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DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in the Graduate School of The Ohio State University

By

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* * * * * *

The Ohio State University
1967

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PUBLICATIONS

"The Use of Photogrammetric Methods to Investigate Surface Movement of the Antarctic Ice Sheet," RF Project 1444, Final Report to N.S.F., Columbus Ohio 1964.
FIELDS OF STUDY

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Studies in Geodesy, Professors U. Uotila, I. Mueller, and R. Rapp

Studies in Photogrammetry, Professors A. Brandenberger, and S. Ghosh

Studies in Adjustment, Professor U. Uotila

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1. INTRODUCTION

1.1 Scope and Objective

The instruments whose function is to provide some information pertaining to the outer orientation of the aerial mapping camera at the moment of exposure are generally termed auxiliary instruments. These instruments can be divided into two major groups:

A. Those which provide directly some orientation parameters or data leading directly to the determination of orientation parameters.

B. Those which provide data from which orientation parameters can be deduced.

The first group can be subdivided into:

1. Instruments providing angular orientation parameters.

2. Instruments providing linear orientation parameters.

The horizon camera is one of the auxiliary instruments which directly provides angular orientation parameters and will be treated here exclusively. The selection of this particular instrument for the present analysis of auxiliary data is based on the following factors:

1. Utilizing the horizon as a source of information is not limited to the horizon camera only. In high oblique aerial photography, in panoramic photography and in some terrestrial photography,
the horizon is an integral part of the picture. Thus the knowledge of how to use information from the horizon may serve a wider field of applications. It should be emphasized in this connection that any visible point in the horizon picture is included here in the general concept of 'horizon'.

2. The horizon camera, unlike most other auxiliary instruments, has been utilized already in actual projects in Finland, Canada, Nigeria, the Netherlands, etc.

3. The horizon camera is a relatively inexpensive, and easy to handle instrument which makes it attractive to photogrammetric organizations.

This work treats the subject of horizon-controlled analytical strip triangulation. The scope of the investigation is purposely limited to the analytical strip triangulation because while several investigations have already been made on the application of horizon data to analog instruments, there are few, if any, works on this subject in connection with the analytical approaches to photogrammetry.

The general trend in former investigations in this area was to find better ways of applying the horizon determined parameters rather than to find new and better ways of determining them in the first place. The present work is intended to fill this gap. Chapter II outlines the conventional methods of determining the parameters from horizon pictures. These methods are compared later with the new suggested approach. The new set of equations
provide the absolute values of the tilt components φ and ω and under certain conditions, also the value of the swing (κ). An analysis is also conducted to determine the effect of κ on the determination of φ and ω in the event that κ cannot be determined. The entire analysis is backed by experiments utilizing fictitious data. Apart from the fact that no adequate horizon photography were available for use in this project, the advantage of using fictitious data is that the results are not limited to the particular set of photographs under investigation but enable a general investigation of the geometry of the suggested method.

Chapter III discusses the application of the horizon-determined parameters in the strip triangulation. A brief study is made of using the parameters for orienting one photograph to the previous one, in the strip formation phase. This approach is not recommended due to the rather limited accuracy of the horizon data at the present stage of the technique. The suggested method involves the determination of the tilt components of the entire model from the horizon-determined parameters. The application of these tilt components, i.e., the rotation of each model in the strip, take place in the adjustment phase rather than during the strip formation. The adjustment method aims at adjusting each individual model without resorting to polynomials. Experiments have been conducted using the fictitious data made available to the University of Illinois by the International Society of Photogrammetry, (Commission III). The results compared favorably with results obtained by conventional adjustment procedures. It is recommended that...
should be focused on developing more sensitive emulsions for the horizon camera, improving the geometry of the camera by using wide angle lenses, and investigating the effectiveness of the suggested method in long strips. These conclusions are embodied in Chapter IV.

1.2 Historical Note

The principles and use of the horizon camera in photogrammetry were originated in Finland in 1928. General Nenonen and Dr. Lofstrom developed and improved the methods of extracting the outer orientation elements φ and ω (i.e., roll and pitch angles) from pictures obtained from the horizon camera built by Zeiss Aerotopograph. The camera photographed the horizon in two directions; along the flight direction and perpendicular to it. The horizon photography was synchronized with the photography taken by the main camera. The horizon pictures were reflected onto the film of the main camera so that the resulting single photograph, \( 18 \times 24 \text{ cm}^2 \) in size, contained an \( 18 \times 18 \text{ cm}^2 \) picture of the area and a \( 6 \times 18 \text{ cm}^2 \) picture of the two horizon sectors.

Professor 0. Von Gruber was very enthusiastic about the concept and termed it the "universal method of photogrammetry." He and Dr. W. Brucklacher further investigated the method between the years 1934-1940 when they were interrupted by World War II. However, the method did not gain much popularity outside Finland and the World War II put a temporary end to the use of the horizon
camera as an auxiliary system. In the late 1950's the investigations were renewed and in 1962 the Wild Company manufactured the HC-1 horizon camera. This camera photographs the horizon in four directions and the images are not reflected to the main camera but are registered separately. This enables the use of two different types of emulsions suitable for the main camera and for the horizon camera, and thus improving the quality of the horizon pictures. The use of four horizon pictures, rather than only two, improved the accuracy and dependability of the determination of the tilt components. These improvements, coupled with the advance of photogrammetric measuring devices and techniques opened new areas for the application of the auxiliary data. Thus, instead of using the data only for rectification purposes as was the case in the 20's, they could now be employed in association with aerial triangulation.

1.3 The WILD HC-1 Horizon Camera

The main characteristics of the horizon camera are:

1. The camera shutter is synchronized with each exposure of the main camera (WILD RC-7a, RC-8, RC-9);

2. Four horizon pictures in directions separated by 90° are photographed simultaneously;

3. The orientation of the horizon camera is independent of the flight direction, that is, the camera can rotate about its vertical axis corresponding to the drift setting of the main camera;
Figure 1  WILD HCl Horizon Camera
(With WILD RC 8 Camera)

Figure 2  One Registration at Horizon Camera
4. The camera is adjustable in height;

5. The horizon is recorded on an 35 mm film with infrared or panchromatic emulsion, together with a counter, note tablet and a second clock;

6. Nine fiducial marks are registered on each picture;

7. The dimensions of the camera are:

   principal distance = 34 mm, picture size = 8 x 24 mm,

   Horizontal field of view = 43° (39°).

   Vertical field of view = 15.95 (14°).

1.4 The Horizon and the Horizon Picture

Using trigonometric leveling formulas (Plane and Geodetic Surveying - Vol. II by D. Clark, 4th edition, 1961), the difference in elevation between two points on the surface of the earth is, in first approximation:

\[ h_2 - h_1 = d \left(1 + \frac{h_1 + h_2}{2R}\right) \cot Z_1 + \frac{1 - K}{2R} d^2, \quad (1-1) \]

where \( K \) is the refraction index and the other terms are graphically explained in Fig. (3). Since \( (h_1 + h_2) \) is a small quantity compared with \( 2R \), the term \( \frac{h_1 + h_2}{2R} \) can be neglected, and thus:

\[ h_2 - h_1 = d \cot Z_1 + \frac{1 - K}{2R} d^2. \quad (1-2) \]
Assuming $h_2 = 0$ (i.e., point 2 is on the true horizon), the equation is then:

$$-h_1 = d \cot Z_1 + \frac{1 - K}{2R} d^2,$$

or

$$\cot Z_1 = -\frac{h_1}{d} - \left( \frac{1 - K}{2R} \right) d.$$  \hspace{1cm} (1-4)

Reversing the direction of observation while $h_2$ is still kept at zero, it follows that

$$h_1 = d \cot 90^\circ + \frac{1 - K}{2R} d^2 = \frac{d^2}{2R} (1 - K),$$  \hspace{1cm} (1-5)

or:

$$d = \sqrt{\frac{2R h_1}{1 - K}}.$$  \hspace{1cm} (1-6)

Substituting equation (1-6) in equation (1-4), the following expression is obtained:

$$-\cot Z_1 = \frac{h_1}{\sqrt{\frac{2R h_1}{1 - K}}} + \left( \frac{1 - K}{2R} \right) \sqrt{\frac{2R h_1}{1 - K}}.$$  

Squaring both sides in the preceding expression; that is,

$$\cot^2 Z_1 = \frac{h_1^2 (1 - K)}{2R h_1} + \frac{(1 + K)^2 \cdot 2R h_1}{4R^2 \cdot (1 - K)} + \frac{2h_1 (1 - K)}{d \cdot 2R}.$$
then:

\[
cot^2 Z_1 = \frac{4 (1 - K) h_1}{2R} , \quad \text{and}
\]

\[
cot Z_1 = 2 \sqrt{\frac{(1 - K) h_1}{2R}} .
\]  

(1-7)

If the true horizon is not observed (i.e., \( h_2 \neq 0 \)), equation (1-2) provides the following relation:

\[
cot Z_1 = \frac{h_2 - h_1}{d} - \frac{(1 - K) d}{2R} .
\]  

(1-8)

Differentiating the above expression with respect to \( h_2 \), it follows that

\[
\frac{\partial \cot Z_1}{\partial h_2} = \frac{1}{d} .
\]

The effect of \( h_2 \) on the distance \( d \) is sufficiently small to be neglected (see following example). Substituting equation (1-6) in the above equation, then

\[
\partial \cot Z_1 = \sqrt{\frac{(1 - K)}{2R h_1}} \cdot \partial h_2
\]  

(1-9)

Equation (1-9) represents the effect of an error in the determination of the apparent horizon elevation upon the determination of
the cot of the zenithal angle $Z$.

Assuming $h_1 = 3$ km (Flight height),

$$ K = 0.14 \text{ (Standard atmospheric refraction coefficient)} $$

and

$$ R = 6375 \text{ km (Approximate radius of earth)}, $$

then it follows from equation (1-6) that: $d = 210$ km, and from equation (1-7) that: $(Z - 90^\circ) = 38$ minutes of arc.

---

**Figure 3** Trigonometric Levelling Scheme
Using the above example, the scale of the horizon in a picture taken with WILD HC-1 is about 1:6200000. For a picture size of 8 x 24 mm, the length of the horizon registered in the picture is about 150 km. If the nadiral aerial mapping camera is a standard one (P = 150 mm. Picture size = 23 x 23 cm) and if the regular 60% forward overlap is used, then the air base between two exposure stations is 1.84 km. From these data it is apparent that eighty-three successive sideway horizon pictures show practically the same part of the horizon.

The apparent horizon on the picture is not necessarily the true one. It might be one of three possibilities:

1. A vapour horizon
2. A line of clouds
3. A mountain ridge, or in general; ground in the distance.

Over water or under special atmospheric conditions, the true horizon may be observed, however, the vapour horizon, a line of clouds or ground in the distance can yield the same information as does the true horizon provided that:

1. There is no change or break of the apparent horizon along the strip
2. The apparent horizon is level all around.

Condition #1 must be maintained whenever the horizon camera is used. Condition #2 is not an absolute necessity in case differential quantities are to be extracted from the horizon pictures. However, no absolute quantities can be obtained if this condition
is not realized, unless some sophisticated methods are provided in an attempt to overcome this obstacle.

If the true horizon is observed, $Z$ is computed from equation (1-7). If an apparent horizon is observed and its elevation is known (such as in the case where a mountain ridge is observed), $Z$ can be computed from equation (1-8). If the elevation of the apparent horizon is not known (such as in the case where a vapour horizon is observed), a nadiral photograph containing the images of a sufficient number of ground control points is utilized; a space resection can be performed and the resulting orientation elements are introduced in equation (2-27), from which $Z$ can be deduced. A more sophisticated method for the above case is suggested in Section 2.33.
2. DETERMINATION OF TILT COMPONENTS

2.1 Conventional Methods

If the airplane is tilted (\(\alpha\)) about its flight axis at the moment of exposure, the front and back horizon pictures will record an inclined horizon with respect to the average line connecting the central fiducial marks (see Figure 4). If in addition, the airplane is tilted (\(\beta\)) about an axis perpendicular to the flight axis, the same horizon pictures will record also a displaced horizon i.e., an upward or downward shift of the horizon with respect to the central fiducial marks. The same, but in a reversed order, will be the case with the left and right pictures. (see Figure 6). The underlying assumptions involved here are:

1. The two tilt components are independent of each other.
2. The X axis of the nadiral camera is parallel to the flight axis.
3. The X axis of the horizon camera is parallel to that of the nadiral one.

2.11 Monocular Methods

Using the above notations, the tilt angle \(\alpha\) can be obtained in one of the following ways:

1. Measuring the inclination of the horizon line with a protractor.
2. Measuring the lengths \(b\) and \(c\) (see Figure 4) with a scale, monocomparator or any other linear measuring device, and substituting these values in the following equation:
\[ \tan \alpha = \frac{b - c}{D}, \quad (2-1) \]

where \( D \) is the distance between fiducial marks 1 and 3. The tilt angle \( \beta \) can be obtained by measuring distance \( a \) (Figure 4) and using the equation:

\[ \tan \beta = \frac{a}{P}, \quad (2-2) \]

where \( P \) is the principal distance of the horizon camera. In order to increase the accuracy of angle determination, various suggestions have been made such as:

1. Use of precise measuring devices (monocomparators)
2. Use of averaged values (for example: \( \tan \beta = \frac{a + b + c}{3P} \))
3. Extending the above equations to include compensations for atmospheric refraction, zenithal angle and the fact that the image of the horizon is actually hyperbolic rather than a straight line.
4. Using only equation (2-2) in the different pictures, for reasons which will be explained later.

These monocular methods yield the absolute tilt components of the aerial camera at the moment of exposure.

---

**Figure 4** Distances on the Horizon Picture
2.12 Stereoscopic Methods

Stereoscopic methods have replaced monocular methods almost exclusively. It was generally accepted that ---

"it is not primarily a question of a direct determination of absolute components of inclination but rather of an indirect measuring of the mutual variations in inclinations of the photographs in a series of pictures." [13]

The most widely used procedure is:

"to chose a reference picture at approximately eight exposure intervals and measure the differences in the position of the horizon between the reference picture and other pictures of the line. The reference horizon picture is placed in the left frame of the stereomicroscope so that the horizon line is perpendicular to the instrument base. The other horizon pictures of the same strip are placed one by one in the right frame in the same position. The horizontal parallaxes of the central fiducial marks \( R_1, R_2, R_3 \) and the horizontal parallaxes of the adjacent points on the horizon line \( h_1, h_2, h_3 \) in corresponding horizon pictures are stereoscopically measured by using the floating mark. The differences of these parallax readings \( R_1 - h_1, R_2 - h_2, R_3 - h_3 \) express the displacement of the horizon line in the measured horizon picture relative to the horizon line in the corresponding reference horizon pictures." [17]

With the above procedure, the relative tilt components can be computed using the following equations (See [17]):

\[
\Delta \alpha = \frac{\left( R_1 - h_1 \right) - \left( R_3 - h_3 \right)}{D \rho^c}, \quad (2-3)
\]

and

\[
\Delta \beta = \frac{\left[ R_1 - h_1 \right] + \left( R_2 - h_2 \right) + \left( R_3 - h_3 \right)}{3P \rho^c}. \quad (2-4)
\]
Application of the law of error propagation to the above two equations indicates that the accuracy of equation (2-4) is superior to that of equation (2-3). Since the angle $\Delta \beta = \Delta \omega$ in the left and right horizon pictures, and $\Delta \phi$ in the front and back horizon pictures, equation (2-4) alone is used.

The results, as explained above, are differences of tilt components and ---

"---to determine the absolute values of $\phi$ and $\omega$, it is necessary that at least one model in each flight line be fully controlled in elevation... Then the absolute $\phi$ and $\omega$ of any subsequent photo in the strip can be computed by adding to the absolute values the differences $\Delta \phi$ and $\Delta \omega$ determined from the horizon pictures." [17]
2.13 Summary of Conventional Methods Studied

The two components of the tilt can be obtained by using any of the horizon pictures. The term "inclination angle" will be used here to denote the angle associated with the length of the horizon picture, and the term "declination angle" will be used to denote the angle associated with the focal length (see Figure 6).

Essentially the horizon equations utilized in the stereoscopic methods do not differ from those utilized in the monocular methods [compare equation (2-3) with (2-1) and equation (2-4) with (2-2)] except for the fact that the former yield absolute quantities, and the later, differential quantities. As a matter of fact, the difference between utilizing stereoscopic or monocular vision is only in the quality of the observation, not in concept. This fact leads to the conclusion that no basic change in data acquisition, regarding the horizon pictures, has been made since the early 20's. Because of this fact, the term "conventional methods" will be used here to denote the methods described so far as well as other methods which do not deviate significantly from the above mentioned ones. On the other hand, the term "new approach" will indicate the method to be suggested in the following chapters. For later comparison between the new approach and the conventional methods, the method used most often, as outlined above, will serve as a reference. The characteristics of this method can be summed up as follows:

1. It utilizes stereoscopic vision and parallax measurements;
2. It utilizes two horizon pictures for each tilt component
determination, preferring the use of declination angle rather than the inclination angle;

3. It yields differential values of tilt components.

2.2 The New Approach

2.2.1 Fundamental Concepts and Relations

The new treatment of horizon camera pictures is based upon a general approach to analytical photogrammetry developed by Inghilleri[5,6]. The underlying principle of this approach is the treatment of the bundle of rays which constitute the photograph as a basic unit, rather than the photograph itself. With such an approach, the question of the type of the photograph utilized loses its significance. Figure 7 demonstrates this statement graphically by showing that the geometry of the rays does not change, regardless of whether a vertical, convergent, panoramic or any other type of photograph is used.
With five cameras (one nadiral and four horizontal) operating simultaneously at each air station, the situation can be viewed as having one bundle of rays, originating from one center and spread from horizon to horizon. Of course, this concept can be realized only when the geometrical relations between the four horizon cameras with respect to each other and to the nadiral one are known.

The measurement of plate coordinates are merely a tool for determining the direction of the rays belonging to the bundle. The conventional direction-cosines are replaced here by direction tangents, each of which can be viewed as a ratio between two direction cosines, a ratio that is dominant in most analytical equations. Thus while in concept there is no difference between the above two approaches, the use of direction tangents, in the opinion of the writer, offers a more compact and simplified set.
of equations. A direction in space, referred to a rectangular right handed coordinate system, passes through point \( O (X_O, Y_O, Z_O) \) and point \( E (X_E, Y_E, Z_E) \) in the direction \( O \rightarrow E \). The projections of this direction on the \( XY \), \( XZ \), \( YZ \) planes can be determined by corresponding pairs of direction tangents (see Figure 8) as follows:

\[
\tan \theta_A = \frac{X_E - X_O}{Y_E - Y_O} = t_A ,
\]

\[
\tan \theta_B = \frac{Y_E - Y_O}{X_E - X_O} = t_B ,
\]

\[
\tan \theta_X = \frac{X_E - X_O}{Z_E - Z_O} = t_X ,
\]

\[
\tan \theta_C = \frac{Z_E - Z_O}{X_E - X_O} = t_C ,
\]

\[
\tan \theta_Y = \frac{Y_E - Y_O}{Z_E - Z_O} = t_Y ,
\]

\[
\tan \theta_Z = \frac{Z_E - Z_O}{Y_E - Y_O} = t_Z .
\]

Given two direction-tangents, not belonging to the same pair, all other direction-tangents can be computed and thus direction in space can be determined by utilizing different couples of direction
tangents. A pair of direction-tangents does not define the quadrant in which $\theta$ is located and therefore, it is necessary also to know the hemisphere within which direction $d$ is located. However, for nadiral and horizon photography, all directions are in the negative hemisphere ($Z<0$). The nature of the "tangent" must be taken into
consideration when choosing the couples of direction-tangents to be used at critical locations where the angle $\theta$ approaches $90^\circ$ or $270^\circ$. Thus for example in near vertical photograph, $t_x$ and $t_y$ can be safely used, for a photograph tilted about its $X$ axis ($\omega$), $t_A$ and $t_Z$ will be used, and for a photograph tilted about its $Y$ axis ($\varphi$), $t_B$ and $t_C$ will be used. However, except for the possible change of couples of direction tangents, the general equations are valid.

The following definitions will be used throughout this treatment:

The measured bundle is defined by the values of the direction tangents of its corresponding rays as deduced from plate measurements and denoted $t_x$, $t_y$, $t_A$, $t_Z$, $t_B$, $t_C$.

The transformed bundle is defined by the values of the direction-tangents of the rays which have undergone a spatial transformation and denoted $t'_x$, $t'_y$, $t'_A$, $t'_Z$, $t'_B$, $t'_C$.

The oriented bundle of rays is defined by the values of the direction-tangents satisfying certain conditions imposed upon them and denoted $T_x$, $T_y$, $T_A$, $T_Z$, $T_B$, $T_C$.

If an internal reference system having its origin at the perspective center (0), is set parallel to the plate coordinate system defined by the plates' fiducial marks, and if the measured plate coordinates are reduced to the principal point of the plate and denoted $x^*$, $y^*$ (see Figure 9), then the "measured" direction-tangents of point E, following equation (2-6) are:
Since $z_E^* = -P$ (principal distance) in the above described internal coordinate system it follows that:

$$t_X = \frac{x_E^* - 0}{z_E^* - 0}, \quad \text{and} \quad t_Y = \frac{y_E^* - 0}{z_E^* - 0}.$$  \hfill (2-8)

The well known spatial transformation formulas read, in matrix notation:

$$\mathbf{C}' = \mathbf{S} + \mathbf{CM},$$ \hfill (2-9)
where:

\[ C' = [X' Y' Z'] = \text{The transformed coordinates}; \]

\[ C = [X Y Z] = \text{The original coordinates}; \]

\[ S = [S_1 S_2 S_3] = \text{The translations along XYZ axes}; \]

\[
M = \begin{bmatrix}
\cos \varphi \cos \kappa & \cos \omega \sin \kappa & \sin \omega \sin \kappa \\
+\sin \omega \sin \varphi \cos \kappa & -\cos \omega \sin \varphi \cos \kappa & \\
-\cos \varphi \sin \kappa & \cos \omega \cos \kappa & \sin \omega \cos \kappa \\
-\sin \omega \sin \varphi \sin \kappa & +\cos \omega \sin \varphi \sin \kappa & \\
\sin \varphi & -\sin \omega \cos \varphi & \cos \omega \cos \varphi
\end{bmatrix}
\]

\[
= \begin{bmatrix}
m_{11} & m_{12} & m_{13} \\
m_{21} & m_{22} & m_{23} \\
m_{31} & m_{32} & m_{33}
\end{bmatrix}
= \begin{bmatrix} M_X & M_Y & M_Z \end{bmatrix} \quad (2-10)
\]

The matrix refers to the right handed coordinate system and rotation convention used in the U.S.,\cite{15} where \( \omega \) is the primary rotation, \( \varphi \) the secondary and \( \kappa \) the tertiary rotation.
Using the above notations, the transformation formulas can be written as:

\[ X^i_E = S_1 + X \cdot m_{11} + Y \cdot m_{21} + Z \cdot m_{31}, \]
\[ Y^i_E = S_2 + X \cdot m_{12} + Y \cdot m_{22} + Z \cdot m_{32}, \]
\[ Z^i_E = S_3 + X \cdot m_{13} + Y \cdot m_{23} + Z \cdot m_{33}. \]  

Equation (2-6) for example can be written as:

\[ \tan \theta^i_X = \frac{X^i_E - X^i_o}{Z^i_E - Z^i_o} = t^i_X. \]

Combining this equation and equation (2-11), one gets:

\[ t^i_X = \frac{(X^i_E - X^i_o) \cdot m_{11} + (Y^i_E - Y^i_o) \cdot m_{21} + (Z^i_E - Z^i_o) \cdot m_{31}}{(X^i_E - X^i_o) \cdot m_{12} + (Y^i_E - Y^i_o) \cdot m_{22} + (Z^i_E - Z^i_o) \cdot m_{33}}. \]

Dividing the numerator and denominator by \((Z^i_E - Z^i_o)\) and using equations (2-6), (2-7), it follows that:

\[ t^i_X = \frac{t_X \cdot m_{11} + t_Y \cdot m_{21} + m_{31}}{t_X \cdot m_{12} + t_Y \cdot m_{22} + m_{32}} = \frac{[t_X \ t_Y \ 1] \cdot \bar{M}_X}{[t_X \ t_Y \ 1] \cdot \bar{M}_Z}, \]

Where \(\bar{M}\) is a partial orientation matrix as will be explained later.

Similarly:

\[ t^i_Y = \frac{t_X \cdot m_{12} + t_Y \cdot m_{22} + m_{32}}{t_X \cdot m_{13} + t_Y \cdot m_{23} + m_{33}} = \frac{[t_X \ t_Y \ 1] \cdot \bar{M}_Y}{[t_X \ t_Y \ 1] \cdot \bar{M}_Z}. \]
Equations (2-12) represent the relations between the transformed and the measured direction tangents. Replacing \( t^i \) by \( T \) and \( t \) by \( t^i \), the following relations between the transformed and oriented direction-tangents are obtained:

\[
\begin{align*}
\frac{[t^i_A \ t^i_Z \ 1]}{[t^i_X \ t^i_Y \ 1]} \cdot \tilde{M}_X &= T_X, \\
\frac{[t^i_A \ t^i_Z \ 1]}{[t^i_X \ t^i_Y \ 1]} \cdot \tilde{M}_Y &= T_Y, \\
\frac{[t^i_A \ t^i_Z \ 1]}{[t^i_X \ t^i_Y \ 1]} \cdot \tilde{M}_Z &= T_Z, \\
\frac{[t^i_B \ t^i_C \ 1]}{[t^i_X \ t^i_Y \ 1]} \cdot \tilde{M}_X &= [1 \ t_B \ t_C] \cdot \tilde{M}_X, \\
\frac{[t^i_B \ t^i_C \ 1]}{[t^i_X \ t^i_Y \ 1]} \cdot \tilde{M}_Y &= [1 \ t_B \ t_C] \cdot \tilde{M}_Y, \\
\frac{[t^i_B \ t^i_C \ 1]}{[t^i_X \ t^i_Y \ 1]} \cdot \tilde{M}_Z &= [1 \ t_B \ t_C] \cdot \tilde{M}_Z.
\end{align*}
\]
\[
\begin{align*}
\left[ t_A^t \quad 1 \quad t_Z^t \right] \cdot \vec{M}_X &= \vec{T}_A , \\
\left[ t_A^t \quad 1 \quad t_Z^t \right] \cdot \vec{M}_Y &= \vec{T}_A , \\
\left[ t_A^t \quad 1 \quad t_Z^t \right] \cdot \vec{M}_Z &= \vec{T}_Z , \\
\left[ t_A^t \quad 1 \quad t_Z^t \right] \cdot \vec{M}_Y &= \vec{T}_Z ,
\end{align*}
\]

(2-13)

\[
\begin{align*}
\left[ 1 \quad t_B^t \quad t_C^t \right] \cdot \vec{M}_Z &= \vec{T}_C , \quad \text{and} \\
\left[ 1 \quad t_B^t \quad t_C^t \right] \cdot \vec{M}_X &= \vec{T}_B .
\end{align*}
\]

Finally, the relations between the measured and oriented direction-tangents can be expressed as follows:

\[
\begin{align*}
\left[ t_X^t \quad t_Y^t \quad 1 \right] \cdot \vec{M}_X &= \vec{T}_X , \\
\left[ t_X^t \quad t_Y^t \quad 1 \right] \cdot \vec{M}_Y &= \vec{T}_X , \\
\left[ t_X^t \quad t_Y^t \quad 1 \right] \cdot \vec{M}_Z &= \vec{T}_Y , \\
\left[ t_A^t \quad 1 \quad t_Z^t \right] \cdot \vec{M}_X &= \vec{T}_A , \\
\left[ t_A^t \quad 1 \quad t_Z^t \right] \cdot \vec{M}_Y &= \vec{T}_A , \\
\left[ t_A^t \quad 1 \quad t_Z^t \right] \cdot \vec{M}_Z &= \vec{T}_Z ,
\end{align*}
\]

(2-14)
The orientation matrices used in equations (2-12), (2-13), (2-14) i.e. $\bar{M}$, $\bar{M}$ and $M$, do not all contain the same values. The $M$ of equation (2-14) contains the rotation elements required to rotate a direction from its measured position all the way to its oriented position, while the $M$ of equations (2-12) and (2-13) contains the elements which yield only a partial rotation of the direction.

2.22 The Horizon Equations - General Problem

For the following investigation, only one couple of direction-tangents will be utilized, namely the couple associated with the left horizon camera. The fundamental equations as deduced from equation (2-14) are:

\[
\begin{align*}
\begin{bmatrix} t_A & t_x & t_z \end{bmatrix} \cdot M_x &= T_A, \\
\begin{bmatrix} t_A & t_x & t_z \end{bmatrix} \cdot M_y &= T_B,
\end{align*}
\]

and

\[
\begin{align*}
\begin{bmatrix} t_A & t_x & t_z \end{bmatrix} \cdot M_z &= T_Z.
\end{align*}
\]  

If direction tangents $T_A$ and $T_Z$ can somehow be obtained, the values of orientation elements $\varphi$, $\omega$, and $\kappa$, contained in the expressions $M_x$, $M_y$, and $M_z$, could then be determined from equation (2-15) and
In the case of terrestrial photogrammetry, $T_A$ and $T_Z$ can be measured with a theodolite. In the case of aerial photogrammetry, utilizing horizon pictures, the situation is more complex. If a point $E$ of known ground coordinates can be observed in the horizon picture (for example: a peak of a mountain), then $T_A$ and $T_Z$ can be computed as follows:

$$T_A = \frac{X_E - X_0}{Y_E - Y_0}, \quad \text{and} \quad T_Z = \frac{Z_E - Z_0}{Y_E - Y_0},$$

where $X_0$ and $Y_0$ are the coordinates of the perspective center of the horizon camera. The latter are generally unknown but a preliminary aerial triangulation can provide these missing data. The following outline of an iterative procedure may take place in this case:

1. Preliminary triangulation in order to determine the coordinates $X_0, Y_0, Z_0$ of the horizon camera perspective center.

2. Computation of $T_A$ and $T_Z$ by means of the above equations, utilizing the data obtained in step 1.

3. Computation of $\varphi$, $\omega$ and $\kappa$ by substituting the data obtained in step 2, in equations (2-15), (2-16).

4. Secondary adjustment of the strip, using the data obtained in step 3, in order to obtain better values of $X_0, Y_0, Z_0$.

Whereas in analog photogrammetry this procedure is quite cumbersome, it is conceivable in analytical photogrammetry.

If point $E$, whose ground coordinates are not known, is observed in the horizon pictures, then the relative positions of its images in the different pictures yield the differences in $\kappa$ between
the corresponding photographs. If the first model in the strip contains two ground control points, the absolute value of $K$ can be deduced. By adding to this value of $K$ the differences in $K$ successively, the corresponding absolute $K$ values can be obtained. The determination of $T_A$, being mainly a function of $K$ [see equation (2-23)] can now be computed. An iterative process could be applied here to get better results.

However, with the presently (1967) available photographic emulsions, the chances of observing a definite point on the horizon picture are rather slim and thus there is actually no way by which $T_A$ or $K$ can rigorously be determined. It is obvious that without the possibility to determine $K$, the other orientation elements cannot be rigorously obtained from the horizon pictures. This can be demonstrated by the hypothetical extreme case where the camera is rotated $90^\circ$ at the moment of exposure, in which case the value derived for $\phi$ corresponds actually to $\omega$ and vice versa.

Due to this problem, equations (2-15), (2-16) cannot be utilized in the way explained above. However, it is possible to set up a relationship between $T_Z$ and $T_A$. This will be the starting point of the following investigation which will assume no observable points on the horizon picture.

2.23 Horizon Equations in First Approximation

From the unit sphere in Figure 10, the following relationships can be deduced:
In the above equations, \( Z \) is the zenithal angle to any point on the horizon. Assuming true horizon and given the flight height, the zenithal angle can be computed [Equation (1-7)]. However, in order to determine \( T_z \), \( T_A \) must also be known and it is readily known only in one case, namely when \( \varphi = \omega = \kappa = 0 \), in which case \( T_A = t_A \). (\( t_A \) = the internal measured direction tangent).

\[
T_x = \tan Z \sin \theta_A, \\
T_y = \tan Z \cos \theta_A, \quad \text{and} \\
T_z = \cot Z \sec \theta_A = \cot Z (1 + \tan^2 \theta_A)^{1/2} \quad (2-17)
\]

\[
= \cot Z (1 + T_A^2)^{1/2}.
\]

Figure 10  Direction Tangents in a Unit Sphere
The error introduced by the substitution of $t_A$ for $T_A$ is investigated below:

If the values of $\varphi$, $\omega$ and $\kappa$ are sufficiently small so that higher order terms may be neglected, then the orientation matrix (2-10) can be written as:

$$M = \begin{bmatrix}
1 & \kappa & -\varphi \\
-\kappa & 1 & \omega \\
\varphi & -\omega & 1
\end{bmatrix}.$$  

(2-18)

Combining equations (2-18) and (2-15), it follows that:

$$\frac{t_A - \kappa + \varphi \cdot t_Z}{\kappa \cdot t_A + 1 - \omega \cdot t_Z} = T_A.$$  

(2-19)

The denominator of equation (2-19) can be written as:

$$1 - (\omega \cdot t_Z - \kappa \cdot t_A).$$

The term in the parentheses is very small when compared to 1 and thus, developing this expression in series, neglecting high order terms, equation (2-19) becomes:

$$(t_A - \kappa + \varphi \cdot t_Z) \left(1 + \omega \cdot t_Z - \kappa \cdot t_A\right) = T_A.$$  

(2-20)

Since the values of orientation elements were assumed to be small the product of any two of them is negligible. Taking this fact into consideration, equation (2-20) could be rewritten as:
\[ t_A - \kappa + \varphi \cdot t_Z - t_A^2 \cdot \kappa + \omega \cdot t_Z \cdot t_A = T_A, \quad (2-21) \]

or in other words:

\[ T_A - t_A = -\kappa + \varphi \cdot t_Z - t_A^2 \cdot \kappa + \omega \cdot t_Z \cdot t_A. \quad (2-22) \]

Equation (2-22) represents the error introduced by substituting \( t_A \) for \( T_A \). Moreover, \( t_Z \) is a rather small quantity when near-vertical photography is assumed and the product of its multiplication with any orientation element is negligible indeed. Thus equation (2-22) becomes:

\[ T_A - t_A = -\kappa \left(1 + t_A^2\right) = \frac{-\kappa}{\cos^2 \theta_A}. \quad (2-23) \]

In the WILD HC-1 horizon camera, \( \theta_A \) MAX. is equal to either side of its zero position (the angular field of the camera = 40°). Thus, the maximum possible error, according to equation (2-23), is:

\[ \left| T_A - t_A \right|_{\text{max.}} = \frac{\kappa}{0.88} \approx \kappa. \quad (2-24) \]

Replacing \( T_A \) by \( t_A \) in equation (2-17), then:

\[ T_Z = \cot Z \left(1 + t_A^2\right)^{\frac{1}{2}}. \]

Applying the law of error propagation to the above equation, the error in \( T_Z \) (\( eT_Z \)) is expressed as function of the error in \( t_A \) (\( et_A \)).
\[ eT_Z = \frac{\cot Z}{2 \sqrt{1 + t_A^2}} \cdot t_A \cdot \epsilon t_A \]  \hspace{1cm} (2-25)

Where \( \epsilon t_A \), according to equation (2-24), equals \( \kappa \).

In summarizing this, in order to determine \( T_Z \), \( T_A \) must be

known. \( T_A \) can be replaced by \( t_A \), which is a known quantity, only if

\( \phi = \omega = \kappa = 0 \) [see equation (2-22)]. If this is not the case, but

the values of the orientation elements are assumed to be small, an

error is committed by replacing \( T_A \) by \( t_A \), the magnitude of which

is in the order of \( \kappa \). Thus the better \( \kappa \) is known, the more reliable

is the determination of \( T_Z \) [equation (2-25)]. Obviously, the value

of \( T_Z \) will be affected also by measuring errors and by errors in the

calibration of the horizon camera. However, at this stage, only

the effect of the assumption that \( T_A = t_A \) is investigated. In this

connection, the importance of knowing \( \kappa \) has thus been demonstrated.

A closer investigation of the influence of \( \kappa \) on the determination

of \( \phi \) and \( \omega \) will now be made.

Combining equations (2-15), (2-16), and (2-17), it follows

that:

\[
\begin{align*}
\frac{[t_A \hspace{1em} 1 \hspace{1em} t_Z] \cdot M_Z}{[t_A \hspace{1em} 1 \hspace{1em} t_Z] \cdot M_Y} &= T_Z = \frac{(1 + T_A^2)^{1/2}}{\tan Z} = \frac{1}{\tan Z} \left( 1 + \left( \frac{[t_A \hspace{1em} 1 \hspace{1em} t_Z] \cdot M_X^2}{[t_A \hspace{1em} 1 \hspace{1em} t_Z] \cdot M_Y} \right)^{1/2} \right) \\
&= (2-26)
\end{align*}
\]
Combining equations (2-26) and (2-18), the following equation is deduced:

\[
\frac{-\phi \cdot t_A + \omega + t_Z}{\kappa \cdot t_A + 1 - \omega \cdot t_Z \tan Z} = \frac{1}{\tan Z} \cdot \left( \frac{(\kappa \cdot t_A + 1 - \omega \cdot t_Z)^2 + (t_A - \kappa + \varphi \cdot t_Z)^2}{(\kappa \cdot t_A + 1 - \omega \cdot t_Z)^2} \right)^{\frac{1}{2}}.
\]

In other words:

\[
-\phi \cdot t_A + \omega + t_Z = \frac{1}{\tan Z} \cdot \left( (\kappa \cdot t_A + 1 - \omega \cdot t_Z)^2 + (t_A - \kappa + \varphi \cdot t_Z)^2 \right)^{\frac{1}{2}},
\]

(2-27)

or, neglecting the products of any two orientation elements:

\[
-\phi \cdot t_A + \omega + t_Z = \frac{1}{\tan Z} \cdot (1 - 2 \omega \cdot t_Z + t_A^2 + 2 t_A \cdot t_Z \cdot \varphi)^{\frac{1}{2}}.
\]

(2-28)

The element \( t_Z \) is rather small in nature, and its multiplication with any orientation element makes the product negligible. Equation (2-28) could thus be further simplified to read:

\[
-\phi \cdot t_A + \omega + t_Z = \frac{1}{\tan Z} \cdot (1 + t_A^2)^{\frac{1}{2}}.
\]

(2-29)

Equation (2-29) does not contain the element \( \kappa \). In developing this equation, second order terms were neglected as stated above. There is no assurance, however, that those neglected terms, being located under the square root, would not turn out to be of first order in the solution. In order to be sure that the approximations applied are acceptable, and in order to be sure that \( \kappa \) indeed does not affect \( \varphi \) and \( \omega \) in the first approximation, a more rigorous approach is
used in the following test:

From equation (2-26) it is evident that

\[
\frac{[t_A \ t_z] \cdot M_Z}{[t_A \ t_z] \cdot M_Y} = \cot Z \left(1 + \frac{T_A^2}{2}\right)\quad (2-30)
\]

Applying series expansion to the term in parentheses, it follows that:

\[
(1 + \frac{T_A^2}{2})^{\frac{1}{2}} = 1 + \frac{T_A^2}{2} - \frac{1}{8} T_A^4 - \cdots \quad (2-31)
\]

Since $\theta_A$ cannot exceed half the angular field of the horizon camera, the maximum value for $t_A$ is 0.36. The third term in the above expansion ($1/8 \cdot T_A^4$) assumes, in the maximum case, the value of 0.00125. Taking into consideration that this term is multiplied by $\cot Z$ in equation (2-30), it becomes obvious that the expansion could be safely limited to the first two terms. In other words, it could be stated that:

\[
(1 + \frac{T_A^2}{2})^{\frac{1}{2}} = (1 + \frac{T_A^2}{2})\quad .\quad (2-32)
\]

Substituting equation (2-32) in equation (2-30), it follows that:

\[
\frac{[t_A \ t_z] \cdot M_Z}{[t_A \ t_z] \cdot M_Y} = \cot Z \cdot \left(1 + \frac{T_A^2}{2}\right)\quad (2-33)
\]
Substituting equation (2-15) in equation (2-33), then

\[
\frac{[t_A \ 1 \ T_Z] \cdot M_Z}{[t_A \ 1 \ T_Z] \cdot M_Y} = \cot Z \left(1 + \frac{([t_A \ 1 \ T_Z] \cdot M_X)^2}{2([t_A \ 1 \ T_Z] \cdot M_Y)^2}\right).
\]  

(2-34)

Considering the following substitutions:

\[
[t_A \ 1 \ T_Z] \cdot M_Z = N_Z,
\]

\[
[t_A \ 1 \ T_Z] \cdot M_Y = N_Y,
\]

\[
[t_A \ 1 \ T_Z] \cdot M_X = N_X.
\]

(2-35)

Equation (2-34) could be rewritten as:

\[
\frac{N_Z}{N_Y} = \cot Z \cdot \left(1 + \frac{N_X^2}{2N_Y}\right),
\]  

(2-36)

or as:

\[
2N_Z \cdot N_Y - 2\cot Z \cdot N_Y^2 - \cot Z \cdot N_X^2 = C.
\]  

(2-37)

Combining equation (2-18) and equation (2-35) it follows that:

\[
N_X = t_A - \kappa + \varphi \cdot t_Z,
\]

\[
N_Y = \kappa \cdot t_A + 1 - \omega \cdot t_Z
\]

and

\[
N_Z = -\varphi \cdot t_A + \omega + t_Z.
\]

(2-38)
Combining equations (2-38) and (2-37), and neglecting the terms of minor magnitude, the following expression is deduced:

$$2(-\varphi t_A + \omega + t_z) - 2 \cot Z (2 \kappa t_A + 1) - \cot Z (t_A^2 - 2 t_A \kappa) = 0$$

which can be reduced to:

$$-\varphi t_A + \omega + t_z = \cot Z \left(1 + \frac{t_A^2}{2}\right)$$  \hspace{1cm} (2-39)

Equation (2-39) confirms the conclusion reached from equation (2-29), i.e., it proves that the effect of \( \kappa \) on the determination of \( \varphi \) and \( \omega \) may be neglected in first approximation.

If equation (2-39) is applied to two points located at an equal distance (i.e., equal \( t_A \)) on either side of the principal point of the horizon picture, then at point 1:

$$\varphi \cdot t_A + \omega + t_{z_1} = \cot Z \left(1 + \frac{t_A^2}{2}\right)$$

and at point 2:

$$-\varphi \cdot t_A + \omega + t_{z_2} = \cot Z \left(1 + \frac{t_A^2}{2}\right)$$  \hspace{1cm} (2-40)

subtracting one equation from the other, it follows that:

$$2 \varphi \cdot t_A + t_{z_1} - t_{z_2} = 0$$
and hence;
\[ \phi^c = \frac{t_{Z2} - t_{Z1}}{2t_A} \cdot \rho^c. \] (2-41)

Since \( t_Z = \frac{z^*}{P} \) and \( t_A = \frac{x^*}{P} \) (see equation (2-8)),
equation (2-41) could be rewritten as:
\[ \phi^c = \frac{z_2^* - z_1^*}{2x^*} \cdot \rho^c, \] (2-42)

where \( z_1^* \), \( z_2^* \) and \( x^* \) are photo-coordinates of points 1 and 2, and \( P \) = principal distance of the horizon camera, (see figure 11 and 12).
Equation (2-42) is the formula used in conventional methods to determine the inclination angle, which, in the left horizon picture utilized here, is \( \phi \).

Moreover, at the principal-point of the horizon picture \( t_A = 0 \), in which case equation (2-39) becomes:
\[ \omega^c = (\cot Z - t_Z) \rho^c. \] (2-43)

The relative tilt component between two adjacent horizon pictures (\( i \) and \( i + 1 \)) could be expressed, according to equation (2-43), as
\[ \Delta \omega^c = \left( \cot Z - t_{Z_i} - (\cot Z - t_{Z_i+1}) \right) \rho^c = (t_{Z_{i+1}} - t_{Z_i}) \rho^c \]

\[ = \frac{z^*_{i+1} - z^*_i}{p} \rho^c = \frac{z^*_d}{p} \rho^c. \]

Equation (2-44) is the conventional formula for the determination of the relative declination angle, which, in the left horizon picture utilized here, is \( \Delta \omega \).

2.24 Horizon Equations in Second Approximation

In the previous treatment, small values of \( \phi \), \( \omega \), \( \kappa \) were assumed. Consequently, the orientation matrix was simplified and products of small terms were neglected while deriving the formulas. The use of near vertical photographs may justify the assumption of small values for \( \phi \) and \( \omega \) but there is no reason to expect small values of \( \kappa \) in this case. Thus it seems proper to further investigate the influence of \( \kappa \) on the determination of \( \phi \) and \( \omega \) by applying a second approximation to \( \kappa \). This is accomplished by using up to second degree terms in the series expansion of \( \cos \kappa \) and \( \sin \kappa \), while leaving only first degree terms for \( \phi \) and \( \omega \) as was done previously. In this case the orientation matrix (2-10) becomes:

\[
M = \begin{bmatrix}
1 - \frac{\kappa^2}{2} & +k & \omega \cdot \kappa - \left(1 - \frac{\kappa^2}{2}\right) \\
-\kappa & 1 - \frac{\kappa^2}{2} & \omega(1 - \frac{\kappa^2}{2}) + \phi \cdot \kappa \\
+\phi & -\omega & 1
\end{bmatrix}
\]
Combining equations (2-45) and (2-35), it follows that:

\[ N_x = t_A \left(1 - \frac{\kappa^2}{2}\right) - \kappa + \phi \cdot t_Z \]

\[ N_y = t_A \cdot \kappa + 1 - \frac{\kappa^2}{2} - \omega \cdot t_Z \]

and

\[ N_z = t_A \left(\omega \cdot \kappa - \phi + \phi \cdot \frac{\kappa^2}{2}\right) + \omega - \omega \frac{\kappa^2}{2} + \phi \cdot \kappa + t_Z \]

Substituting equations (2-46) in equation (2-37) and applying the following approximations:

1. \( \phi \cdot \omega = \phi^2 = \omega^2 = 0 \);
2. \( \kappa^n = 0 \) for \( n > 2 \);
3. \( \cot Z \cdot t_Z = 0 \);
4. \( t_Z^2 = 0 \);
5. \( \cot Z \cdot \kappa^2 = 0 \);
6. \( t_Z \cdot \kappa^2 = 0 \);

then equation (2-37) is reduced to:

\[ 2 \left[ t_A^2 \omega \kappa^2 + 2 t_A \omega \kappa - t_A^2 \phi \kappa - t_A \phi + \omega - \omega \kappa^2 + \phi \kappa^2 t_A + \phi \kappa \right. \\
+ t_Z t_A \kappa + t_Z \right] - 2 \cot Z \left[ 2 t_A \kappa + 1 \right] \\
- \cot Z \left[ t_A^2 - 2 t_A \kappa \right] = 0. \]
Rearranged, equation (2-47) reads as follows:

\[
\omega \left[ 2 t_A^2 \kappa^2 + 4 t_A \kappa + 2 - 2 \kappa^2 \right] + \varphi \left[ -2 t_A^2 \kappa - 2 t_A + 2 t_A \kappa^2 + 2 \kappa \right] + 2 t_z t_A \kappa + 2 t_z - 2 \cot Z t_A \kappa - 2 \cot Z - \cot Z \cdot t_A^2 = 0. \tag{2-48}
\]

Assuming that \( \varphi = 0 \), it follows that:

\[
\omega \left[ 2 t_A^2 \cdot \kappa^2 + 4 t_A \cdot \kappa + 2 - 2 \kappa^2 \right] + 2 t_z t_A \kappa + 2 t_z - 2 \cot Z \cdot t_A \kappa - 2 \cot Z - \cot Z t_A^2 = 0. \tag{2-49}
\]

At the principal point of the horizon picture \( t_A = 0 \), and thus

\[
\omega \left[ 2 - 2 \kappa^2 \right] + 2 t_z - 2 \cot Z = 0,
\]

or

\[
\omega \left[ 1 - \kappa^2 \right] = \cot Z - t_z. \tag{2-50}
\]

The right hand side of equation (2-50) is identical to that of the right hand side of equation (2-34) which expresses \( \omega \) in first approximation. Denoting \( \omega \) in equation (2-43) as \( \omega^* \), equation (2-50) becomes:

\[
\omega \left( 1 - \kappa^2 \right) = \omega^* , \tag{2-51}
\]

or

\[
\omega = \frac{\omega^*}{(1 - \kappa^2)} \approx \omega^* \left( 1 + \kappa^2 \right). \tag{2-52}
\]
Assuming that $\omega = 0$, equation (2-48) could be rewritten as follows:

$$
\phi \left[ 2 t_A^2 \cdot \kappa - 2 t_A + 2 t_A \cdot \kappa^2 + 2 \kappa \right] + 2 t_z \cdot t_A \cdot \kappa + 2 t_z - 2 \cot Z \\
\cdot t_A \cdot \kappa - 2 \cot Z - \cot Z \cdot t_A^2 = 0. 
$$

(2-53)

Dividing equation (2-53) by $2 t_A$, it follows that:

$$
\phi \left[ t_A \cdot \kappa - 1 + \kappa^2 + \frac{\kappa}{t_A} \right] + t_z \cdot \kappa + \frac{t_z}{t_A} - \cot Z \cdot \kappa - \frac{\cot Z}{t_A} \\
\cdot \cot Z \frac{t_A}{2} = 0,
$$

or

$$
\phi \left[ t_A \cdot \kappa - 1 + \kappa^2 + \frac{\kappa}{t_A} \right] + \kappa \left[ t_z - \cot Z \right] + \frac{t_z}{t_A} - \frac{\cot Z}{t_A} \\
\left[ 1 + \frac{t_A^2}{2} \right] = 0.
$$

The last two terms in the left hand side of the above equation are equal to $\phi$ in equation (2-39), [under the assumption that $\omega = 0$].

If $\phi$ in equation (2-39) is denoted as $\phi^*$, then the last equation can be written as:

$$
\phi \left[ t_A \cdot \kappa - 1 + \kappa^2 + \frac{\kappa}{t_A} \right] + \kappa \left[ t_z - \cot Z \right] + \phi^* = 0.
$$

(2-54)

Applying equation (2-54) to two points located at equal distances on either side of the principal point of the horizon picture (i.e.,
equal and opposite $t_A$) then at Point 1:

$$\phi \left[ t_A \kappa - 1 + \kappa^2 + \frac{\kappa}{t_A} \right] = -\kappa \left[ t_{Z_1} - \cot Z \right] - \phi^*, \tag{2-55}$$

and at Point 2:

$$\phi \left[ -t_A \kappa - 1 + \kappa^2 - \frac{\kappa}{t_A} \right] = -\kappa \left[ t_{Z_2} - \cot Z \right] - \phi^*. \tag{2-56}$$

The sum of the above two equations is:

$$\phi \left[ -2 + 2 \kappa^2 \right] = -\kappa \left[ t_{Z_1} + t_{Z_2} - 2 \cot Z \right] - 2\phi^*, \nonumber$$

or

$$-\phi \left[ 1 - \kappa^2 \right] = -k \left[ \frac{t_{Z_1} + t_{Z_2}}{2} - \cot Z \right] - \phi^*. \tag{2-56}$$

The expression in brackets at the right hand side of equation (2-56) can be analyzed as follows:

$$\frac{t_{Z_1} + t_{Z_2}}{2} - \cot Z = \frac{z_1^* + z_2^*}{2P} - \cot Z = \frac{z_1^* + z_2^* - 2P \cot Z}{2P} = \frac{(z_1^* - P \cot Z) + (z_2^* - P \cot Z)}{2P}. \tag{2-57}$$

This means that both point 1 and 2 were given a vertical shift equal to the depression of the horizon (which, on the horizon picture, is equal to $P \cot Z$), with the result that the horizon line is passing now through the principal point of the picture.
Defining

\[(z_1^* - P \cot Z) = Z_1, \quad \text{and} \]
\[(z_2^* - P \cot Z) = Z_2, \]

and recalling that the two points above are located at two opposite sides of the principal point, that \(\omega\) is assumed = 0, and that the horizon line is passing through the principal point, it follows that

\[Z_1 = -Z_2.\]

Consequently, the numerator in equation (2-57) equals zero and thus equation (2-56) can be rewritten as follows:

\[\varphi \left[1 - \kappa^2\right] = \varphi^*, \quad (2-58)\]

or

\[\varphi = \frac{\varphi^*}{(1 - \kappa^2)} \approx \varphi^* (1 + \kappa^2). \quad (2-59)\]

Equations (2-58) and (2-59) are analogous to equations (2-51) and (2-52) derived for \(\omega\).

Equations (2-52) and (2-59) lead to the following conclusions:

The effect of \(\kappa\) on the determination of \(\varphi\) is the same as on the determination of \(\omega\). This effect can be neglected in first approximation only. If \(\alpha\) denotes either \(\varphi\) or \(\omega\), then according to the above equations
\[ \alpha = \alpha^* (1 + \kappa^2) , \]
\[ \alpha = \alpha^* + \alpha^* \kappa^2 , \]
\[ \alpha - \alpha^* = \alpha^* \kappa^2 , \text{ and} \]
\[ \frac{\alpha - \alpha^*}{\alpha^*} = \kappa^2 . \quad (2-60) \]

If the range of tilt for a near vertical photograph is \( \pm 5^\circ \), and if the error of determining the tilt does not exceed \( \pm 2^C \) (the generally accepted value for the accuracy of tilt determination utilizing the horizon camera) then:

\[ \frac{2^C}{500^C} = \kappa^2 , \text{ or} \]
\[ \kappa = \sqrt{0.004} = \pm 0.063 \approx \pm 4^g. \quad (2-61) \]

In other words, if \( \kappa \) is known within \( \pm 4^g \), its effect on the determination of \( \phi \) and \( \omega \) is within the expected accuracy of the horizon camera, and thus can be neglected.

2.25 Linearization of the Horizon Equations

The equation

\[ T_z = \frac{[t_A \quad I \quad t_z] \cdot M_z}{[t_A \quad I \quad t_z] \cdot M_y} = \cot Z (1 + t_A^2)^{\frac{1}{2}} , \]

contains the unknowns \( \phi \) and \( \omega \) (disregarding \( \kappa \)). A minimum of two
such equations is needed to solve for the two unknowns. If more equations are available, i.e. more points are observed along the horizon line in the picture, a least squares adjustment may be applied. In either case, linearization of these equations is necessary for their solution. Written in full, the above equation becomes:

$$T_Z = \frac{t_A (\sin \omega \sin \kappa - \cos \omega \cos \varphi \cos \kappa) + (\sin \omega \cos \kappa + \cos \omega \sin \varphi \sin \kappa)}{t_A (\cos \omega \sin \kappa + \sin \omega \sin \varphi \cos \kappa) + (\cos \omega \cos \kappa - \sin \omega \sin \varphi \sin \kappa)}$$

$$+ \frac{t_z (\cos \omega \cos \varphi)}{t_z (\sin \omega \cos \varphi)} \times (2-62)$$

It follows that:

$$\frac{\partial T_Z}{\partial \varphi} = \left[ t_A (\cos \omega \sin \kappa + \sin \omega \sin \varphi \cos \kappa) + \cos \omega \cos \kappa \right.$$

$$- \sin \omega \sin \varphi \sin \kappa + t_z (-\sin \omega \cos \varphi) \left. \right] \left[ -t_A \cos \omega \cos \varphi \cos \kappa + \cos \omega \cos \varphi \sin \kappa - t_z \cos \omega \sin \varphi \right]$$

$$- t_A \left( \sin \omega \sin \kappa - \cos \omega \sin \varphi \cos \kappa \right) + \sin \omega \cos \kappa$$

$$+ \cos \omega \sin \varphi \sin \kappa + t_z (\cos \omega \cos \varphi) \right]$$

$$\left[ t_A \sin \omega \cos \varphi \cos \kappa - \sin \omega \cos \varphi \sin \kappa + t_z \sin \omega \sin \varphi \right]$$

$$\left[ t_A (\cos \omega \sin \kappa + \sin \omega \sin \varphi \cos \kappa) + \cos \omega \cos \kappa \right.$$

$$- \sin \omega \sin \varphi \sin \kappa + t_z (-\sin \omega \cos \varphi) \left. \right]^2$$

Evaluating the above equation at $\varphi = \omega = \kappa = 0$, the equation is
In the same manner the derivatives with respect to \( \omega \) and \( \kappa \) are:

\[
\left[ \frac{\partial T_z}{\partial \omega} \right]_0 = (1 + t_Z^2), \quad \text{and} \quad \left[ \frac{\partial T_z}{\partial \kappa} \right]_0 = -t_A t_Z.
\]

Linearized (Taylor series), equation (2-62) could be written as:

\[
-t_A \Delta \phi + (1 + t_Z^2) \Delta \omega - t_A t_Z \Delta \kappa + [t_Z - \cot Z (1 + t_A^2) t_Z] = 0.
\]

(2-63)

The part of the equation in square brackets is the known term. The term involving \( \kappa \) should be disregarded because of the inability to observe a definite point on the horizon picture. However, should \( \kappa \) be a known quantity, it would be placed within the square brackets and become a part of the known term.

Disregarding \( \kappa \), equation (2-63) is reduced to:

\[
-t_A \Delta \phi + (1 + t_Z^2) \Delta \omega = [\cot Z (1 + t_A^2)^{\frac{1}{2}} - t_Z].
\]

(2-64)
One observation equation such as equation (2-64) is associated with each observed point along the horizon line. The matrix containing the coefficients of the two unknowns, from \( n \) equations, assumes a size of \( nx2 \) and the following form:

\[
A = \begin{bmatrix}
    t_{A_1} & (1 + t_{Z_1}^2) \\
    t_{A_2} & (1 + t_{Z_2}^2) \\
    t_{A_3} & (1 + t_{Z_3}^2) \\
    \vdots & \vdots \\
    \vdots & \vdots 
\end{bmatrix} .
\]  

(2-65)

2.26 **Weights and Accuracy**

Following the rigorous treatment of least squares adjustment, weights must be assigned, a priori, to the observations, the observations being the known part of the observation equation. The standard error of this known part (\( K_T \)) is obtained by applying the law of error propagation to the right hand side of equation (2-64) as follows:

\[
m_{K_T}^2 = m_T^2 + \frac{1 + t_A^2}{\sin^4 Z} m_Z^2 + \frac{\cot^2 Z \cdot 4 t_A^2}{4 (1 + t_A^2)} m_t^2 .
\]

(2-66)

Considering that the value of \( \cot^2 Z \) is very close to zero and that the value of \( t_A^2 \) cannot exceed 0.09, it follows that the numerator \( \cot^2 Z \cdot 4 t_A^2 \) is negligible indeed with respect to the denominator.
Thus the third term is extremely small and may be considered negligible compared to the second and first terms which are both close to unity. In other words, $m_{KT}$ could be expressed as a function of $m_t$ and $m_z$ only. Moreover, the denominator of the second term is very close to one so that the equation can be further simplified to read:

$$m_{KT}^2 = m_t^2 + (1 + t_A^2) m_z^2.$$  

$m_t$ is affected by lens distortion, film shrinkage, the measuring accuracy of the instrument used and the pointing accuracy. The pointing error is the main contributor to the total error $m_t$.

$m_z$ is affected by errors in determining the flight height, in refraction index evaluation, and in determining the elevation of the apparent horizon. The errors in determining the elevation of the apparent horizon are the main contributors to the total error $m_z$.

Supposing that the error $m_z$ is negligible (as is the case when true horizon is observed), it follows that the error $m_{KT}$ is solely due to $m_t$. In this case, even if $m_t$ is known, there is no sense in computing weights since each observation will be multiplied by the same constant value. Thus in this case a unit weight is applied.

Supposing now that the error $m_t$ is negligible, (as is the case when the horizon line is very sharply defined). In this case, the
Following relationships are valid:

\[ m_{KT}^2 = (1 + t_A^2) m_Z^2 \]  \hspace{2cm} (2-68)

And the weight

\[ W = \frac{1}{(1 + t_A^2) m_Z^2} \]  \hspace{2cm} (2-69)

\[ m_Z \] is a constant term and therefore need not be evaluated. The weight will change only as a function of \( t_A \).

In the case where none of the assumptions above can be made, the weight is

\[ W = \frac{1}{m_{t_Z}^2 + (1 + t_A^2) m_Z^2} \]  \hspace{2cm} (2-70)

And the ratio between the 2 errors involved (\( m_{t_Z} \) and \( m_Z \)) must be evaluated. \( m_{t_Z} \) is mainly due to pointing errors on the horizon line. A reasonable value for it can be obtained only after extensive statistical tests in which several observers point independently on the horizon and a representative sample of the horizon pictures are observed. As for the evaluation of \( m_Z \), if the apparent horizon is a mountain ridge whose elevation can be obtained from a map, \( m_Z \) can be evaluated accordingly. For example an error of 100 m in determining the elevation of the horizon, when the flight height is 3 km, will cause an error of about 1'.5 in \( Z \) determination (see equation (1-9)). In the case of a vapour horizon, \( Z \) can be obtained following the method described in Section 1.4 or in Section 2.33,
and its accuracy will have to be evaluated according to the specific case under consideration. In any event, the higher the flight height is, the less is the effect of the error in the elevation of the horizon on the determination of \( Z \) (see equation (1-9)). It should be pointed out that in general, the computed weights need not be precisely obtained and an intelligent guess will suffice. Moreover, in the new approach the orientation unknowns are solved by utilizing all four horizon pictures in which case \( Z \) need not be computed, provided, of course, that it is equal in all four sectors of the horizon. Likewise, the influence of the error in \( t_z \) on the result will be reduced because more than forty pointings are made to determine the tilt components of each vertical photograph.

Applying the weight (Equation (2-70) to each term in the matrix (2-65), the latter could be rewritten as follows:

\[
A = \begin{bmatrix}
\frac{1}{t_{A_1}} \sqrt{\frac{1}{c_1^2 + (1 + t_{A_1}^2) c_2^2}} & (1 + t_{Z_1}^2) \sqrt{\frac{1}{c_1^2 + (1 + t_{A_1}^2) c_2^2}} \\
\frac{1}{t_{A_2}} \sqrt{\frac{1}{c_1^2 + (1 + t_{A_2}^2) c_2^2}} & (1 + t_{Z_2}^2) \sqrt{\frac{1}{c_1^2 + (1 + t_{A_2}^2) c_2^2}} \\
\vdots & \vdots \\
\end{bmatrix}
\]

(2-71)
2.27 The Linearized Equation Used in the Determination of Tilt Components

The horizon picture used in the analysis so far has been the left one (see Figure 11). The linearized horizon equation associated with it was derived in detail (Section 2.25) and found to be:

\[ -t_A \Delta \phi + (1 + t_Z^2) \Delta \omega + t_Z - \cot Z (1 + t_A^2)^{\frac{1}{2}} = 0, \quad (2-72) \]

where \( t_A = \frac{x^*}{p} \) and \( t_Z = \frac{z^*}{p} \).

Similarly the linearized equation associated with the right picture is as follows:

\[ -t_A \Delta \phi + (1 + t_Z^2) \Delta \omega + t_Z + \cot Z (1 + t_A^2)^{\frac{1}{2}} = 0, \]

where \( t_A = -\frac{x^*}{p} \) and \( t_Z = -\frac{z^*}{p} \). \quad (2-73)
In a direct analogy to the above (see original equations (2-13))
the linearized equation associated with the front picture is:

\[-(1 + t_c^2) \Delta \varphi + t_B \Delta \omega + t_c - \cot Z (1 + t_B^{\frac{1}{2}}) = 0,\]

where \( t_c = \frac{z^*}{p} \) and \( t_B = \frac{y^*}{p} \). \quad (2-74)

Similarly, the linearized equation associated with the back
picture is:

\[-(1 + t_c^2) \Delta \varphi + t_B \Delta \omega + t_c + \cot Z (1 + t_B^{\frac{1}{2}}) = 0,\]

where \( t_B = \frac{-y^*}{p} \) and \( t_c = \frac{-z^*}{p} \). \quad (2-75)

The underlying principle of the new approach is that all
pictures in the system share one reference coordinate system
(Figure 11). It is easy to visualize this principle when one
camera is used (for example: panoramic camera), however when
five cameras are involved as is the case here, the above
principle can be realized only if the orientation between the
cameras and the relations between their corresponding axes are
known. Since it is assumed that the four horizon cameras are
perpendicular to the nadiral camera and to each other in the
fashion shown in Figure 11, and since the axes associated with
each picture are set parallel to the reference system, then when
viewed from the origin of the reference system each picture will
contain a pair of axes belonging to the common system as shown in Figure 12.

The above linear equations (2-72 to 2-75) can be reduced to the ones used in the conventional methods (Section 2.12) in the following way. The equations associated with the left and right picture are:

\[-t_A \Delta \varphi + (1 + t_{z_1}^2) \Delta \omega + t_{z_1} - \cot Z (1 + t_{A}^{2 \frac{1}{2}}) = 0,\]

and

\[-t_A \Delta \varphi + (1 + t_{z_2}^2) \Delta \omega + t_{z_2} + \cot Z (1 + t_{A}^{2 \frac{1}{2}}) = 0.\]  

Assuming $\Delta \varphi = 0$ the sum of the above two equations is:

\[(2 + t_{z_1}^2 + t_{z_2}^2) \Delta \omega + t_{z_1} + t_{z_2} = 0.\]

Thus:

\[\Delta \omega = \frac{-(t_{z_1} + t_{z_2})}{2 + t_{z_1}^2 + t_{z_2}^2} = \frac{-(\frac{z_1^*}{p} - \frac{z_2^*}{p})}{2 + \frac{z_1^{*2}}{p^2} + \frac{z_2^{*2}}{p^2}} = \frac{(z_2^* - z_1^*) p}{2 p^2 + z_1^{*2} + z_2^{*2}}.\]  

(2-77)

In the denominator, the value $(z_1^{*2} + z_2^{*2})$ is negligible with respect to the value of $2p^2$ and thus may be disregarded,
resulting in

\[ \Delta \omega = \frac{(z_2^* - z_1^*) \rho}{2 \rho^2}. \]

Defining:

\[ \frac{z_2^* - z_1^*}{2} = z_m^* \]

it follows that,

\[ \Delta \omega^c = \frac{z_m^*}{\rho^c} \cdot \rho^c \] (2-78)

This is the formula used in the conventional methods, (see equation 2-4). Assuming that \( \Delta \omega = 0 \), the sum of equations (2-76) is

\[ 2 t_A \Delta \varphi + t_{z_1} + t_{z_2} = 0 \]

and thus

\[ \Delta \varphi^c = \frac{t_{z_1} + t_{z_2}}{2 t_A} \cdot \rho^c = \frac{t_{z^*}}{t_A} \cdot \frac{\rho^c}{x^*} \]

(2-79)

This formula is used in the conventional methods (see equation (2-3)). Similar derivations can be made from the equations associated with the front and back pictures, leading to the same results.

The analysis provided above serves two purposes:

1. Checking the linearized equations,
2. Showing the basic differences between the conventional methods and the new approach, as will be further discussed in Section 2.4.

2.28 Data Acquisition

The set of four horizon pictures associated with one vertical photograph is registered on the film in the way sketched in Figure 13.

![Figure 12 Axes of Each Horizon Picture](image)

![Figure 13 A Set of Horizon Pictures in the Comparator](image)
Figure 13 also shows the manner in which the film is inserted in
the comparator (Xₜ Yₜ are the machine axes). The measurements can
be performed on a monocomparator or a stereocomparator. If the
latter is used, another set of horizon pictures is inserted in the
other plate carrier and the dove prisms are used to optically
rotate the images so that stereoscopy will be achieved (see
Figure 5).

It should be emphasized that stereoscopy in the new approach
serves only two purposes:
1. To obtain a better pointing accuracy;
2. To expedite the reading of horizon pictures by recording
simultaneously the coordinates of points at two pictures [At the
WILD STK-1, Xᵢ, Yᵢ, Pₓᵢ, Pᵧᵢ, are recorded simultaneously, in
which case Xᵢ = Xᵢ + Pₓ and Yᵢ = Yᵢ + Pᵧ].

2.29 Data Reduction

The following operational procedure is associated with the
program of data processing (see Section 3.30) developed in this
investigation:
1. Read the coordinates of the three central fiducial marks, from
left to right.
2. Read several points along the horizon line, starting from left.
3. An identification number should be assigned to each point.
From the eight identification digits available on the EK-6 (read
out of the WILD STK-1), the first two digits may contain the number of the left picture, the next two digits may contain the number of the point (in order from 1 to n), the next two digits - the number of the right picture, and the last two digits - the number of the point observed.

On a 9 x 9" plate carrier, 24 sets of pictures can be placed (i.e. 96 horizon pictures), and read successively, thus reducing the time for observations to a minimum.

2.30 Data Processing

2.301 Transformation of Coordinates

If a rigorous treatment is to be applied, the measured coordinates have to be corrected for lens distortion, film shrinkage and refraction. Lens distortion correction cannot be applied unless calibration data are available. Film shrinkage can be compensated for if the precise distance between fiducial marks are known. However, since the horizon picture is rather small in size, the effect of shrinkage is likely to be negligible. Refraction correction is taken care of in the horizon equations (see equation (1-7)). In case enough information is available to compensate for all other possible errors, the measured coordinates will be refined according to the usual analytical procedures.

The machine coordinate system does not coincide with the coordinate system adopted for each of the horizon pictures. The relation between the different systems is shown in Figure 14.
In view of the above, a transformation program has been devised to accommodate each of the individual pictures.

The features of this program are as follows:

a) A provision is made for the case in which the pictures are observed monocularly, or stereoscopically.

b) A provision is made for the cases in which only one picture, two, three or all four pictures in the set are observed.

c) The coordinate origin of each picture is its central fiducial mark.

d) The rotation of the machines' coordinate system to the picture system is computed by utilizing the coordinates of the left and right central fiducial marks.
e) The following transformation equations were derived geometrically. In order to maintain the same transformation equations for all four pictures, the measured coordinate $Y_s$ should change sign in the right and front pictures, before being introduced to the transformation equations.

Back picture

\[
\begin{align*}
  z^* &= X_s \cos \alpha - Y_s \sin \alpha \\
  y^* &= -X_s \sin \alpha - Y_s \cos \alpha
\end{align*}
\]

Left picture

\[
\begin{align*}
  z^* &= X_s \cos \alpha - Y_s \sin \alpha \\
  x^* &= -X_s \sin \alpha - Y_s \cos \alpha
\end{align*}
\]

Right picture

\[
\begin{align*}
  z^* &= X_s \cos \alpha - Y_s \sin \alpha \\
  x^* &= -X_s \sin \alpha - Y_s \cos \alpha
\end{align*}
\]

(2-80)

Front picture

\[
\begin{align*}
  z^* &= X_s \cos \alpha - Y_s \sin \alpha \\
  y^* &= -X_s \sin \alpha - Y_s \cos \alpha
\end{align*}
\]

f) XY in the program refer to coordinates associated with the left carrier;

UV in the program refer to coordinates associated with the right carrier;

The letter S is used to identify every machine coordinate.
The letter B, L, R, F is used to identify the Back, Left, Right, and Front picture respectively. (Example: XSB denotes the machine coordinate X of a point in the back picture located in the left carrier of the instrument).

N, 1, 2, 3, 4 refer also to the Back, Left, Right, and Front pictures respectively.
The letter L and R denote the Left and Right carrier respectively (Example: NL1 denotes the back picture, located in the left carrier).

The computer program is presented in Appendix A.

2.302 Determination of $\omega$ and $\omega$

The program presented here for the determination of $\varphi$ and $\omega$ takes as input the transformed plate coordinates (see Appendix A). The only restriction involved in this program is that the same number of points are read in each horizon picture that is utilized.

The program is so devised as to enable an analysis of the new approach, as will be explained later.

The steps followed in this program are discussed below:

a) Reading of the general data.

The first card contains the following information in the given order: Number of points read in each picture, flight height, radius of earth, principal distance of horizon camera, $\varphi_0$, $\omega_0$, $K_0$, $\sigma$ (as explained in step j below), the horizon pictures involved (inserting the numeral 1 for each of the pictures involved), and
refraction coefficient.

The second card contains the standard deviations of \( t_Z \) and \( Z \). These data are provided for the computation of weights, in accordance with equation (2-70). If unit weights are to be assigned, \( m_{t_Z} \) must be given the value of 63.662 (= \( \rho \)) so that \( W = 1 \) (since the unit of \( m_Z \) is converted in the program to radians and the same is applied also to \( m_{t_Z} \)).

Thus for example,

If unit weights are assigned; then \( m_{t_Z} = 63.662 \) and \( m_Z = 0 \).

If only \( m_Z \) is involved, then \( m_{t_Z} = 0 \) and \( m_Z = 63.662 \).

If it is assumed that \( m_{t_Z} = m_Z \), then \( m_{t_Z} = 63.662 \) and \( m_Z = 63.662 \).

If it is assumed that \( m_{t_Z} = 2m_Z \); then \( m_{t_Z} = 127.324 \) and \( m_Z = 63.662 \).

b) Reading of the measured data.

The 'measured' data are the reduced coordinate \( x^* \), \( z^* \) and \( y^* \), \( z^* \). Provisions are made for the case where one, two, three or all four pictures are utilized.

c) Computation of direction-tangents.

The following relations are deduced from equation (2-6) and (2-7);

\[
T_B^B = -\frac{y^*}{P}, \quad T_A^L = \frac{x^*}{P}, \quad T_A^R = -\frac{x^*}{P}, \quad T_B^F = \frac{y^*}{P},
\]

\[
T_C^B = -\frac{z^*}{P}, \quad T_Z^L = \frac{z^*}{P}, \quad T_Z^R = -\frac{z^*}{P}, \quad T_C^F = \frac{z^*}{P}.
\]
d) Construction of initial orientation matrix (\( M_0 \)).

For first approximation \( \varphi = \omega = \kappa = 0 \), in which case \( M = 1 \) (see matrix (2-10)).

e) Transformation of direction tangents.

The orientation matrix is matrix (2-10).

The transformation equations (2-15) and (2-16) are denoted as

\[
T_A = \frac{[t_A \ 1 \ t_Z] M_X}{[t_A \ 1 \ t_Z] M_Y} = \frac{AN1}{DN},
\]

and

\[
T_Z = \frac{[t_A \ 1 \ t_Z] M_Z}{[t_A \ 1 \ t_Z] M_Y} = \frac{AN2}{DN}.
\]

The subroutine "Transformation of \( TA \) and \( TZ \)", and similarly "Transformation of \( TB \) and \( TC \)", provides the necessary transformation.

f) Number of observation equations and number of unknowns.

Number of observation equations = number of all points read (= \( NQ \)).

Number of unknowns here is fixed and = 2 (= \( NT1 \)).

g) Approximating \( \varphi = \omega = \kappa = 0 \).

This approximation is utilized in every iteration, as explained later.

h) Construction of the general matrix.

The general matrix has the form shown in Figure 15
and contains the normal matrix, (N), the transposed observation matrix (Aᵀ) and the unit matrix (I). The matrices involved are explained below:

1) Construction of Aᵀ matrix.

Since the normal matrix is of dimensions 2 x 2 (having 2 unknowns), the coefficients of Aᵀ should start at the third column and each observation equation will occupy one column thereafter. The coefficients are those of the linearized equation (2-72), (2-73), (2-74) and (2-75). The first picture to be utilized is N₁ (Back) and the first pair of coefficients will be located at row 1 column 3 and row 2 column 3 respectively. The other coefficients will follow in the same way, however a provision is made to shift the locations to the left in case one or more pictures are not utilized. If all pictures are utilized, the dimensions of Aᵀ are: NTI (Number of unknowns) rows, NQ(Number of observation equations) columns.
2) Construction of N matrix.

The coefficients of the normal matrix are the products of the matrices $A^T \cdot A$. The matrix is located to the left of $A^T$. (see Figure 15).

3) Construction of 1 matrix.

Matrix 1 has the same dimensions as the normal matrix, and is located to the right of $A^T$.

At this stage, the general matrix is completed and will be used from now on as one matrix.

i) Computation of the known terms.

In this step the known terms $[L]$ of the linearized equations (2-72), (2-73), (2-74), and (2-75) are computed. The signs of the known terms are determined according to their location at the right hand side of the equations, as requested by the subroutine SOLSIS (step k). The column matrix $[L]$ is not a part of the general matrix constructed previously.

The known terms of iteration $i$ are the residuals of iteration $i - 1$. In view of this, the magnitude of the known terms (i.e. residuals) becomes the criterion for checking the procedure and terminating the iterations.

j) Check

The first step in this phase is to compute the sum of the squares of the residuals, $[WVV]$, the result of which is denoted SVA. The result obtained in the previous iteration is denoted by SVP. In the first iteration no check is made but in the next iteration SVP
will be compared with SVA and if SVA is smaller, it is an indication that the trend is correct and the iteration will continue.

If the two values of SVA and SVP are equal, the iteration will terminate. However, SVA will approach SVP in an asymptotic manner (i.e., at a certain stage it will take many iterations in order to reduce slightly the difference between SVA and SVP). In order to avoid these iterations which add only little significance, a value SIGMA is introduced (it was found that in practice, SIGMA = 0.99 will suffice) and the comparison between SVA and SVP will be done at this level (SVA = SIGMA·SVP).

k) The solution of the general matrix.

The subroutine SOLSIS computes and replaces the previous general matrix with the one shown in Figure 16.

Figure 16 The "SOLSIS" Matrix

1) The solution of the unknowns.

Denoting the unknowns as X, then

\[ \{X\} = \{N^{-1} A^T\} \{L\} \]
\([N^{-1}A^T]\) is taken from step \(k\), and \([L]\) from step \(i\).

m) Construction of the partial orientation matrix \(M_1\) (of first iteration).

The orientation matrix is constructed using \(\phi\) and \(\omega\) as obtained from step \(l\).

n) Transformation of direction tangents.

In a manner similar to that of step \(e\), the transformation is executed, utilizing the values obtained in step \(m\).

o) Construction of the total orientation matrix (as explained below).

\[
M = M_o \cdot M_1
\]

Where \(M_o\) is obtained in step \(d\) and \(M_1\) in step \(m\).

Step \(o\) marks the end of one full iteration. Notice that the second (and further) iterations start at step \(g\), by again approximating \(\phi = \omega = k = 0\). In the present method; \(\phi\) and \(\omega\) are needed only to rotate the direction tangents until they are fully oriented. In step \(n\) of the first iteration, the direction tangents are transformed according to the resulting \(\phi\) and \(\omega\), and thus the directions assume a new position which becomes the initial position for the second iteration. The second iteration will transform the previously obtained direction tangents so that the resulting directions will assume a closer position to the oriented ones. This iteration process goes on until the orientation is finally achieved. This approach simplifies the computations by allowing the beginning
of each iteration to assume the initial values $\varphi = \omega = \kappa = 0$, since each iteration is viewed as the start of a new stage. The product of all partial orientation matrices (step o) is the total orientation matrix for transforming a ray from its initial position to its oriented position; the values of $\varphi$ and $\omega$ contained in this matrix are the final values of these parameters.

p) The final values of $\varphi$ and $\omega$.

$\varphi$ and $\omega$ are computed from the final total orientation matrix. $\varphi$ is computed from row 3 column 1 of this orientation matrix ($\sin \varphi$). Similarly, $\omega$ is computed from row 3 column 2 ($-\sin \omega \cos \varphi$).

q) Precision of results.

In this step the following is computed:

The standard error of unit weight $= \text{ERUW} = \sqrt{\frac{[VV]}{n-u}}$.

The standard error of $\varphi = \text{ETAFl} = \sqrt{\frac{[VV]}{n-u}} \cdot 0.11$.

The standard error of $\omega = \text{ETAOM} = \sqrt{\frac{[VV]}{n-u}} \cdot 0.22$.

The information for computing the degrees of freedom ($n-u$) is obtained in step f. The value of $[VV]$ is computed in step q.

The cofactors are deduced from step k.

The flow chart of the computer program is shown in Figure 17 and the program is presented in Appendix B.
Figure 17 Flow Chart of ϕ and ω Determination Program
2.3 Experimental Work

2.3.1 Preparation of Fictitious Data

A positive rotation of $5.5^\circ$ was arbitrarily assumed around the $X$ axis and a negative rotation of $5.5^\circ$ around the $Y$ axis. The convention of rotation is according to the right handed system. The zenithal angle from an assumed flight height of 3000m was computed by means of equation (1-7) and the distance $a-b$ (see Figure 18) was computed according to the relation:

$$ab = P \cdot \tan \left[ \alpha + (Z - 90^\circ) \right],$$

where $\alpha$ is the declination angle ($\omega$ in left and right pictures, $\varphi$ in back and front pictures). The inclination angle ($= \varphi$ in left and right pictures, $\omega$ in back and front pictures) was plotted from point $b$ by means of a protractor. The graph of the horizon obtained in this way is shown, for all four pictures, in Figure 14. All linear dimensions were enlarged 10 times in order to improve the graphical accuracy. The coordinates of points along the horizon line were read from the graph and transformed by means of the transformation program. Those transformed coordinates were used in the tilt determination program.

Figure 18 Construction of a Fictitious Horizon Picture
2.32 Experiments

37 sets of experiments have been conducted. The characteristics of each of these sets are listed in Table (1). Each of the sets 1-25 utilized the data extracted from the following 6 combinations of horizon camera pictures:

1. Back, left right and front  (Denoted as B.L.R.F.);
2. Back and front only  (Denoted as B.F.);
3. Left and right only  (Denoted as L.R.);
4. Back and left only  (Denoted as B.L.);
5. Back only  (Denoted as B.);
6. Left only  (Denoted as L.).

Experiments 26-37 utilized the data obtained from the first combination of pictures only. Thus, a total of 162 cases were examined. The location of 11 measured points in each horizon picture is shown in Figure 19, the location of 9 measured points are shown in Figure 20 and 3 points in Figure 21.

![Figure 19 Location of 11 Points in the Picture](image1)
![Figure 20 Location of 9 Points in the Picture](image2)
![Figure 21 Location of 3 Points in the Picture](image3)
Table 1  Characteristics of the Various Experiments

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<th>m_{z}</th>
<th>P (mm)</th>
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The particular arrangement of the observed points in Figure 20 was made in attempt to strengthen the geometry of the inclination angle. This angle is associated with the distance between the left and right crosses, which is shorter than the principal distance which is associated with the declination angle. The two additional
points in Figure 19 will check whether more pointings along the horizon line will strengthen the overall geometry. The arrangement in Figure 21 is the conventional one.

2.33 Analysis

In preparing the fictitious data, no attempt was made to prevent the systematic effect of the primary rotation (ω) upon the secondary (φ), because the fictitious data are anyhow systematically affected by the inaccuracy of plotting the tilt components, a situation that can hardly be compensated for. Since the systematic effects cannot be handled by the least squares adjustment unless they are counted for in the mathematical structure, the resulting standard deviation of unit weight \( m_o = \sqrt{\frac{\text{VV}}{n-u}} \) has no meaning in this case. Consequently the standard deviation of φ and ω \[ m_\alpha = m_o \cdot \sqrt{Q_{ii}} \] has no meaning either. However, the resulting cofactors \[ Q = N^{-1} \] are not affected by the above condition but rather by the existing geometrical situation. Thus, assuming arbitrarily that the standard deviation of unit weight is \( \pm 1^\circ \) it follows that \( m_\alpha = \sqrt{Q_{ii}} \) and those values can be used in the following analysis. Such a determination of \( m_\alpha \) is to be used, of course, for comparison purposes only.

In order to further simplify the analysis, \( m_\alpha \) resulting from the first experiment, utilizing all four horizon pictures, was multiplied by a factor so that its value will become unity. All other \( m_\alpha \)'s belonging to the same experiment were reduced to the
corresponding $\tilde{m}_{\alpha_i}$ by multiplying each of them by the same factor. This provides an easier interpretation of the following graphs.

A. The effect of the number of points utilized in each picture on $m_\varphi$ and $m_\omega$.

Experiments 1-15 (see Section 2.32) were divided into five groups according to the weights assigned to the observations. Each group was subdivided into six sections according to the combination of pictures utilized. The first group with its six sections is represented in Figure 22. Observing the graphs it becomes evident that:

1. The standard deviation of each unknown, based on three observations per picture, increases to about twice the standard deviation obtained by observing eleven points per picture.

2. The difference between the values of the standard deviations of the unknowns, resulting from three observations on one or two opposite pictures, is almost twice the value of the difference obtained from eleven observations, but the ratio between the standard deviations is practically the same in both cases.

3. There is an insignificant improvement of the accuracy, from the point of view of geometry, when eleven points are observed, compared with the case in which nine points are observed.

4. The standard deviation of the declination angle is about four times smaller than that of the inclination angle when one picture or two opposite pictures are utilized. The standard
deviation of the declination angle becomes practically the same as that of the inclination angle when two perpendicular or all four pictures are utilized. This statement refers to geometrical considerations only.

5. Utilizing four pictures does not improve the results considerably when compared with the result obtained from utilizing two perpendicular pictures, from a geometrical viewpoint.

6. Simultaneous determination of $\varphi$ and $\omega$ from two perpendicular pictures is not associated with a decrease of accuracy when compared with the determination of each component from the single picture utilizing the declination angle alone. In other words: a poor tilt determination by means of inclination angles does not lessen the accuracy of the results when two perpendicular pictures are utilized.

The remaining five groups of experiments yielded practically the same results and therefore were not plotted. The explanation for this phenomenon is as follows:

\[
W = \frac{1}{m_{tZ}^2 + (1 + t_A^2) m_{Z}^2},
\]

Since $W = \frac{z - \cdot}{z - \cdot + \phi}$, and since $t_A^2$ is a small quantity in the horizon camera used (WILD HC 1), it follows that the weight in this case could be expressed as:

\[
W = \frac{1}{m_{tZ}^2 + m_{Z}^2}.
\]
In the above equation, the denominator is a constant and thus:

\[ W = \frac{1}{C} . \]

Multiplying each term in the observation matrix by the same constant factor will not affect the resulting cofactors.

![Graph](image)

**Figure 22** \( m_\phi \) and \( m_\omega \) (\( m^{(*)} \)) Versus Number of Points Observed Per Picture. (Unit weights assigned to the observations)

B. The effect of the angular field of the horizon camera on \( m_\phi \) and \( m_\omega \).

The graphs in Figure 23 contain the data from experiments 1-3 and 16-25. They are divided into five sections (rows) according to the weights assigned to the observations. Each section contains

\( m^{(*)} \) Computed on the basis of unit variance
three graphs (columns) according to the angular field assigned to a fictitious horizon camera. Since the angular field of $40^\circ$ yields practically the same results for the various experiments (see previous section) only one graph representing this case was plotted.

Analyzing these graphs it is evident that:

1. The ratio between the standard deviations of $\varphi$ and $\omega$ when one or two opposite pictures are utilized, decreased from about 4:1 (angular field = $40^\circ$), to about 1.4:1 (angular field = $90^\circ$), and to about 1.2:1 (angular field = $110^\circ$) in the worst cases.

2. Poor "Z" determination affect the above ratios more than a poor "TZ". Notice that when $m_Z$ is involved, the ratio between the standard deviations of $\varphi$ and $\omega$ is bigger (cases B, C, D in Figure 23) than when $m_{TZ}$ is dominant (case E in Figure 23). A logical explanation for this phenomenon is that since

$$W = \frac{1}{m_{TZ}^2 + (1 + t_A^2) m_Z^2},$$

it follows that if $m_{TZ}$ is the largest value in the denominator (this happens when the horizon is very blurred) then $t_A$, associated with the smaller value $m_Z$ and being the only variable will not affect the result as much.

3. While in the existing HC-1 camera (angular field = $40^\circ$) the ratio between the accuracy of the declination angle obtained by utilizing one picture and that obtained by utilizing all four
Figure 23 $m_\varphi$ and $m_\omega$ Versus Various Combinations of Utilized Horizon Pictures, Assigned Angular Fields and Weights

(*$m_\omega$ Computed on the basis of unit variance)
pictures, is about 1.4:1, it reaches a value of about 1.7:1 when
the angular field is 90° and about 2.2:1 when the angular field is
110°. However, making the same comparison with the inclination
angle, the ratio is 5.7:1 when the angular field is 40°, 2.5:1
when the angular field is 90°, and becomes 2.3:1 when the angular
field is 110°.

C. The effect of flight height or Zenithal distance on $m_o$.

A change of flight height may be viewed as a change in
elevation of the apparent horizon and hence as a change in the
zenithal angle. Thus the abscissa of graph (24) can be either $h$
or the angle $Z$. The ordinate is $m_o$.

$m_o$ would have a meaning as the standard deviation of unit
weight only if accidental errors, following the Gaussian law of
distribution, will be encountered. This is not the case with
the fictitious data being handled here.

If the flight height or the zenithal distance to the apparent
horizon are not known precisely, the resulting orientation
parameters will not be affected by these errors when all four
pictures are utilized, provided the horizon is level in all four
sectors. However, if the resulting oriented direction tangents
are expressed in terms of plate coordinates and plotted, the re-
sulting horizon line will be either above or below the actual one,
and consequently the value of $m_o$ will increase with the increase
of the above constant error. In the experiments presented here,
the values of $h$ were increased from 2,500m to 3,500m with increments
Figure 24 The Effect of an Error in the Value of Flight Height Upon $m_0$
of 125m, while the fictitious data were prepared for \( h = 3,000 \) m. The computed \( m_o \)'s were plotted against the corresponding flight-heights and it is logical to assume that the resulting curve will reach a minimum at the correct flight height. The non-linearity of the curve can be attributed to the fact that the flight height term in the horizon equation is located under the square root. The assymmetry of the results, as obtained in these experiments, can be attributed to the inaccuracy in generating the fictitious data, and as a result, the data may fit best a flight height which was not included in the experiments (As a matter of fact it seems as if the data fit better the case in which \( h = 2,750 \) m rather than \( h = 3,000 \) m, although the differences between the corresponding \( m_o \)'s is very small).

If this trend is confirmed in practice, the following method of determining the zenithal angle \( Z \) is suggested: using a set of four horizon pictures, several values of flight height will be inserted into the horizon equations and the corresponding standard deviation of unit weight (\( m_o \)) will be computed for each case. These values then will be plotted in the same manner as Graph 24, and the zenithal angle or flight height corresponding to the minimum value of \( m_o \) on the curve, will be used for actual computations.
2.34 Conclusions

Specific conclusions have been drawn and reported in Section 2.33. However, the following general conclusions pertain to the entire set of experiments:

1. It is sufficient to observe 9 points per picture rather than the 11 utilized in the experiments.
2. Only unit weights need to be assigned to the observations when the WILD HC-1 horizon camera is used;
3. The wider the angular field of the horizon camera, the smaller the difference between the standard deviations of φ and ω when only one or two opposite pictures are utilized. Thus, with a wide angle horizon camera, the utilization of each available picture for the determination of φ and ω is justified even from a purely geometrical viewpoint.
4. The improvement of the geometry resulting from widening the angular field from 40° to 90° is far more drastic than from 90° to 110°, and therefore the use of the 90° wide angle lens in the horizon camera is recommended in practice.
5. The dependence of the absolute determination of φ and ω upon the zenithal angle is eliminated in case all four pictures are utilized and when the apparent horizon is level. However, a method of determining the zenithal angle to be used in the horizon equations is being suggested.
It should be emphasized once again that both the analysis of the experiments and the conclusions drawn, pertain only to the geometry of the entire setup, not to the overall accuracy which involve the instruments used, the atmospheric conditions during the flight, the quality of the exposures and the like.

2.4 Comparison Between the New Approach and the Conventional Methods

A. Differential vs. absolute tilt values

The conventional methods that have been studied yield differential values of tilt, i.e. the relative change of tilt between corresponding pictures, while the new approach yields the absolute values of tilt for each photograph. The argument in favor of the differential tilt approach is that it is not affected by situations in which one apparent horizon sector is higher than the other or in case the apparent horizon is inclined with respect to the true horizon (see condition 2 in Section 1.4). Also, the flight height and zenithal angles need not be known for the determination of the differential tilt. However, the absolute tilt determination has the following merits:

1. In case the true horizon can be observed, the solution is rigorous while those of the conventional methods studied are approximate. The author is of the opinion that an emulsion capable of penetrating physical obstacles like clouds or haze will eventually
be available, or as expressed by Helava\textsuperscript{4}:

"with the development of aerial films with high infrared sensitivity, light of the longest wave lengths may be used in the horizon cameras. This kind of light penetrates best through the haze of the atmosphere. The chance that the true horizon will be visible on the photographs is thus much greater."

2. In case the apparent horizon has the same elevation all around, it can be used exactly as the true horizon.

3. If the horizon is level and all four pictures are utilized, there is no need to know the zenithal angle and flight height even for the absolute determination of the tilt.

4. In case the apparent horizon is inclined or has different elevations in each sector, the effect of the erroneous absolute results can be eliminated to a large extent by subtracting

\[ \varphi_{n+1} - \varphi_n = \Delta \varphi, \quad \text{and} \]
\[ \omega_{n+1} - \omega_n = \Delta \omega \]

In this way the relative values \( \Delta \varphi_1, \Delta \varphi_2 \ldots \Delta \varphi_n \) are obtained, and from here on the conventional method is followed. Since the first model is absolutely oriented, based on given ground control points, its orientation elements are taken as "correct" quantities and:

\[ \varphi_1 + \Delta \varphi_1 = \varphi_2 \]
\[ \varphi_2 + \Delta \varphi_2 = \varphi_3 \quad \text{etc.} \]

Stated differently; differential values of tilt can be obtained also
by utilizing the new approach. A more rigorous method is, of course, to subtract the corresponding horizon equations rather than their final results and to express these subtractions in terms of $\Delta \varphi$ and $\Delta \omega$. However, such refinements are hardly necessary when compared with the conventional equations which, although providing directly the differential values, can do so only by applying approximations. The manner in which the conventional equation could be obtained, via a series of approximations, from the new set of equations, have already been demonstrated in Sections 2.23 and 2.27. The approximations involved in the conventional equations (2.3) (2.4) are:

1. Neglecting $\kappa$;
2. Using linearized equations only once for the final results rather than as a tool of computing the results in an iterative way;
3. Utilizing measurements of points located at specific geometrical positions with respect to the fiducial marks.

The approximation of neglecting $\kappa$ is rather a serious one since if $\kappa$ does exist, there is no correspondence between $\varphi$ of the reference picture and any other $\varphi_i$. Thus subtracting one from the other will not render the actual relative difference between them. The conventional equations do not provide means for considering this effect in case $\kappa$ is not zero. Utilizing the linearized equations directly for the solution of the unknowns rather than attempting to approach the results as would have been obtained
from the original equations, is unavoidable when the conventional equations are used. This fact becomes clear when observing the orientation matrix (2-10). Such an orientation matrix is associated with the reference picture and with any other picture i, and it seems impossible to factor out $\phi_R$ and $\phi_i$ from these matrices to compute $\Delta \phi$ rigorously.

Utilizing measurements at specific locations limit the number of observations along the horizon line since the only specific locations available there are those adjacent to the central fiducial marks and there are only three such fiducial marks. Since the apparent horizon is not always sharply defined on the picture, observation at three locations only decreases the accuracy of the tilt determination.

It then becomes clear that the approximations involved in subtracting absolute values obtained by the new equations in order to get relative values (in the case of unequal elevation of the apparent horizon) are practically the same as the approximations involved in the conventional determination of relative tilt. This fact leads to a more important conclusion, namely that while both methods will yield approximate results in the case of unleveled apparent horizon, the new approach will yield rigorous results in case the true horizon or an apparent horizon parallel to the true one is observed or in case definite points in the horizon pictures can be observed.
8. Two pictures vs. four pictures

The conventional methods studied utilize the declination angles only i.e. $\omega$ in the left and right pictures and $\varphi$ in front and back pictures, while the new approach can utilize all four pictures for the determination of both $\varphi$ and $\omega$.

It is true that because of geometrical properties of the horizon camera, the error analysis indicates a superior accuracy for the declination angles when compared with that of the inclination angles. However, the following must be taken into consideration:

1. Geometrical properties are of major importance in case the horizon line is sharply defined. Since usually this is not the case in practice, the determination of the horizon line may require more consideration, in which case the following logical rule should be applied: The more determinations available the better is the result. Thus if four pictures are available, each capable of yielding the values of $\varphi$ and $\omega$, all four should be used to get the most reliable results.

2. In case the horizon camera should be provided in the future with wide-angle lenses, as suggested in this report, the difference between the accuracy of the declined and inclined angles will tend to equalize, in which case utilizing all four pictures for the determination of both $\omega$ and $\varphi$ will be justified even from the purely geometrical viewpoint.
C. Stereoscopic observation and parallax measurements vs. coordinates measurements

Stereoscopic vision can be used here to
1) increase the pointing accuracy by better defining the image of the horizon;
2) measure parallaxes between the reference line (1-3 in Figure 4) and the horizon line;
3) increase the efficiency and speed of the coordinate measuring operations.

The conventional methods use stereoscopy for reasons 1 and 2. The new approach uses stereoscopy (optionally) for reasons 1 and 3.

Parallaxes are the measured quantities utilized in the conventional equation (2-4) which therefore yield relative quantities. Coordinates are the measured quantities utilized in the new equations. From a theoretical point of view parallaxes are more accurate because

1. They require only one measurement per location while two measurements are required for the same purpose in case coordinates are utilized. Since every measurement contains an observational error, the resulting precisions differ by $\sqrt{2}$ in favor of the parallax measurement.

2. Mechanically it is possible to manufacture measuring screws of short length (i.e. sufficient for small parallax measurements) with higher accuracy than lead screws which enable scanning of the entire picture for coordinate measurements.
However, those arguments should be examined also from the following points of view:

1. The overall measuring accuracy attained by the parallax bar is inferior to that attained by a comparator even if two coordinate measurements per location are required rather than one.
2. Most comparators employ lead screws of practically the same travel length for coordinates and for parallax measurements. This is the case with the WILD STK-1 for example, and therefore no increase in accuracy can be noted for parallax measurements when compared with coordinate measurement.
3. The instrumental precision is in the order of a few microns. While the differences in accuracy discussed within this section may reach one micron, a negligible quantity with respect to other sources of errors.
4. From an economical point of view it should be remembered that the number of horizon pictures that have to be measured is four times the number of the nadiral photographs, and even if other procedures exist to extract the data from the horizon pictures in a more refined way, by reconstructing individually every horizon model, these procedures seem to be uneconomical on a commercial basis.

The new approach, on the other hand, can utilize stereoscopic measurements to increase the pointing accuracy and to speed up the operations. On the WILD STK-1 for example, the parallaxes are used merely to compute the coordinates of the picture at the
right plate carrier thus the coordinates of two horizon pictures are measured simultaneously, saving operation time without decreasing the accuracy.
3. APPLICATION OF THE AUXILIARY DATA IN AERIAL TRIANGULATION

3.1 Introduction

In the aeropolygon method of aerial triangulation, the formation of the strip is accomplished through the orientation of each photograph relative to the preceding one. However, systematic and accidental errors, involved in this operation, will affect the orientation of the triangulated photographs. Due to the propagation and accumulation of these errors the triangulated photographs will deviate from their true position in space; the longer the strip, the larger is the expected deviation. It then becomes necessary to adjust the strip, an operation which requires some additional data. These data can be in form of control points, auxiliary data or both.

In the present state of photogrammetry, ground control points are necessary for the adjustment and cannot be totally replaced by auxiliary data. Three non-colinear ground control points in the first model of the strip enable the absolute orientation of this model but do not provide means of controlling the errors affecting the succeeding models. If no additional ground control points are available along the strip, the triangulated strip is referred to as a cantilever strip. In this case the auxiliary data can, in a way, compensate for the missing additional control points for controlling the error propagation. In the case where additional control points are available along the strip, some means of controlling the errors and adjusting the strip are provided.
However in this case, generally referred to as the bridged strip, an assumption is made as to the behavior of the errors. The validity of this assumption depends, to a large extent, on the density and distribution of the available ground control points.

It is a well known fact that the accuracy of the orientation parameters obtainable in fully controlled models, exceeds that obtainable by auxiliary data. However in view of the unavoidable error accumulation in aerial triangulation, particularly in long strips with sparse ground control, the auxiliary data, although less accurate, provide a powerful alternative for controlling the errors accumulation. This can be further illustrated by drawing an analogy to the comparison between traversing with a precise theodolite and with a compass. The precise theodolite will measure a direction far better than a compass but its error will propagate along the net, while that of the compass, although less accurate, will remain more or less the same. In a similar fashion the errors of the horizon data will not propagate since they are independently obtained for each exposure.

The horizon pictures yield information pertaining to the orientation of their corresponding nadiral photographs. The different methods of obtaining this information were discussed in the previous chapters. The application of the auxiliary information to the triangulation can take place in different ways, either in the process of strip formation or in the adjustment phase. The various methods are investigated in the following chapters, forming the basis for a new method of adjustment.
3.2. **Incorporation of Auxiliary Data in the Process of Strip Formation**

Model formation within the strip is defined here as a process in which the bundle of rays constituting photograph \( j (j = i + 1) \) is oriented with respect to the already oriented bundle of rays of photograph \( i \), utilizing the horizon-determined orientation elements \( \varphi \) and \( \omega \) associated with this bundle. In case the auxiliary data are not accurate enough, the above process will fail to eliminate the parallaxes within the model, resulting in model deformations as well as discrepancies between models. Thus the criterion in applying the auxiliary data to the formation of a model is the accuracy with which these data can be obtained, compared with the accuracy achieved by the conventional procedures. Many experiments, done by different investigators at different times \([17][13][9]\) lead to the conclusion that the determination of \( \varphi \) and \( \omega \) by utilizing the horizon pictures is, at best, within \( \pm 2^\circ \). This accuracy seems to be quite optimistic, but even if this value is accepted, it is inferior to the accuracy obtained by the conventional procedures of relative orientation \([2]\). However, it is hoped that with a more sensitive emulsion, capable of penetrating haze and other physical obstacles, and with a more advanced horizon camera, the accuracy of determining \( \varphi \) and \( \omega \) will reach the \( \pm 1^\circ \) level.

When the above level of accuracy is reached, the auxiliary data can be utilized for the formation of the model in any of the possible ways suggested below. In order to demonstrate these
possibilities, Inghilleri's general approach to analytical photogrammetry\[6\] will be used. It should be emphasized however that the suggested ways for solving the problem in question are not restricted to this approach only, and that any set of analytical photogrammetry equations and matrices can be used as well.

In the chosen approach, bundle $i$ is said to be oriented if $X^i_o$, $Y^i_o$, $Z^i_o$, (the coordinates of the perspective center) and $T_{X^i_E}$, $T_{Y^i_E}$ (direction tangents to point $E$) are known. To orient bundle $j$ relative to the oriented bundle $i$, the following parallax equation is used:

$$\left(X^j_o - X^i_o\right) \frac{T^j_Y - T^i_Y}{T^j_X - T^i_X} + \left(Y^j_o - Y^i_o\right) + \left(Z^j_o - Z^i_o\right) = 0. \tag{3-1}$$

and the scaling can be expressed as:

$$X^j_o - X^i_o - T^j_X \left(Z^j_o - Z^i_o\right) \frac{T^j_X - T^i_X}{T^i_X - T^j_X} - \left(Z - Z^i_o\right) = 0. \tag{3-2}$$

$T_X$, $T_Y$ in each of the above equations are a function of $\varphi$, $\omega$, and $\kappa$ as expressed in equation (2-14). The linearized equation (3-1), by Taylor series in a general form, is:

$$P_1 \Delta X^i_o + P_2 \Delta Y^i_o + P_3 \Delta Z^i_o + P_4 \Delta X^j_o + P_5 \Delta Y^j_o + P_6 \Delta Z^j_o + P_7 \Delta \varphi^i + P_8 \Delta \omega^i + P_9 \Delta \kappa^i + P_{10} \Delta \varphi^j + P_{11} \Delta \omega^j + P_{12} \Delta \kappa^j + P_0 = 0. \tag{3-3}$$
Similarly, the linearized equation (3-2) is:

\[ H_1 \Delta x^i + H_2 \Delta z^i + H_3 \Delta x^j + H_4 \Delta z^j + H_5 \Delta p^i + H_6 \Delta \omega^i + H_7 \Delta \omega^j + H_8 \Delta \kappa^i \\
+ H_9 \Delta p^j + H_{10} \Delta \omega^j + H_{11} \Delta \kappa^j + H_0 = 0. \] (3-4)

Since bundle \( i \) is assumed to be oriented, all terms with subscript \( i \) in equations (3-3) and (3-4) are known. Applying equation (3-3) to each of the six points in the model, and equation (3-4) to the scale transfer point (Point 2 in Figure 30), the observation matrix becomes:

\[
A = \begin{bmatrix}
\Delta x & \Delta y & \Delta z & \Delta p & \Delta \omega & \Delta \kappa \\

P_4^{(1)} & P_5^{(1)} & P_6^{(1)} & P_{10}^{(1)} & P_{11}^{(1)} & P_{12}^{(1)} \\
P_4^{(2)} & P_5^{(2)} & P_6^{(2)} & P_{10}^{(2)} & P_{11}^{(2)} & P_{12}^{(2)} \\
P_4^{(3)} & P_5^{(3)} & P_6^{(3)} & P_{10}^{(3)} & P_{11}^{(3)} & P_{12}^{(3)} \\
P_4^{(4)} & P_5^{(4)} & P_6^{(4)} & P_{10}^{(4)} & P_{11}^{(4)} & P_{12}^{(4)} \\
P_4^{(5)} & P_5^{(5)} & P_6^{(5)} & P_{10}^{(5)} & P_{11}^{(5)} & P_{12}^{(5)} \\
P_4^{(6)} & P_5^{(6)} & P_6^{(6)} & P_{10}^{(6)} & P_{11}^{(6)} & P_{12}^{(6)} \\
H_3^{(2)} & 0 & H_4^{(2)} & H_9^{(2)} & H_{10}^{(2)} & H_{11}^{(2)}
\end{bmatrix}. \] (3-5)

The transposed vector of the known-terms is:

\[
\begin{bmatrix}
P_0^{(1)} & P_0^{(2)} & P_0^{(3)} & P_0^{(4)} & P_0^{(5)} & P_0^{(6)} & H_0^{(2)}
\end{bmatrix}. \] (3-6)

With this background [for more details see \([6]\)], we will proceed to discuss the various methods of introducing the horizon data in the strip formation.
3.21 Horizon Data as Known Parameters

Introducing the values of $\varphi$ and $\omega$, obtained from the horizon camera, as known parameters of relative orientation, observation matrix (3-5) becomes:

$$AX AY AZ AfC$$

Matrix (3-7) is a four-columns matrix, represented in the above way to facilitate its comparison with matrix (3-5). The dots symbolize coefficients in the above and following matrices.

3.22 Horizon-determined Parameters ($\varphi^h$ and $\omega^h$) as Additional "Observations"

Introducing the values of $\varphi^h$ and $\omega^h$, as obtained from the horizon pictures, in the following observation equations:

$$\varphi_i - \varphi_i^h = V_{\varphi_i}, \quad \text{and}$$
$$\omega_i - \omega_i^h = V_{\omega_i},$$

matrix $A$ becomes:
and the transposed vector of the known terms could be expressed as:

\[
\begin{pmatrix}
\Delta X & \Delta Y & \Delta Z & \Delta \phi & \Delta \omega & \Delta \chi
\end{pmatrix}^T,
\]

where \( \Delta \phi \) and \( \Delta \omega \) are the approximate values of the unknowns.

### 3.23 Horizon Equations as Additional Observation Equations

Using the linearized horizon equations (equations (2-72)) as additional observation equations, matrix A becomes:
It is obvious that if more observation equations of this type are added, the more weight is assigned to the auxiliary data.

It should be noted that the three methods suggested here are applicable also in cases of simultaneous adjustment involving any number and combination of models. Hence, these three approaches can be considered as general patterns. However, as already mentioned, the application of the horizon data to the model formation in the present time is not recommended (see Section 3.2).
3.3 Incorporation of Auxiliary Data in the Adjustment Stage

The outcome of the strip formation is a set of bundles oriented to each other but deviated from their corresponding absolute position in space due to the error accumulation and the effect of earth curvature. The relative orientation of each bundle to its preceding one, can be achieved with a high degree of accuracy, thus reducing to a minimum the effect of model deformation. It is, therefore, recommended that the relative orientation, obtained in a regular way, be kept intact as far as possible at the adjustment stage. This can be achieved by adjusting (i.e. absolutely orienting) each individual model in the strip as a rigid unit. Such an approach is the underlying principle of the following suggested adjustment procedure.

For the sake of clarity, the suggested method will be applied first to the cantilever strip having control points only in its first model. Then the approach will be demonstrated in the case of the bridged strip, where the adjustment method will take full advantage of both the auxiliary data and the additional available ground control points.

3.31 Adjustment of the Cantilever Strip

The absolute orientation equation of a model, in form of matrix notation, is

\[ C_i - (S + \lambda C_i^* M) = 0 \]  

(3-11)
where: \( C_i \) = A set of coordinates of a given ground control point;
\( C_i^* \) = The corresponding observed set of coordinates;
\( \lambda \) = Scale factor \((1 \text{ unknown})\);
\( S \) = A set of 3 translations along the 3 axes \((3 \text{ unknowns})\);
\( M \) = Orientation matrix \((3 \text{ unknowns})\).

The first model in the cantilever strip is to be oriented according to this equation. The next model, or in general, model n consisting of bundles i and j, has to undergo a translation and rotation in order to assume its absolute position, next to model n-1. The information available for this purpose is:

\[
\begin{align*}
X^n, Y^n, Z^n &= \text{coordinates of scale transfer point } P \text{ at model } n, \text{ as obtained in the strip formation;} \\
X^{n-1}, Y^{n-1}, Z^{n-1} &= \text{coordinates of scale transfer point } \bar{P} \text{ at model } n-1, \text{ which is already absolutely oriented;} \\
\varphi_i, \omega_i &= \text{Orientation parameters of bundle } i, \text{ as obtained in the strip formation;} \\
\varphi_j, \omega_j &= \text{Orientation parameters of bundle } j, \text{ as obtained in the strip formation;} \\
\varphi^h_i, \omega^h_i &= \text{Orientation parameters of bundle } i, \text{ as obtained by the horizon camera;} \\
\varphi^h_j, \omega^h_j &= \text{Orientation parameters of bundle } j, \text{ as obtained by the horizon camera}.
\end{align*}
\]

The aforementioned orientation parameters are associated with model n.
3.311 Model Translation

The amount of translation can be obtained by computing the differences between coordinates of the scale transfer point P as obtained in models n and n-1. In other words, referring to Figure 26, it follows that:

\[ S_1 = x^n_P - x^{n-1}_P, \]
\[ S_2 = y^n_P - y^{n-1}_P, \quad \text{and} \]
\[ S_3 = z^n_P - z^{n-1}_P. \]

3.312 Model Rotation

The amount of rotation that model n, as a unit, has to undergo can be computed in any of the following suggested ways:

3.3121 Approximate Determination of Model Rotation

Using the orientation elements associated with one of the bundles constituting model n, for the rotation of the entire model, the components of this rotation are:

\[ \Delta \phi = \phi_j^h - \phi_j, \]
\[ \Delta \psi = \psi_i^h - \psi_i, \]
\[ \Delta \Omega = \omega_j^h - \omega_j. \quad \text{or} \quad \Delta \Omega = \omega_i^h - \omega_i. \]  

3.3122 Determination of Model Rotation from Averaged Values of Orientation Parameters

Utilizing the averaged orientation parameters of both bundles i and j, and referring to Figure 25, the components of the model rotation are:
Figure 25 Model Rotation

Figure 26 Rotation and Translation of the Models in the Strip
3.3123 Refined Determination of Model Rotation

The following method is based upon the realization that the optical axis of the nadiral camera should coincide with that determined by the horizon camera. This condition can be mathematically expressed as:

\[ T_x^m - T_x^h = 0, \]
\[ T_y^m - T_y^h = 0, \]

where \( T_x^m, T_y^m \) are the oriented direction tangents of the camera axis, and \( T_x^h, T_y^h \) are the oriented direction tangents as determined by the horizon camera.

The direction tangents of the camera axis, as obtained in the process of strip formation, are \( t_X^m, t_Y^m \), whose expressions are (see equation (2-12);

\[ t_X^m = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} M_X^m \\ M_Z^m \end{bmatrix}, \]
\[ t_Y^m = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} M_Y^m \\ M_Z^m \end{bmatrix}, \]

Where \( M_X^m, M_Y^m, M_Z^m \) are the columns of the orientation matrix containing \( \phi, \omega, \kappa \) as obtained in the process of strip formation. The values of \( t_X^m \) and \( t_Y^m \) can thus be viewed as the transformed direction tangents of the camera axis, whose initial
direction was precisely nadiral, to the position obtained by the
strip formation (see Figure 27). The rotation from this trans­
formed position to the oriented one is given in the following form:

\[ T_m^X = \begin{bmatrix} t_X^{m} & t_Y^{m} & 1 \\ t_X^{m} & t_Y^{m} & 1 \end{bmatrix} \bar{M}_X \quad \text{and} \quad T_m^Y = \begin{bmatrix} t_X^{m} & t_Y^{m} & 1 \\ t_X^{m} & t_Y^{m} & 1 \end{bmatrix} \bar{M}_Y \] (3-17)

Where \( \bar{M}_X, \bar{M}_Y, \bar{M}_Z \) are the columns of the unknown orientation
matrix (containing the unknowns \( \Delta \phi, \Delta \Omega \)).

On the other hand, the direct transformation from the initial
nadiral position to the oriented direction as determined by the
horizon camera is (see Figure 27) expressed by the following
equations:

\[ T_h^X = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \bar{M}_X^{h} \quad \text{and} \quad T_h^Y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \bar{M}_Y^{h} \] (3-18)

Where \( \bar{M}_X^{h}, \bar{M}_Y^{h}, \bar{M}_Z^{h} \) are the columns of the horizon-determined
orientation matrix (\( \kappa = 0 \)).

Substituting equation (3-17) and equation (3-18) in equation
(3-15), it follows that:

\[ \frac{[t_X^{m} \ t_Y^{m} \ 1]}{[t_X^{m} \ t_Y^{m} \ 1]} \cdot \bar{M}_X - \frac{[0 \ 0 \ 1]}{[0 \ 0 \ 1]} \cdot \bar{M}_X^{h} = 0 \] , and
The only unknowns in equations (3-19) are the orientation elements contained in $\vec{M}_X$, $\vec{M}_Y$, and $\vec{M}_Z$.

The linearized form (Taylor Series) of the above equations is:

$$-(1 + t_{X}^{im})^{2} \Delta \phi + t_{X}^{im} t_{Y}^{im} \Delta \Omega + t_{X}^{im} T_{X}^{h} = 0,$$ and

$$-t_{X}^{im} t_{Y}^{im} \Delta \phi - (1 + t_{Y}^{im}) \Delta \Omega + t_{Y}^{im} T_{Y}^{h} = 0.$$

The set of equations (3-20) is associated with bundle i of model n. An equivalent set can be deduced for bundle j, thus four equations are available for the determination of the two unknowns $\Delta \phi$ and $\Delta \Omega$ by least squares technique. The resulting values will be the amount by which the two bundles associated with the model must rotate.

Theoretically $\kappa$ should be solved for here as well. Each model has two fixed directions in space, associated with the two bundles i and j (those obtained by the horizon camera). When the model undergoes a rotation to fit into these two directions, it undertakes a swing ($\kappa$) as well. However practically the two fixed directions as obtained from the horizon camera are nearly parallel to each other and thus $\kappa$ is insignificantly affected by the rotation of the model.
3.3124 Parabolic Smoothing of Breaks Between Models

Because of the imperfect rectification of each model, discrepancies will remain along the line joining any two adjacent models. This is an unavoidable outcome or a "price" for controlling the error accumulation. A rather known concept can be modified here to fit the purpose of smoothing the discrepancies between models. The suggested method is based upon the assumption that the propagation of errors in aerial triangulation can be expressed in terms of a third degree polynomial such as:

\[ \Delta Z = c_0 + c_1 x + c_2 y + c_3 x^2 + c_4 x^3 + c_5 x y \]

which, for points along the strip axis, becomes:

\[ \Delta Z = c_0 + c_1 x + c_3 x^2 + c_4 x^3 \]
Since
\[ \Delta \phi = \frac{\partial \Delta \gamma}{\partial x}, \]
then it follows that
\[ \Delta \phi = C_1 + 2C_3X + 3C_4X^2, \]
or in a general form:
\[ \Delta \phi = A_1 + A_2X + A_3X^2. \] (3-21)

In a similar way \( \Delta \kappa \) can be expressed in a general form as:
\[ \Delta \kappa = D_0 + D_1X + D_2X^2 \]
\( \Delta \Omega \) is a linear function of \( X \) but since the other two corrections are expressed as a second degree polynomial, the expression for \( \Delta \Omega \) can also be raised to this degree, so that:
\[ \Delta \Omega = B_1 + B_2X + B_3X^2. \] (3-22)

Since it is assumed here that no definite point can be observed in the horizon picture and thus that \( K \) cannot be solved for, only the equations for \( \phi \) and \( \Omega \) will be dealt with. Substituting equation (3-21) and (3-22) in equation (3-20) the following expressions are obtained:
\[ -(1 + t^m_X^2) (A_1 + A_2X + A_3X^2) - t^m_X t^m_Y (B_1 + B_2X + B_3X^2) \]
\[ + t^m_X - t^h_X = 0, \]
\[ - t^m_X t^m_Y (A_1 + A_2X + A_3X^2) - (1 + t^m_Y^2) (B_1 + B_2X + B_3X^2) \]
\[ + t^m_Y - t^h_Y = 0. \] (3-23)
Six unknowns are involved in the above equation \( (A_1, A_2, A_3, B_1, B_2, B_3) \). For every model there are then four equations with six unknowns. In a strip of \( n \) models there will be \( 4n \) equations from which the six unknowns can be solved for (\( X \) can be taken as the machine coordinates of the right nadir point of each model).

Once the coefficients are solved for, equations (3-21) and (3-22) are used once again to compute \( \Phi \) and \( \Omega \) of each model and these elements should be used in the rectification stage. It should be emphasized that this suggested method does not correct discrepancies but rather smooth them in a neat way. The "smoothening curves" do not require given ground control points for the determination of the coefficients and cannot be compared in any way with the "correction curves" which are usually fitted to the measured coordinates in the conventional strip adjustment process.

3.3.13 The Operational Procedure

Model \( n \) has to be rectified in order to assume its absolute position next to the already absolutely oriented model \( n-1 \). The scale transfer point \( P \), (Figure 26) common to models \( n \) and \( n-1 \), has already assumed its new adjusted position by the virtue of its belonging to model \( (n-1) \). Scale transfer point \( Q \), common to models \( n \) and \( n+1 \), will reach its new adjusted position following the rectification of model \( n \). From equation (3-11) it follows that;

\[
\tilde{C}_Q = S + \lambda \tilde{C}_Q^* \quad M, \quad (3-24)
\]
where:

\( S = \) Translations along the three axes, computed from equations (3-12);

\( M = \) Orientation matrix, whose elements were computed following the equations presented in section 3.3.12, except for \( \kappa \) which is assumed as zero;

\( \lambda = \) Scale factor which, in this case, cannot be controlled and therefore is assumed to be unity;

\( C^*_Q = \) The coordinates of point \( Q \) as obtained in the process of strip formation; (see Figure 26)

\( \bar{C}_Q = \) The adjusted coordinates of point \( Q \). These are the only unknowns to be computed in equation (3-24).

Next, model \( n + 1 \) has to undergo a rectification. Denoting scale transfer point \( \bar{Q} \) as \( \bar{P} \), the above procedure is repeated and adjusted coordinates of the adjusted point \( \bar{Q} \) (common to models \( n + 1 \) and \( n + 2 \)) are computed.

### 3.3.2 Adjustment of the Bridged Strip

Bridging can be analytically viewed as the case in which more information exists along the strip, than in the case of the cantilever strip. The additional information, in form of control points, enables an adjustment of the strip. The effectiveness of the adjustment depends upon the density and distribution of the available control points. In the conventional methods, this information is utilized to establish a second or higher degree
polynomials as means of adjustment for the entire strip. The following suggested method, in contrast, is aiming at the adjustment of the individual models in the strip, taking advantage of any available type and amount of information.

Three control points at the first model for example will enable the absolute orientation of this model. The other models will be linked to the first one, each containing 7 unknowns, i.e. three rotations, three translations and a scale factor. If the strip contains n models then at least 7(n-1) equations must be set up. These equations can be of two types:

1. \[ C_i - (S + \lambda C^*_i M) = 0 \]  \hspace{1cm} (3-25)

2. \[ S_n + \lambda_n C^*_n M_n = S_{n+1} + \lambda_{n+1} C^*_{n+1} M_{n+1} \]  \hspace{1cm} (3-26)

The equation of Type I is applied whenever a ground control point is available (as explained in Section 3.31). The equation of Type II is applied as a condition which the Pass points along the line joining the two adjacent models must fulfill. Thus it is required that:

Number of equations of Type I + Number of equations of Type II \( \geq \) total number of unknowns.

The ideal situation is the case in which enough information is available to solve for the seven unknowns of each model in the strip without resorting to various assumptions and approximations. However, there is hardly a need for aerial triangulation in such a case, since the achievement of this goal requires control
points at each model. When the horizon data are available, the amount of unknowns is reduced to five per model, but unless additional auxiliary data and/or ground control points are available, approximations have to be made. This was the case with the cantilever strip where $K$, $S$ and $\lambda$ could not be controlled. When the ground control points are available along the strip, an adjustment for $\lambda$ and $K$ can be applied in linear segments between the control points. Obviously this is an approximation whose effect is lessened by the increase of the number of available control points. In this case, if $\Delta K$ of each model is plotted against $X$ (the distance between any two control points) the resulted curve is a straight line expressed mathematically as:

$$\Delta K = ax + b,$$

but since

$$\Delta K = \frac{d\Delta Y}{dX},$$

then

$$\Delta Y = \int (ax + b) \, dx = \frac{1}{2a} (ax + b)^2,$$

i.e. the linear change in $K$ has a parabolic effect on the resulting adjusted $Y$ coordinate. However this approach should not be confused with the second degree polynomial commonly used in photogrammetry as means of a direct adjustment of point coordinates throughout the entire strip. In long strips where ground control points are sparse, the underlying assumption, that the error accumulation does resemble the degree of polynomial used, is quite doubtful. On the other hand, the auxiliary data will yield corrections to the errors "as they are, not as they should
The parameters which are not solved for by the horizon data will require some approximations, as already explained, but the effect of these approximations is limited to the range between the available control points. Consequently the simultaneous adjustment offered in this method, utilizing both the auxiliary data and any amount of given ground control points, will aim at getting as close as possible to the rigorous solution.

### 3.3.21 Choice of a Coordinate System as a Reference for Computation

The computation of aerial triangulation could be referred to any of the available coordinate systems. Most popular among these are the following two:

1. The geocentric rectangular (also known as universal) coordinate system having its origin at the center of the earth ellipsoid, X and Y axes perpendicular to each other on the equatorial plane and Z axis perpendicular to both.

2. The local rectangular coordinate system whose X and Y are on a plane tangent to the ellipsoid at a selected point which serves as the origin, X points to the East, Y points to the North and Z is in the direction of the normal to the ellipsoid. Since it is more convenient to use the local system whose origin is in the neighborhood of the triangulation, this system is recommended for the work in question. The coordinates of the control points are generally given with reference to either geodetic or a certain map projection system. Formulas are available to transform these coordinates to the local coordinate system. Using these transformed coordinates in the adjustment of the strip,
no further compensation for the earth curvature effect is needed.

The transformation of the coordinates of ground control points, given in a certain map projection system, to the local system is not within the scope of this work. However, if a short strip is triangulated, various simplifications can be made without significantly affecting the accuracy of the results.

An example of such an approximation is the use of the following earth-curvature compensation equations; [15]

\[ \Delta h = \frac{x^2}{2R}, \quad \text{and} \quad \Delta x = \frac{x^3}{6R^2} \]

to be applied to the measured coordinates. Needless to say that this approximation is valid to short strips only, whereas auxiliary data are generally associated with long strips in which case approximations of the above nature may not be tolerated. In general, there is no justification to resort to simplifications when analytical procedures can easily provide a more rigorous approach.

In the case of cantilever strip having control points only at the first model, as explained previously, each model is rectified by the amount of the difference between the triangulated and the horizon-determined rotation angles. This is equivalent to rotating the earth surface, covered by the model, in which case no additional compensation for earth curvature is required. In the case of a bridged strip where both ground control points and horizon data are utilized, as the present method suggests, the
situation is more complex. The values of $\varphi$ and $\omega$, obtained by observing the true horizon, refer to the normal to the geoid passing through the perspective center of the nadiral camera at each exposure station. Thus the horizon data refer to a system which is continually changing from photograph to photograph while the triangulation operation generally takes place in a space defined by one local coordinate system (Figure 28). Therefore, just as the given ground control points have to be transformed into the local system, so have the auxiliary data to be transformed into the local system to maintain congruency of data. Rotating the local system about its Z axis so that its X axis will coincide with the flight direction, only $\varphi$ has to be corrected by the amount of convergence angle $\alpha$

$$
\alpha_i = \frac{d_i}{R} \rho
$$

where $d$ is the distance from the nadir point of the i-th photo to the origin of the local system, and $R$ is the earth's radius at the origin of the system. (Figure 28). The difference between this rough computation of the convergence angle and the more sophisticated methods used in higher geodesy, is in the order of a few seconds of arc for a strip length of 200km., whereas the accuracy of the horizon-determined $\varphi$ is in the order of about $\pm 1'$.5. Therefore, this rough procedure of determining the convergence angle is justified.
3.322 The Adjustment Equations

Considering that all the auxiliary data has already been transformed to the local reference system (as explained in Section 3.321), that $\phi$ and $\omega$ of every model has been computed from the available horizon data (as explained in section 3.3123), and that the strip has already been formed, the next step is the adjustment of the strip. The equations to be used for this purpose...
are (3-25) and (3-26). Equation (3-25), in full reads:

\[
F_X = X - S_X - \lambda [x^* (\cos \phi \cos \kappa) + y^* (-\cos \phi \sin \kappa) + z^* (\sin \phi)] = 0,
\]

\[
F_Y = Y - S_Y - \lambda [x^* (\cos \Omega \sin \kappa + \sin \Omega \sin \phi \cos \kappa) + y^* (\cos \Omega \cos \kappa \\
- \sin \Omega \sin \phi \sin \kappa) + z^* (-\sin \Omega \cos \phi)] = 0,
\]

and

\[
F_Z = Z - S_Z - \lambda [x^* (\sin \Omega \sin \kappa - \cos \Omega \sin \phi \cos \kappa) + y^* (\sin \Omega \cos \kappa \\
+ \cos \Omega \sin \phi \sin \kappa) + z^* (\cos \Omega \cos \phi)] = 0.
\]

\[
(3-29)
\]

Differentiating equation (3-29), with respect to the unknowns and assuming \( \phi = \Omega = \kappa = 0 \), it follows that:

\[
\begin{bmatrix}
\frac{\partial F_X}{\partial x} \\
\frac{\partial F_Y}{\partial x} \\
\frac{\partial F_Z}{\partial x}
\end{bmatrix}_0 = \lambda y^*, \quad
\begin{bmatrix}
\frac{\partial F_X}{\partial y} \\
\frac{\partial F_Y}{\partial y} \\
\frac{\partial F_Z}{\partial y}
\end{bmatrix}_0 = -\lambda x^*, \quad
\begin{bmatrix}
\frac{\partial F_X}{\partial z} \\
\frac{\partial F_Y}{\partial z} \\
\frac{\partial F_Z}{\partial z}
\end{bmatrix}_0 = 0,
\]

\[
\begin{bmatrix}
\frac{\partial F_X}{\partial \lambda} \\
\frac{\partial F_Y}{\partial \lambda} \\
\frac{\partial F_Z}{\partial \lambda}
\end{bmatrix}_0 = -x^*, \quad
\begin{bmatrix}
\frac{\partial F_X}{\partial \alpha} \\
\frac{\partial F_Y}{\partial \alpha} \\
\frac{\partial F_Z}{\partial \alpha}
\end{bmatrix}_0 = -y^*, \quad
\begin{bmatrix}
\frac{\partial F_X}{\partial \phi} \\
\frac{\partial F_Y}{\partial \phi} \\
\frac{\partial F_Z}{\partial \phi}
\end{bmatrix}_0 = -z^*,
\]

\[
\begin{bmatrix}
\frac{\partial F_X}{\partial S_X} \\
\frac{\partial F_Y}{\partial S_X} \\
\frac{\partial F_Z}{\partial S_X}
\end{bmatrix}_0 = -1, \quad
\begin{bmatrix}
\frac{\partial F_X}{\partial S_Y} \\
\frac{\partial F_Y}{\partial S_Y} \\
\frac{\partial F_Z}{\partial S_Y}
\end{bmatrix}_0 = 0, \quad
\begin{bmatrix}
\frac{\partial F_X}{\partial S_Z} \\
\frac{\partial F_Y}{\partial S_Z} \\
\frac{\partial F_Z}{\partial S_Z}
\end{bmatrix}_0 = 0,
\]

\[
\begin{bmatrix}
\frac{\partial F_X}{\partial S_X} \\
\frac{\partial F_Y}{\partial S_Y} \\
\frac{\partial F_Z}{\partial S_Y}
\end{bmatrix}_0 = 0, \quad
\begin{bmatrix}
\frac{\partial F_X}{\partial S_X} \\
\frac{\partial F_Y}{\partial S_Y} \\
\frac{\partial F_Z}{\partial S_Y}
\end{bmatrix}_0 = -1, \quad
\begin{bmatrix}
\frac{\partial F_X}{\partial S_Z} \\
\frac{\partial F_Y}{\partial S_Z} \\
\frac{\partial F_Z}{\partial S_Z}
\end{bmatrix}_0 = 0,
\]

\[
F_{X_0} = X - S_X - \lambda x^*, \quad F_{Y_0} = Y - S_Y - \lambda y^*, \quad F_{Z_0} = Z - S_Z - \lambda z^*.
\]
Thus the linearized set of equation (3-29) is:

\[
\begin{align*}
F_x &= \lambda y^* \Delta \kappa - x^* \Delta \lambda - \Delta S_x + F_{x_0} = 0, \\
F_y &= \lambda y^* \Delta \kappa - y^* \Delta \lambda - \Delta S_y + F_{y_0} = 0, \\
F_z &= -z^* \Delta \lambda - \Delta S_z + F_{z_0} = 0.
\end{align*}
\]  

(3-30)

The corresponding set of linear equations deduced from equation (3-26), in the same manner as equation (3-30) was deduced, is:

\[
\begin{align*}
G_x &= (-\lambda y^* \Delta \kappa + x^* \Delta \lambda + \Delta S_x)_n - (-\lambda y^* \Delta \kappa + x^* \Delta \lambda + \Delta S_x)_{n+1} \\
&\quad + [(S_x + \lambda x^*)_n - (S_x + \lambda x^*)_{n+1}] = 0, \\
G_y &= (\lambda x^* \Delta \kappa + y^* \Delta \lambda + \Delta S_y)_n - (\lambda x^* \Delta \kappa + y^* \Delta \lambda + \Delta S_y)_{n+1} \\
&\quad + [(S_y + \lambda y^*)_n - (S_y + \lambda y^*)_{n+1}] = 0, \\
G_z &= (z^* \Delta \lambda + \Delta S_z)_n - (z^* \Delta \lambda + \Delta S_z)_{n+1} \\
&\quad + [(S_z + \lambda z^*)_n - (S_z + \lambda z^*)_{n+1}] = 0.
\end{align*}
\]  

(3-31)

The adjustment program (Section 3.325) utilizes an iterative process similar to the one used in the program for tilt determination (Appendix B). This program calls for starting each iteration with the initial first approximations assigned to the unknowns.

In the present case, the initial values of the unknowns are:

\[
\begin{align*}
\kappa &= 0, \\
\lambda &= 1, \quad \text{and} \\
S_x &= S_y = S_z = 0.
\end{align*}
\]
Accordingly, the known terms of equation (3-30) become:

\[ F_X = X - x^* \]
\[ F_Y = Y - y^* \]
\[ F_Z = Z - z^* \]

and those of equations (3-31) become:

\[ G_X = x_n^* - x_{n+1}^* \]
\[ G_Y = y_n^* - y_{n+1}^* \]
\[ G_Z = z_n^* - z_{n+1}^* \]

These equations are solved by a least squares adjustment, whose program is given in Appendix C.

3.323 Shift of the Origin of Coordinates to the Center of the Strip and its Effects

In the adjustment program, the origin of the coordinates is shifted to the center of the strip (which need not be exactly determined) and the computations are done with the reduced coordinates to this origin. The reasons for this shift of origin are:

1. To avoid large numerical translations of the coordinates during the adjustment process.
2. To reduce the effect of the elevation upon the planimetry while the models are rotated.
However, because of this shift, the element $\phi$, as determined from the horizon picture, is a rotation about the Y axis located at the center of the strip rather than a rotation about the Y axis located at the center of its corresponding model. As a result, it becomes necessary to shift vertically each model (see Figure 29). $\alpha$, being a rotation about the X axis will not cause a similar shift. If the primary rotation is $\phi$ and X axis of the strip coincides with the flight direction, the amount of this shift is equal to $X \tan \phi$, where $X$ is the distance between the new coordinate origin and the model in question. However, the necessary vertical shift of the model is nothing else but an additional amount of shift to that of $S_z$, and thus will be taken care of by the imposed condition that the elevation of point $P$ in model $n$ should be the same as the corresponding elevation of point $P$ in model $n + 1$. (Equation (3-26)).

Figure 29 The Effect of $\phi$ Rotation About The Y Axis Located at the Center of the Strip
3.324 Points Involved in the Adjustment and Their Function

In previous sections, the term "pass point" was applied to the points located at the common overlap between any two adjacent models. The measured XYZ coordinates of these points (u,m,l in Figure 30) in the two adjacent models enable the linking of these models to each other by imposing the condition that their coordinates in model n be the same as in model n + 1. However, the Z coordinates of pass points u and l lead to the determination of a certain value of Ω, which might disagree with the value of Ω deduced from the horizon camera. Since Ω computed from the horizon data is taken as a fixed, unchangeable quantity, then if both this element and the Z coordinates of the pass points will be involved in the adjustment, it will be left only for the planimetry to compensate for the possible resulting discrepancy in elevation. This can be accomplished by rotating the free element κ, yet κ has to rotate a considerable amount in order to affect Z, which is of course intolerable in the triangulation. Point m (Figure 30) located on the X axis about which Ω rotates, will not contribute to the above problem and hence its XYZ coordinates can be fully utilized in the adjustment. As for the points u and l, their XY coordinates can be utilized while their Z coordinates should not. In this way the model can freely rotate by the amount imposed by the "horizon-determined" φ and Ω, without causing unbearable constrains on the adjustment. For this reason, the third equation in the set of equations (3-31) is
multiplied by a numerically small factor whenever it is applied to a pass point. This factor can be looked upon as a reduced weight.

To sum up: All points in the strip can be divided into four groups: 1. control points;
2. points \((m)\);
3. pass points \((u, l)\);
4. all other points whose coordinates are required.

Only groups 1, 2, and 3 are involved in the adjustment phase; groups 1 and 2 are fully involved while group 3 with the exception of the Z coordinates. The strip coordinates of points belonging to the fourth group will be transformed into the ground coordinate system using the parameters resulted from the adjustment.

![Figure 30 Classification of Points in the Model](image)
3.325 The Strip Adjustment Computer Program

The present program, as already stated, has as its purpose the adjustment of the individual models in the strip, taking advantage of any available type and amount of information. The least squares solution is applied simultaneously to all models in the strip and the parameters of each individual model are then deduced from the simultaneous solution. The technical procedure of performing this task is explained below.

The data deck contains:

1. The general data card: NPIO (= Number of Points Involved in the Operation, as explained in Section 3.324), and NM (= Number of models in the strip). If a control point is also a scale transfer or a pass point, it should be counted twice for the value of NPIO.

2. Cards associated with the various points involved in the operation. Each of these cards contains the following information in the listed order:

   Identification number of the point in question;
   XYZ strip coordinates in model n;
   UWW strip coordinates in model n + 1; [in case of a control point, U, V, W are the given ground coordinates].
   Model number (n) and the adjacent model number (n + 1)
   Type of point: 1 will denote a pass or scale transfer point, and 2 will denote a control point;
   A factor WZ: 1. if the Z coordinate of the point is to be
considered, and 0,001 if not, [see 3.24]

3. A reference card containing the coordinates of a point at the approximate center of the strip.

4. Horizon-determined elements card, containing the tilt components of each model.

5. Cards containing the elements $\kappa$, $\lambda$, $S_\kappa$, $S_\lambda$, $S_z$ of each model. [if these parameters are unknown, the corresponding values assigned to them are: 0, 1, 0, 0, 0].

6. Cards containing the coordinates of the points belonging to group 4, as explained in section 3.324.

The program is following this procedure:

a) The data deck is read, except for the data included in the aforementioned Point 6.

b) The strip coordinates are reduced to the selected center of the strip.

c) The orientation matrix ($M_1$) of each model is constructed, using $\phi$ and $\Omega$ as derived from the horizon pictures. If $\kappa$ is not known, its value is assumed zero.

d) The strip coordinates are transformed using the corresponding orientation matrix ($M$) of the models in which they are located. This transformation is applied twice for each pass point and scale transfer point, since each of them are located at two adjacent models. The coordinates transformation follows equation (2-11).

e) The total number of unknowns ($= 5NM$) and the total number
of observation equations (= 3.NPIO) are computed.

f) The general matrix, shown in Figure 15, has the size (5NM + 3NPIO + 5NM) X (5NM), and contains the sub-matrices $A^T$, $N$, and $I$.

1. The construction of $A^T$ Matrix.

Each point involved in the operation leads to 3 equations, thus the number of columns in the matrix $A^T$ is $= 3 \cdot NPIO$, and number of rows = number of unknowns = 5.N. If the first point read is a control point, the coefficients associated with it will occupy the first 3 columns and will extend from row 1 to row 5. If it is a pass point the corresponding coefficients will occupy the first 3 columns and from row 1 to row 10. The following is an example of a $A^T$ matrix constructed for a case in which 3 models containing 3 control points and 6 pass points (NPIO = 9) are involved [see Figure 31 and 32].

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{3-models-strip.png}
\caption{A 3-Models Strip}
\end{figure}
The coefficients associated with control points are taken from equation (3-30) and those associated with pass and scale transfer points, from equation (3-31).

2. The construction of the normal matrix \( (N) \) and the unit matrix \( (1) \) is exactly the same as explained in Section 2.302.

g) The known terms of the equations associated with control points are computed according to equation (3-32), and those associated with the scale transfer and pass points are computed according to equation (3-33).

h) The check for termination of the iterations, the solution matrix (rendering \( N^{-1}A^T \) and \( Q \)), the solution of the unknowns and the construction of the partial matrix for each model, follow the same instructions as outlined in Section 2.302. It should be emphasized that in the present procedure each iteration shifts the coordinates of the points to a new position which will serve as the initial position for the next iteration. Thus every
iteration starts with the initial first approximation assigned to the parameters, and ends with the resulting increments or partial values of these parameters. These values are used in the partial transformation of the coordinates from the previous to the new position. The final values of the orientation parameters, at the end of the iteration process, are deduced from the increments resulting at each iteration.

i) The standard deviation of unit weight as well as the standard deviation of the unknowns can be computed as explained in Section 2.302.

j) When the iteration process is terminated and the final orientation parameters of every model in the strip, are obtained, the points belonging to group number 4 (see Section 3.324) are read and their coordinates are reduced to the origin of the coordinate system at the center of the strip. These coordinates are transformed to the ground system, utilizing the orientation matrix of the model in which the corresponding points are located. If the points are located in two models, their coordinates will be transformed twice in the same manner.

The computer program is represented in Appendix C and the flow chart, in Figure 33.
Figure 33 Flow Chart for the Strip Adjustment Program
3.4 **Experimental Work**

3.41 **Data Acquisition and Reduction**

The fictitious data used here are taken from a fictitious block prepared by the International Society of Photogrammetry, Commission III. The characteristics of this block, consisting of several strips, are:

- **Cameras:** Wide angle, near vertical.
- **focal length = 152.00 mm.;**
- **format = 23 x 23 cm.;**
- **flight height = 11000 m.;**
- **average terrain elevation = 1000 m.;**
- **overlap = 60%.**

The theoretical data were perturbed to contain both systematic and random errors. The ground control points available in each model were transformed to the local system, thus they were brought to the same reference system of the strip coordinates. In the following experiments, only strip number 1 was used, from photograph number 2 to 14, i.e. 12 models.

The photographs were linked to each other in the regular aeropolygon fashion, based upon an assumed absolutely horizontal model at the beginning of the strip. The first model was then absolutely oriented by means of six ground control points. The resulting parameters of this orientation were used to transform all strip coordinates into ground coordinates. At this stage the adjustment had to be applied. However since no horizon data were available, each of the models in the strip underwent an
absolute orientation based upon the available ground control points contained in each of them, and the resulting $\phi$ and $\Omega$ were treated as the values derived from a horizon camera.

The input of the adjustment program contains the strip coordinates of the points as obtained after the absolute orientation of the first model, as well as the determined values of $\phi$ and $\Omega$ of each model (assumed as the horizon camera data). The initial approximate values of $S_x$, $S_y$, $S_z$, and $\kappa$ were set to zero, and the scale factor to 1. The strip under investigation is shown in Figure 34.

![Diagram of strip under investigation](image)

Figure 34 The Strip Under Investigation

### 3.42 Experiments

The strip was adjusted five times, utilizing different densities and distributions of ground control points as indicated in Figure 35.
Figure 35 Arrangement of Control Points (▲) in the Strip
3.43 Analysis and Conclusions

Figure 36 represents the differences between the transformed strip coordinates (following the absolute orientation of the first model) and the corresponding ground control points. The curves $\Delta X$, $\Delta Y$, and $\Delta Z$ were plotted in the same scale, revealing a systematic appearance of error accumulation. While $Y$ coordinates are strongly affected by error accumulation in this test, the $Z$ coordinates are scarcely affected by it. However in order to test the suggested adjustment program, trying different cases, the curves $\Delta X$, $\Delta Y$, $\Delta Z$ were plotted again, each in a different scale, to enable appropriate visual comparison between the quantities before and after the adjustment [see Figure 38].

Figure 37 represents the discrepancies between the adjusted coordinates and the corresponding ground control points. The adjustment here is based upon the minimum necessary number of ground control points [Experiment 1, Figure 35]. The discrepancies were plotted in a scale approximately 800 times bigger than that used for plotting the planimetric position of the points. The discrepancies in elevation were plotted separately from the discrepancies in planimetry but both share the same scale. The graph indicates smaller residuals in elevation than in planimetry which is expected when the auxiliary elements $\Phi$ and $\Omega$ are utilized in the adjustment. It is also evident that while additional control points at the middle of the strip might improve the planimetric accuracy, they will not affect
Figure 36 Discrepancies (Transformed-Ground Control) Before Adjustment
Discrepancies in Planimetry

Figure 37 Discrepancies (Adjusted-Ground Control) After Adjustment
(scale of discrepancies = 800 times that of the planimetric position of the points)
considerably that of the elevations. The deformation pattern along the upper, central and lower longitudinal cross sections of the strip is about the same and therefore in the following analysis, only the points along the flight axis will be investigated.

The curves shown in Figure 38 represent the discrepancies between the adjusted and the corresponding ground control values. Each of these graphs is plotted in a different scale in order to enable an appropriate visual comparison between the various curves. The dashed curve is associated with the adjustment based upon two cross sections of ground control points, one at the beginning and one at the end of the strip [Experiment 3, Figure 35]; the dash-point curve is associated with the adjustment based upon three cross sections of ground control points, at the beginning, close to the beginning, and at the end of the strip [Experiment 4, Figure 35]. The curves representing the discrepancies before the adjustment (see Figure 36) are superimposed in the Figure for comparison purposes (solid lines).

It is evident that, from this specific set of experiments:

1. An adjustment utilizing an additional cross section of ground control points, located close to the beginning of the strip does not change the situation considerably, and the slight improvement is generally of a local nature.

2. An adjustment utilizing control points evenly
After Transformation

Control: 1, 2, 3, 37, 38, 39

Control: 1, 2, 3, 19, 20, 21, 37, 38, 39

Figure 38 Discrepancies After Adjustment (residuals) Based Upon Various Densities and Distributions of Ground Control Points
Figure 38 (Continued)
distributed along the strip, such as at the beginning, middle and end, has generally a greater effect upon the results than in the previous case, although quantitatively, the differences between the two cases are small.

3. Quantitatively, the effect of additional control points at the middle of the strip on the \( Z \) coordinates is much smaller than the effect on the planimetric coordinates.

In order to further investigate the properties of the suggested adjustment procedure, the strip under investigation underwent a conventional adjustment utilizing a second degree polynomial. This polynomial is the one used regularly by the U.S. Coast and Geodetic Survey\(^{10}\) and whose computer program was modified by D. Moellman, a Ph.D. candidate at the University of Illinois. The polynomials are:

\[
\begin{align*}
x' &= x + bx^2 + cx - 2dxy - ey + f, \\
y' &= y + 2bxy + cy + dx^2 + ex + g, \quad \text{and} \\
z' &= z + ix^2 + jx + lxy + my + n.
\end{align*}
\]

The ground control points utilized are those involved in Experiment 5 (Figure 35), and the differences between the adjusted and ground coordinates were plotted and are represented in Figure 39 (solid line). Also plotted there, for comparison purposes, are the discrepancies as obtained from the new adjustment program (dashed lines). The close proximity between the two curves is expected because the linear adjustment of the
Figure 39 Comparison of Discrepancies Resulting From Different Adjustment Procedures
angular parameters results in a second degree adjustment of the coordinates. This fact is more evident in the following experiment.

The strip under investigation underwent an adjustment utilizing a first degree polynomial (a linear adjustment) based upon ground control points at the beginning and end of the strip only [Experiment 3, Figure 35] and the discrepancies (adjusted-ground coordinates) were plotted and are represented in Figure 40, as a solid line. The corresponding discrepancies obtained from the new program were plotted as well (dashed curve). From the previous experiment it was evident that in the case under investigation, a second degree correction will take care of the errors in Y. In the present experiment the new adjustment indeed provides this correction while the direct application of a first degree polynomial as means of adjustment does not. Moreover, a second degree polynomial cannot be directly applied to this particular arrangement of control points while the new adjustment program can. Thus it is evident from this set of experiments that:

1. The suggested adjustment program does perform its task.

2. The density and distribution of the available ground control points have a bearing on the accuracy of the results in planimetry; the more control points available and the more even is their distribution, the better are the results. However, the suggested method seems to be less sensitive to the pattern of control distribution than the polynomial adjustment.
Figure 40 Comparison of Discrepancies Resulting From Different Adjustment Procedures
3. The elevations are far less dependent upon the density and distribution of the ground control points than the planimetric coordinates are. This is expected when the auxiliary elements $\Phi$ and $\Psi$ are used.

It should be emphasized that the graphs presented in this investigation do not intend to show the difference in accuracy between the two compared methods. The graphs only indicate the characteristics and potentials of each method.

However, it should be kept in mind that the strip under investigation is rather short and thus cannot demonstrate the full advantage of the suggested adjustment procedure. The effectiveness of the new program, utilizing the auxiliary data, should be more pronounced in case of long strips where the assumption that the error accumulation resembles any particular polynomial is doubtful.
4. SUMMARY

The following quotation of Blachut\(^{[1]}\) summarizes the general problem under investigation:

"Experience has proven that using even the most precise triangulation procedure, for instance analytical treatment, it is impossible at present to predict the exact form and magnitude of accumulative errors. In addition, their absolute value is quite large so that in order to guarantee an accuracy dictated by general mapping requirements, the triangulation strip must be restricted to relatively short distances....One would like to have a method which would make it possible to bridge practically unlimited distance and thus to map wide territories without having recourse to field surveying....Auxiliary data present convenient and efficient means of controlling the aerial triangulation over long distances. Generally errors in a triangulated strip do not follow a second order curve, therefore a parabolic correction does not give the best answer. The auxiliary data suggested in my talk permit us to make suitable corrections all along the strip."

The horizon camera is one of the auxiliary instruments which provide some of the necessary information for controlling the aerial triangulation, by yielding the values of the orientation parameters \(\varphi\) and \(\omega\) at each exposure station. A new method of deducing the orientation parameters from the horizon pictures have been established. The properties of this method are:

1. It allows for any number of observations along the horizon line;

2. It is subject to a least-squares solution;

3. It yields rigorous absolute components of tilt whenever the true horizon or a levelled apparent horizon is observed;
4. Relative tilt components can be obtained from the absolute values or directly from the equations whenever this is required;

5. It provides a fast and economical data-reduction and processing procedure, compared with the conventional methods studied;

6. It is not restricted to horizon camera pictures only but can be used in association with terrestrial or high oblique photography as well. Nor is it restricted to observation of the horizon line but also to definite points observable on the horizon picture, in which case both $\varphi$, $\omega$, and $\kappa$ can be obtained.

Although the observation of definite points in the horizon pictures is the most promising means of deducing the three rotational elements of orientation, the present available camera and films are incapable of registering such points in most cases. When definite points cannot be observed, the orientation element $\kappa$ cannot be deduced and an investigation is conducted here to analyze the effect of $\kappa$ on the determination of $\varphi$ and $\omega$. It was found that if $\kappa$ is known to within $\pm 4^\circ$ or if it may be assumed that the value of $\kappa$ does not exceed $4^\circ$, it is possible, at least geometrically, to determine $\varphi$ and $\omega$ with an accuracy of $\pm 2^\circ$, with the currently available equipment viz. Hc-1 (WILD) camera.

It was further demonstrated that, from a geometrical point of view, a wide angle horizon camera will enable the determination of both $\varphi$ and $\omega$ from every single horizon picture, with practically the same degree of accuracy. With the currently available horizon camera, the above condition is reached only when two perpendicular,
or all four horizon pictures, registered in each air station, are utilized.

Restricted to the rather limited accuracy of the horizon-determined orientation parameters, the incorporation of these data in aerial triangulation must be handled with care. Brandenberger[2] comments on the use of auxiliary data:

"If aerial triangulation with auxiliary data should be more accurate than aerial triangulation without auxiliary data (aeropolygon) it is of utmost importance that the auxiliary data have sufficient accuracy. This means that elements of exterior orientation such as $b_x$, $b_y$, $b_z$, $\omega$, $\phi$, $\kappa$ should be obtained with higher accuracy from aerial triangulation with auxiliary data than from the aeropolygon method. Since we have not yet reached this stage...efforts have to be made to increase the inherent accuracy of auxiliary data...The problem encountered here is that it is not so easy to attain with these auxiliary data an accuracy of about $1^\circ$ for $\omega$ and $\phi$ which can be obtained from a direct relative orientation in a first order stereoplotting instrument."

It is, thus, evident that at present the horizon-determined orientation parameters could not be incorporated in the process of strip formation (i.e. in the relative orientation of one photograph to the preceding one) where conventional procedures yield a higher degree of accuracy. It is therefore suggested that the auxiliary data should be incorporated in the adjustment stage. In view of the above discussion it is advisable to establish an adjustment method with the following characteristics:

1. Keeping the conventional relative orientation of each model intact;
2. Not resorting to polynomials alone, especially when long strips are triangulated;
3. Utilizing both the auxiliary data and any available ground control point along the strip.

The suggested method in this presentation meets the above specifications, using the following technique:

1. Each model in the strip is adjusted as a rigid unit.
2. The horizon-determined parameters of each model are viewed as known elements and are not subject to any change in the adjustment process.
3. The other orientation parameters required for adjusting each model, and which are not given apriori, are solved simultaneously for all the models in the strip.
4. The coordinates of the ground control points are incorporated in the above simultaneous solution, thus contributing to the adjustment of the entire strip. The density and distribution of the control points will affect the accuracy of the determination of the parameters which are not deduced from the horizon camera.
5. The orientation parameters of each individual model are deduced from the simultaneous solution, and the adjustment of each model constitute finally the adjustment of the entire strip.

No polynomials are involved in this method, and comparing the results with those obtained by the conventional methods utilizing polynomials, it was found that:

1. In case of cantilever strip, the methods involving auxiliary data, such as the one suggested here, are the only ones capable of providing some means of adjustment.
2. If additional control points are available, enabling the use
of a first degree polynomial only as means of adjustment, the results will be inferior to those obtained by the suggested method.

3. If more control points are available, enabling the use of a second degree polynomial as means of adjustment, the results will be practically the same as those obtained from the new method, provided that:

   a. The strip is short.

   b. The control points are more or less evenly distributed along the strip. The conventional methods are generally more sensitive to the distribution of the control points than the new method.

However, in long strips where ground control points are sparse, and the assumption of error accumulation resembling a certain polynomial is doubtful, the suggested method seems to be the most effective means of adjustment.

In order to further investigate the suggested method, it is recommended that:

1. Horizon-controlled long strips, covering different test areas of known and high precision, be flown and analyzed.

2. A new wide-angle horizon camera should be developed, in line with the conclusions reached in this work.

3. Extensive efforts should be made to develop emulsions capable of penetrating haze and clouds, and registering definite objects in addition to the true horizon.
APPENDIX A

COORDINATE TRANSFORMATION PROGRAM

University of Illinois IBM 7094 Electronic Computer
Modified FORTRAN II COMPILER

600 FORMAT(6I2)
602 FORMAT(214,4F7.3)
604 FORMAT(/,5X,6HPHOT N, I4)
605 FORMAT(5X,2HZ=F10.3,5X,2HY=F10.3)
606 FORMAT(5X,2HZ=F10.3,5X,2HX=F10.3)
607 FORMAT(/,5X,20HSEQUENCE NOT CORRECT)
   DIMENSION X53(30), Y5B(30), XSL(30), YSL(30), XSR(30),
   YSR(30), XSF(30), YSF(30), USR(30), VSR(30), USL(30),
   VSL(30), USR(30), VSR(30), USF(30), VSF(30), Z(30), Y(30), X(30)
10 READ600,NPEP,N1,N2,N3,N4,I$W
   KK=1
   KKK=NPEP+1
IF(N1) 18, 22, 18
18 DO201 = 1 * KKK
20 READ602 * NL1 * NR1 * XSB(I), YSB(I), USB(I), VSB(I)
   DO211 = 1 * KKK
   USB(I) = USB(I) + XSB(I)
21 VSB(I) = VSB(I) + YSB(I)
22 IF(N2) 20, 32, 28
28 DO301 = 1 * KKK
30 READ602 * NL2 * NR2 * XSL(I), YSL(I), USL(I), VSL(I)
   DO311 = 1 * KKK
   USL(I) = USL(I) + XSL(I)
31 VSL(I) = VSL(I) + YSL(I)
32 IF(N3) 38, 42, 38
38 DO401 = 1 * KKK
40 READ602 * NL3 * NR3 * XSR(I), YSR(I), USR(I), VSR(I)
   DO411 = 1 * KKK
   USR(I) = USR(I) + XSR(I)
41 VSR(I) = VSR(I) + YSR(I)
42 IF(N4) 48, 52, 48
48 DO501 = 1 * KKK
50 READ602 * NL4 * NR4 * XSF(I), YSF(I), USF(I), VSF(I)
   DO511 = 1 * KKK
   USF(I) = USF(I) + XSF(I)
51 VSF(I) = VSF(I) + YSF(I)
52 IF(N3) 58, 62, 58
58 DO601 = 1 * KKK
   YSR(I) = - YSR(I)
59 VSR(I) = - VSR(I)
60 IF(N4) 68, 72, 68
68 DO701 = 1 * KKK
   YSF(I) = - YSF(I)
70 VSF(I) = - VSF(I)
72 CONTINUE

IF(N1) 105, 110, 105
105 CALLREDUC(XSB, YSB, Z, Y, KKK)
   WOT6 * 604 * NL1
   WOT6 * 605 * (Z(I) * Y(I) * I = 1 * KKK)
110 IF(N2) 115, 120, 115
115 CALLREDUC(XSL, YSL, Z, X, KKK)
   WOT6 * 604 * NL2
   WOT6 * 606 * (Z(I) * X(I) * I = 1 * KKK)
120 IF(N3) 125, 130, 125
125 CALLREDUC(XSR, YSR, Z, X, KKK)
   WOT6 * 604 * NL3
   WOT6 * 606 * (Z(I) * X(I) * I = 1 * KKK)
130 IF(N4) 135, 140, 135
135 CALLREDUC(XSF, YSF, Z, Y, KKK)
   WOT6 * 604 * NL4
   WOT6 * 605 * (Z(I) * Y(I) * I = 1 * KKK)
140 IF(ISW - 1) 143, 200, 143
143 IF(N1) 145, 150, 145
SUBROUTINE REDUC(XSC,YSC,Z,Y,N)

DIMENSION XSC(120),YSC(120),Z(120),Y(120)
XM=XSC(2)
YM=YSC(2)
DO 10 I=1,N
XSC(I)=XSC(I)-XM
YSC(I)=YSC(I)-YM
10 CONTINUE
X1=XSC(1)-XSC(3)
Y1=YSC(1)-YSC(3)
ALFA=ATN1(X1,Y1)
COSALFA=COS(ALFA)
SINALFA=SIN(ALFA)
DO 20 I=1,N
Z(I)=XSC(I)*COSALFA-YSC(I)*SINALFA
Y(I)=-(XSC(I)*SINALFA+YSC(I)*COSALFA)
20 CONTINUE
RETURN
END
APPENDIX B

TILT COMPONENTS DETERMINATION PROGRAM

666 FORMAT(1H1)
600 FORMAT(I2,F6.0,F7.0,F6.2,3F6.5,F4.2,4I1,F4.2)
601 FORMAT(A6,I4)
610 FORMAT(6X,3HFI=F6.3,5X,3HOM=F6.3,5X)
16HETAO=M,F5.3,5X,6HERUW=F10.8)
612 FORMAT(6X,4HQ11=F20.10,6X,4HQ12=F20.10)
614 FORMAT(6X,4HQ21=F20.10,6X,4HQ22=F20.10)
602 FORMAT(F10.3,F10.3)
620 FORMAT(5X,9HRESIDUALS)
625 FORMAT(6X*3HFI=F20.10)

DIMENSIONZB(20),YB(20),XZ(20),XR(20),
1ZF(20),YF(20),TBB(20),TCB(20),TAL(20),TZL(20),
2TAR(20),TZR(20),TB(20),TC(20),XO(3),TOT(3,3),
3E(20,140),V(20,20),CT(20),PAR(3,3),PR(3,3),D(20)

COMMONE,TBB,TCB,TAL,TZR,TC,T,F,N,NTI,NPEP
1N1,N2,N3,N4,NQ,REFCOE,HF,R,XO,CT,ETATZ,ETAZ
10 READ 600,NPEP,HF,R,P,F1O,OMO,CAO,SIGMA,N1,N2,N3,N4,
1REFCOE
IF(NPEP)11,500,11
11 READ 601,SIGN,NUMB
READ 602,ETATZ,ETAZ
ETATZ=ETATZ/63.662
ETAZ=ETAZ/63.662
DO 14 I=1,20
TBB(I)=0.
TCB(I)=0.
TAL(I)=0.
TZL(I)=0.
TAR(I)=0.
TZR(I)=0.
TB(1)=0.

14 TCF(I)=0.
IF(N1)15,18,15
15 READ 602,(ZB(I),YB(I),I=1,NPEP)
18 IF(N2)25,28,25
25 READ 602,(ZL(I),XR(I),I=1,NPEP)
28 IF(N3)35,38,35
35 READ 602,(ZR(I),XR(I),I=1,NPEP)
38 IF(N4)45,49,45
45 READ 602,(ZF(I),YF(I),I=1,NPEP)
49 IF(N1)50,61,50
50 DO600 I=1,NPEP
TBB(I) = -YB(I)/P
60  TCB(I) = -ZB(I)/P
61  IF(N2)62*66*6?
62  DO65I=1,NPEP
63    TAL(I) = XL(I)/P
65  TZL(I) = ZL(I)/P
66  IF(N3)67*71*67
67  DO70I=1,NPEP
68    TAR(I) = -XR(I)/P
70  TZR(I) = -ZR(I)/P
71  IF(N4)72*81*72
72  DO80I=1,NPEP
73    TBF(I) = YF(I)/P
80  TCF(I) = ZF(I)/P
81  XO(1) = F10
82  XO(2) = OMO
83  XO(3) = CAO
84  CALLMTOR(TOT)
85  CALLTRTBTC(TOT,TBB,TCB,NPEP)
86  CALLTRATZ(TOT,TAL,TZL,NPEP)
87  CALLTRATZ(TOT,TAR,TZR,NPEP)
88  CALLTRTBTC(TOT,TBF,TCF,NPEP)
89  NQ = (N1+N2+N3+N4)*NPEP
90  NTI = 2
91  IRIP = 0
1000  XO(1) = 0.
92  XO(2) = 0.
93  XO(3) = 0.
94  DO90I=1,2
95  DO90J=1,120
96  E(I,J) = 0.
97  CALLPARDER
98  CALLMTCNOR
99  DO92I=1,NTI
100  IJ1 = NTI + NQ + 1
101  E(I,IJ1) = 1.
102  CALLKNOTE
103  DO95I=1,NQ
104  D(I) = CT(I)
105  SV = 0.
106  DO105I=1,NQ
107  V2(I) = CT(I)**2
108  SV = SV + V2(I)
109  SVA = SV
110  IF(IRIP)110*112*110
111  IF(IRIP-50)111*140*140
112  TEMP = SIGMA*SVP
113  IF(SVA-TEMP)112*14J*140
114  SVP = SVA
115  IRIP = IRIP + 1
116  NE1 = NQ + NTI*2
117  CALLSOLSIS(NTI,NE1,E)
NN1 = NT1 + N0 + 1
NN2 = NN1 + 1
Q11 = E(1, NN1)
Q12 = E(1, NN2)
Q21 = E(2, NN1)
Q22 = E(2, NN2)
DO120 I = 1, NT1
DO120 K = 1, N0
KT = K + NT1
120 X0(I) = X0(I) + E(I, KT) * CT(K)
CALL TRAN(PAR)
CALL TRBC(PAR, TGB, TCB, NPEP)
CALL TRAZ(PAR, TAR, TZR, NPEP)
CALL TRIBIC(PAR, TGB, TCB, NPEP)
DO132 I = 1, 3
DO132 J = 1, 3
135 PR(I, J) = 0
DO135 I = 1, 3
DO135 K = 1, 3
DO135 J = 1, 3
135 PR(I, K) = PR(I, K) + TOT(I, J) * PAR(J, K)
DO138 I = 1, 3
DO138 J = 1, 3
138 TOT(I, J) = PR(I, J)
GOTO 100C0
140 CONTINUE
S = NQ - NT1
ERUW = SQRT(SVP/S)
ETA = ERUW * SQRT(Q11)
EATOM = ERUW * SQRT(Q22)
ARG = TOT(3, 1)
X0(1) = ARSIN(ARG)
) 
ARG = -TOT(3, 2) / COS(X0(1))
X0(2) = ARSIN(ARG)
X0(1) = X0(1) * 63.662
X0(2) = X0(2) * 63.662
ETA = ETA * 63.662
EATOM = EATOM * 63.662
WOT6 * 645
WOT6 * 601 * SIGN * NUMB
WOT6 * 610 * (X0(1), X0(2), ETA, ETAOM, ERUW)
WOT6 * 620
WOT6 * 625 * (D(I), I = 1, NQ)
WOT6 * 612 * Q11, Q12
WOT6 * 614 * Q21, Q22
WOT6 * 666
GO TO 10
500 STOP
END
SUBROUTINE KNOTE

DIMENSION (20,140),TBX(20),TCB(20),TAL(20),TZL(20),
1TAR(20),TZR(20),TCF(20),TBF(20),XO(3),CT(80)
COMMON,EBB,TCB,TAL,TZL,TAR,TZR,TCF,TBF,NTI,NPEP,N1,
1N2,N3,N4,NQ,REFC0E,RF R,XO,CT,ETATZ,ETAZ
TANA=2.5*SQRT(((1.-REFC0E)*RF)/(2.*R))
IF(N1)10,12,10
10 DO11=1,NPEP
K=I+NI
W=SQRT(1./((ETATZ**2+(1.+TBB(I)**2)*ETAZ**2)))
11 CT(K)=-(TBB(I)-TANA*SQRT(1.+TBB(I)**2))*W
12 IF(N2)20,22,20
20 DO21=1,NPEP
K=I+NI+NI
W=SQRT(1./((ETATZ**2+(1.+TAL(I)**2)*ETAZ**2)))
21 CT(K)=-(TZL(I)+TANA*SQRT(1.+TAL(I)**2))*W
22 IF(N3)30,32,30
30 DO31=1,NPEP
K=I+(NI+NI)*NPEP
W=SQRT(1./((ETATZ**2+(1.+TAR(I)**2)*ETAZ**2)))
31 CT(K)=-(TZR(I)+TANA*SQRT(1.+TAR(I)**2))*W
32 IF(N4)40,42,40
40 DO41=1,NPEP
K=I+(NI+NI+N3)*NPEP
W=SQRT(1./((ETATZ**2+(1.+TBF(I)**2)*ETAZ**2)))
41 CT(K)=-(TCF(I)+TANA*SQRT(1.+TBF(I)**2))*W
42 RETURN
END

SUBROUTINE PARDER

DIMENSION (20,140),TBX(20),TCB(20),TAL(20),TZL(20),
1TAR(20),TZR(20),TCF(20),TBF(20),XO(3),CT(80)
COMMON,EBB,TCB,TAL,TZL,TAR,TZR,TCF,TBF,NTI,NPEP,N1,
1N2,N3,N4,NQ,REFC0E,RF R,XO,CT,ETATZ,ETAZ
IF(N1)40,42,40
10 E(1,K)=TBB(I)*W
10 E(2,K)=TBB(I)*W
10 E(2,K)=TBB(I)*W
10 E(2,K)=TBB(I)*W

SUBROUTINE VTOR(TP)

DIMENSION (20,140), TBB(20), TCB(20), TAL(20), TZL(20),
TAR(20), TZR(20), TCF(20), TBF(20), XO(3), CT(80)

COMMON *TBB, TCB, TAL, TAL, TAR, TZR, TCF, TBF, NTI, NPEP, N1,
N2, N3, N4, NO, REFCOE, HF, R, XO, CT, ETAZ, ETAZ

SF = SIN(XO(1))
CF = COS(XO(1))
SO = SIN(XO(2))
CO = COS(XO(2))
SK = SIN(XO(3))
CK = COS(XO(3))

TP(1,1) = CK*CF
TP(2,1) = -SK*CF
TP(3,1) = SF
TP(1,2) = SK*CK+CK*SF*SO
TP(2,2) = CK*CO-SK*SO*SF
TP(3,2) = -CF*SO
TP(1,3) = -CK*CO*SF+SK*SO
TP(2,3) = CK*SO+SK*CO*SF
TP(3,3) = CF*CO

RETURN
END

SUBROUTINE MTCNOR

DIMENSION (20,140), TBB(20), TCB(20), TAL(20), TZL(20),
TAR(20), TZR(20), TCF(20), TBF(20), XO(3), CT(80)

COMMON *TBB, TCB, TAL, TAL, TAR, TZR, TCF, TBF, NTI, NPEP, N1,
N2, N3, N4, NO, REFCOE, HF, R, XO, CT, ETAZ, ETAZ

NO = NO + NTI

JK = 0
DO160 I = 1 * NIT1
JK = JK + 1
KN1 = NIT1 + 1
DO130 J = JK * NIT1
DO210 K = KN1 * NO
IF(I - J) 140 * 130 * 130
140 E(J, I) = E(I, J)
130 CONTINUE
160 CONTINUE
RETURN
END

SUBROUTINE SOLSIS(NCOL, NKOL, D)
DIMENSION D(20 * 140)
DO 220 K = 1, NCOL
JK = K + 1
DO 210 I = 1, NCOL
DO 200 J = JK * NKOL
IF(D(K, J)) 170 * 200 * 170
170 IF(I - K) 180 * 200 * 180
180 IF(D(I, K)) 190 * 200 * 190
190 D(I, J) = D(I, J) - D(K, J) * (D(I, K) / D(K, K))
200 CONTINUE
210 CONTINUE
DO 220 J = JK * NKOL
D(K, J) = D(K, J) / D(K, K)
220 CONTINUE
RETURN
END

SUBROUTINE TRATZ(TT, TA, TZ, NPEP)
DIMENSION TA(120), TZ(120), TT(3, 3)
DO10 I = 1, NPEP
AN1 = TA(I) * TT(1, 1) + TT(2, 1) + TZ(I) * TT(3, 1)
AN2 = TA(I) * TT(1, 3) + TT(2, 3) + TZ(I) * TT(3, 3)
DN = TA(I) * TT(1, 2) + TT(2, 2) + TZ(I) * TT(3, 2)
TA(I) = AN1 / DN
TZ(I) = AN2 / DN
10 CONTINUE
RETURN
END
SUBROUTINE TRTBTC(TT, TB, TC, NPEP)

DIMENSION TB(120), TC(120), TT(3,3)
DO10 I = 1, NPEP
AN1 = TT(1,3) + TB(I)*TT(2,3) + TC(I)*TT(3,3)
AN2 = TT(1,2) + TB(I)*TT(2,2) + TC(I)*TT(3,2)
DN = TT(1,1) + TB(I)*TT(2,1) + TC(I)*TT(3,1)
TB(I) = AN2/DN
TC(I) = AN1/DN
10 CONTINUE
RETURN
END
APPENDIX C

STRIP ADJUSTMENT PROGRAM

DIMENSION NAME(60), X(60), Y(60), Z(60), U(60), V(60),
1W(60), MODI(60), MODI1(60), NTYPE(60), FI(15), OM(15),
2CAPPA(15), ALAMDA(15), S1(15), S2(15), S3(15), XO(7, 15),
3TOT(3, 3), TOTMOD(3, 3, 15), X(7), AMAT(3, 3), E(60, 255),
4V2(180), CT(180), CORR(75), PARMOD(3, 3, 15), PAR(3, 3),
5SPR(3, 3, 15), ERN(75), WZ(60), X0T(7, 15),

COMMON NP10, MODI, MODI1, NTYPE, NTI, XO, Y, Z, U, V, Z,

1INQ.CT.WZ

600 FORMAT(6I2)
602 FORMAT(I5, 6F10.3, 2I2, I1, F6.3)
604 FORMAT(I6F6.3, F6.3)
606 FORMAT(F6.3, F9.6, 3F9.2)
608 FORMAT(4F8.5, 3F10.2, I2)
610 FORMAT(5F10.3)
611 FORMAT(I5, 3F10.3)
612 FORMAT(I5, 6F10.2, 2I2, I1)
613 FORMAT(F15.5)
614 FORMAT(5F15.5)

1 READ600*NP1O*NMI
   READ602*(NAME(I), X(I), Y(I), Z(I), U(I), V(I), W(I)),
1MODI(I), MODI1(I), NTYPE(I), WZ(I), I=1*NP10)
   WOT6*602*(NAME(I), X(I), Y(I), Z(I), U(I), V(I), W(I)),
1MODI(I), MODI1(I), NTYPE(I), WZ(I), I=1*NP10)
   READ610*XREF, YREF, ZREF
   DO7I=1*NP1O
   X(I)=X(I)-XREF
   Y(I)=Y(I)-YREF
   Z(I)=Z(I)-ZREF
   U(I)=U(I)-XREF
   V(I)=V(I)-YREF

7 W(I)=W(I)-ZREF
   READ604*(FI(I), OM(I), I=1*NM)
   READ606*(CAPPA(I), ALAMDA(I), S1(I), S2(I), S3(I), I=1*NM)
   DO10N=1*NM
   XO(1*N)=FI(N)/63.662
   XO(2*N)=OM(N)/63.662
   XO(3*N)=CAPPA(N)/63.662
   XO(4*N)=ALAMDA(N)
   XO(5*N)=S1(N)
   XO(6*N)=S2(N)
   XO(7*N)=S3(N)
XOT(1,N) = XO(1,N)
XOT(2,N) = XO(2,N)
XOT(3,N) = XO(3,N)
XOT(4,N) = XO(4,N)
XOT(5,N) = XO(5,N)
XOT(6,N) = XO(6,N)
XOT(7,N) = XO(7,N)
A = XO(1,N)
B = XO(2,N)
C = XO(3,N)
CALL MTOR(TOT,A,B,C)
DO20I = 1,3
DO20J = 1,3
TOTMOD(I,J,N) = TOT(I,J)
20 CONTINUE
10 CONTINUE
DO100I = 1, NPI0
MI = MODI(I)
DO52K = 1,7
52 X1(K) = XO(K,MI)
DO54J1 = 1,3
DO54J2 = 1,3
54 AMAT(J1,J2) = TOTMOD(J1,J2,MI)
XA = X(I)
YA = Y(I)
ZA = Z(I)
CALL TRCOOR(X1,XN,YN,ZN,XA,YA,ZA,AMAT)
X(I) = XN
Y(I) = YN
Z(I) = ZN
NTYP = NTYPE(I)
IF(N TYP - 1) 80, 56, 8C
56 MI1 = MODI1(I)
DO58K = 1,7
58 X1(K) = XO(K,MI1)
DO60J1 = 1,3
DO60J2 = 1,3
60 AMAT(J1,J2) = TOTMOD(J1,J2,MI1)
XA = U(I)
YA = V(I)
ZA = W(I)
CALL TRCOOR(X1,XN,YN,ZN,XA,YA,ZA,AMAT)
U(I) = XN
V(I) = YN
W(I) = ZN
80 CONTINUE
100 CONTINUE
IRIP = 0
1000 NTI = 5*N
NQ = 3*NPI0
DO120I = 1,60
DO120J = 1,255
120 E(I,J)=0.
    CALLPARDER
    NQQ=NQ
    NTII=NTI
    CALLMTCNOR(E,NQQ,NTII)
    DO140 I=1,NTI
    IJ1=NTI+NQ+1
140 E(I+IJ1)=1.
    CALLKNOTE
    SV=0.
    DO145 I=1,NQ
    V2(I)=CT(I)**2
145 SV=SV+V2(I)
    SVA=SV
    IF(IRIP)146,150,146
146 IF(IRIP-50)147,220,220
147 TEMP=0.98*SVP
150 SVP=SVA
    IRIP=IRIP+1
    NE1=NQ+NTI*
    CALLSOLISI(NTI,NE1,E)
    DO152 I=1,NTI
152 CORR(I)=0.
    DO160 I=1,NTI
    KT=K+NTI
160 CORR(I)=CORR(I)+E(I,KT)*CT(K)
    DO165 I=1,NM
    I3=I*5-4
    I4=I3+1
    I5=I3+2
    I6=I3+3
    I7=I3+4
    XO(1,I)=0.
    XO(2,I)=0.
    XO(3,I)=CORR(I3)
    XO(4,I)=1.+CORR(I4)
    XO(5,I)=CORR(I5)
    XO(6,I)=CORR(I6)
    XO(7,I)=CORR(I7)
    XOT(3,I)=XOT(3,I)+CORR(I3)
    XOT(4,I)=XOT(4,I)+CORR(I4)
    XOT(5,I)=XOT(5,I)+CORR(I5)
    XOT(6,I)=XOT(6,I)+CORR(I6)
    XOT(7,I)=XOT(7,I)+CORR(I7)
165 CONTINUE
    DO185 N=1,NM
    A=0.
    B=0.
    C=XO(3,N)
    CALLMTOR(PAP,A,B,C)
DO180I = 1,3
DO180J = 1,3
PARMOD(I*J,N) = PAR(I,J)

180 CONTINUE
185 CONTINUE
DO190I = 1, NP10
MI = MODI(I)
DO192K = 1,7
192 XI(K) = X0(K,MI)
DO194J1 = 1,3
DO194J2 = 1,3
194 AMAT(J1, J2) = PARMOD(J1, J2, MI)
XA = X(I)
YA = Y(I)
ZA = Z(I)
CALL TRCOORD(X1, XN, YN, ZN, XA, YA, ZA, AMAT)
X(I) = XN
Y(I) = YN
Z(I) = ZN
NTYP = NTYPE(I)
IF (NTYP = 1) 190, 196, 190
196 MI1 = MODI1(I)
DO198K = 1, 7
198 XI(K) = X0(K, MI1)
DO200J1 = 1,3
DO200J2 = 1,3
200 AMAT(J1, J2) = PARMOD(J1, J2, MI1)
XA = U(I)
YA = V(I)
ZA = W(I)
CALL TRCOORD(X1, XN, YN, ZN, XA, YA, ZA, AMAT)
U(I) = XN
V(I) = YN
W(I) = ZN
190 CONTINUE
DO219N = 1, NM
DO212I = 1, 3
DO212J = 1, 3
212 PR(I, J, N) = 0.
DO215I = 1, 3
DO215K = 1, 3
DO215J = 1, 3
DO218I = 1, 3
DO218J = 1, 3
218 TOTMOD(I, J, N) = PR(I, J, N)
219 CONTINUE
GOTO 1000
220 CONTINUE
NE1 = NO + NTI*2
CALL SOLSIS(NTI, NE1, E)
DO222I = 1, NP10
\[ X(I) = x(I) + x_{\text{REF}} \]
\[ Y(I) = y(I) + y_{\text{REF}} \]
\[ Z(I) = z(I) + z_{\text{REF}} \]
\[ U(I) = u(I) + u_{\text{REF}} \]
\[ V(I) = v(I) + v_{\text{REF}} \]

W(I) = w(I) + w_{\text{REF}}

WOT6 \times 612 \times (\text{NAME}(I), x(I), y(I), z(I), u(I), v(I), w(I),
MODI(I) + MODII(I) + NTYPE(I), nZ(I), \text{NAME}(I)) = \text{NAME}(I, NPIO)

\text{DO}230 I = 1, N\text{M}
\text{ARG} = - \text{TOTMOD}(3, 1, I)
\text{XO}(1, I) = \text{ARSIN}(-\text{ARG})
\text{XO}(1, I) = \text{XO}(1, I) + \text{XREF}
\text{ARG} = - \text{TOTMOD}(3, 2, I) / \text{COS}(X01)
\text{XO}(2, I) = \text{ARSIN}(-\text{ARG})
\text{XO}(2, I) = \text{XO}(2, I) / \text{COS}(X01)

\text{DO}230 X(3, I) = \text{ARSIN}(-\text{ARG})
\text{DOF} = \text{NQ} - \text{NTI}
\text{EM1W} = \text{SORT}((\text{SVA} / \text{DOF}))
\text{WOT6} \times 613, \text{EM1W}
\text{DO}240 I = 1, \text{NTI}
\text{IJ} = \text{NTI} + \text{NQ} + 1

\text{DO}240 \text{ERI}(I) = \text{SORT}((\text{ERI}(I), I = 1, \text{NTI}))
\text{WOT6} \times 600, \text{TRIP}

\text{READ}612, \text{NME}, \text{XX}, \text{YY}, \text{ZZ}, \text{UU}, \text{VV}, \text{WW}, \text{MDI}, \text{MDI1}, \text{NYPE}
\text{WOT6} \times 612, \text{NME}, \text{XX}, \text{YY}, \text{ZZ}, \text{UU}, \text{VV}, \text{WW}, \text{MDI}, \text{MDI1}, \text{NYPE}
\text{XX} = \text{XX} - \text{XREF}
\text{YY} = \text{YY} + \text{YREF}
\text{ZZ} = \text{ZZ} - \text{ZREF}
\text{UU} = \text{UU} - \text{XREF}
\text{VV} = \text{VV} - \text{YREF}
\text{WW} = \text{WW} - \text{ZREF}
\text{IF}(\text{NME}) \text{IF}(310, 300, 310)
\text{DO}310 \text{K} = 1, 7
\text{DO}320 \text{X}(K) = \text{XOT}(K, \text{MDI})
\text{DO}330 \text{J} = 1, 3
\text{DO}330 \text{J2} = 1, 3

\text{DO}330 \text{AMAT(J1, J2)} = \text{TOTMOD}(\text{J1}, \text{J2}, \text{MDI})
\text{CALLTRCOOR(X1, XN, YN, ZN, XX, YY, ZZ, AMAT)}
\text{XXN} = \text{XX} + \text{XREF}
\text{YNN} = \text{YN} + \text{YREF}
\text{ZZN} = \text{ZN} + \text{ZREF}
\text{WOT6} \times 611, \text{NME}, \text{XX}, \text{YY}, \text{ZZN}
\text{IF}(\text{NYPE} = 1) \text{IF}(305, 340, 305)
\text{DO}340 \text{K} = 1, 7
\text{DO}350 \text{X}(K) = \text{XOT}(K, \text{MDI})
\text{DO}360 \text{J} = 1, 3
\text{DO}360 \text{J2} = 1, 3

\text{DO}360 \text{AMAT(J1, J2)} = \text{TOTMOD}(\text{J1}, \text{J2}, \text{MDI})
\text{CALLTRCOOR(X1, XN, YN, ZN, UU, VV, WW, AMAT)}
\text{XXN} = \text{XX} + \text{XREF}
\text{YNN} = \text{YN} + \text{YREF}
SUBROUTINE PARDER

DIMENSION MODI(60), MODI1(60), NTYPE(60), E(60, 255), W(60),
XO(7, 15), X(60), Y(60), Z(60), U(60), V(60), W(60), CT(180)
COMMON NPIO, MODI, MODI1, NTYPE, NTI, XO, X, Y, Z, U, V, W, ENQ,
CT, WZ

DO1001 = 1, NPIO

NTYP = NTYPE(I)
NODI = MODI(I)
NODI1 = MODI1(I)
KC1 = I * 3 - 2 + NTI
KC2 = KC1 + 1
KC3 = KC1 + 2
IF (NTYP - 1) > 25, 15, 25

15
KR1 = NODI * 5 - 4
KR2 = KR1 + 1
KR3 = KR1 + 2
KR4 = KR1 + 3
KR5 = KR1 + 4
KR6 = KR1 + 5
KR7 = KR1 + 6
KR8 = KR1 + 7
KR9 = KR1 + 8
KR10 = KR1 + 9

E(KR1, KC1) = -XO(4 * NODI) * Y(I)
E(KR2, KC1) = X(I)
E(KR3, KC1) = 1.0
E(KR4, KC1) = 0.
E(KR5, KC1) = 0.
E(KR6, KC1) = +XO(4 * NODI1) * V(I)
E(KR7, KC1) = -U(I)
E(KR8, KC1) = -1.
E(KR9, KC1) = 0.
E(KR10, KC1) = 0.
E(KR1, KC2) = -XO(4 * NODI) * X(I)
E(KR2, KC2) = Y(I)
E(KR3, KC2) = 0.
E(KR4, KC2) = 1.
E(KR5, KC2) = 0.
E(KR6, KC2) = X0(4 * NODI1) * U(I)
E(KR7, KC2) = -V(I)
E(KR8, KC2) = 0.
E(KR9, KC2) = -1.
\( E(KR10, KC3) = 0. \)
\( E(KR1, KC3) = 0. \)
\( E(KR2, KC3) = Z(I) \cdot WZ(I). \)
\( E(KR3, KC3) = 0. \)
\( E(KR4, KC3) = 0. \)
\( E(KR5, KC3) = 1 \cdot WZ(I). \)
\( E(KR6, KC3) = 0. \)
\( E(KR7, KC3) = W(I) \cdot WZ(I). \)
\( E(KR8, KC3) = 0. \)
\( E(KR9, KC3) = 0. \)
\( E(KR10, KC3) = -1 \cdot WZ(I). \)
GOTO 100

25  \( KR1 = NODI \cdot 5 - 4 \)
KR2 = KR1 + 1
KR3 = KR1 + 2
KR4 = KR1 + 3
KR5 = KR1 + 4
\( E(KR1, KC1) = \chi(4 \cdot NODI) \cdot Y(I). \)
\( E(KR2, KC1) = -X(I). \)
\( E(KR3, KC1) = -1.0. \)
\( E(KR4, KC1) = 0. \)
\( E(KR5, KC1) = 0. \)
\( E(KR1, KC2) = -X0(4 \cdot NODI) \cdot X(I). \)
\( E(KR2, KC2) = -Y(I). \)
\( E(KR3, KC2) = 0. \)
\( E(KR4, KC2) = -1. \)
\( E(KR5, KC2) = 0. \)
\( E(KR1, KC3) = 0. \)
\( E(KR2, KC3) = Z(I). \)
\( E(KR3, KC3) = 0. \)
\( E(KR4, KC3) = 0. \)
\( E(KR5, KC3) = -1. \)

100 CONTINUE
RETURN
END

SUBROUTINE KNOTE

DIMENSION MODI(60), MODI1(60), NTYPE(60), E(60, 255), WZ(60),
XO(7, 15), X(60), Y(60), Z(60), U(60), V(60), W(60), CT(180),
COMMON NPIO, MODI, MODI1, NTYPE, NTI, XO, X, Y, Z, U, V, W, E, NQ,
ICT, WZ
DO 100 I = 1, NPIO
NTYP = NTYPE(I)
NODI = MODI(I)
NODI1 = MODI1(I)
KC1 = I * 3 - 2
KC2 = KC1 + 1
KC3 = KC1 + 2
IF (NTYP - 1) 25, 10, 25
10 \( CT(KC1) = 1 \times X(I) - 1 \times U(I) \)
\( CT(KC2) = 1 \times Y(I) - 1 \times V(I) \)
\( CT(KC3) = 1 \times Z(I) - 1 \times W(I) \)
\( CT(KC1) = -CT(KC1) \)
\( CT(KC2) = -CT(KC2) \)
\( CT(KC3) = -CT(KC3) \times WZ(I) \)
GOTO 100

25 \( CT(KC1) = U(I) - X(I) \times 1. \)
\( CT(KC2) = V(I) - Y(I) \times 1. \)
\( CT(KC3) = W(I) - Z(I) \times 1. \)
\( CT(KC1) = -CT(KC1) \)
\( CT(KC2) = -CT(KC2) \)
\( CT(KC3) = -CT(KC3) \)
100 CONTINUE
RETURN
END

SUBROUTINE TRCOORD(X1,XN,YN,ZN,XA,YA,ZA,AMAT)
DIMENSION X1(7), AMAT(3,3)
XN = X1(5) + X1(4) \times (XA \times AMAT(1,1) + YA \times AMAT(2,1) + ZA \times AMAT(3,1))
YN = X1(6) + X1(4) \times (XA \times AMAT(1,2) + YA \times AMAT(2,2) + ZA \times AMAT(3,2))
ZN = X1(7) + X1(4) \times (XA \times AMAT(1,3) + YA \times AMAT(2,3) + ZA \times AMAT(3,3))
RETURN
END

SUBROUTINE SOLSIS(NCOL,NKOL,D)
DIMENSION D(60,255)
DO 220 K = 1, NCOL
JK = K + 1
DO 210 I = 1, NCOL
DO 200 J = JK, NKOL
IF (D(K,J)) \times 170, 200, 170
170 IF (I-K) \times 180, 200, 180
180 IF (I*K) \times 190, 200, 190
190 D(I,J) = D(I,J) - D(K,J) \times (D(I,K) / D(K,K))
200 CONTINUE
210 CONTINUE
DO 220 J = JK, NKOL
D(K,J) = D(K,J) / D(K,K)
220 CONTINUE
RETURN
END
SUBROUTINE MTCNOR(E, NQ, NTI)

DIMENSION E(60,255)
NO=NQ+NTI
JK=0
DO160 I=1, NTI
   JK=JK+1
   KN1=NTI+1
   DO130 J=JK, NTI
      DO110 K=KN1, NO

110 E(I,J)=E(I,J)+E(I,K)*E(J,K)
      IF(I-J)140, 130, 130
140 E(J,I)=E(I,J)
130 CONTINUE
160 CONTINUE
RETURN
END

SUBROUTINE MTOR(TP, FI, OM, CA)

DIMENSION TP(3,3)
SF=SIN(FI)
CF=COS(FI)
SO=SIN(OM)
CO=COS(OM)
SK=SIN(CA)
CK=COS(CA)
TP(1,1)=CK*CF
TP(2,1)=-SK*CF
TP(3,1)=SF
TP(1,2)= SK*CO+CK*SF*SO
TP(2,2)=CK*CO-SK*SO*SF
TP(3,2)=-CF*SO
TP(1,3)=CK*CO*SF+SK*SO
TP(2,3)=CK*SO+SK*CO*SF
TP(3,3)=CF*CO
RETURN
END
REFERENCES


7. I.T.C. Notes, Aerial Triangulation. Delft, Holland


