}(\delta_\alpha) : \alpha \in I\}\). Then \(\mathcal{U}^* = \mathcal{U}(\delta^*)\) where \(\delta^* = \sup\{\delta_\alpha : \alpha \in I\}\).

Proof. Since \(\delta_\alpha \leq \delta^*\) for all \(\alpha \in I\), I.2.2 implies that \(\mathcal{U}(\delta_\alpha) \subseteq \mathcal{U}(\delta^*)\) for all \(\alpha \in I\). Therefore \(\mathcal{U}^* \subseteq \mathcal{U}(\delta^*)\).

Since \(\mathcal{U}(\delta_\alpha) \subseteq \mathcal{U}^\ast\) for all \(\alpha \in I\), I.2.3 implies that \(\delta(\mathcal{U}(\delta_\alpha)) = \delta_\alpha \leq \delta(\mathcal{U}^\ast)\). But then \(\delta^* \leq \delta(\mathcal{U}^\ast)\). Again by I.2.2, \(\mathcal{U}(\delta^*) \subseteq \mathcal{U}(\delta(\mathcal{U}^\ast))\). But by 0.3.13 (2), \(\mathcal{U}(\delta(\mathcal{U}^\ast)) \subseteq \mathcal{U}^\ast\). Thus \(\mathcal{U}(\delta^*) \subseteq \mathcal{U}^\ast\) and the theorem is proved.
1.2.12. Corollary. Suppose $I \neq \emptyset$, \{$d_{\alpha}: \alpha \in I\} \subseteq \Pi(X)$. Then $\tau(\sup\{d_{\alpha}: \alpha \in I\}) = \sup\{\tau(d_{\alpha}): \alpha \in I\}$.

Proof. Let $U^* = \sup\{u(d_{\alpha}): \alpha \in I\}$. It is well known that $\tau(U^*) = \sup\{\tau(u(d_{\alpha})): \alpha \in I\}$. From 0.3.15 (1), $\tau(U^*) = \tau(\delta(U^*)) = \tau(\sup\{d_{\alpha}: \alpha \in I\})$ by 1.2.11. But $\tau(U(d_{\alpha})) = \tau(\delta)\tau(\delta)$ again by 0.3.15 (1) and thus $\tau(\sup\{d_{\alpha}: \alpha \in I\}) = \sup\{\tau(d_{\alpha}): \alpha \in I\}$.

1.2.13. Theorem. Suppose \{$u_{\alpha}: \alpha \in I\} \subseteq U(X)$ is non-empty, Then $S(\inf\{u_{\alpha}: \alpha \in I\}) = \inf\{S(u_{\alpha}): \alpha \in I\}$.

Proof. For convenience let $S(\inf\{u_{\alpha}: \alpha \in I\}) = S_1$ and $\inf\{S(u_{\alpha}): \alpha \in I\} = S_*$. Since $\inf\{u_{\alpha}: \alpha \in I\} \subseteq u_{\alpha}$ for all $\alpha \in I$, by 1.2.3, $S_1 \leq S(u_{\alpha})$ for all $\alpha \in I$. Hence $S_1 \leq S_*$. We will now show that $S_* \leq S_1$. We know that $S_* \leq S(u_{\alpha})$ for all $\alpha \in I$. Then $u(S*) \subseteq u(S(u_{\alpha}))$ by 1.2.2. By 0.3.13, $u(S(u_{\alpha})) \subseteq u_{\alpha}$ for all $\alpha \in I$. Thus $u(S*) \subseteq u_{\alpha}$ for all $\alpha \in I$ and hence $u(S*)$ is contained in $\inf\{u_{\alpha}: \alpha \in I\}$. Then by 1.2.3, $S(u(S*)) = S* \leq S_1$.

1.2.14. Remark. Although we do have certain lattice properties preserved under the functors from $U$ to $\mathcal{P}$ and $\mathcal{P}$ to $\tau$ (see previous theorems) some properties are not. We show in 1.4 that generally $S(\sup\{u_{\alpha}: \alpha \in I\})$ is not equal to $\sup\{S(u_{\alpha}): \alpha \in I\}$. That $\tau(\inf\{S_{\alpha}: \alpha \in I\})$ is not equal to $\inf\{\tau(S_{\alpha}): \alpha \in I\}$ follows from the fact that...
inf\{\sigma(\delta_\alpha): \alpha \in I\} need not be completely regular although 
\sigma(\inf\{\delta_\alpha: \alpha \in I\}) must be completely regular.

1.2.15. **Remark.** For any p-space \((X, \mathcal{S})\) let \(D(\mathcal{S})\) 
be the gauge of \(\mathcal{S}\) (see 0.3.20). Then for \(A, B \subseteq X\), \(A \sim B\) 
iff \(D(d)(A, B) = 0\) for all \(d \in D(\mathcal{S})\). In [20] Mrowka 
defines several proximity constructions by specifying a 
class of pseudometrics on the given set. It is never 
explicitly stated in this paper how the proximity relation 
is to be determined from the family of pseudometrics. Let 
\(D^*\) be a family of pseudometrics on a given set \(X\). We have 
the following possibilities:

(1) Define a relation \(\mathcal{S}^*\) on \(X\) by: \(A \sim^* B\) iff 
\(D(d)(A, B) = 0\) for all \(d \in D^*\).

(2) Define a relation \(\mathcal{S}^*\) on \(X\) by: \(\mathcal{S}^*\) is equal to 
sup\{\mathcal{S}(d): d \in D^*\}.

(3) Define \(\mathcal{U} \subseteq \mathcal{U}(X)\) where \(\mathcal{U} = \sup\{\mathcal{U}_d: d \in D^*\}\) and 
define a relation \(\mathcal{S}^*\) where \(\mathcal{S}^* = \mathcal{S}(\mathcal{U})\).

As the next example shows the relation defined in (1) 
need not be a proximity relation. Although the relations in 
(2) and (3) are p-relations they are not generally equal 
(see I.4). One solution to this problem is to give nec-
essary and sufficient conditions for a family of pseudo-
metrics on a set \(X\) to determine a p-relation as defined in 
(1) above. This we do in I.2.17.
1.2.16. Example. Let \( X = \{1,2\} \times \{1,2\} \). Define pseudometrics \( d_i, i = 1,2 \) on \( X \) as follows:

\[
d_1((a,b),(c,d)) = \begin{cases} 
0 & \text{if } a = c \\
1 & \text{if } a \neq c 
\end{cases}
\]

\[
d_2((a,b),(c,d)) = \begin{cases} 
0 & \text{if } b = d \\
1 & \text{if } b \neq d 
\end{cases}
\]

Then \( D^* = \{d_i: i = 1,2\} \) is a family of pseudometrics for \( X \).

For \( A, B \subseteq X \) define \( A \succ B \) iff \( D(d_i)(A,B) = 0 \) for \( i = 1,2 \).

We claim that \( \succ \) is not a \( p \)-relation. For, let \( A = \{(1,1)\}, B = \{(2,1)\}, \) and \( C = \{(1,2)\} \). Then \( A \succ (B \cup C) \). To show this, \( D(d_1)(A,B \cup C) = \min \{d_1((1,1),(2,1)), d_1((1,1),(1,2))\} \) and since for each \( i \) at least one of the above numbers is zero, \( D(d_1)(A,B) = 0 \) for \( i = 1,2 \). Hence \( A \succ (B \cup C) \). But \( D(d_1)(A,B) = 1 \) and \( D(d_2)(A,C) = 1 \); \( A \nsubseteq B \) and \( A \nsubseteq C \). Since \( A \nsubseteq (B \cup C) \) but \( A \nsubseteq B \) and \( A \nsubseteq C \), (2) of 0.3.1 is not true and \( \succ \) is not a \( p \)-relation.

1.2.17. Theorem. Suppose \( D^* \) is a non-empty family of pseudometrics on \( X \). Define \( A \preceq^* B \) iff \( D(d)(A,B) = 0 \) for all \( d \in D^* \). Then \( \preceq^* \) is a \( p \)-relation iff given \( A \subseteq X, d_i \in D^* \), \( \varepsilon_i > 0, i = 1,2 \), then there is a \( d^* \in D^* \) and an \( \varepsilon > 0 \) such that \( S_{d^*}(A) \subseteq S_{d_1}^{\varepsilon_i}(A) \) where \( S_{d}^{\varepsilon}(A) = \{x: d(x,A) < \varepsilon\} \). \( S_{d}^{\varepsilon}(\emptyset) = \emptyset \).

Proof. Suppose \( \preceq^* \) is a \( p \)-relation. Let \( \subseteq^* \) be the \( p \)-neighborhood relation for \( \preceq^* \). Suppose \( A \subseteq X, d_1 \in D^* \), \( \varepsilon_1 > 0, i = 1,2 \). Let \( A_1^* = S_{d_1}^{\varepsilon_1}(A) \). Then \( A \preceq^* A_1^* \) since \( D(d_1)(A, \subseteq^* A_1^*) > \varepsilon_1 > 0 \). Suppose there is no \( d \in D, \varepsilon > 0 \).
such that $S^e_d(A) \subseteq A^*_1$ for $i = 1,2$. Then we claim that
$A \notin \star \cap \{A^*_1: i = 1,2\}$ thus contradicting (4) of 0.3.8. We assume then that for all $\varepsilon > 0$, for all $d \in D^*$, $S^e_d(A)$ and $C(A^*_1)$ are not disjoint for at least one $i$, $i = 1,2$. Then given $\varepsilon > 0$, $d \in D^*$, $D(d)(A, \cup C A^*_1) < \varepsilon$ since $S^e_d(A) \cap \cup C A^*_1 \neq \emptyset$. But this implies that for all $d \in D^*$, $D(d)(A, \cup C A^*_1) = 0$ or that $A \notin \star \cap \{A^*_1: i = 1,2\}$. We thus have a contradiction and therefore there is an $\varepsilon > 0$ and a $d \in D$ such that $S^e_d(A) \subseteq S^e_{d_1}(A)$ for $i = 1,2$.

To show that the condition is sufficient to make $\delta^*$ a $p$-relation, we show that conditions (1) through (5) of 0.3.1 are satisfied.

(1) $A \delta^* B$ iff $B \delta^* A$ is a direct result of the fact that $D(d)(A, B) = D(d)(B, A)$.

(2) Suppose $A \delta^* B_1$ and $A \delta^* B_2$. Then there are $d_1 \in D^*$, $i = 1,2$, $\varepsilon_i > 0$, such that $D(d_1)(A, B_1) > \varepsilon_i$, that is, $S^e_{d_1}(A) \cap B_1 = \emptyset$. Then from the hypothesis, there is a $d \in D^*$ and an $\varepsilon > 0$ such that $S^e_d(A) \subseteq S^e_{d_1}(A)$. Thus $S^e_d(A) \cap B_1 = \emptyset$ for $i = 1,2$ and $D(d)(A, B_1 \cup B_2) > \varepsilon > 0$. Thus $A \delta^* (B_1 \cup B_2)$.

If $A \delta^* (B_1 \cup B_2)$ then $D(d)(A, B_1 \cup B_2) > \varepsilon > 0$ for some $d \in D^*$, some $\varepsilon > 0$. Since $D(d)(A, B_1 \cup B_2) = \min\{D(d)(A, B_i): i = 1,2\}$, we have $D(d)(A, B_1) > \varepsilon > 0$ for $i = 1,2$ and $A \delta^* B_i$ for $i = 1,2$.

(3) Since $D(d)(x, x) = 0$ for all $d \in D^*$, for all $x \in X$, $\{x\} \delta^* \{x\}$. 
(4) Since $D(d)(A,\varnothing) = \infty$ for all $d \in D^*$, $A \varnothing^* \varnothing$ for all $A \subseteq X$.

(5) Suppose $A \varnothing^* B$. Then there is a $d \in D^*$ and an $\varepsilon > 0$ such that $D(d)(A, B) = \varepsilon > 0$. Let $C = S^g_d(A)$, $D = S^g_d(B)$. Then $C \cap D = \varnothing$, $A \subseteq C$, and $B \subseteq D$. Since $D(d)(A, C) > \varepsilon / \lambda > 0$ and $D(d)(B, D) > \varepsilon / \lambda > 0$, $A \varnothing^* \subseteq C$ and $B \varnothing^* \subseteq D$.

1.2.18. Remark. The condition for this theorem is suggested by the 4'th condition of 1.1.2. Both conditions are suggested by condition G.U.5 of [4].

1.3 Quotient Maps in Proximity Spaces

1.3.0. Remark. The concept of proximal quotient spaces and quotient maps is first mentioned by Katetov in [15], where existence is proved. They are studied again by Poljakov [23] where it is announced that a factorization theorem for $p$-continuous maps similar to the known decomposition theorems for uniformly continuous and continuous functions holds. This theorem is also credited to Katetov by Poljakov. In this section we define the concept of $p$-quotient maps, give an explicit characterization of $p$-quotients, and prove another characterization of $p$-quotients.

1.3.1. Definition. Suppose $(X, \delta)$ and $(X^*, \delta^*)$ are $p$-spaces and $f \in [X,X^*]$, $f$ onto. Then $f$ will be called a proximal quotient ($p$-quotient) map iff $\delta^* = \sup \{ \delta^1 : f \in p[(X, \delta),(X^*, \delta^1)] \}$. If $f$ is a $p$-quotient map then
(X*, δ*) will be called the quotient space of (X, δ) by f.

1.3.2. Remark. For any space (X, δ), any set X*, and any onto function \( f \in [X, X^*] \), \( \{S^i : f \in p[(X, \delta), (X^*, \delta')] \} \) contains \( \delta_m \) for \( X^* \) and thus there is always a quotient of \( (X, \delta) \) by \( f \) from I.2.7. The following theorem shows that the sup of the above family is actually a member of that family and hence a p-quotient map is one in which the range has the largest proximity making the function proximally continuous.

1.3.3. Theorem. Suppose \( f : (X, \delta) \to (X^*, \delta^*) \) is a p-quotient map (see 0.3.3). Then \( f \in p[(X, \delta), (X^*, \delta^*)] \).

Proof. Let \( A, B \subseteq X \). We will show that \( A \delta B \) implies \( f[A] \delta^* f[B] \). Using I.2.7, we must show that if \( \{A^i_1\} \) and \( \{B^i_j\} \) are finite covers from below of \( f[A] \) and \( f[B] \) respectively then there is an \( i^* \) and a \( j^* \) such that \( A^i_1 \delta^* B^i_j \) for all \( \delta^* \) such that \( f \in p[(X, \delta), (X^*, \delta')] \).

We have that the unions of \( \{f^{-1}[A^i_1]\} \) and \( \{f^{-1}[B^i_j]\} \) contain \( A \) and \( B \) respectively and thus are \( \delta \) near in \( X \). Then by (2) of 0.3.1, there is an \( i^* \) and a \( j^* \) such that \( f^{-1}[A^i_1] \) and \( f^{-1}[B^i_j] \) are \( \delta \) near in \( X \). But this implies that \( ff^{-1}[A^i_1] = A^i_1 \) and \( ff^{-1}[B^i_j] = B^i_j \) are \( \delta^* \) near in \( X^* \) for all \( \delta^* \) for which \( f \in p[(X, \delta), (X^*, \delta')] \). Thus \( f[A] \delta^* f[B] \).

1.3.4. Remark. Theorem I.3.6 gives a characterization of p-quotient maps in terms of the respective
proximal neighborhood relations. The following concept will be used.

1.3.5. Definition. Suppose \((X, \mathcal{S})\) is a \(p\)-space. By a scale of sets in \((X, \mathcal{S})\) we will mean a collection \(C = \{C_\alpha : \alpha \in D\}\) where \(D\) is the set of diadic rationals in \((0,1)\) with the property that \(\alpha, \beta \in D\) and \(\alpha < \beta\) implies that \(C_\alpha \subseteq C_\beta\).

1.3.6. Theorem. Suppose \((X, \mathcal{S})\) is a \(p\)-space, \(X^*\) a set, and \(f \in [X, X^*], f\) onto. For \(A^*, B^* \subseteq X^*\) define \(A^* \sqsubset^* B^*\) iff there is a collection of sets \(\{C_\alpha^* : \alpha \in D\}\) such that:

(a) \(\{f^{-1}[C_\alpha^*] : \alpha \in D\}\) is a scale in \((X, \mathcal{S})\), and
(b) \(f^{-1}[A^*] \subseteq f^{-1}[C_\alpha^*] \subseteq f^{-1}[B^*]\) for all \(\alpha \in D\).

Then \(\mathcal{S}^*\) determined by the relation \(\sqsubset^*\) (see 0.3.8) is a \(p\)-relation and \((X^*, \mathcal{S}^*)\) is the quotient space of \((X, \mathcal{S})\) by \(f\).

Proof. We first show that \(\sqsubset^*\) actually defines a proximity relation by showing that it satisfies conditions (1) through (6) of 0.3.8.

(1) To show that \(X^* \sqsubset^* X^*\) we simply note that the collection \(C = \{C_\alpha : \alpha \in D\}\) where \(C_\alpha = X^*\) satisfies conditions (a) and (b) above.

(2) That \(A^* \sqsubset^* B^*\) implies \(A^* \subseteq B^*\) is obvious from (b) above and the fact that \(f\) is onto.

(3) If \(A^* \subseteq B^* \sqsubset^* C^* \subseteq D^*\) then the collection which shows that \(B^* \sqsubset^* C^*\) also shows that \(A^* \sqsubset^* D^*\).
(4) Suppose \( A^* \subseteq B^*_k \) for \( 1 \leq k \leq n \). To show that \( A^* \subseteq B^*_k \), let \( C_k = \{ C(\alpha, k) : \alpha \in D \} \) be the collection corresponding to \( B^*_k \) which satisfies (a) and (b) above. Letting \( C_\alpha = \bigcap \{ C(\alpha, k) : 1 \leq k \leq n \} \), it is obvious from (4) of 0.3.8 that \( f^{-1}[A^*] \subseteq f^{-1}[C_\alpha] \) for all \( \alpha \in D \).

Suppose \( \alpha, \beta \in D \) and \( \alpha < \beta \). Then \( f^{-1}[C(\alpha, k)] \) is proximally contained in \( f^{-1}[C(\beta, k)] \) for all \( k \). Then by (3) of 0.3.8, \( f^{-1}[C_\alpha] \subseteq f^{-1}[C_\beta] \) for all \( k \) and by (4) of 0.3.8 we have \( f^{-1}[C_\alpha] \subseteq f^{-1}[\bigcap B^*_k] \). Finally, \( f^{-1}[C(\alpha, k)] \) is proximally contained in \( f^{-1}[B^*_k] \) implies \( f^{-1}[C_\alpha] \subseteq f^{-1}[B^*_k] \) for all \( k \) by (3) of 0.3.8; and thus \( f^{-1}[C_\alpha] \subseteq f^{-1}[\bigcap B^*_k] \) by (4) of 0.3.8. The collection \( \bigcap' = \{ C_\alpha : \alpha \in D \} \) then satisfies (a) and (b) above and thus implies that \( A^* \subseteq \bigcap B^*_k \). We note that the converse of this statement follows from (3) proved above.

(5) Suppose \( A^* \subseteq B^* \) and \( C \) is the collection implied by the definition. For each \( \alpha \in D \) let \( C^*_\alpha = C_{1-\alpha} \) for \( C_{1-\alpha} \in C \). Then \( \alpha < \beta \) implies that \( 1-\beta < 1-\alpha \) which implies

\[ f^{-1}[A^*] \subseteq f^{-1}[C_{1-\beta}] \subseteq f^{-1}[C_{1-\alpha}] \subseteq f^{-1}[B^*]. \]

Using the collection \( C^* = \{ C^*_\alpha : \alpha \in D \} \), property (5) of 0.3.8, and the compliment operator applied to (*) above we have \( C_B^* \subseteq C_A^* \).

(6) Let \( A^* \subseteq B^* \) and \( C = \{ C_\alpha : \alpha \in D \} \) the indicated collection. Define \( C^* = C_{1/2} \). Define \( M = \{ M_\alpha : \alpha \in D \} \) where \( M_\alpha = C_{\alpha/2} \) for \( 0 < \alpha < 1 \). Define \( N = \{ N_\alpha : \alpha \in D \} \) where
\[ N_\alpha = C^*(\alpha+1)/2 \] for \( 0 < \alpha < 1 \). Then \( \mathcal{M} \) and \( \mathcal{N} \) are the collections which satisfy (a) and (b) to show that \( A* \subseteq C* \subseteq B* \).

We next show that \( f \in p[(X, S),(X*, S*)] \) and that \( S* \) is the largest proximity relation for \( X* \) for which this is true. Since \( A* \subseteq B* \) implies \( f^{-1}[A*] \subseteq f^{-1}[B*] \) we have that \( f \in p[(X, S),(X*, S*)] \).

Finally, suppose \( f \in p[(X, S),(X*, S*)] \). We will show that \( S' \leq S* \) by showing that \( A* \subseteq B* \) implies \( A* \subseteq B* \) (see I.2.2). If \( A* \subseteq B* \), then using condition (6) of 0.3.8 and mathematical induction we can construct a collection of sets \( C = \{C_\alpha : \alpha \in D\} \) such that \( A* \subseteq C_\alpha \subseteq C_\beta \subseteq B* \) when \( \alpha < \beta \). But applying \( f^{-1} \) to this string of inclusions gives us conditions (a) and (b), since \( f \) is \( p \)-continuous. This implies that \( A* \subseteq B* \).

1.3.7. Remark. We can think of a quotient space as the largest structure on the range of the function \( f \) which makes \( f \) proximally continuous. Of some interest is the case where the image space possesses a structure and a structure is then imposed on the domain. We examine this case now for its own interest and for its use in I.3.12. The definition given here is part of the folklore of the theory.

1.3.8. Theorem. Suppose \((X*, S*)\) is a \( p \)-space and \( X \) is a set. Suppose \( f \in [X, X*] \). For \( A, B \subseteq X \) define \( A \subseteq B \) iff there are far sets \( A* \) and \( B* \) in \( X* \) such that \( A \subseteq f^{-1}[A*] \) and
$B \in f^{-1}[B^*]$. Then $\delta$ is a proximity relation for $X$ and is the smallest proximity relation on $X$ for which $f$ is $p$-continuous.

**Proof.** It is only necessary to show that (2) and (5) of 0.3.1 hold since (1), (3), and (4) are obvious.

(2) Suppose $A \delta (B \cup C)$. Then there are far sets $A^*$ and $D^*$ in $X^*$ such that $A \subseteq f^{-1}[A^*]$ and $B \cup C \subseteq f^{-1}[D^*]$. Then $B$ and $C \subseteq f^{-1}[D^*]$ and thus $A \overline{\delta} B$ and $A \overline{\delta} C$. Conversely suppose $A \overline{\delta} B$ and $A \overline{\delta} C$. Then there are far sets $A^*_i$, $B^*$ and far sets $A^*_2$, $C^*$ in $X^*$ such that $A \subseteq f^{-1}[A^*_i]$, $i = 1, 2$, and $B \cup C \subseteq f^{-1}[B^*] \cup f^{-1}[C^*]$. Since $A^*_1 \cap A^*_2 \overline{\delta} B^* \cup C^*$, $A \overline{\delta} (B \cup C)$.

(5) Suppose $A \overline{\delta} B$. Then there are far sets $A^*$ and $B^*$ in $X^*$ such that $A \subseteq f^{-1}[A^*]$ and $B \subseteq f^{-1}[B^*]$. Since $A^* \overline{\delta} B^*$, there are disjoint sets $C^*$ and $D^*$ such that $A^* \subseteq C^*$ and $B^* \subseteq D^*$. Then $f^{-1}[C^*]$ and $f^{-1}[D^*]$ are disjoint, $f^{-1}[A^*] \delta f^{-1}[C^*]$ and $f^{-1}[B^*] \delta f^{-1}[D^*]$. Letting $C$ be $f^{-1}[C^*]$ and $D = f^{-1}[D^*]$, condition (5) is proved.

From the fact that $p$-continuity is equivalent to: $A^* \overline{\delta}^* B^*$ implies $f^{-1}[A^*] \delta f^{-1}[B^*]$, we have that $f$ is $p$-continuous.

Suppose $f \in p[(X, \delta^i), (X^*, \delta^*)]$ and $A \delta^i B$. Suppose $A \overline{\delta} B$. Then there exists far sets $A^*$ and $B^*$ in $X^*$ such that $f[A] \subseteq A^*$ and $f[B] \subseteq B^*$. But then, since $f$ is $p$-continuous, $f^{-1}[A^*] \overline{\delta^i} f^{-1}[B^*]$ which implies that $A \overline{\delta^i} B$. 
Since this is not possible, \( A \delta' B \) implies \( A \delta B \), and \( \delta \) is therefore the smallest proximity relation for which \( f \) is \( p \)-continuous.

**I.3.9. Remark.** When discussing properties in \( T \) or \( U \) many authors call the structure induced on the domain by a function the weak topology or uniformity. Since we will be using the term "weak" in a different sense later, we avoid this label. We will speak rather of the proximity relation induced by \( f \) on \( X \).

**I.3.10. Theorem.** Suppose \( f \in [X, X^*] \) and \( (X^*, \delta^*) \) is a \( p \)-space. Suppose \( \delta \in \Pi(X) \). Then \( \delta \) is the proximity relation induced by \( f \) on \( X \) iff

1. \( f \in \text{p}[\text{p}(X, \delta), (X^*, \delta^*)] \), and
2. for any \( p \)-space \( (Y, \delta') \) and any \( g \in [Y, X] \), \( g \in \text{p}[Y, X] \) iff \( f \circ g \in \text{p}[Y, X^*] \).

**Proof.** Let us first assume that \( \delta \) is the \( p \)-relation induced by \( f \) on \( X \). Then from I.3.8, we know that \( f \) is in \( \text{p}[X, X^*] \), and hence condition (1) holds. To show condition (2), let \( (Y, \delta') \) be any \( p \)-space and let \( g \in [Y, X] \). Then if \( g \in \text{p}[Y, X] \), it follows from the fact that the composition of \( p \)-continuous functions is \( p \)-continuous that \( f \circ g \in \text{p}[Y, X^*] \). Suppose, on the other hand, that \( f \circ g \in \text{p}[Y, X^*] \). Let \( A \) and \( B \subseteq X \) and \( A \overline{\delta} B \). Then there are \( A^* \) and \( B^* \), far sets in \( X^* \) such that \( A \subseteq f^{-1}[A^*] \) and \( B \subseteq f^{-1}[B^*] \). But then \( g^{-1}f^{-1}[A^*] \supseteq f^{-1}[A] \) and \( g^{-1}f^{-1}[B^*] \supseteq f^{-1}[B] \). Since \( A^* \overline{\delta^*} B^* \)
and \( f \circ g \) is \( p \)-continuous, \( g^{-1}f^{-1}[B^*] \supseteq g^{-1}f^{-1}[A^*] \) and thus \( g^{-1}[A] \supseteq g^{-1}[B] \); therefore \( g \) is \( p \)-continuous.

Now suppose that conditions (1) and (2) hold. We will show that \( \delta \) is the \( p \)-relation induced on \( X \) by \( f \).

Since \( f \in p[(X, S)(X^*, \delta^*)] \) we show that \( \delta \leq \delta_o \) for any \( \delta_o \) such that \( f \in p[(X, \delta_o), (X^*, \delta^*)] \). Indeed, consider \((Y, \delta') = (X, \delta_o) \) and \( g = i_X \). Then \( f \circ i_X = f \) which is in \( p[(X, \delta_o), (X^*, \delta^*)] \) implies \( i_X \in p[(X, \delta_o), (X^*, \delta^*)] \). That this implies that \( \delta \leq \delta_o \) is obvious.

**I.3.11. Remark.** We are now in a position to prove that a similar theorem serves to characterize \( p \)-quotient spaces.

**I.3.12. Theorem.** Suppose \( f \in [X, X^*] \), \( f \) is onto, and \((X, S)\) is a \( p \)-space. Suppose \( \delta^* \in \mathcal{T}(X^*) \). Then \( f \) is a \( p \)-quotient map iff

\[
(1) \quad f \in p[(X, S),(X^*, \delta^*)],
\]

\[
(2) \quad \text{for any } p \text{-space } (Y, S') \text{ and any } g \in [X^*, Y], \quad g \in p[X^*, Y] \text{ iff } g \circ f \in p[X, Y].
\]

**Proof.** Suppose that conditions (1) and (2) are satisfied. Suppose \( \delta^*_o \in \mathcal{T}(X^*) \) and \( f \in p[(X, S),(X^*, \delta^*_o)] \), then \( \delta^*_o \leq \delta^* \). For, if we let \((Y, S') = (X^*, \delta^*_o) \) and let \( g = i_{X^*} \), we have \( i_{X^*}f \in p[(X, S),(X^*, \delta^*_o)] \) and thus \( i_{X^*} \in p[(X^*, \delta^*),(X^*, \delta^*_o)] \), and thus \( \delta^*_o \leq \delta^* \). Therefore conditions (1) and (2) are sufficient to make \( f \) a \( p \)-quotient map, and thus to make \((X^*, \delta^*)\) the quotient space.
That (1) is necessary follows from I.3.3. To show that (2) is necessary, let \((Y, \delta')\) be any \(p\)-space, \(g \in [X^*, Y]\). Then \(g \in p[(X^*, \delta^*), (Y, \delta')]\) implies \(g \circ f \in p[(X, \delta), (X^*, \delta^*)]\) since the composition of \(p\)-continuous functions is \(p\)-continuous. Now suppose \(g \circ f \in p[(X, \delta), (Y, \delta')]\). Let \(\delta^{**}\) be the \(p\)-relation induced on \(X^*\) by the function \(g\) (see I.3.8, I.3.9). Then \(g\) is in \(p[(X^*, \delta^{**}), (Y, \delta')]\). We claim that \(f \in p[(X, \delta), (X^*, \delta^{**})]\). This follows directly from I.3.10 and the fact that \(g \circ f \in p[(X, \delta), (Y, \delta')]\). Then since \(\delta^*\) is the largest \(p\)-relation on \(X^*\) for which \(f\) is \(p\)-continuous, \(\delta^{**} \leq \delta^*\). Thus for \(A^*, B^* \subseteq X^*\), \(A^* \delta^* B^*\) implies that \(A^* \delta^{**} B^*\). But \(g \in p[(X^*, \delta^{**}), (Y, \delta')]\) implies that \(g[A^*] \delta' g[B^*]\). Thus \(A^* \delta^* B^*\) implies that \(g[A^*]\) is \(\delta'\) near to \(g[B^*]\), or \(g \in p[(X^*, \delta^*), (Y, \delta')]\).

I.3.13. Remark. It is well known that a similar theorem holds for quotient maps in topological spaces, and it is easily shown that a similar theorem holds for quotients in uniform spaces. Since quotients are characterized by similar properties in the categories \(\mathcal{U}\) and \(\mathcal{P}\), one might expect some relationship between quotients in these categories. The next several paragraphs describe this relationship.

I.3.14. Remark. For the reader unfamiliar with quotients of uniform spaces we note that if \((X, \mathcal{U})\) is a uniform space, \(X^*\) a set, \(f \in [X, X^*]\), and \(f\) is onto, then \(\mathcal{U}^*\) is the quotient uniform structure on \(X^*\) iff \(\mathcal{U}^*\) is
\[ \{ U^* \in X^* \times X^* : \text{there is a normal sequence} \ \{ U^*_n \} \ \text{in} \ 2^{X^*} \times X^* \ \text{such that} \ \text{(ff)}^{-1}[U^*_n] \in \mathcal{U} \ \text{for all} \ n \in \mathbb{N} \ \text{and} \ U^*_1 \circ U^*_1 \subseteq U^* \} \]

[13]. A uniformly continuous function will be called a uniform quotient map if it is onto and if the range has the uniform quotient structure determined by the given function.

I.3.15. Theorem. Suppose \( f : (X, \mathcal{U}) \rightarrow (X^*, \mathcal{U}^*) \) is a uniform quotient map. Then \( f : (X, \mathcal{U}(\mathcal{S}(\mathcal{U}))) \rightarrow (X^*, \mathcal{U}(\mathcal{S}(\mathcal{U}^*))) \) is a uniform quotient map.

Proof. Let \( \mathcal{U} \in \mathcal{U}(X^*) \) be the quotient uniform structure induced by \( f \) and \( (X, \mathcal{U}(\mathcal{S}(\mathcal{U}))) \). Then \( \mathcal{U} \) is totally bounded since \( \mathcal{U}(\mathcal{S}(\mathcal{U})) \) is totally bounded (see [6], p. 203). We also know that \( f \in \mathcal{U}[(X, \mathcal{U}), (X^*, \mathcal{U}^*)] \) implies \( f \in \mathcal{S}[(X, \mathcal{U}(\mathcal{S}(\mathcal{U}))), (X^*, \mathcal{S}(\mathcal{U}^*))] \) which implies that \( f \) is in \( \mathcal{U}[(X, \mathcal{U}(\mathcal{S}(\mathcal{U}))), (X^*, \mathcal{U}(\mathcal{S}(\mathcal{U}^*)))]. \) Therefore \( \mathcal{U}(\mathcal{S}(\mathcal{U}^*)) \subseteq \mathcal{U}. \)

We claim that \( \mathcal{U} \subseteq \mathcal{U}^* \). Let \( V \in \mathcal{U} \). Then there is a normal sequence \( \{ V_n \} \subseteq 2^{X^*} \times X^* \) such that \( V_1 \circ V_1 \subseteq V \) and \( (ff)^{-1}[V_n] \in \mathcal{U}(\mathcal{S}(\mathcal{U})). \) But \( \mathcal{U}(\mathcal{S}(\mathcal{U})) \subseteq \mathcal{U}, \) therefore each \( (ff)^{-1}[V_n] \in \mathcal{U}. \) Since this characterizes \( V \) being in the quotient structure of \( (X, \mathcal{U}) \) by \( f \) (I.3.14) we have that \( V \in \mathcal{U}^*. \)

To complete the proof, we show that \( \mathcal{S}(\mathcal{U}) = \mathcal{S}(\mathcal{U}^*) \) which is \( \mathcal{S}(\mathcal{U}(\mathcal{S}(\mathcal{U}^*))) \). Then, since \( \mathcal{U}(\mathcal{S}(\mathcal{U}^*)) \) is the only totally bounded uniform structure which generates
δ(U*), and U is totally bounded we can conclude that
U = U(δ(U*)).

To show that δ(U) = δ(U*), suppose $A^* \subseteq (U*)$ for $A^*, B^* \subseteq X^*$. Then $V[A^*] \cap V[B^*] \neq \emptyset$ for all $V \in U$ and therefore $U*[A^*] \cap U*[B^*] \neq \emptyset$ for all $U* \in U(δ(U*))$ since $U(δ(U*)) \subseteq U$, and thus $A^* \subseteq (U*) B^*$. Now suppose $A^* \subseteq (U*) B^*$. Then $U*[A^*] \cap U*[B^*] \neq \emptyset$ for all $U* \in U$.

But $U \subseteq U^*$, and therefore $V[A^*] \cap V[B^*] \neq \emptyset$ for all $V \in U$ and $A^* \subseteq (U) B^*$. Since $δ(U) = δ(U*)$ we have completed the proof.

I.3.16. Theorem. If $f:(X, U) \rightarrow (X^*, U^*)$ is a uniform quotient map, then $f:(X, δ(U))(X^*, δ(U*))$ is a p-quotient map.

Proof. Let $(X^*, δ*)$ be the quotient of $(X, δ(U))$ by $f$. Then $f \in u[(X, U), (X^*, U*)]$ implies that $f$ is in $p[(X, δ(U)), (X^*, δ(U*))]$ and therefore $δ(U*) \subseteq δ*$. To show that $δ* \subseteq δ(U*)$ we first observe that $f$ is in $p[(X, δ(U)), (X^*, δ*)]$ implies $f \in u[(X, U(δ(U))), (X^*, U(δ*))]$. By I.3.15, $U(δ(U*))$ is the quotient of $(X, U(δ(U)))$ by $f$. Therefore $U(δ(U*)) \supseteq U(δ*)$. Then $δ(U*) \supseteq δ*$. Since $δ* = δ(U*)$, the theorem is proved.

I.3.17. Theorem. Suppose $(X, δ)$ and $(X^*, δ*)$ are p-spaces, $f \in [X, X^*]$, and $f$ is onto. Then $f:(X, δ) \rightarrow (X^*, δ*)$ is a p-quotient map iff $f:(X, U(δ)) \rightarrow (X^*, U(δ*))$ is a uniform quotient map.
Proof. That \( f: (X, \mathcal{U}(S)) \rightarrow (X^*, \mathcal{U}(S^*)) \) is a uniform quotient implies that \( f: (X, S) \rightarrow (X^*, S^*) \) is a p-quotient map follows from 1.3.16 and the fact that \( S(\mathcal{U}(S)) = S \).

Now suppose \( f: (X, S) \rightarrow (X^*, S^*) \) is a p-quotient map. Let \((X^*, \mathcal{U}^*)\) be the quotient of \((X, \mathcal{U}(S))\) by \( f \). Then, by 1.3.16, \( f: (X, S(\mathcal{U}(S))) \rightarrow (X^*, S(\mathcal{U}^*)) \) is a p-quotient map. Since \( S(\mathcal{U}(S)) = S \), we must have \( S(\mathcal{U}^*) = S^* \). But \( \mathcal{U}^* \) totally bounded and \( S(\mathcal{U}^*) = S^* \) imply that \( \mathcal{U}^* \) is \( \mathcal{U}(S^*) \). This completes the proof.

1.3.18. Remark. Although one might also expect some similar relationship to exist between topological quotients and p-quotients, this is not the case as the following example shows.

1.3.19. Example. A p-quotient map need not be a topological quotient map. For, let \( X = [0,1] \) with the usual uniform structure \( \mathcal{U} \). Let \( X^* = \{a,b,c\} \). Define \( f: X \rightarrow X^* \) as follows: \( f(0) = a \), \( f(x) = b \) for \( 0 < x < 1 \), and \( f(1) = c \). Using 1.3.14 we can see that the only subset of \( X^* \times X^* \) which can be in the quotient uniform structure of \((X, \mathcal{U})\) by \( f \) is the set \( X^* \times X^* \). Thus the quotient \( \mathcal{U}^* \) of \((X, \mathcal{U})\) on \( X^* \) is the trivial uniformity. But the fact that both spaces are totally bounded implies by 1.3.17 that \( S(\mathcal{U}^*) \) is the p-quotient of \( S(\mathcal{U}) \). Thus \( \mathcal{T}(S(\mathcal{U}^*)) \) is the trivial topology. But \( \{b\} \) is open in the quotient topology on \( X^* \), completing the proof.
1.4 Total Proximity Spaces

1.4.0. Remark. That there is an intimate relationship between proximity spaces and uniform spaces was discovered early in the history of the theory. Many authors pointed out that for any p-space \((X, \delta)\) there is a smallest uniformity (actually \(\mathcal{U}(\delta)\)) which generates \(\delta\) \([1,11,25]\). In \([25]\) Smirnov proved that, for a metric space there is a largest uniformity generating the proximity and asked if there is always a largest uniformity which generates a given proximity. Various authors have given examples to show that this is not true \([4,15,19]\), as well as necessary and/or sufficient conditions for the existence of such a uniformity \([5,19,22]\). We abuse somewhat the notation of \([5]\) in calling spaces \((X, \delta)\) for which a largest uniformity generating \(\delta\) exists total p-spaces. (Poljakov uses the term regular in \([22]\) and \([23]\).) We list the properties of these spaces and where possible quote references to proofs in the literature.

1.4.1. Definition. A proximity space \((X, \delta)\) will be called total iff \(\mathcal{P}(\delta)\) is a filter.

1.4.2. Remark. That 1.4.1 is equivalent to the intuitive definition given in 1.4.0 follows from the definition of \(\mathcal{P}(\delta)\) in 1.1.5 and the fact that, if \(\mathcal{P}(\delta)\) is a filter then \(\mathcal{P}(\delta) = \sup\{\mathcal{U}: \mathcal{U} \in \mathcal{P}(\delta)\}\).
1.4.3. Theorem. (Alfsen and Njastad) If $(X, \mathcal{S})$ is a $p$-space, $\mathcal{U} \in \Pi(\mathcal{S})$, and there is a base $\mathcal{B}$ for $\mathcal{U}$ for which the subset relation is a linear order (total order), then $(X, \mathcal{S})$ is a total $p$-space and $\varphi(\mathcal{S}) = \mathcal{U}$.

Proof. The proof of this theorem is given in [5].

1.4.4. Corollary. (Smirnov) If $(X, \mathcal{S})$ is a pseudometrizable $p$-space and $\mathcal{U}_d$ is the uniformity of the pseudometric then $(X, \mathcal{S})$ is a total $p$-space and $\varphi(\mathcal{S}) = \mathcal{U}_d$.

Proof. We note that subset is a linear order on the base $\{U_\varepsilon; \varepsilon > 0\}$ for $\mathcal{U}_d$. The result follows then from 1.4.3.

1.4.5. Remark. Proofs of 1.4.4 independent of 1.4.3 can be found in Smirnov [25] and Alfsen and Njastad [4]. The former uses the concept of uniform $\mathcal{S}$-covers and the latter generalized uniform spaces. Both seem to be based on a lemma of Efremovich [10], a generalization of which also serves to prove 1.4.3 (see [4], p. 243).

1.4.6. Theorem. (Poljakov-Leader) Suppose $(X, \mathcal{S})$ is a $p$-space. Suppose $D(\mathcal{S})$ is the gauge of $\mathcal{S}$ (see 0.3.20). Then $(X, \mathcal{S})$ is total iff $d_1$ and $d_2 \in D(\mathcal{S})$ implies that $d_1 \lor d_2 \in D(\mathcal{S})$.

Proof. The proof of this theorem can be found in [19].

1.4.7. Remark. Examples of $p$-spaces which are not total exist. The following is essentially the example given by Leader in [19].
1.4.8. Example. A p-space need not be total. For, let \( X = N \times N \) where \( N \) is the natural numbers. For \( i = 1, 2 \), define \( \delta_i \) on \( X \) as follows: \( A \delta_i B \) iff \( \prod_1[A] \cap \prod_1[B] \) is not empty where \( \prod_1 \) is the \( i \)'th projection map. Define \( \delta \) to be \( \delta_1 \vee \delta_2 \). Define

\[
d_1((a,b),(c,d)) = \begin{cases} 0 & \text{if } a = c \\ 1 & \text{if } a \neq c \end{cases}, \quad d_2((a,b),(c,d)) = \begin{cases} 0 & \text{if } b = d \\ 1 & \text{if } b \neq d \end{cases}.
\]

Then both \( d_1 \) and \( d_2 \) are in \( D(\delta) \), for if \( A \delta B \) then \( A \delta_1 B \) for \( i = 1, 2 \). But \( A \delta_1 B \) implies \( D(d_1)(A,B) = 0 \). Let \( d \) be \( d_1 \vee d_2 \). Then \( D(d)(\Delta, \mathcal{C}_\Delta) = 1 \). But choosing any \( \{D_1\} \) a finite cover from below of \( \Delta \) and \( \{C_j\} \) of \( \mathcal{C}_\Delta \) implies that there is an \( i \) and a \( j \) such that \( D_1 \delta_k D_j \) for \( k = 1, 2 \). Then \( \Delta \delta \mathcal{C}_\Delta \) but \( D(d)(\Delta, \mathcal{C}_\Delta) = 1 \). Thus, by 1.4.6, \( (X, \mathcal{S}) \) is not total.

I.5. Fine Proximity Spaces

I.5.0. Remark. For any compact metric space \( (X, d) \) and any metric space \( (X^*, d^*) \) a map \( f \in [X, X^*] \) is uniformly continuous iff it is continuous. This property of compact metric spaces has been generalized to uniform spaces (see [14]), and such spaces are called fine uniform spaces. A fine uniform space turns out to be one in which the uniformity is the largest which is compatible with its topology. In this section we introduce the concept of fine p-spaces and show that, in a sense, it is equivalent to the concept of fine uniform spaces. The definition of fine p-spaces as well as part of the equivalence mentioned can
be found in Čech [8]. Theorem 1.5.3 seems to be new but is an obvious extension of the property mentioned above for uniform spaces.

**I.5.1. Definition.** We will call a p-space \((X, \mathcal{S})\) a fine p-space iff \(\mathcal{S} = \sup \{\mathcal{S}': \mathcal{T}(\mathcal{S}') = \mathcal{T}(\mathcal{S})\}\).

**I.5.2. Remark.** The existence of a fine proximity for any completely regular topology follows from the fact that \(\mathcal{T}(\sup \{\mathcal{S}': \mathcal{T}(\mathcal{S}') = \mathcal{T}(\mathcal{S})\}) = \mathcal{T}(\mathcal{S})\) by I.2.12.

**I.5.3. Theorem.** Suppose \((X, \mathcal{S})\) is a p-space. Then \((X, \mathcal{S})\) is a fine p-space iff given any \((X^*, \mathcal{S}^*)\) and any \(f \in [X, X^*] ; f \in p[(X, \mathcal{S}), (X^*, \mathcal{S}^*)]\) iff \(f\) is an element of \(t[(X, \mathcal{T}(\mathcal{S})), (X^*, \mathcal{T}(\mathcal{S}^*))]\).

**Proof.** Suppose \((X, \mathcal{S})\) is a fine p-space. Let \((X^*, \mathcal{S}^*)\) be any p-space and let \(f \in [X, X^*]\). That \(f\) is continuous if \(f\) is proximally continuous has already been mentioned (0.3.14). Suppose \(f \in t[(X, \mathcal{T}(\mathcal{S})), (X^*, \mathcal{T}(\mathcal{S}^*))]\). Let \(\mathcal{T}\) be the proximity relation induced on \(X^*\) by \(f\) and \((X^*, \mathcal{S}^*)\) (see I.3.8, I.3.9). Then \(f \in p[(X, \mathcal{T}), (X^*, \mathcal{T}(\mathcal{S}^*))]\). We need only show that \(\mathcal{T}(\mathcal{T}\mathcal{S}) \subseteq \mathcal{T}(\mathcal{S})\) for then we will have that \(\mathcal{T}(\mathcal{T}\mathcal{S}) = \mathcal{T}(\mathcal{T}\mathcal{S}) \cup \mathcal{T}(\mathcal{S})\) by I.2.12 and \(\mathcal{T}(\mathcal{T}\mathcal{S}) \subseteq \mathcal{T}(\mathcal{S})\) will then imply that \(\mathcal{T}(\mathcal{T}\mathcal{S}) = \mathcal{T}(\mathcal{S})\). Then, since \((X, \mathcal{S})\) is fine, we must have \(\mathcal{T}\mathcal{S} \subseteq \mathcal{T}\mathcal{S} \mathcal{S} \subseteq \mathcal{S}\). Then \(A \mathcal{S} B\) implies \(A \mathcal{T} B\) which implies \(f[A] \mathcal{S}^* f[B]\) since \(f\) is in \(p[(X, \mathcal{T}), (X^*, \mathcal{T}(\mathcal{S}^*))]\), and hence \(f \in p[(X, \mathcal{S}), (X^*, \mathcal{S}^*)]\).
To show that \( \mathcal{T}(\mathcal{S}) \subseteq \mathcal{T}(\mathcal{S}^{*}) \), we will first show that if \( A \subseteq X \) is closed in \( \mathcal{T}(\mathcal{S}) \) then \( A = f^{-1}[A^{*}] \) where \( A^{*} \) is closed in \( \mathcal{T}(\mathcal{S}^{*}) \cap f[X] \). For, suppose \( A \) is closed in \( \mathcal{T}(\mathcal{S}) \) and \( x \notin A \). Then \( \{x\} \nsubseteq A \) and therefore there are far sets \( A^{*} \) and \( B^{*} \) in \( X^{*} \) such that \( x \in f^{-1}[A^{*}] \) and \( A \subseteq f^{-1}[B^{*}] \).

But this implies that \( \{f(x)\} \nsubseteq f[A] \) since \( f(x) \) is in \( f^{-1}[A^{*}] \subseteq A^{*} \), \( f[A] \subseteq f^{-1}[B^{*}] = B^{*} \). Since \( \{f(x)\} \neq f[A] \) we know that \( f(x) \notin f[A] \), or \( A = f^{-1}[A] \).

We claim that \( f[A] \) is closed in \( \mathcal{T}(\mathcal{S}^{*}) \cap f[X] \). Suppose \( \{x^{*}\} \subseteq f[A] \). Consider \( f^{-1}(x^{*}) \). If there is an \( x \in f^{-1}(x^{*}) \) such that \( \{x\} \nsubseteq A \) then \( \{f(x)\} = \{x^{*}\} \nsubseteq f[A] \).

Therefore \( x \in f^{-1}(x^{*}) \) implies \( \{x\} \nsubseteq A \) which implies \( x \in A \) and thus \( f(x) = x^{*} \in f[A] \).

Now, since \( A \subseteq X \) closed relative to \( \mathcal{T}(\mathcal{S}) \) implies \( A = f^{-1}[A^{*}] \) where \( A^{*} \) is closed relative to \( \mathcal{T}(\mathcal{S}^{*}) \cap f[X] \), and since \( f \in \mathcal{T}(X, \mathcal{T}(\mathcal{S})) \), \( (X^{*}, \mathcal{T}(\mathcal{S}^{*})) \), we have that \( A \) must be closed in \( \mathcal{T}(\mathcal{S}) \). Thus \( \mathcal{T}(\mathcal{S}) \subseteq \mathcal{T}(\mathcal{S}) \), completing this part of the proof by the above remark.

Now suppose that \( (X, \mathcal{S}) \) is a p-space satisfying the given conditions for any \( (X^{*}, \mathcal{S}^{*}) \) and any \( f \). Suppose \( \mathcal{S}' \) is in \( \mathcal{T}(X) \) and \( \mathcal{T}(\mathcal{S}) = \mathcal{T}(\mathcal{S}') \). We claim that \( \mathcal{S}' \subseteq \mathcal{S} \).

The map \( i_{X}: (X, \mathcal{T}(\mathcal{S})) \rightarrow (X, \mathcal{T}(\mathcal{S}')) \) is continuous. Then \( i_{X} \in \mathcal{P}(X, \mathcal{T}(\mathcal{S}'), (X, \mathcal{T}(\mathcal{S}')) \) by hypothesis. Therefore \( A \subseteq B \) implies \( i_{X}[A] \subseteq i_{X}[B] \), or \( A \subseteq B \). Thus \( \mathcal{S}' \subseteq \mathcal{S} \).
I.5.4. Remark. The following theorem shows that fine p-spaces are generated by fine uniform spaces and that fine p-spaces generate fine uniform spaces.

I.5.5. Theorem. If \((X, \mathcal{S})\) is a p-space then \((X, \mathcal{S})\) is a fine p-space iff there exists a fine uniform space \((X, \mathcal{U})\) such that \(\mathcal{S}(\mathcal{U}) = \mathcal{S}\).

Proof. Suppose \((X, \mathcal{U})\) is a fine uniform space. Then we claim that \((X, \mathcal{S}(\mathcal{U}))\) is a fine p-space. We use the characterization of fine p-spaces given in I.5.3. Let \((X^*, \mathcal{S}^*)\) be any p-space and let \(f:(X, \mathcal{S}(\mathcal{U})) \to (X^*, \mathcal{S}^*)\) be continuous. Then \((X, \mathcal{U})\) is fine implies that \(f\) is in \(\mathcal{U}[(X, \mathcal{S}(\mathcal{U})), (X^*, \mathcal{S}^*)]\). Hence \(f \in \mathcal{P}[(X, \mathcal{S}(\mathcal{U})), (X^*, \mathcal{S}^*)]\), and thus \((X, \mathcal{S}(\mathcal{U}))\) is fine.

Now suppose \((X, \mathcal{S})\) is fine. Since \(\mathcal{T}(\mathcal{S})\) is a completely regular topological space there is a fine uniformity \(\mathcal{U}\) for \(\mathcal{T}(\mathcal{S})\). Now \(\mathcal{S}(\mathcal{U}) \in \mathcal{T}(X)\) and \(\mathcal{T}(\mathcal{S}(\mathcal{U})) = \mathcal{T}(\mathcal{S}) = \mathcal{T}(\mathcal{S})\) and \((X, \mathcal{S})\) fine implies \(\mathcal{S}(\mathcal{U}) \subseteq \mathcal{S}\). But \((X, \mathcal{U})\) fine and \(\mathcal{T}(\mathcal{U}(\mathcal{S})) = \mathcal{T}(\mathcal{S})\) which is \(\mathcal{T}(\mathcal{U})\) implies that \(\mathcal{U}(\mathcal{S}) \subseteq \mathcal{U}\). Hence \(\mathcal{S}(\mathcal{U}(\mathcal{S})) \subseteq \mathcal{S}(\mathcal{U})\) by I.2.3 and thus \(\mathcal{S} \subseteq \mathcal{S}(\mathcal{U})\).

I.5.6. Corollary. If \((X, \mathcal{S})\) is a fine p-space then \((X, \mathcal{S})\) is total.

Proof. Letting \((X, \mathcal{U})\) be the fine uniform space for which \(\mathcal{T}(\mathcal{S}) = \mathcal{T}(\mathcal{U})\), we first remark that \(\mathcal{U} \in \mathcal{T}(\mathcal{S})\). Let \(\mathcal{U}'\) be any uniformity such that \(\mathcal{S}(\mathcal{U}') = \mathcal{S}\). Then, since \(\mathcal{T}(\mathcal{S}(\mathcal{U}')) = \mathcal{T}(\mathcal{U}') = \mathcal{T}(\mathcal{S})\), we must have \(\mathcal{U}' \subseteq \mathcal{U}\).
since \((X, \mathcal{U})\) is fine. But this implies that \(\mathcal{U}(\delta) \subseteq \mathcal{U}\). Thus \(\mathcal{U} = \mathcal{U}(\delta)\), and \((X, \delta)\) is total.

1.5.7. Corollary. If \((X, \delta)\) is a p-space and \(\mathcal{U}(\delta)\) is compact, then \((X, \delta)\) is fine and hence total.

Proof. As a matter of fact \(\mathcal{U}(\delta)\) compact implies that there is a unique uniformity \(\mathcal{U}\) such that \(\mathcal{U}(\mathcal{U})\) is \(\mathcal{U}(\delta)\). Hence there is a unique proximity \(\delta(\mathcal{U})\) which generates \(\mathcal{U}(\delta)\) and \(\delta(\mathcal{U}) = \delta\).

1.6 Complete Proximity Spaces

1.6.0. Remark. A metric space is complete iff any sequence which satisfies the Cauchy criterion converges. A uniform space is complete iff every Cauchy net converges. Completeness of p-spaces seems to be a natural topic for discussion and indeed has been studied by many authors \([2,3,18,25,26,27]\). Most of these authors have been primarily concerned with completions of p-spaces which we will not discuss in this paper since the literature on the subject is extensive. Although most of the notions of completeness and completion in the literature coincide many different approaches are used. We offer here a theory of completeness independent of completions (as many of the discussions in the literature are not). Our discussion is equivalent to the theories in the literature and is based on the concept of a p-Cauchy net which is defined with the use of p-structures (1.1). Proofs will be given to most of
the theorems although the theorems which are so indicated appear in the literature.

I.6.1. **Definition.** Suppose \((X, \delta)\) is a \(p\)-space. Suppose \((T, D)\) is a net in \(X\) (see 0.2.2.). Then \((T, D)\) will be called a \(p\)-Cauchy net iff for every \(P \in \mathcal{P}(\delta)\) there is a \(d(P) \in D\) such that \(d_1, d_2 > d(P)\), then \((T(d_1), T(d_2)) \in P\).

I.6.2. **Remark.** One can also define a \(p\)-Cauchy filter in the obvious way. The centered \(c\)-systems of Smirnov [26] are simply \(p\)-Cauchy filters. The following theorem is similar to a theorem proved by Smirnov about centered \(c\)-systems [26, Theorem 9].

I.6.3. **Theorem.** If \((X, \delta)\) and \((X^*, \delta^*)\) are \(p\)-spaces, \(f \in p[X, X^*]\), and \((T, D)\) is a \(p\)-Cauchy net in \(X\), then \((f \circ T, D)\) is a \(p\)-Cauchy net in \(X^*\).

**Proof.** Let \(P^* \in \mathcal{P}(\delta^*)\). Then \(f\) \(p\)-continuous implies that \((fxf)^{-1}[P^*] \in \mathcal{P}(\delta)\). Thus there is a \(d^* \in D\) such that \(d_1, d_2 > d^*\) implies \((T(d_1), T(d_2)) \in (fxf)^{-1}[P^*]\). Then for \(d_1, d_2 > d^*,\) \((f(T(d_1)), f(T(d_2))) \in P^*\).

I.6.4. **Remark.** The definition of a \(p\)-Cauchy net and a uniform Cauchy net are similar. The following paragraphs describe the relationship between the two concepts.

I.6.5. **Theorem.** Suppose \((X, \delta)\) is a \(p\)-space and \((T, D)\) is a \(p\)-Cauchy net in \(X\). Suppose \(\mathcal{U} \in \Pi(\delta)\). Then \((T, D)\) is a uniform Cauchy net in \((X, \mathcal{U})\).

**Proof.** From I.1.5, \(\mathcal{U} \in \Pi(\delta)\) implies \(\mathcal{U} \in \mathcal{P}(\delta)\), and the theorem then follows easily.
I.6.6. Remark. It seems easier to prove the existence of a net \((T,D)\) which is Cauchy relative to some \((X,\mathcal{U})\) and not \(p\)-Cauchy in \((X,\mathcal{S}(\mathcal{U}))\) than to find a specific example. We will construct such a proof after we have discussed totally bounded proximity spaces in I.7 (see I.7.13). The proof of theorem I.6.8 is based on the following lemma of Alfsen and Njastad. The reader interested in its proof is referred to [5].

I.6.7. Lemma. (Alfsen and Njastad) Suppose \(U\) and \(V\) are subsets of \(X \times X\), where \(X \neq \emptyset\). Suppose \(V\) is symmetric, contains \(\Delta_X\), and \(V \circ V \circ V \circ V \subseteq U\). Suppose \((T,D)\) is a net in \(X \times X\) and that \(D\) is a linearly ordered directed set. Suppose \(T(d) \notin U\) for each \(d \in D\). Then there exists a subnet \((T \circ j, E)\) of \((T,D)\) such that \((\prod_1 [T(j(e))] \cup \prod_2 [T(j(e))])\) is not in \(V\) whenever \(e_1\) and \(e_2 \in E\).

I.6.8. Theorem. Suppose \((X,\mathcal{S})\) is a \(p\)-space and \((T,D)\) is a net in \(X\), \(D\) a linearly ordered set. Then \((T,D)\) is \(p\)-Cauchy iff for every pair of subnets \((T \circ j_1, E_1)\), \(i = 1, 2\), of \((T,D)\), \(T[j_1[E_1]] \subseteq T[j_2[E_2]]\).

Proof. Suppose \((T,D)\) is \(p\)-Cauchy in \((X,\mathcal{S})\). Then for \(P \in \mathcal{P}(\mathcal{S})\) there is a \(d(P) \in D\) such that for \(d_1\) and \(d_2\) greater than or equal to \(d(P)\), \((T(d_1), T(d_2)) \in P\). Let \((T \circ j_1, E_1)\), \(i = 1, 2\), be any two subnets of \((T,D)\). Then there is an \(e_i \in E_i\) such that \(e_i \geq e_i^*\) implies that \(j(e_i) \geq d(P)\), \(i = 1, 2\). Thus \((T(j_1(e_i^*)), T(j_2(e_2^*))) \in P\). Then \(T[j_2(e_2^*)] \subseteq P[T[j_1[E_1]]]\). Since \(T[j_2(e_2^*)]\) is
in $T[j_2[E_2]]$, we have that, for all $P \in \mathcal{P}(S)$, $P[T[j_1[E_1]]]$ and $P[T[j_2[E_2]]]$ are not disjoint, and therefore we have.

$T[j_1[E_1]] \cap T[j_2[E_2]]$.

We will prove the converse by contraposition. Suppose $(T,D)$ is not $p$-Cauchy. Then there is a $P \in \mathcal{P}(S)$, $P$ symmetric such that for any $d \in D$ there are indices $d_1(d)$ and $d_2(d) \in D$, $d_1(d) \geq d$, such that $(T(d_1(d)), T(d_2(d))) \notin P$. Define a net $(S,D)$ into $X \times X$ such that $S(d)$ is the pair $(T(d_1(d)), T(d_2(d)))$. Let $P^*$ be a symmetric element of $\mathcal{P}(S)$ such that $P^* \circ P^* \circ P^* \circ P^* \subseteq P$. We then have $S(d)$ is not in $P$ for each $d \in D$. Then by I.6.7 there is a subnet $(S \circ j, E)$ of $(S,D)$ such that $(\pi_1(S \circ j(e_1)), \pi_2(S \circ j(e_2)))$ is not in $P^*$ for any $e_1$ and $e_2 \in E$. Then we claim that $(\pi_1 \circ S \circ j, E)$, $i = 1, 2$, are subnets of $(T,D)$ and that their ranges are far as subsets of $X$.

To see the former, for $i = 1, 2$, define $k_i : E \rightarrow D$ such that $k_i(e) = d_1(j(e))$. Let $d \in D$. Since $(S \circ j, E)$ is a subnet of $(S,D)$, there is an $e^* \in E$ such that $j(e) \geq d$ whenever $e \geq e^*$. Then for each $i$, $d_1(d) \geq d$ implies that $d_1(j(e)) \geq j(e) \geq d$. Thus, for $d \in D$ and for each $i$, there is an $e^* \in E$ such that $e \geq e^*$ implies that $k_i(e) \geq d$. Thus the maps $k_i$ have the subnet property. We show now that $\pi_1 \circ S \circ j(e) = T \circ k_1(e)$. $S \circ j(e) = S(j(e))$ which is the pair $(T(d_1(j(e))), T(d_2(j(e))))$. Thus $\pi_1 \circ S \circ j(e)$ is $T(d_1(j(e))) = T \circ k_1(e)$. Thus $(T \circ k_1, E) = (\pi_1 \circ S \circ j, E)$ for $i = 1, 2$ are subnets of $(T,D)$. 
Finally, suppose the ranges of the above subnets are near in \( X \). Then there is an \( x \in X \) such that \( x \) is in
\[ P'[\pi_i \circ S \circ j[E]] \] for \( i = 1, 2 \), where \( P' \in \mathcal{P}(S) \), \( P' \) symmetric, and \( P' \circ P' \subseteq P^* \). Then there is an \( e_1 \) and an \( e_2 \in E \) such that
\[ (\pi_1(S \circ j(e_1)), x) \] and
\[ (\pi_2(S \circ j(e_2)), x) \in P'. \]
Then
\[ (\pi_1(S \circ j(e_1)), \pi_2(S \circ j(e_2))) \in P' \circ P' \subseteq P^*. \]
But this is impossible by I.6.7. Therefore \((T, D)\) not \( p\)-Cauchy implies that there are subnets \((T \circ k_1[E_1], i = 1, 2)\) of \((T, D)\) such that
\[ (T \circ k_1[E_1]) \notin (T \circ k_2[E_2]). \]

I.6.9. Corollary. Suppose \( D \) is a linearly ordered set and \((T, D)\) is a net in \( X \). Suppose \((X, S)\) is a \( p\)-space and that \( \mathcal{U} \in \pi(S) \). Then \((T, D)\) is Cauchy in \((X, \mathcal{U})\) iff \((T, D)\) is \( p\)-Cauchy in \((X, S)\).

Proof. We have already proved the if part of the theorem in I.6.5.

Now suppose \((T, D)\) is not \( p\)-Cauchy in \((X, S)\). Then there are subnets \((T \circ k_1[E_1], i = 1, 2)\) of \((T, D)\) such that their respective ranges are far in \( X \). Thus, there must be a \( \mathcal{U} \in \mathcal{U} \) such that
\[ U[T \circ k_1[E_1]] \cap U[T \circ k_2[E_2]] = \emptyset. \]
But if \((T, D)\) is Cauchy relative to \((X, \mathcal{U})\), an argument similar to the one used to prove the first part of I.6.8 shows that
\[ U[T \circ k_1[E_1]] \cap U[T \circ k_2[E_2]] \neq \emptyset. \]
Therefore, \((T, D)\) not \( p\)-Cauchy in \((X, S)\) implies \((T, D)\) is not Cauchy in \((X, \mathcal{U})\), and the theorem is proved.
I.6.10. **Corollary.** Suppose \((T,N)\) is a sequence in \(X\). Suppose \((X,\mathcal{S})\) is a \(p\)-space and \(\mathcal{U} \in \mathcal{P}(\mathcal{S})\). Then \((T,N)\) is \(p\)-Cauchy in \((X,\mathcal{S})\) iff \((T,N)\) is Cauchy in \((X,\mathcal{U})\).

**Proof.** \(N\) is linearly ordered and hence I.6.9 applies.

I.6.11. **Remark.** In light of I.1.17 (1) and I.1.2 (2), the proof of the following theorem is the same as the proof of the analog for uniform spaces and is therefore omitted.

I.6.12. **Theorem.** Suppose \((X,\mathcal{S})\) is a \(p\)-space and \((T,D)\) is a net in \(X\). Suppose \((T,D)\) converges relative to \(\mathcal{P}(\mathcal{S})\). Then \((T,D)\) is \(p\)-Cauchy.

I.6.13. **Definition.** A proximity space \((X,\mathcal{S})\) will be called proximally complete (\(p\)-complete) iff every \(p\)-Cauchy net (equivalently, every \(p\)-Cauchy filter) converges.

I.6.14. **Lemma.** A \(p\)-Cauchy filter \(\mathcal{F}\) in \((X,\mathcal{S})\) converges to \(x\) iff \(x \in cF\) for all \(F \in \mathcal{F}\).

**Proof.** We recall that a filter \(\mathcal{F}\) in a topological space converges to a point \(x\) iff \(\mathcal{N}(x)\), the neighborhood filter of \(x\), is contained in \(\mathcal{F}\). We also note that in a \(p\)-space \(\mathcal{N}(x) = \{P[x] : P \in \mathcal{P}(\mathcal{S})\}\). Now, if \(\lim \mathcal{F} = x\) then \(x\) is a cluster point of \(\mathcal{F}\) and hence \(x \in cF\) for all \(F \in \mathcal{F}\).

If \(x \in cF\) for all \(F \in \mathcal{F}\), let \(P \in \mathcal{P}(\mathcal{S})\). Let \(P^* \in \mathcal{P}(\mathcal{S})\) be symmetric such that \(P^* \circ P^* \circ P^* \subseteq P\). Then \(\mathcal{F}\) \(p\)-Cauchy implies that there is an \(F \in \mathcal{F}\) such that \(F \times F \subseteq P^*\). Then \(cF \times cF \subseteq P^* \circ P^* \circ P^* \subseteq P\). But \(x \in cF\) implies \(\{x\} \times cF \subseteq P\) and hence \(cF \subseteq P[x]\). Hence \(P[x] \in \mathcal{F}\), and \(\mathcal{F}\) converges to \(x\).
1.6.15. Remark. In [26] Smirnov proves that a $p$-space $(X, \mathcal{S})$ satisfies his definition of completeness ($(X, \mathcal{S})$ is equal to its completion) iff every centered closed $c$-system (closed $p$-Cauchy filter) has a non-empty intersection ([26], Theorem 3'). That this is equivalent to our definition follows from the preceding lemma.

1.6.16. Remark. As an example of a $p$-complete space we give the following theorem, due to Smirnov.

1.6.17. Theorem. (Smirnov) Suppose $(X, \mathcal{S})$ is a pseudometrizable $p$-space and $d$ is the indicated pseudometric. Then $(X, \mathcal{S})$ is $p$-complete iff $(X, d)$ is complete as a metric space.

Proof. This theorem follows directly from the fact that $\mathcal{S} = \mathcal{U}_d$ (see 1.4.4).

1.6.18. Remark. As a matter of fact, the same argument shows the following stronger result.

1.6.19. Theorem. If $(X, \mathcal{S})$ is a total $p$-space (I.4.1) and $\mathcal{U}$ is the largest uniformity for which $\mathcal{S}(\mathcal{U}) = \mathcal{S}$, then $(X, \mathcal{S})$ is $p$-complete iff $(X, \mathcal{U})$ is complete.

1.6.20. Theorem. (Smirnov) If $(X, \mathcal{U})$ is a complete uniform space, then $(X, \mathcal{S}(\mathcal{U}))$ is a $p$-complete $p$-space.

Proof. Let $(T, D)$ be a $p$-Cauchy net in $(X, \mathcal{S}(\mathcal{U}))$. Then by 1.6.5 $(T, D)$ is Cauchy in $(X, \mathcal{U})$. Therefore $(T, D)$ converges in the topology of $\mathcal{U}$ which is the topology of $\mathcal{S}(\mathcal{U})$. 
1.6.21. Remark. That the converse of 1.6.20 is not true will be shown in 1.7.13.

1.6.22. Theorem. (Smirnov) If \((X, \mathcal{E})\) is a \(p\)-complete \(p\)-space and \(A \subseteq X\), \(A\) closed, then \((A, \mathcal{E}\,|\,A)\) is a \(p\)-complete \(p\)-space.

Proof. Suppose \((T,D)\) is a \(p\)-Cauchy net in \((A, \mathcal{E}\,|\,A)\). Since \(\mathcal{P}(\mathcal{E}) \cap (A \times A) \subseteq \mathcal{P}(\mathcal{E}\,|\,A)\) (I.1.14), we have \((T,D)\) is \(p\)-Cauchy in \((X, \mathcal{E})\). Hence it converges. Since \(A\) is closed, its limit is in \(A\).

1.7 Totally Bounded Proximity Spaces

1.7.0. Remark. In metric spaces and uniform spaces the concept of compactness is equivalent to the two concepts of completeness and total boundedness (precompactness). It is therefore logical to ask if there is a corresponding concept of total boundedness for proximity spaces. As usual, the basic work in this area was done by Smirnov [26], [27]. He defined a \(p\)-space to be totally bounded iff its completion is compact, and showed, among other things, that this is equivalent to the fact that every uniform \(\mathcal{E}\)-cover has a finite refinement ([26], Theorem 10). That this is equivalent to our definition follows from the fact that the class of all uniform \(\mathcal{E}\)-covers corresponds to the set \(\mathcal{P}(\mathcal{E})\).

1.7.1. Definition. A \(p\)-space \((X, \mathcal{E})\) will be called proximally totally bounded (\(p\)-totally bounded) iff for each \(P \in \mathcal{P}(\mathcal{E})\) there is a finite set \(A \subseteq X\) such that \(P[A] = X\).
I.7.2. Theorem. A p-space \((X, \mathcal{S})\) is p-totally bounded iff \(\mathcal{P}(\mathcal{S}) = \mathcal{U}(\mathcal{S})\).

Proof. Clearly, if \(\mathcal{P}(\mathcal{S}) = \mathcal{U}(\mathcal{S})\) then \((X, \mathcal{S})\) is p-totally bounded since every \(U \in \mathcal{U}(\mathcal{S})\) has the necessary property.

To show the converse we need only show that \(\mathcal{P}(\mathcal{S})\) is a subset of \(\mathcal{U}(\mathcal{S})\) since the reverse inclusion is always true by I.1.5. Since \(U \in \mathcal{U}(\mathcal{S})\) iff there is a finite p-cover \(\{A_1\}\) of \(X\), (see 0.3.10, 0.3.12), such that \(\bigcup (A_1 \times A_1) \subseteq U\), we will show that for every \(P \in \mathcal{P}(\mathcal{S})\) such a collection exists. Let \(P \in \mathcal{P}(\mathcal{S})\) and let \(P^* \in \mathcal{P}(\mathcal{S})\) be symmetric such that \(P^* \circ P^* \circ P^* \subseteq P\). Since \(P^* \in \mathcal{P}(\mathcal{S})\) there is a symmetric \(P' \in \mathcal{P}(\mathcal{S})\) such that \(P' \circ P' \subseteq P^*\). There is a finite set \(A\) such that \(P'[A] = X\). Let \(A\) be indexed, \(A = \{a_i\}\). Then, by letting \(A_i = P'[a_i]\), we claim that \(\bigcup (P^*[A_i] \times P^*[A_i]) \subseteq P\). Since \(\{P^*[A_i]\}\) is clearly a finite p-cover of \(X\), the theorem will then be proved. Let \((x, y)\) be in \(P^*[A_i] \times P^*[A_i]\) for some \(i\). Then there are \(b_1\) and \(b_2\) in \(A_i\) such that \((x, b_1)\) and \((y, b_2)\) \(\in P^*\). But \(b_1\) and \(b_2 \in A_i\) implies \((a_i, b_1)\) and \((a_i, b_2)\) \(\in P'\), and thus \((b_1, b_2) \in P^*\). Then \((x, y) \in P^* \circ P^* \circ P^* \subseteq P\).

I.7.3. Corollary. If a p-space is p-totally bounded then there is a unique uniformity generating \(\mathcal{S}\) and hence the p-space is total.
I.7.4. Theorem. Suppose \((X, \mathcal{S})\) is \(p\)-totally bounded, \((X^*, \mathcal{S}^*)\) is a \(p\)-space, and \(f \in p[X, X^*]\). Then the \(p\)-space \((f[X], \mathcal{S}^*|f[X])\) is \(p\)-totally bounded.

Proof. Let \(\mathcal{S}^{**} = \mathcal{S}^*|f[X]\). Then for \(A^*\) and \(B^*\) in \(f[X]\), \(A^* \subseteq B^*\) iff \(A^* \subseteq B^*\). Thus \(f \in p[(X, \mathcal{S}), (f[X], \mathcal{S}^{**})]\). Hence, by I.1.11, for each \(P^* \in \mathcal{P}(\mathcal{S}^{**})\), \((fxf)^{-1}[P^*] \in \mathcal{P}(\mathcal{S})\).

Now suppose \(P^* \in \mathcal{P}(\mathcal{S}^{**})\). Then there is a finite set \(A \subseteq X\) such that \((fxf)^{-1}[P^*][A] = X\). We assert that \(P^*[f[A]] = f[X]\). To see this, let \(x^* \in f[X]\) and \(x \in f^{-1}(x^*)\). Then there is an \(a \in A\) such that \((a, x) \in P^*\).

I.7.5. Example. A simple example illustrates the fact that total boundedness of uniform spaces and total boundedness of \(p\)-spaces are different concepts. Let \(X = \mathbb{R}\) the real line, and \(d\) be the Euclidean metric on \(\mathbb{R}\). Let \(\mathcal{U}_d\) be the metric uniform structure. Let \(\mathcal{S} = \mathcal{S}(\mathcal{U}_d)\).

Then \((X, \mathcal{U}(\mathcal{S}))\) is totally bounded. Since \((\mathbb{R}, \mathcal{U}_d)\) is complete, if \(\mathcal{U}(\mathcal{S})\) were equal to \(\mathcal{U}_d\), \((\mathbb{R}, \mathcal{U}_d)\) would be compact. Since this is not the case, \(\mathcal{U}_d = \mathcal{P}(\mathcal{S}) \neq \mathcal{U}(\mathcal{S})\), and hence \((\mathbb{R}, \mathcal{S})\) is not \(p\)-totally bounded, although \((\mathbb{R}, \mathcal{U}(\mathcal{S}))\) is totally bounded as a uniform space.

I.7.6. Remark. Theorem I.7.8 should probably be credited to Smirnov for reasons discussed in I.7.0.

I.7.7. Definition. A \(p\)-space \((X, \mathcal{S})\) will be called compact \iff \((X, \mathcal{T}(\mathcal{S}))\) is compact.
1.7.8. **Theorem.** A p-space \((X, \mathcal{S})\) is compact iff \((X, \mathcal{S})\) is p-complete and p-totally bounded.

**Proof.** We will appeal to the well known fact that a uniform space is compact iff it is complete and totally bounded.

Suppose \((X, \mathcal{S})\) is p-complete and p-totally bounded. Then by 1.7.2, \(\mathcal{U}(\mathcal{S}) = \wp(\mathcal{S})\). But \((X, \mathcal{S})\) is p-complete then implies that \((X, \mathcal{U}(\mathcal{S}))\) is complete. Then \((X, \mathcal{U}(\mathcal{S}))\) is complete and totally bounded. Hence \(\mathcal{T}(\mathcal{U}(\mathcal{S}))\) which is \(\wp(\mathcal{S})\) is compact.

Suppose \(\mathcal{T}(\mathcal{S})\) is compact. Then from Isbell [14] we know that there is a unique \(\mathcal{U} \in \mathcal{U}(X)\) such that \(\mathcal{T}(\mathcal{U}) = \mathcal{T}(\mathcal{S})\). Further, \((X, \mathcal{U})\) is complete and totally bounded. By 1.5.7, \(\mathcal{S}(\mathcal{U}) = \mathcal{S}\). Since \(\mathcal{U}\) is unique, \(\wp(\mathcal{S}) = \mathcal{U} = \mathcal{U}(\mathcal{S})\) since \(\mathcal{U}\) is totally bounded. Then by 1.7.1, \((X, \mathcal{S})\) is p-totally bounded and by 1.6.20 \((X, \mathcal{S})\) is p-complete.

1.7.9. **Remark.** A subspace of a totally bounded uniform space is totally bounded. The next example shows that this is not true for p-totally bounded p-spaces. The following theorem is due to Efremovich.

1.7.10. **Theorem.** Suppose \((X, \mathcal{S})\) is a compact p-space. Then for \(A\) and \(B\) subsets of \(X\), \(A \subseteq B \) iff \(cA \cap cB \not= \emptyset\).
Proof. Clearly $cA \cap cB \neq \emptyset$ implies $cA \subseteq cB$ and this implies $A \subseteq B$ by 0.3.15 (3).

Let $(X, \mathcal{U})$ be the unique uniformity for which $\mathcal{E}(\mathcal{U})$ is $S$. Then $cA$ compact and $cB$ compact and $cA \cap cB = \emptyset$ implies that there is a $U \in \mathcal{U}$ such that $U[cA] \cap U[cB] = \emptyset$. Then $cA \subseteq cB$ and hence $A \subseteq B$.

1.7.11. Example. Let $(N, \mathcal{S}_M)$ be the natural numbers with the discrete proximity. Consider $(N^*, \mathcal{U}^*)$ the uniform completion of $(N, \mathcal{U}(\mathcal{S}_M))$. $(N^*, \mathcal{U}^*)$ is compact and hence $(N^*, \mathcal{E}(\mathcal{U}^*))$ is $p$-totally bounded. By 0.3.16 we have $(N, \mathcal{E}(\mathcal{U}^*)|N) = (N, \mathcal{E}(\mathcal{U}^* \cap (N \times N))) = (N, \mathcal{E}(\mathcal{U}(\mathcal{S}_M)))$ which is $(N, \mathcal{S}_M)$. Hence $(N, \mathcal{S}_M)$ is a $p$-subspace of $(N^*, \mathcal{E}(\mathcal{U}^*))$. That $(N, \mathcal{S}_M)$ is not $p$-totally bounded follows from the fact that $\mathcal{P}(\mathcal{S}_M) = \{P: \Delta_N \subseteq P\}$, and there is no finite set $A$ such that $\Delta_N[A] = N$.

1.7.12. Remark. In 1.6.6 we mentioned that the converse of 1.6.5 is not true. We are now in a position to illustrate this statement.

1.7.13. Example. Again consider the space $(N, \mathcal{S}_M)$. Since $\mathcal{P}(\mathcal{S}_M) = \{P: \Delta_N \subseteq P\}$, a net in $N$ is $p$-Cauchy iff it is eventually constant. Since all such nets converge, this space is $p$-complete. Now consider the uniform space $(N, \mathcal{U}(\mathcal{S}_M))$. If this space were complete, the fact that it is totally bounded would imply that it is compact. Since this is not the case, there is some net $(T,D)$ in $N$ which
is Cauchy relative to \( \mathcal{U}(\mathcal{S}_M) \) but which does not converge and is therefore not p-Cauchy.

**1.8 Products in Proximity Spaces**

**1.8.0. Remark.** There is a natural way to define a product proximity structure for the product of proximity spaces. This definition corresponds to taking the proximity of the product of the totally bounded uniform structures which come from the p-relations on the factors. The problem with this definition is that it is not equivalent to taking products of metric spaces. In [20] Mrowka indicates that some other proximity relation may be defined on the product of proximity spaces. In [22] Poljakov defines a "strong" product of p-spaces. We shall give a definition for strong and weak products of p-spaces equivalent to those in the literature. Many of the properties of products are announced (although not proved) in [22]. These will be so indicated. Throughout this section if \( \{X_\alpha : \alpha \in I\} \) is a non-empty family of non-empty sets then \( X \) will denote \( \times \{X_\alpha : \alpha \in I\} \).

**1.8.1. Definition.** Suppose \( \{(X_\alpha, \mathcal{S}_\alpha); \alpha \in I\} \) is a non-empty family of p-spaces and \( X = \times \{X_\alpha : \alpha \in I\} \).

For each \( \alpha \in I \) let \( \mathcal{S}_\alpha \) be the proximity induced on \( X \) by the \( \alpha \) projection map (see I.3.8, I.3.9). Define \( \mathcal{S}_W = \sup \{\mathcal{S}_\alpha : \alpha \in I\} \). The space \( (X, \mathcal{S}_W) \) will be called the weak product of the spaces \( (X_\alpha, \mathcal{S}_\alpha) \) and will be denoted \( (X, \mathcal{S}_W) = \times \mathcal{S}_W \{X_\alpha, \mathcal{S}_\alpha : \alpha \in I\} \).
1.8.2. Remark. In [14] Isbell defines a product structure for a family of objects in a category as follows: Suppose \((X, [\cdot, \cdot], \circ)\) is a category and \(\{X_\alpha : \alpha \in I\}\) is a non-empty indexed family of objects. An object \(X\) will be called a product of \(\{X_\alpha : \alpha \in I\}\) iff there exists a family \(\{p_\alpha \in [X, X_\alpha] : \alpha \in I\}\) such that

(1) If \(X^*\) is in \(X\), \(f, g \in [X^*, X]\) and \(f \neq g\), then there is an \(\alpha\) such that \(p_\alpha \circ f \neq p_\alpha \circ g\).

(2) If \(X^*\) is in \(X\) and \(\{f_\alpha : \alpha \in I\}\) is such that \(f_\alpha \in [X^*, X_\alpha]\), then there is an \(F \in [X^*, X]\) such that \(p_\alpha \circ F = f_\alpha\) for all \(\alpha \in I\).

He also indicates that any two products of a given family are isomorphic in the given category. We will show that the weak product of a family of \(p\)-spaces is indeed the categorical product. The following lemma will be used.

1.8.3. Lemma. Suppose \(X\) is a set, for all \(\alpha \in I\) \((X_\alpha, \delta_\alpha)\) are \(p\)-spaces, and for each \(\alpha \in I\), \(g_\alpha \in [X, X_\alpha]\). For each \(\alpha \in I\) let \(\Phi_\alpha\) be the \(p\)-relation induced on \(X\) by \(g_\alpha\). Let \(\Phi = \sup\{\Phi_\alpha : \alpha \in I\}\). Then for any \(p\)-space \((X^*, \delta^*)\) and any \(f \in [X^*, X]\), \(f \in p[(X^*, \delta^*), (X, \Phi)]\) iff \(g_\alpha \circ f \in p[(X^*, \delta^*), (X_\alpha, \delta_\alpha)]\) for all \(\alpha \in I\).

Proof. We first note that all \(g_\alpha\) are \(p\)-continuous with respect to \(\Phi\) by virtue of the fact that \(\Phi \geq \Phi_\alpha\) and I.3.8.
Suppose $g^\alpha \circ f \in p[(X^*, S^*)]$. Let $A^*, B^* \subseteq X^*$ and $A^* \subseteq B^*$. By I.3.10, $g^\alpha \circ f$ in $p[(X^*, S^*), (X^*_\alpha, \delta^*_\alpha)]$ implies that $f \in p[(X^*, S^*), (X^*, \varphi^*_\alpha)]$ for each $\alpha \in I$. Let $\{A_i^\alpha\}, \{B_j^\alpha\}$ be any finite covers from below of $f[A^*]$ and $f[B^*]$ respectively. Then $A^* \subseteq \bigcup f^{-1}[A_i^\alpha]$ and $B^* \subseteq \bigcup f^{-1}[B_j^\alpha]$. Hence, since $A^* \subseteq B^*$, there is an $i^*$ and a $j^*$ such that $f^{-1}[A_i] \subseteq f^{-1}[B_j^\alpha]$. But this implies that $A_i^\alpha \subseteq B_j^\alpha$ for all $\alpha \in I$ and hence $f[A^*] \subseteq f[B^*]$. Thus $g^\alpha \circ f \in p[(X^*, S^*), (X^*_\alpha, \delta^*_\alpha)]$ for all $\alpha \in I$ implies that $f \in p[(X^*, S^*), (X^*, \varphi^*_\alpha)]$. That the converse of this statement is true follows from the fact the composition of $p$-continuous functions is $p$-continuous.

I.8.4. Theorem. If $\{(X^*_\alpha, \delta^*_\alpha) : \alpha \in I\}$ is a non-empty family of $p$-spaces, then the weak product $(X, S^w)$ is the categorical product of the family $\{X^*_\alpha, \delta^*_\alpha : \alpha \in I\}$ in the category $P$.

Proof. For each $\alpha \in I$ let $p^\alpha$ be the $\alpha$ projection map. We assert that $\{p^\alpha : \alpha \in I\}$ satisfies (1) and (2) of I.8.2. Clearly, each $p^\alpha$ is $p$-continuous from the remark made in the proof of I.8.3.

That this family satisfies (1) is obvious since if $(X^*, S^*)$ is a $p$-space, $f$ and $g \in p[(X^*, S^*), (X, S^w)]$ and $f$ is not $g$, then there is an $x^* \in X^*$ such that $f(x^*) \neq g(x^*)$. But $f(x^*) \neq g(x^*)$ iff there is an $\alpha$ such that $p^\alpha(f(x^*))$ is not equal to $p^\alpha(g(x^*))$. 
Suppose \((X^*, \delta^*)\) is any \(p\)-space, \(\{f_\alpha: \alpha \in I\}\) any family of maps such that \(f_\alpha \in P[(X^*, \delta^*), (X_\alpha, \delta_\alpha)]\). Define \(F: X^* \to X\) such that \(F(x^*)(\alpha) = f_\alpha(x^*)\). The fact that \(F\) is \(p\)-continuous follows from the preceding lemma and the definition of \(\delta_w\).

1.8.5. Example. As we mentioned, the weak product of pseudometrizable \(p\)-spaces does not correspond to the product of their metric structures. We can show this by considering the space of 1.4.8. Clearly the \(\delta\) constructed on \(N \times N\) in this example is the weak product of the discrete proximity relation on \(N\). This discrete proximity is metrizable and the product of the two metrics on \(N \times N\) will again be discrete. But we have already shown that \((N \times N, \delta)\) is not the discrete proximity.

1.8.6. Remark. We introduce the following definition due originally to Mrowka and Poljakov, [20] and [22].

1.8.7. Definition. Suppose \(\{(X_\alpha, \delta_\alpha): \alpha \in I\}\) is a non-empty family of \(p\)-spaces. Let \(\mathcal{G}\) be the collection of all maps \(\nu\) from \(I\) into \(\bigcup \{\Pi(\delta_\alpha): \alpha \in I\}\) such that \(\nu(\alpha) \in \Pi(\delta_\alpha)\). For each \(\nu \in \mathcal{G}\) let \(\nu = \times \{\nu(\alpha): \alpha \in I\}\) and let \(\delta_\nu = \delta(\nu)\). Define \(\delta_s = \sup\{\delta_\nu: \nu \in \mathcal{G}\}\). The set \(X\) with the \(p\)-relation \(\delta_s\) will be called the strong product of the spaces \((X_\alpha, \delta_\alpha)\) and will be denoted by \((X, \delta_s) = \times_s \{\langle X_\alpha, \delta_\alpha\rangle: \alpha \in I\}\).
I.8.8. Remark. Before proceeding with the discussion of $\mathcal{G}_S$ we show that $\mathcal{G}_W$ may also be characterized in terms of uniformities on the factor spaces. We remark that the above definition is stated as a theorem in [22]. However, since we have the characterization of supremums given in I.2.7, nothing is gained by giving Poljakov's original definition.

I.8.9. Theorem. Suppose $\{(X_\alpha, \mathcal{G}_\alpha) : \alpha \in I\}$ is a non-empty family of p-spaces. Then $\mathcal{U}(\mathcal{G}_W) = \bigvee_{\alpha \in I} \mathcal{U}(\mathcal{G}_\alpha)$.

Proof. Since $\mathcal{U}(\mathcal{G}_\alpha) : \alpha \in I$ is totally bounded it is sufficient to show that $\mathcal{G}(\bigvee_{\alpha \in I} \mathcal{U}(\mathcal{G}_\alpha)) = \mathcal{G}_W$. Let $\mathcal{G}^* = \mathcal{G}(\bigvee_{\alpha \in I} \mathcal{U}(\mathcal{G}_\alpha))$. From the construction of $\mathcal{G}_W$ we know that it is the smallest p-structure on $X$ for which each $\mathcal{T}_\alpha$ is p-continuous. Since each of the $\mathcal{T}_\alpha$ is in $\mathcal{U}(X, \bigvee_{\alpha \in I} \mathcal{U}(\mathcal{G}_\alpha), (X_\alpha, \mathcal{G}(\mathcal{G}_\alpha)))$, we have that each $\mathcal{T}_\alpha \in \mathcal{U}(X, \mathcal{G}^*, (X_\alpha, \mathcal{G}_\alpha))$ and hence $\mathcal{G}^* \geq \mathcal{G}_W$. We also have that $\mathcal{T}_\alpha \in \mathcal{U}(X, \mathcal{G}(\mathcal{G}_W), (X_\alpha, \mathcal{G}(\mathcal{G}_\alpha)))$ and hence $\bigvee_{\alpha \in I} \mathcal{U}(\mathcal{G}_\alpha) : \alpha \in I \subseteq \mathcal{G}(\mathcal{G}_W)$ and therefore $\mathcal{G}^* \leq \mathcal{G}_W$.

I.8.10. Theorem. Suppose $\{(X_\alpha, \mathcal{G}_\alpha) : \alpha \in I\}$ is a non-empty family of p-spaces. Then three topologies $\mathcal{T}(\mathcal{G}_W)$, $\mathcal{T}(\mathcal{G}_S)$, and $\bigvee_{\alpha \in I} \mathcal{T}(\mathcal{G}_\alpha)$ are equal.

Proof. Since, for an indexed family of uniform spaces $\{(X_\alpha, \mathcal{U}_\alpha)\}$, $\mathcal{T}(\bigvee \mathcal{U}_\alpha) = \bigvee \mathcal{T}(\mathcal{U}_\alpha)$, and $\mathcal{T}(\mathcal{G}) = \mathcal{T}(\mathcal{U}(\mathcal{G}))$, the theorem follows from I.8.9, I.8.7, and I.2.12.
I.8.11. Theorem. (Poljakov) Suppose \( \{(X_\alpha, \delta_\alpha) : \alpha \in I\} \) is a non-empty family of p-spaces. Then \( \delta_{\mathcal{W}} \leq \delta_S \).

Proof. Define \( \gamma^*: I \rightarrow \bigcup_{\alpha \in I} (\delta_\alpha) : \alpha \in I \) where \( \gamma^*(\alpha) = \mathcal{U}(\delta_\alpha) \). Then \( \gamma^* \in \mathcal{S} \). Further, by I.8.9, \( \delta_{\gamma^*} = \delta_{\mathcal{W}} \). Since \( \delta_S = \sup \{ \delta_U : \forall \in \mathcal{S} \} \), \( \delta_{\mathcal{W}} \leq \delta_S \).

I.8.12. Remark. The following theorem is announced by Poljakov. It shows that, in those cases where a product metric exists, the strong proximal product and the proximity of the product metric coincide.

I.8.13. Theorem. (Poljakov) Suppose \( \{(X_\alpha, \delta_\alpha) : \alpha \in I\} \) is a non-empty family of total p-spaces. Suppose that for each \( \alpha \in I \), \( \forall (\delta_\alpha) = \mathcal{U}_{\alpha}^* \). Then \( \delta_S = \delta(\times \{ \mathcal{U}_{\alpha}^*: \alpha \in I \}) \).

Proof. Let \( \gamma^*: I \rightarrow \bigcup_{\alpha \in I} (\delta_\alpha) : \alpha \in I \) such that \( \gamma^*(\alpha) = \mathcal{U}_{\alpha}^* \). Then \( \gamma^* \in \mathcal{S} \) and \( \delta_{\gamma^*} = \delta(\times \{ \mathcal{U}_{\alpha}^*: \alpha \in I \}) \). Since \( \delta_S \geq \delta_\nu \) for all \( \nu \in \mathcal{S} \), \( \delta_S \geq \delta_{\gamma^*} \).

Suppose \( A \subseteq \mathcal{F} \) and \( \{A_i\} \) and \( \{B_j\} \) are finite covers from below of \( A \) and \( B \) respectively. Then there is an \( i^* \) and a \( j^* \) such that \( A_{i^*} \subseteq \mathcal{F} \) \( B_{j^*} \) by (2) of 0.3.1. We will show that \( A_{i^*} \subseteq \mathcal{F} \subseteq \bigcap_{i \leq k \leq n} [U_{\alpha_k}] \). Let \( U \subseteq \mathcal{S} \) and \( \{ \bigcap_{k \leq i \leq n} [U_{\alpha_k}] \} \). Let \( U_{\gamma^*} \subseteq \mathcal{S} \) for all \( k \), \( U_{\gamma^*} \subseteq \mathcal{S} \) for all \( k \), \( U \subseteq \mathcal{S} \). Since \( A_{i^*} \subseteq \mathcal{F} \subseteq \bigcap_{i \leq k \leq n} [U_{\alpha_k}] \). Then, since \( \forall (\alpha_k) \subseteq \mathcal{U}_{\alpha_k}^* \) for all \( k \), \( U \subseteq \mathcal{S} \). Since \( A_{i^*} \subseteq \mathcal{F} \subseteq \bigcap_{i \leq k \leq n} [U_{\alpha_k}] \), \( U[A_{i^*}] \subseteq \bigcap_{i \leq k \leq n} [U_{\alpha_k}] \). This will be true for each \( U \subseteq \mathcal{S} \) and thus \( A_{i^*} \subseteq \mathcal{F} \subseteq \bigcap_{i \leq k \leq n} [U_{\alpha_k}] \).
Theorem. Suppose \( \{ (X_n, S_n) : n \in \mathbb{N} \} \) is a family of pseudometrizable p-spaces. Then \( (X, S_S) \) is pseudometrizable.

Proof. Let \( \bigcup_n^* = \varphi(S_n) \) for \( n \in \mathbb{N} \). Then \( (X_n, S_n) \) pseudometrizable implies \( \bigcup_n^* \) is pseudometrizable, and hence \( \chi[\bigcup_n^* : n \in \mathbb{N}] \) is pseudometrizable. But \( \delta(\chi[\bigcup_n^* : n \in \mathbb{N}]) \) is \( S_S \) by 1.8.13 and hence \( S_S \) is pseudometrizable.

Remark. Poljakov has stated necessary and sufficient conditions for \( S_w = S_S \) in the case of the product of two spaces (iff at least one is p-totally bounded). We will give a generalization of this theorem. We first prove a lemma.

Lemma. Suppose \( \{ (X_\alpha, S_\alpha) : \alpha \in I \} \) is a non-empty family of p-spaces. For any \( \alpha \in I \), any \( A \subseteq X_\alpha \), and any \( P_\alpha \subseteq \varphi(S_\alpha) \), \( A \subseteq_w (T_\alpha \times T_\alpha)^{-1}[P_\alpha][A] \), where \( \subseteq_w \) is the p-neighborhood relation for \( S_w \).

Proof. Since \( P_\alpha[\pi_\alpha[A]] \) is a \( S_\alpha \) p-neighborhood of \( \pi_\alpha[A] \), there is a \( U_\alpha \subseteq \varphi(S_\alpha) \) such that \( U_\alpha[\pi_\alpha[A]] \) is contained in \( P_\alpha[\pi_\alpha[A]] \). Then \( U^* = (T_\alpha \times T_\alpha)^{-1}[U_\alpha] \), \( U^* \subseteq \varphi(S_w) \) by 1.8.9 and \( U^*[A] \subseteq (T_\alpha \times T_\alpha)^{-1}[P_\alpha][A] \) and the conclusion follows.

Remark. The following lemma and its proof can be found in Isbell [14], page 24.

Lemma. A uniform space \( (X, \mathcal{U}) \) is totally bounded iff there is no countably infinite subset of \( X \) which is a discrete subspace of \( (X, \mathcal{U}) \).
1.8.19. Theorem. Suppose \( \{ (X_\alpha, S_\alpha) : \alpha \in I \} \) is a non-empty family of p-spaces. Then the following two conditions are equivalent.

(a) \( S_w = S_s \).

(b) There is at most one \( \alpha \in I \) such that \( (X_\alpha, S_\alpha) \) is not p-totally bounded.

Proof. (b) \( \rightarrow \) (a) If \( (X_\alpha, S_\alpha) \) is p-totally bounded for all \( \alpha \in I \), then \( \forall (\alpha) \neq \emptyset (S_\alpha) \) for all \( \alpha \in I \). This implies \( S_w = S_s \) by 1.8.9 and 1.8.13.

Suppose there is exactly one \( \alpha \in I \) such that \( (X_\alpha, S_\alpha) \) is not p-totally bounded. Using the lemma of Alfsen and Njastad quoted in the proof of I.1.4, we will show that \( S_w \supset S_s \) and thus \( S_w = S_s \) by 1.8.11. We will do this by showing that \( A \subseteq S B \) implies \( A \subseteq S_w B \). Suppose \( A, B \subseteq X \) and \( A \subseteq S B \). Then by 1.2.9 there are finite collections \( \{ A_1 \} \) and \( \{ B_j \} \) such that \( \bigcup A_1 = A \), \( \bigcap B_j = B \) and for any ordered pair \( (i,j) \) \( A_i \subseteq S B_j \) for some \( \gamma \in \Gamma \) (see 1.8.7). That is, for any ordered pair \( (i,j) \) there is a \( \gamma \in \Gamma \), an \( n \in N \), a collection \( \{ \alpha_k : 1 \leq k \leq n \} \), and sets \( U_{\alpha_k} \in \gamma (\alpha_k) \) (all dependent on \( (i,j) \)) such that if \( U_{(1,j)} \) is the set \( \bigcap \left\{ (\Pi_{\alpha_k} \times \Pi_{\alpha_k})^{-1}[U_{\alpha_k}] : 1 \leq k \leq n \right\} \) then \( U_{(1,j)}[A_1] \subseteq B_j \). We claim that \( A_i \subseteq S_w B_j \) for each pair \( (i,j) \). We consider two cases.

1) Suppose \( (i,j) \) is such that \( \alpha \neq \{ \alpha_k \} \). Then \( \gamma (\alpha_k) = \cup (S_{\alpha_k}) \) for each \( k \). By 1.8.9 we then have that...
$U(i,j) \in \mathcal{U}(\delta_w)$. Since $U(i,j)[A_1] \subseteq B_j$, $A_1 \subseteq_w B_j$.

(2) Suppose $(i,j)$ is such that $\alpha \in \{\alpha_k\}$, say $\alpha = \alpha_1$. Then let $U' = \cap \{ (\pi_{\alpha_k} \times \pi_{\alpha_k})^{-1}[U_{\alpha_k}]: 2 \leq k \leq n\}$. For each $\alpha_k$ such that $k \geq 2$, $\nu(\alpha_k) = \mathcal{U}(\delta_{\alpha_k})$ and therefore $U' \in \mathcal{U}(\delta_w)$ by I.8.9. Therefore $A_1 \subseteq_w U'[A_1]$. Now $U_{\alpha_1} \in \nu(\alpha_1) \subseteq \mathcal{P}(\delta_{\alpha_1})$. Then from I.8.16 we can conclude that $A_1 \subseteq_w (\pi_{\alpha_1} \times \pi_{\alpha_1})^{-1}[U_{\alpha_1}][A_1]$. Using the lemma of Alfsen and Njastad mentioned earlier we have that $A_1 \subseteq_w ((\pi_{\alpha_1} \times \pi_{\alpha_1})^{-1}[U_{\alpha_1}] \cap U')[A_1] = U[A_1] \subseteq B_j$. Thus $A_1 \subseteq_w B_j$.

We now have that $A \subseteq_s B$ implies the existence of finite collections $\{A_1\}$ and $\{B_1\}$, $A = \cup A_1$, $B = \cap B_1$. But then for each $i$, $A_1 \subseteq_w \cap B_j$ by (4) of 0.3.8. Then $A_1 \delta_w \subseteq (\cap B_j)$ for each $i$ and thus $\cup A_1 \delta_w \subseteq (\cap B_j)$ by (2) of 0.3.1. From the fact that $A = \cup A_1$, $B = \cap B_1$ we have $A \delta_w \subseteq B$ and thus $A \subseteq_w B$, completing this part of the proof.

(a) $\rightarrow$ (b) Assume that $\alpha_1 \neq \alpha_2$ and that $(X_{\alpha_1}, \delta_{\alpha_1})$ is not $p$-totally bounded for $i = 1, 2$. We will show that $\delta_w \neq \delta_s$. By I.7.2, for $i = 1, 2$, there are uniformities $U_{\alpha_i} \neq \mathcal{U}(\delta_{\alpha_i})$ such that $U_{\alpha_i} \in \mathcal{P}(\delta_{\alpha_i})$. From (3) of 0.3.13 we know that the $U_{\alpha_i}$ are not totally bounded uniformities. From I.8.17 we then have, for $i = 1, 2$, the existence of sets $A_i = \{a^i_n: n \in N\} \subseteq X_{\alpha_i}$, and entourages $V_i \in U_{\alpha_i}$ such that $V_i \cap (A_i \times A_i) = \Delta A_i$. Without loss of generality we assume that the indexing for each $i$ is such that $a^i_m \neq a^i_n$.
iff \( m \neq n \). We define sets \( A^* \) and \( B^* \subset X \) as follows:

\[
A^* = \{ x \in X : \text{there is an } n \in \mathbb{N} \text{ such that } x(\alpha_1) = a^1_m \text{ for } i = 1,2 \},
\]

\[
B^* = \{ x \in X : \text{there are } m, n \in \mathbb{N}, m \neq n, \text{ such that } x(\alpha_1) = a^1_m, x(\alpha_2) = a^2_n \}.
\]

We claim that \( A^* \subseteq B^* \) and \( A^* \subseteq B^* \).

To show that \( A^* \subseteq B^* \), let \( I_0 \subseteq I \), \( I_0 \) finite. Let \( U = \bigcap \{ (\mathbb{N}^\alpha \times \mathbb{N}^\alpha)^{-1}[U_\alpha] : \alpha \in I_0 \} \) where each \( U_\alpha \in \mathcal{U}(\mathcal{S}_\alpha) \). The collection of all such \( U \)'s is a base for \( \mathcal{U}(\mathcal{S}_w) \). Therefore, to show \( A^* \subseteq B^* \) we show that \( U[A^*] \cap U[B^*] \neq \emptyset \) for all such \( U \). For each \( \alpha \in I \), \( \alpha \neq \alpha_1, i = 1,2 \), fix \( x_\alpha \in X_\alpha \).

We consider two cases.

(1) Suppose at most one \( \alpha_i, i = 1,2 \), is in \( I_0 \). Then define \( a^*, b^*, x^* \in X \) as follows: (i) if neither \( \alpha_1 \) nor \( \alpha_2 \in I_0 \) let \( a^* \) be such that \( a^*(\alpha) = x_\alpha \) for \( \alpha \neq \alpha_1 \), \( i = 1,2 \); \( a^*(\alpha_1) = a^1_1 \); choose \( b^* \) such that \( b^*(\alpha) = x_\alpha \) for \( \alpha \neq \alpha_1 \), \( b^*(\alpha_1) = a^1_1 \) and \( b^*(\alpha_2) = a^2_2 \). (ii) if \( \alpha_i \in I_0 \) and \( \alpha_j \notin I_0 \), \( i \neq j \), \( i, j \in \{1,2\} \), choose \( a^* \) and \( b^* \) such that \( a^*(\alpha) = b^*(\alpha) = x_\alpha \) for \( \alpha \neq \alpha_k, k = 1,2 \); \( a^*(\alpha_1) \) equal to \( b^*(\alpha_1) = a^1_1 \), \( a^*(\alpha_j) = a^j_1 \), \( b^*(\alpha_j) = a^j_2 \). In each case choose \( x^* = a^* \). Then \( a^* \in A^* \), \( b^* \in B^* \), and \( x^* \in X \).

Further, in each case \( a^*(\alpha) = b^*(\alpha) = x^*(\alpha) \) for all \( \alpha \in I_0 \). Therefore \( (a^*, x^*), (b^*, x^*) \in U \) and thus we have \( x^* \in U[A^*] \cap U[B^*] \).
(2) Suppose \( \alpha_1 \) and \( \alpha_2 \in I_0 \). Choose \( a_{n_1}^2 \in A_2 \) such that there is an \( m \neq n^* \) for which \( a_{m}^2 \in \mathcal{U}_{A_2}[a_{n_1}^2] \). We note that if such a choice were not possible, \( \mathcal{U}_{A_2} \cap (A_2 \times A_2) \) would be \( \Delta_{A_2} \), which would imply that \( \mathcal{U} (\delta_{A_2}) \) is not totally bounded by 1.8.18. Let \( x^* \in X \) such that \( x^*(\alpha) = x_{\alpha} \) for \( \alpha \neq \alpha_1, i = 1,2; x^*(\alpha_1) = a_{n_1}^1 \), and \( x^*(\alpha_2) = a_{m}^2 \). Then \( x^* \in \mathcal{U}[A^*] \cap \mathcal{U}[B^*] \). For, if we let \( a^* \) be such that \( a^*(\alpha) \) is \( x_{\alpha} \) for \( \alpha \neq \alpha_1, i = 1,2, a^*(\alpha_1) = a_{n_1}^1 \); and let \( b^* = x^* \), then \( (b^*, x^*) \in U \). We claim that \( (a^*, x^*) \) is also in \( U \), since for all \( \alpha \in I_0, \alpha \neq \alpha_2; a^*(\alpha) = x^*(\alpha) \) and thus \( (a^*(\alpha), x^*(\alpha)) \in U_{\alpha}; \) for \( \alpha_2, (a^*(\alpha_2), x^*(\alpha_2)) = (a_{n_1}^2, a_{m}^2) \) which is in \( U_{A_2} \) by construction. Thus \( \mathcal{U}[A^*] \cap \mathcal{U}[B^*] \neq \emptyset \).

From cases (1) and (2) we then have, for \( \mathcal{U} \in \mathcal{U}(\delta_w), \mathcal{U}[A^*] \cap \mathcal{U}[B^*] \neq \emptyset \) and therefore \( A^* \notin S \mathcal{U} B^* \).

Finally, we will show that \( A^* \overline{\delta_S} B^* \). Define \( \gamma \in \mathcal{S} \) such that \( \gamma(\alpha) = \mathcal{U}(\delta_{\alpha}) \) for \( \alpha \neq \alpha_1, i = 1,2; \) and \( \gamma(\alpha_1) = U_{\alpha_1} \) mentioned earlier. We claim that \( A^* \overline{\delta_{\gamma}} B^* \) and therefore \( A^* \overline{\delta_S} B^* \). As mentioned earlier there are\ entourages \( V_i \in \mathcal{U}_{\alpha_1} \) such that \( V_i \cap (A_1 \times A_1) = \Delta_{A_1} \), for \( i = 1,2 \). For \( i = 1,2, \) let \( V_i^* \in \mathcal{U}_{\alpha_1} \) be symmetric such that \( V_i^* \circ V_i^* \subseteq V_i \). Let \( U^* = \cap \{ \bigcap_{i=1}^{2} (\mathcal{T}_{A_1} \times \mathcal{T}_{A_1})^{-1}[V_i^*]; i = 1,2 \} \). Then \( U^* \in \mathcal{U}_{\gamma} \). We claim that \( U^*[A^*] \cap U^*[B^*] = \emptyset \). Suppose there is an \( x \in U^*[A^*] \cap U^*[B^*] \). Then there would be an \( a^* \in A^* \) and a \( b^* \in B^* \) such that \( (a^*, x^*) \) and \( (b^*, x^*) \in U^* \). Now \( a^*(\alpha_1) = a_{n_1}^1 \) and \( a^*(\alpha_2) = a_{n}^2 \) for some \( n \), \( b^*(\alpha_1) = a_{p}^1 \) and \( b^*(\alpha_2) = a_{q}^2 \) for some \( p \) and \( q \), \( p \neq q \). If \( (a^*, x^*) \) and
and \((b^*, x^*) \in U^*\) then \((a^*(\alpha_1^*), x^*(\alpha_1^*))\) and \((b^*(\alpha_1^*), x^*(\alpha_1^*))\) are in \(V^*_1\) for \(i = 1, 2\). Then \((a^1_n, a^1_p) \in V^*_1 \circ V^*_1 \subseteq V_1\) and \((a^2_n, a^2_q) \in V^*_2 \circ V^*_2 \subseteq V_2\). Since \(V_i \cap (A_i \times A_i) = \Delta_{A_i}\) for \(i = 1\) and 2, we would then have \(a^1_n = a^1_p\) and \(a^2_n = a^2_q\). But then, since distinct indices are associated with distinct points, \(q = n\) and \(p = n\) and hence \(p = q\) which is a contradiction. Therefore \(U^*[A^*] \cap U^*[B^*] = \emptyset\). Thus \(A^* \bar{\sigma} B^*\) which implies \(A^* \bar{\sigma}_S B^*\) by 1.2.2 and 1.8.7.

\section*{I.8.20. Remark} We remark that the idea for the second half of this proof is based on an exercise in [14] (12 (b), page 34) where it is indicated that the uniform product of two non-totally bounded uniform spaces need not be in \(\Pi(\sigma_w)\) for the associated proximities.

\section*{I.8.21. Remark} Poljakov has stated that the weak product of \(p\)-complete \(p\)-spaces is \(p\)-complete. That this implies that the strong product is also \(p\)-complete follows from I.8.10 and I.8.11. We supply a proof of Poljakov's result.

\section*{I.8.22. Theorem} (Poljakov) Suppose \(\{(X_\alpha, \sigma_\alpha) : \alpha \in I\}\) is a non-empty family of \(p\)-complete \(p\)-spaces. Then \((X, \sigma_w)\) is \(p\)-complete.

\textbf{Proof.} Let \((T, D)\) be a net in \(X\) such that \((T, D)\) is \(p\)-Cauchy relative to \(\sigma_w\). We claim that for each \(\alpha \in I\), \((\cap_\alpha \circ T, D)\) is \(p\)-Cauchy in \(\sigma_\alpha\). This follows from I.6.3 and the definition of \(\sigma_w\). Therefore each \((\cap_\alpha \circ T, D)\) converges
to some \( x_\alpha \). Letting \( x \in X \) such that \( x(\alpha) = x_\alpha \) and noting that, by 1.8.10, \((T,D)\) converges to \( x \) completes the proof.

**1.8.23. Corollary.** Suppose \( \{ (X_\alpha, S_\alpha) : \alpha \in \Gamma \} \) is a non-empty family of \( p \)-complete \( p \)-spaces. Then \((X, S_s)\) is \( p \)-complete.

**Proof.** Since \( \delta_w \leq s_s \), \( \mathcal{P}(\delta_w) \subseteq \mathcal{P}(s_s) \) by 1.2.2. Hence \((T,D)\) \( p \)-Cauchy relative to \( s_s \) implies that \((T,D)\) is \( p \)-Cauchy relative to \( \delta_w \). By 1.8.22, there is an \( x \in X \) such that \((T,D)\) converges to \( x \) relative to \( \mathcal{T}(\delta_w) \). Since \( \mathcal{T}(\delta_w) = \mathcal{T}(s_s) \) by 1.8.10, \((T,D)\) converges to \( x \) relative to \( \mathcal{T}(s_s) \) and \((X, S_s)\) is \( p \)-complete.
CHAPTER II

WEAK AND STRONG HYPERPROXIMITIES

II.1 Definition of Weak and Strong Hyperproximities

II.1.1. Definition. Suppose \((X, \delta)\) is a p-space and \(\mathcal{U}(\delta)\) is as in 0.3.12. Then, by 0.4.4, \((\hat{X}, \mathcal{U}(\hat{\delta}))\) is a uniform space. Define \(\hat{\delta}_w\), a proximity relation on \(\hat{X}\) by, \(\hat{\delta}_w = \delta(\mathcal{U}(\hat{\delta}))\). The space \((\hat{X}, \hat{\delta}_w)\) will be called the weak hyperproximity space determined by \((X, \delta)\).

II.1.2. Theorem. For any p-space \((X, \delta)\), \(\mathcal{U}(\hat{\delta}_w)\) is \(\mathcal{U}(\delta)\).

Proof. From 0.4.5 we know that \((\hat{X}, \mathcal{U}(\hat{\delta}))\) is a totally bounded uniform space. We also know from 0.3.13 that \(\mathcal{U}(\hat{\delta}_w)\) is the only totally bounded uniformity in \(\Pi(\hat{\delta}_w)\). Since \(\mathcal{U}(\delta) \in \Pi(\hat{\delta}_w)\) and is totally bounded, \(\mathcal{U}(\delta) = \mathcal{U}(\hat{\delta}_w)\).

II.1.3. Theorem. Let \(i:X \rightarrow \hat{X}\) such that \(i(x) = \{x\}\). Then for any p-space \((X, \delta)\), \(i\) is a p-isomorphism (see 0.3.2) from \((X, \delta)\) into \((\hat{X}, \hat{\delta}_w)\).

Proof. That \(i\) is one-to-one and onto \(i[X]\) is obvious. Suppose \(A, B \subseteq X\). Then \(A \cap B\) implies that \(U[A] \cap U[B] \neq \emptyset\) for all \(U \in \mathcal{U}(\delta)\). Then for each symmetric \(U \in \mathcal{U}(\delta)\), \(H(U)[i[A]] \cap H(U)[i[B]] \neq \emptyset\). Since, by 0.4.4, the
symmetric U's in $\mathcal{U}(\mathcal{S})$ form a base for $\mathcal{U}(\mathcal{S})$, the $H(U)$'s where $U$ is symmetric form a base for $\mathcal{U}(\mathcal{S})$, and hence $i[A] \triangleleft W i[B]$. Conversely, $A \triangleleft B$ implies $U[A] \cap B = \emptyset$ for some symmetric $U \in \mathcal{U}(\mathcal{S})$. But $H(U)[i[A]] \cap i[B] \neq \emptyset$ implies $(\{a\}, \{b\}) \in H(U)$ where $a \in A$ and $b \in B$, and hence $(a, b) \in U$, and thus $U[A] \cap B \neq \emptyset$. Therefore $A \triangleleft B$ implies that $i[A]$ is $\triangleleft W$ far from $i[B]$.

**II.1.4. Theorem.** Suppose $(X, \mathcal{S})$ and $(X^*, \mathcal{S}^*)$ are p-spaces and $f \in [X, X^*]$. Then $f \in p[X, X^*]$ iff $\hat{f}$ is in $p[(\hat{X}, \hat{\mathcal{S}}_W), (\hat{X}^*, \hat{\mathcal{S}}^*_W)]$.

**Proof.** Suppose $f \in p[X, X^*]$. Suppose $Q, \Omega \in X$ and $Q \triangleleft W \Omega$. Suppose $U^* \in \mathcal{U}(\mathcal{S}^*)$. Then by 0.3.12, $(fxf)^{-1}[U^*] \in \mathcal{U}(\mathcal{S})$. Then there is an $A \in Q$, $B \in \Omega$, and $C \subset X$ such that $A \cup B \in (fxf)^{-1}[U^*][C]$ and $C \subset (fxf)^{-1}[U^*][A]$ and $(fxf)^{-1}[U^*][B]$. Thus $f[A] \cup f[B] \subset U^*[f[C]]$ and $f[C] \subset U^*[f[A]] \cap U^*[f[B]]$. Since $f[A] \in \hat{f}[Q]$, $f[B] \in \hat{f}[\Omega]$, and $(f[A], f[C])$, $(f[B], f[C]) \in H(U^*)$, $\hat{f}[Q] \triangleleft W \hat{f}[\Omega]$.

Now suppose $f \in p[(\hat{X}, \hat{\mathcal{S}}_W), (\hat{X}^*, \hat{\mathcal{S}}^*_W)]$. Let $A, B \in X$ and $A \triangleleft B$. Then $i[A] \triangleleft W i[B]$ and hence $\hat{i}[i[A]] \triangleleft W \hat{i}[i[B]]$. Then $(i^*)^{-1}[\hat{i}[i[A]]] \triangleleft W (i^*)^{-1}[\hat{i}[i[B]]]$. But $(i^*)^{-1} \circ i(x) = (i^*)^{-1}[f[i x]] = (i^*)^{-1}[\{f(x)\}] = f(x)$; hence $f[A] \triangleleft W f[B]$.

**II.1.5. Remark.** The correspondence $\triangleleft W$ from the category of p-spaces to the category of p-spaces is thus a covariant functor.
II.1.6. Remark. It is well known that, if \((X, \mathcal{U})\) is a pseudometrizable uniform space, then \((\hat{X}, \mathcal{U})\) is also pseudometrizable. As we will show later (II.2.12) if \((X, \mathcal{S})\) is a pseudometrizable p-space, \((\hat{X}, \mathcal{S}_w)\) may not be pseudometrizable. It is for this reason that we introduce the following definition.

II.1.7. Definition. Suppose \((X, \mathcal{S})\) is a p-space and \(\Pi(\mathcal{S})\) is as in 0.3.10. Define \(\mathcal{S}_s = \sup \{ \delta(\mathcal{U}); \mathcal{U} \in \Pi(\mathcal{S}) \}\). The space \((\hat{X}, \mathcal{S}_s)\) will be called the strong hyperproximity space determined by \((X, \mathcal{S})\).

II.1.8. Remark. In [20] Mrowka suggests a definition for a proximity relation on the space of closed bounded subsets of a p-space using the gauge \(D(\mathcal{S})\). Our definition is similar to Mrowka's definition although in light of I.2.15 they may not agree. No results of Mrowka's definition are obtained in [20].

II.1.9. Theorem. Suppose \((X, \mathcal{S})\) is a total p-space and \(\mathcal{P}(\mathcal{S}) = \mathcal{U}\). Then \(\mathcal{S}_s = \delta(\mathcal{U})\).

Proof. Since \(\mathcal{U} \in \Pi(\mathcal{S})\), by definition \(\mathcal{S}_s > \delta(\mathcal{U})\). Suppose \(\mathcal{O}, \mathcal{G} \in X\) and \(\mathcal{O} \mathcal{S}(\mathcal{U}) \mathcal{G}\). Let \(\mathcal{Q}_i\) and \(\mathcal{G}_j\) be finite covers from below of \(\mathcal{O}\) and \(\mathcal{G}\) respectively. Then \(\mathcal{Q}_i \mathcal{S}(\mathcal{U}) \mathcal{G}_j\) for some \(i^*\) and \(j^*\). We claim that \(\mathcal{Q}_i \mathcal{S}(\mathcal{U}_o) \mathcal{G}_j\) for all \(\mathcal{U}_o \in \Pi(\mathcal{S})\) and hence by I.2.7 \(\delta(\mathcal{U}) > \mathcal{S}_s\). To see this, let \(\mathcal{U}_o \in \Pi(\mathcal{S})\). Then \(\mathcal{Q}_i \mathcal{S}(\mathcal{U}_o) \mathcal{G}_j\) iff \(\mathcal{H}(\mathcal{U}_o)[\mathcal{Q}_i]\cap \mathcal{G}_j \neq \emptyset\) for all symmetric \(\mathcal{U}_o \in \mathcal{U}_o\). But \(\mathcal{U}_o \in \mathcal{P}(\mathcal{S}) = \mathcal{U}\), and the fact that
\( Q_{i*} \subseteq (\hat{\mathcal{U}}) \land j* \) implies \( \mathcal{H}(U^*_0) \land (Q_{i*}) \land (j*) \neq \emptyset \) for all symmetric \( U_0 \in \mathcal{U}_0 \) and hence \( Q_{i*} \subseteq (\hat{\mathcal{U}}^*_0) \land (j*) \) for all \( U_0 \in \mathcal{U}(\delta) \).

**II.1.10. Corollary.** Suppose \((X, \mathcal{S})\) is a pseudometrizable p-space. Then \((X, \hat{\mathcal{S}})\) is also pseudometrizable and \( \mathcal{P}(\hat{\mathcal{S}}) = \mathcal{P}(\mathcal{S}) \).

**Proof.** Let \( d \) be any pseudometric for \( X \) for which \( \delta(d) = \mathcal{S} \). Then \( \mathcal{P}(\mathcal{S}) \) is \( \mathcal{U} \) and by II.1.9, \( \hat{\mathcal{S}} = \delta(\hat{\mathcal{U}}_d) \).

Since \( \mathcal{U}_d \) has a countable base, so does \( \hat{\mathcal{U}}_d \). Hence \( \hat{\mathcal{U}}_d \) is pseudometrizable and thus so is \( \delta(\hat{\mathcal{U}}_d) \). The second part of the conclusion follows from I.4.4.

**II.1.11. Remark.** The following theorems show that the correspondence \( \wedge_s \) is a covariant functor from the category of proximity spaces to the category of proximity spaces.

**II.1.12. Theorem.** For any p-space \((X, \mathcal{S})\) the map \( i \) defined in II.1.3 is a p-isomorphism from \((X, \mathcal{S})\) into \((X, \hat{\mathcal{S}})\).

**Proof.** We again use the characterization of supremum in I.2.7. Suppose \( A \subseteq B, A, B \subseteq X \). Let \( \{Q_i\} \) and \( \{G_j\} \) be finite covers from below of \( i[A] \) and \( i[B] \) respectively. Since each \( Q_i \) or \( G_j \) is in \( i[A] \) or \( i[B] \) they are subsets of \( i[X] \). Let \( A_i = i^{-1}[Q_i], B_j = i^{-1}[G_j] \). Then \( A \subseteq B \) implies that there is an \( i* \) and a \( j* \) such that \( A_{i*} \subseteq B_{j*} \).

Thus, for any \( U \in \mathcal{U}(\mathcal{S}) \) and any symmetric \( U \in \mathcal{U} \), \( U[A_{i*}] \cap U[B_{j*}] \neq \emptyset \). Arguing as in II.1.3, this implies
\( H(U)[\alpha_{i*}] \cap H(U)[\beta_{j*}] \neq \emptyset \) for all symmetric \( U \in \mathcal{U} \), for all \( \mathcal{U} \in \Pi(\mathcal{S}) \). Hence \( \alpha_{i*} \in \mathcal{S}^*(U) \), \( \beta_{j*} \) for each \( \mathcal{U} \in \Pi(\mathcal{S}) \).

Then by I.2.7, \( i[A] \in \mathcal{S} \), \( i[B] \).

Suppose \( A \in \mathcal{S} \mathcal{B} \). Then there is a symmetric \( P \in \mathcal{P}^*(\mathcal{S}) \) such that \( P[A] \cap B = \emptyset \). Hence, there is a \( \mathcal{U} \in \Pi(\mathcal{S}) \) such that \( P \in \mathcal{U} \). We claim \( i[A] \) is \( \mathcal{S}^*(U) \) far from \( i[B] \) and hence \( \mathcal{S} \) far also. Again arguing as in II.1.3, \( P[A] \cap B = \emptyset \) implies \( H(P)[i[A]] \cap i[B] = \emptyset \) and thus \( i[A] \) is then \( \mathcal{S}^*(U) \) far from \( i[B] \), thus completing the proof.

II.1.13. Theorem. Suppose \((X, \mathcal{S})\) and \((X^*, \mathcal{S}^*)\) are \( \mathcal{P}\)-spaces and \( f \in [X, X^*] \). Then \( f \in \mathcal{P}[X, X^*] \) iff \( \hat{f} \) is in \( \mathcal{P}[(X, \hat{\mathcal{S}}), (X^*, \hat{\mathcal{S}}^*)] \).

Proof. Suppose \( f \in \mathcal{P}[X, X^*] \). Suppose \( \alpha \in \mathcal{S} \), \( \mathcal{B} \). Let \( \{Q_1^*\} \) and \( \{\mathcal{B}_j^*\} \) be finite covers from below of \( \hat{f}[\alpha] \) and \( \hat{f}[\mathcal{B}] \) respectively. Then letting \( \alpha_1 = \hat{f}^{-1}[Q_1^*] \) and \( \beta_j = \hat{f}^{-1}[\mathcal{B}_j^*] \) we have \( U \alpha_1 \mathcal{S}, U \mathcal{B}_j \). Then there are indexes \( i^* \) and \( j^* \) such that \( \alpha_{i^*} \mathcal{S}, \mathcal{B}_{j^*} \). We claim that \( \alpha_{i^*} \mathcal{S}^*(U) \mathcal{B}_{j^*} \) for all \( \mathcal{U} \in \Pi(\mathcal{S}^*) \) and hence by I.2.7, \( \hat{f}[\alpha] \in \mathcal{S}^*(U) \). To show that \( \alpha_{i^*} \mathcal{S}^*(U) \mathcal{B}_{j^*} \), we show that \( \alpha_{i^*} \mathcal{S}^*(U) \mathcal{B}_{j^*} \) and \( \hat{f}[\mathcal{B}_{j^*}] \) and the result follows since \( \alpha_{i^*} \mathcal{S}^*(U) \mathcal{B}_{j^*} \). Let \( \mathcal{U} \in \Pi(\mathcal{S}^*) \) and let \( U \in \mathcal{U} \), \( U^* \) symmetric. Then \( f \in \mathcal{P}[X, X^*] \) implies that \( U = (fxf)^{-1}[U^*] \in \mathcal{P}(\mathcal{S}) \). Hence there is a \( \mathcal{U} \in \Pi(\mathcal{S}) \) such that \( U \in \mathcal{U} \). Then \( \alpha_{i^*} \mathcal{S}^*(U) \mathcal{B}_{j^*} \) implies \( \alpha_{i^*} \mathcal{S}^*(U) \mathcal{B}_{j^*} \). Arguing as in II.1.4, \( H(U)[\alpha_{i^*}] \cap H(U)[\beta_{j^*}] \neq \emptyset \) implies
\( H(U^*)[\hat{f}(Q_{1*})] \cap H(U^*)[\hat{f}(G_{j*})] \neq \emptyset. \) Therefore, we have 
\( \hat{f}(Q_{1*}) \in (U^*)^* \hat{f}(G_{j*}) \) for all \( U^* \in \Pi(\delta^*) \) and by the above remarks \( \hat{f} \in p[(x, \hat{s}), (x^*, \hat{s}^*)]. \)

The converse follows from the same argument used in II.1.4 since \( i \) is a \( p \)-isomorphism from \((x, s)\) into \((x, \hat{s})\) and \( i^* \) is a \( p \)-isomorphism from \((x^*, s^*)\) into \((x^*, \hat{s}^*)\).

**II.1.4. Lemma.** Suppose \((x, s)\) is a \( p \)-space. Let \( \hat{\varphi}(s) = \{ U \subseteq x \times x : H(P) \subseteq U \) for some \( P \in \varphi(s) \}. \) Then \( \hat{\varphi}(s) \subseteq \varphi(\hat{s}). \)

**Proof.** From (3) of I.1.2, it is sufficient to show that \( H(P) \in \varphi(\hat{s}) \) for all \( P \in \varphi(s) \). Suppose \( P \in \varphi(s) \). Then there is a \( U \in \Pi(\delta) \) such that \( P \subseteq U \). There is a normal sequence \( \{ U_n \} \) such that \( U_\infty \subseteq U \) and \( U_n \subseteq U \) for all \( n \). From 0.4.3, \( \{ H(U_n) \} \) is a normal sequence in \( x \) and \( H(U_1) \circ H(U_1) \subseteq H(P). \) We claim that \( Q \subseteq_{s} H(U_n)[Q] \) for all \( Q \subseteq \hat{x} \) for all \( n \in \mathbb{N} \), hence satisfying condition (5) of I.1.2. Consider \( H(U_{n+1})[Q] \cap H(U_{n+1})[Q \cdot H(U_n)[Q]] \). If this set is non-empty, the fact that \( H(U_{n+1}) \circ H(U_{n+1}) \) is contained in \( H(U_n) \) implies that there is an \( A \in Q \) and a \( B \in \mathcal{C}(H(U_n)[Q]) \) such that \( (A, B) \in H(U_n) \). But then \( B \in H(U_n)[Q] \) which is not possible. Hence we have \( Q \) is \( \delta(Q) \) far from \( \mathcal{C}(H(U_n)[Q]) \) and therefore \( Q \subseteq_{s} H(U_n)[Q] \), for all \( Q \subseteq \hat{x} \), and for all \( n \in \mathbb{N} \). Thus \( \{ H(U_n) \} \) is a normal sequence with the properties necessary to imply that \( H(P) \in \varphi(\hat{s}). \)
II.1.15. Remark. We close this section with a brief discussion of the relationship between $\hat{\delta}_w$ and $\hat{\delta}_S$.

II.1.16. Theorem. For any space $(X, \mathcal{S})$, $\hat{\delta}_w \leq \hat{\delta}_S$.

Proof. Since $\mathcal{U}(\mathcal{S}) \in \mathcal{W}(\mathcal{S})$, $\mathcal{S}(\mathcal{U}(\mathcal{S})) \leq \hat{\delta}_S$ by definition of $\hat{\delta}_S$. But $\mathcal{S}(\mathcal{U}(\mathcal{S})) = \hat{\delta}_w$ by definition of $\hat{\delta}_w$.

II.1.17. Theorem. Suppose $(X, \mathcal{S})$ is a p-space. Then $\hat{\delta}_w = \hat{\delta}_S$ iff $(X, \mathcal{S})$ is p-totally bounded.

Proof. From 1.7.2 $(X, \mathcal{S})$ is p-totally bounded implies that $\mathcal{P}(\mathcal{S}) = \mathcal{U}(\mathcal{S})$ or that $\mathcal{U}(\mathcal{S}) = \{\mathcal{U}(\mathcal{S})\}$. Then $\hat{\delta}_w = \hat{\delta}_S$ follows from the definitions.

We will show the converse by contraposition. Suppose $(X, \mathcal{S})$ is not p-totally bounded. Then there is a $P \in \mathcal{P}(\mathcal{S})$, $P$ symmetric, such that $P[A] = X$ for no finite set $A \subseteq X$. Let $\mathcal{Q}_P \subseteq X$, $\mathcal{Q}_P = \{A: A \subseteq X$ and $A$ finite$\}$. We claim that $\mathcal{H}(P)[\mathcal{Q}_P]$ is a $\hat{\delta}_S$ p-neighborhood of $\mathcal{Q}_P$ and that $\mathcal{H}(P)[\mathcal{Q}_P]$ is not a $\hat{\delta}_w$ p-neighborhood of $\mathcal{Q}_P$. To show that $\mathcal{H}(P)[\mathcal{Q}_P]$ is a $\hat{\delta}_S$ p-neighborhood of $\mathcal{Q}_P$, we note that by II.1.14 $\mathcal{H}(P) \in \mathcal{P}(\hat{\delta}_S)$ and hence $\mathcal{H}(P)[\mathcal{Q}_P]$ is a $\hat{\delta}_S$ p-neighborhood of $\mathcal{Q}_P$ by I.1.13.

To show that $\mathcal{H}(P)[\mathcal{Q}_P]$ is not a $\hat{\delta}_w$ p-neighborhood of $\mathcal{Q}_P$, suppose that it is. Then by 0.3.11 there is a $U \in \mathcal{U}(\mathcal{S})$ such that $\mathcal{H}(U)[\mathcal{Q}_P] \subseteq \mathcal{H}(P)[\mathcal{Q}_P]$. But, since $\mathcal{U}(\mathcal{S})$ is a totally bounded uniform space, there is a finite set $A_0$ such that $U[A_0] = X$. Since $A_0 \subseteq \mathcal{Q}_P$ and $X \subseteq U[A_0]$, $A_0 \subseteq U[X]$, we have $(A_0, X) \subseteq \mathcal{H}(U)$. Thus $X \in \mathcal{H}(U)[A_0] \subseteq \mathcal{H}(U)[\mathcal{Q}_P] \subseteq \mathcal{H}(P)[\mathcal{Q}_P]$. 


Thus there is a set $A^*_\delta \in \mathcal{Y}$ such that $(A^*_\delta, X) \in H(P)$. But this implies $X \subseteq P[A^*_\delta]$. Since this is not possible for any $A \in \mathcal{Y}$, $H(P)[\mathcal{Y}]$ is not a $\hat{\delta}_w^p$-neighborhood of $\mathcal{Y}$ and $\hat{\delta}_w^p \neq \hat{\delta}_s^p$.

**II.1.18. Remark.** Using the above theorem it is not hard to show examples where $\hat{\delta}_w^p \neq \hat{\delta}_s^p$. The space $(N, \delta_M)$ is not $p$-totally bounded and thus $(\hat{N}, (\hat{\delta}_M)_w)$ is not equal to $(\hat{N}, (\hat{\delta}_M)_s)$.

### II.2 Properties of $\hat{\delta}_w^p$ and $\hat{\delta}_s^p$.

**II.2.1. Remark.** In this section we study the proximal properties of $(X, \delta)$ and how they relate to the proximal properties of $(\hat{X}, \hat{\delta}_w^p)$ and $(\hat{X}, \hat{\delta}_s^p)$.

**II.2.2. Lemma.** For any $p$-space $(X, \delta)$, $\varphi(\hat{\delta}_s^p | i[X])$ is equal to $\varphi(\hat{\delta}_s^p) \cap (i[X] \times i[X]) = (iX)^{\varphi(\delta)}$.

**Proof.** That $\varphi(\hat{\delta}_s^p | i[X]) = (iX)^{\varphi(\delta)}$ follows from I.1.11 and II.1.12.

From I.1.14 we can see that $\varphi(\hat{\delta}_s^p) \cap (i[X] \times i[X])$ is contained in $\varphi(\hat{\delta}_s^p | i[X])$. Let $\varphi \in \varphi(\hat{\delta}_s^p | i[X])$. Then $P^* = (iX)^{-1}[\varphi] \in \varphi(\delta)$. Then $H(P^*) \in \varphi(\hat{\delta}_s^p)$ by II.1.14 and thus so is $H(P^*) \cup \varphi$. Therefore $(H(P^*) \cup \varphi) \cap (i[X] \times i[X])$ is in $\varphi(\hat{\delta}_s^p) \cap (i[X] \times i[X])$. We claim that $\varphi$ is equal to $(H(P^*) \cup \varphi) \cap (i[X] \times i[X])$. We need only show that $(H(P^*) \cup \varphi) \cap (i[X] \times i[X]) \subseteq \varphi$. Let $(\{a\}, \{b\}) \in H(P^*)$. Then $a \in P^*[b]$ and $b \in P^*[a]$ and $(a, b) \in P^* = (iX)^{-1}[\varphi]$. Therefore $(i(a), i(b)) = (\{a\}, \{b\}) \in \varphi$, and the lemma is proved.
11.2.3. Theorem. Suppose \((X, \hat{\delta}_S)\) is a total \(p\)-space. Then \((X, \delta)\) is also total.

Proof. Let \(P_1\) and \(P_2 \in \mathcal{P}(\hat{\delta})\). We will show that \(P_1 \cap P_2 \in \mathcal{P}(\delta)\). Since \(\hat{\delta}_S\) is total, and \(H(P_1) \in \mathcal{P}(\hat{\delta}_S)\), \(i = 1, 2\), by 11.1.14, there is a \(\varphi^* \in \mathcal{P}(\hat{\delta}_S)\) such that \(\varphi^* \subseteq H(P_1) \cap H(P_2)\). By 11.2.2, \(\varphi^* \cap (i[X] \times i[X])\) is in \((iX) \{\varphi(\hat{\delta})\}\). Let \(P^* \in \mathcal{P}(\delta)\) be symmetric such that \(\varphi^*\) contains \((iX)[P^*]\). Then we claim that \(P^* \in P_1 \cap P_2\). Let \((a, b) \in P^*\). Then \((\{a\}, \{b\}) \in \varphi^* \cap (i[X] \times i[X])\) which is contained in \(H(P_1) \cap H(P_2)\). Thus \(b \in P_1[a]\) and \(a \in P_1[b]\) for \(i = 1, 2\). Thus \((a, b) \in P_1 \cap P_2\). Therefore \(P^* \subseteq P_1 \cap P_2\) and by property (3) of 1.1.2, \(P_1 \cap P_2 \in \varphi(\delta)\). \(\varphi(\delta)\) is therefore a filter and \((X, \delta)\) is total.

11.2.4. Theorem. Suppose \((X, \hat{\delta}_W)\) is total and that \(\varphi(\hat{\delta}_W)\) has a linearly ordered base. Then \((X, \delta)\) is total and \(\varphi(\delta)\) has a linearly ordered base.

Proof. Let \(U_0\) be the linearly ordered base for \(\varphi(\hat{\delta}_W)\). Let \(U\) be the uniformity generated on \(i[X]\) by the uniform base \(U_0 \cap (i[X] \times i[X])\). Then by 0.3.16, we have \(\delta(U) = \hat{\delta}_W|i[X]\). By 1.4.3, \(\hat{\delta}_W|i[X]\) is total and \(\varphi(\hat{\delta}_W|i[X]) = U\). But then \(\delta\) is total and \(\varphi(\delta)\) which is \((iX)^{-1}[U]\) has a linearly ordered base.

11.2.5. Theorem. If \((X, \delta)\) is total and \(\varphi(\delta)\) has a linearly ordered base, then \((X, \hat{\delta}_S)\) is total and \(\varphi(\hat{\delta}_S)\) has a linearly ordered base.
Proof. By II.1.9, $\hat{S}_s = S(\hat{\varphi}(S))$. Since $\varphi(S)$ has a linearly ordered base, so does $\hat{\varphi}(S)$. Hence the conclusion using I.4.3.

II.2.6. Corollary. If $(X, \hat{S}_w)$ is total and $\varphi(\hat{S}_w)$ has a linearly ordered base, then $(X, \hat{S}_s)$ is total and $\varphi(\hat{S}_s)$ has a linearly ordered base.

Proof. Combine II.2.5 and II.2.4.

II.2.7. Remark. Theorem II.2.5 seems to be all that can be said about the totality of $(X, S)$ implying the totality of $(X, \hat{S}_s)$, although we would conjecture that $(X, S)$ total need not imply that $(X, \hat{S}_w)$ need be total. Until more is known about total $p$-spaces counterexamples or proofs concerning totality may continue to be elusive.

II.2.8. Theorem. Suppose $(X, \hat{S}_s)$ is $p$-totally bounded. Then $(X, S)$ is $p$-totally bounded.

Proof. If $(X, \hat{S}_s)$ is $p$-totally bounded, then $\varphi(\hat{S}_s)$ is a totally bounded uniform space. Therefore $\varphi(\hat{S}_s) \cap (1[X] \times 1[X])$ is a totally bounded uniform space. But by II.2.2, $\varphi(\hat{S}_s|1[X]) = \varphi(\hat{S}_s) \cap (1[X] \times 1[X])$ and thus $(1[X], \hat{S}_s|1[X])$ is $p$-totally bounded. Using the fact that $(1[X], \hat{S}_s|1[X])$ and $(X, S)$ are $p$-isomorphic and I.7.4, $(X, S)$ is $p$-totally bounded.

II.2.9. Corollary. Suppose $(X, \hat{S}_s)$ is $p$-totally bounded. Then $(\hat{X}, \hat{S}_w) = (\hat{X}, \hat{S}_s)$.

Proof. This follows from II.1.17 and II.2.8.
II.2.10. Remark. We have already remarked in II.1.10 that the strong hyperproximity space of a pseudometrizable p-space is pseudometrizable. We now show that this is not the case for the weak hyperproximity space. The following lemma is due to Caulfield and a proof can be found in [7].

II.2.11. Lemma. Suppose \((X,\mathcal{U})\) is a uniform space. Define \(g \in [(X \times X), \hat{X}]\) by \(g(x,y) = \{x, y\} \setminus \{x, y\}^+\). Then \(g\) is a uniform isomorphism from \((X \times X, \mathcal{U} \times \mathcal{U})\) into \((X, \mathcal{U})\).

II.2.12. Example. Suppose that it is true that \((X, \mathcal{S})\) pseudometrizable implies that \(\hat{(X, \mathcal{S}_w)}\) is pseudometrizable. Then \(\hat{(X, (\mathcal{S}_w)_w)}\) will also be pseudometrizable by applying the theorem again. Since \(\mathcal{U}(\mathcal{S}_w) = \mathcal{U}(\mathcal{S})\), we have \(\mathcal{U}(\mathcal{S}_w) = \mathcal{U}(\mathcal{S})\) is in \(\mathcal{V}((\mathcal{S}_w)_w)\). From II.2.11 the uniform spaces \((X \times X, \mathcal{U}(\mathcal{S}) \times \mathcal{U}(\mathcal{S}))\) and \((g[X \times X], \hat{\mathcal{U}(\mathcal{S})}) \cap (g[X \times X] \times g[X \times X])\) are isomorphic as uniform spaces and thus \((X \times X, \mathcal{S}_w \mathcal{S}_w)\) and the space \((g[X \times X], (\hat{\mathcal{S}_w}_w) \mid g[X \times X])\) are p-isomorphic. Thus if \((X, (\hat{\mathcal{S}_w}_w)_w)\) is pseudometrizable, then so is the space \((g[X \times X], (\hat{\mathcal{S}_w}_w) \mid g[X \times X])\) and hence \((X \times X, \mathcal{S}_w \mathcal{S}_w)\) is pseudometrizable.

In example I.4.8, the space defined is \((N \times N, \mathcal{S}_M \times \mathcal{S}_M)\). It is shown in this example that \(\mathcal{S}_M \times \mathcal{S}_M\) is not even total much less pseudometrizable. Therefore the conjecture "\((X, \mathcal{S})\) pseudometrizable implies \(\hat{(X, \mathcal{S}_w)}\) is pseudometrizable" must be false.
II.2.13. Remark. In order to discuss hypercompleteness we will briefly examine convergence of nets in hyperproximity spaces. The concepts introduced here were used in uniform spaces by Isbell [14].

II.2.14. Lemma. Suppose \((X, \delta)\) is a p-space and \((T, D)\) is a net ranging in \(X\). Suppose \(A \subseteq X\). Then \((T, D)\) converges to \(A\) relative to \(\hat{\delta}\) iff for all \(P \in \mathcal{P}(\delta)\) there is a \(d^* \in D\) such that \(d \geq d^*\) implies \((A, T(d)) \in H(P)\).

Proof. Since \(\hat{\delta} = \sup\{\delta(\hat{u}): \hat{u} \in \pi(\delta)\}\), \(\pi(\hat{\delta})\) is sup\(\{\pi(\delta(\hat{u})): \hat{u} \in \pi(\delta)\}\) by I.2.12. Suppose \((T, D)\) converges to \(A\) relative to \(\hat{\delta}\). Then \((T, D)\) converges to \(A\) relative to \(\delta(\hat{u})\) for all \(\hat{u} \in \pi(\delta)\). Given \(P \in \mathcal{P}(\delta)\), there is a \(\hat{u} \in \pi(\delta)\) such that \(P \in \mathcal{Q}_\hat{u}\). Thus \(H(P) \in \mathcal{Q}_\hat{u}\) and \(H(P)[A]\) is a \(\delta(\hat{u})\) neighborhood of \(A\). Then there is a \(d^* \in D\) such that \(d \geq d^*\) implies \(T(d) \in H(P)[A]\) or that \((A, T(d)) \in H(P)\).

Since \(A \in \mathcal{Q} = \pi(\hat{\delta})\) iff there is a finite collection \(\{Q_1\}\) such that \(Q_1 \subseteq \pi(\hat{u}_1)\) and \(A \subseteq \cap \cap \subseteq \mathcal{Q}\), \(A \subseteq \mathcal{Q} \subseteq \pi(\hat{\delta})\) iff there is a finite collection \(\{U_1 \subseteq \mathcal{Q}_1: U_1 \subseteq \pi(\delta_1)\}\) and \(A \subseteq \cap \mathcal{H}(U_1)[A] \subseteq \mathcal{Q}\). Then assuming that for all \(P \in \mathcal{P}(\delta)\) there is a \(d^* \in D\) such that \(d \geq d^*\) implies \((A, T(d)) \in H(P)\), we claim that \((T, D)\) converges to \(A\). For, \(A \in \mathcal{Q} \subseteq \pi(\hat{\delta})\) implies that there is a finite collection \(\{U_1\}\), \(U_1 \subseteq \mathcal{Q}_1\), and hence \(U_1 \subseteq \mathcal{P}(\delta)\), such that \(\cap \mathcal{H}(U_1)[A] \subseteq \mathcal{Q}\). For each \(i\) let \(d^*_i\) be such that \(d \geq d^*_i\) implies \((A, T(d)) \in H(U_i)\). Letting \(d^* \geq d^*_i\) for all \(i\), \(d \geq d^*\) implies \(T(d) \in H(U_i)[A]\).
for all \( i \), and hence \( d \geq d^* \) implies \( T(d) \in Q \). Thus \( (T, D) \) converges to \( A \).

**II.2.15. Lemma.** Suppose \( (T, D) \) is a net ranging in \( \hat{\mathcal{S}}_D \) and \( (X, \mathcal{S}) \) is a p-space. Then if \( (T, D) \) converges to \( A \) in \( \hat{\mathcal{S}}_D \) then \( (T, D) \) also converges to \( cA \) in \( \hat{\mathcal{S}}_D \).

**Proof.** We will give the proof for \( \hat{\mathcal{S}}_D \). The proof for \( \hat{\mathcal{S}}_w \) is similar using \( U \in \mathcal{U}(\mathcal{S}) \) instead of \( P \in \mathcal{P}(\mathcal{S}) \).

Let \( P \in \mathcal{P}(\mathcal{S}) \) and \( P^* \in \mathcal{P}(\mathcal{S}) \) symmetric such that \( P^* \supset P \). Then, by II.2.14, there is a \( d^* \in D \) such that \( d \geq d^* \) implies \( (T(d), A) \in H(P^*) \). We claim that \( (T(d), cA) \) is in \( H(P) \) for all \( d \geq d^* \). Clearly \( T(d) \in P^*[A] \) for all \( d \geq d^* \) implies that \( T(d) \in P[cA] \) for all \( d \geq d^* \). Now \( cA \in P[T(d)] \) for all \( d \geq d^* \). For, \( x \in cA \) implies \( P^*[x] \cap A \neq \emptyset \). But then \( x \in P^*[A] \subseteq P^*[P^*[T(d)]] \subseteq P[T(d)] \). Thus \( (T(d), cA) \in H(P) \) for all \( d \geq d^* \). It then follows that \( (cA, T(d)) \in H(P) \) for all \( d \geq d^* \) and hence \( (T, D) \) converges to \( cA \) by II.2.14.

**II.2.16. Definition.** Suppose \( (X, \mathcal{S}) \) is a p-space and \( (T, D) \) is a net ranging in \( \hat{\mathcal{S}}_D \). Define \( C(T, D) \) to be

\[
\{ x : x \in 0 \in \mathcal{U}(\mathcal{S}) \text{ implies that for each } d \in D \text{ there is a } d^* \geq d, d^* \in D, \text{ such that } T(d^*) \cap 0 \neq \emptyset \}.
\]

**II.2.17. Theorem.** Suppose \( (T, D) \) converges in \( \hat{\mathcal{S}}_D \) (or \( \hat{\mathcal{S}}_w \)), then \( (T, D) \) converges to \( C(T, D) \) in \( \hat{\mathcal{S}}_D \) (or \( \hat{\mathcal{S}}_w \)).

**Proof.** Again we will give the proof for \( \hat{\mathcal{S}}_D \) leaving the other proof for the reader. Suppose \( (T, D) \) converges to \( A \) where \( A \subseteq X \). Then \( A \neq \emptyset \) unless \( (T, D) \) is eventually \( \emptyset \) since \( H(P)[\emptyset] = \{ \emptyset \} \). Note that in this case \( C(T, D) = \emptyset \).
If \( A \neq \emptyset \), we will show that \( cA = C(T,D) \) and hence, by II.2.15, \((T,D)\) converges to \( C(T,D) \). Let \( x \in cA \) and let \( P \in \wp(\mathcal{S}) \) be symmetric. We claim that, given any \( d \in D \) there is a \( d^* \supseteq d \) such that \( T(d^*) \cap P[x] \neq \emptyset \). There is a \( d_0 \in D \), \( d_0 \supseteq d \), such that \( d^* \supseteq d_0 \) implies that \( cA \subseteq P[T(d^*)] \), by II.2.15, and hence \( x \in P[T(d')] \) or \( P[x] \cap T(d') \neq \emptyset \). Choose \( d^* \supseteq d_0 \) to complete this part of the proof showing that \( cA \subseteq C(T,D) \).

Now, suppose \( x \in C(T,D) \), and let \( P \in \wp(\mathcal{S}) \). Let \( P^* \) be symmetric and \( P^* \cap P^* \subseteq P \). By II.2.14 there is a \( d_0 \in D \) such that \( d \supseteq d_0 \) implies \( T(d) \subseteq P^*[A] \). There is a \( d^* \supseteq d_0 \) such that \( T(d^*) \cap P^*[x] \neq \emptyset \). Then \( x \in P^*[T(d^*)] \subseteq P^*[P^*[A]] \) and thus \( x \in P[A] \) which implies \( x \in cA \).

II.2.18. Definition. A p-space \((X, \mathcal{S})\) will be called \( w\)-hypercomplete iff \((X, \mathcal{S}_w)\) is p-complete. \((X, \mathcal{S})\) will be called \( s\)-hypercomplete iff \((X, \mathcal{S}_s)\) is p-complete.

II.2.19. Theorem. Suppose the p-space \((X, \mathcal{S})\) is \( w\)-hypercomplete. Then \((X, \mathcal{S})\) is \( s\)-hypercomplete.

Proof. Suppose \((T,D)\) is p-Cauchy in \((X, \mathcal{S}_s)\). Then \( \mathcal{S}_w \subseteq \mathcal{S}_s \) implies by I.2.2 that \( \wp(\mathcal{S}_w) \subseteq \wp(\mathcal{S}_s) \) and thus \((T,D)\) is p-Cauchy in \((X, \mathcal{S}_w)\). Hence, by II.2.17 and hypothesis, \((T,D)\) converges to \( C(T,D) \) in \( \mathcal{S}_w \). We claim that \((T,D)\) converges to \( C(T,D) \) in \( \mathcal{S}_s \). Let \( P \in \wp(\mathcal{S}) \). Then there is a \( U \in \wp(\mathcal{S}) \) such that \( U[C(T,D)] \subseteq P[C(T,D)] \). For this \( U \), there is a \( d' \in D \) such that \( d \supseteq d' \) implies that \( T(d) \subseteq U[C(T,D)] \) and hence for all \( d \supseteq d' \), \( T(d) \subseteq P[C(T,D)] \).
We now claim that there is a $d'' \in D$ such that $d \geq d''$ implies $C(T,D) \subseteq P[T(d)]$. Let $P^* \in \mathcal{P}(S)$ be symmetric such that $P^* \circ P^* \subseteq P$. Since $(T,D)$ is $\hat{\delta}_s$ $p$-Cauchy there is a $d'' \in D$ such that $d_1$ and $d_2 > d''$ implies $(T(d_1), T(d_2)) \in H(P^*)$. Suppose $x \in C(T,D)$. Then there is a $d_{p*} > d''$ such that $P^*[x] \cap T(d_{p*}) \neq \emptyset$. Then $x \in P^*[T(d_{p*})]$. Since $T(d_{p*})$ is a subset of $P^*[T(d)]$ for all $d \geq d''$, we have that $x$ is in $P^*[P^*[T(d)]]$ for all $d \geq d''$. Thus for all $d \geq d''$, $C(T,D)$ is contained in $P[T(d)]$.

Choose $d^* > d'$ and $d''$. Then, for all $d \geq d^*$, we have $T(d) \subseteq P[C(T,D)]$ and $C(T,D) \subseteq P[T(d)]$. But this implies that $(T(d), C(T,D)) \in H(P)$ and thus $(T,D)$ converges to $C(T,D)$ in $\hat{\delta}_s$, by II.2.14.

**II.2.20. Theorem.** Suppose $(X, S)$ is $s$-hypercomplete. Then $(X, S)$ is $p$-complete.

**Proof.** Suppose $(T,D)$ is $p$-Cauchy in $(X, S)$. Then $(1 \circ T,D)$ is a $p$-Cauchy net in $(i[X], \hat{\delta}_s | i[X])$ and hence in $(X, \hat{\delta}_s)$ by II.2.2. Then there exists $A \subseteq X$ such that $(1 \circ T,D)$ converges to $A$ in $\hat{\delta}_s$. Since $1(T(d)) \neq \emptyset$ for any $d \in D$, $A \neq \emptyset$. Let $a \in A$. We claim that $(T,D)$ converges to $a$. Let $P \in \mathcal{P}(S)$. Then there is a $d* \in D$ such that $d \geq d*$ implies $(1(T(d)), A) \in H(P)$. But then for $d \geq d^*$, $T(d) \subseteq P[A]$ and $A \subseteq P[T(d)]$. Hence $(T(d), a) \in P$ and $(T,D)$ converges to $a$.

**II.2.21. Corollary.** Suppose $(X, S)$ is $w$-hypercomplete. Then $(X, S)$ is $p$-complete.
Proof. Using II.2.19 and II.2.20 we arrive at the conclusion.

II.2.22. Theorem. Suppose \((X, \delta)\) is a pseudometrizable p-space. Suppose \((X, \delta)\) is p-complete. Then \((X, \delta)\) is s-hypercomplete.

Proof. Suppose \(d\) is the pseudometric such that \(\delta(d)\) is \(\delta\). From I.6.17 we know that \((X, \delta)\) is p-complete iff the pseudometric space \((X, d)\) is complete. From Caulfield's Thesis [7] we have "\((X, U)\) complete and pseudometrizable implies that \((\hat{X}, U)\) is complete". Using II.1.10 we have \(\varphi(\hat{\delta}_S) = \hat{\varphi(\delta)}\). Since the uniform space \((\hat{X}, \varphi(\delta))\) is complete, and \(\delta(\varphi(\delta)) = \hat{\delta}_S\) by II.1.9, \((\hat{X}, \hat{\delta}_S)\) is p-complete by I.6.20.

II.2.23. Remark. Although we know from II.2.12 that \((\hat{X}, \hat{\delta}_w)\) need not be pseudometrizable when \((X, \delta)\) is, this example does not show whether or not the weak hyperproximity space of a complete pseudometrizable p-space must be p-complete. The next example shows that \((\hat{X}, \hat{\delta}_w)\) need not be p-complete if \((X, \delta)\) is a p-complete pseudometrizable p-space.

II.2.24. Example. We consider the space \((N, \delta_M)\). Since this space is discrete, it is pseudometrizable and complete. We claim that \((\hat{N}, (\hat{\delta}_M)_w)\) is not p-complete. We define a sequence \((T, N)\) ranging in \(\hat{N}\) as follows:

\[ T(n) = \{m \in N: m \geq n\}. \]
We first show that \((T,N)\) does not converge in \((\hat{N}, (\hat{\delta}_M)_w)\). From II.2.17 we know that, if \((T,N)\) converges, it converges to \(G(T,N)\). Since \(x \in \sigma(T(\delta_M))\) for all \(x \in N\), \(x \in G(T,N)\) iff \(x \in T(n_j)\) for every \(n_j\) in \(\{n_j\}\) some subsequence of \(N\). Since, given any \(x \in N\), \(x\) is only in a finite number of \(T(n)\)'s, we can conclude that \(G(T,N) = \emptyset\). But \((T,N)\) converges to \(\emptyset\) implies that for each \(U \in \mathcal{U}(\delta_M)\) there is an \(n_0 \in N\) such that \(T(n) \subseteq U[\emptyset] = \emptyset\) for all \(n > n_0\). Since \(T(n) = \emptyset\) for no \(n \in N\), \((T,N)\) does not converge.

We next show that \((T,N)\) is \(p\)-Cauchy in \((\hat{N}, (\hat{\delta}_M)_w)\). From I.6.9 and II.1.2 it is sufficient to show that \((T,N)\) is Cauchy in \(\mathcal{U}(\hat{\delta})\). Let \(U \in \mathcal{U}(\delta_M)\). Then there is a \(p\)-cover \(\{B_i: 1 \leq i \leq n\}\) of \(N\) such that \(\bigcup B_i \times B_i: 1 \leq i \leq n \subseteq U\). Since \(B_i \subseteq B_1\) for any \(B_i \subseteq N\), we can say: there is a finite cover \(\{B_i\}\) of \(N\) such that \(\bigcup B_i \times B_i \subseteq U\). Assume that the indexing of the \(B_i\)'s is such that for \(1 \leq i \leq j^*\) each \(B_i\) is infinite and for \(j^* < i \leq n\) \(B_i\) is finite. Let \(m^* = \max\{x: x \in B_i\text{ for some }1, j^* < i \leq n\} + 1\). Then we claim that, for \(m_1, m_2 > m^*,\) \((T(m_1), T(m_2)) \in H(U)\). Let \(m_1, m_2 > m^*\).

Then we claim that \(T(m^*) \subseteq U[T(m_1)]\) for \(i = 1, 2\). Then since \(T(m_j) \subseteq T(m^*)\) for \(m_j > m^*,\) \(T(m_j) \subseteq U[T(m_1)]\) for \(i, j = 1, 2\), and hence \((T(m_1), T(m_2)) \in H(U)\). To see that \(T(m^*) \subseteq U[T(m_1)]\) let \(k \in T(m^*)\). Then \(k \in B_{i^*}\) for some \(i^*\) such that \(1 \leq i^* \leq j^*\). Since \(k > m^*\). Now \(T(m_1) \cap B_{i^*}\) must contain some \(k_0\), for if not \(B_{i^*} \subseteq \{n: 1 \leq n \leq m_1\}\) and \(B_{i^*}\) would then be finite. Thus
which implies that \((k_0, k) \in B_1^* \times B_1^* \subseteq U\). Hence \(k \in U[k_0] \subseteq U[T(m_1)]\).

**II.2.25. Remark.** The previous example also serves to illustrate the fact that \(s\)-hypercompleteness is not equivalent to \(w\)-hypercompleteness since \((N, (\hat{\mathcal{M}})_s^*)\) is \(p\)-complete by II.2.22.

**II.2.26. Theorem.** Suppose \((X, \mathcal{S})\) is a \(p\)-space. Then \((\hat{X}, \hat{\mathcal{S}}_w^*) = (\hat{X}, \hat{\mathcal{S}}_s^*)\) and they are compact iff \((X, \mathcal{S})\) is compact.

**Proof.** Suppose \((X, \mathcal{S})\) is compact. Then \(\mathcal{U}(\mathcal{S}) = \mathcal{T}(\mathcal{S})\) and \((X, \mathcal{U}(\mathcal{S}))\) is complete. \(\mathcal{U}(\mathcal{S}) = \mathcal{T}(\mathcal{S})\) implies that \(\hat{\mathcal{S}}_w^* = \hat{\mathcal{S}}_s^*\). \((\hat{X}, \mathcal{U}(\mathcal{S}))\) is compact from 0.4.5. Hence \(\mathcal{S}(\mathcal{U}(\mathcal{S})) = \mathcal{S}_w^* = \mathcal{S}_s^*\) is compact.

Suppose \((\hat{X}, \hat{\mathcal{S}}_w^*) = (\hat{X}, \hat{\mathcal{S}}_s^*)\) is compact. Then by II.2.8 \((X, \mathcal{S})\) is \(p\)-totally bounded and by II.2.21 \((X, \mathcal{S})\) is \(p\)-complete and therefore \((X, \mathcal{S})\) is compact by I.7.8.

**II.3 Proximal Quotients of \(p\)-Complete Spaces**

**II.3.0. Remark.** It is well known that under certain conditions the uniform quotient of a complete metric space is complete (see [16], page 203). Examples exist of complete uniform spaces and quotient maps for which the image is not complete. In this section, using an example of Ginsburg and Isbell [12], we will show that the \(p\)-quotient of a \(p\)-complete \(p\)-space need not be complete and give some sufficient conditions for the \(p\)-quotient of a \(p\)-complete \(p\)-space to be \(p\)-complete.
II.3.1. Example. In [12] Ginsburg and Isbell give an example of a uniform space \((X, \mathcal{U})\), a uniform space \((Y, \mathcal{U}')\), and a map \(f: X \to Y\) which is a uniform quotient map with the following properties:

1. \((X, \mathcal{U})\) is complete,
2. \((Y, \mathcal{U}')\) is not complete,
3. \(\mathcal{U}'\) is the only uniformity compatible with \(\mathcal{U}(\mathcal{U}')\).

From I.3.16 we can conclude that \(f: (X, \mathcal{S}(\mathcal{U})) \to (Y, \mathcal{S}(\mathcal{U}'))\) is a \(p\)-quotient map. From (2) and (3) above we can conclude that \(\mathcal{P}(\mathcal{S}(\mathcal{U}')) = \mathcal{U}'\) and hence \((Y, \mathcal{S}(\mathcal{U}'))\) is not \(p\)-complete although I.6.20 tells us that \((X, \mathcal{S}(\mathcal{U}))\) is \(p\)-complete.

II.3.2. Definition. A function \(f: (X, \mathcal{S}) \to (X^*, \mathcal{S}^*)\) which is onto will be called \(w\)-proximally open iff the function \(f: (X, \mathcal{S}\mathcal(U)) \to (X^*, \mathcal{S}\mathcal(U)(\mathcal{S}^*))\) is uniformly open (see 0.2.7).

II.3.3. Remark. Suppose \(X\) and \(X^*\) are sets, \(f \in [X,X^*]\). Then \(f^{-1}\) generally denotes a subset of \(X^* \times X\), and \(f^{-1}(y)\) is really \(f^{-1}[y] = \{x: (y,x) \in f^{-1}\}\). But \(f^{-1}\) can also be thought of as a map \(f^{-1}: X^* \to \hat{X}\) where \(f^{-1}(x^*) = \{x: f(x) = x^*\}\). If \(f \in [X,X^*]\) we will denote by \(\hat{f}\) the element of \([X^*, \hat{X}]\) usually denoted by \(f^{-1}\).

II.3.4. Theorem. Suppose \(f \in [X,X^*]\) is onto and \((X, \mathcal{S})\) and \((X^*, \mathcal{S}^*)\) are \(p\)-spaces. Then \(f \in p[(X^*, \mathcal{S}^*),(\hat{X}, \mathcal{S}_w)]\) iff \(f\) is \(w\)-proximally open.
Proof. We know that $f \in p[(X^*, \delta^*), (\hat{X}, \hat{\delta}_w)]$ iff $f \in u[(X^*, \mathcal{U}(\delta^*), (\hat{X}, \mathcal{U}(\hat{\delta}_w))]$ by 0.3.9 and 0.3.12. Suppose $f$ is $p$-continuous. Then given $U \in \mathcal{U}(\hat{\delta}_w), (fxf)^{-1}[U] \in \mathcal{U}(\delta^*)$. Let $U \in \mathcal{U}(\delta)$ and let $U' \subseteq U$, $U'$ symmetric. Then $H(U')$ is in $\mathcal{U}(\hat{\delta}_w)$ and therefore $U^* = (fxf)^{-1}[H(U')] \in \mathcal{U}(\delta^*)$. We claim that $U^*[f(x)] \subseteq f[U[x]]$ for all $x \in X$. For, let $x \in X$ and $x^* \in U^*[f(x)]$. Then $(f(x), x^*) \in U^*$ and thus we have $(f(f(x)), f(x^*)) \in H(U')$. Then $f^{-1}(x^*) \subseteq U^*[f^{-1}f(x)]$ and $f^{-1}f(x) \subseteq U'[f^{-1}(x^*)]$. Then $x \in f^{-1}f(x)$ implies that there is a $y \in f^{-1}(x^*)$ such that $(y, x) \in U'$. Then $(x, y) \in U' \subseteq U$ and $y \in U[x]$. Hence $f(y) = x^* \in f[U[x]]$.

Conversely, suppose $f$ is $w$-proximally open. Let $U \in \mathcal{U}(\delta)$. We claim that $(fxf)^{-1}[H(U)] \in \mathcal{U}(\delta^*)$. Let $U' \in \mathcal{U}(\delta)$ be symmetric and $U' \subseteq U$. Then there is a $U^*$ symmetric in $\mathcal{U}(\delta^*)$ such that $U^*[f(x)] \subseteq f[U'[x]]$ for all $x \in X$. We will show that $U^* \subseteq (fxf)^{-1}[H(U)]$. Suppose $(x^*, y^*) \in U^*$. Then $y^* \in U^*[f(x)]$ for all $x \in f^{-1}(x^*)$. Thus $y^* \in f[U'[x]]$ for all $x \in f^{-1}(x^*)$. Hence for all $x \in f^{-1}(x^*)$ there is a $y \in f^{-1}(y^*)$ such that $(x, y)$ and hence $(y, x) \in U'$. Then $f^{-1}(x^*) \subseteq U'[f^{-1}(y^*)]$. Using a symmetric argument we can show that $f^{-1}(y^*) \subseteq U'[f^{-1}(x^*)]$. Hence $(f(x^*), f(y^*))$ is in $H(U') \subseteq H(U)$. Thus $(f^{-1}f(x^*), f^{-1}f(y^*)) = (x^*, y^*)$ is an element of $(fxf)^{-1}[H(U)]$. 
II.3.5. Corollary. Suppose $f \in [X, X^*]$ is onto where $(X, \mathcal{S})$ and $(X^*, \mathcal{S}^*)$ are $p$-spaces. Suppose $f : (X, \mathcal{S}) \rightarrow (X^*, \mathcal{S}^*)$ is $w$-proximally open and $p$-continuous. Then $\mathcal{S}^*$ is the proximity generated by the function $f$ on $X^*$ (see I.3.9).

Proof. By I.3.8, we must show that for $A^*, B^* \subseteq X^*$, $A^* \mathcal{S}^* B^*$ iff there are sets $A, B \subseteq X$ such that $A$ is $\mathcal{S}_w$ far from $\varnothing$ and $A^* \subseteq f^{-1}[A]$, $B^* \subseteq f^{-1}[B]$.

Since $f$ is $p$-continuous from the previous theorem, we know that $\varnothing, A \subseteq X, A \mathcal{S}_w$ far from $\varnothing$ and $A^* \subseteq f^{-1}[A]$, $B^* \subseteq f^{-1}[B]$ implies that $A^* \mathcal{S}^* B^*$.

To show the converse, let $A^* \mathcal{S}^* B^*$. We claim that $f[A^*]$ is $\mathcal{S}_w$ far from $f[B^*]$. If $A^* \mathcal{S}^* B^*$ then there is a $U^* \in \mathcal{U}(\mathcal{S}^*)$ such that $U^*[A^*] \cap U^*[B^*] = \varnothing$. Since $f$ is $p$-continuous, $(fxf)^{-1}[U^*] = U \in \mathcal{U}(\mathcal{S})$. We claim that $H(U)f[A^*] \cap H(U)f[B^*] = \varnothing$. For suppose $C \subseteq X$ such that $C$ is an element of both of the above. Then there exists $a^* \in A^*, b^* \in B^*$ such that $C \subseteq U[f^{-1}(a^*)] \cap U[f^{-1}(b^*)]$ and $f^{-1}(a^*), f^{-1}(b^*) \subseteq U[C]$. If $C \subseteq U[f^{-1}(a^*)] \cap U[f^{-1}(b^*)]$ then $f[C] \subseteq U*[a^*] \cap U*[b^*]$ which implies that $C = \varnothing$. But $C = \varnothing$ implies that $f^{-1}(a^*), f^{-1}(b^*) \subseteq U[\varnothing] = \varnothing$. Since $f$ is onto, this is impossible and so $f[A^*]$ is $\mathcal{S}_w$ far from $f[B^*]$.

II.3.6. Theorem. Suppose $f : (X, \mathcal{S}) \rightarrow (X^*, \mathcal{S}^*)$ is onto, $w$-proximally open, and $p$-continuous. Then $f$ is a $p$-quotient map.
Proof. We use the characterization of \( p \)-quotient maps given in I.3.12. Let \( (Y, S') \) be any \( p \)-space and \( g \) any element of \([X^*, X]\). We must show that the \( p \)-continuity of \( g \) implies the \( p \)-continuity of \( g \circ f \).

Suppose \( V \in \mathcal{U}(S') \). Then \( U = ((g \circ f)x(g \circ f))^{-1}[V] \) is in \( \mathcal{U}(S) \). Since \( f \) is \( w \)-proximally open there is a \( U^* \) in \( \mathcal{U}(S^*) \) such that \( U^*[f(x)] \subseteq f[U[x]] \) for all \( x \in X \). We claim that \( U^* \) is contained in \((fxf)[U] \) and hence \((g \circ xg)^{-1}[V] \) is in \( \mathcal{U}(S^*) \). Let \((x^*, y^*) \in U^* \). Then \( x^* = f(x) \) for some \( x \in X \) and thus \( y^* \in U^*[f(x)] \subseteq f[U[x]] \). Then there is a \( y \in X \) such that \( f(y) = y^* \) and \( y \in U[x] \). Then \((x, y) \in U \) and hence \((f(x), f(y)) \) which is \((x^*, y^*) \in (fxf)[U] \).

II.3.7. Theorem. Suppose \((X, S) \) is \( w \)-hypercomplete. Suppose \( f: (X, S) \to (X^*, S^*) \) is onto, \( w \)-proximally open, and continuous. Then \((X^*, S^*) \) is \( p \)-complete.

Proof. Suppose \((T, D) \) is a \( p \)-Cauchy net in \((X^*, S^*) \).
Then from II.3.4 and I.6.3, \((f \circ T, D) \) is a \( p \)-Cauchy net in \((X^*, S_w^* \)). Then \((X, S) \) \( w \)-hypercomplete implies that there is a \( C, C = C(f \circ T, D) \subseteq X \) such that \((f \circ T, D) \) converges to \( C \) in \( S_w \). Using the fact that \( f \) is onto, \( f(T(d)) \neq \emptyset \) for all \( d \in D \) and hence \( C \neq \emptyset \). Letting \( c \in C \) we claim that \((T, D) \) converges to \( f(c) \). Let \( U^* \in \mathcal{U}(S^*) \) and let \( U^* \) be open and symmetric such that \( U^* \circ U^* \subseteq U^* \). Then \(((fxf)^{-1}[U^*]) [c] \) is open in \( \mathcal{T}(\mathcal{S}) \) since \( f \), and therefore \( fxf \), is continuous. Then, since \( c \in C(f \circ T, D) \), given \( d \in D \) there is a \( d^*(d) \geq d \).
and an $x \in X$ such that $\{x\} \subseteq ((fx) \cap [f(T(d^*))]) \cap f(T(d^*))$. Then $(f(o), f(x)) = (f(o), T(d^*)) \in U_o^*$. Since $(T, D)$ is $p$-Cauchy there is a $d(U_o^*) \in D$ such that $d_1$ and $d_2 > d(U_o^*)$ implies $(T(d_1), T(d_2)) \in U_o^*$. Then we have for all $d > d(U_o^*)$, $(f(o), T(d^*(d(U_o^*))))$ and $(T(d^*(d(U_o^*))), T(d)) \in U_o^*$ and therefore, $(f(o), T(d)) \in U^*$. Thus $(T, D)$ converges to $f(o)$.

**II.3.8. Remark.** By adding the condition of $p$-continuity of $f$ in the above theorem we have a sufficient condition for the quotient of a $p$-complete $p$-space to be $p$-complete. It is possible to prove theorems similar to those above for mappings which we will call $s$-proximally open maps.

**II.3.9. Definition.** A function $f: (X, S) \rightarrow (X^*, S^*)$ which is onto will be called $s$-proximally open iff given $P \in \mathcal{P}(S)$ there is a $P^* \in \mathcal{P}(S^*)$ such that $P^*[f(x)] \subseteq f[P[x]]$ for all $x \in X$.

**II.3.10. Remark.** We note that for total $p$-spaces $s$-proximally open is equivalent to saying $f$ is uniformly open relative to the uniformities $\mathcal{P}(S)$ and $\mathcal{P}(S^*)$.

**II.3.11. Theorem.** Suppose $(X, S)$ and $(X^*, S^*)$ are $p$-spaces and $f \in [X, X^*]$, $f$ onto. Then $f \in p[(X^*, S^*), (X, \hat{S}_S)]$ iff $f$ is $s$-proximally open.

**Proof.** Suppose $f \in p[(X^*, S^*), (X, \hat{S}_S)]$. Let $P \in \mathcal{P}(S)$. Then there is a $\mathcal{Q} \in \mathcal{U}(S)$ such that $P \in \mathcal{Q}$. Let $U \in \mathcal{Q}$ be symmetric such that $U \subseteq P$. By II.1.14, $U \in \mathcal{Q} \subseteq \mathcal{P}(S)$ implies $H(U) \subseteq \mathcal{P}(\hat{S}_S)$. From I.1.14, $f$ proximally continuous implies
$U^* = (f \cdot f)^{-1}[H(U)] \in \wp(\delta^*)$. An argument similar to that given in the first paragraph of the proof of II.3.4 shows that $U^*[f(x)] \subseteq f[P[x]]$. Thus $f$ is $s$-proximally open.

Conversely, suppose $f$ is $s$-proximally open. Let $A^*, B^* \subseteq X^*$ such that $A^* \in \delta^* B^*$. We will show that $f[A^*] \in \delta f[B^*]$. Suppose $\{Q_1\}$ and $\{Q_j\}$ are finite covers from below of $f[A^*]$ and $f[B^*]$ respectively. Then $f^{-1}[Q_1]$ and $f^{-1}[Q_j]$ are finite covers from below of $A^*$ and $B^*$. Then there is an $i^*$ and a $j^*$ such that $f^{-1}[Q_{i^*}] \in \delta f^{-1}[Q_{j^*}]$. Let $U \in \wp \in \wp(\delta)$ and let $U' \in \wp$ be symmetric such that $U' \subseteq U$. Then there is a symmetric $P^* \in \wp(\delta^*)$ such that $P^*[f(x)]$ is contained in $f[U'[x]]$ for all $x \in X$. Since $P^* \in \wp(\delta^*)$, $P^*[f^{-1}[Q_{i^*}]] \cap P^*[f^{-1}[Q_{j^*}]] \neq \emptyset$. Using an argument similar to the one in II.3.4, $P^* \subseteq (f \cdot f)^{-1}[H(U)]$. Then letting $a^* \in f^{-1}[Q_{i^*}]$, $b^* \in f^{-1}[Q_{j^*}]$, $x^* \in X^*$ such that $(a^*, x^*)$ and $(b^*, x^*) \in P^*$, $(f(a^*), f(x^*))$ and $(f(b^*), f(x^*)) \in H(U)$. Thus $H(U)[Q_{i^*}] \cap H(U)[Q_{j^*}] \neq \emptyset$ for all $U \in \wp$, for all $U \in \wp(\delta)$. Then $Q_{i^*} \in \delta(\wp) Q_{j^*}$ for all $U \in \wp(\delta)$ and hence we have $f[A^*] \in \delta f[B^*]$.

II.3.12. Corollary. Suppose $(X, \delta)$ and $(X^*, \delta^*)$ are $p$-spaces, $f \subseteq [X, X^*]$, and $f$ is onto. Suppose $f: (X, \delta) \rightarrow (X^*, \delta^*)$ is $s$-proximally open and $p$-continuous. Then $\delta^*$ is the proximity generated by the function $f$ on $X^*$.

Proof. From II.3.11, $f$ is $p$-continuous. Then $a, b \in \hat{X}$, $a \in \delta^* \in \wp(\delta)$, and $A^* \subseteq f^{-1}[a]$, $B^* \subseteq f^{-1}[b]$ implies that $A^* \in \delta^* B^*$. 
Now suppose $A^* \not\subseteq B^*$. We will show that $f[A^*]$ is far from $f[B^*]$. If $A^* \not\subseteq B^*$ then there is a $P^* \in \mathcal{P}(S^*)$ such that $P^*[A^*] \cap P^*[B^*] = \emptyset$. Since $f$ is $p$-continuous, $P = (fxf)^{-1}[P^*] \in \mathcal{P}(S)$. Thus there is a $U \subseteq \tau(S)$ such that $P \subseteq U$. We claim that $f[A^*] \subseteq (\bar{U}) f[B^*]$ and thus $f[A^*]$ is far from $f[B^*]$. To see this we claim that $H(P)[f[A^*]] \cap H(P)[f[B^*]] = \emptyset$. The proof of this fact is the same as the proof of the similar fact given in II.3.4.

II.3.13. Theorem. Suppose $f:(X,S) \to (X^*,S^*)$ is onto, $p$-continuous, and $s$-proximally open. Then $f$ is a $p$-quotient map.

Proof. The proof here is exactly like the proof of II.3.6, replacing $\mathcal{U}(S)$ and $\mathcal{U}(S^*)$ with $\mathcal{P}(S)$ and $\mathcal{P}(S^*)$.

II.3.14. Theorem. Suppose $(X,S)$ is $s$-hypercomplete. Suppose $f:(X,S) \to (X^*,S^*)$ is onto, continuous, and $s$-proximally open. Then $(X^*,S^*)$ is $p$-complete.

Proof. In light of I.1.12, we can repeat the proof of II.3.7 replacing $\mathcal{U}(S)$ and $\mathcal{U}(S^*)$ with $\mathcal{P}(S)$ and $\mathcal{P}(S^*)$.

II.3.15. Remark. We note that II.3.7 and II.3.14 are direct generalizations of Corollary 37, Chapter 6, of Kelley [16].
11.3.16. Theorem. Suppose \( f \in [X, X^*] \) is onto and \((X, S)\) and \((X^*, S^*)\) are \(p\)-spaces. Suppose \( f \) is \(s\)-proximally open. Then \( f \) is \(w\)-proximally open.

Proof. If \( f \) is \(s\)-proximally open then by II.3.11,
\[ f \in p[(X^*, S^*), (X, S)] . \]
Thus \( A^* S^* B^* \) implies that
\[ f[A^*] \supseteq f[B^*] \]. Since \( S^*_w \subseteq S^*_s \) by II.1.16, we then have
\[ f[A^*] \supseteq f[B^*] \]. But this is sufficient to show that
\[ f \in p[(X^*, S^*), (X, S)] \] which by II.3.4 is equivalent to
\( f \) being \(w\)-proximally open.

11.3.17. Remark. The question of whether or not the converse of II.3.16 is true is still open although we have
the following partial result.

11.3.18. Theorem. Suppose \((X, S)\) is discrete, \( f \) is in \([X, X^*]\) and onto, and \((X^*, S^*)\) is a \(p\)-space. Then \( f \) is \(w\)-proximally open iff \( f \) is \(s\)-proximally open.

Proof. That \(s\)-proximally open implies \(w\)-proximally open has already been mentioned in II.3.16.

Suppose \( f \) is \(w\)-proximally open and \( A^*, B^* \subseteq X^* \) such that
\[ f[A^*] \] is \(S^*_s\) far from \(f[B^*]\). Then letting \( A \) be the set
\[ \{ x : x \in f(a^*) \text{ for some } a^* \in A^* \} \] and \( B = \{ x : x \in f(b^*) \text{ for some } b^* \in B^* \} \), \( A \cap B = \emptyset \). For, if \( x \in A \cap B \) then \( f(x) \in f[A] = A^* \) and \( f(x) \in f[B] = B^* \). Then \( f(f(x)) \in f[A^*] \cap f[B^*] \) and thus
\[ f[A^*] \supseteq f[B^*] \]. Therefore \( A \cap B = \emptyset \) and \( A \cup B \). Then
there is a \( U \in \mathcal{U}(S) \) such that \( U[A] \cap U[B] = \emptyset \). We claim
that \( H(U)f[A^*] \cap f[B^*] = \emptyset \). Suppose \( b^* \in B^* \), \( a^* \in A^* \) and
and \((f(a^*), f(b^*)) \subseteq H(U)\). Then \(f(a^*) \subseteq U[f(b^*)] \subseteq U[B]\). Since \(f(a^*) \subseteq A\), we would then have \(U[A] \cap U[B] \supseteq f(a^*) \neq \emptyset\), since \(f\) is onto. Hence \(H(U)[f[A^*]] \cap f[B^*] = \emptyset\) and thus \(f[A^*]\) is \(\hat{S}_w\) far from \(f[B^*]\).

Now suppose \(A^* \supseteq B^*\). Then if \(f[A^*]\) is \(\hat{S}_g\) far from \(f[B^*]\) it will also be \(\hat{S}_w\) far from \(f[B^*]\) which is a contradiction since \(f \in p[(X^*, S^*), (X, \hat{S}_w)]\). Thus \(f\) will be in \(p[(X^*, S^*), (X, \hat{S}_g)]\) and hence \(s\)-proximally open by \(\text{II.3.11}\).
CHAPTER III

WEAK AND STRONG PROXIMITY CONSTRUCTIONS
FOR FUNCTION SPACES

III.1 Weak and Strong Proximities of Pointwise Convergence

III.1.0. In this section we discuss proximities of pointwise convergence for $[X, X^*]$. Throughout this chapter proximities of pointwise convergence will be genericly denoted by $\mathcal{Q}$. Hence $\mathcal{Q}_W$ will be the weak proximity of pointwise convergence, etc. Since the set $[X, X^*]$ and the set $\times\{X^*: x \in X\}$ are equal, it is possible to compare the proximities defined in 1.8 with those defined here and this will be done in this section. It is suggested that the reader refer back to 0.5 to review the definitions of the topologies and uniformities concerned.

III.1.1. Definition. Suppose $X$ is a non-empty set and $(X^*, S^*)$ is a $p$-space. Let $\mathcal{U}(p.c., \mathcal{U}(S^*))$ be as in (1) of 0.5.3. Define $\mathcal{Q}_W = S(\mathcal{U}(p.c., \mathcal{U}(S^*)))$. The proximity relation $\mathcal{Q}_W$ will be called the weak proximity of pointwise convergence for $[X, X^*]$.

III.1.2. Remark. Since $\mathcal{Q}_W$ is the proximity of $\mathcal{U}(p.c., \mathcal{U}(S^*))$ the following theorem is not surprising and needs little proof.
III.1.3. Theorem. Suppose X is a non-empty set and 
\((X^*, S^*)\) is a p-space. Then \(([X, X^*], \phi_w) = ([X, X^*], S_w)\)
which is \(\times_{x \in X^*} \{ (X^*, S^*) : x \in X \}\).

Proof. From 1.8.9, \(S_w\) is the proximity of the
uniform space \(\times \{ (X^*, \cup (S^*)) : x \in X \}\). It is well known that
\(\times \{ (X^*, \cup (S^*)) : x \in X \}\) = \(([X, X^*], \cup (p.c., \cup (S^*)))\). Since
\(\phi_w = \delta (\cup (p.c., \cup (S^*)))\) and \(\delta (\cup (p.c., \cup (S^*)) = S_w\), the
theorem is proved.

III.1.4. Definition. Suppose X is a non-empty set and
\((X^*, S^*)\) is a p-space. For each \(\cup* \in \Pi (S^*)\), define
\(\phi (\cup*)\) to be the proximity of the uniformity \(\cup (p.c., \cup*); \phi (\cup*) = \delta (\cup (p.c., \cup*))\). Define \(\phi_s\) to be the relation
\(\sup \{ \phi (\cup*); \cup* \in \Pi (S^*)\}\). The proximity relation \(\phi_s\)
will be called the strong proximity of pointwise convergence
for \([X, X^*]\).

III.1.5. Remark. We should mention several things
about III.1.4. First we note the obvious fact that \(\phi_w\)
is \(\phi (\cup (S^*))\). Second, our definition for \(\phi_s\) is a natural
definition in the sense that it is the proximity of the
uniform construction in the case of metric spaces (see
III.1.10). We also note that the analog of III.1.3 for
\(\phi_s\) is not generally true although we have certain partial
results as the following paragraphs show.
III.1.6. Definition. For a non-empty set $X$ and a $p$-space $(X^*, S^*)$ define $\varphi^*$, a proximity relation on $[X, X^*]$ as follows: $\varphi^* = \sup \{ \delta_v : v \in G \text{ and } v \text{ is a constant map} \}$ where $G$ and $v$ are as in I.8.7.

III.1.7. Lemma. For a non-empty set $X$ and a $p$-space $(X^*, S^*)$, $\delta_w \leq \varphi^* \leq S^*$.  
Proof. Since $\delta_v \leq \delta_S$ for all $v \in G$, we have that $\sup \{ \delta_v : v \in G \text{ and } v \text{ is constant} \} \leq \delta_S$. Also, $\delta_w = \delta_{\nu^0}$ where $\nu^0(x) = U(S^*)$ for all $x \in X$. Since $\nu^0 \in G$ and constant, $\delta_w \leq \varphi^*$. Therefore $\delta_w \leq \varphi^* \leq \delta_S$.

III.1.8. Theorem. Suppose $X$ is non-empty and $(X^*, S^*)$ is a $p$-space. Then $\varphi_S = \varphi^*$.  
Proof. We first note that for any $U^* \in \Pi(S^*)$, $\varphi(U^*)$ is $\delta_{\nu^*}$ where $\nu^* : X \rightarrow \Pi(S^*)$ and $\nu^*(x) = U^*$ for all $x \in X$. The proof of this statement is the same argument as in III.1.3.

Since $\varphi(U^*) \leq \delta_{\nu^*}$, $\varphi(U^*) \leq \sup \{ \delta_v : v \in G \text{ and } v \text{ is constant} \}$ for each $U^* \in \Pi(S^*)$. Therefore $\varphi_S \leq \varphi^*$. Also, $\delta_{\nu^*} \leq \varphi(U^*)$ implies $\delta_{\nu^*} \leq \sup \{ \varphi(U^*) ; U^* \in \Pi(S^*) \}$ for each constant $\nu^* \in G$ and hence $\varphi^* \leq \varphi_S$.

III.1.9. Remark. From the above we have that $\delta_w = \varphi_w \leq \varphi_S \leq \delta_S$. Under certain conditions we can show that $\varphi_S = \delta_S$.

III.1.10. Theorem. Suppose $X$ is a non-empty set and $(X^*, S^*)$ is total. Then $\varphi_S = S(U(p.c., \varphi(S^*))$. 
Proof. Since $\varphi(\mathcal{S}^*) \in \bigcap(\mathcal{S}^*)$, we have $\varphi(\varphi(\mathcal{S}^*))$ is smaller than $\varphi_s$. But $\varphi(\varphi(\mathcal{S}^*)) = \delta(\cup(p.o., \mathcal{P}(\mathcal{S}^*)))$ by definition.

We claim that $\varphi(\varphi(\mathcal{S}^*)) > \varphi_s$. Suppose $\mathcal{\mathcal{T}}$ and $\mathcal{J}$ are subsets of $[X, X^*]$ and $\mathcal{T} \varphi(\varphi(\mathcal{S}^*)) \mathcal{J}$. Let $\{\mathcal{T}_i\}$ and $\{\mathcal{J}_j\}$ be finite covers from below of $\mathcal{T}$ and $\mathcal{J}$ respectively. Then there is an $i^*$ and a $j^*$ such that $\mathcal{T}_{i^*}$ is $\varphi(\varphi(\mathcal{S}^*))$ near $\mathcal{J}_{j^*}$ by (2) of 0.3.1. Let $\mathcal{U}^* \in \bigcap(\mathcal{S}^*)$.

Let $\{x_k: 1 \leq k \leq n\} \subseteq X$. $U_{i_k}^* \in \mathcal{U}^*$ for $1 \leq k \leq n$. Then $U_{i_k}^* \in \varphi(\mathcal{S}^*)$ for $1 \leq k \leq n$. Therefore $W = \bigcup_{i_k} W(x_k, U_{i_k}^*) \subseteq \cup(p.o., \mathcal{P}(\mathcal{S}^*))$.

Hence $W[\mathcal{T}_{i^*}] \cap W[\mathcal{J}_{j^*}] \neq \emptyset$. But this implies that $\mathcal{T}_{i^*} \varphi(\mathcal{U}^*) \mathcal{J}_{j^*}$. Since this will be true for all $\mathcal{U}^*$ in $\bigcap(\mathcal{S}^*)$, $\mathcal{T} \varphi_s \mathcal{J}$. Hence $\varphi(\varphi(\mathcal{S}^*)) > \varphi_s$.

III.1.11. Corollary. Suppose $X$ is a non-empty set and $(X^*, \mathcal{S}^*)$ is total. Then $\varphi_s = \varphi^* = \mathcal{S}_s$.

Proof. From III.1.10, $\varphi_s = \delta(\cup(p.o., \mathcal{P}(\mathcal{S}^*)))$ which is $\varphi(\varphi(\mathcal{S}^*))$. From I.8.13, $\mathcal{S}_s = \delta(\{x: \varphi(\mathcal{S}^*): x \in X\})$.

But $\delta(\{x: \varphi(\mathcal{S}^*): x \in X\}) = \delta_{\nu^*}$ where $\nu^*: X \rightarrow \bigcap(\mathcal{S}^*)$ such that $\nu^*(x) = \varphi(\mathcal{S}^*)$, and $\delta_{\nu^*} = \varphi(\varphi(\mathcal{S}^*))$.

The conclusion then follows.

III.1.12. Theorem. Suppose $\text{card}(X) \geq 2$ and $(X^*, \mathcal{S}^*)$ is a p-space. Then $\varphi_w = \varphi_s$ iff $(X^*, \mathcal{S}^*)$ is p-totally bounded.

Proof. If $(X^*, \mathcal{S}^*)$ is p-totally bounded, then I.8.19 implies that $\mathcal{S}_w = \varphi_w \leq \varphi_s \leq \mathcal{S}_s = \mathcal{S}_w$ and hence $\varphi_w = \varphi_s$.

This is independent of the cardinality condition on $X$.\]
Suppose \((X^*, \mathcal{S}^*)\) is not \(p\)-totally bounded. Then there is a \(U^* \in \Pi(\mathcal{S}^*)\) such that \(U^*\) is not a totally bounded uniformity. Let \(x_1\) and \(x_2 \in X\) such that \(x_1 \neq x_2\). Let \(\gamma : X \to \Pi(\mathcal{S}^*)\) such that \(\gamma(x_1) = \gamma(x_2) = U^*\) and for all \(x \neq x_1, i = 1, 2, \gamma(x) = U(\mathcal{S}^*)\). An argument similar to the one used to prove I.8.19 shows that \(\mathcal{S}_w \neq \mathcal{S}_y\) and \(\mathcal{S}_w \neq \mathcal{S}_y^*\). Define \(\gamma^*: X \to \Pi(\mathcal{S}^*)\) such that \(\gamma^*(x) = U^*\) for all \(x \in X\). Then \(U_y \subset U_{\gamma^*}\) and hence \(\mathcal{S}_w \subset \mathcal{S}_y \subset \mathcal{S}_{\gamma^*}\) and therefore \(\mathcal{S}_w \neq \mathcal{S}_{\gamma^*}\). Since \(\Phi^* = \Phi_s > \mathcal{S}_{\gamma^*}\) \(\mathcal{S}_w \neq \Phi_s\) and hence the theorem is proved.

III.1.13. Theorem. Suppose \(X\) is a non-empty set and \((X^*, \mathcal{S}^*)\) is a \(p\)-space. Then \(\mathcal{T}(\Phi_w) = \mathcal{T}(\Phi_s)\) which is \(\mathcal{T}(p, \mathcal{T}(\mathcal{S}^*))\).

Proof. Since \(\mathcal{S}_w < \Phi_s < \mathcal{S}_s\) and we know from I.8.10 that \(\mathcal{T}(\mathcal{S}_w) = \mathcal{T}(\mathcal{S}_s)\), we have that \(\mathcal{T}(\mathcal{S}_w) = \mathcal{T}(\Phi_s)\). Since \(\mathcal{S}_w = \Phi_w\), we have that \(\mathcal{T}(\Phi_w) = \mathcal{T}(\Phi_s)\).

It is well known that \(\mathcal{T}(p, \mathcal{T}(\mathcal{U}(\mathcal{S}^*)))\) is \(\mathcal{T}(\mathcal{U}(p, \mathcal{T}(\mathcal{U}(\mathcal{S}^*))))\). But \(\mathcal{T}(p, \mathcal{T}(\mathcal{U}(\mathcal{S}^*))) = \mathcal{T}(p, \mathcal{T}(\mathcal{S}^*))\) and \(\mathcal{T}(\mathcal{U}(p, \mathcal{T}(\mathcal{S}^*))) = \mathcal{T}(\Phi_w)\) by III.1.1. Hence \(\mathcal{T}(\Phi_w) = \mathcal{T}(\Phi_s) = \mathcal{T}(p, \mathcal{T}(\mathcal{S}^*))\).

III.1.14. Theorem. Suppose \(\mathcal{T} \in [X, X^*]\) and \(\mathcal{T} \neq \emptyset\). Then \(\Phi_w |_{\mathcal{T}} \subset \Phi_s |_{\mathcal{T}}\). If \((X^*, \mathcal{S}^*)\) is \(p\)-totally bounded then \(\Phi_w |_{\mathcal{T}} = \Phi_s |_{\mathcal{T}}\).

Proof. From III.1.9 we have \(\Phi_w \subset \Phi_s\). Then for \(H, \Phi \subset \mathcal{T}\) and \(\Phi \subset \Phi_s |_{\mathcal{T}}\) implies \(H \subset \Phi_s |_{\mathcal{T}}\). Then \(H \subset \Phi_w |_{\mathcal{T}}\) and hence \(H \subset \Phi_s |_{\mathcal{T}}\).
If \((X^*, S^*)\) is \(p\)-totally bounded then \(\mathcal{G}_w = \mathcal{G}_s\). Then
for \(\mathcal{H} \subseteq \mathcal{G}_w\), \(\mathcal{H} \subseteq \mathcal{G}_s\) iff \(\mathcal{H} \subseteq \mathcal{G}_w\) iff \(\mathcal{H} \subseteq \mathcal{G}_s\)
iff \(\mathcal{H} \subseteq \mathcal{G}_s\).

**III.1.15. Remark.** We close this section with an examination of the proximal properties of \(\mathcal{G}_w\) and \(\mathcal{G}_s\).

**III.1.16. Theorem.** Suppose \(X\) is a non-empty set and \((X^*, S^*)\) is a \(p\)-complete \(p\)-space. Then the spaces \([[X, X^*], \mathcal{G}_w]\) and \([[X, X^*], \mathcal{G}_s]\) are \(p\)-complete.

**Proof.** That \([[X, X^*], \mathcal{G}_w]\) is complete follows from 1.8.22 and III.1.3.

Suppose \((T, D)\) is a \(p\)-Cauchy net in \([[X, X^*], \mathcal{G}_s]\).
Since \(\mathcal{G}_w \subseteq \mathcal{G}_s\), \(\mathcal{G}(\mathcal{G}_w) \subseteq \mathcal{G}(\mathcal{G}_s)\). Hence \((T, D)\) is \(p\)-Cauchy in \([[X, X^*], \mathcal{G}_w]\). Since \([[X, X^*], \mathcal{G}_w]\) is \(p\)-complete there is an \(f \in [X, X^*]\) such that \((T, D)\) converges \(f\) relative to \(\mathcal{G}(\mathcal{G}_w)\).
But \(\mathcal{G}(\mathcal{G}_w) = \mathcal{G}(\mathcal{G}_s)\) by III.1.13, and therefore \((T, D)\) converges to \(f\) relative to \(\mathcal{G}(\mathcal{G}_s)\). Thus \([[X, X^*], \mathcal{G}_s]\) is \(p\)-complete.

**III.1.17. Lemma.** Suppose \((T, D)\) is a net ranging in \([X, X^*]\). Then \((T, D)\) converges to \(f\) relative to \(\mathcal{G}(\mathcal{G}_w)\)
(or \(\mathcal{G}(\mathcal{G}_s)\)) iff \((\mathcal{G}_s \circ T, D)\) converges to \(f(x)\) for all \(x \in X\).

**Proof.** This follows from III.1.13 and the fact that a net in a product space converges iff each of its projections converge.
III.1.18. Theorem. Suppose \( X \) is a non-empty set and \((X^*, S^*)\) is a separated p-space. Suppose \( \mathfrak{T} \subseteq \mathbb{[X, X^*]} \) and \( \mathfrak{T} \neq \emptyset \). Then \((\mathfrak{T}, \mathfrak{g}_S|_{\mathfrak{T}})\) is compact iff

1. \( c^*(\mathfrak{T}[x]) \) is compact for all \( x \in X \), and
2. \( \mathfrak{T} \) is closed in \( ([X, X^*], \mathfrak{g}_S) \).

Proof. Suppose \( \mathfrak{T} \) is compact. Then it is well known that the map \( e_x : \mathbb{[X, X^*]} \to X^* \), where \( e_x(f) = f(x) \) is continuous relative to \( \mathfrak{T}(p, \mathfrak{T}(S^*)) \) and \( \mathfrak{T}(S^*) \). Then \( e_x|_{\mathfrak{T}} \) is compact and hence closed in \((X^*, S^*)\) since \( \mathfrak{T}(S^*) \) is Hausdorff. But \( e_x|_{\mathfrak{T}} = \mathfrak{T}[x] \) by 0.5.4, and thus we have

1. That \( \mathfrak{T} \) is closed in \( \mathbb{[X, X^*]} \) follows from the fact that \( \mathfrak{T}(\mathfrak{g}_W) \) is Hausdorff and hence so is \( \mathfrak{T}(\mathfrak{g}_S) \).

Suppose conditions (1) and (2) are satisfied. Then 
\[ \{c^*(\mathfrak{T}[x]) : x \in X\} \subseteq \mathbb{[X, X^*]} . \] Since each \( c^*(\mathfrak{T}[x]) \) is compact, \( \{c^*(\mathfrak{T}[x]) : x \in X\} \) is a compact subspace of \( ([X, X^*], \mathfrak{g}_S) \). Since \( \mathfrak{T} \subseteq \{c^*(\mathfrak{T}[x]) : x \in X\} \) and \( \mathfrak{T} \) is closed \( \mathfrak{T} \) is compact.

III.1.19. Remark. Since \( \mathfrak{T}(\mathfrak{g}_W) = \mathfrak{T}(\mathfrak{g}_S) \), the following corollary needs no proof.

III.1.20. Corollary. Suppose \( X \) is a non-empty set and \((X^*, S^*)\) is a separated p-space. Suppose \( \mathfrak{T} \subseteq \mathbb{[X, X^*]} \) and \( \mathfrak{T} \neq \emptyset \). Then \((\mathfrak{T}, \mathfrak{g}_W|_{\mathfrak{T}})\) is compact iff

1. \( c^*(\mathfrak{T}[x]) \) is compact for all \( x \in X \), and
2. \( \mathfrak{T} \) is closed in \( ([X, X^*], \mathfrak{g}_W) \).
III.2 Weak and Strong Proximities of Proximal Convergence

III.2.0. Remark. In this section we discuss proximities of proximal convergence for \([X, X^*]\). Throughout this chapter proximities of proximal convergence will be denoted generically by \(\sigma\).

III.2.1. Definition. Suppose \(X\) is a non-empty set and \((X^*, S^*)\) is a \(p\)-space. The uniformity of uniform convergence associated with \(U(S^*)\) is denoted by \(U(u.o., U(S^*))\) as in 0.5.3 (2). Define \(\sigma_w\) to be the relation \(\delta(U(u.o., U(S^*)))\). The proximity relation \(\sigma_w\) will be called the weak proximity of proximal convergence for \([X, X^*]\).

III.2.2. Definition. Suppose \(X\) is a non-empty set and \((X^*, S^*)\) is a \(p\)-space. For each \(U^* \in \pi(S^*)\) let \(\sigma(U^*) = \delta(U(u.o., U^*))\). Define a proximity relation \(\sigma_s\) as follows: \(\sigma_s = \sup \{\sigma(U^*): U^* \in \pi(S^*)\}\). The relation \(\sigma_s\) will be called the strong proximity of proximal convergence for \([X, X^*]\).

III.2.3. Remark. We make the obvious remark that \(\sigma(U(S^*)) = \sigma_w\). The following theorem shows that the strong proximity of proximal convergence is the proximity of \(U(u.o., \mathcal{P}(S^*))\) if the space \((X^*, S^*)\) is total. Thus, in the case of metric spaces, \(\sigma_s\) is the proximity of the usual uniformity of uniform convergence which is metrizable. That this is generally not true for \(\sigma_w\) follows from III.2.8.
III.2.4. Theorem. Suppose \((X^*, S^*)\) is total. Then 
\[ \sigma_s = \sigma(\mathcal{P}(S^*)) \]

Proof. Since \(\mathcal{P}(S^*) \subseteq \pi(S^*)\) when \((X^*, S^*)\) is total, 
\[ \sigma(\mathcal{P}(S^*)) \subseteq \sigma_S \] by definition. To show \(\sigma_S \leq \sigma(\mathcal{P}(S^*))\), 
suppose \(\tau_i, j \in [X, X^*]\) and \(\tau_i \subseteq \mathcal{P}(\mathcal{P}(S^*))\). Then, if 
\(\{\tau_i\}\) and \(\{j, j\}\) are finite covers from below of \(\tau_i\) and \(\tau_j\) respectively, there is an \(i^*\) and a \(j^*\) such that \(\tau_i \subseteq \mathcal{P}(S^*)\) near \(\tau_j\). We claim that \(\tau_i \subseteq \sigma(\mathcal{P}(S^*))\) 
for all \(\tau_i \in \pi(S^*)\) and therefore \(\tau_i \subseteq \sigma_S\). Let \(\tau_i \in \pi(S^*)\) and let \(U^* \in \mathcal{P}(S^*)\). Then \(U^* \in \mathcal{P}(S^*)\). Therefore 
\(W(U^*)[\tau_i] \cap W(U^*)[\tau_i] \neq \emptyset\). Since this is true for all 
\(U^* \in \mathcal{P}(S^*)\), for all \(U^* \in \pi(S^*)\), \(\tau_i \subseteq \mathcal{P}(S^*)\) 
for all \(U^* \in \pi(S^*)\).

III.2.5. Theorem. Suppose \(\tau \in [X, X^*]\), \(\tau \neq \emptyset\). Then 
\(\tau, \sigma_S | \tau \neq (\tau, \sigma_S | \tau)\).

Proof. Since \(\mathcal{U}(S^*) \subseteq \pi(S^*)\), \(\sigma(\mathcal{U}(S^*)) \leq \sigma_S\). But 
by III.2.3 \(\sigma(\mathcal{U}(S^*)) = \sigma_w\). Thus \(\sigma_w \leq \sigma_S\) on \([X, X^*]\) 
and hence on any non-empty subspace \(\tau \subseteq [X, X^*]\).

III.2.6. Remark. The following lemma will be used 
to prove III.2.8 and again to prove III.2.13. The author 
has not seen the result in the literature.

III.2.7. Lemma. Suppose \(X\) and \(X^*\) are non-empty sets and 
\(\text{card}(X) \geq \text{card}(X^*)\). Suppose \(\mathcal{U}^*\) and \(U^* \in \mathcal{U}(X^*)\), 
\(\mathcal{U}^* \subseteq U^*\), and \(U^* \neq U^*\). Then \(\sigma(\mathcal{U}(u.c., U^*))\) is 
contained in \(\tau(\mathcal{U}(u.c., U^*))\) and \(\sigma(\mathcal{U}(u.c., U^*))\) is 
not equal to \(\sigma(\mathcal{U}(u.c., U^*))\).
Proof. Since \( \mathcal{U} \subseteq \mathcal{V} \) we know that \( \mathcal{U}(u.c., \mathcal{U}*) \) is contained in \( \mathcal{U}(u.c., \mathcal{V}) \) and therefore \( \mathcal{T}(\mathcal{U}(u.c., \mathcal{U}*)) \) is contained in \( \mathcal{T}(\mathcal{U}(u.c., \mathcal{V})) \).

To show that \( \mathcal{T}(\mathcal{U}(u.c., \mathcal{U}*)) \neq \mathcal{T}(\mathcal{U}(u.c., \mathcal{V})) \), we first note that \( \text{card}(X) \geq \text{card}(X*) \) implies the existence of a function \( g: X \rightarrow X* \) which is single valued and onto. Since \( \mathcal{U} \subseteq \mathcal{V} \) and \( \mathcal{U} \neq \mathcal{V} \), there is a \( V^* \subseteq \mathcal{V} \) such that \( V^* \subseteq \mathcal{U} \).

That is, for each \( U^* \subseteq \mathcal{U} \) there is a \( V^* \subseteq \mathcal{V} \) such that \( (x(U^*), y(U^*)) \in X^* \times X^* \) such that \( (x(U^*), y(U^*)) \in U^* \) and \( (x(U^*), y(U^*)) \in V^* \).

We define an indexed set of functions \( \{ f_{U^*}: U^* \subseteq \mathcal{U}^* \} \), \( f_{U^*}:X^* \rightarrow X^* \) as follows:

\[
f_{U^*}(x^*) = x^* \text{ if } x^* \neq y(U^*) \text{ and } f_{U^*}(y(U^*)) = x(U^*).
\]

Consider the set \( \mathcal{U} \) to be directed as follows: \( U^*_1 \supseteq U^*_2 \) iff \( U^*_1 \subseteq U^*_2 \). Then letting \( T(U^*) = f_{U^*} \circ g \), \( (T, \mathcal{U}^*) \) is a net of functions in \( [X, X^*] \). We claim that \( (T, \mathcal{U}^*) \) converges to \( g \) in \( \mathcal{T}(\mathcal{U}(u.c., \mathcal{U}*)) \) but not in \( \mathcal{T}(\mathcal{U}(u.c., \mathcal{V})) \).

We first show that, given \( U^*_1 \subseteq \mathcal{U} \), there is a \( U^*_0 \subseteq \mathcal{U} \) symmetric such that \( U^*_0 \supseteq U^*_1 \) implies \( T(U^*_1) \in W(U^*_1) \), that is, \( (T, \mathcal{U}^*) \) converges to \( g \) in \( \mathcal{T}(\mathcal{U}(u.c., \mathcal{U}*)) \). Given \( U^*_1 \subseteq \mathcal{U} \), choose \( U^*_0 = U^*_1 \cap (U^*_1)^{-1} \). Then \( U^*_0 \supseteq U^*_1 \) implies \( U^*_0 \supseteq U^*_1 \). We claim that \( (T(U^*_1), g) \in W(U^*_1) \) or that \( (T(U^*_1)(x), g(x)) \in U^*_1 \) for all \( x \in X \). For \( x \) such that \( g(x) \neq y(U^*) \), \( T(U^*)(x) = g(x) \) and hence \( (T(U^*)(x), g(x)) \in U^*_1 \). For \( x \) such that \( g(x) = y(U^*) \), \( f_{U^*}(g(x)) = f_{U^*}(y(U^*)) = x(U^*) \). But \( (x(U^*), y(U^*)) \in U^* \subseteq U^*_0 \).

Thus for all \( x \in X \), \( (T(U^*)(x), g(x)) \in U^*_1 \) and hence \( (T, \mathcal{U}^*) \) converges to \( g \) in \( \mathcal{T}(\mathcal{U}(u.c., \mathcal{U}*)) \).
We now show that $(T, U^*)$ does not converge to $g$ in $\tau(U(u.c., U^*))$. Consider the entourage $V^*_0$. We claim that $T(U^*) \notin W(V^*_0)[g]$ for no $U^* \in U^*$. For, let $U^* \in U^*$ be symmetric and consider $(T(U^*), g) = (f_{U^*} \circ g, g)$. We need only show the existence of an $x \in X$ such that the pair $(f_{U^*}(g(x)), g(x)) \notin V^*_0$. Choosing any $x_0 \in g^{-1}(y(U^*))$, which is non-empty since $g$ is onto, we have $(f_{U^*}(g(x_0)), g(x_0))$ which is $(x(U^*), y(U^*)) \notin V^*_0$ by construction.

III.2.8. Theorem. Suppose $X$ is a non-empty set, $(X^*, S^*)$ a p-space, and card($X$) $\geq$ card($X^*$). Then $\tau = \sigma_s$ iff $(X^*, S^*)$ is p-totally bounded.

Proof. If $(X^*, S^*)$ is p-totally bounded then $\tau(S^*)$ is $\{U(S^*)\}$. Hence $\sigma_s = \sup\{\sigma(U^*): U^* \in \tau(S^*)\}$ is $\sigma(U(S^*))$ which is $\tau$. We note that this part of the theorem is true without the cardinality conditions on $X$ and $X^*$.

We prove the converse as follows. If $\tau = \sigma_s$ then $\sigma(\tau) = \sigma(\sigma_s)$. We know from III.2.1 that $\sigma(\tau)$ is $\sigma(U(u.c., U(S^*)))$. We also know from III.2.2 that $\sigma(\sigma_s)$ is $\sup\{\sigma(U(u.c., U^*)): U^* \in \tau(S^*)\}$. Since $U(S^*) \in U^*$ for all such $U^*$, we then have $\sigma(U(u.c., U^*))$ equal to $\sigma(U(u.c., U(S^*)))$ for all $U^* \in \tau(S^*)$. From III.2.7 and the fact that card($X$) $\geq$ card($X^*$) we know that $\sigma(U(u.c., U^*)) \neq \sigma(U(u.c., U(S^*)))$ unless $U(S^*)$ is equal to $U^*$. But this will imply that $\tau(S^*) = \{U(S^*)\}$. Hence $(X^*, S^*)$ is p-totally bounded.
III.2.9. Example. If we drop the cardinality condition from III.2.8 the theorem is no longer true. One can easily show that, if $X$ is a one point space $([X,X^*],\sigma_w)$ and $([X,X^*],\Sigma_s)$ are both p-isomorphic to $(X^*,\mathcal{S}^*)$ under the same mapping, and hence equal. This will be true whether or not $(X^*,\mathcal{S}^*)$ is p-totally bounded.

III.2.10. Remark. In section 8 of [18] Leader defines two types of convergence for functions whose ranges are contained in proximity spaces. (1) A net of functions $(T,D)$ from a set $X$ into a p-space $(X^*,\mathcal{S}^*)$ is said to converge in proximity to a function $f$ iff $A \subseteq X$, $B^* \subseteq X^*$, and $f[A] \not\subseteq B^*$ implies the existence of a $d^* \in D$ such that $d^* \not> d^*$ implies $T(d)[A] \not\subseteq B^*$. (2) A net of functions $(T,D)$ is said to converge uniformly to $f$ iff for every pseudometric $r^*$ in the gauge of $\mathcal{S}^*$, $r^*(f(x),T(d)(x))$ converges to 0 uniformly for all $x \in X$. He proves that convergence of the second type implies convergence of the first type and then conjectures that the converse need not be true. We will show that convergence of the first type is implied by convergence in $\mathcal{T}(\sigma_w)$ and that convergence of the second type is equivalent to convergence in $\mathcal{T}(\Sigma_s)$. We will then give an example to show that $\mathcal{T}(\sigma_w) \neq \mathcal{T}(\Sigma_s)$, thus proving that Leader's conjecture is correct.

III.2.11. Theorem. Suppose $X$ is a non-empty set and $(X^*,\mathcal{S}^*)$ is a p-space. Suppose $(T,D)$ is a net ranging in $[X,X^*]$. Then if $(T,D)$ converges to $f \in [X,X^*]$ relative to
\( \mathfrak{T}(\mathcal{S}_w) \), then \((T,D)\) converges in proximity (as defined in III.2.10) to \(f\).

**Proof.** Suppose \((T,D)\) converges to \(f\) in \(\mathfrak{T}(\mathcal{S}_w)\). Then given \(U^* \in \mathcal{U}(S^*)\) there is a \(d^* \in D\) such that \(d > d^*\) implies \((f,T(d)) \in W(U^*)\). Suppose \(A \subseteq X^*, B^* \subseteq X^*,\) and \(f[A] \subseteq B^*\). Then there is a \(U^* \in \mathcal{U}(S^*)\) such that \(U^*[f[A]] \cap B^* = \emptyset\).

Let \(U_o \subseteq \mathcal{U}(S^*)\) be symmetric such that \(U_o \circ U_o \subseteq U^*\). Let \(d^* \in D\) such that \(d > d^*\) implies \((T(d),f) \in W(U_o^*)\). Then \((T(d)(a),f(a)) \in U_o^*\) for all \(a \in A\) and therefore \((f(a),T(d)(a))\) is in \(U_o^*\) for all \(a \in A\). Then \(U_o^*[T(d)[A]] \subseteq U_o^*[U_o^*[f[A]]]\) which is contained in \(U_o^*[f[A]]\). But this implies that \(B^*\) and \(U_o^*[T(d)[A]]\) are disjoint. Then for all \(d > d^*\), \(T(d)[A]\) is \(S^*\) far from \(B^*\). Thus \((T,D)\) converges in proximity to \(f\).

**III.2.12. Theorem.** Suppose \(X\) is a non-empty set and \((X^*, S^*)\) is a \(p\)-space. Suppose \((T,D)\) is a net ranging in \([X,X^*]\). Then \((T,D)\) converges to \(f\) in \(\mathfrak{T}(\mathcal{S}_g)\) iff \((T,D)\) converges to \(f\) uniformly (as defined in III.2.10).

**Proof.** Suppose \((T,D)\) converges to \(f\) in \(\mathfrak{T}(\mathcal{S}_g)\). Let \(r^* \in D(S^*)\). Let \(U^*\) be the metric uniformity on \(X^*\) generated by \(r^*\). Then \(U^* \subseteq \mathcal{P}(S^*)\). We claim that, for all \(\varepsilon > 0\) there is a \(d^* \in D\) such that \(d > d^*\) implies \(r^*(T(d)(x),f(x))\) is less than \(\varepsilon\) for all \(x \in X\) or equivalently, \((T(d)(x),f(x))\) is in \(V_\varepsilon\) for all \(d > d^*\) for all \(x \in X\). Since, for \(\varepsilon > 0\), \(V_\varepsilon \subseteq U^* \subseteq \mathcal{P}(S^*)\), there is a \(U_o^* \subseteq \mathcal{P}(S^*)\) such that \(V_\varepsilon \subseteq U_o^*\).

Then \(W(V_\varepsilon) \subseteq \mathcal{U}(u.o., U_o^*)\). Since \((T,D)\) converges to \(f\) in \(\mathfrak{T}(\mathcal{S}_g)\) = \(\sup \{ \mathfrak{T}(\mathcal{S}(U^*)) : U^* \subseteq \mathcal{P}(S^*) \}\), it converges to \(f\)
in \( \tau(\mathfrak{U}^*_0) = \tau(\mathfrak{U}(u.c.,\mathfrak{U}^*_0)) \). Hence, there is a \( d^* \in D \) such that \( d > d^* \) implies \( (T(d),f) \in W(V_E) \) and therefore \( (T(d)(x),f(x)) \in V_E \) for all \( x \in X \).

Suppose \( (T,D) \) converges to \( f \) uniformly. Since 
\[
\tau(\mathfrak{U}_s) = \sup\{\tau(\mathfrak{U}(u^*_*)): u^*_* \in \Pi(\mathfrak{S}_*)\} \quad \text{which is equal to} \\
\sup\{\tau(\mathfrak{U}(u.c.,u^*_*)): u^*_* \in \Pi(\mathfrak{S}_*)\},
\]
we have that \( \tau(\mathfrak{U}_s) \) is \( \tau(\sup\{\mathfrak{U}(u.c.,u^*_*): u^*_* \in \Pi(\mathfrak{S}_*)\}) \) by a well known theorem about uniform spaces. Thus, it is sufficient to show that for every \( P \in \mathfrak{P}(\mathfrak{S}_*) \) there is a \( d^* \in D \) such that \( d > d^* \) implies \( (T(d),f) \in W(P) \) since \( \{W(P); P \in \mathfrak{P}(\mathfrak{S}_*)\} \), which is \( \{W(U^*_*)\}: there is a \( u^*_* \in \Pi(\mathfrak{S}_*) \) such that \( U^*_* \in \mathfrak{U}(\mathfrak{S}_*) \} \) is a subbase for \( \sup\{\mathfrak{U}(u.c.,u^*_*): u^*_* \in \Pi(\mathfrak{S}_*)\} \). Let \( P \in \mathfrak{P}(\mathfrak{S}_*) \). Then there is a normal sequence \( \{P_n\} \subset \mathfrak{P}(\mathfrak{S}_*) \) such that \( P_1 \cap P_1 \subseteq P_1 \). The sequence \( \{P_n\} \) is a base for a pseudometrizable uniform space \( (X^*_*,U^*_*) \). Let \( r_* \) be the implied pseudometric. Since each \( P_n \in \mathfrak{P}(\mathfrak{S}_*) \), \( U^*_* \in \mathfrak{P}(\mathfrak{S}_*) \) and hence \( \mathfrak{S}(U^*_*) = \mathfrak{S}(r_*^*) \subseteq \mathfrak{S}_* \) and thus \( r_* \in D(\mathfrak{S}_*) \). Then for \( \varepsilon = 1 \) there is a \( d^* \in D \) such that \( d > d^* \) implies that 
\[
(T(d)(x),f(x)) \in V_{d^*} \quad \text{for all} \ x \in X.
\]
But \( V_{d^*} \subseteq P_1 \subseteq P \). Thus for all \( d > d^* \), \( (T(d),f) \in W(P) \), thus completing the proof.

III.2.13. Example. Let \( R \) be the real numbers and let \( (N, \mathcal{S}_M) \) be the integers with the discrete proximity. Then

1. \( \text{card}(R) > \text{card}(N) \).
2. \( \mathcal{P}(\mathcal{S}_M) \not= \mathcal{U}(\mathcal{S}_M) \).
(3) \( \mathcal{U}(\mathcal{S}_M) \subseteq \mathcal{P}(\mathcal{S}_M) \), and
(4) \( \mathcal{P}(\mathcal{S}_M) \) is a uniformity.

Thus by III.2.4 \( \tau_s = \mathcal{S}(\mathcal{U}(u.c., \mathcal{P}(\mathcal{S}_M))) \) and hence \( \tau(\tau_s) \) is \( \tau(\mathcal{U}(u.c., \mathcal{P}(\mathcal{S}_M))) \). From III.2.1 \( \tau(\tau_w) \) is equal to \( \tau(\mathcal{U}(u.c., \mathcal{U}(\mathcal{S}_M))) \). By III.2.7, the given conditions, \( \tau(\mathcal{U}(u.c., \mathcal{U}(\mathcal{S}_M))) \neq \tau(\mathcal{U}(u.c., \mathcal{P}(\mathcal{S}_M))) \) and hence \( \tau(\tau_w) \neq \tau(\tau_s) \). There is, then, a net \((T,D)\) in \([X,X^*]\) which converges in \( \tau(\tau_w) \) but not in \( \tau(\tau_s) \). Since convergence in \( \tau(\tau_w) \) implies convergence in proximity and convergence in \( \tau(\tau_s) \) is equivalent to uniform convergence, the two concepts are not equivalent and hence Leader's conjecture (see III.2.10) is correct.

**III.2.14. Remark.** The concept of convergence of functions into a proximity space seems to have been studied first by Mrowka [21] and by Leader [18]. The first part of III.2.8, or at least the implication that \( \tau(\tau_w) \) is the same as \( \tau(\tau_s) \) is similar to Theorem 16 of [18]. Parts of III.2.15 and III.2.16 of this paper, although proved independently, can be obtained from Theorems 17 and 18 of [18] and III.2.12 of this paper. The next two theorems state that the weak or strong proximal limit of a net of continuous functions is continuous, and that the weak or strong proximal limit of a net of p-continuous functions is p-continuous.
III.2.15. Theorem. Suppose \((X, \mathcal{T})\) is a topological space, \((X^*, S^*)\) a p-space. Let \(c_w\) and \(c_s\) be the closure operators for \(\mathcal{T}(\sigma_w)\) and \(\mathcal{T}(\sigma_s)\) respectively. Then 
\[
c_w(t[X,X^*]) = c_s(t[X,X^*]) = t[X,X^*].
\]

Proof. Since \(\mathcal{T}(\sigma_w) \subseteq \mathcal{T}(\sigma_s)\) by III.2.5, we have 
\[
c_s(t[X,X^*]) \subseteq c_w(t[X,X^*]).
\]
But from III.2.1 we know that 
\[
\mathcal{T}(\sigma_w) = \mathcal{T}(\mathcal{U}(uc.o, \mathcal{U}(S^*))).
\]
Hence by the known facts about \(\mathcal{T}(\mathcal{U}(uc... \mathcal{U}(S^*)))\), we have 
\[
t[X,X^*] \subseteq c_s(t[X,X^*])
\]
which is contained in \(c_w(t[X,X^*]) \subseteq t[X,X^*]\).

III.2.16. Theorem. Suppose \((X, S)\) and \((X^*, S^*)\) are p-spaces. Suppose \(c_w\) and \(c_s\) are as in III.2.15. Then 
\[
c_w(p[X,X^*]) = c_s(p[X,X^*]) = p[X,X^*].
\]

Proof. We will show that \(p[X,X^*]\) is closed in \(\mathcal{T}(\sigma_w)\) and the rest of the theorem will then follow from III.2.5 as above. We claim that 
\[
[X,X^*] \cap C(p[X,X^*])
\]
is open in \(\mathcal{T}(\sigma_w)\). Let \(g \in [X,X^*] \cap C(p[X,X^*])\). Then there is a \(U^*\) in \(\mathcal{U}(S^*)\) such that 
\[
(g \times g)^{-1}[U^*] \subseteq \mathcal{U}(S)
\]
Let \(U_0^* \subseteq \mathcal{U}(S^*)\) be symmetric such that 
\[
U_0^* \circ U_0^* \circ U_0^* \subseteq U^*.
\]
Then \(W(U_0^*)[g]\) is a subset of 
\[
[X,X^*] \cap C(p[X,X^*])
\]
and suppose \(f \in W(U_0^*)[g]\) and suppose \(f\) is p-continuous. Then 
\[
(f x f)^{-1}[U_0^*] \subseteq \mathcal{U}(S),
\]
and 
\[
(f x f)^{-1}[U_0^*] \subseteq (g x g)^{-1}[U^*].
\]
To see this, let \((x,y)\) be in 
\[
(f x f)^{-1}[U_0^*].
\]
Then \((f(x), f(y)) \in U_0^*\), and also \((g(x), f(x))\) and \((g(y), f(y)) \in U_0^*\) from the fact that \(f \in W(U_0^*)[g]\). Then 
\[
(g(x), g(y)) \in U^*.
\]
Hence \((f x f)^{-1}[U_0^*] \subseteq (g x g)^{-1}[U^*]\) which implies 
\[
(g x g)^{-1}[U^*] \subseteq \mathcal{U}(S),
\]
which is a contradiction. Therefore, 
\[
[X,X^*] \cap C(p[X,X^*])
\]
is open in \(\mathcal{T}(\sigma_w)\).
III.2.17. Theorem. For a non-empty set $X$ and a $p$-space $(X^*, S^*)$, $\Phi_w \leq \sigma_w$ and $\Phi_w \leq \sigma_s$.

Proof. Suppose $\Phi, H \subseteq [X, X^*]$ and $\Phi \subseteq \sigma_w \subseteq H$. Then for any $U^* \in \mathcal{U}(S^*)$, $W(U^*)[\sigma_T] \cap H \neq \emptyset$. For $1 \leq i \leq n$, let $x_i \in X$ and $U_i^* \in \mathcal{U}(S^*)$. Then $\cap U_i^* \in \mathcal{U}(S^*)$. Hence there is an $f \in [X, X^*]$ such that $f \in W(\cap U_i^*)[\sigma_T] \cap H$. Then since $W(\cap U_i^*) \subseteq \cap W(x_i, U_i^*), f \in \cap W(x_i, U_i^*)[\sigma_T] \cap H$. Thus $\Phi \subseteq \Phi_w \subseteq H$, and $\Phi_w \leq \sigma_w$. That $\Phi_w \leq \sigma_s$ follows from the above and III.2.5.

III.2.18. Theorem. A net $(T, D)$ ranging in $[X, X^*]$ converges to $f$ relative to $\sigma_w$ (or $\sigma_s$) iff $(T, D)$ is $p$-Cauchy relative to $\sigma_w$ (or $\sigma_s$) and $(T, D)$ converges to $f$ relative to $\sigma(T(\Phi_w))$.

Proof. That $(T, D)$ converging to $f$ relative to $\sigma_w$ or $\sigma_s$ implies that $(T, D)$ is $p$-Cauchy and converges to $f$ relative to $\sigma(T(\Phi_w))$ follows from I.6.12 and III.2.17.

Suppose $(T, D)$ is $p$-Cauchy relative to $\sigma_s$ (the proof for $\sigma_w$ is similar) and converges pointwise to $f$. Let $P^* \in \mathcal{P}(S^*)$ and let $P_0^*$ be closed and symmetric such that $P_0^* \subseteq P^*$. Then $(T, D)$ $p$-Cauchy implies the existence of a $d^* \in D$ such that $d_1$ and $d_2 > d^*$ imply $T(d_2)(x) \in P_0^*[T(d_1)(x)]$ for all $x \in X$. Since $(W \circ T, D)(\text{see III.1.17})$ converges to $f(x)$ and $P_0^*[T(d_1)(x)]$ is closed, $f(x) \in P_0^*[T(d_1)(x)]$ for all $x \in X$. Hence, for all $d \geq d^*$, $(T(d)(x), f(x)) \in P_0^*$ and thus $(f, T(d)) \in W(P^*)$. Then $(T, D)$ converges to $f$ relative to $\sigma_s$ using an argument similar to the one used in III.2.12.
III.2.19. Theorem. Suppose \((X^*, S^*)\) is a p-complete p-space. Then \(([X,X^*], \sigma_s)\) and \(([X,X^*], \sigma_w)\) are p-complete p-spaces.

Proof. From III.1.16 we know that \(([X,X^*], \phi_w)\) is p-complete. Let \((T,D)\) be p-Cauchy in \(\sigma_s\). Then from III.2.17 \(\phi_w \subseteq \sigma_s\) and hence \(\phi(\phi_w) \subseteq \phi(\sigma_s)\). Thus \((T,D)\) is p-Cauchy in \(\phi_w\). Thus it converges to some \(f\) relative to \(\sigma(\phi_w)\). But \((T,D)\) p-Cauchy relative to \(\sigma_s\) and converging to \(f\) relative to \(\sigma(\phi_w)\) implies, from III.2.18, that \((T,D)\) converges to \(f\) in \(\sigma(\sigma_s)\). A similar proof shows the same result for \(([X,X^*], \sigma_s)\).

III.2.20. Corollary. Suppose \((X, S)\) is a p-space, \((Y, \tau)\) is a topological space, and \((X^*, S^*)\) is a p-complete p-space. Then \((p[X,X^*], \sigma_w), (p[X,X^*], \sigma_s), (t[Y,X^*], \sigma_w),\) and \((t[Y,X^*], \sigma_s)\) are all p-complete where \(\sigma_w\) and \(\sigma_s\) represent the appropriate subspace uniformities in each case.

Proof. Using III.2.19, III.2.15, III.2.16, and I.6.22 the result is immediate.

III.3 Weak and Strong Proximities of Proximal Convergence on Compacta

III.3.1. Definition. Suppose \((X, \tau)\) is a topological space and \((X^*, S^*)\) is a p-space. Let \(\mathcal{K} = \{K \subseteq X: K \text{ compact}\}\). Define \(\gamma_w = \mathcal{S}(\mathcal{U}(\text{u.c.c}, \mathcal{U}(S^*)))\). The proximity relation \(\gamma_w\) will be called the weak proximity of proximal convergence on compacta for \([X,X^*]\).
III.3.2. Definition. Suppose \((X, \mathcal{T})\) is a topological space and \((X^*, \mathcal{S}^*)\) is a \(p\)-space. For each \(U^* \in \mathcal{P}(\mathcal{S}^*)\) let \(\mathcal{G}(U^*) = \mathcal{S}(U(u.c., C, U^*))\). Define \(\mathcal{G}_s\) to be the relation \(s = \sup\{\mathcal{G}(U^*); U^* \in \mathcal{P}(\mathcal{S}^*)\}\). The proximity relation \(\mathcal{G}_s\) will be called the strong proximity of proximal convergence on compacta for \([X, X^*]\).

III.3.3. Remark. In this section \(\mathcal{G}\) will be used generically to denote proximities of proximal convergence on compacta. We observe the obvious facts that \(\mathcal{G}(U(\mathcal{S}^*))\) is \(\mathcal{G}_w\) and that \(\mathcal{G}(\mathcal{G}_w) = \mathcal{G}(U(u.c., C, U(\mathcal{S}^*)))\). We again show that in the case of total spaces \(\mathcal{G}_s\) is the proximity of the usual uniform construction for \(\mathcal{P}(\mathcal{S}^*)\).

III.3.4. Theorem. Suppose \((X, \mathcal{T})\) is a topological space, \((X^*, \mathcal{S}^*)\) is a total \(p\)-space. Then \(\mathcal{G}_s = \mathcal{G}(\mathcal{P}(\mathcal{S}^*))\).

Proof. Since \(\mathcal{P}(\mathcal{S}^*) \in \mathcal{P}(\mathcal{S}^*)\), \(\mathcal{G}(\mathcal{P}(\mathcal{S}^*)) \leq \mathcal{G}_s\) follows from III.3.2.

Suppose \(\mathcal{T}, \mathcal{H} \in [X, X^*]\) and \(\mathcal{G}(\mathcal{P}(\mathcal{S}^*)) \mathcal{H}\). Then, if \(\{U^*_1\}\) and \(\{U^*_j\}\) are finite covers from below of \(\mathcal{T}\) and \(\mathcal{H}\) respectively, there is an \(i^*\) and a \(j^*\) such that \(\mathcal{T}_{i^*}\) is \(\mathcal{G}(\mathcal{P}(\mathcal{S}^*))\) near \(\mathcal{H}_{j^*}\). We claim that \(\mathcal{T}_{i^*} \mathcal{G}(U^*) \mathcal{H}_{j^*}\) for all \(U^* \in \mathcal{P}(\mathcal{S}^*)\) and hence \(\mathcal{T} \mathcal{G}_s \mathcal{H}\). Let \(U^* \in \mathcal{P}(\mathcal{S}^*)\).

Let \(\{U_k^*\}\) be a finite collection, \(U_k^* \in U^*\), and for each \(k\) let \(K_k \subseteq X\) be compact. We claim that \(\bigcap W(K_k, U_k^*)[\mathcal{T}_{i^*}] \cap \mathcal{H}_{j^*}\) is not empty and hence \(\mathcal{T}_{i^*} \mathcal{G}(U^*) \mathcal{H}_{j^*}\). Since \(U^* \in \mathcal{P}(\mathcal{S}^*)\), \(U^* \in \mathcal{P}(\mathcal{S}^*)\). Hence each \(U_k^* \in \mathcal{P}(\mathcal{S}^*)\). Then, since
Thus completing the proof.

**III.3.5. Lemma.** Suppose \( \mathcal{Q} \subseteq [X, X^*] \). Then \( \mathcal{Q} \) is open in \( \mathcal{U}(\mathcal{G}_S) \) iff for all \( g \in \mathcal{Q} \) there is a finite collection \( \{P_i^*; 1 \leq i \leq n\}, P_i^* \in \mathcal{U}(S^*) \), and a finite collection \( \{K_i; 1 \leq i \leq n\}, K_i \subseteq X \) compact, such that:

\[
(*) \quad \bigcap_i W(K_i, P_i^*)[g] = \emptyset
\]

Proof. \( \gamma_S = \sup \{ \mathcal{U}(\mathcal{G}^*) ; \mathcal{U}^* \in \mathcal{U}(S^*) \} \) and therefore \( \mathcal{U}(\mathcal{G}_S) = \sup \{ \mathcal{U}(\mathcal{G}^*) ; \mathcal{U}^* \in \mathcal{U}(S^*) \} \) by I.2.11. Thus \( \mathcal{U}(\mathcal{G}_S) = \sup \{ \mathcal{U}(u.c., C, \mathcal{U}^*) ; \mathcal{U}^* \in \mathcal{U}(S^*) \} \) which is \( \sup \{ \sup \{ \mathcal{U}(u.c., C, \mathcal{U}^*) ; \mathcal{U}^* \in \mathcal{U}(S^*) \} \} \). Thus \( \mathcal{Q} \) is open in \( \mathcal{U}(\mathcal{G}_S) \) iff \( g \in \mathcal{Q} \) implies there is a \( W \in \sup \{ \mathcal{U}(u.c., C, \mathcal{U}^*) \} \) where \( \mathcal{U}^* \) ranges in \( \mathcal{U}(S^*) \) such that \( W[g] \subseteq \mathcal{Q} \).

Now suppose that for each \( g \in \mathcal{Q} \) there are finite collections \( \{P_i^*\} \) and \( \{K_i\} \) satisfying (*) above. For each \( i \) there is a \( \mathcal{U}_i^* \in \mathcal{U}(S^*) \) such that \( P_i^* \subseteq \mathcal{U}_i^* \) and hence \( W(K_i, P_i^*) \subseteq \mathcal{U}(u.c., C, \mathcal{U}_i^*) \). Thus \( \bigcap W(K_i, P_i^*) \) is in the uniformity \( \sup \{ \mathcal{U}(u.c., C, \mathcal{U}^*) ; \mathcal{U}^* \in \mathcal{U}(S^*) \} \). Since

\[
(\bigcap W(K_i, P_i^*))[g] = \bigcap (W(K_i, P_i^*)[g])
\]

\( \mathcal{Q} \) is open in

\( \mathcal{U}(\mathcal{G}_S) \).

Conversely, if \( \mathcal{Q} \) is open in \( \mathcal{U}(\mathcal{G}_S) \), for each \( g \in \mathcal{Q} \) there is a \( W \in \sup \{ \mathcal{U}(u.c., C, \mathcal{U}^*) ; \mathcal{U}^* \in \mathcal{U}(S^*) \} \) such that \( W[g] \subseteq \mathcal{Q} \). But \( W \in \sup \{ \mathcal{U}(u.c., C, \mathcal{U}^*) ; \mathcal{U}^* \in \mathcal{U}(S^*) \} \) implies that there is a finite collection \( \{ \mathcal{U}_j^* \} \in \mathcal{U}(S^*) \) and sets \( W_j \in \mathcal{U}(u.c., C, \mathcal{U}_j^*) \) such that \( \bigcap W_j \subseteq W \) and hence \( \bigcap W_j[g] \subseteq \mathcal{Q} \).
For each $j$, there is a finite collection $\{U^*_j, i \in U^*_j\}$ and a finite collection $\{K^*_j, 1 \in X; K^*_j, 1 \text{ is compact}\}$ such that
$$\bigcap \mathcal{W}(K^*_j, 1, U^*_j, 1) \subseteq \mathcal{W}_j.$$ Then
$$\bigcap \bigcap \mathcal{W}(K^*_j, 1, U^*_j, 1)[g] \subseteq \mathcal{Q}.$$ Noting that for each $i$ and $j U^*_j, 1 \in \mathcal{C}(S^*)$, the above inclusion is equivalent to $(\ast)$. 

III.3.6. Theorem. Suppose $(X, \mathcal{T})$ is a topological space and $(X^*, S^*)$ is a p-space. The following topological spaces are equal.

1. $(t[X, X^*], \mathcal{T}(\gamma_w) \cap t[X, X^*])$
2. $(t[X, X^*], \mathcal{T}(\gamma_s) \cap t[X, X^*])$
3. $(t[X, X^*], \mathcal{T}(\mathcal{C}, \mathcal{T}, \mathcal{T}(S^*)) \cap t[X, X^*])$

Proof. The method of proof is to show that $(1) = (3)$ and that $(2) = (3)$. We remark that it is well known that $(t[X, X^*], \mathcal{T}(U(u.c., C, U(S^*)) \cap t[X, X^*])$ is equal to $(3)$. But, since $\mathcal{T}(U(u.c., C, U(S^*)) = \mathcal{T}(\gamma_w)$ this proves that $(1) = (3)$. We now give the proof for $(2) = (3)$. We shorten the notation as follows:

\[ \mathcal{T}(c) = \mathcal{T}(c, \mathcal{T}, \mathcal{T}(S^*)) \text{ restricted to } t[X, X^*] \text{ and } \mathcal{T}^*(\gamma_s) = \mathcal{T}(\gamma_s) \cap t[X, X^*]. \]

To show that $\mathcal{T}(c) \subseteq \mathcal{T}^*(\gamma_s)$, let $K \subseteq X$ compact, $0^* \subseteq X^*$ open, and consider $(K, 0^*)$. We claim that $(K, 0^*) \in \mathcal{T}^*(\gamma_s)$ and hence $\mathcal{T}(c) \subseteq \mathcal{T}^*(\gamma_s)$. Let $g \in (K, 0^*)$. Then $g[K] \subseteq 0^*$. Since $g$ is continuous, $g[K]$ is compact and therefore there is a $P^* \in \mathcal{F}(S^*)$ such that $P^*[g[K]] \subseteq 0^*$ and $P^*$ is symmetric. Let $f \in W(K, P^*)[g]$. Then $f(x) \in P^*[g(x)]$ for all $x \in K$, or $f(x) \in P^*[g[K]] \subseteq 0^*$ for all $x \in K$. Thus $g \in W(K, P^*)[g] \subseteq (K, 0^*)$. By III.3.5, $(K, 0^*)$ is then in $\mathcal{T}^*(\gamma_s)$ and $\mathcal{T}(c) \subseteq \mathcal{T}^*(\gamma_s)$. 

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We now show that $\mathfrak{T}^*([S, S]) \subseteq \mathfrak{T}(c)$. Let $g \in \pi[X, X*]$ and consider $W(K, P^*)[g]$. We claim that there are compact sets $K \subseteq X$ and open sets $O_i \subseteq X^*$, $1 \leq i \leq n$, such that $g \in \bigcap \{(K_1, O_1^*): 1 \leq i \leq n\} \subseteq W(K, P^*)[g]$.

By III.3.5 this is sufficient to show that $\mathfrak{T}^*([S, S]) \subseteq \mathfrak{T}(c)$. Let $P^{**} \in \mathcal{P}(S^*)$, $P^*$ closed and symmetric, such that $P^{**} \circ P^{**} = P^*$ and let $P^* \in \mathcal{P}(S^*)$ be open such that $P^* \subseteq P^{**}$. Since $g[K]$ is compact and $\{P^*_i[g(x)]: x \in X^*\}$ is an open cover of $g[K]$, there are elements $x_i \in X$, $1 \leq i \leq n$, such that $\{P^*_i[g(x_i)]: 1 \leq i \leq n\}$ is a cover of $g[K]$. Define $K_i = K \cap g^{-1}[P^*_i[g(x_i)]]$. Since $P^*_i[g(x_i)]$ is closed for each $i$, each $K_i$ is compact. Let $O_i^* = \text{int} P^{**}[P^*_i[g(x_i)]]$. Then $g \in \bigcap \{(K_1, O_1^*): 1 \leq i \leq n\}$. For, if $x \in K_i$ then $g(x)$ is in $P^*_i[g(x_i)]$. Since $P^*_i[g(x_i)] \subseteq O_i^*$, $g[K_i] \subseteq O_i^*$. We now claim that $\bigcap \{(K_1, O_1^*): 1 \leq i \leq n\} \subseteq W(K, P^*)[g]$. Let $f$ be in $\bigcap \{(K_1, O_1^*): 1 \leq i \leq n\}$. Then $x \in K_i$ implies $x \in K_i$ for some $i$. Then $f(x) \in P^*_i[P^*_i[g(x_i)]]$ and $g(x) \in P^*_i[g(x_i)]$. Then $(f(x), g(x)) \in P^{**} \circ P^{**} \circ P^* \subseteq P^*$ and thus $f \in W(K, P^*)[g]$.

**III.3.7. Theorem.** Suppose $\mathfrak{T} \subseteq [X, X^*]$, $\mathfrak{T} \not\neq \emptyset$. Then $\mathfrak{T}_W \cap T \subseteq \mathfrak{T}_S \cap T$.

**Proof.** Since $\mathfrak{T}(\cup(S^*)) \subseteq \mathfrak{T}_S$ by III.3.2 and $\mathfrak{T}_W$ is $\mathfrak{T}(\cup(S^*))$ by III.3.3, the result follows.

**III.3.8. Theorem.** If $(X^*, S^*)$ is a p-totally bounded p-space, then for any non-empty $\mathfrak{T} \subseteq [X, X^*]$, $\mathfrak{T}_W \cap T = \mathfrak{T}_S \cap T$. 


Proof. If \((X*, S*)\) is p-totally bounded, \(\Pi (S*)\) is \(\{U(S*)\}\). Hence \(\gamma_S = \sup\{\gamma(U*): U* \in \Pi (S*)\} = \gamma(\cup (S*))\) which is \(\gamma_W\). Since \(\gamma_W = \gamma_S\), \(\gamma_W^{\sigma_T} = \gamma_S^{\sigma_T}\) for any non-empty \(\sigma_T \subseteq [X,X^*]\).

III.3.9. Remark. In order to discuss the completeness properties of \(\gamma_W\) and \(\gamma_S\) we establish some relationships between \(\gamma_W^{\sigma_T}\), \(\gamma_S^{\sigma_T}\), and \(\sigma_W^*\).

III.3.10. Theorem. For any topological space \((X, \mathcal{O})\) and any p-space \((X^*, S^*)\), and for any non-empty \(\sigma_T \subseteq [X,X^*]\), \(\sigma_W^* \subseteq \gamma_S^{\sigma_T}\) and \(\sigma_W^{\sigma_T} \subseteq \gamma_W^{\sigma_T}\).

Proof. Let \(J, \sigma_T \subseteq \sigma_T\) such that \(J \in \gamma_W^{\sigma_T}\). For \(1 \leq i \leq n\), let \(x_i \in X\) and \(U_i^* \in \mathcal{U}(S*)\). Let \(K = \bigcup \{x_i\}\). Then \(K\) is compact. Let \(U^* = \bigcap U_i^*\), then \(U^* \in \mathcal{U}(S*)\). Therefore there is an \(f \in \gamma_W^{\sigma_T}\) such that \(f \in \bigcup (W(K,U^*)[\sigma_T] \cap W(K,U^*)[\sigma_T])\). We claim that \(f \in \bigcap W(x_i, U_i^*)[\sigma_T] \cap \bigcap W(x_i, U_i^*)[\sigma_T]\). Let \((g, f)\) and \((h, f) \in W(K, U^*)\). Then \((g(x_i), f(x_i)), (h(x_i), f(x_i)) \in U_i^*\) and thus in \(U_i^*\) for each \(i\). Thus \((g, f)\) and \((h, f)\) are in \(\bigcap W(x_i, U_i^*)\). This establishes the inequality \(\sigma_W^{\sigma_T} \subseteq \gamma_W^{\sigma_T}\).

Since \(\gamma_W^{\sigma_T} \subseteq \gamma_S^{\sigma_T}\) by III.3.7, \(\sigma_W^{\sigma_T} \subseteq \gamma_S^{\sigma_T}\).

III.3.11. Theorem. A net \((T, D)\) ranging in \([X,X^*]\) converges to \(f\) relative to \(\sigma_T(\gamma_S)\) (or \(\sigma_T(\gamma_W)\)) iff \((T, D)\) is p-Cauchy relative to \(\gamma_S\) (or \(\gamma_W\)) and converges to \(f\) in \(\sigma_T(\sigma_W^*)\).

Proof. If \((T, D)\) converges to \(f\) in \(\sigma_T(\gamma_S)\) (or \(\sigma_T(\gamma_W)\)) then III.3.10 and I.6.12 imply that \((T, D)\) converges relative to \(\sigma_T(\sigma_W^*)\) and is p-Cauchy.
We will show the converse for $\gamma_s$. The proof for $\gamma_w$ is similar. Suppose $(T,D)$ converges to $f$ pointwise and is $p$-Cauchy relative to $\gamma_s$. Let $K$ be compact, $K \subseteq X$, $P^*$ be in $\mathcal{P}(S^*)$, and $P_0^* \in \mathcal{P}(S^*)$ such that $P_0^*$ is closed and symmetric such that $P_0^* \subseteq P^*$. Then there is a $d^* \in D$ such that $d_1$ and $d_2 > d^*$ implies $T(d_1)(x) \in P_0^*[T(d_2)(x)]$ for all $x \in K$. Arguing as in III.2.18, for all $x \in K$ and for all $d > d^*$, $T(d)(x) \in P_0^*[f(x)]$. Hence $T(d) \in W(K,P^*)[f]$ for all $d > d^*$. Since $\bigcap W(K_i,P_i^*)[f] = \bigcap [W(K_i,P_i^*)[f]]$ for $K_i \subseteq X$ and $P_i^* \in \mathcal{P}(S^*)$, $1 \leq i \leq n$, and using III.3.5, the above remark is sufficient to show that $(T,D)$ converges to $f$ relative to $\tau(\gamma_s)$.

III.3.12. Corollary. Suppose $(X^*, S^*)$ is $p$-complete. Then $([X,X^*], \gamma_w)$ and $([X,X^*], \gamma_s)$ are $p$-complete.

Proof. By III.1.16, $([X,X^*], \Phi_w)$ is $p$-complete. If $(T,D)$ is $p$-Cauchy relative to $\gamma_w$ or $\gamma_s$ then III.3.10 shows that it is $p$-Cauchy relative to $\Phi_w$. Hence it converges to some $f \in [X,X^*]$ relative to $\tau(\Phi_w)$. III.3.11 then completes the proof.

III.3.13. Remark. We define the concept of equicontinuity to conform to the usual definition in the metric case.

III.3.14. Definition. For a topological space $(X, \tau)$ and a $p$-space $(X^*, S^*)$, a family of single valued functions $\mathcal{F}$ will be called equicontinuous iff $P^* \in \mathcal{P}(S^*)$ and $x \in X$
Implies the existence of an \( 0 \in \mathfrak{Q} \) such that \( x \in 0 \) and 
\[ f[0] \subseteq F^*[f(x)] \text{ for every } f \in \mathfrak{O}. \]

**III.3.15. Theorem.** Suppose \( \mathfrak{Q} \subseteq [X, X^*] \), \( \mathfrak{Q} \neq \emptyset \), and 
\( \mathfrak{Q} \) is equicontinuous. Then \( \mathfrak{Q} = \emptyset \).

**Proof.** Suppose \( \mathfrak{Q} \neq \emptyset \) and \( \mathfrak{Q} \subseteq \mathfrak{Q} \). Let \( \mathfrak{Q} \) and \( \mathfrak{Q} \) be finite covers from below of \( \mathfrak{Q} \) and \( \mathfrak{Q} \) respectively. Then there is an \( \alpha \) and a \( \beta \) such that 
\[ \mathfrak{Q} \alpha \subseteq \mathfrak{Q} \beta \text{ for each } \mathfrak{Q} \in \mathfrak{Q}(\alpha) \text{ and } \mathfrak{Q} \neq \emptyset. \]

We claim that 
\[ \mathfrak{Q} \alpha \subseteq \mathfrak{Q} \beta \text{ for each } \mathfrak{Q} \in \mathfrak{Q}(\alpha) \text{ and } \mathfrak{Q} \neq \emptyset. \]

Let \( \mathfrak{Q} \in \mathfrak{Q}(\alpha) \). For \( 1 \leq k \leq n \), let \( K_k \subseteq X \) be compact and \( U_k \subseteq \mathfrak{Q}_k \). Let \( U_k \subseteq \mathfrak{Q}_k \) be symmetric such that 
\[ U_k \subseteq U_k \subseteq U_k^* \subseteq U_k. \]
Then \( U_k \subseteq \mathfrak{Q}(\alpha) \) and \( \mathfrak{Q} \) equicontinuous implies that for all \( x_k \in K_k \) there is an \( 0 \) such that 
\( x_k \in 0 \) and 
\[ (f(x_k), f(y)) \in U_k^* \text{ for all } y \in 0, \text{ and for all } f \in \mathfrak{Q}. \]

Since each \( K_k \) is compact there exists finite sets \( \{x_k, j\} \) such that \( K_k \subseteq \cup_j \{0, j\} \). Since \( \mathfrak{Q} \subseteq \mathfrak{Q} \), there is an \( f \in \mathfrak{Q} \) such that 
\[ f \in \cap W(x_k, j, U_k^* \mathfrak{Q}) \cap \mathfrak{Q} \text{ for all } k \text{ and } j. \]

We claim that 
\[ f \in \cap W(K_k, U_k^* \mathfrak{Q}) \cap \mathfrak{Q}. \]

There is a \( g \in \mathfrak{Q} \) such that 
\[ (g, f) \in \cap W(x_k, j, U_k^* \mathfrak{Q}). \]
Let \( x_k \in K_k \). Then there is a \( j \) such that \( x_k \in 0, j \), and hence 
\[ (f(x_k, j), f(x_k)) \in U_k^* \text{ and } (f(x_k, j), g(x_k)) \in U_k^*. \]
Also \( (g(x_k), f(x_k)) \in U_k^* \). Then 
\[ (g(x_k), f(x_k)) \in U_k^* \cap U_k^* \subseteq U_k^*. \]
Hence \( (g(x_k), f(x_k)) \in U_k^* \) for each \( x_k \in K_k \), and for \( 1 \leq k \leq n \). Then 
\[ f \in \cap W(K_k, U_k^* \mathfrak{Q}) \cap \mathfrak{Q}. \]
We now show that $\mathcal{S}_s \subseteq \mathcal{S}_S$. Let $\mathcal{V}, \mathcal{W} \subseteq \mathcal{T}$ and let $\mathcal{J} : \mathcal{V} \to \mathcal{W}$. Let $\{\mathcal{J}_i\}$ and $\{\mathcal{H}_j\}$ be finite covers from below of $\mathcal{J}$ and $\mathcal{H}$ respectively. Then there is an $i^*$ and a $j^*$ such that $\mathcal{J}_{i^*} \subseteq (\mathcal{U}^*) \mathcal{H}_{j^*}$ for all $\mathcal{U}^* \in \mathcal{T}(\mathcal{S}^*)$. We claim that $\mathcal{J}_{i^*} \subseteq (\mathcal{U}^*) \mathcal{H}_{j^*}$ for all $\mathcal{U}^* \in \mathcal{T}(\mathcal{S}^*)$. Let $\mathcal{U}^*$ be in $\mathcal{T}(\mathcal{S}^*)$. For $1 \leq k \leq n$, let $x_k \in X$ and $U_k^* \subseteq \mathcal{U}^*$. Then if $U^* = \bigcap U_k^*$, $U^* \subseteq \mathcal{U}^*$ and $K = \bigcup \{x_k\} \subseteq X$ is compact. Hence $W(K, U^*) \subseteq \mathcal{J}_{i^*} \mathcal{H}_{j^*} \neq \emptyset$. But $W(K, U^*) \subseteq \bigcap W(x_k, U_k^*)$ and therefore $\bigcap W(x_k, U_k^*) \subseteq \mathcal{J}_{i^*} \mathcal{H}_{j^*} \neq \emptyset$. Therefore we have $\mathcal{J}_{i^*} \subseteq (\mathcal{U}^*) \mathcal{H}_{j^*}$. Thus $\mathcal{V} \subseteq \mathcal{W}$.

**III.3.16. Theorem.** Suppose $(\mathcal{X}^*, \mathcal{S}^*)$ is p-totally bounded. Suppose $\mathcal{Q}^*$ is a non-empty subset of $[X, X^*]$ and is equicontinuous. Then $\mathcal{Q}_s \mathcal{Q}^* = \mathcal{Q}_s \mathcal{Q}^* = \mathcal{Q}_w \mathcal{Q}^* = \mathcal{Q}_w \mathcal{Q}^*$.

**Proof.** From III.3.7, III.3.10, and III.3.15 we have $\mathcal{Q}_w \mathcal{Q}^* \subseteq \mathcal{Q}_w \mathcal{Q}^* \subseteq \mathcal{Q}_s \mathcal{Q}^* \subseteq \mathcal{Q}_w \mathcal{Q}^*$. But, by III.1.14, $\mathcal{Q}_w = \mathcal{Q}_s$ and hence $\mathcal{Q}_w \mathcal{Q}^* = \mathcal{Q}_s \mathcal{Q}^*$, completing the list of equalities.

**III.3.17. Lemma.** Suppose $(\mathcal{X}, \mathcal{Q})$ is a topological space and $(\mathcal{X}^*, \mathcal{S}^*)$ is a p-space. Suppose $\mathcal{Q}^* \subseteq [X, X^*]$, $\mathcal{Q}^* \neq \emptyset$, and $\mathcal{Q}^*$ is equicontinuous. Then $c(\mathcal{Q}^*)$, the closure of $\mathcal{Q}^*$ relative to $\mathcal{Q}^* (\mathcal{Q}_w)$, is also equicontinuous.

**Proof.** Suppose $P^* \in c(\mathcal{S}^*)$ and $x \in X$. Let $P^*_o$ be symmetric such that $P^*_o P^*_o P^*_o \subseteq P^*$. Then there is an $0 \in \mathcal{Q}^*$ such that $f[0] \subseteq P^*_o[f(x)]$ for all $f \in \mathcal{Q}^*$. We claim that $g[0] \subseteq P^*[g(x)]$ for all $g \in c(\mathcal{Q}^*)$. Since $g \in c(\mathcal{Q}^*)$, there is a net $(T, D)$ ranging in $\mathcal{Q}^*$ such that $(\mathcal{V}_x, T, D)$ converges
to \( g(x) \) for each \( x \in X \). Let \( y \in 0 \). Since \( T(d) \in \mathcal{F} \), we have 
\((T(d)(x), T(d)(y)) \in P_o^* \) for all \( d \in D \). Also, since \((\bigcap_x T,D)\) converges to \( g(x) \), there is a \( d^* \in D \) such that \((T(d^*)(x), g(x))\) is in \( P_o^* \) and \((T(d^*)(y), g(y)) \in P_o^* \). Then \((g(x), g(y)) \in P^* \) or \( g(y) \in P^*[g(x)] \).

**III.3.18. Remark.** The following Ascoli-type theorem is similar to one found in Kelley [16].

**III.3.19. Theorem.** Suppose \((X, \mathcal{O})\) is a topological space and \((X, \mathcal{O})\) is locally compact (every point has a compact neighborhood). Suppose \((X^*, \mathcal{O}^*)\) is a separated p-space and \( \mathcal{F} \) is a non-empty subset of \( t[X, X^*] \). Then \((\mathcal{F}, \mathcal{O}, \mathcal{O}^*)\) is compact iff

1. \( \mathcal{F} \) is closed in \((t[X, X^*], \mathcal{O}(\mathcal{O}) \cap t[X, X^*])\).
2. \( \mathcal{O}[x] \) is compact for all \( x \in X \), and
3. \( \mathcal{O} \) is equicontinuous.

**Proof.** Suppose \((\mathcal{F}, \mathcal{O}, \mathcal{O}^*)\) is compact. Then, since \( \mathcal{O}_w \leq \mathcal{O}_s \) by III.3.10, \( \mathcal{O}(\mathcal{O}_w) \\mathcal{O}(\mathcal{O}) \) and thus \((\mathcal{F}, \mathcal{O}, \mathcal{O}^*)\) is compact. Then since \( \bigcap_x ([X, X^*], \mathcal{O}(\mathcal{O}_w)) \to (X^*, \mathcal{O}(\mathcal{S}^*)) \) is continuous for each \( x \in X \), \( \bigcap_x [\mathcal{O}^*] = \mathcal{F}[x] \) is compact thus proving (2). We note that (1) follows from III.1.20.

Suppose \( x \in X \) and \( P^* \in \mathcal{O}(\mathcal{S}^*) \). Let \( P^*_o \in \mathcal{O}(\mathcal{S}^*) \) be open and symmetric such that \( P^*_o \circ P^*_o \circ P^*_o \leq P^* \). Since \( x \in X \), there is a \( K \) compact and an \( \mathcal{O} \in \mathcal{O} \) such that \( x \in \mathcal{O} \subseteq K \). Since \( P^*_o \in \mathcal{O}(\mathcal{S}^*) \) there is a \( U^* \in \bigcap(\mathcal{S}^*) \) such that \( P^*_o \in U^* \).

Then \( W(K, P^*_o) \in \mathcal{U}(\text{u.o., C, U}^*) \subseteq \mathcal{O}(\mathcal{O}_s) \). Since \((\mathcal{F}, \mathcal{O}, \mathcal{O}^*)\) is compact, there is a finite collection of functions \( \{ f_i \} \)
such that \( \mathcal{A} \subseteq \cup W(K, P_0)[f_1] \). Since each \( f_1|_0 \) is continuous, there is an \( 0_1 \in \mathcal{A} \), \( x \in 0_1 \subseteq 0 \) such that \( f_1[0_1] \subseteq P_0[f_1(x)] \).

Let \( 0^* = \cap 0_1 \). Then \( x \in 0^* \in \mathcal{A} \), and \( 0^* \subseteq K \). If \( y \in 0^* \) then \( (f_1(x), f_1(y)) \in P_0^* \). If \( f \in \mathcal{A} \), then \( (f, f_1) \in W(K, P_0^*) \) for some \( i \) and hence \( (f(x), f_1(x)), (f(y), f_1(y)) \in P_0^* \). Then \( (f(x), f(y)) \) is in \( P^* \) and \( f(y) \in P_0^*[f(x)] \) for all \( f \in \mathcal{A} \). Hence \( \mathcal{A} \) is equicontinuous.

Suppose conditions (1), (2), and (3) hold. Since \( \mathcal{A}[x] \) is compact, \( \times \{ \mathcal{A}[x] \colon x \in X \} \) is compact in \( \mathcal{T}(\mathcal{A}[x]) \).

Since \( c(\mathcal{A}) \) (relative to \( \mathcal{T}(\mathcal{A}[x]) \)) is a closed subset of \( \times \{ \mathcal{A}[x] \colon x \in X \} \), \( c(\mathcal{A}) \) is compact in \( \mathcal{T}(\mathcal{A}[x]) \). Since \( c(\mathcal{A}) \) is equicontinuous by III.3.17 above, \( \mathcal{T}(\mathcal{A}[x]) \cap c(\mathcal{A}) \) is contained in \( \mathcal{T}(\mathcal{A}[x]) \cap c(\mathcal{A}) \) and hence \( (c(\mathcal{A}), \mathcal{T}(\mathcal{A}[x]) \cap c(\mathcal{A})) \) is compact. By (2) \( \mathcal{A} \) is a closed subset of \( (c(\mathcal{A}), \mathcal{T}(\mathcal{A}[x]) \cap c(\mathcal{A})) \) and hence \( (\mathcal{A}, \mathcal{T}(\mathcal{A}[x]) \cap \mathcal{A}) \) is compact.


