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AN EXPOSITORY PRESENTATION OF FINITE
GEOMETRIES AS A RESOURCE FOR TEACHERS.

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Mathematics

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AN EXPOSITORY PRESENTATION OF FINITE GEOMETRIES

AS A RESOURCE FOR TEACHERS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

by

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Higher Education. Professor Karl M. Openshaw

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1.1 Background of the Study

In recent years mathematicians have made frequent use of modular number systems in elementary textbooks, and extensive use of finite fields in presentations of modern algebra. The relation of certain finite geometries to Euclidean geometry is somewhat similar to that between modular arithmetic and ordinary arithmetic.

In the year 1964 Professor Leslie H. Miller started to introduce this writer to a certain 25-point finite geometry. After attending his advanced geometry course in the spring quarter of 1965, the importance of finite geometries was more fully realized.

Students were led to discover several new theorems in the class. Finally, some advanced theorems of Euclidean geometry were discovered using ideas from finite geometries. Later, Professor Miller made some applications of finite geometries to solve problems arising in geometric transformations and in projective geometry.

The exploration of finite geometries provides a method of analysis which can be used to solve different problems. Thus, it is useful in research work for advanced geometry and can be used
to suggest new theorems. In a systematic study of finite geometry, methods of inductive reasoning lead to conjectures which can be tested either by investigating all cases or by the use of deductive logic. This process contributes to an understanding of the meaning of a proof.

In addition to the 25-point geometry, Professor Miller has during the past two years introduced other finite geometries containing 9, 13, 31, 49, and 57 points. All of these are useful in solving certain problems which cannot conveniently be investigated in 25-point geometry.

Experience suggests that it is worthwhile for college students and prospective mathematics teachers to study this interesting branch of mathematics. There are many advantages in the study of finite geometries over the ordinary Euclidean geometry. Can it be introduced to high school students or even elementary students?

Although it would be interesting to know about the values of finite geometries for the elementary and secondary mathematics curriculum of the future, a thorough study of this question is beyond the scope of this study. Rather, examples are presented from several Euclidean-type finite geometries in the hope that many teachers will welcome and try to use such examples in their teaching.

1.2 Statement of the Problem

It is assumed that a collection of examples from the study of finite geometries will be useful to teachers and textbook
writers of the future. These examples will be chosen to illustrate such possibilities as the following:

1. Using finite geometries as examples of mathematical systems. Examples are:
   a. 25-point, Euclidean-type geometry
   b. 49-point, Euclidean-type geometry
   c. 9-point, Euclidean-type geometry

2. Using systems that involve axioms, definitions, and theorems.

3. Using systems that may be introduced in several different ways, for example, through listing of points on lines (blocks), by postulates or by the use of coordinates.

4. Showing how to use inductive reasoning based on observations in a finite geometry to make hypotheses about Euclidean plane geometry.

5. Showing how to use a discovery method in finite geometry.

6. Showing the similarity of some elementary portions of the finite geometries to the informal geometry of the present school curriculum.

7. Illustrating that finite geometries can provide a method of analysis which can be used to solve difficult problems and that they are useful in research work for teachers.

8. Showing that the meaning of a proof can be illustrated by investigating all cases or by using deductive logic in finite geometries.
9. Showing that the material of finite geometries can be used at many levels.

10. Showing that finite geometries can be used to illustrate sets, mapping, functions, transformations, coordinate systems, modular numbers, binary operations, and other topics which are now considered to be appropriate for modern mathematics courses.

1.3 Design of the Study

There are many available references to finite projective geometries. This study, however, is primarily concerned with non-projective finite geometries which have properties similar to Euclidean geometry. The available literature concerning this type of geometry is quite limited. The major references are listed below.


This thesis contains no finite geometry but is a reference for Chapter VII.


This text contains no finite geometry but is a suitable reference for the advanced Euclidean geometry used in this study.


The present study contains seven chapters. Most of the material in Chapter II is contained in some of the above references. Material in the remaining chapters has not, in general, appeared in print previously.

Some topics in this study as 49 or 57-Point Geometries, Transformations and Properties of a Postulate System are part of Professor Miller's recent inventions.

In order to organize material related to finite geometries so that it may be of maximum benefit to teachers, research workers and text book writers, this dissertation utilizes the following pattern.

Chapter II discusses a 25-point geometry and shows how it is related to Euclidean geometry at the high school level.

Chapter III extends properties of 25-point and 31-point geometry to include more advanced topics sometimes considered in a college geometry course.
Chapter IV introduces a coordinate system which uses modular arithmetic. The coordinate system for 25 (or 31)-point geometry differs from that discussed by Martha Heidlage [6].

The basic structure of 25 (or 31)-point geometry is illustrated in Chapter V by constructing analogous geometries with 9, 13, 49, and 57 points. The basic idea of the 9-point geometry is found in Dr. Bennett's article [1] "Modular Geometry". Examples show similarities and differences between finite geometries and Euclidean geometry.

The study of geometric transformations is assuming increasing importance in mathematics. Chapter VI considers several geometric transformations and illustrates how study of finite transformations can lead to conjectures about corresponding transformations in Euclidean space.

This study ends with a chapter containing several applications which show how finite geometries can be used to illustrate the inductive reasoning that is characteristic of discovery methods.

1.4 Significance of the Study

It is hoped that the readers of this dissertation, teachers and textbook writers, will find among the illustrations provided

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↑[6]—This symbol indicates the reference numbered 6 on p. 4. It also agrees with the numbers in the bibliography at the end of this study.
content that is interesting and well adapted to achieving their objectives.

In particular:

1. Some of the material presented may be directly useful to the curriculum builder.

2. Some of the results may suggest further research to be undertaken by the reader himself.

3. Some of the methods employed may help the reader to stimulate his students to undertake original research in mathematics.

This writer would consider any one or combination of these effects to be a significant outcome of this study.
2.1 Introduction

With the present emphasis on set properties, it may be appropriate to say that plane geometry is the study of properties of subsets of points in the plane. Usually the subsets considered include an infinite number of points since, for example, the Euclidean line contains an unlimited number of points. There are, however, many finite sets of points which have interesting properties.

As an illustration consider three points designated by the symbols A, B, and C. Define a line as a set containing exactly two of these unordered points. So there are three lines: \{A, B\}, \{A, C\}, and \{B, C\}. Although this system contains only three points it may possess some interesting properties. For example, two points determine one line and two lines determine one point. If a triangle is defined as a set of three non-colliner points, the system contains exactly one triangle. One may define binary operations which are closed on this set of points. For example, let the "product" of two distinct points be the third point and let the product of two identical points be the same point.

A twenty-five point finite geometry which has many properties
in common with Euclidean geometry has been studied by several mathematicians and mathematics educators. This geometry contains 30 lines with 5 points on each line. One convenient starting point for a discussion of this geometry is to list the 30 lines and show which points are on each line. The three blocks in Figure 2.1 contain this information in a convenient form from which additional information is easily obtained. The material in the next few pages is adapted from the article "Geometric Diversions: a 25-point Geometry" by Arthur F. Coxford, Jr. [3] which appeared in the December 1964 issue of the Mathematics Teacher.

2.2 Definitions

Let the first 25 letters of the English alphabet be arranged in three blocks as shown.

Figure 2.1

<table>
<thead>
<tr>
<th>Block I</th>
<th>Block II</th>
<th>Block III</th>
</tr>
</thead>
<tbody>
<tr>
<td>A B C D E</td>
<td>A I L T W</td>
<td>A H O Q X</td>
</tr>
<tr>
<td>F G H I J</td>
<td>S V E H K</td>
<td>N P W E G</td>
</tr>
<tr>
<td>K L M N O</td>
<td>G O R U D</td>
<td>V D F M T</td>
</tr>
<tr>
<td>P Q R S T</td>
<td>Y C F N Q</td>
<td>J L S U C</td>
</tr>
<tr>
<td>U V W X Y</td>
<td>M P X B J</td>
<td>R Y B I K</td>
</tr>
</tbody>
</table>

Make the following definitions:

**Point:** Each of the 25 letters is called a point.
Line: Each row or column in any block is called a line. (There are 30 lines each containing 5 points and each of the 25 points is on 6 lines.)

Parallel lines: Two lines are parallel if they have no point in common. (Lines are parallel only if they are rows or columns in the same block.)

Perpendicular lines: Two lines are perpendicular if one is a row and the other is a column in the same block.

Row distance: If two points are in a common row, the distance between the points is the least number of steps between the points where, when reaching the end of a row counting is continued by jumping to the beginning of the same row. (For example, AB = CD = AE and AC = AD = CE. The notation AB = CD indicates that AB is equal to CD or that AB is congruent to CD.)

Column distance: If two points are in a common column, the distance between the points is the least number of steps between the points. (Thus, AF = PK = AU and AK = PF = AP.)

Distance notation: In a row, distances of one and two steps are designated by 1 and 2. In a column, distances of one and two steps are designated by 1' and 2'. (The four distances are distinct. AI = 1 and AM = 1' but AI ≠ AM.)

Triangle: A set of three points is called a triangle if the points are not collinear.

Circle: A circle is the set of points a given distance from a given point. (Each circle is composed of 6 points.)
The circle with center R and radius 2' contains points C, H, L, X, V, and N.)

**Midpoint:** A point is the midpoint of a given segment if it is on the line through the segment and is equidistant from the endpoints of the segment. (The midpoint of AB is D.)

For this 25-point geometry a reader can verify many results which correspond to those in Euclidean geometry. For example, the Euclidean postulate that two points determine exactly one line is a valid theorem in 25-point geometry, since any two of the 25 points occur simultaneously in one and only one row or column of the three given blocks.

Additional definitions can be formulated so that the vocabulary of 25-point geometry resembles that of Euclidean geometry. For example it is easy to use distances to define equilateral, isosceles, scalene, and congruent triangles. Properties of parallel and perpendicular lines can be used to define parallelograms, rectangles, right triangles and altitudes of triangles. Properties of midpoints can be used to define medians in a triangle and perpendicular bisectors of segments. Later discussions will include suggested definitions for other terms.

2.3 Properties

The theorem that the medians of any triangle are concurrent can be proved by testing for all possible triangles. It is not difficult to show that there are exactly 2000 triangles in the 25-point geometry. Perhaps students would show interest in
one deductive proof of this theorem if they were asked to verify it by testing 2000 cases.

Example 2.3-1

Show that the medians of a specific triangle are concurrent.

Let the vertices of the triangle be L, H, M. Since Y is the midpoint of LH, W is the midpoint of HM, and O is the midpoint of ML, the medians are LW, HO, and MY respectively. Since the lines AILTW, AHOQX, and A3GYM have the point A in common the three medians are concurrent at A.

The diagram in the above example is included to aid in following the steps but it is not part of the verification since, in finite geometry, a line is a set of isolated points.

The following list contains elementary results which are valid for 25-point geometry. A student should illustrate some of these by examples.

1. Two points determine exactly one line.
2. Either two distinct lines are parallel (both in rows or columns of the same block) or they determine exactly one point.

3. Through a given point not on a given line there is exactly one line parallel to the given line.

4. Through a given point there is exactly one line perpendicular to a given line.

5. Each line segment has a unique midpoint.

6. There are equilateral, isosceles, scalene, and right triangles.

7. The number of points of intersection of a line and a circle is always none, one or two. (If a line and a circle have exactly one point in common, the line is said to be tangent to the circle.)

8. The three altitudes of a triangle are concurrent at the orthocenter.

9. The perpendicular bisectors of the sides of a triangle meet at the circumcenter. (This provides a method for finding the circle through three given points.)

10. The three medians of a triangle are concurrent at the centroid.

11. The line through the midpoints of two sides of a triangle is parallel to the third side.

12. The centroid, orthocenter, and circumcenter are collinear.

13. Two tangents to a circle from a common point are equal.

14. If two circles intersect, the line of centers bisects their common chord at right angles.
15. If two circles are tangent the point of tangency is on the line of centers.

16. A line is tangent to a circle if it is perpendicular to the radius drawn from the point of tangency.

17. A line perpendicular to a tangent at its point of tangency passes through the center of the circle.

18. If a diameter of a circle is perpendicular to a chord, it bisects the chord.

19. In the same circle or in equal circles, if chords are equal, they are equidistant from the center(s).

20. If a line is perpendicular to one of two parallel lines it is perpendicular to the other.

21. Two lines parallel to the same line are parallel.

22. If r and s are parallel lines, then their mid-parallel bisects every segment that has one endpoint in r and one endpoint in s.

23. A point on the perpendicular bisector of a segment is equidistant from the endpoints of the segment and a point not on the perpendicular bisector of a segment is not equidistant from the endpoints.

24. If each of two points is equidistant from the ends of a segment, the two points determine the perpendicular bisector of the segment.

Let a quadrilateral be defined as an ordered set of four points no three collinear. These points are called vertices and the four lines joining consecutive vertices are called sides.
The two lines joining non-adjacent vertices are called diagonals. If all sides are equal, the quadrilateral is a rhombus. If both pairs of opposite sides are parallel, the quadrilateral is a parallelogram. If consecutive sides of a parallelogram are perpendicular the parallelogram is a rectangle. If two and only two sides of a quadrilateral are parallel, the figure is called a trapezoid.

25. The diagonals of a rhombus are perpendicular. (BFLH is an example of a rhombus.)

26. The line joining the midpoints of the non-parallel sides of a trapezoid is parallel to the bases.

27. The midpoints of consecutive sides of a quadrilateral are the vertices of a parallelogram.

28. Opposite sides of a parallelogram are congruent.

29. If two opposite sides of a quadrilateral are parallel and congruent the quadrilateral is a parallelogram.

30. If the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram.

31. The diagonals of a rectangle are congruent.

32. If the diagonals of a parallelogram are congruent, the parallelogram is a rectangle.

33. Every rhombus is a parallelogram.

34. A parallelogram is a rhombus if and only if its diagonals are perpendicular.

35. The circumcenter of a rectangle is the point of intersection of the diagonals.
36. The circumcenter of a right triangle is the midpoint of the hypotenuse.

37. If $Z_1Z_2$ is a diameter of a circle and $Z$ is any point distinct from $Z_1$ and $Z_2$ then $Z$ is on the circle if and only if $Z_1Z Z_2$ is a right angle.

Example 2.3-2

Show that midpoints of consecutive sides of a quadrilateral are vertices of a parallelogram.
The same four vertices L, H, S, and R determine three different quadrilaterals LHSR, LSHR, and LHRS where, for example, in quadrilateral LHSR consecutive sides are LH, HS, SR, and RL. The midpoints of these four sides are Y, K, P, and E respectively. It is obvious that Y, K, P, and E are vertices of a parallelogram. Similarly the vertices of parallelogram CKME are the midpoints of consecutive sides of quadrilateral LSHR and the vertices of parallelogram YMPC are the midpoints of consecutive sides of quadrilateral LHRS.

2.4 Justification of Definitions

For a better understanding of 25-point geometry one can examine questions such as: How was the arrangement of points in the three blocks of Figure 2.1 determined and why were definitions chosen in the given way?

The arrangement of points in blocks I, II, and III will be discussed later. At present it is noted that the points in blocks II and III obviously follow some pattern and that it is this pattern which enables a student to verify the theorems which have been listed.

As author has the privilege of making definitions so that they satisfy his purpose. Perhaps, for example, two writers would make completely different definitions for a quadrilateral in 25-point geometry. As a general criterion however, one may call a 25-point geometry definition satisfactory if it assigns properties similar to those in Euclidean geometry.
Perhaps a reader will feel that the definition given for distance seems to be unnecessarily complicated. The following discussion will show that this definition is consistent with the definition of midpoint of a segment and with the arrangement of points in blocks I, II, and III.

Recall the theorem that lines joining the endpoints of a diameter of a circle to any point on the circumference are perpendicular. One can use this theorem to find all points which should be on the circle with diameter KM. Line KM is a row in block I and L is the midpoint of KM so that L should be the center of the circle. In block II the rectangle MJKS has KM as a diagonal so that points J and S should be on the circle. Applying this method to block III shows that points J and I should also be on the circle.

Thus it is expected that points K, M, J, S, T, and I (and no others) should compose the circle with KM as a diameter and with L as center. If all radii of a circle are congruent, distances must be defined so that $LK = LM = LJ = LS = LT = LI$. Note that each of the radii is equal to 1 by the definition of Section 2.2.

The above illustration exhibits one circle with center at L. If a second circle has center L but does not contain any of the points K, M, J, S, T, and I it is expected that the second circle will contain 6 additional points of the 25. Since one can find exactly four distinct circles with center L by this method, it
seems that there should be exactly four distinct distances in the 25-point geometry.

If a valid statement from elementary plane Euclidean geometry is given and it is asked whether or not the statement is valid in 25-point geometry the answer may be yes or no but in some cases there is no clear answer. A satisfactory answer to this question, but one that would be hard to apply, is that a statement which can be proved using certain postulates is valid in any other system which contains all of these postulates.

Some Euclidean results are not valid for 25-point geometry. In Euclidean geometry each line through the center of a circle intersects the circle twice. In 25-point geometry there are six lines through the center of any circle, but three of these lines fail to intersect the circle.

Consider the Euclidean theorem "Angles inscribed in the same arc are congruent." One cannot say that the statement is valid in 25-point geometry because angles and arcs have not been defined. There is, however, a possibility that angles and arcs can be defined so that this statement is valid. In following chapters attempts will be made to define Euclidean terms for finite geometries so that additional known theorems of Euclidean geometry are valid for the finite geometries. Examples of terms to be defined include the measure of the angle between two lines, the interior angles of a triangle, directed segments, convex quadrilateral, and conic.

In general Chapter II has considered analogies, in 25-point
geometry, of material found in most high school geometry courses. Since one objective of this study is to encourage inductive reasoning (or discovery methods) it is suggested that an interested reader can now investigate finite geometry results which correspond to advanced topics found in college courses in Euclidean geometry. Chapter III illustrates this possibility by showing the details of several specific examples.
CHAPTER III

ADVANCED TWENTY-FIVE POINT GEOMETRY

3.1 Introduction

A student seeking ideas for extending the material of Chapter II might consult a college level geometry text. The present chapter contains examples of advanced theorems which are valid in finite geometry. Sometimes it is necessary to make appropriate definitions before these theorems are meaningful and the process of formulating precise definitions should help to develop an appreciation of the structure of mathematics. The major topics of this chapter are summarized in the following paragraphs.

In Euclidean geometry, the theorem of Pythagoras is essential in developing elementary properties of analytic geometry. Although the finite geometry of Chapter II contains right triangles, it is not clear that the theorem of Pythagoras is valid. Since there are no squares in 25-point geometry, the theorem of Pythagoras must be interpreted as a relation concerning lengths of line segments. The next section provides a method for addition and multiplication of the four numbers associated with line segments so that the theorem of Pythagoras is valid. This
will, in a later chapter, provide a basis for the introduction of coordinate geometry.

Directed segments and ratios are also introduced. One new point is added to each line and two parallel lines have this special point in common. The six new points are called ideal points and the six ideal points are said to be on the new ideal line. This extended geometry of 31 points and 31 lines has many properties of a finite projective geometry. As examples of the use of directed segments and ideal elements, the theorems of Ceva and Menelaus are discussed.

The degree measure of the angle between two lines is defined so that the interior angles of a triangle have a sum of 180°. The angle concept leads to a criterion for determining if four points are vertices of a convex quadrilateral.

A non-degenerate conics is defined as a set of six points, no three collinear. The number of ideal points on the conic identifies it as an ellipse, parabola, or hyperbola.

In 31-point geometry the maximum number of points with the property that no three are collinear is six. Thus, if a polygon is defined as a set of n points no three collinear, there are no polygons with more than six sides. This chapter ends with a discussion of the number of triangles, quadrilaterals, pentagons,
hexagons, ellipses, parabolas, and hyperbolas in 25-point (or in 31-point) geometry.

3.2 Distance Properties

The theorem of Pythagoras states that in any right triangle the sum of the squares of the two sides including the right angle is equal to the square of the third side (the hypotenuse). In 25-point geometry a student can identify right triangles and find the lengths of each side but cannot say that the theorem of Pythagoras is valid because multiplication and addition have not been defined for the four lengths designated by \(1, 1', 2, \text{ and } 2'\) in Chapter II. Is it possible to define multiplication and addition for these numbers so that the theorem of Pythagoras will be valid?

In any right triangle if one side of the right angle is in a row in one of the three blocks on page 9 the other side must be in a column of the same block. The following diagram shows four right triangles which exhibit all possible combinations of different lengths for the sides and the hypotenuse.
Can addition and multiplication be defined so that
\[ l'^2 + l^2 = 2'^2, \quad l'^2 + 2^2 = 1^2, \quad 2'^2 + l^2 = 2^2, \quad \text{and} \quad 2'^2 + 2^2 = l'^2? \]

The following discussion will assign real numbers which correspond to the row and column steps used for the lengths of line segments. Let the integer 1 represent one step in a row and the integer 2 represent two steps in a row. How can one assign a meaningful real number to represent one step in a column? Because there are five points in each line and the last element in a line is followed by the first element in the same line when computing steps between points, it seems reasonable to consider using arithmetic modulo 5.

Define these sums and products: \( 1 + 1 = 2, \quad 2 + 2 = 4, \quad \) \( l^2 = 1, \quad 2^2 = 4 \) and re-examine the previous four examples of the theorem of Pythagoras. The second, \( l'^2 + 2^2 = 1^2, \) now becomes \( l'^2 + 4 = 1 \) or \( l'^2 = 1 - 4 = 2 \pmod{5}. \) This suggests representing a unit step in a row by the real number \( \sqrt{2}. \) If \( l' = \sqrt{2}, \) the desired equality \( 2'^2 + l^2 = 2^2 \) becomes \( 2'^2 = 4 - 1 = 3 \) which suggests choosing \( \sqrt{3} \) to represent \( 2'. \) Note, however, that \( \sqrt{3} = \sqrt{5 + 2} = \sqrt{3} = 2 \sqrt{2} \pmod{5}. \) If \( l' \) is represented by \( \sqrt{2}, \) \( 2' \) can be represented by \( 2 \sqrt{2} \) so that \( l' + l' = 2'. \)

Now make the additional definition \( l' = \sqrt{2}, \quad 2' = 2 \sqrt{2} \) and check to determine if \( l'^2 + l^2 = 2'^2 \) and \( 2'^2 + 2^2 = l'^2. \) Since \( l'^2 + l^2 = 2 + 1 = 3 \) and \( 2'^2 = 3 = 3 \pmod{5} \) then \( l'^2 + l^2 = 2'^2; \)
likewise since $2^2 + 2^2 = 8 + 4 = 2 \pmod{5}$ and $1'^2 = 2$, then $2^2 + 2^2 = 1'^2$.

Applying the theorem of Pythagoras to 25-point geometry it is only necessary to define multiplication for numbers of the form $a \times a$, where $a = 1, 2, 1'$ and $2'$. It is easy, however, to extend this multiplication to a larger set which is closed (see Figure 3.1).

Figure 3.1

Multiplication Table

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</table>

The previous remark that $\sqrt{5} = 2 \sqrt{2} \pmod{5}$ can lead to other similar relations. For example $\sqrt{2} = \frac{\sqrt{2}}{1} = \frac{\sqrt{2} \cdot \sqrt{5}}{\sqrt{5}} = \frac{1}{\sqrt{5}} \pmod{5}$.

Thus it seems that $1'$ can be equally well represented by $\sqrt{2}$ or by $\frac{1}{\sqrt{5}}$. The representation of $1'$ by $\frac{1}{\sqrt{5}}$ and of $2'$ by $\frac{2}{\sqrt{5}}$ is
used in Figure 3.2 to indicate a geometric method for interpreting some of the properties of 25-point geometry. In Figure 3.2 the 25 points of block I (Figure 2.1), are placed in a repeated rectangular array so that if the Euclidean distance between two consecutive columns is one unit, the distance between two consecutive rows is $\frac{1}{\sqrt{2}}$.

Figure 3.2
The arrangement in Figure 3.2 exhibits many properties of the 25-point geometry. For example, if any two points are joined by a straight line the line passes through five and only five points and those points are the five points on the line as determined in blocks I, II, and III of Figure 2.1. Also it is possible to use Figure 3.2 to find the midpoint of a given line segment. For example consider the segment IO. By choosing points in different positions one can locate I and O so that exactly one point, V, is between them. However for other locations of I and O one finds these ordered arrangements: IO, ICVPO, and IPCO. This raises a question as to whether or not it is possible to tell which are the interior points of a given line segment.

Figure 2.1 shows that there are six lines through each point. From Figure 3.2 it is possible to tell the distance between any two points without using blocks I, II, or III. For example, consider the distance between N and V. Since N and V are not in a common row or column, select a right triangle such as NXV with NV as hypotenuse. Then NX = 2' and XV = 2 so \((NV)^2 = 2'^2 + 2^2 = 3 + 4 = 2\). Since NV = \(\sqrt{2}\) the length of segment NV is 1'.

Martha Heidlage, on page 109 of her article in the *Mathematics Teacher* of February, 1965 [6], states that the elements in block II (Figure 2.1) are obtained from the corresponding elements in block I by a rotation of 60° in a lattice.
about point A and elements in block III one obtained by a similar rotation of the elements in block II. This rotation is clearly evident in Figure 3.2. Each of the 24 points other than A occurs on one of the four given circles and a rotation of 60° about A carries it into the next point on the circle as indicated by the arrows. For example Q is the element in the 4th row and 2nd column of block I. Figure 3.2 shows that Q becomes C in a 60° rotation about A so C is the element in the 4th row and 2nd column of block II. Likewise C is rotated into L so L is the element in the 4th row and 2nd column of block III.

For the four circles shown in Figure 3.2 the radii, using Euclidean distances, are \( \frac{2}{\sqrt{3}} \), 2, \( \frac{4}{\sqrt{3}} \), and 4. Each circle contains 6 points and these points are those which are on circles with center A and with radii \( 2', 2, 1', \) and 1 in the 25-point geometry.

3.3 Directed Segments

A central idea of the present chapter is the introduction of geometry more advanced than is usually found in a high school text. In a college geometry course, which contains generalizations of elementary Euclidean geometry, it is usual to introduce directed line segments and to define an ideal point as a point of intersection of two parallel Euclidean lines. These extended concepts are used in the theorems of Ceva and Menelaus which are generalizations of familiar high school theorems.

Directed distances are now introduced in the rows and columns of blocks I, II and III of Figure 2.1. The positive
direction is to the right in a row and upward in a column. Using \( \overrightarrow{AB} \) to represent the directed distance from A to B we have these examples: \( \overrightarrow{AB} = 1, \overrightarrow{BA} = -1 = 4 \text{ (mod 5)}, \overrightarrow{AC} = 2, \overrightarrow{AD} = 3, \overrightarrow{AE} = 4, \overrightarrow{EA} = -4 = 1, \overrightarrow{AF} = -1' = 4', \overrightarrow{AP} = -3' = 2', \overrightarrow{OS} = -3' = 2', \text{ etc.} \)

If \( Z_1, Z_2, Z_3 \) are collinear, \( \overrightarrow{Z_1Z_2} + \overrightarrow{Z_2Z_3} = \overrightarrow{Z_1Z_3} \). This result is illustrated by an example.

\[
\overrightarrow{AK} + \overrightarrow{KF} = -2' + 1' = -1' = \overrightarrow{AF}
\]

To each line of the twenty-five point geometry (Figure 2.1) one additional point is added as shown in Figure 3.3. These new points, designated by the symbols \( I_1, I_2, I_3, I_4, I_5, I_6 \) are called ideal points. Now parallel lines intersect in a common ideal point and the six ideal points are said to be on the ideal line. This addition provides a geometry with 31 points and 31 lines with 6 points on each line and 6 lines through each point. Distance is not defined between two points if one point is an ideal point.

**Figure 3.3**

<table>
<thead>
<tr>
<th>Block I</th>
<th>Block II</th>
<th>Block III</th>
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<tbody>
<tr>
<td>I₁  I₁  I₁  I₁  I₁</td>
<td>I₃  I₃  I₃  I₃  I₃</td>
<td>I₅  I₅  I₅  I₅  I₅</td>
</tr>
<tr>
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<tr>
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<td>Block I</td>
<td>Block II</td>
<td>Block III</td>
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</table>
As in Euclidean geometry, if three points $Z_1$, $Z_2$, and $Z$ are collinear, $Z$ is said to divide the segment $Z_1Z_2$ in the ratio $r$ if $\frac{Z_1Z}{Z_2Z} = r$ where $Z_1Z$ and $Z_2Z$ are directed segments. If $Z$ is the ideal point on $Z_1Z_2$, $\frac{Z_1Z}{Z_2Z}$ is assigned the value of $-1 = 4$. The ratio $\frac{Z_1Z}{Z_2Z}$ is not defined if either $Z_1$ or $Z_2$ are ideal points.

For this 31-point geometry, any given line segment is divided in a specific ratio by any of the remaining four points on the line through the segment. Using modulo 5 arithmetic and directed segments, it is easy to verify that these four points divide the segment in the ratios 1, 2, 3, and 4. For example consider the segment $\overline{AR}$ in block III, Figure 3.3

\[
\frac{AJ}{JR} = \frac{-3'}{-1'} = 3;
\]

\[
\frac{AV}{VR} = \frac{-2'}{-2'} = 1;
\]

\[
\frac{AN}{NR} = \frac{-1'}{-3'} = 2;
\]

and

\[
\frac{A_5I_5}{I_5R} = -1 = 4, \text{ by definition. (See Figure 3.4)}
\]

It should be noted that if $\frac{Z_1Z}{Z_2Z} = r$, $\frac{Z_2Z}{Z_1Z} = \frac{1}{r}$. It is also possible to evaluate the ratio $\frac{Z_1Z_2}{Z_2Z_3}$ for three collinear
### Figure 3.4

Multiplication Table for Directed Segments (mod 5)

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points $Z_1$, $Z_2$, and $Z_3$ without use of directed segments by the following method.

If $Z_2$ is the ideal point on $Z_1Z_3$, $Z_1Z_2/Z_2Z_3 = 4$; if $Z_2$ is the midpoint of $Z_1Z_3$, then $Z_1Z_2/Z_2Z_3 = 1$; if $Z_1$ is the midpoint of $Z_2Z_3$, $Z_1Z_2/Z_2Z_3 = 2$; if $Z_3$ is the midpoint of $Z_1Z_2$, $Z_1Z_2/Z_2Z_3 = 3$.

With these definitions of ratios, given two ordinary points and a ratio $r$, $r \in \{1, 2, 3, 4\}$, there is exactly one point dividing the given segment in this ratio.

3.4 Theorem of Menelaus

In Euclidean geometry, the Theorem of Menelaus can be stated as follows. If $P_1$, $P_2$, and $P_3$ are the vertices of a triangle and $P_1'$, $P_2'$, and $P_3'$ are three other points such that point $P_1'$ is on $P_2P_3$, point $P_2'$ is on $P_3P_1$, and point $P_3'$ is on $P_1P_2$, then $P_1'$, $P_2'$, and $P_3'$ are collinear if and only if

$$\frac{P_1P_3}{P_1P_2} \cdot \frac{P_2P_1}{P_2P_3} \cdot \frac{P_3P_2}{P_3P_1} = -1$$
Example 3.4-1

Illustration in 31-point geometry: (a) Verification of the "only if" case of the Theorem of Menelaus for given triangle OSU, having collinear points F, L, and R on OS, SU, and OU. It is desired to verify the relation.

\[ \frac{OF}{FS} \cdot \frac{SL}{LU} \cdot \frac{UR}{RO} = -1 = 4 \pmod{5}. \]
By the definition of directed segments, or by use of Figure 3.4,

\[
\frac{OF}{FS} = \frac{-2'}{-1'} = \frac{2'}{1'} = 2,
\]

\[
\frac{SL}{LU} = \frac{-1}{2} = \frac{4}{2} = 2,
\]

and

\[
\frac{UR}{RO} = \frac{-1}{-1} = 1.
\]

Therefore

\[
\frac{OF}{FS} \cdot \frac{SL}{LU} \cdot \frac{UR}{RO} = 2 \cdot 2 \cdot 1 = 4
\]

(b) A verification for the "if" case is made using the definition of ratios without use of Figure 3.4. The given data consists of \(\triangle YML\) with \(A\) on \(YH\), \(O\) on \(ML\), and \(H\) on \(LY\) so that

\[
\frac{YA}{AM} \cdot \frac{MO}{OL} \cdot \frac{LH}{HY} = 4
\]
(The fact that this product is 4 is easily verified. Since M is the midpoint of YA, \( \frac{YA}{AM} = 3 \). Likewise, O is the midpoint of ML so \( \frac{MO}{OL} = 1 \), and Y is the midpoint of LH so \( \frac{LH}{HY} = 3 \).

The conclusion that A, O, and H are collinear is valid since points A, O, and H are found in the same row in block III, Figure 3.3.

3.5 Theorem of Ceva

In Euclidean geometry, if \( P_1, P_2, \) and \( P_3 \) are the vertices of a triangle and \( P_1', P_2', \) and \( P_3' \) are three other points such that point \( P_1' \) is on \( P_2P_3 \), point \( P_2' \) is on \( P_3P_1 \), and point \( P_3' \) is on \( P_1P_2 \), then the three lines \( P_1P_1', P_2P_2', \) and \( P_3P_3' \) are concurrent if and only if

\[
\frac{P_1P_3'}{P_1P_2} \cdot \frac{P_2P_1'}{P_2P_3} \cdot \frac{P_3P_2'}{P_3P_1} = 1
\]
Example 3.5-1

Illustration of Ceva's Theorem in 31-point geometry:

(a) "Only if" case: Given triangle USA with $I_1$ on AU, L on US, and M is on SA so that AL, SI_1, and UM meet at I, it is to be shown that

$$\frac{AI_1}{I_1U} \cdot \frac{UL}{LS} \cdot \frac{SM}{MA} = 1$$

Since
\[
\frac{\overline{AT_1}}{\overline{T_1U}} = 4, \quad \frac{\overline{UL}}{\overline{LS}} = 3, \quad \text{and} \quad \frac{\overline{SM}}{\overline{MA}} = 3
\]

then

\[
\frac{\overline{AT_1}}{\overline{T_1U}} \cdot \frac{\overline{UL}}{\overline{LS}} \cdot \frac{\overline{SM}}{\overline{MA}} = 4 \cdot 3 \cdot 3 = 36 = 1
\]

(b) Proof: "If case": Given triangle HLK with Y on HL, M on LK, and E on KH, so that

\[
\frac{\overline{HY}}{\overline{YL}} \cdot \frac{\overline{LM}}{\overline{MK}} \cdot \frac{\overline{KE}}{\overline{EH}} = 1 \cdot 2 \cdot 3 = 6 = 1
\]

it is to be verified that HM, LE, and KY are concurrent.

Now H and M are on line CHMRW, L and E are on line LERFX, and K and Y are on line RYBIK. Since point R is on each of these three lines the lines are concurrent.

Example 3.5-2

If the circle with center M and radius 1' is inscribed in triangle CJF with tangent points U on CJ, H on JF, and Y on FC, are CH, FU, and JY concurrent?
Solution: Since

\[ \frac{CU}{UJ} = 2, \quad \frac{JH}{HF} = 1 \]

and

\[ \frac{FY}{YC} = 3 \]

it follows that

\[ \frac{CU}{UJ} \cdot \frac{JH}{HF} \cdot \frac{FY}{YC} = 2 \cdot 1 \cdot 3 = 6 = 1 \]

Thus CH, FU, and JY are concurrent by Ceva's theorem. That the point of concurrency is \( I_1 \), is easily established. A physical interpretation of this example and the fact that JY, HC, and FU meet in an ideal point is suggested in Figure 3.5 where the circle with center M and radius 1' is shown in two positions, to illustrate
the fact that it is tangent to the sides of triangle CJF at points U, H, and Y.

3.6 Concurrent Perpendiculars

In Euclidean geometry let L, M, and N be points on sides BC, CA, and AB respectively of triangle ABC. The three lines perpendicular to BC, CA, and AB at points L, M, and N are concurrent if and only if

\[(BL)^2 + (CM)^2 + (AN)^2 = (LC)^2 + (MA)^2 + (NB)^2\]

(For proof see [9], page 67.)

Example 3.6-1

Verification of the "only if" case of the theorem of concurrent perpendiculars for triangle ALC in 31-point geometry.

Lines NJ, ND, and NT are concurrent at N with NJ \perp LC, ND \perp CA, and NT \perp AL. It is necessary to verify the relation

\[(LJ)^2 + (CD)^2 + (AT)^2 = (JC)^2 + (DA)^2 + (TL)^2 \quad (1)\]
Thus the two members of equation (1) are equal.

A verification of "if" case when the given data consists of triangle ALC with W on AL, U on LC, and B on CA, so that

\[(AW)^2 + (LU)^2 + (CB)^2 = (WL)^2 + (UC)^2 + (BA)^2 \quad (2)\]

(Since \(AW = 4, LU = 2, CB = -1, WL = -2, UC = 1\) and \(BA = -1\) both members of (2) have the value of 1.)
The perpendicular to AL at W is WKDQJ in block II, (Fig. 3.3). The perpendicular to LC at U is QEMUI in block III. The perpendicular to CA at B is BGIQV in block I. The conclusion that the three perpendiculars at W, U, and B are concurrent is valid, since the lines meet in the common point Q.

Example 3.6-2

By combining the two portions of the previous example one can illustrate another result showing how two sets of points on sides of a triangle satisfying the theorem of concurrent perpendiculars can be used to find a third set of points satisfying this theorem.

Lines NJ and QB meet at V, ND, and QW meet at D, and NT and QU meet at U. The perpendicular from V to AL is IVOCP which cuts AL at I, the perpendicular from D to LC is HPDLY which cuts LC at L, the perpendicular from U to CA is AFKPU which cuts CA at A. Since these three lines are concurrent at point P,
3.7 Angles

If triangles are defined as similar when corresponding sides have equal ratios, it is possible to have analogies of Euclidean theorems about similar triangles in 25-point geometry. Although angles have not been defined it seems natural to require that corresponding angles in similar triangles be equal.

From a study of numerous triangles and circles it seems reasonable to define a finite number of angles and to assign numerical values to these angles. For 25-point geometry angles of $30^\circ$, $60^\circ$, $90^\circ$, $120^\circ$, and $150^\circ$ are associated with lines and triangles in the following definitions.

(1) The angle between two lines is $90^\circ$ if one line is a row and the other is a column in the same block.

(2) The angle between two lines is $60^\circ$ or $120^\circ$, if both lines are rows in different blocks or if both lines are columns in different blocks.
(3) The angle between two lines is \(30^\circ\) or \(150^\circ\), if the lines are rows and columns in different blocks.

(4) Let three lines intersect to form a triangle. The angles of the triangle are assigned numerical values so that the sum of the interior angles is \(180^\circ\).

Example 3.7-1

In \(\triangle P N Q\) the interior angle at vertex \(P\) is \(60^\circ\) since the angle between lines \(N P\) and \(Q P\) is \(60^\circ\) or \(120^\circ\) and if \(120^\circ\) is chosen as the interior angle the interior angles at the vertex \(N\) and \(Q\) would have to be \(30^\circ\) but these angles must also be either \(60^\circ\) or \(120^\circ\). The only choice in which the sum of the interior angles can be \(180^\circ\) is for each angle to be \(60^\circ\). Note that this choice of \(60^\circ\) is consistent with calling \(\triangle P N Q\) equilateral since \(PQ = QN = NP = 1\).

![Diagram of \(\triangle P N Q\) with \(60^\circ\) angles at each vertex.]

Example 3.7-2

In \(\triangle A V T\) the angle between lines \(A V\) and \(V T\) is \(90^\circ\), since \(AV\) is a column in block III and \(VT\) is a row in the same block.
The angle between AV and AT is either 30° or 150° since AV is a column in block III and AT is a row in block II. The choice of 150° for the interior angle at vertex A would produce a triangle containing angles with a sum greater than 180° so the interior angles at A and T are assigned values of 30° and 60°.

Example 3.7-3

For any triangle each interior angle, not a right angle, has two possible sizes but only one choice for each angle leads to the sum of the angles being 180°.

In triangle TRJ the angles at T and J must be either 30°
or 150°. However no 25-point geometry triangle can contain an interior angle of 150° since this would require angles smaller than 30° at the remaining vertices. Hence for triangle TRJ the only choice of interior angle is \( \angle TRJ = 120°, \angle RJT = 30°, \) and \( \angle JTR = 30° \).

The three examples given above include all possible combinations of interior angles in triangles so that for any triangle the measures of each interior angle is easily and uniquely determined. The number of degrees in an exterior angle of a triangle is formed by subtracting the number of degrees in the corresponding interior angle from 180°.

Triangles with corresponding angles equal are called similar. Similar triangles also have corresponding sides proportional. Here it is convenient to consider only the set of numbers 1, 2, 1', 2' both as lengths of sides in triangle, and for ratios of corresponding sides. In Figure 3.1 numbers 3, 4, 3', and 4', when interpreted as lengths of segments, are replaced by 2, 1, 2', 1' to obtain Figure 3.6 shown below.

Figure 3.6

<table>
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<tr>
<th>+</th>
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<th>1'</th>
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<td>2'</td>
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<td>1'</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>
For another interpretation of obtaining Figure 3.6 from Figure 3.1 note that it is possible to define \(|a| = -a\) when \(a\) is negative. Then \(|3| = |-2| = 2\).

Example 3.7

Triangles AKL and JRB are similar, because corresponding angles are equal. Using Figure 3.6,

\[
\frac{AK}{JR} = \frac{2'}{1'} = 2, \quad \frac{KL}{RB} = \frac{1}{2} = 2, \quad \text{and}
\]

\[
\frac{LA}{BJ} = \frac{2}{1} = 2
\]

Given three non-collinear points \(P_1, P_2, P_3\) the measure of angle \(P_1P_2P_3\) is by definition, the measure of the interior angle at \(P_2\) in triangle \(P_1P_2P_3\).
If $P_1$, $P_2$, $P_3$, and $P_4$ are distinct points on a circle, then angles $P_1P_2P_1$ and $P_1P_3P_4$ are either equal or supplementary, if supplementary $P_2$ and $P_3$ are said to be on opposite sides of chord $P_1P_4$.

Example 3.7-5

Points $F$, $M$, $U$, $N$ are on the circle with center $A$ and radius 1' (see Figure 3.2). From triangle $FMN$ and the above definition it follows that $\angle FMN = 60^\circ$. Likewise $\angle FUN = 60^\circ$. 
Since $\angle FMN$ and $\angle FUN$ are equal, $M$ and $U$ are said to be on the same side of chord $FN$.

If $P_1$, $P_2$, $P_3$, and $P_4$ are distinct points for which angles $\angle P_1P_2P_4$ and $\angle P_1P_3P_4$ are either equal or supplementary, points $P_1$, $P_2$, $P_3$, and $P_4$ are on a circle.

Example 3.7-6

Since $\angle BKV = 120^\circ$, $\angle BOV = 60^\circ$, the points $B$, $K$, $O$, and $V$ are on a circle. The circle has center $M$ and radius 2. Since $K$ and $O$ are on opposite sides of $BV$, quadrilateral $BKVO$ is called convex.

Example 3.7-7

Note that $\angle PRX = 30^\circ$, $\angle PYX = 30^\circ$. Points $P$, $R$, $X$, and $Y$ are on the circle with center $Q$ and radius 1 and $R$ and $Y$ are on
the same side of chord PX. This information does not show whether or not PRYX is a convex quadrilateral.

3.8 Convex Quadrilaterals

Essentially one recognizes whether a given polygon in Euclidean geometry is convex or concave by looking at a diagram. For 25-point geometry no suitable diagram is available so a definition of a convex polygon clearly must be independent of "intuitive" concepts.

By definition a quadrilateral, in 25-point geometry, is an ordered set of four points no three collinear. The quadrilateral $P_1P_2P_3P_4$ contains four sides, the lines $P_1P_2$, $P_2P_3$, $P_3P_4$, and $P_4P_1$. Since for the same points $P_1$, $P_2$, $P_3$, and $P_4$ quadrilaterals $P_1P_2P_3P_4$ and $P_2P_3P_4P_1$ contain the same sides these quadrilaterals are called equivalent. Quadrilateral $P_1P_2P_3P_4$ and $P_1P_3P_2P_4$, which have the same vertices, do not contain the same sides so they are not equivalent.

The four letters $P_1$, $P_2$, $P_3$, and $P_4$ can be arranged in order in 24 ways but for any given order there are seven other orders corresponding to equivalent quadrilaterals. If $P_1P_2P_3P_4$ is a quadrilateral the same vertices, in other orders, determine two other quadrilaterals so no two of the three quadrilaterals are equivalent. For example $P_1P_2P_3P_4$, $P_1P_3P_2P_4$, and $P_1P_2P_4P_3$ are three quadrilaterals with different sets of sides if no three of points $P_1$, $P_2$, $P_3$, and $P_4$ are collinear.
In Figure 3.7 the Euclidean quadrilaterals illustrated in (a), (b), and (c) are called convex, concave, and intersecting. Rearranging vertices of convex quadrilateral ABCD into orders ACBD and ABDC gives two different quadrilaterals both intersecting. Rearranging the vertices of concave quadrilateral EFGH in this way gives EGFH and EFHG both of which are concave. Rearrangement of points IJKL of the intersecting quadrilateral leads to the convex quadrilateral IJKL.

For 25-point geometry the following definitions are designed to differentiate among cases like (a), (b), and (c) above so
that a given quadrilateral can be classified as convex, concave or intersecting.

Consider the two triangles $P_1P_2P_3$ and $P_1P_3P_4$ and the corresponding interior angles. By definition the interior angles at $P_2$ and $P_4$ in quadrilateral $P_1P_2P_3P_4$ are the angles at $P_2$ and $P_4$ in triangles $P_1P_2P_3$ and $P_1P_3P_4$. The interior angles at $P_1$ and $P_3$ in the quadrilateral are the sums of the two angles at these vertices in the two triangles. By this definition any 25-point quadrilateral has interior angles at each vertex and the sum of the four interior angles is $360^\circ$.

The quadrilateral $P_1P_2P_3P_4$ is called convex if and only if at each vertex, the interior angles of $P_1P_2P_3P_4$ are equal to the interior angles of $P_2P_3P_4P_1$. Quadrilateral $P_1P_2P_3P_4$ is concave if only one interior angle of $P_1P_2P_3P_4$ is equal to the corresponding angle in $P_2P_3P_4P_1$ and intersecting if no corresponding angles are equal.

Example 3.8-1

![Diagram of PRYX and RYXP quadrilaterals with interior angles labeled.]
In both PRYX and RYXP $\angle P = 60^\circ$, $\angle R = 60^\circ$, $\angle Y = 120^\circ$, $\angle X = 120^\circ$ so PRYX is convex. Since PRYX is convex it is to be expected that PYRX and PYXR are intersecting quadrilaterals. This will be verified for PYXR.

In PRYX
- $P = 90^\circ$, $X = 210^\circ$
- $R = 30^\circ$, $Y = 30^\circ$

In YXRP
- $P = 30^\circ$, $X = 30^\circ$
- $R = 90^\circ$, $Y = 210^\circ$
Since none of angles in PYXR are equal to the corresponding angles in YXRP, PYXR is an intersecting quadrilateral.

Example 3.3-2

Quadrilateral AKIJ is concave.

\[
\begin{align*}
\text{In AKIJ} & \quad \text{In KIJA} \\
\angle A &= 60^\circ, \angle I = 240^\circ, \angle J = 30^\circ & \angle A &= 60^\circ, \angle K = 90^\circ \\
\angle K &= 30^\circ & \angle J &= 90^\circ, \angle I = 120^\circ
\end{align*}
\]

Ptolemy's Theorem: The vertices of a convex quadrilateral are cyclic if and only if the sum of the products of the two pairs of opposite sides is equal to the product of the diagonals.

Illustration in 25-point geometry: (A) "Only if" case:
The given quadrilateral FRMN is cyclic since the four points are on the circle with A as a center and radius 2'. Quadrilateral FRMN is convex since angles FNM and FRM are
supplementary. Then FM and RN are the two diagonals.

\[
(FR)(MN) + (RM)(NF) = (FM)(RN)
\]

\[
(FR)(MN) + (RM)(NF) = l' \cdot 1 + l' \cdot 1 = l' + l' = 2'
\]

\[
(FM)(RN) = 1 \cdot 2' = 2'
\]

therefore

\[
(FR)(MN) + (RM)(NF) = (FM)(RN)
\]

(B) "If" case: Since PXYR is a convex quadrilateral such that

\[
(PX)(YR) + (XY)(RP) = (PY)(XR)
\]

then points P, X, Y, and R are cyclic.
It is easy to verify that points $P$, $X$, $Y$, $R$ are cyclic since they are on the circle with center $Q$ and radius 1.

3.9 Conics

In Euclidean geometry five points, no three collinear, determine one conic and no line cuts a conic in more than two points. For a tentative definition in 31-point geometry consider a conic as a set of five or more points no three on a line. The next example shows that such sets of points can exist.

Example 3.9-1

If one choose $P$ and $Q$ as two points on a conic, then the four points $R$, $S$, $T$, and $I_2$ on the same line in block I cannot be on the conic. Choose $K$ as the third point on the conic. Using line $KP$ one eliminates the points $A$, $F$, $I_1$, and $U$ in block I. From line $KQ$ (block II), eliminate the points
W, D, J, and I₃. Choose V as the fourth point on the conic.
Using line PV (block II), eliminate the points C, I, O, and I₃.
Using the line VQ (block I), eliminate L, G, and B. Using VK (block II), eliminate E, H, and I₄. Choose N as the fifth point.
Using NK (block I), eliminate M, using NQ (block II), eliminate Y. Using line NV (block III) eliminate I₅, using line NP, eliminate I₆. The conic can contain X, the only remaining point. So PQRSTVNX is a conic by the definition.

The above process may be repeated for other choices of points but in each case it is found that exactly 6 points, of the 31, are on any conic and that the number of ideal points on a conic can be 0, 1, or 2. This suggests a revised definition of a conic.

In 31-point geometry, a conic in a set of six points no three on a line. The conic is called an ellipse, parabola or hyperbola if the number of ideal points on the conic is none, one or two respectively.
The conic PQKVN is discussed above is an ellipse. It is possible to find points on an hyperbola by constructing a conic starting with 2 ideal points. The next example shows that \( I_1 I_4 \) EXQW is an hyperbola.

Example 3.9-2

If one choose \( I_1 \) and \( I_4 \) as two points on a conic, and choose E as the third point on the conic, then points J, O, T, Y in block I can not be on the conic. Using line \( EI_4 \) one eliminates the points S, U, H, K in block II. Choose X as the fourth point on the conic. Using line \( XI_1 \) (block I) eliminate the points D, I, N, S. Using line \( XI_4 \) (block II), eliminate points M, P, and B. Using line \(XE \) (block II) eliminate \( I_3 \), L, R, and F. Choose Q as the fifth point. Using \( QI_1 \) (block I), eliminate G, using \( QE \) in (block III), eliminate \( I_5 \) and U. Using line \( QI_4 \) in block II, eliminate C, using line \( QX \) in block III.
eliminate A and I₆. The conic can contain W, the only remaining point. Thus I₁I₄EXQW is an hyperbola by the definition.

It is also possible to construct a parabola by starting with one ideal point and choosing other points with some care. An example of a parabola is KLRDTI₁.

A well known theorem of projective geometry (Pascal's theorem) states that if a hexagon is inscribed in a conic the three pairs of opposite sides intersect in collinear points. An illustration of this theorem is contained in the following example.

Example 3.9-3

The hexagon AG₁BI₆F, inscribed in conic ABFGI₄I₆, satisfies Pascal's theorem. Line AG in block II and line BI₆ in block III contain the common point Y, so opposite sides AG and BI₆ meet at Y. Likewise opposite sides GI₄ and I₆F meet at D and opposite sides I₄B and FA meet at P. Points Y, D, and P are on line I₅HYLDP in block III.

3.10 Polygons

One may wish to determine the number of triangles, quadrilaterals, pentagons, and hexagons in 25-point geometry. There are many methods for explaining the existence of 2000 triangles in 25-point geometry. The simplest way is as follows.
(1) Select the first point in 25 ways.

(2) Select the second point in 24 ways.

(3) Select the third point in 20 ways (5 of the 25 points are ineligible, since they lie on the same line as the first two selected points.)

(4) The total number of triangles is \( \frac{25 \times 24 \times 20}{3!} = 2000 \).

In step 4, division by \( 3! = 6 \) removes the repetition caused by counting each triangle in the six ways in which the vertices can be ordered.

One can find the same number, using 31-point geometry, as in the following method.

(1) Using the above method one can find there are \( \frac{31 \times 30 \times 25}{3!} \) triangles in the 31-point geometry.

(2) The number of triangles with one ideal vertex can be found by choosing the ideal point in 6 ways, the second point in 25 ways, and the third point in 20 ways. The total is \( \frac{6 \times 25 \times 20}{2!} \).

(3) The number of triangles with two ideal vertices can be found by choosing two ideal points in \( \frac{6 \times 5}{2!} \) ways and the third point in 25 ways. The total is \( \frac{6 \times 5 \times 25}{2!} \). The number of triangles in 25-point geometry can be found by subtracting the number of triangles having ideal vertices from the total number of triangles in the 31-point geometry, that is
The first of the above methods can be used in finding the number of quadrilaterals in 25-point geometry. The total number of quadrilaterals is \( \frac{25 \cdot 24 \cdot 20 \cdot 13}{4!} = 6500 \).

One may also find the number of quadrilaterals in 25-point geometry using 31-point geometry. The total number of quadrilaterals in 31-point geometry is \( \frac{31 \cdot 30 \cdot 25 \cdot 16}{4!} \). The number of quadrilaterals having two ideal vertices is \( \frac{6 \cdot 5 \cdot 25 \cdot 16}{2! \cdot 2!} \) and the number with one ideal vertex is \( 6 \cdot \left( \frac{25 \cdot 20 \cdot 12}{3!} \right) \).

Thus the total number of quadrilaterals without ideal vertices is \( \frac{31 \cdot 30 \cdot 25 \cdot 16}{4!} - \frac{6 \cdot 5 \cdot 25 \cdot 16}{2! \cdot 2!} - \frac{6 \cdot 25 \cdot 20 \cdot 12}{3!} = 25 \cdot (620 - 120 - 240) = 6500 \).

It is difficult to use the first of the above methods to determine the number of pentagons in 25-point geometry. After four points, no three collinear, have been chosen the number of points eligible for the fifth vertex depends on the location of the first four points. If the four points determine a parallelogram, four points are eligible; if they determine exactly one pair of parallel lines, five points are eligible; and if they determine no parallel lines, six points are eligible. Thus the
number of pentagons in 25-point geometry is larger than
\[ \frac{25 \cdot 24 \cdot 20 \cdot 13 \cdot 4}{5!} = 4800 \] and is smaller than \[ \frac{25 \cdot 24 \cdot 20 \cdot 13 \cdot 6}{5!} = 7800. \]

The difficulty suggested in the above paragraph does not arise in counting the number of pentagons in 31-point geometry since ordinary and ideal points need not be treated in different manners. Thus the number of pentagons (or hexagons) in 25-point geometry can be calculated by finding the number in 31-point geometry and deleting those with one or two ideal vertices.

The number of pentagons in 31-point geometry is

(1) \[ \frac{31 \cdot 30 \cdot 25 \cdot 16 \cdot 6}{5!} \] since the fifth point is chosen after eliminating 25 points including 4 vertices, 3 diagonal points and 3 additional points on each of the six lines determined by the first four points. (This result is given by W. L. Edge in his article "31-point Geometry", Mathematical Gazette, XXXIX, May, 1955 [5].)

(2) The number of pentagons which have two ideal vertices is

\[ \frac{6 \cdot 5}{2!} \cdot \frac{25 \cdot 16 \cdot 6}{3!}. \]

(3) The number of pentagons which have only one ideal vertex is

\[ \frac{6 \cdot 25 \cdot 20 \cdot 12 \cdot (31 - 27)}{4!}. \]

(4) The total number of pentagons in 25-point geometry is

\[ \frac{31 \cdot 30 \cdot 25 \cdot 16 \cdot 6}{5!} - \frac{6 \cdot 5}{2!} \cdot \frac{25 \cdot 16 \cdot 6}{3!} - \frac{6 \cdot 25 \cdot 20 \cdot 12 \cdot 4}{4!} = \]

\[ = (6)(25)(124 - 40 - 40) = (150)(44) = 6600. \]
The number of ellipses, parabolas, and hyperbolas in 31-point geometry will be computed next. The number of hexagons having exactly one ideal vertex is \( \frac{6 \cdot 25 \cdot 20 \cdot 12 \cdot 2}{5!} = 600 \). In choosing points on a hexagon with only one ideal vertex there are 6 choices for the ideal vertex, 25 choices for the second point, 20 choices for the third and 12 choices for the fourth. At this stage four points have not been discarded because they are on lines through two vertices but a choice of one of these points requires that one other also is on the hexagon so only two different hexagons can be obtained. Since the set of hexagons with one ideal vertex is also the set of parabolas in 31-point geometry, exactly 600 parabolas exist in 31-point geometry.

In a similar way it can be shown that the number of hexagons with two ideal vertices is \( \frac{6 \cdot 5 \cdot 25 \cdot 16 \cdot 6 \cdot 1}{2! \cdot 4!} = 1500 \).

Thus there are 1500 hyperbolas in 31-point geometry.

The total number of hexagons in 31-point geometry is \( \frac{31 \cdot 30 \cdot 25 \cdot 16 \cdot 6 \cdot 1}{6!} = 3100 \). The number of ellipses in 31-point geometry can be found by subtracting the parabolas and hyperbolas from the hexagons. Thus there are \( 3100 - 600 - 1500 = 1000 \) ellipses. Since ellipses contain no ideal vertices, they coincide with the number of hexagons in 25-point geometry and there are 1000 hexagons in 25-point geometry.

The accuracy of computations that involve several numbers
can often be checked by finding a relation, independent of the method of computation, which the number must satisfy. For example, if a student measures the sides of a right triangle with a ruler the accuracy can be tested by substituting his results in the theorem of Pythagoras.

A check for some of the numbers obtained above is now considered. Any 25-point pentagon is either a parabola or it consists of five points on an ellipse. There are 1000 distinct ellipses in 25-point geometry. Successive elimination of each point on specific ellipse will result in six pentagons. Thus to the 1000 ellipses, 6000 pentagons correspond. By adding the 600 pentagons, which are identical to the 600 parabolas, one obtains the previous total of 6600 pentagons in 25-point geometry.

The introduction of coordinates into geometry provides a powerful method for the discovery of unknown results. Chapter IV contains coordinate systems both for 25-point geometry and for 31-point geometry. It is suggested that a reader consider the problem of assigning coordinates to 25-point geometry before looking at the method used in Chapter IV. There is no apparent reason why satisfactory coordinates cannot be chosen in more than one way.
CHAPTER IV

COORDINATE SYSTEMS

4.1 Introduction

A coordinate system is especially useful in geometry if one wishes to discover an unknown result. For example, the equation satisfied by the set of points twice as far from a given line as from a given point can be found, using rectangular coordinates, without any prior knowledge that the set forms a conic.

At many levels, modern geometry uses various transformations. With a coordinate system a transformation may be represented by equations and an analytical study of the images of lines, circles, and other curves in the transformation leads to theorems concerning properties of the transformation.

In the present chapter a coordinate system corresponding to the cartesian coordinate system of Euclidean geometry is related to 25-point geometry. This system cannot be used satisfactorily to represent the ideal points in 31-point geometry. By assigning homogeneous coordinates to the points of 31-point geometry this deficiency can be overcome.

A coordinate system for 25-point geometry has been discussed by Martha Heidlage in the Mathematics Teacher of February 1965, on page 109 [6]. In her paper the same unit is used to represent
a distance corresponding to one step in a row or to one step in a column in blocks I, II, and III of Figure 2.1. Since row distances and column distances are not equal, her approach requires a modification of the theorem of Pythagoras in finding the distances between two points whose coordinates are known.

In this chapter a coordinate system is chosen so that vertical and horizontal steps are not represented by the same unit. With this approach the theorem of Pythagoras does not require modification and equations of lines, circles, and conics are closely analogous to the corresponding equations of analytic Euclidean geometry. Computations in this system employ modulo five arithmetic.

4.2 Rectangular Coordinates

Consider the coordinate system shown in Figure 4.1 where the primed numerals indicate column distances and arithmetic operations use the multiplication table, Figure 4.2, combined with modulo five numbers.

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<th>B(1, 4')</th>
<th>C(2, 4')</th>
<th>D(3, 4')</th>
<th>E(4, 4')</th>
</tr>
</thead>
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<td>H(2, 3')</td>
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<td>N(3, 2')</td>
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<tr>
<td>P(0, 1')</td>
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<td>R(2, 1')</td>
<td>S(3, 1')</td>
<td>T(4, 1')</td>
</tr>
<tr>
<td>U(0, 0')</td>
<td>V(1, 0')</td>
<td>W(2, 0')</td>
<td>X(3, 0')</td>
<td>Y(4, 0')</td>
</tr>
</tbody>
</table>
To emphasize the choice of 0, 1, 2, 3, 4 as abscissa and 0', 1', 2', 3', 4' as ordinates, the coordinates of a general point will be represented by (x, y').

With this coordinate system numerous problems can be solved directly. For example, define the midpoint of a segment with coordinates (x₁, y₁'), (x₂, y₂') as the point \( \left( \frac{x_1 + x_2}{2}, \frac{y_1' + y_2'}{2} \right) \).

Then the coordinates of the midpoint of \( R(2, 1') \) and \( X(3, 0') \) is \( \left( \frac{2 + 3}{2}, \frac{1' + 0'}{2} \right) = (0, 3') \), so that F is the midpoint of RX.

Let the equation of the line through \( (x_1, y_1') \) and \( (x_2, y_2') \) be \( y' - y_1' = \frac{x - x_1}{x_2 - x_1} \) or \( y' - y_1' = \frac{y_2' - y_1'}{x_2 - x_1} \cdot (x - x_1) \). The

![Multiplication Table](image)

<table>
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<tr>
<th>x</th>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>1'</th>
<th>2'</th>
<th>3'</th>
<th>4'</th>
</tr>
</thead>
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<td>4</td>
<td>1'</td>
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<td>1'</td>
<td>3</td>
<td>1</td>
<td>4</td>
<td>2'</td>
</tr>
</tbody>
</table>
equation of the line through R(2, 1') and X(3, 0') is
\[ y' - 1' = \frac{0' - 1'}{3 - 2} (x - 2) \] or \[ y' - 1' = -1' (x - 2) \].

The point L(1, 2') is on line RX and substitution of \( x = 1 \) and \( y' = 2' \) in this equation gives \( 2' - 1' = -1' (1 - 2) \) or \( 1' = 1' \). Likewise the coordinates of F(0, 3') satisfy the equation since \( 3' - 1' = -1' (0 - 2) \) or \( 2' = 2' \).

Lines which are columns in block I have equations \( x = 0 \), \( x = 1 \), \( x = 2 \), \( x = 3 \), and \( x = 4 \). Other lines have equations of the form \( y' - y'_1 = m(x - x'_1) \) where the "slope" \( m \) is 0', 1', 2', 3', or 4'. Lines with equal slopes are parallel. Lines of slope 0' are perpendicular to lines which are columns of block I. Otherwise two lines are perpendicular if the product of their slopes is \(-1 = 4 \pmod{5}\).

Is \( (x - a)^2 + (y' - b')^2 = r^2 \) the equation of a circle for \( a \in \{0, 1, 2, 3, 4\} \), \( b' \in \{0', 1', 2', 3', 4'\} \), \( r \in \{1, 2, 1', 2'\} \)?

Example 4.2-1

Consider \( (x - 2)^2 + (y' - 3')^2 = (1')^2 \). Is this the circle with center (2, 3') and radius 1'? That is, do the points C(2, 4'), M(2, 2'), T(4, 1'), U(0, 0'), P(0, 1'), Y(4, 0') satisfy the equation \( (x - 2)^2 + (y' - 3')^2 = (1')^2 \)? The answer is yes as can be verified by direct substitution. This verification is shown for the two points C and M.
For C: \((2 - 2)^2 + (4' - 3')^2 = (1')^2;\)
\(0 + (1')^2 = (1')^2\)

For M: \((2 - 2)^2 + (2' - 3')^2 = (1')^2;\)
\(0 + (-1')^2 = 1'^2\)

Example 4.2-2

Can the equation \(x^2 + y'^2 - 2x + 3'y' + 2 = 0\) be changed to the form \((x - a)^2 + (y' - b')^2 = r^2?\)

\((x^2 - 2x + 1) + y'^2 + 3'y' + (4')^2 =\)
\(-2 + 1 + (4')^2 = -2 + 1 + 2\)
\((x - 1)^2 + (y' + 4')^2 = (1)^2\)

or \((x - 1)^2 + (y' - 1')^2 = 1^2\)

Thus the original equation in this example seems to represent a circle with center \((1, 1')\) and radius 1. One point on the circle with this center and radius is \(R(2, 1').\)

Substituting \(R\) into the original equation gives

\(2^2 + (1')^2 - 2(2) + 3'(1') + 2 =\)
\(4 + 2 - 4 + 1 + 2 = 5 = 0\)

Definition: In 25-point geometry the equation of the circle with center \((a, b')\) and radius \(r\) can be written in the form

\((x - a)^2 + (y' - b')^2 = r^2\) where
\[ a \in \{0, 1, 2, 3, 4\}, \quad b' \in \{0', 1', 2', 3', 4'\} \]

and

\[ r \in \{1, 2, 1', 2'\} \]

In Euclidean geometry \( \frac{x^2}{b^2} + \frac{y^2}{b^2} = 1 \) is the equation of a special type of ellipse for all positive numbers \( a \) and \( b \). This suggests finding values of \( a \) and \( b \), if any, for which

\[ \frac{x^2}{a^2} + \frac{(y')^2}{b^2} = 1 \]

is the equation of an ellipse in 25-point geometry.

Several examples suggest that every equation of the form

\[ \frac{(x - h)^2}{a^2} + \frac{(y' - k)^2}{b^2} = 1 \]

represents an ellipse if \( a^2, b^2 \in \{1, 4\} \) or if \( a^2, b^2 \in \{2, 3\} \) with \( h \in \{0, 1, 2, 3, 4\} \) and

\[ k \in \{0', 1', 2', 3', 4'\}. \]

Example 4.2-3

Consider \( \frac{(x - 0)^2}{1} + \frac{(y' - 4')^2}{4} = 1 \). This is an ellipse with center \( A(0, 4') \) since points \( B(1, 4'), E(4, 4'), M(2, 2'), (3, 2'), R(2, 1') \) and \( S(3, 1') \) satisfy the equation.

For \( B(1, 4') \):

\[ \frac{(1 - 0)^2}{1} + \frac{(4' - 4')^2}{4} = 1 \]

For \( E(4, 4') \):

\[ \frac{(4 - 0)^2}{1} + \frac{(4' - 4')^2}{4} = 16 = 1 \]
For $M(2, 2')$: \[
\frac{(2 - 0)^2}{1} + \frac{(2' - 4')^2}{4} = 4 + \frac{3}{4} = \frac{19}{4} = \frac{4}{4} = 1
\]

For $N(3, 2')$: \[
\frac{(3 - 0)^2}{1} + \frac{(2' - 4')^2}{4} = 9 + \frac{3}{4} = 4 + \frac{3}{4} = 1
\]

For $R(2, 1')$: \[
\frac{(2 - 0)^2}{1} + \frac{(1' - 4')^2}{4} = 4 + \frac{3}{4} = 1
\]

For $S(3, 1')$: \[
\frac{(3 - 0)^2}{1} + \frac{(1' - 4')^2}{4} = 9 + \frac{3}{4} = 1
\]

Example 4.2-4

Consider \[\frac{(x - 0)^2}{3} + \frac{(y' - 4')^2}{2} = 1\]. This is an ellipse with center $A(0, 4')$ since points $F(0, 3')$, $U(0, 0')$, $L(1, 2')$, $Q(1, 1')$, $O(4, 2')$, and $T(4, 1')$ satisfy the equation.

Example 4.2-5

Is $3x^2 + 2y'^2 - 6x - 4'y' + 1 = 0$ an ellipse? One may change the equation into the other form, using the method of completing the square.
\[ 3(x^2 - 2x + 1) + 2[y_2^2 - 2'y_1 + (l')^2] = -1 + 3 + 4 \]
\[ 3(x - 1)^2 + 2(y' - 1')^2 = 6 \]
\[ \frac{(x - 1)^2}{2} + \frac{(y' - 1')^2}{3} = 1 \]

This is the ellipse with center \( Q(l, l') \) passing through points \( B, G, N, O, X, \) and \( Y. \)

From Chapter III it is known that there are 1000 ellipses in 25-point geometry. By symmetry there should be 40 ellipses with a given point as center. Not all of these with center at \((0, 0')\) have equations of the form \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) since not all have major and minor axes parallel to the coordinate axes. In the earlier discussion of Chapter II it was pointed out that a rotation of 60° can turn the elements of block I (Figure 1.1) into the corresponding elements of block II and that a rotation of 120° can turn the elements of block I into those of block III.

It can then be conjectured that for each ellipse with center \((0, 0')\) and equation of the form \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) rotations of 60° and 120° will produce ellipses with different equations except for the case in which the original ellipse is a circle.

The equation \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) is satisfied by 6 points on an ellipse.
provided the ordered pair \((a^2, b^2)\) is one of \((1, 1), (2, 2), (3, 3), (4, 4), (1, 4), (2, 3), (3, 2)\) or \((4, 1)\). For the first four sets of values of \((a^2, b^2)\) the ellipses are circles.

Rotation of 60° about point \(U(0, 0')\) for the other four ellipses will produce 4 new ellipses (since the circles rotate into themselves). Thus rotations of 60° and 120° combined with the 8 ellipses of the form \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\) give 16 ellipses. How can one find equations of the other ellipses with center \((0, 0')\)?

In Euclidean geometry if \(ax^2 + bxy + cy^2 + dx + cy + f = 0\) represents an ellipse, \(b^2 - 4ac < 0\). In 25-point geometry one can obtain some values of \(b^2 - 4ac\) from the following table:

<table>
<thead>
<tr>
<th>Equation of the ellipse</th>
<th>The value of (b^2 - 4ac)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x^2 + y^2 = 1)</td>
<td>(\Delta = b^2 + ac \pmod{5})</td>
</tr>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>(x^2 + 4y^2 = 1)</td>
<td>4</td>
</tr>
<tr>
<td>(2x^2 + 2y^2 = 1)</td>
<td>4</td>
</tr>
<tr>
<td>(2x^2 + 3y^2 = 1)</td>
<td>1</td>
</tr>
</tbody>
</table>

This suggests the conjecture that \(ax^2 + b'xy' + cy'^2 = 1\) is an ellipse if \(b'^2 + ac\) is 1 or 4.

Example 4.2-6

Is \(3x^2 + 2'xy' + 2y'^2 + 1 = 0\) an equation of an ellipse?

This equation can be written as \(2x^2 + 3'xy' + 3y'^2 = 1\) where
\( b'^2 + ac = 3 + 1 = 4 \). One can obtain the coordinates of all the points which satisfy the equation as follows:

Let \( y' = 1' \), then \( 2x^2 + 3'\(1'\)x + 3(1')^2 = 1 \)

or \( 2x^2 + x = 0 \) so that \( x = 0 \) or \( 2 \)

Thus points \( P(0, 1') \) and \( R(2, 1') \) satisfy the equation. Similarly, let \( y' = 2' \), \( y' = 3' \) and \( y' = 4' \) to find points \( M(2, 2') \), \( I(3, 3') \), \( A(0, 4') \), and \( D(3, 4') \).

If \( y' = 0 \), \( 2x^2 = 1 \), \( x = 2' \) or \( 3' \) but no point of 25-point geometry has coordinates \( (2', 0) \) or \( (3', 0) \).

Since no three of the points \( P, R, M, I, A, \) and \( D \) are collinear they form an ellipse and the above analysis shows that only these six points satisfy the equation \( 2x^2 + 3'xy' + 3y'^2 = 1 \).

Perhaps all ellipses with center at \( (0, 0') \) have equations of the form \( ax^2 + b'xy' + cy'^2 = 1 \) with \( a, c \in \{0, 1, 2, 3, 4\} \), \( b' \in \{0', 1', 2', 3', 4'\} \), but all equations of this form may not be ellipses. How many such choices of \( a, b', \) and \( c \) are possible?

**Conjecture:** The 20 cases in which \( \Delta = 1 \) represent ellipses.

Let \( \Delta = \) The 20 cases in which \( \Delta = 4 \) represent ellipses.

\( b'^2 + ac \) The 30 cases in which \( \Delta = 3 \) represent hyperbolas.

(See Figure 4.3) The 30 cases in which \( \Delta = 2 \) represent hyperbolas.

The 25 cases in which \( \Delta = 0 \) represent two parallel lines or no locus.

Let each of the 125 cases be assigned a case number as shown in
Figure 4.3. Case (1), in which $A = 0$, has the corresponding equation:

$$0 \cdot x^2 + 0 \cdot xy' + 0 \cdot y'^2 = 1$$

Figure 4.3

Various Values of $\Delta = b'^2 + ac (\mod 5)$

<table>
<thead>
<tr>
<th>a</th>
<th>c</th>
<th>$ac$</th>
<th>Case Number</th>
<th>$\Delta = b'^2 + ac$</th>
</tr>
</thead>
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<tr>
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<td>0</td>
<td>0</td>
<td>(1)</td>
<td>(1) 0 (26) 2 (51) 3 (76) 3 (101) 2</td>
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<tr>
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<td>0</td>
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<td>(2) 0 (27) 2 (52) 3 (77) 3 (102) 2</td>
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<td>0</td>
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<td>0</td>
<td>(4)</td>
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</tr>
<tr>
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<td>0</td>
<td>(5)</td>
<td>(5) 0 (30) 2 (55) 3 (80) 3 (105) 2</td>
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<td>0</td>
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<td>(6) 0 (31) 2 (56) 3 (81) 3 (106) 2</td>
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<tr>
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<td>1</td>
<td>(7)</td>
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<td>2</td>
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<td>(8) 2 (33) 4 (58) 0 (83) 0 (108) 4</td>
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<td>(9)</td>
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<td>(12) 2 (37) 4 (62) 0 (87) 0 (112) 4</td>
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<td>0</td>
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</tr>
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<td>(20)</td>
<td>(20) 2 (45) 4 (70) 0 (95) 0 (120) 4</td>
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<td>3</td>
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</tr>
<tr>
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<td>4</td>
<td>1</td>
<td>(25)</td>
<td>(25) 1 (50) 3 (75) 4 (100) 4 (125) 3</td>
</tr>
</tbody>
</table>

There is no locus for this equation. Case (7), in which $\Delta = 1$, has the corresponding equation: $x^2 + y'^2 = 1$. 
The locus is a circle. Other examples representing the above conjecture are shown below.

Example 4.2-7

For case 59, where \( a = 1, b' = 2', c = 3 \) and \( b'^2 + ac = 1 \) test the conjecture that \( x^2 + 2'xy' + 3y'^2 = 1 \) represents an ellipse.

Let \( y' = 0 \), then \( x = 1 \) or \( 4 \), so the equation is satisfied by the two points \((1, 0')\)
and \((4, 0')\)

Let \( y' = 1' \), \( x^2 + 4x + 1 = 1, x = 0 \) or \( x = 1 \)
then two more points are \((0, 1')\) and \((1, 1')\)

Let \( y' = 2' \), \( x^2 + 3x + 4 = 1, x^2 + 3x + 3 = 0 \);
then \( x = \frac{-3 + \sqrt{2}}{2} \) is not the abscissa of any point in 25-point geometry.

Let \( y' = 3' \), \( x^2 + 2x + 4 = 1, x^2 + 2x + 3 = 0 \);
there is no such \( x \)

Let \( y' = 4' \), \( x^2 + x + 1 = 1, \) so \( x = 0 \), or \( x = 4 \)
and two points on locus are \((0, 4')\) and \((4, 4')\)

The above six pairs of coordinates represent points \( V(1, 0'), Y(4, 0'), P(0, 1'), Q(1, 1'), A(0, 4'), \) and \( E(4, 4') \).
They are on the ellipse with center $U(0, 0')$.

Example 4.2-8

For case 95 where $a = 3$, $c = 4$, $b' = 3'$, $b'^2 + ac = 0$, test the conjecture that the equation $3x^2 + 3'xy' + 4y'^2 = 1$ represents two lines or no locus.

Let $x = 0$, then $4y'^2 = 1$, $y'^2 = \frac{1}{4} = 4$, and $y'$ does not exist.

Likewise no permissible value of $y'$ exists for $x = 1, 2, 3$ or 4 so there is no locus associated with the equation $3x^2 + 3'xy' + 4y'^2 = 1$.

Example 4.2-9

For case 72 where $a = 4$, $c = 1$, $b' = 2'$ and $b'^2 + ac = 2$, test the hypothesis that the equation $4x^2 + 2'xy' + y'^2 = 1$ represents an hyperbola.

Let $x = 0$, $y'^2 = 1$, $y' = 1$ or 4 (no point).

Let $x = 1$, $4 + 2'y + y'^2 = 1$, $y' = \frac{-2' + 1'}{2}$ (no point).

Let $x = 2$, $1 + 4'y + y'^2 = 1$, $y' = 0$ or $y' = 1$, then, two points $(2, 0')$ and $(2, 1')$ are obtained.

Let $x = 3$, $1 + l'y + y'^2 = 1$, $y' = 0'$ or $4'$, then, two points $(3, 0')$ and $(3, 4')$ are obtained.
Let $x = 4$, $4 + 3'y' + y'^2 = 1$, $y'^2 + 3'y + 3 = 0$,

$$y' = \frac{-3' \pm 1}{2} \text{ (no point)}$$

Example 4.2-10

Test case 74 where $a = 4$, $b' = 2'$, $c = 3$ and $b'^2 + ac = 0$

to see if the equation $4x^2 + 2'xy' + 3y'^2 = 1$ represents two
lines or no locus. By the previous methods it is easily verified
that this equation is satisfied by the ten points on the two
parallel lines ATLW and MPXBJ. These lines have equations
$2x + 3'y' + 1 = 0$ and $3x + 2'y' + 1 = 0$ and multiplication gives
$4x^2 + 2'xy' + 3y'^2 = 1$.

Since any conic of the form $ax^2 + bxy' + cy'^2 = 1$ is sym­
metric about the point $U(0, 0')$ this family of equations does
not include parabolas. It may be conjectured that any conic
in 25-point geometry has an equation of the form

$$ax^2 + b'xy' + cy'^2 + dx + e'y' + f = 0$$

where $a$, $c$, $d$, and $f$ are in the set 0, 1, 2, 3, 4, and $b'$ and $e'$.  

are in the set $0'$, 1', 2', 3', 4'.

For this equation it is conjectured that if $b'^2 + ac = 0 \text{ (mod 5)}$
then points which satisfy the equation are on a parabola unless
they form a degenerate conic.

Example 4.2-11

Find all points which satisfy the equation

$$3x^2 + 1'xy' + y'^2 + 2x + 4 = 0.$$
Let \( y' = 0, \ 3x^2 + 4 = 0, \ x = \frac{3 \pm 1}{1} = 4 \text{ or } 2 \)

Let \( y' = 1', \ 3x^2 + 4x + 1 = 0, \ x = 3, 4 \)

Let \( y' = 2', \ 3x^2 + x + 2 = 0, \ x = \frac{4 \pm \sqrt{2}}{1} \) (no point)

Let \( y' = 3', \ 3x^2 + 3x + 2 = 0, \ x = \frac{2 \pm 0}{1} = 2 \)

Let \( y' = 4', \ 3x^2 + 1 = 0, \ x^2 = \frac{4}{3} = 3 \) (no point)

The five points \( H, S, T, W, Y \) are on a parabola. These points are symmetric about the line VDFMT, so \( T \) is the vertex and the sixth point on the parabola is the ideal point \( I_6 \).

4.3 Homogeneous Coordinates

Homogeneous coordinates may be assigned to the points of 31-point geometry to provide a suitable representation for ideal points. In a homogeneous coordinate system, the points are designated by ordered triples of numbers \((x, y', z)\) where

\[
\begin{align*}
    x &\in \{0, 1, 2, 3, 4\}, \ y' \in \{0', 1', 2', 3', 4'\}, \\
    z &\in \{0, 1, 2, 3, 4\}
\end{align*}
\]

(1)

\((x, y', z) = (0, 0', 0)\) and

\((x, y', z) = (nx, ny', nz)\) for

\(n \in \{1, 2, 3, 4\}\)

(2)

(3)

The numbers \(x, y',\) and \(z\) are called homogeneous coordinates of the point associated with the class \((x, y', z)\). When \(z \neq 0\), one may choose \(n = \frac{1}{z_1}\) and associate with each point \((x_1, y'_1, z_1)\)
a number triple \((x_2, y'_2, 1)\) where \(x_2 = \frac{x_1}{z_1}\) and \(y'_2 = \frac{y'_1}{z_1}\).

The point \((x_1, y'_1, z_1)\) is identical with the point \((x_2, y'_2, 1)\)
and this is the point assigned coordinates \((x_2, y'_2)\) in Section 4.2. When \(z = 0\), there exist points that cannot be represented in the form \((x, y', 1)\) and therefore cannot be represented using non-homogeneous coordinates. Thus the ideal points in 3l-point geometry cannot be represented by cartesian coordinates but they have homogeneous coordinates of the form \((x, y', 0)\) as shown in Figure 4.4.

Figure 4.4

<table>
<thead>
<tr>
<th>A(0, 4', 1)</th>
<th>B(1, 4', 1)</th>
<th>C(2, 4', 1)</th>
<th>D(3, 4', 1)</th>
<th>E(4, 4', 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>F(0, 3', 1)</td>
<td>G(1, 3', 1)</td>
<td>H(2, 3', 1)</td>
<td>I(3, 3', 1)</td>
<td>J(4, 3', 1)</td>
</tr>
<tr>
<td>K(0, 2', 1)</td>
<td>L(1, 2', 1)</td>
<td>M(2, 2', 1)</td>
<td>N(3, 2', 1)</td>
<td>O(4, 2', 1)</td>
</tr>
<tr>
<td>P(0, 1', 1)</td>
<td>Q(1, 1', 1)</td>
<td>R(2, 1', 1)</td>
<td>S(3, 1', 1)</td>
<td>T(4, 1', 1)</td>
</tr>
<tr>
<td>U(0, 0', 1)</td>
<td>V(1, 0', 1)</td>
<td>W(2, 0', 1)</td>
<td>X(3, 0', 1)</td>
<td>Y(4, 0', 1)</td>
</tr>
</tbody>
</table>

Most properties of homogeneous coordinates follow from corresponding properties of non-homogeneous coordinates. For example, the equation of a line through points \((x_1, y'_1)\) and \((x_2, y'_2)\) can be written in the form
The coordinates of ideal points in Figure 4.4 can be found by the following method. Suppose that the ideal point \( I_k \) is on the line through \( U(0, 0', 1) \) and \( Z(a, b', 1) \). Then \( I_k \) is \( (a, b', 0) \), since

\[
\begin{vmatrix}
0 & 0' & 1 \\
a & b' & 1 \\
a & b' & 0 \\
\end{vmatrix} = \begin{vmatrix}
a & b' \\
a & b' \\
\end{vmatrix} = 0
\]

Each ideal point can have four different sets of equivalent co­ordinates, but in Figure 4.4 these are chosen so that \( x = 1 \) except for \( I_1 \) which is assigned coordinates \((0, 1', 0)\).

In 3l-point geometry a line is the set of six points whose coordinates \((x, y', z)\) satisfy a linear homogeneous equation of the form \(ax + b'y' + cz = 0\), where

\[a \in \{0, 1, 2, 3, 4\}, \ b' \in \{0', 1', 2', 3', 4'\} \text{ and } \ c \in \{0, 1, 2, 3, 4\}\]

Points \((x_1, y'_1, z'_1)\), \((x_2, y'_2, z'_2)\), and \((x_3, y'_3, z'_3)\) are
collinear if and only if the determinant of their homogeneous coordinates is zero.

\[
\begin{vmatrix}
x_1 & y_1 & z_1 \\
x_2 & y_2 & z_2 \\
x_3 & y_3 & z_3
\end{vmatrix} = 0
\]

Example 4.3-1

The equation of the line through P and U is

\[
\begin{vmatrix}
x & y & z \\
0 & 1 & 1 \\
0 & 0 & 1
\end{vmatrix} = 0
\]

so that \(x(1')(1) = 0\) or \(x = 0\).

It is evident that points A, F, K, and I₁ (which have \(x\) coordinates of zero) also satisfy this equation.

Example 4.3-2

The equation of line I₁ LERFX, in block II, is

\[
\begin{vmatrix}
x & y' & z \\
4 & 4' & 1 \\
3 & 0' & 1
\end{vmatrix} = 0
\]

This gives \(4'x + 3y' - 2'z - 4y' = 0\)

or \(4'x - y' - 2'z = 0\)

or \(3x + 4'y' + z = 0\)

That point R(2, 1', 1) is on this line can be checked by
substituting the coordinates into the equation $3 \cdot 2 + 4 \cdot 1 + 1 = 6 + 3 + 1 = 10 = 0$.

Example 4.3-3

The equation of the line AILTW, in block III, is

<table>
<thead>
<tr>
<th>x</th>
<th>y'</th>
<th>z</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4'</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>3'</td>
<td>1</td>
</tr>
</tbody>
</table>

That is $4'x + 3y' - 2'z - 3'x = 0$, $1'x + 3y' - 2'z = 0$, $4'x + 2y' + 2'z = 0$, or $2x + 3'y' + z = 0$.

The equation of a line $ax + b'y' + cz = 0$ can be expressed in alternate forms by multiplying by constants 2, 3, or 4.

If $c \neq 0$, division by $c$ will give an equation of the form $dx + e'y' + z = 0$.

If $c = 0$, division by $a$ will give an equation of the form $x + f'y' = 0$. Although it is not necessary to write equations in a specified form, usually coefficients will be chosen so that in the equation $ax + b'y' + cz = 0$, $a \in \{0, 1, 2, 3, 4\}$, $b' \in \{0, 1', 2', 3', 4'\}$, and $c \in \{0, 1\}$.

The ordered numbers $[a, b', c]$ are called line coordinates, the ordered numbers $(x, y', z)$ are called point coordinates, and the point $(x, y', z)$ and line $[a, b', c]$ are said to be incident.
if they satisfy the linear equation $ax + b'y' + cz = 0$.

Example 4.3-4

If $x = 1, y = 3'$ are two coordinates of a point on $2x + 3'y' + z = 0$, what is the other coordinate?

Solution: Substituting $x = 1, y = 3'$ into the above equation gives $2 + (3')^2 + z = 0$ or $z = 0$. Therefore this is the ideal point $I_2$ with homogeneous coordinates $(1, 3', 0)$.

Example 4.3-5

For the line SV\$EHK, the equation is

$$\begin{vmatrix} x & y' & z \\ 4 & 4' & 1 \\ 2 & 3' & 1 \end{vmatrix} = 4'x + 4 \cdot 3'z + 2y' - 2 \cdot 4'z - 3'x - 4y' = 0,$$

or $4x + 1'y' + z = 0$

In rectangular coordinates for Euclidean geometry all conics, including degenerate cases and imaginary conics, have equations of the form $ax^2 + bxy + cy^2 + dx + ey + f = 0$ where the coefficients are real numbers. Changing this equation to homogeneous coordinates and considering the multiplication table of Figure 4.2, it seems logical to call the set of points satisfied by the equation $ax^2 + b'xy' + cy'^2 + dxz + e'y'z + fz^2 = 0$ a conic if $a, c, d,$ and $f$ are in the set $\{0, 1, 2, 3, 4\}$ and if $b'$ and $e'$ are in the set $\{0', 1', 2', 3', 4'\}$.

The details of a study which would show exact conditions
of the coefficients which give specific conics or degenerate cases are not pursued in this study. It is however conjectured that if the above equation represents a non-degenerate conic then the type of conic is determined by the discriminant
\[ b'^2 - 4ac = b'^2 + ac \pmod{5} \]
as follows:

- \[ b'^2 + ac = 0 \] parabola
- \[ b'^2 + ac = 1 \] ellipse
- \[ b'^2 + ac = 2 \] hyperbola

Example 4.3-6

What locus is represented by \( x^2 + y'^2 + xz + 4'y'z = 0 \)?

Here \( b'^2 + ac = 1 \). Now this equation can be written in the alternate forms

\[
\begin{align*}
x^2 - 4xz + y'^2 - l'y'z &= 0 \quad \text{or} \\
x^2 - 4zx + (2z)^2 + y'^2 - l'zy' + (3,z)^2 &= 4z^2 + 3z^2 = 2z^2 \quad \text{or} \\
(x - 2z)^2 + (y' - 3'z)^2 &= 2z^2
\end{align*}
\]

This equation seems to represent the circle with center at \((2, 3', 1)\) and radius \(1'\). From Figure 4.4 it is found that the six points on this circle are \(C(2, 4', 1), M(2, 2', 1), T(4, 1', 1), U(O, O', 1), Y(4, 0', 1)\) and \(P(O, 1', 1)\), and
substitution shows that the coordinates of each point satisfy the equation.

Example 4.3-7

Consider the equation

\[ x^2 + 2'xy' + 2y'^2 + xz + 2'y'z + z^2 = 0 \]

Since \( b'^2 + ac = 3 + 2 = 0 \) (mod 5) this equation is expected to represent a parabola. If so the equation will be satisfied by 6 points no three on a line with one point an ideal point.

Let \( z = 1 \) and \( x = 0 \), then \( 2y'^2 + 2'y' + 1 = 0 \), \( y' = 2 \)

Hence \((0, 2', 1)\) is on the curve

When \( z = 1 \) and \( x = 1 \), then \( 2y'^2 + 4'y' + 3 = 0 \), \( y' = 2' \) or \( 1' \)

Thus \((1, 1', 1)\) and \((1, 2', 1)\) are on the curve

When \( z = 1 \) and \( x = 2 \), then

\[ 2y'^2 + 1'y' + 2 = 0 \], \( y' = \frac{4' + 1}{4} \) (no point on graph)

When \( z = 1 \) and \( x = 3 \), then

\[ 2y'^2 + 3'y' + 3 = 0 \], \( y' = \frac{2' + \sqrt{3} + 1}{4} = \frac{2' + 2}{4} \) (no point on graph)

When \( z = 1 \) and \( x = 4 \), then

\[ 2y'^2 + 1 = 0 \], \( y' = 1' \) or \( 4' \)

So \((4, 1', 1)\) and \((4, 4', 1)\) are on the graph
If \( z = 0 \), \( x = 0 \), \( 2y'^2 = 0 \), which gives no point

If \( z = 0 \), \( x = 1 \); \( 1 + 2'y' + 2y'^2 = 0 \), \( y' = 2' \)

Thus \((1, 2', 0)\) is on curve

If \( z = 0 \), \( x = 2 \), \( y'^2 + 2'y' + 2 = 0 \), \( y' = 4 \)

Thus \((2, 4', 0)\) is on curve, but this is the same as \((1, 2', 0)\)

Likewise for \( z = 0 \) and for \( x = 3 \) or \( x = 4 \), the same ideal point \((1, 2', 0)\) is obtained.

The equation does represent a parabola.

The method illustrated in the above example can be used to find all points which satisfy an equation of the form

\[ ax^2 + b'xy' + cy'^2 + dxz + e'y'z + fz^2 = 0 \]

when \( a, c, d, \) and \( f \) are in the set \( \{0, 1, 2, 3, 4\} \) and \( b' \) and \( e' \) are in the set \( \{0', 1', 2', 3', 4'\} \). From the number of ordinary ideal points which satisfy the equation it is easy to classify the conic as a non-degenerate parabola, ellipse, hyperbola, or as intersecting lines, parallel lines, coincident lines, or a point conic.

Homogeneous coordinates can be used to find simultaneous solutions of two equations. Since the two equations contain three variables one of them can be arbitrarily assigned a constant value. If \( z \) is assigned the value of zero, all ideal
points satisfying both equations can be determined. If \( z \) is assigned the value of one, all other simultaneous solutions can be found.

Example 4.3-8

Find all simultaneous solutions of this system of equations

\[
\begin{align*}
2x + 3'y' + 4z &= 0 \\
2x + 3'y' + 3z &= 0
\end{align*}
\]

First choose \( z = 1 \) to obtain the system

\[
\begin{align*}
2x + 3'y' &= -4 \\
2x + 3'y' &= -3
\end{align*}
\]

which is clearly not satisfied by any values of \( x \) and \( y \).

Next let \( z = 0 \) to obtain the system

\[
\begin{align*}
2x + 3'y' &= 0 \\
2x + 3'y' &= 0
\end{align*}
\]

which is satisfied by \( x = 1, y' = 3' \), by \( x = 2, y' = 1' \), by \( x = 3, y' = 4' \), and by \( x = 4, y' = 2' \). Each of these four values of \( x \) and \( y' \) when combined with \( z = 0 \) give equivalent representations of the same ideal point \((1, 3', 0)\). Hence the two lines represented by the original equations intersect at
the ideal point $I_4$ and the only simultaneous solution is $(x, y', z) = (1, 3', 0)$.

Example 4.3-9

Find all points on the conic

$$3x^2 + l'xy' + y'^2 + 2xz + 4z^2 = 0$$

and also on the line

$$x + l'y' + z = 0$$

First choose $z = 1$ and solve the system

$$3x^2 + l'xy' + y'^2 + 2x + 4 = 0$$
$$x + l'y' + 1 = 0$$

Solving the second equation for $y'$ gives $y' = 2'x + 2'$ and substitution in the first equation shows that $x$ must satisfy the quadratic equation $x^2 + 2x + 1 = 0$ which has only the solution $x = 4$ in the set $\{0, 1, 2, 3, 4\}$. For $x = 4$, $y' = 0'$ so the only simultaneous solution of the two equations for which $z = 1$ is: $x = 4$, $y' = 0'$, $z = 1$.

Next let $z = 0$ and solve the system

$$3x^2 + l'xy' + y'^2 = 0$$
$$x + l'y' = 0$$

The second equation gives $y' = 2'x$ and substitution in the first
shows that \( x \) must satisfy \( 3x^2 + 4x^2 + 3x^2 = 0 \) or \( 0x^2 = 0 \) which is, of course, true for \( x = 0, 1, 2, 3, \) or \( 4. \) The root \( x = 0 \) leads to \( x = 0, y' = 0', z = 0 \) which represents no point. The root \( x = 1 \) leads to \( x = 1, y = 2', z = 0. \) The roots 2, 3, and 4 give the same point. Thus the only points on the original line and conic are points \( Y(4, 0', 1) \) and \( I_6(1, 2', 0). \)

The conic in this example is the parabola considered in Example 4.2-11 of Section 4.2 using non-homogeneous coordinates. Thus \( I_6 \) is the ideal point on the axis of the parabola.

Chapter IV has introduced coordinate systems for 25 and for 31-point geometries and has shown some applications. In the remainder of this study the choice of synthetic methods or of analytic methods will depend on the topic under consideration. In Chapter V several finite geometries not discussed previously are introduced and appropriate coordinates are assigned to the points in these geometries.
CHAPTER V

OTHER FINITE GEOMETRIES

5.1 Introduction

The number of finite geometries is unlimited. An interested student can create a geometry by starting with a finite set of points and adopting a set of postulates. If the postulates are chosen to follow some pattern it may be possible to observe additional properties which suggest theorems.

In Chapter II (Figure 2.1) the points of 25-point geometry are arranged in three blocks which are useful in finding parallels, perpendiculars, midpoints, and distances. This chapter contains a discussion of two finite geometries in which points can also be arranged in convenient blocks. (A person who is familiar with projective geometry can construct a related finite geometry by deleting one line from a finite projective geometry.) Finite geometries can also be generated by assigning coordinates to points in a square array and using properties of analytic geometry to define parallel, perpendicular, midpoint, etc. In the following discussion other finite geometries will be introduced by starting with elements in a square array and rearranging the elements into other square arrays to exhibit properties similar to those in the three blocks for 25-point geometry.
With the three blocks of 25-point geometry each point was on six lines and each line contained 5 points. If a similar system contains K blocks, each point will be on 2K lines but each line will contain only 2K-1 points. (If an ideal point is added to each line the number of points and lines will then be equal.) An example of a finite geometry containing 121 points has been given by Alonzo Church [2].

The above remarks show that finite geometries of the type under discussion have an odd number of points on each line. This choice makes it easy to define the midpoint of a given segment. (If each line contained four points, symmetry could not be used to define the midpoint of a segment but, if each point were on five lines, symmetry could be used to define angle bisectors.)

In this chapter two finite geometries will be developed; in one the points can be exhibited in two square blocks and in the other the points will be displayed in four square blocks.

5.2 Nine-point Geometry

Let three points be on each line and represent the points by letters A through I arranged in a 3 by 3 block, Figure 5.1.

Figure 5.1

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>E</td>
<td>F</td>
</tr>
<tr>
<td>G</td>
<td>H</td>
<td>I</td>
</tr>
</tbody>
</table>
As in Chapter II, this block can be endlessly repeated so that each lattice point of any square array can be assigned the proper letter. (See Figure 5.2).

Figure 5.2

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>E</td>
<td>F</td>
<td>D</td>
<td>E</td>
<td>F</td>
<td>D</td>
<td>E</td>
<td>F</td>
</tr>
<tr>
<td>G</td>
<td>H</td>
<td>I</td>
<td>G</td>
<td>H</td>
<td>I</td>
<td>G</td>
<td>H</td>
<td>I</td>
</tr>
<tr>
<td>A</td>
<td>B</td>
<td>C</td>
<td>A</td>
<td>B</td>
<td>C</td>
<td>A</td>
<td>B</td>
<td>C</td>
</tr>
<tr>
<td>D</td>
<td>E</td>
<td>F</td>
<td>D</td>
<td>E</td>
<td>F</td>
<td>D</td>
<td>E</td>
<td>F</td>
</tr>
<tr>
<td>G</td>
<td>H</td>
<td>I</td>
<td>G</td>
<td>H</td>
<td>I</td>
<td>G</td>
<td>H</td>
<td>I</td>
</tr>
</tbody>
</table>

If any two points in this lattice are joined by an Euclidean line, the line passes through points represented by exactly three letters. In following the method of Chapter II, it seems reasonable to form a second block, containing letters A through I, so each possible combination of two letters appears in exactly one row or one column in one of blocks I or II.

Figure 5.3

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>A</th>
<th>F</th>
<th>H</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>E</td>
<td>F</td>
<td>E</td>
<td>G</td>
<td>C</td>
</tr>
<tr>
<td>G</td>
<td>H</td>
<td>I</td>
<td>I</td>
<td>B</td>
<td>D</td>
</tr>
</tbody>
</table>

Block I  Block II
In Figure 5.3 elements in block II are chosen so the three elements in left to right diagonals in Figure 5.2 appear in columns and elements in right to left diagonals appear in rows. Other arrangements of elements in block II seem, at present, equally valid. For example, rows and columns could be interchanged. If in the following discussion of distance it seems that a different arrangement of elements is desirable such a change can be made.

Following the pattern used in Chapter II these definitions are made for 9-point geometry: Each row and column of blocks I and II is called a line so that there are 12 lines of 3 points each. Two lines with no point in common are called parallel. Two lines are perpendicular if one is a row and one is a column in the same block. The midpoint of a line segment is the third point on the line determined by the end points.

Consider the problem of defining a circle. Suppose for example a circle has FG for a diameter. From the second column of block II (Figure 5.3) it seems reasonable to expect that point B should be the center. From block I it seems that, if an "angle" inscribed in the semicircle with diameter FG is a right angle that D and I should also be on the circle. Thus it seems that the circle with FG as diameter should consist of the four points F, G, D, and I. Also if "radii" of the circle are equal it seems that segments BF, BG, BD, and BI should be equal. It is noted that these segments occur in a row and in a column of block II.
Now consider the circle with center $B$ passing through point $A$. From block I it seems that $C$ should be on this circle and from block II it appears that a circle with $AC$ as diameter should pass through $E$ and $H$. As before it seems that distances $BA$, $BC$, $BE$, $BH$ should be equal. Since two different circles with the same center should not have the same radii the following definition for distance in the 9-point geometry will make $BF = BG$, $BA = BC$, $BF \neq BA$.

Let $X$ and $Y$ represent any two distinct points in 9-point geometry. Then $XY = 1$ if $X$ and $Y$ are in a row or column in block I and $XY = 1'$ if $X$ and $Y$ are in a row or column in block II. The distances $1$ and $1'$ are not equal.

A circle is the set of points equally distant from a fixed point called the center of the circle. From this definition it follows that there are 18 circles in 9-point geometry.

If a polygon is defined as a set of $n$ distinct points no three collinear it is possible to find polygons in 9-point geometry. If a three-point polygon is called a triangle there are $\frac{9 \cdot 8 \cdot 6}{3!} = 72$ distinct triangles. The number of four-point polygons (quadrilaterals) is $\frac{9 \cdot 8 \cdot 6 \cdot 3}{4!} = 54$. There are no polygons with more than 4 sides since it is impossible to select five points without three of them being on a line.

Several examples of triangles should convince a reader that every triangle in 9-point geometry contains two
perpendicular lines. Each triangle will also have two equal sides. Thus in 9-point geometry all triangles are called right isosceles triangles.

Consider the medians of the particular triangle ACF. The three medians are lines AIE, CHD, FBG which are parallel.

This seems to indicate that medians of a triangle are not concurrent. (It may suggest a valid reason for adding ideal points to the line of 9 point geometry.) For triangle ACF the three altitudes obviously meet at point C.

The perpendicular bisectors of sides of triangle ACF are BEH, IGH, HCD which meet as expected at the midpoint H of the hypotenuse AF. Point H is the center of the circle through A, C, and F. The radius of this circle is $AH = 1'$ and from block II we see that the fourth point on the circle is D.

In Euclidean geometry a conic is determined by 5 points, no three collinear. It seems pointless to define a general
conic in 9-point geometry since no conic could contain more than 4 points.

5.3 Thirteen-point Geometry

To each line of 9-point geometry is added a fourth point, called the ideal point on the line, so that two parallel lines contain a common ideal point. The four ideal points are represented by letters W, X, Y, and Z and the arrangement of these points on lines of 9-point geometry is shown in Figure 5.4.

![Figure 5.4](image)

<table>
<thead>
<tr>
<th>W</th>
<th>W</th>
<th>W</th>
<th>Y</th>
<th>Y</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>D</td>
<td>C</td>
<td>X</td>
<td>A</td>
<td>F</td>
</tr>
<tr>
<td>D</td>
<td>E</td>
<td>F</td>
<td>X</td>
<td>E</td>
<td>G</td>
</tr>
<tr>
<td>G</td>
<td>H</td>
<td>I</td>
<td>X</td>
<td>I</td>
<td>B</td>
</tr>
</tbody>
</table>

**Block I**

**Block II**

Example 5.3-1

Show that the midpoints of consecutive sides of a quadrilateral are vertices of a parallelogram.

(A) Let AGIC be a given quadrilateral. Point D is the midpoint of AG, H is the midpoint of GI, F is the midpoint of IC, and B is the midpoint of CA.
Prove that DHFB is a parallelogram.

Proof: Points B and D are on the line IBD, third row in block II (Figure 5.3) and H and F are on the line AFH, first row in block II. There is no common point on these so BD is parallel to HF. Points D and H are on the line HCD, third column in block II and F and B are on the line FGB, second column in block II. There is no common point on these lines so DH is parallel to FB. This completes the proof that DHFB is a parallelogram.

If the given quadrilateral is intersecting and if the intersecting sides bisect each other the parallelogram formed by segments joining midpoints of consecutive sides degenerates into four lines segments on the same line. This case is illustrated by an example.

(B) AHDE is an intersecting quadrilateral. Point F is the midpoint of AH, C is the midpoint of HD, F is the midpoint of DE, and I is the midpoint of EA.
It is easy to show that three points F, C, and I are collinear since CFI is the third column in block I, Figure 5.1.

Example 5.3-2

How many squares are there in 9-point geometry?

In each of blocks I and II there are 3 ways for choosing two rows and 3 ways for choosing two columns so the four lines form a square. Therefore total number of squares is $2 \times 3 \times 3 = 18$. The remaining quadrilaterals are non-square in form. It will be shown that the number of non-square quadrilaterals is 36.

If one chooses A and B for the first two vertices, then C is eliminated. Choose E for the third point, H and I are eliminated. If D is chosen for the fourth vertex the quadrilateral will be a square. If F or G is chosen, the quadrilateral will
be non-square. Therefore the total number of squares is
\[ \frac{9 \times 8 \times 6 \times 1}{4!} = 18. \]
The total number of non-square quadrilaterals is
\[ \frac{9 \times 8 \times 6 \times 2}{4!} = 36. \]

Example 5.3-3

If lines joining corresponding vertices of two triangles are concurrent, the intersections of corresponding sides are collinear. (Theorem 1 on p. 128 in [9]. This theorem is illustrated in 13-point geometry.

Given triangles AHI and CED, with CA, DI, and EH meeting at B, it is to be shown that the intersections of corresponding sides AH and CE; IH and DE; CD and AI are collinear.

Proof:
Since line AFHZ and line EGCZ have common point Z therefore AH and CE meet at Z (block II, Figure 5.4).

Likewise AI and CD meet at Y (block II, Figure 5.4), and HI and ED meet at X (block I, Figure 5.4). Since Z, Y, and X are on the ideal line, they are collinear.

Example 5.3-4

Construct a circle passing through a given point and tangent to a given line. How many solutions exist? In Euclidean geometry one can find an infinite number of solutions. This example is considered in 9-point geometry.

Given line IBD and point C in block II, Figure 5.3. Construct a circle passing through C and tangent to the line IBD.

(A) Draw line HCD, which is perpendicular to IBD at D. Use H, the midpoint of CD as center and l' as radius draw a circle. Circle FCAD with center H and radius l' is tangent to the line IBD.

(B) Join points C and B. Find the perpendicular bisector of line segment CB which is ADG, since A is the midpoint of BC. Then at B draw the perpendicular to line IBD which is line FGB. The desired circle has center on both lines ADG and FGB. Since G is the common point of the two lines, G is the center of a solution circle and the radius is l'. The circle BEFC passes through C and is tangent to IBD at B.
Likewise one can find circle CAGI, with center E, radius 1', passing through C and tangent to the line IBD at Point I.

Thus, there are 3 solutions for this problem.

5.4 Forty-nine and Fifty-seven Point Geometries

Figure 5.5 contains four blocks each containing 49 ordinary points represented by symbols 1A, 1B, ... 7G and each block contains two of the eight ideal points represented by O5, OT, ... OZ. Details of the construction of these blocks are omitted but several of the procedures used are briefly discussed below.

When the eight ideal elements are not considered, the remaining elements shown in blocks I, II, III, and IV form a 49-point geometry. For the 49-point geometry, elements were arranged in a square array as shown in block I. This array was repeated both vertically and horizontally. The Euclidean line joining any two points in this expanded array passes through exactly seven distinct points and these points are said to be on the line determined by the original two points.

The total number of distinct lines is found by dividing the number of combinations of 49 points taken two at a time by the number of combinations of 7 points taken two at a time. This quotient is 56 and blocks I through IV have the 49 points arranged so that each of the 56 rows and columns represents one of these lines.
<table>
<thead>
<tr>
<th>Block I</th>
<th>Block II</th>
</tr>
</thead>
<tbody>
<tr>
<td>OS OS OS OS OS OS OS</td>
<td>OU OU OU OU OU OU OU</td>
</tr>
<tr>
<td>1A 1B 1C 1D 1E 1F 1G OT</td>
<td>1A 6C 4E 2G 7B 5D 3F OV</td>
</tr>
<tr>
<td>2A 2B 2C 2D 2E 2F 2G OT</td>
<td>3C 1E 6G 4B 2D 7F 5A OV</td>
</tr>
<tr>
<td>3A 3B 3C 3D 3E 3F 3G OT</td>
<td>5E 3G 1B 6D 4F 2A 7C OV</td>
</tr>
<tr>
<td>4A 4B 4C 4D 4E 4F 4G OT</td>
<td>7G 5B 3D 1F 6A 4C 2E OV</td>
</tr>
<tr>
<td>5A 5B 5C 5D 5E 5F 5G OT</td>
<td>2B 7D 5F 3A 1C 6E 4G OV</td>
</tr>
<tr>
<td>6A 6B 6C 6D 6E 6F 6G OT</td>
<td>4D 2F 7A 5C 3E 1G 6B OV</td>
</tr>
<tr>
<td>7A 7B 7C 7D 7E 7F 7G OT</td>
<td>6F 4A 2C 7E 5G 3B 1D OV</td>
</tr>
<tr>
<td><strong>Block III</strong></td>
<td><strong>Block IV</strong></td>
</tr>
<tr>
<td>OW OW OW OW OW OW OW</td>
<td>OY OY OY OY OY OY OY</td>
</tr>
<tr>
<td>1A 6B 4C 2D 7E 5F 3G OX</td>
<td>1A 7C 6E 5G 4B 3D 2F OZ</td>
</tr>
<tr>
<td>2C 7D 5E 3F 1G 6A 4B OX</td>
<td>3B 2D 1F 7A 6C 5E 4G OZ</td>
</tr>
<tr>
<td>3E 1F 6G 4A 2B 7C 5D OX</td>
<td>5C 4E 3G 2B 1D 7F 6A OZ</td>
</tr>
<tr>
<td>4G 2A 7B 5C 3D 1E 6F OX</td>
<td>7D 6F 5A 4C 3E 2G 1B OZ</td>
</tr>
<tr>
<td>5B 3C 1D 6E 4F 2G 7A OX</td>
<td>2B 1G 7B 6D 5F 4A 3C OZ</td>
</tr>
<tr>
<td>6D 4E 2F 7G 5A 3B 1C OX</td>
<td>4F 3A 2C 1E 7G 6B 5D OZ</td>
</tr>
<tr>
<td>7F 5G 3A 1B 6C 4D 2E OX</td>
<td>6G 5E 4D 3F 2A 1C 7E OZ</td>
</tr>
</tbody>
</table>
Blocks II, III, and IV were determined so that certain desirable patterns are present. For example, the columns in block II contain elements in diagonals from upper left to lower right in block I while the rows in block II are the diagonals of block I from upper right to lower left. The fact that a row and a column in block II form perpendicular lines then has a counterpart in block I.

Note that the first column in block III contains the points on the line through 1A and 2C in the repeated array based on block I. The other columns in block III contain all lines parallel to the line 1A2C. The first row in block III contains points 1A and 6B. In the repeated array based on block I it is possible to choose points 1A, 2C, and 6B so that the Euclidean line through 1A and 2C is perpendicular to the line through 1A and 6B. The other rows in block III contain all lines parallel to 1A6B. The columns in block IV contain the points on the diagonals from upper left to lower right of block III.

The midpoint of a segment can be determined from the repeated array based on block I. The individual elements are arranged in blocks II, III, and IV so that midpoints are preserved. One other problem considered in arranging elements in blocks I, II, III, and IV was that of defining distances in a useful way. When distances are defined as follows, the theorem of Pythagoras is valid for all right triangles.
If $P_1$ and $P_2$ are two distinct points in a row or column of blocks I or II, the distance $P_1P_2 = 1$, 2, or 3 depending on the least number of steps in the row or column from $P_1$ to $P_2$ where the seven points in the line are arranged in cyclic order. The distance between two points in a line found in block III or IV is $1'$, $2'$, or $3'$ if the least number of steps between the points is 1, 2, or 3.

If the theorem of Pythagoras is to be valid, relations connecting 1, 2, and 3 with $1'$, $2'$, and $3'$ can be determined. For example, consider the right triangle $1A1B6B$. Here $1A1B = 1$ and $1B6B = 2$ are sides and $1A6B = 1'$ is the hypotenuse.

If $(1A1B)^2 + (1B6B)^2 = (1A6B)^2$ then $1 + 4 = (1')^2$ or $1' = \sqrt{5}$.

Thinking of $1' = \sqrt{5}$, $2' = 2\sqrt{5}$, $3' = 3\sqrt{5}$ and using modulo seven arithmetic, one obtains these relations: $1^2 = 1$, $2^2 = 4$, $3^2 = 2$, $1'^2 = 5$, $2'^2 = 6$, $3'^2 = 3$.

Example 5.4-1

Consider right triangle $3G4A3B$ with sides $3G4A = 3$, $4A3B = 3$ and hypotenuse $3G3B = 2$; then

$3^2 + 3^2 = 9 + 9 = 4 = 2^2$.

Example 5.4-2

Consider right triangle $1A4G1E$ with sides $1A4G = 3'$, $4G1E = 2'$ and hypotenuse $1A1E = 3$. Then

$3'^2 + 2'^2 = 3 + 6 = 2 = 3^2$. 
With the given definition for distances, the set of eight points the same distance from a given point is called a circle. In earlier discussions it was noted that the fact that a right angle inscribed in a semicircle has its vertex on the circle can be used to justify a definition of distance. One example illustrating the consistency of the above definition will be given.

Example 5.4-3

What additional points are on the circle with 3G and 6B as end points of a diameter? From block I it is seen that 3B and 6G determine a rectangle with 3G6B as a diagonal and therefore 3B and 6G are on the circle. Likewise from block II, points 7C and 2F are on this circle. Since 3G and 6B are in the first row of block III and 1A is the midpoint, 1A is expected to be the midpoint of the circle. From the proper rectangle in block IV, 2C, and 7F are found to be points on the circle. It is found that each of the eight segments joining the center 1A to the points on the circle represent steps of one unit in blocks III or IV so the radius of this circle is 1'.

In 49-point geometry, polygons are all defined as sets of n points no three on a line. The eight points on a circle form an octagon. There are no polygons with more than eight sides. The problem of determining the number of polygons with three to eight sides will not be considered further.

Most of the theorems which were valid for 25-point geometry
are also valid for 49-point geometry. Since there is a wider variety of triangles in 49-point geometry, it may be of more interest to verify a theorem about a triangle in this system rather than in the 25-point geometry.

In advanced Euclidean geometry a well known theorem involves the so-called nine-point circle. Since a circle in 25-point geometry contains only six points and a circle in 49-point geometry contains only eight points it may seem that this theorem would apply only in a finite geometry with more than 49-points, but it may also be of interest to see how this theorem differs in 25-point and 49-point geometry.

Example 5.4-4

In Euclidean geometry for any triangle the midpoints of the sides, the feet of the altitudes and the midpoints of the segments joining the orthocenter to the vertices lie on a circle.

In 25-point geometry, these nine points are found for triangle ART.
The midpoints of sides are V, S, and W as shown. The feet of altitudes are V, P, and L. Lines TV, RL, and AP contain the common point F, which is the orthocenter of ART.

Three midpoints from F to the vertices are: P, L, and M. These nine points V, S, W, V, P, L, P, L, and M give only six distinct points since points V, P, and L occur twice. Thus the "nine-point" theorem is valid for this triangle in 25-point geometry since these points are on the circle with center T and radius 1.

In 49-point geometry for triangle 1C7E6F the corresponding nine points are found below.

The three midpoints of the sides are 4D, 3B, and 7A. The feet of altitudes are 5B, 5G, and 3G. The three altitudes 7E3G, 105G, and 6F5B have a common point 2D, which is the orthocenter of Δ1C7E6F.
The midpoints of segments joining the orthocenter to the vertices are 5G, 1A, and 4E. The point 5G occurs twice in the set of nine points and the "nine-point" circle passes through the eight distinct points in this set since each is on the circle with center 4A and radius 3.

In 31-point geometry every non-degenerate conic is a set of six points no three collinear. Since circles in 49-point geometry contain eight points it seems that in 57-point geometry non-degenerate conics should contain eight points. For a non-degenerate conic no three of these eight points should be collinear. It is not evident that these conditions are sufficient to determine a conic, but in the next example five points no three on a line are given and all sets of eight points which include these five and satisfy the property that no three are collinear will be found.

Example 5.4-5

The five points 1A, 1B, 2B, 2C, and 3A satisfy the property that no three are on a line. To choose a sixth point which satisfies this condition one can eliminate all points on a line through two of these five points. When this elimination has been performed only the seven points 3F, 3G, 4C, 5C, 7C, 7E, and OY remain eligible for choice as the sixth point. The results of using each of these seven points as a sixth point satisfying the above condition are summarized below.
<table>
<thead>
<tr>
<th>First Five Points</th>
<th>Sixth Point</th>
<th>Additional Points so no three are collinear</th>
</tr>
</thead>
<tbody>
<tr>
<td>1A, 1B, 2B, 2C, 3A</td>
<td>3F</td>
<td>7C, 7E</td>
</tr>
<tr>
<td>1A, 1B, 2B, 2C, 3A</td>
<td>3G</td>
<td>None</td>
</tr>
<tr>
<td>1A, 1B, 2B, 2C, 3A</td>
<td>4C</td>
<td>None</td>
</tr>
<tr>
<td>1A, 1B, 2B, 2C, 3A</td>
<td>5C</td>
<td>None</td>
</tr>
<tr>
<td>1A, 1B, 2B, 2C, 3A</td>
<td>7C</td>
<td>3F, 7E</td>
</tr>
<tr>
<td>1A, 1B, 2B, 2C, 3A</td>
<td>7E</td>
<td>3F, 7C</td>
</tr>
<tr>
<td>1A, 1B, 2B, 2C, 3A</td>
<td>0Y</td>
<td>None</td>
</tr>
</tbody>
</table>

For the given five points 1A, 1B, 2B, 2C, and 3A there is only one set \{3F, 7C, 7E\} of three additional points so that the eight points satisfy the condition that no three are collinear. This suggests the possibility that the following definition may be appropriate.

In 57-point geometry a non-degenerate conic is a set of eight points no three collinear. The conic is called an ellipse, parabola, or hyperbola according as the number of ideal points on the conic is none, one, or two, respectively.

5.5 Coordinate Systems

Appropriate homogeneous coordinates for points in 13-point geometry and in 57-point geometry are shown in Figures 5.5 and 5.6. For 9 and 49-point geometries corresponding rectangular coordinates are obtained by using the first two given coordinates for each ordinary point.
**Figure 5.5**

<table>
<thead>
<tr>
<th>Ideal Points</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>A(0,2,1) B(1,2,1) C(2,2,1)</td>
<td>W(0,1,0) is on AD</td>
</tr>
<tr>
<td>D(0,1,1) E(1,1,1) F(2,1,1)</td>
<td>K(1,0,0) is on AB</td>
</tr>
<tr>
<td>G(0,0,1) H(1,0,1) I(2,0,1)</td>
<td>Y(1,2,0) is on AE</td>
</tr>
<tr>
<td></td>
<td>Z(1,1,0) is on AF</td>
</tr>
</tbody>
</table>

**Figure 5.6**

<table>
<thead>
<tr>
<th>Ideal Points</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1A(0,6,1) 1B(1,6,1) 1C(2,6,1) 1D(3,6,1) 1E(4,6,1) 1F(5,6,1) 1G(6,6,1)</td>
<td></td>
</tr>
<tr>
<td>2A(0,5,1) 2B(1,5,1) 2C(2,5,1) 2D(3,5,1) 2E(4,5,1) 2F(5,5,1) 2G(6,5,1)</td>
<td></td>
</tr>
<tr>
<td>3A(0,4,1) 3B(1,4,1) 3C(2,4,1) 3D(3,4,1) 3E(4,4,1) 3F(5,4,1) 3G(6,4,1)</td>
<td></td>
</tr>
<tr>
<td>4A(0,3,1) 4B(1,3,1) 4C(2,3,1) 4D(3,3,1) 4E(4,3,1) 4F(5,3,1) 4G(6,3,1)</td>
<td></td>
</tr>
<tr>
<td>5A(0,2,1) 5B(1,2,1) 5C(2,2,1) 5D(3,2,1) 5E(4,2,1) 5F(5,2,1) 5G(6,2,1)</td>
<td></td>
</tr>
<tr>
<td>6A(0,1,1) 6B(1,1,1) 6C(2,1,1) 6D(3,1,1) 6E(4,1,1) 6F(5,1,1) 6G(6,1,1)</td>
<td></td>
</tr>
<tr>
<td>7A(0,0,1) 7B(1,0,1) 7C(2,0,1) 7D(3,0,1) 7E(4,0,1) 7F(5,0,1) 7G(6,0,1)</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Ideal Points</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>OS(0,1,0) is on 1A2A</td>
<td></td>
</tr>
<tr>
<td>OT(1,0,0) is on 1A1B</td>
<td></td>
</tr>
<tr>
<td>OU(1,6,0) is on 1A3C</td>
<td></td>
</tr>
<tr>
<td>OV(1,1,0) is on 1A6C</td>
<td></td>
</tr>
<tr>
<td>OW(1,3,0) is on 1A2C</td>
<td></td>
</tr>
<tr>
<td>OX(1,2,0) is on 1A6B</td>
<td></td>
</tr>
<tr>
<td>OY(1,5,0) is on 1A3B</td>
<td></td>
</tr>
<tr>
<td>OZ(1,4,0) is on 1A7C</td>
<td></td>
</tr>
</tbody>
</table>
The equation of the line through points \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\) can be written in the form:

\[
\begin{vmatrix}
    x & y & z \\
    x_1 & y_1 & z_1 \\
    x_2 & y_2 & z_2
\end{vmatrix} = 0.
\]

This can be used to find the equations of the thirteen lines in 13-point geometry. The lines and corresponding equations are listed below.

<table>
<thead>
<tr>
<th>Line</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>GDAW</td>
<td>(x = 0)</td>
</tr>
<tr>
<td>HEBW</td>
<td>(x + 2z = 0)</td>
</tr>
<tr>
<td>IFCW</td>
<td>(x + z = 0)</td>
</tr>
<tr>
<td>ABCX</td>
<td>(y + z = 0)</td>
</tr>
<tr>
<td>DEFX</td>
<td>(y + 2z = 0)</td>
</tr>
<tr>
<td>GHIX</td>
<td>(y = 0)</td>
</tr>
<tr>
<td>AEIY</td>
<td>(x + y + z = 0)</td>
</tr>
<tr>
<td>BFGY</td>
<td>(x + y = 0)</td>
</tr>
<tr>
<td>CDHY</td>
<td>(2x + 2y + z = 0)</td>
</tr>
<tr>
<td>AFHZ</td>
<td>(2x + y + z = 0)</td>
</tr>
<tr>
<td>BDIZ</td>
<td>(x + 2y + z = 0)</td>
</tr>
<tr>
<td>CEGZ</td>
<td>(x + 2y = 0)</td>
</tr>
<tr>
<td>WXYZ</td>
<td>(z = 0)</td>
</tr>
</tbody>
</table>

The line \(ax + by + cz = 0\) can be represented by the line coordinates \([a, b, c]\). For 13-point geometry modulo three arithmetic is used so that \([a, b, c] = [2a, 2b, 2c]\). The point \((x, y, z)\) and the line \([a, b, c]\) are incident if and only if \(ax + by + cz = 0 \pmod{3}\).
Any two ordinary lines \([a_1, b_1, c_1]\) and \([a_2, b_2, c_2]\)
are parallel if and only if \(a_1b_2 = b_1a_2\) or if
\[a_1b_2 + 2b_1a_2 = 0 \text{ (mod 3)}\]. Any two ordinary lines \([a_1, b_1, c_1]\)
and \([a_2, b_2, c_2]\) are perpendicular if and only if
\[a_1a_2 + b_1b_2 = 0 \text{ (mod 3)}\].

Example 5.5-1

Lines CD and BF are clearly parallel since both are
columns in block II, Figure 5.3. Coordinates of those two
lines are \([2, 2, 1]\) and \([1, 1, 0]\) so that
\[a_1b_2 + 2b_1a_2 = 2 + 1 = 0 \text{ (mod 3)}\].

Lines AE and BD with coordinates \([1, 1, 1]\) and \([1, 2, 1]\)
are perpendicular and \(a_1a_2 + b_1b_2 = 1 + 2 = 0 \text{ (mod 3)}\).

The above discussion of point and line coordinates can
be applied to 57-point geometry. Details are omitted but an
example showing that two perpendicular lines of 57-point
gometry satisfy similar conditions will be given.

Example 5.5-2

In Figure 5.4 lines 1A6B and 1A2C are found in the first
row and column of block IV. Points 1A, 2C, and 6B have co-
ordinates 1A(0, 6, 1), 2C(2, 5, 1), and 6B(1, 1, 1) so that
equation of 1A6B is

\[
\begin{vmatrix}
  x & y & z \\
  0 & 6 & 1 \\
  1 & 1 & 1
\end{vmatrix} = 6x + y + z + 6x = 5x + y + z = 0 \text{ (mod 7)}.
\]
The equation of 1A2C is \[ \begin{vmatrix} x & y & z \\ 0 & 1 & 1 \\ 2 & 5 & 1 \end{vmatrix} = x + 2y + 2z = 0 \text{ (mod 7)}. \]

Lines \([a_1, b_1, c_1]\) and \([a_2, b_2, c_2]\) in 57-point geometry are perpendicular if \(a_1a_2 + b_1b_2 = 0 \text{ (mod 7)}\). In this example \(a_1a_2 + b_1b_2 = 5 + 2 = 0\).

Example 5.5-3

Does the equation \((x - 1)^2 + (y - 1)^2 = 1\) represent a circle in 9-point geometry? If so, how many points are on this circle with center \(E(1, 1)\) and radius 1?

One can substitute the coordinates of each point into the above equation. For \(A(0, 2)\): \(1^2 + 1^2 = 1\), so this equation is not satisfied by point A. For \(B(1, 2)\): \(0^2 + 1^2 = 1\), therefore it is satisfied by point B. Following this process one can find that there are four points which satisfy this equation: the points are \(B(1, 2)\), \(D(0, 1)\), \(F(2, 1)\) and \(H(1, 0)\).

Example 5.5-4

Is \(Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0\) an equation of a conic in 57-point geometry? Can the coefficients \(A\) through \(F\) be determined so that this equation represents the conic discussed in Example 5.4-5?

By substituting coordinates of the five points
1A(0, 6), 1B(1, 6), 2B(1, 5), 2C(2, 5), 3A(0, 4) into the above equation, one obtains the equations which follow:

For 1A(0, 6); \[36c + 6E + F = 0\] (1)

or \[C + 6E + F = 0 \pmod{7}\] (mod 7)

For 1B(1, 6): \[A + 6B + 36c + D + 6E + F = 0\] (2)

or \[A + 6B + C + D + 6E + F = 0\]

For 2B(1, 5): \[A + 5B + 25c + D + 5E + F = 0\] (3)

or \[A + 5B + 4c + D + 5E + F = 0\]

For 2C(2, 5): \[4A + 10B + 25c + 2D + 5E + F = 0\] (4)

or \[4A + 3B + 4c + 2D + 5E + F = 0\]

For 3A(0, 4): \[16c + 4E + F = 0\] or \[2C + 4E + F = 0\] (5)

To solve this system of equations one can start by choosing \(F = 1\). (If the resulting system is inconsistent start again by letting some other coefficient be 1.)

\[C + 6E + 1 = 0\] (A)

\[A + 6B + C + D + 6E + 1 = 0\] (B)

\[A + 5B + 4c + D + 5E + 1 = 0\] (C)

\[4A + 3B + 4c + 2D + 5E + 1 = 0\] (D)

and

\[2C + 4E + 1 = 0\] (E)
Repeated elimination of variables leads to the solution set
A = 1, B = 2, C = 5, D = 1, and E = 6. Is the resulting equa-
tion \( x^2 + 2xy + 5y^2 + x + 6y + 1 = 0 \) satisfied by the coordin­
ates of the remaining three points 7C(2, 0), 7E(4, 0) and
3F(5, 4)?

For 7C(2, 0): \( 4 + 2 + 1 = 0 \)

For 7E(4, 0): \( 2 + 4 + 1 = 0 \)

For 3F(5, 4): \( (5)^2 + 2(5)(4) + 5(4^2) + (5) + 6(4) + 1 = 4 + 5 + 3 + 5 + 3 + 1 = 21 = 0 \text{ (mod 7)} \)

Therefore 7C, 7E, and 3F also satisfy the equation. Since one
knows that the above eight points are on a conic, the equa-
tion \( x^2 + 2xy + 5y^2 + x + 6y + 1 = 0 \) is the equation of a conic.

The above example suggests that any conic in 57-point
geometry can be represented in homogeneous coordinates by an
equation of the form \( ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0 \)
(mod 7) with coefficients in the set \( \{0, 1, 2, 3, 4, 5, 6\} \).

If an equation of this form is given it may represent a
degenerate conic since, for example, it could represent two
intersecting lines. However for any given equation of this form
it is relatively easy to find all points which satisfy the
equation. If the equation represents two parallel lines this
will be obvious from the solution set. If the equation
represents a non-degenerate conic the solution set will contain
exactly eight points, no three collinear, and the number of ideal points in this set will show if the conic is an ellipse, parabola, or hyperbola.

There are many definitions and numerous approaches to geometry. In 1872, Felix Klein gave this famous definition of geometry: "A geometry is the study of those properties of a set S which remain invariant when the elements of S are subjected to the transformations of some transformation group."

The present study does not stress group properties but Chapter VI considers several transformations associated with each of the various finite geometries which have been introduced. Transformations to be studied include distance preserving, angle preserving, and line-preserving transformations (each of which forms a group under a suitably defined product). The inversion transformation is introduced as an example of a transformation in which the image of a line need not be a line. (In an inversion the image of a line is a line or a circle and the image of a circle is a circle or a line.)
CHAPTER VI

TRANSFORMATIONS

6.1 Introduction

There are differences of opinion about the proper subject matter content of geometry courses both at the high school and college level. One viewpoint is that properties of triangles and circles are of relatively little interest and importance while properties of transformations and convex sets are more interesting and of more use in other mathematics courses. A set is called convex if all points on the segment joining any two points of the set are also in the set. Unless some criterion is developed to tell what points are interior points of a segment, convex sets seem to have no application in finite geometry. The related concept of a convex polygon, which was discussed in Chapter III for 25-point geometry, can be extended to other finite geometries.

The present chapter is primarily devoted to a study of transformations in finite geometries. A geometric transformation in a plane is a correspondence which assigns to each point in the plane a related point of the plane. An important property of a transformation in a finite geometry is that it is possible to
define a transformation by listing all points and corresponding image points.

In a study of a transformation it is usual to attempt to find properties which are invariant in the transformation. For example in many transformations, lengths of line segments and magnitudes of angles are invariant. If a transformation is defined by directions for finding the image of every point, and a list of all points and corresponding images is prepared, it is easy to make conjectures about images of lines, circles, and conics by considering the set of points which are images of points on these curves.

The number of possible transformations in a finite geometry must be finite but even for a geometry with only nine points the number of one-to-one transformations is $9!$. A transformation chosen at random from this set is not likely to have many interesting properties. If a transformation is chosen according to some pattern the transformation may contain unknown properties and the discovery of such properties can increase a student's interest and ability in mathematics.

In Section 6.2 certain transformations using 9 or 13-point geometry will be carried out in some detail to illustrate methods which can be used in geometries with more points but which become so tedious that they are not likely to be attempted unless a high speed computer is available.

Transformations usually considered in elementary geometry include translation, rotation, line reflection, and point
reflection. For 13-point geometry it will be shown that point reflections, translations, and rotations can be carried out by successive line reflections. In particular it will be shown that any given triangle can be carried into any triangle congruent to the given one by no more than three successive line reflections. This might motivate a student to attempt to prove the corresponding result for Euclidean geometry. Another transformation of elementary geometry involves a change of scale in which the image of a figure is similar to but not congruent to the original. This transformation will be considered for finite geometries.

At a more advanced level a student may be introduced to the class of transformations in which lines and circles become lines and circles, but the image of a given line may be either a line or a circle. The inversion transformation, which has this interesting property, will be discussed. The introduction of inversion for a finite geometry makes it easy for a student to discover theorems about images of lines and circles and to make conjectures about the transformation in Euclidean geometry. An interested student can manufacture his own transformations. Several examples of unusual transformations will be given in Chapter VII.

6.2 Distance Preserving Transformations in 13-Point Geometry

In Euclidean geometry, line reflections, point reflections, translations, and rotations turn a line segment into a segment having the same length as the original. For such a transformation
the image of a triangle is a triangle congruent to the given one. This means that angles between lines are also invariant. (In a line reflection the order of angles in a triangle is reversed.)

Figure 6.1 shows the arrangement of points in blocks I and II for the 13-point geometry discussed in Section 5.3. Each of the twelve ordinary lines is represented by letters (a) through (l) as shown.

Figure 6.1

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th>(j)</th>
<th>(k)</th>
<th>(l)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
</tr>
<tr>
<td>(a)</td>
<td>A</td>
<td>B</td>
<td>C</td>
<td>X</td>
<td>(g)</td>
</tr>
<tr>
<td>(b)</td>
<td>D</td>
<td>E</td>
<td>F</td>
<td>X</td>
<td>(h)</td>
</tr>
<tr>
<td>(c)</td>
<td>G</td>
<td>H</td>
<td>I</td>
<td>X</td>
<td>(i)</td>
</tr>
</tbody>
</table>

Let $n$ be any ordinary line and $P$ be any ordinary point of 13-point geometry. The point $P'$ is the image of $P$ in a reflection in line $n$ if and only if the perpendicular bisector of $PP'$ is line $n$. If $Q$ is an ideal point and $n$ is an ordinary line, the image of $Q$ may be found as follows. Let $P'_1$ and $P'_2$ be the images of $P_1$ and $P_2$ where $P_1$ and $P_2$ are any ordinary points on a line through $Q$. The ideal point, $Q'$, on line $P'_1P'_2$ is the image of $Q$.

By the above definition any point on line $n$ is invariant in
the reflection in line \( n \). Also the ideal point on lines perpendicular to \( n \) is invariant. The definition makes it possible to list the images of all points in reflections in any of the twelve ordinary lines. Reflection in the ideal line is not defined although for completeness it could be defined as the transformation in which all points are invariant (the identity transformation).

Example 6.2-1

Reflection in line (a).

Point \( A B C D E F G H I W X Y Z \)

Image \( A B C G H I D E F W X Z Y \)

It is easy to verify that in any line reflection the line of reflection, all lines perpendicular to the line of reflection, and the ideal line are invariant lines; the image of any other line is a line and the image of a circle is a circle with the same radius as the original.

For reference purposes the images of all points in 13-point geometry in reflections in the twelve lines are listed in Figure 6.2.

The twelve reflections shown in Figure 6.2 are numbered for future reference. The notation \( L_j \) adjacent to the number 10 shows that the elements in this row were obtained from corresponding points at the top of the table by reflection in the line labeled (\( j \)) in Figure 6.1. It is noted that in a line reflection the two ideal points in the block containing the line of
reflection remain invariant while the other two ideal points are interchanged.

It is a simple task to find images of all 13 lines of 13-point geometry in any of the twelve line reflections. The image of each line is a line. In a given line reflection there are five invariant lines: all points on the line of reflection are invariant, two points on the ideal line are invariant, and there are two invariant points on each of the three lines perpendicular to the line of reflection. For the other eight, six give image lines intersecting the original and two give lines parallel to the
original line. In any line reflection the length of a line segment is unchanged. The reflected image of a circle is a circle with radius unchanged. A circle is invariant if its center is on the line of reflection. Some of these statements are illustrated in the following example.

Example 6.2-2

Apply reflection $L_h$ (Figure 6.2).

<table>
<thead>
<tr>
<th>Point Set</th>
<th>Image Set</th>
</tr>
</thead>
<tbody>
<tr>
<td>Line</td>
<td>A B I Y</td>
</tr>
<tr>
<td></td>
<td>I E A Y (same line)</td>
</tr>
<tr>
<td>Line</td>
<td>W X Y Z</td>
</tr>
<tr>
<td></td>
<td>X W Y Z (same line)</td>
</tr>
<tr>
<td>Line</td>
<td>G H I X</td>
</tr>
<tr>
<td></td>
<td>G D A W (perpendicular line)</td>
</tr>
<tr>
<td>Line</td>
<td>I B D Z</td>
</tr>
<tr>
<td></td>
<td>A F H Z (parallel line)</td>
</tr>
<tr>
<td>Circle</td>
<td>B I A F</td>
</tr>
<tr>
<td></td>
<td>F A I B (same circle)</td>
</tr>
<tr>
<td>Circle</td>
<td>A B D I</td>
</tr>
<tr>
<td></td>
<td>I F H E (circle)</td>
</tr>
<tr>
<td>Triangle</td>
<td>F D I</td>
</tr>
<tr>
<td></td>
<td>B H A (congruent triangle)</td>
</tr>
</tbody>
</table>

Transformations can be generated by a succession of individual transformations. The notation $T = T_1 \cdot T_2 = T_1 T_2$ is often used to indicate that $T$ is the transformation obtained by first applying transformation $T_2$ to obtain image points and then subjecting these image points to the transformation $T_1$. For example using Figure 6.2 and the transformation $T = L_b \cdot L_e$, point A becomes C in the reflection $L_e$ and C becomes I in the reflection $L_b$ so in
the transformation \( L_b \cdot L_e \) the image of \( A \) is \( I \). The complete transformation \( L_b \cdot L_e \) is shown in the following table.

\[
T = L_b \cdot L_e
\]

<table>
<thead>
<tr>
<th>Point</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>H</th>
<th>I</th>
<th>W</th>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>Image</td>
<td>I</td>
<td>H</td>
<td>G</td>
<td>F</td>
<td>E</td>
<td>D</td>
<td>C</td>
<td>B</td>
<td>A</td>
<td>W</td>
<td>X</td>
<td>Y</td>
<td>Z</td>
</tr>
</tbody>
</table>

The "product" transformation \( T = L_b \cdot L_e \) is not one of the original twelve line reflections. The product of any two of the twelve line reflections will be a transformation but the same product transformation can be obtained in more than one way. The next table shows the transformation \( L_h \cdot L_j \).

\[
T = L_h \cdot L_j
\]

<table>
<thead>
<tr>
<th>Point</th>
<th>A</th>
<th>D</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>H</th>
<th>I</th>
<th>W</th>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>Image</td>
<td>I</td>
<td>H</td>
<td>G</td>
<td>F</td>
<td>E</td>
<td>D</td>
<td>C</td>
<td>B</td>
<td>A</td>
<td>W</td>
<td>X</td>
<td>Y</td>
<td>Z</td>
</tr>
</tbody>
</table>

Since the images of all 13 points in the transformation \( L_b \cdot L_e \) coincide with the corresponding images in the transformation \( L_h \cdot L_j \), these transformations are called equivalent or equal. In finding the product of line reflections of Figure 6.2, two reflections can be chosen in 144 different orders. In the following discussion all different transformations obtained by this method will be listed.

Note that reflection in a line followed by reflection in the same line produce the identity transformation in which each point coincides with its image. Each product of two line
reflections has at least one invariant point and the set of all such product transformations can be classified into subsets having the same number of invariant points.

The identity transformation is the only transformation in 13-point geometry with 13 invariant points. In the transformation \( T = L_x L_y \), where \( x \) and \( y \) represent perpendicular lines (Figure 6.1), there are five invariant points; the four ideal points and one ordinary point. For example, the images of points A through Z, in order, in the product \( L_k L_i \) are: CDA, IHG, FED, WXYZ where B, W, X, Y, and Z are invariant. Transformations with exactly five invariant points are called point reflections. The transformation \( L_k L_i \) is the point reflection in point B.

The product of two line reflections in parallel lines gives a transformation with no invariant ordinary point but with four invariant ideal points. For example, \( L_a L_b \) gives the ordered images

\[
\text{DEF GHI ABC WXYZ}
\]

If the product of two line reflections contains no invariant ordinary point the resulting transformation is called a translation. As a special case the identity transformation is considered a translation.

The product of two reflections when the lines of reflection are neither parallel nor perpendicular is a transformation with one invariant ordinary point and no invariant ideal points. Such
a transformation is called a rotation in the invariant ordinary point. Both point reflections and the identity transformation are considered as special rotations. An example of a rotation is \( LkLf \) which gives ordered images

\[
\begin{array}{cccc}
BII & CIF & AGD & XWZY \\
\end{array}
\]

Figures 6.3 and 6.4 show all product transformations which can be obtained by successive reflections in two of the lines of Figure 6.2.

The identity transformation assigned number 13 and represented by the notation \( I(0) \), appears both as a translation in Figure 6.3 and as a rotation in Figure 6.4. In Figure 6.3 the transformation numbered 17 is represented by the symbol \( T(E) \).
Figure 6.4

Rotations

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>I(0)</td>
<td>ABC</td>
<td>DEF</td>
<td>GHI</td>
<td>WXYZ</td>
</tr>
<tr>
<td>22</td>
<td>R(A1)</td>
<td>AGD</td>
<td>BHE</td>
<td>CIF</td>
<td>XWZY</td>
</tr>
<tr>
<td>23</td>
<td>R(A2)</td>
<td>ACB</td>
<td>GHI</td>
<td>DFE</td>
<td>WXYZ</td>
</tr>
<tr>
<td>24</td>
<td>R(A3)</td>
<td>ADG</td>
<td>CFI</td>
<td>BEH</td>
<td>XWZY</td>
</tr>
<tr>
<td>25</td>
<td>R(B1)</td>
<td>EBH</td>
<td>FCI</td>
<td>DAG</td>
<td>XWZY</td>
</tr>
<tr>
<td>26</td>
<td>R(B2)</td>
<td>CBA</td>
<td>IHG</td>
<td>FED</td>
<td>WXYZ</td>
</tr>
<tr>
<td>27</td>
<td>R(B3)</td>
<td>HBE</td>
<td>GAD</td>
<td>ICF</td>
<td>XWZY</td>
</tr>
<tr>
<td>28</td>
<td>R(C1)</td>
<td>IFC</td>
<td>GDA</td>
<td>HEB</td>
<td>XWZY</td>
</tr>
<tr>
<td>29</td>
<td>R(C2)</td>
<td>BAC</td>
<td>HGI</td>
<td>EDF</td>
<td>WXYZ</td>
</tr>
<tr>
<td>30</td>
<td>R(C3)</td>
<td>FIC</td>
<td>EHB</td>
<td>DGA</td>
<td>XWZY</td>
</tr>
<tr>
<td>31</td>
<td>R(D1)</td>
<td>FCI</td>
<td>DAG</td>
<td>EBH</td>
<td>XWZY</td>
</tr>
<tr>
<td>32</td>
<td>R(D2)</td>
<td>GIH</td>
<td>DFE</td>
<td>ACB</td>
<td>WXYZ</td>
</tr>
<tr>
<td>33</td>
<td>R(D3)</td>
<td>EHB</td>
<td>DGA</td>
<td>FIC</td>
<td>XWZY</td>
</tr>
<tr>
<td>34</td>
<td>R(E1)</td>
<td>GDA</td>
<td>HEB</td>
<td>IFC</td>
<td>XWZY</td>
</tr>
<tr>
<td>35</td>
<td>R(E2)</td>
<td>IHG</td>
<td>FED</td>
<td>CBA</td>
<td>WXYZ</td>
</tr>
<tr>
<td>36</td>
<td>R(E3)</td>
<td>CFI</td>
<td>BEH</td>
<td>ADG</td>
<td>XWZY</td>
</tr>
<tr>
<td>37</td>
<td>R(F1)</td>
<td>BHE</td>
<td>CIF</td>
<td>AGD</td>
<td>XWZY</td>
</tr>
<tr>
<td>38</td>
<td>R(F2)</td>
<td>HGI</td>
<td>EDF</td>
<td>BAC</td>
<td>WXYZ</td>
</tr>
<tr>
<td>39</td>
<td>R(F3)</td>
<td>GAD</td>
<td>ICF</td>
<td>HBE</td>
<td>XWZY</td>
</tr>
<tr>
<td>40</td>
<td>R(G1)</td>
<td>HEB</td>
<td>IFC</td>
<td>GDA</td>
<td>XWZY</td>
</tr>
<tr>
<td>41</td>
<td>R(G2)</td>
<td>DFE</td>
<td>ACB</td>
<td>GHI</td>
<td>WXYZ</td>
</tr>
<tr>
<td>42</td>
<td>R(G3)</td>
<td>ICF</td>
<td>EBE</td>
<td>GAD</td>
<td>XWZY</td>
</tr>
<tr>
<td>43</td>
<td>R(H1)</td>
<td>CIF</td>
<td>AGD</td>
<td>BHE</td>
<td>XWZY</td>
</tr>
<tr>
<td>44</td>
<td>R(H2)</td>
<td>FED</td>
<td>CBA</td>
<td>IHG</td>
<td>WXYZ</td>
</tr>
<tr>
<td>45</td>
<td>R(H3)</td>
<td>DGA</td>
<td>FIC</td>
<td>EHB</td>
<td>XWZY</td>
</tr>
<tr>
<td>46</td>
<td>R(I1)</td>
<td>DAG</td>
<td>EBH</td>
<td>FCI</td>
<td>XWZY</td>
</tr>
<tr>
<td>47</td>
<td>R(I2)</td>
<td>EDF</td>
<td>BAC</td>
<td>HGI</td>
<td>WXYZ</td>
</tr>
<tr>
<td>48</td>
<td>R(I3)</td>
<td>BEH</td>
<td>ADG</td>
<td>CFI</td>
<td>XWZY</td>
</tr>
</tbody>
</table>
which shows that this translation carries point A into point E.

Example 6.2-3

Using Figure 6.4 one can easily verify the equalities

\[
\begin{align*}
R(D_1) &= R(D_1) \\
R(D_1) \cdot R(D_1) &= [R(D_1)]^2 = R(D_2) \\
[R(D_1)]^3 &= R(D_1) \cdot R(D_2) = R(D_3) \\
[R(D_1)]^4 &= R(D_1) \cdot R(D_3) = [R(D_2)]^2 = I(0)
\end{align*}
\]

The above example suggests that the transformation \(R(D_1)\) represents a rotation of \(90^\circ\) in \(D\) while \(R(D_2)\) and \(R(D_3)\) represent rotations of \(180^\circ\) and \(270^\circ\) in \(D\). In general the notation \(R(P_q)\) represents a rotation in point \(P\) through \(90^\circ\), \(180^\circ\), or \(270^\circ\) for \(q = 1, 2, \text{ or } 3\).

Example 6.2-3 suggests that the four rotations (including the identity) about a point form a group in the product transformation. Consider the rotations \(I(0), R(A_1), R(A_2), R(A_3)\). This set is closed under "multiplication". It contains the identity element \(I(0)\). Each element has an inverse element. For example, the multiplicative inverse of \(R(A_1)\) is \(R(A_3)\) since \(R(A_1)R(A_3) = I(0)\) and \(R(A_3)R(A_1) = I(0)\). That the associative property is valid can be shown by trying all cases. One illustration follows.

\[
\begin{align*}
R(A_1)[R(A_3)R(A_2)] &= R(A_1)R(A_1) = R(A_2) \\
[R(A_1)R(A_3)]R(A_2) &= I(0)R(A_2) = R(A_2)
\end{align*}
\]
The above discussion suggests finding other subsets of the 48 transformations in Figure 6.2, 6.3, and 6.4 which form groups. For example, is the set of all rotations closed under multiplication? If the set of 48 transformations is not closed under multiplication, additional transformations can be generated by computing products of transformations in this set. Some translations followed by line reflections produce translations but others produce transformations which are not reflections, translations, or rotations.

Since $\LaT(D)$ has ordered image points

\begin{align*}
\text{GHI} & \quad \text{DEF} & \quad \text{ABC} \\
\text{BCE} & \quad \text{KIG} & \quad \text{EFB}
\end{align*}

it follows that $\LaT(D) = \LaT(B)$. However, $\LaT(B)$ gives the image set

\begin{align*}
\text{BCE} & \quad \text{KIG} & \quad \text{EFB}
\end{align*}

which is not one of the 48 previous transformations. Likewise, $\LaR(B_2) = L \text{e but}, \LaR(D_1)$ gives a new transformation.

The six transformations called transflections, shown in Figure 6.5, are new products of translations and reflection in line (a). The 13 transformations called roto-flections in Figure 6.6 are new products of rotations and reflection in line (a).

In Figures 6.2 - 6.6 are listed 72 transformations obtained by reflections in one, two or three lines. If this set of 72 transformations is not closed under multiplication then new
### Figure 6.5

**Transfections**

<table>
<thead>
<tr>
<th>Points</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>H</th>
<th>I</th>
<th>W</th>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>49. LaT(B)</td>
<td>B</td>
<td>C</td>
<td>A</td>
<td>H</td>
<td>I</td>
<td>G</td>
<td>E</td>
<td>F</td>
<td>D</td>
<td>W</td>
<td>X</td>
<td>Z</td>
<td>Y</td>
</tr>
<tr>
<td>50. LaT(C)</td>
<td>C</td>
<td>A</td>
<td>B</td>
<td>I</td>
<td>G</td>
<td>H</td>
<td>F</td>
<td>D</td>
<td>E</td>
<td>W</td>
<td>X</td>
<td>Z</td>
<td>Y</td>
</tr>
<tr>
<td>51. LaT(E)</td>
<td>H</td>
<td>I</td>
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### Figure 6.6

**Rotoflections**

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<td>69. LaR(I1)</td>
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<td>70. LaR(I2)</td>
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</tbody>
</table>
product transformations can be generated. In the next section it will be proved that this set is closed.

6.3 Transformation Groups

A binary operation and a set of elements compose a group if:

a) the operation is associative in the set,

b) the operation is closed in the set,

c) an identity element is in the set, and

d) each element has an inverse in the set.

Certain subsets of the 72 numbered transformations of Section 6.2 form groups under the product operation. The smallest such subset consists of the identity element \( I(0) \). The four rotations about one point form a group under multiplication. The set of line reflections do not form a group both because the set is not closed under multiplication and because the identity is not in the set. Figure 6.7 shows the product of any two of the translations considered in Section 6.2. The body of the table in this figure contains products of the form \( T = T_2T_1 \).

From Figure 6.7 it is apparent that translation is closed under multiplication; that the product set contains the identity \( I(0) \) and that for each element \( T_1 \) there is an inverse element \( T_2 \) so that \( T_1T_2 = T_2T_1 = I(0) \). That the associative property \( T_1(T_2T_3) = (T_1T_2)T_3 \) holds for all possible translations can be verified by trying all cases. At present it will be assumed
that the product of any number of line reflections is associative; a proof of this will be mentioned later. It is also noted from Figure 6.7 that the product of translations is commutative so that translations form an Abelian group under multiplication.

It is possible to make a table similar to Figure 6.7 showing all products of the 72 transformations of Section 6.2. The table would clearly indicate whether or not the product is closed in this set, whether or not each element has a unique inverse, and whether or not the product is commutative. The problem of closure can be settled by finding answers to a related set of questions. What kind of transformation is obtained by a product

<table>
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<td>T(E)</td>
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</table>
of the type $T_2T_1$ where $T_1$ and $T_2$ represent any reflection, translation, rotation, transflection, or rotoflection?

Since distances are preserved in a line reflection any transformation which is the product of line reflections must likewise preserve distances. Is it true, conversely, that any distance preserving transformation in 13-point geometry can be expressed as the product of line reflections? If so, how many distance preserving transformations exist and for what value of $k$ is it true that every distance preserving transformation can be expressed as a product of not more than $k$ line reflections? Non-collinear points $A'$, $B'$, and $C'$ are images of $A$, $B$, and $C$ in a distance preserving transformation if and only if the following equalities hold for lengths of segments: $AB = A'B'$, $AC = A'C'$, and $BC = B'C'$.

For a distance preserving transformation in which $A'$, $B'$, and $C'$ are images of non-collinear points $A$, $B$, and $C$ let $P$ be any point in the plane and let $P'$ be its image. Since $P'$ lies on circles with non-collinear points $A'$, $B'$, and $C'$ as centers, $P'$ is uniquely determined. If $P$ is an ideal point let $X$ be a point on $AP$; then $P'$ is the ideal point on the unique line $A'X'$. Thus three points and corresponding images determine a unique distance preserving transformation.

Two distance preserving transformations are equivalent if any triangle has the same image in both transformations. To find the total number of distance preserving transformations
in 13-point geometry it is sufficient to calculate the total number of triangles congruent to a given one, considering the order of corresponding vertices. In considering congruent triangles in 13-point geometry the vertices are restricted to the nine ordinary points.

Each triangle in 9-point geometry is a right isosceles triangle. That triangle $\triangle ABE$ has a right angle at $B$ is apparent from the points shown in block I. In any distance preserving

\[
\begin{array}{cccc}
W & W & W & Y & Y & Y \\
A & B & C & X & A & F & H & Z \\
D & E & F & X & D & G & C & Z \\
G & H & I & X & I & B & D & Z \\
\end{array}
\]

block I  Block II

transformation the image of triangle $\triangle ABE$ will be a triangle in which the perpendicular sides are also a row and a column in block I. Consider all possible triangles congruent to triangle $\triangle ABE$. The angle at the image of $E$ must be a right angle. Choose for $B'$, the image of $B$, any of the nine points in block I. The image of $A$ must be in a row or column through $B'$. Thus, there are four choices for $A'$. Now $E'$ must be on a row or column through $B'$ but cannot be on $A'B'$ so there are two choices for $E'$. Thus the total number of triangles congruent to $\triangle ABE$, with vertices in corresponding order, is $9 \times 4 \times 2 = 72$.

In Section 6.2 are shown 72 distinct transformations
which are products of not more than three line reflections. Since each is distance preserving, there are no other distance preserving transformations in 13-point geometry. This shows that the set of 72 transformations is closed under multiplication and, therefore, forms a group. It also shows that no new transformations can be obtained by taking products of more than three line reflections. It likewise shows that any given triangle can be transformed into any triangle congruent to the given one by no more than three line reflections.

Example 6.3-1

Triangle AID and IDF are congruent. Find the distance preserving transformation which carries AID into IDF. For a mechanical type solution find the transformation of Section 6.2 which turns points A, I, and D into points, I, B, and F. It is easy to find that this transformation is $R_{E2}$ (number 35) or a reflection in point E.

Triangles ABE and AEC are both right isosceles triangles with corresponding sides proportional but not equal. Neither is the image of the other in a distance preserving transformation. It is possible to define an angle preserving transformation in which the points A, B, and E have points A, E, and C as images. The transformation given below carries each point of block I, Figure 6.1 into the corresponding point of block II.

73. A(0) AFH EGC IBD YZWX
The product of any of the 72 distance preserving transformations and this angle preserving transformation number 73, will be an angle preserving transformation. Thus by computing all of these products one can obtain 72 transformations which preserve angles but do not preserve lengths. Of course the distance preserving transformations also preserve angles and it is easy to verify that these are the only 144 angle preserving transformations in 13-point geometry.

Example 6.3-2

Given any two triangles in 31-point geometry with vertices ordered so equal angles are in the same position, there is exactly one angle preserving transformation in which the second is the image of the first.

Let triangle FBD be the image of triangle EDA in an angle preserving transformation. The vertices of FBD in block II occupy the same positions as points B, H, and I in block I. It is easy to find the distance preserving transformation in which B, H, and I are images of E, D, and A. This is the rotation R(G3), number 42.

42. R(G3) ICF HBE GAD XWZY

73. A(0) API IGC ISD YZX

The product A(0)R(G3) provides the angle preserving transformation in which the image of triangle EDA is triangle FBD.
For 25-point geometry there are 30 line reflections, 25 translations, 125 rotations, etc. For 49-point geometry there are 56 line reflections. It is possible but impractical to list image points in all of these transformations. For any finite geometry it is easy to define line reflections and products of line reflections. The product of reflections in two parallel lines are called translations and the product of reflections in intersecting lines are called rotations. The product of any number of line reflections is a distance preserving transformation.

Figure 6.8

<table>
<thead>
<tr>
<th>A B C D E</th>
<th>A I L T W</th>
<th>A II O Q X</th>
</tr>
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<tr>
<td>F G H I J</td>
<td>S V E H K</td>
<td>N P W E G</td>
</tr>
<tr>
<td>K L M N O</td>
<td>G O R U D</td>
<td>V D F N T</td>
</tr>
<tr>
<td>P Q R S T</td>
<td>Y C F N Q</td>
<td>J L S U C</td>
</tr>
<tr>
<td>U V W X Y</td>
<td>M P X B J</td>
<td>R Y B I K</td>
</tr>
</tbody>
</table>

Figure 6.8 repeats the familiar blocks for 25-point geometry. The notation \( L(P_1P_2) \) is used to represent reflection in the line through points \( P_1 \) and \( P_2 \). In listing the image points in a reflection the ordered points correspond to the original points A through Y. As examples, reflections about
lines AI and AN are completely determined by the following lists.

As before let \( I(0) \) be the identity transformation.

\[
I(0): \quad ABCDE \quad FGHIJ \quad KLMNO \quad PRQST \quad UVWXYZ \\
L(AN): \quad AILTW \quad MPXBJ \quad YCFNQ \quad GORUD \quad SVEHK \\
L(AI): \quad AHOQX \quad RYBIK \quad JLSUC \quad VDFMT \quad NPIWEG
\]

The product \( L(AI) \cdot L(AN) \), which is easily computed, is listed below.

\[
[L(AI)L(AN): \quad AILTW \quad SVEHK \quad GORUD \quad YCFNQ \quad MPXBJ.]
\]

The ordered elements in \( L(AI)L(AN) \) coincide with the ordered elements in the rows of block II. Early it was noted that the elements in block II are obtained by a rotation of \( 60^\circ \) about \( A \) of the corresponding elements in block I. That \( L(AI)L(AN) \) is a rotation of \( 60^\circ \) in point \( A \) can be illustrated by applying the transformation \( L(AI)L(AN) \) six times in succession to produce the identity transformation.

Since, in Euclidean geometry, repeated reflections in two intersecting lines is a rotation through twice the angle between the lines, the procedure illustrated above can be used to define the angle between two lines in a systematic way. For example, the two undirected angles between lines \( AI \) and \( AN \) would be assigned values of \( 30^\circ \) and \( 150^\circ \). In Chapter 7 the problem of
defining the interior angles of a 49-point geometry triangle will be considered.

6.4 Projective Transformations

In previous sections distance preserving and angle preserving transformation were discussed, especially for 13-point geometry. All of the 144 angle preserving transformations of 13-point geometry have the property that the image of any line is a line. Is the set of all transformations which turn lines into lines identical to the set of all angle preserving transformations?

Consider the transformation $LaR(Hl)$, numbered 67, in Figure 6.6. In this distance preserving transformation the image of $EIB$ is $DHF$. In the transformation $LaR(Hl)$ the image of $F$ is $G$. Is it possible to have a line preserving transformation in which the image of $E$, $I$, and $B$ are $D$, $H$, and $F$ but in which the image of $F$ is not $G$? In an effort to construct such a transformation one can arbitrarily decide that the image of $F$ is to be $B$. The diagram of Figure 6.9 shows $E$, $I$, $B$, and $F$ with corresponding images in parentheses. Since $EI$ and $FB$ intersect at $Y$, the proper intersection is labeled $Y$ in the diagram. Since images of lines are lines, the image point of $Y$ must be the intersection of $DH$ and $BF$ which is $Y$. Likewise $EF$ and $BI$ intersect at $D$ and the image of $D$ must be on $DB$ and $HF$ which is $Z$. This process can be continued to find images of all 13 points as shown.
This illustration shows that there are line preserving transformations which do not preserve distances or angles. Note that the transformation turns the ideal line WXYZ into the ordinary line EIYA and turns the ordinary line CDHY into the ideal line. This illustration suggests the possibility that four points and corresponding image points may uniquely determine a transformation in which all lines have lines as images.
This conjecture will be tested in the following example for 31-point geometry.

Example 6.4-1

Find the line preserving transformation which changes PROJ into GEOM, in 31-point geometry. Using the procedure illustrated above one can obtain the images shown below.

Points
A B C D E F G H I J K L M N O P Q R S T U V W X Y Z 1 2 3 4 5 6

Images
Q S R T I 2 B J X I 4 M V C I 3 I O G I E W N I U F A K L P D Y I H 5

The transformation turns the ideal line I , X ^ I g into the ordinary line LPDYI5H and turns the ordinary line UEMII5Q into the ideal line.

The transformations discussed above are examples of projective transformations. If homogeneous coordinates are used, point P'(x',y',z') is the image of point P(x,y,z) in a projective transformation if

\[
x' = a_{11}x + a_{12}y + a_{13}z \\
y' = a_{21}x + a_{22}y + a_{23}z \\
z' = a_{31}x + a_{32}y + a_{33}z
\]

where the determinant of the nine coefficient is not zero. These equations can be solved for x, y, and z expressed as linear functions of x', y', and z'. Since the image of any line
AX + BY + CZ = 0 will be of the form A'X' + B'Y' + C'Z' = 0, the projective image of any line is also a line. Likewise the conic $Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz = 0$ has for its projective image another conic. The image of a circle need not be a circle. It is well-known in projective geometry that four points, no three collinear, and the corresponding four image points determine a projective transformation uniquely. The next example illustrates a procedure for finding the equations of a projective transformation determined by four points and the corresponding image points. The transformation is the same as that found in Example 6.4-1 without use of analytic geometry.

Example 6.4-2

Find the projective transformation which changes PROJ into GEOM, in 3l-point geometry. Homogeneous coordinates for the points of 3l-point geometry are repeated below for convenience.

<table>
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<tr>
<th>A(0,4,1)</th>
<th>B(1,4,1)</th>
<th>C(2,4,1)</th>
<th>D(3,4,1)</th>
<th>E(4,4,1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>F(0,3,1)</td>
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<td>H(2,3,1)</td>
<td>I(3,3,1)</td>
<td>J(4,3,1)</td>
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<td>N(3,2,1)</td>
<td>O(4,2,1)</td>
</tr>
<tr>
<td>P(0,1,1)</td>
<td>Q(1,1,1)</td>
<td>R(2,1,1)</td>
<td>S(3,1,1)</td>
<td>T(4,1,1)</td>
</tr>
<tr>
<td>U(0,0,1)</td>
<td>V(1,0,1)</td>
<td>W(2,0,1)</td>
<td>X(3,0,1)</td>
<td>Y(4,0,1)</td>
</tr>
<tr>
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<td></td>
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<td></td>
</tr>
</tbody>
</table>
Numerical coefficients $a_{11}$ through $a_{33}$ are to be determined in the equations

$$
x' = a_{11}x + a_{12}y + a_{13}z
$$

$$
y' = a_{21}x + a_{22}y + a_{23}z
$$

$$
z' = a_{31}x + a_{32}y + a_{33}z
$$

so that images of points P, R, O, and J are G, E, O, and M. In the computation it must be remembered that $(p,q,r) = (Kp,Kq,Kr)$ for $K \neq 0$ and for 31-point geometry the arithmetic is modulo five. In substituting coordinates of a point and its image in the above equations the coordinates of original points will be substituted as given but the coordinates of an image points will each contain the same unknown factor. That is, since the image of P(0,1,1) is G(1,3,1), the set $(x,y,z)$ is replaced by $(0,1,1)$ and $(x',y',z')$ is replaced by $(k,3k,k)$ where the value of $k$ is to be determined. This process gives the following 12 equations where $k$, $m$, $n$, and $p$ must have values of 1, 2, 3, or 4.

$$
P \to G
$$

(0,1,1) to (1,3,1) gives

$$
k = a_{12} + a_{13}
$$

$$
3k = a_{22} + a_{23}
$$

$$
k = a_{32} + a_{33}
$$

$$
R \to E
$$

(2,1,1) to (4,4,1) gives

$$
4m = 2a_{11} + a_{12} + a_{13}
$$

$$
4m = 2a_{21} + a_{22} + a_{23}
$$

$$
m = 2a_{31} + a_{32} + a_{33}
$$
This set of 12 equations contains 13 unknowns. In the process of solution one of the letters k, m, n, or p can be assigned the value of 1. The solution of these equations is tedious but not difficult. The set

\[
\begin{align*}
  k &= a_{12} + a_{13} \\
  4m &= 2a_{11} + a_{12} + a_{13} \\
  4n &= 4a_{11} + 2a_{12} + a_{13}
\end{align*}
\]

is solved for a, b, and c to give

\[
\begin{align*}
  a_{11} &= 2k + 2m \\
  a_{12} &= 1/2 (2k + 4m + 3n) = k + 2m + 4n \\
  a_{13} &= 3m + n
\end{align*}
\]

In the same manner \(a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}\) are expressed in terms of \(k, m,\) and \(n\). These results are substituted in the last three equations of the original 12 to obtain 3 linear equations containing \(k, m, n,\) and \(p\). In this set \(p\) is assigned the value of 1 to determine \(k = 4, m = 1,\) and \(n = 2\). Values of
all through $a_{33}$ are computed to give the following transformation equations

$$x' = 4y$$

$$y' = x + 3y + 4z$$

$$z' = x + 4y$$

That these formulas are correct is easily verified by using them to show that the images of $P$, $R$, $O$, and $J$ are $G$, $E$, $O$, and $M$. The formulas can be used to find all images listed in Example 6.4-1.

Substitution of the coordinates of $A(0,4,1)$ gives

$$x' = 4.4 = 1$$

$$y' = 0 + 3.4 + 4.1 = 1$$

$$z' = 0 + 4.4 = 1$$

so that the image of $A$ is $Q(1,1,1)$. Substitution of the coordinates of $B(1,4,1)$ gives

$$x' = 4.4 = 1$$

$$y' = 1 + 3.4 + 4.1 = 2$$

$$z' = 1 + 4.4 = 2$$

Since the points $(1,2,2)$ and $(3,1,1)$ are identical the image of $B$ is $S(3,1,1)$. The formulas for $x'$, $y'$, and $z'$ can also be solved for $x$, $y$, and $z$ to give the inverse transformation. The inverse
transformation can be used to find the image of a given curve in the projective transformation which changes points $P, R, O,$ and $J$ into points $G, E, O,$ and $M$.

6.5 Matrices as Transformations

Since the projective transformation

$$
\begin{align*}
x' &= a_{11}x + r_{12}y + a_{13}z \\
y' &= a_{21}x + a_{22}y + a_{23}z \\
z' &= a_{31}x + a_{32}y + a_{33}z
\end{align*}
$$

is determined by the values of the nine coefficients the transformations can be, and often is, represented by the matrix

$$
\begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix}
$$

and any matrix of this form for which the associated determinant is not zero will be called a projective transformation.

In Section 6.4 it was indicated that four points, no three collinear, and the corresponding image points are sufficient to determine a projective transformation uniquely. If any reflection, translation, rotation, or angle preserving transformation is given, it is possible to find images of four points and to use the method of the previous section to find the equation of the corresponding special projective transformation. Since in
any angle preserving transformation the image of an ideal point is an ideal point it is always possible to write the equation of an angle preserving transformation so that the corresponding matrix has the form

\[
\begin{pmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  0 & 0 & 1
\end{pmatrix}
\]

A special class of transformations, such as translations, will correspond to certain additional restrictions on the elements of this matrix. The form of a matrix representing a translation can be found by deriving equations of a general translation in a Cartesian coordinate system and then changing to homogeneous coordinates. However, if a student were to compute the corresponding matrices for the twelve translations in 13-point geometry he might be able to make a correct conjecture about the form of a translation matrix for other finite (or infinite) geometries.

If one is familiar with properties of matrices, this knowledge can provide useful shortcuts in a study of transformations. For example, if a given projective transformation is represented by a matrix, the inverse transformation is represented by the inverse of the matrix. If two angle preserving transformations are applied in succession to find a product transformation, the matrix of the product transformation is the product of the
matrices representing the original transformations. The proof that
the product of projective transformations is associative is
equivalent to the well-known fact that multiplication of third
order matrices is associative.

The following examples show some particular matrices as­
sociated with the distance preserving transformations previously
discussed for 13-point geometry and illustrate the statement
that the product of two transformations corresponds to the
product of the matrices.

The homogeneous coordinates for 13-point geometry, in
Figure 5.5, are as follows:

\[
\begin{align*}
A(0,2,1) & \quad B(1,2,1) \quad C(2,2,1) \quad \text{Ideal Points} \\
D(0,1,1) & \quad E(1,1,1) \quad F(2,1,1) \quad W(0,1,0) \quad X(1,0,0) \\
G(0,0,1) & \quad H(1,0,1) \quad I(2,0,1) \quad Y(1,2,0) \quad Z(1,1,0)
\end{align*}
\]

Example 6.5-1

Consider the matrix

\[
\begin{pmatrix}
2 & 0 & 2 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

where

\[
\begin{pmatrix}
2 & 0 & 2 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

Equations of the corresponding transformation are

\[
\begin{align*}
x' & = 2x + 2z \\
y' & = y \\
z' & = z
\end{align*}
\]
From this transformation the image of points can be found as follows:

\[
\begin{align*}
A(0,2,1); & \quad A' = (2,2,1) = C \\
B(1,2,1); & \quad B' = (1,2,1) = B \\
C(2,2,1); & \quad C' = (0,2,1) = A \\
D(0,1,1); & \quad D' = (2,1,1) = F \\
E(1,1,1); & \quad E' = (1,1,1) = E \\
F(2,1,1); & \quad F' = (0,1,1) = D \\
G(0,0,1); & \quad G' = (2,0,1) = I \\
H(1,0,1); & \quad H' = (1,0,1) = H \\
I(2,0,1); & \quad I' = (0,0,1) = G
\end{align*}
\]

This is the line reflection, i.e., number 5 of Section 6.2.

Example 6.5-2

Consider the matrix

\[
\begin{pmatrix}
0 & 2 & 0 \\
2 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

whose determinant is not zero. A person familiar with matrix manipulations can find image points by multiplication of this matrix and a column vector representing a point. For example, since

\[
\begin{pmatrix}
0 & 2 & 0 \\
2 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 \\
1 \\
1
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}
\]

the image of A(0,2,1) is H(1,0,1). Image points found by this transformation are listed on the following page.
This is also a line reflection, $L_k$, number 11 of Section 6.2.

Any translation or rotation is the product of two line reflections. Since the reflections found in Example 6.5-1 and 6.5-2 are with respect to lines which intersect at point $B$, the product of these reflections is a rotation in point $B$. In Example 6.5-3 it is shown that the matrix of the rotation is obtained by multiplying matrices for transformations $L_e$ and $L_k$.

Example 6.5-3

The product of the matrices in Examples 6.5-1 and 6.5-2 is now computed, using modulo 3 arithmetic.

$$
L_eL_k = \begin{pmatrix}
2 & 0 & 2 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 2 & 0 \\
2 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix} =
\begin{pmatrix}
0 & 1 & 2 \\
2 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
$$

The corresponding equations

$$
x' = y + 2z
$$

$$
y' = 2x
$$

$$
z' = z
$$

determine image points as follows:

Points A B C D E F G H I
Images II B E I C F G A D I C F
This is the rotation \( R(B3) \), number 27 of Section 6.2.

It is possible to introduce 13-point geometry by defining points and lines as ordered triples of numbers, in the set \( \{0,1,2\} \), which satisfy certain properties. If \((x,y,z)\) represents a point and \([u,v,w]\) represents a line, the point and line are incident if

\[
(u \ v \ w) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0
\]

Three distinct points are collinear if

\[
\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0
\]

and three distinct lines are concurrent if

\[
\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} = 0
\]

6.6 Inversion

In Euclidean geometry point \( P' \) is called the image of point \( P \) in an inversion with respect to the circle with center \( O \) and radius \( r \) if \( O, P, \) and \( P' \) are collinear and if \( OP \cdot OP' = r^2 \). Here \( OP \) and \( OP' \) are directed segments so that \( P \) and \( P' \) are on the same side of \( O \). The transformation is self-inverse since if
$P'$ is the image of $P$, $P$ is the image of $P'$. Every ideal point has for its image the center $O$ and conversely.

The points on the circle of inversion are invariant. Points inside the circle have images outside and conversely. In an inversion any figure composed of lines and circles has for its image another figure composed of lines and circles but the image of a line may be either a line or a circle.

Given a circle of inversion and a point $P$, the image point $P'$ can be constructed in different ways. The following method (which can be used as a definition for inversion) applies equally well to Euclidean geometry or to finite geometries.

Given a circle with center $O$ and a point $P$, the image point $P'$ in the inversion with respect to circle $O$ is found as follows. If and only if $P$ is on the circle of inversion, $P$ and $P'$ coincide. If $P$ is outside circle $O$, let tangent lines from $P$ meet the circle in points $R$ and $S$; then $RS$ and $OP$ intersect in $P'$, the image of $P$. If $P$ is inside circle $O$ let the perpendicular to $OP$ through $P$ intersect the circle at $R$ and $S$. Then tangents to the circle at $R$ and $S$ intersect in the image point $P'$. If $P$ is an ideal point, $P'$ is the center $O$. The image of the center $O$ is the set of all ideal points.

Example 6.6-1

Find image points in an inversion in 13-point geometry. Let $ACEH$ be the circle of inversion with center $B$ and radius 1.
All points and corresponding images are shown below.

<table>
<thead>
<tr>
<th>Points</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>H</th>
<th>I</th>
<th>W</th>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>Images</td>
<td>A</td>
<td>W</td>
<td>X</td>
<td>Y</td>
<td>Z</td>
<td>C</td>
<td>I</td>
<td>E</td>
<td>G</td>
<td>F</td>
<td>H</td>
<td>D</td>
<td>D</td>
</tr>
</tbody>
</table>

Example 6.6-2

For 31-point geometry, let $UAREYH$ be the circle of inversion with center $M$ and radius $1'$. The image of $M$ is the ideal line; other corresponding images are shown below.

| Points  | A   | B   | C   | D   | E   | F   | G   | H   | I   | J   | K   | L   | M   | N   | O   | P   | Q   | R   | S   | T   | U   | V   | W   | X   | Y   | Z   | 1   | 2   | 3   | 4   | 5   | 6   |
|---------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| Images  | A   | J   | W   | F   | E   | D   | S   | H   | Q   | B   | L   | K   | O   | N   | X   | I   | G   | V   | U   | T   | C   | P   | M   | M   | M   | M   | M   | M   | M   | M   | M   | M   | M   | M   |

Example 6.6-3

In 49-point geometry, if $6A5E1F4E3A4D1C5D$ is the circle of inversion with center at $1A$, the images of all the points are listed next.

| Points  | 1A  | 2A  | 3A  | 4A  | 5A  | 6A  | 7A  | 1B  | 2B  | 3B  | 4B  | 5B  | 6B  | 7B  | 1C  | 2C  | 3C  | 4C  | 5C  | 6C  | 7C  | 1D  | 2D  | 3D  | 4D  | 5D  | 6D  | 7D  | 1E  | 2E  | 3E  | 4E  | 5E  |
|---------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| Images  | 5A  | 3A  | 7A  | 2A  | 6A  | 4A  | 1B  | 3C  | 4F  | 5G  | 4G  | 5F  | 6C  | 1C  | 2C  | 3C  | 4C  | 5C  | 6C  | 7C  | 1D  | 2D  | 3D  | 4D  | 5D  | 6D  | 7D  | 1E  | 2E  | 3E  | 4E  | 5E  | 1F  | 2F  | 3F  | 4F  | 5F  | 6F  | 7F  | 1G  | 2G  | 3G  | 4G  | 5G  | 6G  | 7G  |

| Points  | 6E  | 7E  | 1F  | 2F  | 3F  | 4F  | 5F  | 6F  | 7F  | 1G  | 2G  | 3G  | 4G  | 5G  | 6G  | 7G  | 2F  | 2D  | 1F  | 6E  | 2G  | 3B  | 6B  | 7G  | 3E  | 1D  | 3F  | 4C  | 53  | 4B  | 5C  | 6F  |
From the definition of inversion it follows that the image of a line through the center of inversion is the same line. In Euclidean geometry the image of a line not cutting the circle of inversion must be completely inside the circle of inversion. The next examples show images of lines not cutting the circle of inversion for the inversions discussed in Examples 6.6-1, 6.6-2, and 6.6-3.

Example 6.6-4

In the inversion of Example 6.6-1, line IEAY becomes circle DAEB with center I and radius 1'.

Example 6.6-5

In the inversion of Example 6.6-2, line Z3IV0CP turns into MQTNWX, a circle with center P and radius 1.

Example 6.6-6

In the inversion of Example 6.6-3, line 2A2E2C2D2E2F2G0T inverts into circle 5A3C6D7E7D6E3F1A with center 3A and radius 2.

Since the inversion transformation is self-inverse, the previous three examples illustrate the fact that the inverse of a circle through the center of inversion is a line not through the center of inversion. The following examples contain images of circles which do not pass through the center of inversion. One such circle is the circle of inversion which is, of course,
invariant. The examples will indicate that there are other invariant circles in an inversion.

Example 6.6-7

In the inversion of Example 6.6-1 the circle CDEI with center F and radius 1 becomes CIED which is the same circle. Circle ACGI with center E and radius 1' turns into another circle ACDF with center H and radius 1'.

Example 6.6-8

In the inversion of Example 6.6-2 circle NOPRXY with center Q and radius 1 is invariant since the ordered points N, O, P, R, X, and Y become O, N, X, R, P, and Y. Circle ABISUV with center N and radius 1' turns into another circle AJQGUT with center K and radius 2'.

Example 6.6-9

In the inversion of Example 6.6-3 circle 2D2E4F4G5B5G7D7E with center IA and radius 3' is invariant, but circle 3A5A2B6B2D6D3E5E with center 4C and radius 1' turns into the circle 3A2A3C5F7E2C7F5E with center 6B and radius 3'.

Inversion can be used in solving the general problem of Appollonius (constructing a circle tangent to three given circles)
or in solving special cases in which some or all of the given circles are replaced by lines or points.

Example 6.6-10

In 31-point geometry find all circles passing through point M, tangent to circle QSOUYK, and tangent to line ANVJRI_5.

Solution: If M is used as center of a circle of inversion, the given circle and line have circles for images; a solution circle becomes a line tangent to the two new circles. Reinversion of a common tangent line gives a solution.

Let the circle with center M and radius 1' be chosen as a circle of inversion. Example 6.6-2 shows all image points in this inversion. Circle QSOUYK becomes circle IGNUYL and line ANVJRI_5 becomes circle ACTBRM. Each new circle has six tangent lines; any common tangent will lead to a solution. There are four common tangents: AFKPU_1, IVOCIT_2, LERFXI_3, and XGTCKI_5.

Reinversion of the four tangent lines gives as solutions of the original problem the four circles ADLXUM, QTNXOM, KERDPM, and PSYWLM.

The next and concluding chapter contains discussions and examples showing additional applications of finite geometries. Sections 7.5 and 7.6 introduce transformations in which images of lines are neither lines nor circles.
CHAPTER VII

APPLICATIONS

7.1 Introduction

Emphasis in previous chapters has been placed on the use of inductive methods to formulate reasonable conjectures. It is hoped that some persons will be encouraged to extend previous materials to obtain new results. Some of the topics discussed earlier seem so elementary that they might be appropriate for classes in elementary school. Perhaps some interested teachers could prepare material on finite geometry especially for elementary, junior high, senior high, or college students and evaluate the effectiveness of this type of geometric instruction.

The type of finite geometry discussed can be related to many of the mathematical topics suggested as appropriate in modern high school programs. Use of analytic geometry in finite geometry has been frequently illustrated in Chapter IV, V, and VI. Section 6.5 has shown how properties of matrices and determinants are applicable in finite geometry. A major goal in mathematics is developing the function concept. The transformations of finite geometry are functions which give correspondences between sets of points. In modern high school geometry there is increased emphasis on properties of transformations; the study of
transformations in finite geometry can clarify the concept of transformations as it appears in Euclidean or projective geometries. Problems such as determining the number of quadrilaterals in 49-point geometry or of finding the probability that three distinct points in 25-point geometry determine a right triangle are related to probability which is sometimes recommended as a high school topic.

In the same way that the structure of arithmetic can be illustrated using a modular number system, it seems possible to illustrate many of the important ideas of trigonometry by use of a finite geometry so that trigonometric tables can be shown completely and that theorems can be proved by testing all cases.

It seems likely that computers will become more important in school work in the near future. Surely problems in finite geometry can be solved on computers and an interested student might design a simple computer to be associated with a particular finite geometry.

This chapter contains several examples of applications of finite geometries. The first application demonstrates a generalization of previous work such as might be expected from a student. In section 3.7 angles were introduced for 25-point geometry. A corresponding development for 49-point geometry is contained in section 7.2.

In advanced Euclidean geometry there is a method for constructing the roots of a given quadratic equation with rational coefficients and real roots. An analogue of this method is used
in section 7.3 to show how a given quadratic equation can be solved in a modulo seven number system by applying properties of 49-point geometry.

Section 7.4 contains a discussion of mathematical systems which satisfy three given postulates. The problem of discovering theorems which can be proved using these postulates is related to finite geometries through the finding of models which satisfy the given postulates.

In section 7.5 a transformation which turns points into lines and lines into points is discussed. Results in finite geometry are used to formulate conjectures in Euclidean Geometry.

Section 7.6 contains a discussion of a point transformation which turns lines into conics. Properties of this transformation can be discovered by use of finite geometries.

7.2 Angles in 49-point Geometry

For 49-point geometry (see Figure 5.5) can we define the angle between two lines in a reasonable way? For example, consider the first row in block I and the first row in block III. Repeated reflection about these two lines should be a rotation about 1A. By examining properties of this rotation it may be possible to answer the above question.

Let L(1A1B) represent reflection in line 1A1B and L(1A6B) represent reflection in line 1A6B. The rotation produced by reflection first in line 1A6B followed by reflection in 1A1B is L(1A1B) • L(1A6B). It is easy to tabulate the 49 images of points
in each of these reflections but, at present, images of individual points in the desired rotation will be found directly from Figure 5.5

For example, the image of 3A in the rotation is found by reflecting 3A in line 1A6B to obtain 5E and then reflecting 5E in line 1A1B to obtain 4E. Thus in the rotation, the image of 3A is 4E. By the same method it is found that the image of 4E is 1F; the image of 1F is 5E; the image of 5E is 6A; the image of 6A is 5D; the image of 5D is 1C; the image of 1C is 4D; and the image of 4D is 3A. This result shows that the circle with center 1A and radius 2, contains the points 3A, 4E, 1F, 5E, 6A, 5D, 1C, and 4D. From the order of these points in block I and II it is reasonable to say that L(1A1B)*L(1A6B) is a rotation about 1A of minus 45 degrees. Thus, neglecting direction, it seems reasonable to define the measure of the angles between lines 1A1B and 1A6B as 22 1/2° and 157 1/2°.

In Figure 7.1, the line through 1A6B is sketched so that reflection, in the Euclidean sense, in line 1A6B followed by reflection in line 1A1B gives the correct images for points in rotation of -45° about 1A. In the rotation L(1A1B)*L(1A6B) the image of 6B is 3B and Figure 7.1 shows line 1A3B sketched to suggest that a rotation of -45° about 1A turns line 1A6B into line 1A3B. Points 2F, 3G, 6G, 7F, and 7C are placed in Figure 7.1 by the same method.

From Figure 7.1 it is possible to define the smaller angle between two lines which intersect at point 1A. Since two lines
which do not intersect at 1A are parallel to lines through 1A, this definition can be extended to give the angle between any two lines of 49-point geometry. The table in Figure 7.2 shows the smaller of the angles between two lines when the positions of the lines as rows or columns in block I, II, III, and IV of Figure 5.5 are known.
Figure 7.2. Smaller angle of intersection (in degrees) of two lines in 49-point geometry.

<table>
<thead>
<tr>
<th>Row in Block</th>
<th>Column in Block</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>I</td>
</tr>
<tr>
<td>II</td>
<td>II</td>
</tr>
<tr>
<td>III</td>
<td>III</td>
</tr>
<tr>
<td>IV</td>
<td>IV</td>
</tr>
<tr>
<td>I</td>
<td>0 45 22(\frac{1}{2}) 67(\frac{1}{2})</td>
</tr>
<tr>
<td>II</td>
<td>45 0 22(\frac{1}{2}) 22(\frac{1}{2})</td>
</tr>
<tr>
<td>III</td>
<td>22(\frac{1}{2}) 22(\frac{1}{2}) 0 45</td>
</tr>
<tr>
<td>IV</td>
<td>67(\frac{1}{2}) 22(\frac{1}{2}) 45 0</td>
</tr>
<tr>
<td>I</td>
<td>90 45 67(\frac{1}{2}) 22(\frac{1}{2})</td>
</tr>
<tr>
<td>II</td>
<td>45 90 67(\frac{1}{2}) 67(\frac{1}{2})</td>
</tr>
<tr>
<td>III</td>
<td>67(\frac{1}{2}) 67(\frac{1}{2}) 90 45</td>
</tr>
<tr>
<td>IV</td>
<td>22(\frac{1}{2}) 67(\frac{1}{2}) 45 90</td>
</tr>
</tbody>
</table>

The table of angles in Figure 7.2 can be used to determine the "interior" angles of a given triangle in 49-point geometry by requiring that the sum of the angles be 180°.

Example 7.2-1

Find the interior angles in triangle 4E4C7A
In the diagram suggesting triangle 7A4E4C the symbol II^C indicates that 7A4E is a column in block II while the symbol 2' shows the length of side 4C7A.

From Figure 7.2 it is seen that lines 4E4C and 4C7A intersect at angles of 22 1/2° or 157 1/2°. Since the smallest possible interior angle of a triangle is 22 1/2°, no interior angle can exceed 135°. Thus the interior angles at 4C is 22 1/2°. Likewise the interior angle at 7A is 22 1/2°. Since 7A4E and 4E4C intersect at angles of 45° or 135°. The only choice for the interior angle at 4E is 135°, since the sum of the three angles must be 180°.

In the triangle 4E4C7A two sides are equal and the interior angles, as determined, have the property that angles opposite equal sides are equal.

Example 7.2-2

Assign values to the interior angles of triangle 4B2A7D.
The interior angle at $7D$ is $90^\circ$. The interior angles at $4B$ and $2A$ are both $45^\circ$.

Example 7.2-3

Find the angles in triangle $2D^4A^2G$.

Figure 7.2 shows the angle at $2D$ is $67 \frac{1}{2}^\circ$ or $112 \frac{1}{2}^\circ$. The angle at $4A$ is $45^\circ$ or $135^\circ$. The angle at $2G$ is $22 \frac{1}{2}^\circ$. 
or 157 1/2°. The only possible choice for the interior angles at 2D, 4A, and 2G is 112 1/2°, 45° and 22 1/2°.

Example 7.2-4

Find the interior angles in triangle 1F6B2E.

The interior angles at 1F, 6B, and 2E are 67 1/2°, 22 1/2°, and 90°.

The definition of interior angles in triangle for 49-point geometry can be used in several ways. For example, two triangles with corresponding angles equal can be defined as similar. As
in 25-point geometry, one can use angles in triangles to define a convex polygon.

7.3 Solution of Quadratic Equations Using 49-Point Geometry

The equation to be solved is $ax^2 + bx + c = 0 \pmod{7}$. It can be shown

in Euclidean Geometry, that if the circle with points $(0,1)$ and $(p,q)$ as ends of a diameter cuts the $x$-axis at points $(r,0)$ and $(s,0)$ then $r$ and $s$ are roots of the equation $x^2 - px + q = 0$. 
This process will be illustrated using modulo 7 arithmetic.

Example 7.3-1

What are the roots of the equation $3x^2 + x + 4 = 0 \pmod{7}$?

The equation can be written as $x^2 + 5x + 6 = 0 \pmod{7}$ or $x^2 - 2x + 6 = 0$ and comparison with $x^2 - px + q = 0$ shows $p = 2, q = 6$. The point 1C of Figure 5.6 has coordinates $(2,6)$ and the coordinates $(0,1)$ correspond to point 6A. The center of the circle with 6A and 1C as ends of a diameter is

7B and the radius is 3. The eight points on the circle are 1A, 1C, 3B, 4B, 6A, 6C, 7E, and 7F. Since 7E and 7F are on the
X-axis and have coordinates (4,0) and (5,0) perhaps, 4 and 5 are roots of the original quadratic equation.

Check: \(3x^2 + x + 4 = 0\)

Let \(x = 4\); \(3(4)^2 + 4 + 4 = 6 + 4 + 4 = 0 \pmod{7}\)

Let \(x = 5\); \(3(5)^2 + 5 + 4 = 5 + 5 + 4 = 0 \pmod{7}\)

Therefore there are two distinct solutions for this equation.

Example 7.3-2

Solve the equation \(4x^2 + 5x + 2 = 0 \pmod{7}\).

This can be written as \(x^2 + 3x + 4 = 0\) or \(x^2 - 4x + 4 = 0\) where \(p = 4\), and \(q = 4\) corresponds to point 3E(4,4). The circle with 6A and 3E as ends of a diameter, has center 1C and radius 1. This circle meets the X-axis only at the point 7C whose coordinates are (2,0).
Perhaps 2 is a double root of the original quadratic equation.

Check: \[ 4x^2 + 5x + 2 = 0 \pmod{7} \]

Let \( x = 2 \); \[ 4(2)^2 + 5(2) + 2 = 2 + 3 + 2 = 0 \pmod{7} \]

Since the original equation is equivalent to \((x - 2)(x - 2) = 0\), 2 is a double root.

Example 7.3-3

Is there any solution for the equation \( 5x^2 + 2x + 2 = 0 \pmod{7} \)?

This can be written as \( x^2 + 6x + 6 = 0 \) or \( x^2 - x + 6 = 0 \) where \( p = 1, q = 6 \) corresponds to point 1B(1,6). The points on circle with 6A and 1B as ends of a diameter, center 7E and radius 3' are 1A, 1B, 3D, 3F, 4D, 4F, 6A, and 6B. Since this circle does not intersect the X-axis the equation seems to have no solution.
That the equation has no solution in the set \( \{0, 1, 2, 3, 4, 5, 6\} \) can be verified by substitution.

7.4 Properties of a Postulate System

The three postulates considered in this section were discussed by Mr. James Satterfield, a graduate student at the Ohio State University in a classroom report in 1966. The purpose of his report was to illustrate methods for discovering theorems. In the present section it will be shown that models related to 9-point geometry and 25-point geometry can be constructed so that the three postulates are satisfied. From a study of properties of models it is possible to make inferences about possible theorems in the system.

Let the symbols \( a, b, c, \) etc. represent any elements in a given set. Assume that a binary operation, called multiplication and represented by symbols \( ab, a \cdot b, \) or \( (a)(b) \), is closed on the set and that the following three postulates are satisfied. Assume the usual properties of equality.

\[
\begin{align*}
P&\cdot I \quad a a = a \\
P&\cdot II \quad (a b)(c d) = (a c)(b d) \\
P&\cdot III \quad a(b a) = b
\end{align*}
\]

A postulate system must be consistent; that is, the validity of all but one of the postulates must not imply that the remaining postulate is not valid. It is also desirable that the postulates of a system be independent; that is, the validity one
of the postulates should not be provable when this postulate is deleted from the system. A single example of a model which satisfies all of the postulates guarantees consistency. Independence can be demonstrated for a system of three postulates by three models in which different combinations of two of the postulates are true with the remaining postulate false.

That the three postulates of this section are consistent is demonstrated by considering a set containing only one element, say $A$. Then by the closure property $AA = A$ and the three postulates are obviously satisfied.

An interesting exercise results from considering all closed binary operations on a set of exactly two elements. Since this leads to four products each with two possible values there are exactly 16 such operations. For each of the 16 operations, simple calculations will show if each of postulates I, II, and III is true or false. Those calculations show that no binary operation on a set of two elements satisfies all three of the postulates. This suggests attempting to find binary operations with more than two elements which do satisfy the postulates. For a set of three elements there are 9 products each with three possible values; this gives $3^9$ possible closed operations, so that a study of each in turn is clearly impractical. Since postulates I and III are relatively simple, one approach is to
construct models which satisfy postulate I and III and then test to see if these models satisfy postulate II.

Example 7.4-1

Given three distinct elements A, B, and C define the product of any two elements so that postulates I and III are satisfied.

In multiplication tables shown here, and in later examples, the product ab is shown in the row to the right of a and in the column below b. The table of Figure 7.3 shows that the product aa = a. For the product AB there are three choices A, B, or C. It will be shown that choices AB = A or AB = B lead to contradictions using postulates I and III so the only acceptable choice of AB is C.

![Figure 7.3](image)

Assume that AB = A. Then B(AB) = BA = A by P•III. Likewise B(BA) = BA = B and A(BA) = AB = B which contradicts the assumption AB = A. Assume that AB = B. Then by P•III, B(AB) = BB = A which contradicts postulate I. Thus AB = C.

Using P•III, B(AB) = BC = A and C(BC) = CA = B, so these products are placed in Figure 7.3. Similar reasoning shows that only the unique arrangement of products shown in Figure 7.3 can
satisfy both postulates I and III for the three distinct elements A, B, and C.

Example 7.4-2

Test the binary operation of Figure 7.3 to see if postulate II is valid.

A single counterexample is sufficient to show that \((ab)(cd) = (ac)(bd)\) is not valid. To prove the postulate valid by testing all cases is tedious even in this example and is not practical for operations on larger sets of elements.

Assign coordinates to elements A, B, and C as follows: A(0,0), B(1,0), and C(2,0). Let \(p\) and \(q\) represent elements with coordinates \((x_p, 0)\) and \((x_q, 0)\). Define the product \(pq = r\) where \(r\) has coordinates \((x_r, 0)\) with

\[ x_r = \frac{x_p + x_q}{2} \pmod{3} \text{ or } x_r = 2x_p + 2x_q \pmod{3} \]

This definition which has the interpretation, the product of two "points" is the midpoint of the segment joining them in modulo three arithmetic, produces the multiplication Table of Figure 7.3. Postulates I, II, and III can now be verified for all possible products from this formula.

To verify postulate I let \(a\) be the point \((x_1, 0)\). Note that \(aa\) is the point with coordinates \((x_2, 0)\) where

\[ x_2 = 2x_1 + 2x_1 = x_1 \pmod{3} \text{ so } aa = a. \]

To test postulate II let
the x coordinates of a, b, c, and d be $x_1, x_2, x_3,$ and $x_4$. Then 

(ab) has coordinates $(2x_1 + 2x_2, 0)$, (cd) has coordinates

$(2x_3 + 2x_4, 0)$; and (ab)(cd) has coordinates

$(4x_1 + 4x_2 + 4x_3 + 4x_4, 0) = (x_1 + x_2 + x_3 + x_4, 0)$. In the same manner the coordinates of (ac)(bd) are found to be

$(x_1 + x_2 + x_3 + x_4, 0)$ which shows that (ab)(cd) = (ac)(bd).

To test postulate III note that (ba) has coordinates 

$(2x_1 + 2x_2, 0)$ and that a(ba) has coordinates $(2x_1 + 4x_2 + 4x_1, 0) = (x_2, 0)$ so that a(ba) = b.

Example 7.4-3

Given four distinct elements A, B, C, and D define a closed product operation so that postulates I and III are valid.

Figure 7.4 shows a multiplication table which was constructed to satisfy postulates I and III; after assigning values

aa = a, the product AB was chosen to be D. Use of postulates I and III then uniquely determined the remaining products.

**Figure 7.4**

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Figure 7.5 shows a second multiplication table which was constructed to satisfy both postulates I and III.

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It will next be shown that properties of 9-point and 25-point geometries can be used to define multiplication for sets of 9 and for sets of 25 elements so that postulates I, II, and III are valid. Let elements A through I represent the 9-points shown in blocks I and II of Figure 5.5 which are repeated below:

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Define pq = r where p and q are elements of the set A, B, C, D, E, F, G, H, and I and r is the midpoint of the segment determined by points p and q. Thus if points p, q, and r have coordinates \((x_p, y_p)\), \((x_q, y_q)\) and \((x_r, y_r)\), then \(x_r = 2x_p + 2x_q \pmod{3}\) and \(y_r = 2y_p + 2y_q \pmod{3}\).
The postulate \( aa = a \) is verified by noting that
\[
(2x_1 + 2x_1, 2y_1 + 2y_1) = (4x_1, 4y_1) = (x_1, y_1) \pmod{3}.
\]

To test the postulate \( (ab)(cd) = (ac)(bd) \) let \( a, b, c, \) and \( d \) have coordinates \((x_1, y_1), (x_2, y_2), (x_3, y_3), \) and \((x_4, y_4)\).
The coordinates of \((ab)\) are \((2x_1 + 2x_2, 2y_1 + 2y_2)\). The coordinates of \((cd)\) are \((2x_3 + 2x_4, 2y_3 + 2y_4)\). The coordinates of \((ab)(cd)\) are \((x_1 + x_2 + x_3 + x_4, y_1 + y_2 + y_3 + y_4)\). Since the coordinates of \((ab)(cd)\) are also \((x_1 + x_2 + x_3 + x_4, y_1 + y_2 + y_3 + y_4)\), \((ab)(cd) = (ac)(bd)\) for all combinations of the nine elements.

Postulate III is valid since the coordinates of \((ab)(ba)\) are
\[
(2x_1 + 4x_2 + 4x_1, 2y_1 + 4y_2 + 4y_1) = (6x_1 + 4x_2, 6y_1 + 4y_2) = (x_2, y_2).
\]

The products of this multiplication are shown in the table of Figure 7.6.

It is also possible to define the product of any two of the points of 25-point geometry so that postulates I, II, and III are valid. The table of Figure 7.7 contains such products where \( ab = c \) if point \( b \) is rotated 60° counterclockwise about point \( a \) to obtain point \( c \). In Figure 2.1 each element in block II can be obtained by rotating the corresponding element in block I about point \( A \). (Rotation about another point can be obtained by a translation of elements in blocks I and II.)
Since for the elements in 25-point geometry the product ab was defined as the point obtained by rotating the point represented by b through 60° about the point represented by a it is possible to compute equations of this transformation, using the coordinate system of Figure 4.1. If pq = r where p, q, and r represent points with coordinates (xp, yp), (xq, yq), (xr, yr) then xr and yr are given by the following equations.

\[
x_r = 3x_p + 3x_q + 4'y_p + 1'y_q \quad (\text{mod } 5)
\]
\[
y_r = 1'x_p + 4'x_q + 3y_p + 3y_q \quad (\text{mod } 5)
\]

(In these equations, the multiplication table of Figure 4.2 is used.) These equations combined with routine computations,
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verify that each of the postulates: $aa = a$, $(ab)(cd) = (ac)(bd)$, and $a(ba) = b$ is valid for the operation tabulated in Figure 7.7.

The stated purpose of this section was to find theorems whose validity followed from the original set of three postulates. The several models of operations which satisfy these postulates show that some properties of common operations are not implied by the postulates and suggest conjectures about some properties which may depend on the postulates.

Consider any closed operation for which

- P·I $aa = a$
- P·II $ab)(cd) = (ac)(bd)$
- P·III $a(ba) = b$

Simple examples show that the operation need not be commutative that it need not be associative, and that no identity element need exist. In constructing previous models it was noted that if $AB = C$ then $BC = A$ and $CA = B$; this is now proved. In each of the theorems I - VI, the "multiplication" operation is any closed operation which satisfies postulates I, II, and III.

Theorem I: If $ab = c$, then $bc = a$ and $ca = b$.

Proof: Let $ab = c$. By the uniqueness of an operation $b(ab) = bc$. By P·III, $b(ab) = a$. By properties of the equality relation, $bc = a$. Likewise $c(bc) = ca = b$.

Theorem II: $a(bd) = (ab)(ad)$

Proof: By P·I, $a(bd) = (aa)(bd)$

By P·II, $(aa)(bd) = (ab)(ad)$. 

Theorem III: $(ab)c = (ac)(bc)$

Proof: By P·I, $(ab)c = (ab)(cc)$

By P·II, $(ab)(cc) = (ac)(bc)$. 
Theorem IV: \( a(ba) = (ab)a = b \)

Proof: By theorem II, \( a(ba) = (ab)(aa) \)

By P•I, \( (ab)(aa) = (ab)a \)

By P•III, \( a(ba) = b \)

Some binary operations have related inverse operations and some operations have a cancellation property. The previous models of operations satisfying postulates I, II, and III do not have the same element repeated in any row or column of a product table and thus exhibit the cancellation property. This suggests that inverse operations may also exist.

Theorem V: Cancellation property

If \( ba = ca \) then \( b = c \) and if \( ab = ac \) then \( b = c \).

Proof: Assume \( ba = ca \). Then \( a(ba) = a(ca) \).

By P•III, \( b = c \).

Assume \( ab = ac \). Then \( (ab)a = (ac)a \).

By theorem IV, \( b = c \).

A notation for inverse operations will be introduced. Let \((x)\) represent a closed binary operation on a set \( S \). The equality \( a(x)b = c \) contains three ordered elements. For this operation one can ask if it is possible to find the second element \( b \) when the third element \( c \) and the first \( a \) are given. If \( b \) is uniquely determined when \( c \) and \( a \) are any elements of the set \( S \), this produces an operation \( \bar{x} \) for which \( c \bar{x}a = b \). The operation \( \bar{x} \) will be called the primary inverse of operation \((x)\).

Let \( a(x)b = c \). If \( a \) is uniquely determined when \( c \) and
b are elements of S, there is an operation \( \chi \), called the secondary inverse of operation \((x)\), so that \( c \chi b = a \).

**Theorem VI:** Let \((x)\) be a closed operation which satisfies postulate III, then inverse operations \( \bar{\chi} \) and \( \breve{\chi} \) exist;
\[
a \bar{\chi} b = a (x) b \quad \text{and} \quad a \breve{\chi} b = b (x) a.
\]

**Proof:** This is a restatement of theorem I which was proved using only postulate III. Let \( a (x) b = c \) so \( b (x) c = a \) and \( c (x) a = b \).

Let \( p, q, \) and \( r \) be specific elements of \( S \) for which \( p (x) q = r \). By definitions of \( \bar{\chi} \) and \( \breve{\chi} \), \( r \bar{\chi} p = q \) and \( r \breve{\chi} q = p \).

By theorem I, \( q = r (x) p \) and \( p = q (x) r \) so \( r (x) p = r \bar{\chi} p \) and \( r \breve{\chi} q = q (x) r \).

Expressing these results with \( a \) and \( b \) representing any elements of set \( S \) gives
\[
a \bar{\chi} b = a (x) b \quad \text{and} \quad a \breve{\chi} b = b (x) a.
\]

**Theorem VII:** If an operation satisfying postulate III is commutative the primary and secondary inverses are the same and each is identical with the original operation.

**Proof:** From theorem VI, \( a \bar{\chi} b = a (x) b \) and \( a \breve{\chi} b = b (x) a \). Since \( x \) is commutative, \( a (x) b = b (x) a \).

Hence \( a \bar{\chi} b = a (x) b = a \breve{\chi} b \).

In the operation of Figure 7.7, \( ab = c \) if point \( b \) is rotated \( 60^\circ \) counterclockwise about \( a \) to obtain point \( c \). The corresponding operation in the Euclidean plane can suggest
relationships since the same point can be obtained by different combinations of permissible rotations.

The diagram indicates a few points which can be expressed in terms of two original points \( a \) and \( b \). Obviously as more points are added, the expressions for these points in terms of \( a \) and \( b \) become more complex. Partly to simplify computation and partly to create a mathematical system with two related binary operations, an operation called addition is defined in terms of the multiplication operation.

A binary operation called addition and represented by the symbol "+" is defined for all elements of a set \( S \), in terms of a multiplication operation which satisfies postulates I, II, and III, so that \( a + b = (ab)(ba) \).
For the multiplication table of Figure 7.5, the corresponding addition table is shown in Figure 7.8. For addition in the set of four elements of Figure 7.5 it is noted that addition is identical with the multiplication operation.

**Figure 7.8**

<table>
<thead>
<tr>
<th>+</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>A</td>
<td>A</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>B</td>
<td>B</td>
<td>B</td>
<td>B</td>
<td>B</td>
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<tr>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
</tr>
<tr>
<td>D</td>
<td>D</td>
<td>D</td>
<td>D</td>
<td>D</td>
</tr>
</tbody>
</table>

Simple examples show that addition need not be commutative nor associative, that an identity element may not exist, and that the cancellation property need not hold.

For the 25-element multiplication of Figure 7.7 where multiplication is a $60^\circ$ rotation of the second element about the first, addition has a simple geometric interpretation. $a + b = c$ if point $b$ is the midpoint of the segment joining $a$ to $c$.

$$(b + a) = (ba)(ab)$$

$$(ab)(ba) = (a + b)$$
This observation leads to conjectures including $a + a = a$, $a + b = a$ and $(a + b)(b + a) = ab + ba$. The preceding conjecture suggests the possible generalization $(a + b)(c + d) = ac + bd$.

**Theorem VIII: $a + a = a$**

**Proof:** $a + a = (aa)(aa) = aa = a$

**Theorem IX: $(a + b) + b = a$**

**Proof:**

$(a + b) + b = \[(a + b)b\][b(a + b)]] \quad \text{Definition}$

$= \[(a + b)(ba)b\][b(ab)(ba)] \quad \text{Definition}$

$= \[(a + b)[(ba)b][b(ab)(ba)] \quad \text{Theorems II & III}$

$= \[(a + b)[(ba)[b(ab)] \quad \text{Postulate III}$

$= \[(a + b)[(ba)](ab) \quad \text{Theorems II & III}$

$= [b(ba)][(ab)b] \quad \text{Postulate III}$

$= [b(ab)][(ab)b] \quad \text{Postulate II}$

$= aa = a \quad \text{Postulates IV & I}$

**Theorem X: $(a + b)(c + d) = ac + bd$**

**Proof:** $(a + b)(c + d) = [(ab)(ba)][(cd)(dc)] \quad \text{Definition}$

$= [(ab)(cd)][(ba)(dc)] \quad \text{Postulate II}$

$= [(ac)(bd)][(bd)(ac)] \quad \text{Postulate II}$

$= ac + bd$

Since $ac + bd = [(ac)(bd)][(bd)(ac)] \quad \text{Definition}$
Theorem XI: If multiplication is commutative
\[ a + b = ab = b + a \]

Proof: 
\[
\begin{align*}
a + b &= (ab)(ba) & \text{Definition} \\
&= (ab)(ab) & \text{Hypothesis} \\
&= ab & \text{Postulate I}
\end{align*}
\]

Likewise
\[
\begin{align*}
b + a &= (ba)(ab) = (ab)(ab) = ab
\end{align*}
\]

Multiplications which satisfy postulates I, II, and III and additions as defined need not be associative so it seems unlikely that a distributive property holds. By theorem X, \(a(b + c)\) and \((a + b)c\) can be changed to alternate forms.

Theorem XII: \(a(b + c) = ab + ac\)

Proof: 
\[
\begin{align*}
a(b + c) &= (a + a)(b + c) & \text{Theorem VIII} \\
&= ab + ac & \text{Theorem X}
\end{align*}
\]

Theorem XIII: \((a + b)c = ac + bc\)

Proof: 
\[
\begin{align*}
(a + b)c &= (a + b)(c + c) & \text{Theorem VIII} \\
&= ac + bc & \text{Theorem X}
\end{align*}
\]

None of the multiplication operations investigated in preparing the material of this section satisfied both of postulates I and III but failed to satisfy postulate II. The question of the independence of postulate II seems appropriate for further study.

The extensive table of Figure 7.7 can be used in preparing code messages. Any given letter of the alphabet, except Z, can
be located in the body of the table in any of 25 different locations and the two letters whose product is the given letter can be substituted for the letter. To decode the message one replaces two consecutive letters by the single product. That SITROINAPUNASSKHTMAVMS is one of the many code names for Mississippi illustrates how the code disguises the fact that the same letter may be repeated.

It is possible to construct other models which satisfy postulates I, II, and III. In particular a multiplication table for products of 49 elements is suggested by 49-point finite geometry. At least one such table exists and would be more useful for code messages than a table with only 25 elements.

7.5 A Point To Line Transformation

Ruth Anne Prince has used 31-point and 57-point geometries to help formulate conjectures about properties of a particular transformation [11]. Her thesis may contain the first published account of using a finite non-projective geometry to discover Euclidean geometry theorems. The present section contains a summary of portions of Chapter 2 of Miss Prince's work.

Let ABC be a given triangle. Let P be any point not on a line through two vertices. Relate to point P a line p by the following construction (Figure 7.9). Let AP cut BC at L, BP cut CA at M, and CP cut AB at N. Let LM cut AB at S, MN cut BC at R, and NL cut CA at T. (Then, by the theorems of Ceva and Menelaus, R, S, and T are collinear.) The line through R, S,
and T, say line p, is the image of point P in a transformation designated as the T-transformation. The point P is also the image of line p in the T-transformation. Given line p cutting sides of triangle ABC in points R, S, and T, points L, N, and M are constructed so these points and R, S, and T divide the sides of ABC internally and externally in the same ratios. Lines AL, BM, and CN intersect at point P. This construction can be made
using only a straight-edge (see Miller, College Geometry, page 43 in [9]).

Limiting cases of the above construction suggest extending the definition to cover special points and lines. The T-image of a point on a side, but not a vertex, of triangle ABC is the side through the point. The image of a vertex is the set of all lines through the vertex. The image of a side of triangle ABC is the set of all points on the side. The image of a line through a vertex, but not a side, is the vertex.

A primary purpose of Miss Prince's study was to discover properties of the T-transformation. For example, by computing the T-images of all points on a line for a finite geometry, Miss Prince made the conjecture that the resulting configuration is a line conic. She proved this, and other conjectures, by use of analytic geometry.

Figure 7.10 contains the three blocks considered previously for 31-point geometry. The numbers, 1-30 shown to the left of and below the rows and columns, represent the corresponding lines. Number 31 represents the ideal line.

For any triangle in 31-point geometry it is possible to tabulate the images of all points and lines in the T-transformation. Figure 7.11 contains this information for the specific triangle with vertices A, D, and V.

Example 7.5-1

Find the T-image, with respect to triangle ADV, of the set of all lines through point H.
Solution: From Figure 7.10, the lines through H are those numbered 2, 8, 12, 19, 21, and 30. The corresponding image points, from Figure 7.11, are: $I_4$, $Q$, $V$, $P$, $A$, and $D$. The image points form a parabola that passes through the vertices of triangle ADV.

Example 7.5-2

Use Figure 7.11 to find the image of the set of all lines through point R.
Solution: The images of lines 4, 8, 13, 18, and 22 are K, Q, D, W, and I. The image of line 26, a side of triangle ADV, is the set of points A, R, J, V, N, and I. The image is a pair of lines; one is side AV of triangle ADV, the other is a line through vertex D. (Thus the image is a degenerate conic through the vertices of triangle ADV.)

Figure 7.11
T-Images for Triangle ADV

<table>
<thead>
<tr>
<th>Point</th>
<th>Line</th>
<th>Point</th>
<th>Line</th>
<th>Point</th>
<th>Line</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1,6,11,16,21,26</td>
<td>K</td>
<td>4</td>
<td>U</td>
<td>10</td>
</tr>
<tr>
<td>B</td>
<td>1</td>
<td>L</td>
<td>14</td>
<td>V</td>
<td>5,7,12,17,24,26</td>
</tr>
<tr>
<td>C</td>
<td>1</td>
<td>M</td>
<td>24</td>
<td>W</td>
<td>18</td>
</tr>
<tr>
<td>D</td>
<td>1,9,13,20,24,30</td>
<td>N</td>
<td>26</td>
<td>X</td>
<td>28</td>
</tr>
<tr>
<td>E</td>
<td>1</td>
<td>O</td>
<td>15</td>
<td>Y</td>
<td>25</td>
</tr>
<tr>
<td>F</td>
<td>24</td>
<td>P</td>
<td>19</td>
<td>I_1</td>
<td>27</td>
</tr>
<tr>
<td>G</td>
<td>23</td>
<td>Q</td>
<td>8</td>
<td>I_2</td>
<td>1</td>
</tr>
<tr>
<td>H</td>
<td>3</td>
<td>R</td>
<td>26</td>
<td>I_3</td>
<td>22</td>
</tr>
<tr>
<td>I</td>
<td>29</td>
<td>S</td>
<td>31</td>
<td>I_4</td>
<td>2</td>
</tr>
<tr>
<td>J</td>
<td>26</td>
<td>T</td>
<td>24</td>
<td>I_5</td>
<td>26</td>
</tr>
</tbody>
</table>

Examples 7.5-1 and 7.5-2 suggest that the T-image of all lines through a given point is a point conic passing through the
vertices of the original triangle. This conjecture is easily tested for triangle ADV in 31-point geometry.

Example 7.5-3

Test the conjecture of the above paragraph for several more points, using Figure 7.11.

(a) The image of all lines through vertex A is the set of points on lines AD and AV.

(b) The image of all lines through point K is the hyperbola ADVHI₁I₃.

(c) The image of all lines through point N, the midpoint of side AV, is a set of parallel lines; one is side AV, the other is parallel to AV through D.

(d) The image of all lines through the ideal point I₁ is the ellipse ADVQSU.

In her thesis, Miss Prince tabulates all points and lines and the corresponding T-transformation image points for triangle 1A1B2D in 57-point geometry. From finite geometry examples, she formulates conjectures which lead to valid theorems in the Euclidean plane (or in the Euclidean plane augmented by ideal points). Several of these theorems are quoted below.

If P is a point not on a line through two vertices of the triangle which determines the T-transformation, the image of all lines through P is a non-degenerate point conic through the vertices of the triangle, for any other point the corresponding image is a degenerate conic through the vertices.
If a line does not pass through a vertex of the triangle which determines the T-transformation, the image of all points on the line is a line conic (the set of lines tangent to a point conic); the sides of the triangle belong to the image set.

For a given triangle ABC, a point P has for its T-image a line p. The T-image of all lines through P is a conic k. The point P and line p are pole and polar with respect to the conic k.

In a study of properties of a point transformation, invariant points and curves are usually discussed. The T-transformation obviously has no invariant points or lines but if the vertices of a triangle have the sides of the triangle as images, invariant triangles will exist. Miss Prince found that there are invariant triangles in the T-transformation for 57-point geometry but that no invariant triangles exist in 31-point geometry. Methods of analytic geometry were used to test the existence of invariant triangles in Euclidean geometry. The analysis showed that there are no real invariant triangles since the coordinates of a vertex of an invariant triangle must contain \( \sqrt{-3} \). This analysis explains the lack of invariant triangles in 31-point geometry, since \( \sqrt{-3} \) (mod 5) is not an integer, and the appearance of invariant triangles in 57-point geometry where \( \sqrt{-3} = 2 \) (mod 7).

In classifying the type of conic associated with the T-image of the set of lines through a point in the plane of a
triangle which determines the T-transformation it is found that an ellipse is tangent to the sides of the triangle at the three midpoints of the sides. The T-image of the set of all lines through a given point inside this ellipse, on this ellipse, or outside this ellipse is, respectively, an ellipse, parabola, or hyperbola.

It is well-known that five points, no three collinear, determine a unique conic. Elementary theorems of projective geometry show how to construct additional points on the conic through five given points using a straight-edge only. The T-transformation can be applied to solve this problem. Let points A, B, C, D, and E, no three collinear, be given. Construct lines d and e, the T-images of D and E with respect to triangle ABC. Let d and e intersect at point X. Every line through X has for its T-image a point on the conic through A, B, C, D, and E.

Example 7.5-4

Use the information in the above paragraph, and Figure 7.11, to find the sixth point on the conic through A, D, V, X, and I_4.

Solution: For triangle ADV the T-images of X and I_4 are lines 28 (QIUMEI_5) and 2 (FGHIJI_2). Of the six lines through the intersection I, only line 22 does not correspond to a given
point. Since the image of line 22 is point $I_3$, the sixth point on the conic is $I_3$.

Example 7.5-5

For 31-point geometry show that there is a conic tangent to the sides of triangle ADV at the midpoints of the sides.

Solution: Let the midpoints $E$, $M$, and $N$ be on the conic. Eliminate all other points on lines $EM$, $EN$, and $MN$. Also eliminate the remaining points on lines $EA$, $MD$, and $NA$ to make each side of the triangle tangent to the conic. Choose $H$ as a possible fourth point on the conic. Eliminate points on lines $HE$, $HM$, and $HN$. This leaves only points $X$ and $Y$. The ellipse $EHMNXY$ is tangent to the sides of triangle ADV at the midpoints of the sides.

Example 7.5-6

Verify that for triangle ADV in 31-point geometry the T-image of all points on line $KLMNOI_2$ is a line conic.

Solution: Figure 7.10 and 7.11 show that the images of points on the given line are the lines $PQRSTI_2$, $YCFNI_4$, $VTMFDI_6$, $ARJVN_5$, $MPXBJI_4$, and $ABCDI_2$. Note that the only point on line $PQRSTI_2$ which is not on the remaining five lines is $S$. This method shows that the six lines, in the given order,
are tangent to points \( S, Y, X, E, I_5, \) and \( I_6 \) of the hyperbola \( SYXEI_5I_6 \).

Example 7.5-7

Formulate a conjecture about the image of the ideal line in the \( T \)-transformation.

Solution: From Figure 7.11 the image of the ideal line is point 3, the centroid of triangle \( ABD \). From the original definition of the \( T \)-transformation it is simple to verify that the image of the ideal line is the centroid of the given triangle.

In Figure 7.9 if points \( R, S, \) and \( T \) are the ideal points on lines \( BC, AB, \) and \( CA \), points \( L, M, \) and \( N \) are midpoints of the sides and \( P \) is the centroid of triangle \( ABC \).

7.6 A Non-linear Transformation
Associated with a Triangle

In 1962 the writer completed a master's thesis concerned with properties of a transformation associated with a triangle [8]. At that time she was not aware of the existence of the type of finite geometry discussed in the present study. This section contains several results from the earlier thesis and suggests how finite geometry might have been helpful in finding properties of the transformation.

The theorem quoted below was suggested by a student Dale Fry, at the Ohio State University. A transformation defined in terms of this transformation is called the \( F \)-transformation.
Theorem I: Let $P$ be a point in the plane of triangle $ABC$. Let perpendiculars from $P$ cut $BC$, $CA$, and $AB$ at $A'$, $B'$, and $C'$, respectively (Figure 7.12). Then perpendicular lines from $A$, $B$, and $C$ to lines $B'C'$, $C'A'$, and $A'B'$ meet in a point $P'$. 

Figure 7.12

(A proof of this theorem can be based on Theorem 1, page 67 in [9].)

Definition: In Theorem 1, the transformation which carries point $P$ into $P'$ is called the F-transformation.

Remark: By a limiting case of the above definition the F-image of any point on a side of triangle $ABC$ is the opposite vertex. The construction of Theorem 1, however, defines neither the image of a vertex of $ABC$ nor the image of a given ideal.
point. Finite geometry will be used to suggest a suitable
definition for these cases and to illustrate the formation of
conjectures about theorems related to the F-transformation.

F-images can be determined in finite geometries. To il­
thesize the use of the above definition, the F-image of
point A will be computed with respect to triangle LHM of 31-
point finite geometry. The three blocks of 31-point geometry
are repeated below in Figure 7.13, for reference.

Figure 7.13

<table>
<thead>
<tr>
<th>I_1</th>
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<th>I_1</th>
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<th>I_3</th>
<th>I_3</th>
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<th>I_3</th>
<th>I_5</th>
<th>I_5</th>
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</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
<td>C</td>
<td>D</td>
<td>E</td>
<td>I_2</td>
<td>I_2</td>
<td>I_2</td>
<td>I_2</td>
<td>I_2</td>
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<td>V</td>
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<td>X</td>
<td>Y</td>
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</tr>
</tbody>
</table>

To find the F-image of A with respect to triangle LHM
first drop perpendiculars from A to the sides of triangle LHM.
AH l LH at H, AC l MH at C, AK l LM at K. Then drop perpendic­
ulars to sides of triangle HCK from points H, M, and L.
Line FGHJ is perpendicular to CH at H.
Line MTFDV is perpendicular to KC at T.
Line LERFX is perpendicular to KH at E.

Since these three lines intersect at F, the F-image of A is F.

In the same way the F-image of any ordinary point, not on a side of triangle LHM, can be computed. By definition, if a point is on the side of triangle LHM, but not a vertex, the F-image is the opposite vertex. The computed images are listed below. Note that $I_1$ is the ideal point on side HM so the image of $I_1$ is the third vertex, L.
The above table shows no image for the vertices H, L, and M and for ideal points I₃, I₄, and I₆. An interesting property is noted in the table; the image of A is F, the image of F is A; the image of B is U, the image of U is B; the image of C is L, the image of L is missing; the image of D is M the image of M is missing; the image of E is I₄, the image of I₄ is missing.

In each case in which the image is not missing, if the F-image of a point Z is Z', then F-image of Z' is Z. The observation that the F-transformation seems to be self-inverse leads to the additional definition which provides F-images for all points.

Definition: The F-image of a vertex of triangle ABC is the set of all points on the opposite side. The F-image of an ideal point, not on a side of triangle ABC is the ordinary point which has the ideal point as its F-image.

By the definition the F-image of vertex H, for triangle LHM in 3L-point geometry, is the set of points on line KLMNOI₂. The F-image of the ideal point I₃ is point T. The complete set of image points for triangle LHM is shown in Figure 7.14.
Example 7.6-1

Discuss the F-image of points on the ideal line.

Solution: For triangle LHM, Figure 7.14 shows that the F-image of the ideal line is the set of points HMELGT. This is the circle through the vertices of triangle LHM.

Conjecture 1: In Euclidean geometry, the F-image with respect to triangle ABC of a point on the circumcircle of triangle ABC is an ideal point.

Proof: Let P be a point on the circumcircle of triangle ABC. Let perpendiculars from P cut sides AB, BC, and CA at C', A', and B'. By the theorem of Simson (Miller, College Geometry, page 36 in [9]) points C', A', and B' are collinear. Then lines through A, B, and C perpendicular to lines B'C',
C'A', and A'B' are perpendicular to the same line so they intersect in a common ideal point, P', which is the F-image of P.

Example 7.6-2

For triangle LHM find F-images of several lines.

Solution: From Figure 7.14 it is found that the line AFKPU1 has for its F-image the ellipse FAHMEL.

The image of line PQPSTI is the parabola MJLVH.

The image of line XGTCKI5 is the hyperbola II6I3LHM.

The image of line VDFMTI6 is the set of intersecting lines SMAYGI3 and HPDLYI5.

The results of Example 7.6-2 lead to a conjecture about the F-image of any line.

Conjecture 2: The F-image of a line is a conic through the vertices of the triangle which determines the transformation.
If the line does not cut the circumcircle of the triangle, the F-image is an ellipse; if the line is tangent to the circumcircle, the image is a parabola; if the line cuts the circumcircle in two points, the image is an hyperbola; if the line passes through a vertex of the triangle, the image is a degenerate conic.

This conjecture may be tested using other triangles in 3l-point or in 57-point geometry. It can be proved correct by use of analytic geometry. Figure 7.14 suggests that the F-transformation is self inverse; this can also be proved by analytic geometry.

If these conjectures are valid then, given five points, no three collinear, other points on the conic through these points can be constructed as follows. Choose three given points as vertices of a triangle. Find F-images of the other two points. Find the line through these image points. The F-images of points on this line are points on the desired conic.

Example 7.6-3

Given five points 1B, 2C, 3A, 3F, and 7E on a conic in 57-point geometry can one find the remaining three points by using the conjectures about properties of F-transformation?

Solution: Choose any three convenient points out of the five points such as 1B, 2C, and 3F to form a triangle. Then find the F-images of points 3A and 7E with respect to the
triangle 1B2C3F. The F-images of 3A and 7E are 7C and 5B. The F-images of all the points on the line 5B7C are listed below.

<table>
<thead>
<tr>
<th>Points on 5B7C</th>
<th>3A</th>
<th>5B</th>
<th>7C</th>
<th>2D</th>
<th>4E</th>
<th>6F</th>
<th>1G</th>
<th>0Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>F-images</td>
<td>7C</td>
<td>7E</td>
<td>3A</td>
<td>2C</td>
<td>3F</td>
<td>1A</td>
<td>1B</td>
<td>2B</td>
</tr>
</tbody>
</table>

W.R.T. 1B2C3F

These eight images 1A, 3A, 1B, 2B, 2C, 7C, 3F, 7E are on an ellipse; they are the points on the conic previously discussed in Example 5.4-5.

This example supports previous conjectures about properties of the F-transformation. Mathematical proofs of these conjectures can be carried out by using a suitable coordinate system and deriving formulas for the F-transformation.

Although the use of examples from finite geometry to suggest conjectures in Euclidean geometry can be fruitful, the method has obvious disadvantages. Figure 7.14 shows that there are no invariant points in the F-transformation determined by triangle LHM. For each Euclidean triangle the corresponding F-transformation has four invariant points which are the incenter and the three excenters of the triangle. The F-transformation for an equilateral triangle in 31-point geometry does have four invariant points since angle bisectors exist for this triangle.

At the beginning of this chapter it was stated that a student might invent interesting geometric transformations. One class of transformations related to a triangle locates the image of a given point as the intersection of three lines determined
by the given point and the triangle. Because of the extra symmetry, such transformations related to equilateral triangles may be of special interest. Three examples are listed below:

1. Given triangle ABC and point P let perpendiculars from P cut the sides BC, CA, and AB at points L, M, and N. Let the circle through L, M, and N cut the corresponding sides in the second set of points L', M', and N'. Then perpendiculars to the sides at points L', M', and N' meet at P', the image of P.

2. Given triangle ABC and point P let lines AP, BP, and CP cut the sides of ABC at points L, M, and N. Let the circle through L, M, and N cut the sides in additional points L', M', and N'. Then AL', BM', and CN' intersect at P', the image of P.

3. Given triangle ABC and point P draw lines AP, BP, and CP. Reflect line AP about a bisector of angle A, reflect BP about a bisector of angle B, and reflect CP about a bisector of angle C. The three reflected lines meet in point P', the image of P.

Each of transformations I, II, and III is defined so that the transformation must agree with the corresponding inverse transformation. Each transformation is well defined for most of the points in a plane but some exceptional points require special attention. Some properties for each of these transformations can be conjectured by a study of the transformation for an equilateral triangle in 31-point geometry. Valid conjectures which might be discovered this way include the fact that
transformations I and III are identical and that both are identical with the F-transformation.

The reader will recall that the statement of the problem included ten ideas which might be illustrated by the examples developed in the dissertation. It is hoped that these ideas have been sufficiently illustrated; but, by its very nature, the dissertation is open-ended. Other illustrations might have been given. There is certainly room for further investigation. The writer hopes that such investigations will be made; and that her dissertation will serve her purpose of interesting some readers in conducting such investigations.
BIBLIOGRAPHY


