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DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

James Terrence Flynn, B.S. in E.E., M.Sc.

*****

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CHAPTER I
INTRODUCTION

A. Purpose of Investigation

The effects on propagation of thin permittivity profiles having greatly varying parameters have received relatively little attention. This problem is of particular importance in re-entry boundary layers in which the electron density changes by several orders of magnitude in a fraction of a wavelength. Figure 1 shows a typical plasma region and its permittivity profile. The importance of considering boundary layer effects has increased as laboratory and space-flight measurements have become more precise.

The analysis performed here was required to explain measurements made in a plasma-loaded waveguide. It will be shown that plane wave solutions for oblique incidence on an infinite plasma region of finite thickness are amenable to the geometry of the waveguide problem. Only the situation of transverse electric fields is considered in this work, since the TE$_{01}$ mode was used in making the measurements. Higher order effects involving diffraction and TM waves were not needed to obtain an adequate analysis. The TE$_{01}$ mode was analyzed using an equivalent form composed of TEM waves.
Reflection and transmission coefficients were found exactly for a quite general plasma profile. This general profile was fitted to a qualitatively known profile derived from gas flow considerations in a shock tube experiment. In this way the plasma parameters were synthesized from the matching of the measured data by the reflection and transmission coefficients found for the general profile. The
results have been compared with calculations for single plasma slabs and multilayer approximations substituting pure air for the plasma boundary regions. It was found that the multilayer approximation is useful to explain the measurements made at low shock velocities, but that a complete description of the magnitude and phase of the measurements requires consideration of the inhomogeneous boundary layers. The theoretical results agreed with measurements showing that the TEM analysis was justified and that the solutions for the general model were quite useful for explaining the experimental results.

B. Application of Results

Exact experimental measurements of the profile of the plasma boundary layer are difficult to obtain because the layer is so thin. However, the effects of the layer are apparent. Propagation measurements cannot be fully explained by a completely homogeneous slab model. Thus, it is desirable to have a sufficiently general theoretical model for a plasma profile that accurately explains the propagation effects that are actually measured.

In this work the approach to the diagnostic problem is to match experimental measurements of reflection and transmission coefficients by exact calculations for a general profile that approximates the actual plasma distribution. Shock tube generated plasmas in a symmetrical, rectangular configuration were used to prove the utility of this
application of the theoretical work. Asymmetrical plasmas, typical of re-entry problems, are also discussed with regard to the diagnosis problem.

C. **Summary of the Literature**

In this work a contribution is made by concentrating on a special case, specifically, oblique incidence of a TE wave on thin plasma profiles. Little analysis on such a problem is available from the literature. Some recent contributions to plasma propagation problems can certainly be mentioned, however.

Important summaries of analysis techniques are available in the works of Budden, Brekhovskikh and Wait. Richmond has applied the WKB technique and also the numerical methods of finite differences to achieve results for both plane layers and cylindrical configurations. The finite difference method has been verified by the exact solutions of the Gaussian profile in the present work. Stickler was able to obtain simple expressions for upper bounds on the field intensities in an inhomogeneous region. These are applicable to electrically thick layers, which are not of concern here.

The work of Albini and Jahn has been often cited in recent literature. Their work with trapezoidal distributions of electron density indicates the specific shape is not as significant as the width of the inhomogeneity. A gradual "ramp" is shown to be more
closely matched to free space than is a sharp discontinuity. Their explanation of the periodicity of the reflection coefficient when plotted versus width of the ramp as evidence of interference effects from reflections at the points of discontinuity of the first derivative of the wave number is in conflict with the results of some other authors.

The controversy about reflections from discontinuity in the first derivative of $k(Z)$ has not been resolved. It has been discussed by Lin and Wetzel* and also by Brekhovskikh. By smoothing the distribution in the region of the discontinuity they show that the reflection coefficient is unchanged, indicating reflection is from the bulk of the distribution not from the discontinuities. It appears that further study of the significance of discontinuities in first order and higher derivatives is needed.

Another difficulty has been pointed out by Brekhovskikh. In an inhomogeneous medium it is not valid to attempt to divide the total field which satisfies Maxwell's equations into incident and reflected traveling waves. There is no unique way to effect such a superposition of two waves in an inhomogeneous medium, since the individual waves in themselves do not satisfy Maxwell's equations as they do for a homogeneous medium.
The physical picture of the distributions to be considered in the present work is that of a continuous inhomogeneous region having a finite range and asymptotic to free space at the edges. The difficulties concerning discontinuities are strictly avoided by considering plasma distributions with at least the first derivative continuous. The incident and reflected waves were determined from the behavior of the total field in the region where the plasma is so thin that the permittivity is close to that of free space. This region will be called the asymptotic region.

The work of Epstein\textsuperscript{11} as presented by Brekhovskikh with regard to a special case, the sech\textsuperscript{2} distribution, will be examined and extended to include a plateau region with sech\textsuperscript{2} edges.

The papers by G. P. Bein\textsuperscript{12} and A. H. Kritz\textsuperscript{13} involving multi-layer and numerical integration solutions are examples of recent work utilizing digital computers. Their results are not compared directly in this work, since they involve frequencies and plasma conditions that are not of immediate interest.
A. General Considerations

We will first analyze plane wave incidence on an infinite region. For the case where \( k = k(z) \) and the field components are chosen for a TEM wave

\[
E_x(y, z) \\
E_y = E_z \equiv 0.
\]

The field is solenoidal and the vector wave equation

\[
\nabla^2 \vec{E} + k^2(z) \vec{E} = 0
\]

reduces to the scalar equation

\[
\left( 1 \right) \quad \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + k^2(z) E_x = 0.
\]

The wave number profile may be chosen subject to the physical restrictions that the plasma is linear, isotropic, subjected to no significant external magnetic fields and is not collision dominated. A re-entry type plasma is such a plasma.

In the present work only the TEM case outlined above will be considered. From Eq. (1) is it observed that
\[ \frac{\partial^2 E_x}{\partial y^2} = -k_y E_x = -k^2(z) \sin^2 \theta E_x, \]

so that propagation at an angle \( \theta \) with respect to the \( z \) axis is described by

\[ \frac{d^2 E_x}{dz^2} + k^2(z) (1 - \sin^2 \theta) E_x = 0. \]

We adopt the time convention \( e^{-i\omega t} \) and specify the complex permittivity of the plasma as

\[ \epsilon_r(z) = 1.0 - \frac{\omega^2_p(z) (1.0 - i\nu/\omega)}{\Omega^2} \]

where

\[ \Omega^2 = \omega^2 + \nu^2, \]

\( \nu \) is the collision frequency (rad/sec), and \( \omega_p \) is the plasma frequency

\[ \omega_p^2(z) = \frac{e^2 N_e(z) / \epsilon_0 m_e}{\Omega^2}. \]

The electron density is a function of position along the \( z \) axis; the collision frequency is considered to be a constant.

Two examples of electron density profiles have been chosen for analysis, the Gaussian and Epstein distributions. The wave number in the inhomogeneous region is

\[ k^2(z) = \omega^2 \mu_0 \epsilon(z) = k_0^2 \epsilon_r(z). \]

For the Gaussian distribution

\[ k^2(z) = k_0^2 \left[ 1, 0 - \frac{\omega_p^2 \max(1, 0 - i\nu/\omega) \exp(-cz^2)}{\Omega^2} \right] \]
For the Epstein distribution

\[ k^2(z) = k_o^2 \left[ 1 - 4M \frac{\exp(mz)}{(1 + \exp(mz))^2} \right], \]

where \( m \) is a constant, and \( 4M \) is equal to the complex number

\[ \frac{\omega_{p \text{ max}}}{\Omega^2} (1 - i \nu/\omega). \]

These two functions are continuous and analytic at all points. Furthermore, for large \(|z|\) the wave number \( k^2(z) \sim k_o^2 \). This feature will be referred to as being asymptotic to free space conditions. It can be seen that points \(|z_o|\) can be found in these symmetrical distributions such that \( k^2(z) \) is arbitrarily close to \( k_o^2 \), and the region \(|z| > z_o\) will be referred to as the asymptotic region.

Initially, it will be assumed that the plasma parameters are known so that the wave equation may be expressed and reflection and transmission coefficients found for a general distribution having the above Gaussian form. In a later section the diagnosis problem in which the plasma parameters are found from measured reflection and transmission data will be discussed.

Provided all the plasma parameters in \( k(z) \) are known, Eq. (2) can be solved exactly using the method of Frobenius if no closed form solutions are available. No tabulated solutions are available for the equation with the Gaussian and the Epstein distributions outlined above.
However, asymptotic series relationships are available for the Epstein distribution. The power series solutions for the Gaussian distribution may be evaluated with great accuracy on a computer.

Since \( k(z) \) is asymptotic to the free space wave number, a plane wave solution to Eq. (2) is required in the asymptotic region. Thus any solution, whether in asymptotic series form or in general power series form, is required by the physics of the problem to be asymptotic to a plane wave solution. This is shown schematically in Fig. 1. A solution in the general form \( AE_1(z) + BE_2(z) \), where \( E_1(z) \) and \( E_2(z) \) are independent solutions of Eq. (2) and A and B are complex constants, should be asymptotic to \( \exp(ikz) \) on the transmission side of the plasma region.

In the present work the series solutions are matched to the required plane wave forms in the asymptotic region where \( k(z) \) is close to the free space value. The amplitude of the transmitted wave is assumed to be unity and the matching technique then provides the amplitudes of the incident and reflected waves. Finally, ratios representing the reflection and transmission coefficients are given. It will be shown that evaluation of the exact series at two points is the only requirement for finding the reflection and transmission coefficients.
In addition to the physical requirements for the asymptotic form of the solution the mathematical aspects can be discussed. A theorem and corollary discussed by Hartman\textsuperscript{14} are of particular interest in discussing the wave equation; they will be outlined here. In the following differential equations

\begin{equation}
(i) \quad w'' + k_0^2 w = 0
\end{equation}

and

\begin{equation}
(ii) \quad u'' + k^2(z) u = 0
\end{equation}

let $k_0^2$ be a constant and let $k^2(z)$ be a continuous, complex-valued function for $0 \leq z < \infty$ satisfying

\[
\int_{\infty}^{\infty} |w(z)|^2 |k_0^2 - k^2(z)| \, dz < \infty
\]

for every solution $w(z)$ of (i). Let $u_0(z)$ and $v_0(z)$ be linearly independent solutions of (i). Then for every pair of constants $\alpha, \beta$, there exists at least one solution $u(z)$ of (ii) satisfying

\[
u(z) = \alpha u_0(z) + \beta v_0(z),
\]

\[
u'(z) = \alpha u_0'(z) + \beta v_0'(z), \text{ as } z \to \infty.
\]

A corollary may be written to show the connection to propagation problems. For the equation $w'' + k_0^2 w = 0$ choose the solution $w(z) = \exp(ik_0z)$. Then $|w(z)|^2 = 1$. Also choose independent

\[\ldots\]
solutions for this differential equation

\[ u_0 = \cos k_0 z \]
\[ v_0 = \sin k_0 z. \]

Applying these chosen solutions to the above theorem, the following statements can be made.

Let \( k^2(z) \) be a continuous, complex-valued function on \( 0 \leq z < \infty \) satisfying

\[ \int_{0}^{\infty} |k_0^2 - k^2(z)| \, dz < \infty. \]

Then if \( \alpha, \beta \) are constants, there exists a solution \( u(z) \) of

\[ u'' + k^2(u) = 0 \]

satisfying the asymptotic relation

\[ u \sim \alpha \cos k_0 z + \beta \sin k_0 z. \]

This last equation may be rewritten

\[ u \sim Y e^{i k_0 z} + \delta e^{-i k_0 z} \]

where \( Y \) and \( \delta \) are complex constants. In the present work series solutions about the origin will be obtained for Eq. (7). These exact series expressions will be related to the above asymptotic forms so that reflection and transmission coefficients may be obtained.
Convergence of series solutions of Eq. (2) is guaranteed if the singular point about which the series expansion is taken is a regular singular point. A dilemma may arise if the solution converges but the general asymptotic behavior is unknown. However, it will be shown that by matching the series solutions of Eq. (2) to the above asymptotic forms the plane wave solutions required in the asymptotic region can be found.

B. The Gaussian Distribution

Assuming the electron density to have a Gaussian profile, the previously defined wave number is written

\( k^2(z) = k_0^2 [1 - a(1.0 - ib) \exp(-cz^2)] \)

where

\[
\begin{align*}
    a &= \frac{\omega_p^2 \max}{(\omega^2 + \nu^2)} \\
    \omega_p^2 \max &= e^2 N_\max / \epsilon_0 m_e \\
    b &= \nu / \omega 
\end{align*}
\]

We will assume these values are known so that for the present the problem may be analyzed to find the reflection and transmission coefficients. The inverse diagnosis problem in which the plasma parameters are found from measured reflection and transmission data will be discussed in Chapter III.
Propagation at an angle with respect to the z axis can be related to the angle $\theta_1$ of the incident plane wave in the asymptotic region using Snell's law. The wave equation is

$$\frac{d^2 E_x}{dz^2} + k_o^2 \left[ 1 - a(1-ib) \exp(-cz^2) - \sin^2\theta_1 \right] E_x = 0.$$  

(9)

The geometry is depicted in Fig. 1. To simplify the equation we rewrite it in the form

$$\frac{d^2 E_x}{dz^2} + \left[ a_1 + a_2 \exp(-cz^2) \right] E_x = 0,$$

where the constants have their obvious meaning.

Equation (9) has an essential singularity at $\pm \infty$. The transformation $\zeta = cz^2$ moves the singularity to the origin and thus the solution for large $\zeta$ may be obtained. Applying the transformation, the following equation results

$$\frac{d^2 E_x}{d\zeta^2} + \frac{c_1}{\zeta} \frac{dE_x}{d\zeta} + \frac{1}{\zeta^2} \left[ \zeta c_2 + \zeta c_3 \exp(-\zeta) \right] E_x = 0$$

(10)

$$c_1 = 1/2$$

$$c_2 = k_o^2 (1 - \sin^2\theta_1)/4c$$

$$c_3 = -k_o^2 a(1-ib)/4c$$

The singularity at the origin is a regular singular point. The exponential function may be expanded in a power series about the origin, and the method of Frobenius applied to obtain exact series solutions.
\[ \sum_{n=0}^{\infty} a_n \zeta^n \quad \text{and} \quad \sum_{n=0}^{\infty} b_n \zeta^n \]

where

\[ r_1, r_2 = 0, 1/2. \]

Carrying out the power series solutions in the \( \zeta \)-plane the recursion relations for the coefficients are

\begin{equation}
\begin{aligned}
(11) & \quad r = 0 \\
a_0 & = 1 \\
a_n & = -\left\{ c_2 a_{n-1} + c_3 \sum_{j=0}^{n-1} (-1)^j a_{n-1-j}/j! \right\}/[n(n-1)+nc_1]
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
(12) & \quad r = 1/2 \\
b_0 & = 1 \\
b_n & = -\left\{ c_2 b_{n-1} + c_3 \sum_{j=0}^{n-1} (-1)^j b_{n-1-j}/j! \right\}/[(n+1-c_1)(n-c_1) \\
& \quad + (n+1-c_1)c_1] .
\end{aligned}
\end{equation}

These recursion formulas show that each coefficient depends on all lower order coefficients. The general theory of convergence of such recursion relationships lies in the domain of the theory of finite differences. No closed form asymptotic relationships were found from these expressions. However, it is possible to apply the theorem and corollary mentioned in the previous section.
As shown in the following development, by effecting a normalized solution for the transmitted field in the form of a plane wave of unit amplitude, the reflection and transmission coefficients may be found. This asymptotic normalization is a form of boundary condition in that the exact series expression for the field is forced to fit a unit amplitude plane wave for large $\zeta$.

The two independent solutions of Eq. (10) may be expressed in the $\zeta$-plane as

$$u_1 = \sum_{n=0}^{\infty} a_n \zeta^n$$

$$u_2 = \zeta^{1/2} \sum_{n=0}^{\infty} b_n \zeta^n.$$

In the $z$-plane we denote these solutions by

$$v_1 = \sum_{n=0}^{\infty} a_n c^n z^{2n}$$

$$v_2 = \sqrt{c} z \sum_{n=0}^{\infty} b_n c^n z^{2n}.$$

Since the coefficient $c_3$ in Eq. (10) is generally complex, the power series coefficients $a_n$ and $b_n$ are complex. In the asymptotic region where $\zeta$ is large, the series solutions are evaluated at two points and a linear combination is found that is asymptotic to
\[ \exp(ik\sqrt{\xi/c}) = \exp(ikz) \]

where \( k = k_0 \cos \theta_i \).

A linear combination \( Au_1 + Bu_2 \) may be used to arrive at the unique solution having the asymptotic form stated in the theorem mentioned previously. The individual independent solutions may be written asymptotically

(13) \[ u_1 \sim a \cos(k\sqrt{\xi/c} + \theta_1) + ib \sin(k\sqrt{\xi/c} + \theta_2) \]

(14) \[ u_2 \sim c \sin(k\sqrt{\xi/c} + \delta_1) + id \cos(k\sqrt{\xi/c} + \delta_2). \]

The real numbers \( a, b, c, d \) and \( \theta_1, \theta_2, \delta_1, \delta_2 \) are determined from the values of \( u_1 \) and \( u_2 \) found at two points where \( \xi \) is large. An additional step is needed to effect the matching of the solution \( Au_1 + Bu_2 \) to the unit amplitude transmitted wave. Since

(15) \[ E(\xi) = Au_1 + Bu_2 \sim A\left[a \cos(k\sqrt{\xi/c} + \theta_1) + ib \sin(k\sqrt{\xi/c} + \theta_2)\right] + B\left[c \sin(k\sqrt{\xi/c} + \delta_1) + id \cos(k\sqrt{\xi/c} + \delta_2)\right] \]

it is required that

(16) \[ \frac{A}{B} = \frac{-i(ce^{-i\delta_1} + de^{-i\delta_2})}{(ae^{-i\theta_1} - be^{-i\theta_2})} \]

so that \( Au_1 + Bu_2 \sim \exp(ik\sqrt{\xi/c}) \). The ratio \( A/B \) is used so \( A \) and \( B \) need not be individually evaluated. Thus the parameters needed to match to the transmitted wave have been obtained.
The expression for the total field where \( z < 0 \) is now found so that the amplitude of the incident and reflected waves may be obtained.

Figure 2 shows that the solution to Eq. (10) lies in the right half of the \( \zeta \)-plane. The transmitted wave is a function of \( \zeta = |\zeta| e^{i\theta} \) and the incident and reflected waves are functions of \( \zeta = |\zeta| e^{i2\pi} \). In other words, the real \( \zeta \) axis can be split to describe the result of folding the negative real \( z \) axis into the right half plane through the transformation \( \zeta = cz^2 \). To describe the total field for \( z < 0 \) the \( \zeta \) dependence is \( \zeta = |\zeta| e^{i2\pi} \), and \( \sqrt{\zeta} \) becomes \( |\zeta|^{1/2} e^{i\pi} = -|\zeta|^{1/2} \). Thus for \( z < 0 \) the total field can be written.
\[ E(\xi) = A u_1 + B u_2 \]
\[ = A \left[ a \cos(-k\sqrt{\xi/c} + \theta_1) + ib \sin(-k\sqrt{\xi/c} + \theta_2) \right] 
- B \left[ c \sin(-k\sqrt{\xi/c} + \delta_1) + id \cos(-k\sqrt{\xi/c} + \delta_2) \right]. \]

This expression may be put in a more convenient form by substituting the equivalent forms for the sines and cosines in terms of \( \exp(ik\sqrt{\xi/c}) \) and \( \exp(-ik\sqrt{\xi/c}) \).

\[ Au_1 + Bu_2 \sim \left[ A \left( \frac{a}{2} e^{-i\theta_1} - \frac{b}{2} e^{-i\theta_2} \right) + iB \left( -\frac{c}{2} e^{-i\delta_1} - \frac{d}{2} e^{-i\delta_2} \right) \right] e^{ik\sqrt{\xi/c}} 
+ \left[ A \left( \frac{a}{2} e^{i\theta_1} + \frac{b}{2} e^{i\theta_2} \right) + iB \left( \frac{c}{2} e^{i\delta_1} - \frac{d}{2} e^{i\delta_2} \right) \right] e^{-ik\sqrt{\xi/c}}. \]

Equation (16) may be substituted to give

\[ E(-z) = -iB \left[ c e^{-i\delta_1} + d e^{-i\delta_2} \right] e^{ik\sqrt{\xi/c}} \]
\[ + B \left[ -i\left( ae^{i\theta_1} + be^{i\theta_2}\right) \right] \left( \frac{c}{2} e^{-i\delta_1} + \frac{d}{2} e^{-i\delta_2} \right) \]
\[ + i \left( \frac{c}{2} e^{i\delta_1} - \frac{d}{2} e^{i\delta_2} \right) \left[ e^{i\xi/c} \right]. \]

The ratio of the coefficients gives the reflection coefficient

\[ V = \frac{1}{2} \left[ \frac{ae^{i\theta_1} + be^{i\theta_2}}{ae^{-i\theta_1} + be^{-i\theta_2}} - \frac{ce^{i\delta_1} - de^{i\delta_2}}{ce^{-i\delta_1} + de^{-i\delta_2}} \right]. \]

Similarly, the transmission coefficient is the ratio of transmitted amplitude to the first term of Eq. (17). Thus,
We observe the result that the reflection and transmission coefficients of a symmetrical $k(z)$ which reduces asymptotically to $k_0 \cos \theta_1$ may be expressed in terms of magnitudes and phases found for the asymptotic solutions.

If the inhomogeneous profile is lossless, then the solutions $u_1$ and $u_2$ and consequently $v_1$ and $v_2$ are real functions and the coefficients $b$ and $d$ in the asymptotic forms are zero. Then

\begin{equation}
V = \frac{1}{2} [e^{i2\delta_1} - e^{i2\delta_1}]
\end{equation}

\begin{equation}
D = \frac{1}{2} [e^{i2\theta_1} + e^{i2\theta_1}]
\end{equation}

for the lossless case.

Calculations have been performed for several Gaussian profiles with a wide variety of widths and depths. The series solutions converged, but required so many terms that a computer calculation was necessary. It was found that a practical approximation to the asymptotic region was $\zeta = 6$. By evaluating the series at two points around $\zeta = 6$ from 50 to 100 terms were needed in the series. Of course, the number of terms required is determined by the width and depth of the $k(z)$ profile.
As an example consider the permittivity profile

\[ \varepsilon_r(z) = 1.0 - 1.2 \times (1.0 - 10.1) e^{-23.0 z^2} \]

as plotted in Fig. 3. We find for \( f = 10 \text{ GHz} \) and normal incidence that

\[ u_1 = -0.005419 - 10.1216 \text{ at } \zeta = 6.0, 87 \text{ terms} \]
and \[ u_1 = -0.2154 - 10.1064 \text{ at } \zeta = 7.0, 146 \text{ terms}; \]
\[ u_2 = 0.9232 - 10.06421 \text{ at } \zeta = 6.0, 86 \text{ terms} \]
and \[ u_2 = 0.6974 - 10.06015 \text{ at } \zeta = 7.0, 144 \text{ terms}. \]

These values were matched to the forms

\[ u_1 = a \cos(k_0 \sqrt{\frac{\zeta}{c}} + \theta_1) + ib \sin(k_0 \sqrt{\frac{\zeta}{c}} + \theta_2) \]
and

\[ u_2 = c \sin(k_0 \sqrt{\frac{\zeta}{c}} + \delta_1) + ib \cos(k_0 \sqrt{\frac{\zeta}{c}} + \delta_2). \]

The results were

\[ a = 0.97219 \quad \theta_1 = -1.14260 \]
\[ b = 0.13436 \quad \theta_2 = 2.43209 \]
\[ c = 1.32051 \quad \delta_1 = -0.35158 \]
\[ d = 0.06527 \quad \delta_2 = 0.60362. \]

The phase reference plane was taken at \( \zeta = 6 \text{ or } z = -0.511 \). Substituting these values into Eqs. (18) and (19),

\[ V = 0.6189 e^{+i17.5^\circ} \]
\[ D = 0.6006 e^{-i75.9^\circ}. \]

The computation time was about 3 seconds on the IBM 7094. It is
important to note that if the time convention $e^{+i\omega t}$ had been used, the complex conjugate of $\epsilon_r(z)$ would be used and the complex conjugate of the above coefficients obtained.

$$\epsilon_r(z) = 1.0 - 1.2(1.0 - 1.0) e^{-23z^2}$$

Fig. 3--Gaussian profile.

A numerical method used by Richmond$^5$ for solving for the total field in an inhomogeneous medium was verified by the exact series expressions derived here. The method utilized the difference
equation approximation to the differential equation. Thus,

\[ \frac{d^2 E}{dz^2} = -k_o^2 (\epsilon_r(z) - \sin^2 \theta_i) E \]

is approximated by

\[ \frac{E_{n+1} - 2E_n + E_{n-1}}{h^2} = -k_o^2 (\epsilon_n - \sin^2 \theta_i) E_n \]

where the field is evaluated at points \( n, n+1, n-1 \) and \( h \) is the distance between the points.

The equation

\[ E_{n+1} = [2 - k_o^2 h^2 (\epsilon_n - \sin^2 \theta_i)] E_n - E_{n-1} \]

follows from (23). The field throughout the profile is found by first assuming a traveling wave of unit amplitude having spatial dependence \( e^{ik_0z} \) in the region far from the origin. Evaluating this traveling wave at two points separated by \( h \) gives \( E_0 \) and \( E_1 \) from which \( E_2 \cdots E_n \) may be found by working back through the profile to the \( z < 0 \) side. The standing wave form of the field gives the reflection coefficient. The reflection and transmission coefficients may be found using this method, but not as rapidly as the exact series method in which the field need be evaluated at only two points. The same reflection and transmission coefficients were found by the difference method and the series method.
The difference equation method provides other useful results that can be verified by the exact series method. The value of the field at any point can be calculated by both the series and difference methods and compared. At the origin, the field value from the series solutions is the coefficient $A$ and the slope is $\sqrt{c}B$. These were evaluated and found to be exactly the values given by the difference equation method. It is also possible to determine by the difference method where on the transmission side the solution is not a plane wave of unit amplitude. In the above example, this occurs at $z = 0.46$, showing that a choice of $\zeta = 6$ or $z = +0.511$ is sufficiently far into the asymptotic region to justify matching to the plane wave solution at $\zeta = 6$.

C. **Homogeneous Slab with Gaussian Boundary Layers**

A problem of practical importance is the propagation through a plasma slab that has a thin inhomogeneous layer on its surfaces. The work of the previous section for the Gaussian layer may be adapted to this problem.

The geometry of the problem is shown in Fig. 4. It is necessary to match boundary conditions at $\zeta z_1$ in addition to finding the parameters of the asymptotic solution. In the inhomogeneous region $z > z_1$ the equation
must be solved. Here, the transformation \( \zeta = c(z + z_1)^2 \) is used and the series solutions are again evaluated at two points in the asymptotic region. The linear combination of solutions \( A u_1 + B u_2 \) is asymptotic to \( e^{i k \sqrt{\zeta / c}} \) and Eq. (16) for A and B still holds.

Fig. 4—Plateau with inhomogeneous boundary profiles.
At $z_1$ the continuity of the field and its derivative must be satisfied by the solutions. The field may be written in the $z$-plane as

$$A v_1 + B v_2 = A \sum_{n=0}^{\infty} a_n c^n (z-z_1)^{2n} + B \sqrt{c} z \sum_{n=0}^{\infty} b_n c^n (z-z_1)^{2n}.$$  

Now

$$\left. (A v_1 + B v_2) \right|_{z=z_1} = A$$

and

$$\frac{d}{dz} \left. (A v_1 + B v_2) \right|_{z=z_1} = \sqrt{c} B.$$  

The field inside the homogeneous plateau region will have the form

$$E(z) = Pe^{ikz} + Q e^{-ikz}$$

where

$$k = k_0 (\epsilon_T - \sin^2 \theta_i)^{1/2}.$$  

Therefore at $z = z_1$ the following equations must be satisfied

$$A = Pe^{ikz_1} + Q e^{-ikz_1}$$

$$\sqrt{c} B/ik = Pe^{ikz_1} - Q e^{-ikz_1}.$$  

From these we may find $P$ and $Q$ in terms of $A$ and $B$.

\[(25) \]  \[P = \frac{e^{-ikz_1}}{2} (A + \sqrt{c} B/ik) \]

\[(26) \]  \[Q = \frac{e^{ikz_1}}{2} (A - \sqrt{c} B/ik).\]
At the other edge of the homogeneous region where \( z = -z_1 \) we must also match the same boundary conditions and find coefficients \( F \) and \( G \) so that in the inhomogeneous layer

\[
E(z) = Fv_1 + Gv_2, \quad z < -z_1.
\]

Thus,

\[
F = Pe^{-ikz_1} + Qe^{ikz_1}
\]

\[
G = \frac{ik}{\sqrt{c}} [Pe^{-ikz_1} - Qe^{ikz_1}]
\]

Substituting the above expressions for \( P \) and \( Q \) we find \( F/G \) in terms of \( A/B \).

\[
\frac{F}{G} = \frac{\frac{A}{2B} \left[ X + \frac{1}{X} \right] + \frac{\sqrt{c}}{ik} \left[ X - \frac{1}{X} \right]}{\frac{i}{\sqrt{c}} \left[ X - \frac{1}{X} \right] + \frac{1}{2} \left[ X + \frac{1}{X} \right]}
\]

where

\[
X = e^{-ik2z_1}.
\]

Employing the asymptotic forms of \( v_1 \) and \( v_2 \),

\[
Fv_1 + Gv_2 \sim \left[ F \left( \frac{a}{2} e^{-i\theta_1} - \frac{b}{2} e^{-i\theta_2} \right) + iG \left( -\frac{c}{2} e^{-i\delta_1} - \frac{d}{2} e^{-i\delta_2} \right) \right] e^{ik_0z}
\]

\[
+ \left[ F \left( \frac{a}{2} e^{i\theta_1} + \frac{b}{2} e^{i\theta_2} \right) + iG \left( \frac{c}{2} e^{i\delta_1} - \frac{d}{2} e^{i\delta_2} \right) \right] e^{-ik_0z}.
\]
The reflection coefficient is, therefore,

\[ V = \frac{F}{G} \left( a e^{i\theta_1} + b e^{i\theta_2} + i(c e^{i\delta_1} - d e^{i\delta_2}) \right) \]

\[ \frac{F}{G} \left( a e^{-i\theta_1} - b e^{-i\theta_2} + i(-c e^{-i\delta_1} - d e^{-i\delta_2}) \right) \]

The transmission coefficient is the ratio of the amplitudes of the transmitted wave to the incident wave which gives

\[ D = \left( \frac{a e^{i\theta_1} + b e^{i\theta_2}}{2} \right) + iB \left( \frac{-c e^{i\delta_1} + d e^{i\delta_2}}{2} \right) \]

\[ \frac{F}{G} \left( \frac{a e^{-i\theta_1} - b e^{-i\theta_2}}{2} + iG \left( \frac{-c e^{-i\delta_1} - d e^{-i\delta_2}}{2} \right) \right) \]

Since \( F \) and \( G \) can be expressed in terms of \( A \) and \( B \), \( D \) may be written finally as

\[ D = \frac{A}{B} \left( a e^{i\theta_1} + b e^{i\theta_2} + i(-c e^{i\delta_1} + d e^{i\delta_2}) \right) \]

\[ \left[ \frac{A}{2B} \left( X + \frac{1}{X} \right) + \frac{\sqrt{c}}{2ik} \left( X - \frac{1}{X} \right) \right] \left( a e^{i\theta_1} - b e^{-i\theta_2} \right) \]

\[ + i \left[ \frac{A}{2B} \left( \frac{k}{\sqrt{c}} \right) \left( X - \frac{1}{X} \right) + \frac{1}{2} \left( X + \frac{1}{X} \right) \right] \left( -c e^{-i\delta_1} - d e^{-i\delta_2} \right) \]

D. **Homogeneous Slab with Epstein Boundary Layers**

As an example of a plasma profile for which the wave equation may be put in the form of a differential equation of mathematical physics, the symmetrical Epstein layer has been considered. The mathematical details are outlined in Appendix A. It can be seen that the asymptotic behavior of \( \text{sech}^2 z \) is similar to that of the Gaussian, although the Gaussian decays much more quickly. The
chief advantage to the Epstein profile is that asymptotic series are available for the solutions to the wave equation.

An attempt was made to form a plateau plus Epstein boundary configuration in which the boundary was close to the Gaussian near the plateau edge. Of course, the slower decay of the Epstein profile made a wider plasma model overall.

As shown in the appendix a typical shock tube profile was analyzed for the Epstein boundary profiles. At an angle of incidence of 41° and a frequency of 10 GHz the results were:

$$V = 0.8293 e^{-113.8^\circ}$$
$$D = 0.3566 e^{1125.2^\circ}.$$ 

A plasma profile having the same plateau dimensions but with Gaussian boundary profiles gave the results

$$V = 0.742 e^{1196.7^\circ}$$
$$D = 0.424 e^{-152.7^\circ}.$$ 

The greater width of the sech² profile gave a higher reflection coefficient and a lower transmission coefficient.
E. A Re-entry Model

Many propagation problems occur in which the plasma distribution is asymmetrical. Such a description is especially pertinent to the plasma sheath generated about a vehicle re-entering the earth's atmosphere. In this situation the electron density builds up rapidly in a short space near the skin of the vehicle and then gradually tapers away. A typical plasma distribution adapted from calculations by Swift and Evans\(^\text{15}\) is shown in Fig. 5.

![Fig. 5 -- A re-entry profile.](image-url)
A useful approximation to an asymmetric plasma region can be obtained by using two Gaussians. The methods used previously to evaluate the solutions of the wave equation in the asymptotic region may be used for this case to obtain reflection and transmission coefficients for TE waves incident on such a profile. On the transmission side of the plasma region the series solutions are $u_1$ and $u_2$ where asymptotically,

$$u_1 \sim a_1 \cos(k\sqrt{\xi/c_+}+\theta_{11}) + ib_1 \sin(k\sqrt{\xi/c_+}+\theta_{12})$$

$$u_2 \sim c_1 \sin(k\sqrt{\xi/c_+}+\delta_{11}) + id_1 \sin(k\sqrt{\xi/c_+}+\delta_{12})$$

$$Au_1 + Bu_2 \sim e^{ik\sqrt{\xi/c_+}}$$

$$A = -i \frac{c_1 e^{-i\delta_{11}} + d_1 e^{-i\delta_{12}}}{a_1 e^{-i\theta_{11}} - b_1 e^{-i\theta_{12}}}$$

$$E(z) \sim \left[ A \left( \frac{a_1}{2} e^{i\theta_{11}} + \frac{b_1}{2} e^{i\theta_{12}} \right) + iB \left( -\frac{c_1}{2} e^{i\delta_{11}} + \frac{d_1}{2} e^{i\delta_{12}} \right) \right] e^{ik\sqrt{\xi/c_+}}.$$}

On the incidence side of the plasma region the solutions are $u_3$ and $u_4$ where asymptotically,

$$u_3 \sim a_2 \cos(-k\sqrt{\xi/c_-}+\theta_{21}) + ib_2 \sin(-k\sqrt{\xi/c_-}+\theta_{22})$$

$$u_4 \sim +c_2 \sin(-k\sqrt{\xi/c_-}+\delta_{21}) + id_2 \cos(-k\sqrt{\xi/c_-}+\delta_{22})$$

By matching boundary conditions at the origin the coefficients for a linear combination of solutions $u_3$ and $u_4$ may be found. Thus
for $z < 0$

$$Au_3 + \sqrt{c_+/c_-} Bu_4 \sim A[a_2 \cos(-k\sqrt{\zeta/c_-} + \theta_{21}) + ib_2 \sin(-k\sqrt{\zeta/c_-} + \theta_{22})] - B\sqrt{c_+/c_-} [c_2 \sin(-k\sqrt{\zeta/c_-} + \delta_{21}) + id_2 \cos(k\sqrt{\zeta/c_-} + \delta_{22})].$$

From this asymptotic form of the field which is the same as derived previously for the symmetrical Gaussian distribution the reflection coefficient is obtained

$$V = \frac{(A/B)\sqrt{c_-/c_+}(a_2 e^{i\theta_{21}} + b_2 e^{i\theta_{22}}) + i(c_2 e^{i\delta_{21}} - d_2 e^{i\delta_{22}})}{(A/B)\sqrt{c_-/c_+}(a_2 e^{-i\theta_{21}} - b_2 e^{-i\theta_{22}}) + i(-c_2 e^{-i\delta_{21}} - d_2 e^{-i\delta_{22}}).}$$

The transmission coefficient can be similarly obtained from the asymptotic expressions for the traveling wave in the positive-$z$ direction for $z > 0$ and $z < 0$.

$$D = \frac{(A/B)(a_1 e^{i\theta_{11}} + b_1 e^{i\theta_{12}}) + i(-c_1 e^{i\delta_{11}} + d_1 e^{i\delta_{12}})}{(A/B)\sqrt{c_-/c_+}(a_2 e^{i\theta_{21}} - b_2 e^{-i\theta_{22}}) + i(-c_2 e^{-i\delta_{21}} - d_2 e^{-i\delta_{22}}).}$$

The determination of $V$ and $D$ for the asymmetric Gaussian distribution requires finding 16 parameters from the asymptotic forms of the solutions. These depend on the two parameters describing the plasma and the steepness of the two Gaussian distributions. A numerical example is given in Chapter III, Section B.
CHAPTER III
PLASMA DIAGNOSTICS

A. Shock Tube Measurements

The results of the previous chapter on infinite plasma regions will be adapted to analyze a waveguide loading problem. At the Ohio State University Shock Tube facility\textsuperscript{16} it was necessary to determine if the plasma generated was adequately described by the parameters predicted by gas dynamics theory.\textsuperscript{17} The effects and dimensions of the plasma flow boundary layers were also to be investigated. Measurements using probes and spectroscopy had indicated plasma densities generated were within the order of magnitude predicted by the theory. These measurements of the bulk properties of the plasma were to be verified and a measure of collision frequency obtained by microwave techniques.

The experimental arrangement shown in Fig. 6 was designed to create a rectangular plasma slug in a waveguide. The block diagram of Fig. 7 shows the equipment used to obtain reflection and transmission coefficients in the plasma loaded waveguide. The four-probe slotted line measured both amplitude and phase of the reflection coefficient; only amplitude of transmission was measured.
Fig. 6—"Parallel-plate" test region (or plasma channel).
Fig. 7--Block diagram of the experimental setup.
The measurements and data reduction were performed in 1964 by Messrs. R. J. Plugge and P. Bohley.\textsuperscript{18}

The apertures cut in the side-walls of the X-band waveguide were small so that the propagation in the characteristic $\text{TE}_{01}$ mode was essentially undisturbed. This fact was verified by measurement and theory. Measurements showed only 0.1 dB loss caused by the radiation from the plasma ports. Theoretically, the scattering coefficient of two rectangular $H$-plane holes was considered. This has been discussed by Peake,\textsuperscript{19} who has summarized the work of Collin\textsuperscript{20} and Cohn\textsuperscript{22} for various waveguide perturbations.

The following expression for a shunt admittance is obtained by Peake,

$$y = -\frac{2\Gamma}{1 + |\Gamma|^2},$$

where

$$\Gamma = -j\frac{w^3 \pi^2}{3a^3 b \beta_{g01}}$$

and $w$ is the hole width, $a$ and $b$ are the waveguide dimensions and $\beta_{g01}$ is the wavenumber in the guide for the $\text{TE}_{01}$ mode. The radiation from the plasma ports can be represented by an equivalent shunt conductance found by calculating the radiation from a magnetic dipole on an infinite sheet. This is found to be negligibly small for the accuracy required, as was verified by the laboratory measurements.
The value of $y$ calculated for the waveguide and plasma ports in air when combined with the admittance of the unmatched dielectric plug gave a total input impedance close to the measured value. It appears that the analysis of the problem using TE waveguide modes is entirely valid, since the measured data can be matched rather well. Edge diffraction effects which would generate TM modes appear to be so small that no measurable effect can be found. As a good theoretical approximation reflection and transmission coefficients for the TE$_{01}$ mode in the waveguide may be found by analyzing oblique incidence of TEM plane waves on an infinite plasma region of finite thickness. Measurements were made at two frequencies (9 GHz and 10 GHz) so that any proposed theoretical model for the plasma distribution could be verified independently of frequency.

Aerodynamic calculations were performed to obtain the electron density and collision frequency profiles of the plasma flow between the parallel flat plates. The results of these calculations are shown in Figs. 8 and 9 for a typical set of shock tube parameters. It was also determined by aerodynamic calculations that the general shape of the profiles and particularly the thickness of the boundary layer should change very little over the range of shock velocities used. Although the shape of the profile was predictably constant, the
actual value of electron density could vary from the gas dynamics predictions, thus the diagnosis was necessary.

\[ u = 12.32 \text{ kft/sec} \]
\[ V_0 = 13.7 \text{ kft/sec,} \]
\[ \Lambda_0 = 75 \mu\text{sec/ft} \]
\[ P_1 = 1\text{mm Hg} \]
\[ T_0 = 300^\circ \text{k} \]
\[ T_1 = 4260^\circ \text{k} \]

Fig. 8--Electron density profile.

The angles of plane wave TEM propagation that represents the $\text{TE}_{01}$ mode in the waveguide were calculated from the formula

\[ k_c = k_0 \sin \theta, \text{ where } k_0 = \frac{2\pi}{\lambda_0}, \text{ and } k_c = \pi/b. \]
mode at 9 GHz and 10 GHz are 46.75° and 41.0°, respectively.

Analysis of a TEM plane wave incident at these angles on a plasma region is essentially equivalent to the situation in the waveguide.

The results of the shock tube measurements are shown in Figs. 10 through 15. These are taken from Reference 18. The data obtained at 10 GHz has been averaged over many test shots.
Fig. 10--Insertion loss versus reciprocal shock wave speed (10 GHz).
Fig. 11--Real part of reflection coefficient (10 GHz).

Fig. 12--Imaginary part of reflection coefficient (10 GHz).
Fig. 13--Insertion loss variation with time for representative shots (9 GHz).

and should be quite reliable. A Smith chart display of the reflection coefficient and input impedance measured at 10 GHz is presented in Fig. 16.

In order to interpret the measurements a more detailed discussion of the waveguide loading is required. As shown in Fig. 6 dielectric plugs were sealed into the waveguide so that the shock tube
could be evacuated. These windows were mounted flush with the edge of the plasma ports and were cut one-half wavelength thick so that there was no mismatch in the guide. Originally it had been assumed that the plasmas generated would be very lossy so that there would be little reflection of the microwave energy from the window on the transmission side of the plasma region. Each of the dielectric windows was therefore made to match the waveguide at one of two frequencies, so that the generator could

Fig. 14--Variation of real part of reflection coefficient with time (9 GHz).
Fig. 15—Variation of imaginary part of reflection coefficient with time (9 GHz).

be placed on either side and tuned to face a matched window. Unfortunately, the window on the transmission side could be seen by the waveguide when low velocity shocks were run. This fact is shown by the Smith chart, Fig. 16, where the impedance at an inverse velocity of 78 μsec/ft differs little from that with air in the test region. Further evidence is provided by the previously mentioned calculation of the small effect of the plasma ports. Combining this small effect with the impedance of the mismatched window translated toward the generator shows close agreement
with the measured impedance with no plasma present. Later discussion of theory and experiment will show that for high shock velocity the plasma was indeed sufficiently dense so that the mismatched window had only a small effect on the measurements.

Having obtained the measurements of reflection and transmission, it was necessary to determine the plasma parameters generated by the shock tube. This diagnosis problem was approached by several techniques. In addition to simple homogeneous slab approximations, a general model utilizing a homogeneous slab with inhomogeneous boundary layers was analyzed.

The general model proposed was that of a homogeneous plateau with Gaussian boundary layers which was discussed in the previous chapter. Several parameters describing the plasma and the geometry may be varied to match the measured reflection and transmission coefficients. A few calculations provided enough intuition to specify the proper set of parameters and finally calculate a matching set of coefficients. This trial method could easily be improved for quick calculation by programming the computer to perform the decision making process to arrive at the proper set of parameters.

It was found that the shape of the electron density profile predicted by the aerodynamic calculations was also specified by the microwave calculations. The collision frequency was considered
a constant in the plasma region. The general form of the complex permittivity in the plasma was \( \varepsilon_r = 1.0 - a(1.0 - ib) \), with thickness \( d \), and the boundary layers had the \( z \) dependence \( \exp[-c(z^+z_0)^2] \).

Since the measured reflection coefficient had a phase reference at the interface of the plasma port and the dielectric window, it was convenient to evaluate the asymptotic solution of the wave equation at that point. Any other point could have been chosen if the proper phase adjustment was included in the calculations. At the interface the permittivity predicted by the model was always within \( 10^{-3} \) of the actual value 1.0.

Table 1 for the data at 10 GHz shows that good agreement of theoretical results of the general model and the measurements was obtained for inverse shock velocities lower than 73 \( \mu \)sec/ft. The complex conjugate of the measured data was used to be consistent with an \( e^{-i\omega t} \) time convention. The aforementioned difficulty with reflection from the dielectric window on the transmission side was evident at inverse velocity higher than 73 \( \mu \)sec/ft. It is also important to note that at 10 GHz an error of 0.1 inch in the point of phase reference gives a phase error of 23°; thus some small allowance for experimental error in phase must realistically be considered.
Fig. 16--Impedance at various inverse shock velocities (10 GHz).
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### TABLE 1 (continued)

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**Key**

(M) measured  
(G) Gaussian plus plateau model  
(S) homogeneous slab model  
(L) multilayer model including dielectric window
### TABLE 2

Plasma Parameters

<table>
<thead>
<tr>
<th>Inverse Shock Velocity (μ/sec/ft)</th>
<th>Theoretical</th>
<th>Diagnosis</th>
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<tr>
<td></td>
<td>Ne(cm⁻³)</td>
<td>νc(sec⁻¹)</td>
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<tr>
<td>68</td>
<td>1.12₁³</td>
<td>2.1¹⁰</td>
</tr>
<tr>
<td>69</td>
<td>1.07₁³</td>
<td>2.06¹⁰</td>
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<td>70</td>
<td>1.0₁³</td>
<td>2.00¹⁰</td>
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<tr>
<td>71</td>
<td>9.0₁²</td>
<td>1.90¹⁰</td>
</tr>
<tr>
<td>72</td>
<td>5.0₁²</td>
<td>1.85¹⁰</td>
</tr>
<tr>
<td>73</td>
<td>4.4₁²</td>
<td>1.80¹⁰</td>
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<tr>
<td>74</td>
<td>3.2₁²</td>
<td>1.75¹⁰</td>
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<tr>
<td>78</td>
<td>1.30₁²</td>
<td>1.55¹⁰</td>
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</tbody>
</table>

Superscripts indicate power of ten by which number is to be multiplied.

(G) - Gaussian plus plateau model
(L) - Multilayer model

The general trend of the theoretical results is that the electron density is 1/3 to 1/4 the nominal value predicted by the shock tube theory of Feldman. Table 2 compares the theoretical results with the predicted values. Figure 17 presents the same data. Such a reduction in electron density from the theoretical values may be explained by contamination effects. Contaminants may significantly lower the gas temperature and the electron density. Other shock tube experiments have shown that a long period of outgassing is required to reduce the contaminants absorbed in the shock tube walls.
Fig. 17--Electron density and collision frequency.
A simplification of the plasma distribution was made in order to obtain a theoretical approximation for inverse shock velocities greater than 73 μsec/ft. Using the plateau thickness of 0.14 inch which was used successfully for the general model and neglecting the inhomogeneous plasma in the boundary regions, oblique incidence on the multilayer system of Fig. 18 was analyzed. The magnitudes of the measured reflection and transmission coefficients were successfully matched with the plasma parameters shown in Tables 1 and 2. Again, the plasma density was found to be 1/4 the predicted value. The accuracy of the matching is not as good as with the general model, but the trend of the calculations is consistent with the theoretical results obtained using the general model for lower inverse shock velocities.

The measured and theoretical data at 9 GHz are shown in Table 3. Only a few data points were available at this frequency, so the data are more sporadic than those at 10 GHz. Comparing Tables 3 and 1, it can be seen that similar parameters were obtained for the general model at 9 GHz and 10 GHz, indicating the model is general and accurately describes the actual plasma density distribution.
As a final demonstration of the effects of the plasma boundary layer, the reflection and transmission coefficients for a homogeneous plasma slab were calculated. The plasma slab had the same parameters as the plateau region described previously, i.e., the slab was the same as the general model with the boundary layers removed. Using the same phase reference and the same angle of incidence the data of Table 1 were obtained for the homogeneous slab. It is apparent that both reflection and transmission measurements cannot be matched with the same multilayer model. For the low loss plasma (a and b small) the inhomogeneous boundary layer
Fig. 18--Multilayer approximation to waveguide loading.

evidently has little effect on the magnitude of reflection and transmission, but does affect the phase. It is also apparent that even for very dense plasmas (low inverse shock velocity) the multilayer and single slab models give somewhat different results, showing that the mismatched dielectric window does have some effect.

It has been shown by Bachynski that if the inhomogeneous profile were normalized with respect to a slab having the same maximum electron density and the same number of electrons in a cross section that the more gradual transition of the inhomogeneous
profile gives a lower reflection coefficient than the abrupt transition of the slab. This normalization was carried out for the Gaussian plus plateau profiles described here and the reflection coefficient was found for a wide variety of steepnesses of the Gaussian. In all cases, it was found that the inhomogeneous profile did indeed have a lower reflection coefficient than the equivalent homogeneous slab.

B. Re-entry Model

An example of an asymmetrical electron density distribution in a plasma is that generated by a vehicle re-entering the earth's atmosphere. A typical distribution involves a relatively thin boundary region in which the electron density builds up to a maximum and a region in which the electron density tapers off from the maximum. Such an electron distribution was studied by Swift and Evans, and their model based on aerodynamic calculations has been used for Fig. 5.

Since the Gaussian distribution is continuous everywhere and has zero slope at the maximum, a logical approximation to the asymmetric aerodynamic model is readily obtained using two Gaussians of different slopes. As shown in Fig. 5 two Gaussians have been chosen to fit the model in the region near the maximum.
At 10 GHz and assuming a constant collision frequency, the reflection and transmission coefficients for a TE wave at oblique incidence can be readily determined using the formulas for the asymmetric Gaussian distribution previously derived.

Swift and Evans as well as Bein have studied the reflection from this model for plane waves at 244.3 MHz. At such a frequency well below the plasma frequency, the attenuation is very high. The series method proposed in this work to evaluate the transmitted field in the asymptotic region requires an inordinate number of terms, so the method is not practical for VHF waves incident on this profile. However, at 10 GHz a useful calculation can be performed. With the time convention \( e^{-i\omega t} \) and at normal incidence, the two Gaussians are 

\[
1 - 0.363(1.0 - i0.0028) \exp(-5.386z^2) \quad \text{and} \\
1 - 0.363(1.0 - i0.0028) \exp(-52.58z^2)
\]

It was found that

\[
V = 0.5892 e^{i21.9^\circ} \\
D = 0.5547 e^{-i63.0^\circ}
\]

The phase reference was taken on the incidence side of the plasma at the point of evaluation in the asymptotic region, \( z_o = 0.41 \) inch.

The plasma diagnosis problem for such a re-entry situation could be attacked in the following manner. It is probable that only a reflection coefficient could be measured. Aerodynamic
calculations should be used to predict the general shape of the plasma profile over the region perpendicular to the microwave aperture. The Gaussian or any other suitable continuous function could be used to form an approximate model. As pointed out in Section E of Chapter III, there are 16 parameters in the model using two Gaussian functions. These parameters can be varied in a systematic fashion to find a set which gives a matching reflection coefficient. Measured data at two or more frequencies would give a definite deterministic problem for matching with a mathematical model.
CHAPTER IV
CONCLUSIONS

The plasma diagnosis problem utilizing plane wave propagation at oblique incidence on an inhomogeneous medium which has a continuous permittivity profile has been treated in general terms. The wave equation can be transformed to recognizable forms of the differential equations of mathematical physics for certain inhomogeneities. Asymptotic forms of the solutions, if they exist in closed form, may be utilized to evaluate the amplitudes of the fields and reflection and transmission coefficients may be derived. The dilemma presented when closed form solutions and asymptotic behavior of the solutions cannot be determined has been resolved by demonstrating that the field may be evaluated in the region away from the influence of the inhomogeneity and the reflection and transmission coefficients are determined thereby.

The Gaussian distribution was chosen as an example of a continuous, analytic inhomogeneity for which no closed form solutions are known. Expressions for the reflection and transmission coefficients in terms of the parameters of the solution in the asymptotic region were derived for the Gaussian and Gaussian
plus plateau distributions. The shape of the plasma profile provides a set of parameters which may be chosen so that reflection and transmission coefficients may be matched to measurements.

Shock tube experimental measurements, when used for diagnosis of the plasma, showed that the plasma profile was in agreement with the shape predicted by aerodynamic calculations. The maximum electron density in the plateau region was found to be 1/3 to 1/4 of the density predicted in the shock tube literature. A multilayer model was used to explain a mismatch in the waveguide that caused undesired reflections when a low-loss plasma was in the waveguide.

Epstein's work on the $\text{sech}^2$ inhomogeneity was extended to include a flat plateau region. This was presented as an example of the use of asymptotic solutions.

An important result of the present study is the demonstration of a useful method of solving for reflection and transmission coefficients for continuous analytic profiles when closed form solutions are not available. Also, confirmation of the plasma density profiles in a parallel plate experiment has been verified by microwave measurements. The effects of the boundary layer of the plasma on the measurements has been clearly demonstrated. Homogeneous slab approximations can be used to match the measured data in magnitude, but only the general inhomogeneous model accounts for
the phase properly. The proper set of parameters in the general model will give good agreement with both measured reflection and transmission; neglecting the boundary profile produces erroneous results.
APPENDIX A
HOMOGENEOUS SLAB WITH EPSTEIN BOUNDARY LAYERS

The propagation of a wave in a region having the wave number

\begin{equation}
\mathbf{k}(z) = k_0 \left[ 1 - N \frac{e^{mz}}{1 + e^{mz}} - 4M \frac{e^{mz}}{(1 + e^{mz})^2} \right]
\end{equation}

where \( m, M, \) and \( N \) are constants was first studied by Epstein.\(^{25}\) His results have been generalized and applied to propagation at oblique incidence by Brekhovskikh.\(^2\) With an extension of this result we consider the homogeneous slab with Epstein boundary layers.

The relationships between the wave equation

\begin{equation}
\frac{d^2\mathbf{E}}{dz^2} + (\mathbf{k}(z) - k_0^2 \sin^2 \theta_j) \mathbf{E} = 0
\end{equation}

and the hypergeometric equation

\begin{equation}
\frac{d^2 F}{d\xi^2} - (\alpha + \beta + 1) \frac{\xi - Y}{\xi(1 - \xi)} \frac{dF}{d\xi} - \frac{\alpha \beta}{\xi(1 - \xi)} F = 0
\end{equation}

have been precisely stated by Brekhovskikh and also by Budden.\(^1\)

The pertinent transformations are

\begin{equation}
\mathbf{E} = r_0^{-1} m^{-\frac{1}{2}} \frac{Y - 1}{2} (1 - \xi) \frac{1 + \alpha + \beta - Y}{2} F
\end{equation}
and $\xi = -e^{rz}$ where $r_0$ is a constant. By defining

$$K_1 = -\frac{1}{4} + \frac{2}{k_0} (\sin^2 \theta - 1)/m^2$$

$$K_2 = 0$$

$$K_3 = -4k_0 \frac{M}{m^2}$$

$$\alpha = 1 + \sqrt{1 + 4k_3}$$

$$\gamma = 1 + \sqrt{1 + 4k_1}$$

$$\beta = \alpha + \gamma - 1$$

the parameters of the inhomogeneous plasma are related to the constants of the hypergeometric equation, since

$$4M = \omega^2_{p_{\text{max}}} \frac{(1, 0 - i\nu/\omega) / (\nu^2 + \nu^2)}{
u^2}$$

and $m$ is the steepness of the profile. For the geometry shown in Fig. 4

$$\xi = \begin{cases} 
-e^{m(z-z_1)}, & z_1 < z < \infty \\
-e^{m(z+z_1)}, & -\infty < z < -z_1 
\end{cases}$$

and in the boundary regions

$$k^2(z) = k_0^2 \left[ 1 - 4M \frac{e^{m(z+z_1)}}{(1 + e^{m(z+z_1)})^2} \right]$$

$$= k_0^2 \left[ 1 - M \text{sech}^2 \frac{m(z+z_1)}{2} \right].$$

From the list of 24 solutions to Eq. (3) the following six may be chosen.\textsuperscript{25}
\[ u_3 = (-\xi)^{-\alpha} F(\alpha + 1 - \gamma; \alpha + 1 - \beta; 1/\xi) \]
\[ = (1-\xi)^{-\alpha} F(\alpha, \gamma - \beta; \alpha + 1 - \beta; 1/1 - \xi) \]

\[ u_1 = F(\alpha, \beta; \gamma; \xi) \]
\[ = (1-\xi)^{-\alpha} F(\alpha, \gamma - \beta; \gamma; \xi/\xi - 1) \]

\[ u_5 = \xi^{1-\gamma} F(\alpha + 1 - \gamma, \beta + 1 - \gamma; 2 - \gamma; \xi) \]
\[ = \xi^{1-\gamma} (1-\xi)^{\gamma-\alpha-1} F(\alpha+1-\gamma, 1-\beta, 2-\gamma; \xi/\xi - 1) \]

The hypergeometric series by which these solutions are formed is

\[ F(\alpha, \beta; \gamma; \xi) = 1 + \frac{\alpha \beta}{\gamma} \xi + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma + 1)} \xi^2 \]
\[ + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma + 1)(\gamma + 2)} \xi^3 + \ldots \]

A very great advantage of the use of such hypergeometric functions is that simple asymptotic forms may be derived. In this problem then, an exact expression for the solution to the wave equation as \( z \to \infty \) may be found. There is no necessity to resort to expressing the solution in a quasi-asymptotic region as was done with the Gaussian distribution.

The transformation (5) gives \(-1 > \xi > -\infty\) for the region of the transmitted wave \( z_1 < z < \infty \). In this region the solution \( u_3 \) is utilized to express the field as

\[ E = r^{-1}_o m^{-1/2} \xi^{\gamma - 1/2} \frac{1 + \alpha + \beta - \gamma}{2} (1-\xi) \frac{1}{u_3} \]
Using the first expression for \( u_3 \),

\[
z \to \infty, \xi \to -\infty
\]

(7) \[
E \sim r_o^{-1} m^{-\frac{1}{2}} (-1)^{\frac{\gamma-1}{2}} \frac{\beta-\alpha}{2} (e^{m(z-z_1)}) = r_o^{-1} m^{-\frac{1}{2}} (-1)^{\frac{\gamma-1}{2}} \frac{\beta-\alpha}{2} e^{\frac{2}{\sqrt{k_0-a^2}(z-z_1)}}.
\]

At \( z = z_1 \) the field and its derivative must be continuous. The second expression for \( u_3 \) may be used near \( z = z_1 \), and \( \xi = -1 \).

Expressing \( E \) as

\[
E = g(z) u_3(z),
\]

then

\[
\frac{dE}{dz} = g(z) \left[ m \left( \gamma - 1 + \frac{\alpha + \beta + 1 - \gamma}{4} \right) u_3(z) + \frac{du_3}{dz} \right]
\]

and

\[
\frac{du_3}{dz} \bigg|_{z = z_1} = -m \left[ \frac{\alpha (2)^{-\alpha-1}}{4(\alpha+1-\beta)} F(\alpha+1, \gamma-\beta+1; \alpha+2-\beta; 1/2) \right].
\]

The field in the homogeneous slab with \( k = k_o \left( \epsilon_r - \sin \theta \right)^{\frac{1}{2}} \) is

\[
E = C e^{ikz} + D e^{-ikz},
\]

and by matching boundary conditions at \( z_1 \), it may be shown that

\[
C = g(z_1) (e^{-ikz_1/2}) \left\{ u_3(z_1) + \frac{1}{ik} m(\gamma - 1 + (\alpha + \beta + 1 - \gamma)/4) \left. \frac{du_3}{dz} \right|_{z = z_1} \right\}
\]

\[
u_3(z_1) + \frac{1}{ik} \left( \frac{du_3}{dz} \bigg|_{z = z_1} \right)
\]
and
\[ D = g(z_1) (e^{-ikz_1/2}) \begin{cases} u_3(z_1) - (1/ik) m(\gamma - 1 + (\beta + 1 - \gamma/4) u_3(z_1) \\ -(1/ik) \frac{du_3}{dz} \bigg|_{z = z_1} \end{cases} \]

At the other boundary where \( z = -z_1 \), the coefficients \( C \) and \( D \) are again used to evaluate the field.

In the Epstein layer on the incidence side of the slab where 
\(-\infty < z < -z_1\) and \(0 > \xi > -1\) a linear combination of solutions \( u_1 \) and \( u_2 \) may be chosen to represent the field. Thus
\[ E(z) = f(z) [P u_1 + Q u_2] \]

where
\[ f(z) = \pi^{-1} m^{\frac{1}{2}} (1 + e^{m(z+z_1)}) \frac{\gamma - 1}{2} (1 + e^{m(z+z_1)}) \frac{\alpha + \beta + 1 - \gamma}{2}. \]

The coefficients \( P \) and \( Q \) are found by matching boundary conditions at \( z = -z_1 \). For the derivative of the field we must evaluate
\[ \frac{du_1}{dz} \bigg|_{z = -z_1} = -m \left[ (\alpha + 1 - \gamma - 1) F(\alpha, \gamma - 1; \gamma + 1/2, 1) - (2)^{-\alpha} \frac{\alpha(\gamma - 1)}{4\gamma} F(\alpha + 1, \gamma - \beta + 1; \gamma + 1, 1/2) \right] \]
\[ \frac{du_2}{dz} \bigg|_{z = -z_1} = -m \left[ (1 - \gamma)(-1)^{\gamma - 1} - (1)^{\gamma - 1} - (1 - \gamma)(\gamma - 1)(2)^{-\alpha - 2} \right] \]
\[ \frac{F(\alpha + 1 - \gamma, 1 - \beta; 2 - \gamma + 1/2) - (1 - \gamma)(\gamma - 1)(2)^{-\alpha - 2}}{F(\alpha + 2 - \gamma, 2 - \beta; 3 - \gamma; 1/2)} \].
From these equations the boundary conditions at $z = -z_1$ may be written.

$$Ce^{-ikz_1} + De^{ikz_1} = f(-z_1)[Pu_1 + Qu_5]$$

$$Ce^{-ikz_1} - De^{ikz_1} = \frac{1}{ik} \left[ P \left( u_1 \frac{df}{dz} + f \frac{du_1}{dz} \right) + Q \left( u_5 \frac{df}{dz} + f \frac{du_5}{dz} \right) \right]$$

The coefficients $P$ and $Q$ may be found by substituting the expressions for $C$ and $D$.

The asymptotic behavior of the expression for the field for $z < -z_1$ can be determined from the behavior of the hypergeometric functions $u_1$ and $u_5$. Since as $z \to -\infty$, $\xi \to 0$, it is seen that

$$u_1 \to 1, \quad u_2 \to \xi^{(1 - \gamma)}.$$  

Expressing the field in the $z$-plane as

$$E(z) = r_o^{-1} m^{-\frac{1}{2}} \left( -e^{m(z+z_1)} \right)^{\frac{\gamma - 1}{2}} \left( 1 + e^{m(z+z_1)} \right)^{\frac{\alpha + \beta + 1 - \gamma}{2}} [Pu_1 + Qu_5],$$

the asymptotic behavior of the field is a standing wave

$$E \sim r_o^{-1} m^{-\frac{1}{2}} \left[ P(-1)^{\frac{\gamma - 1}{2}} e^{m(z+z_1)} + Q(-1)^{\frac{1 - \gamma}{2}} e^{-m(z+z_1)} \right]$$

where

$$\gamma - 1 = \frac{2}{m} \sqrt{a^2 - k_o^2}.$$

The reflection coefficient is the ratio of the coefficients of the two traveling waves and is given by

$$V = \frac{Q}{P(-1)^{1 - \gamma}}.$$
The transmission coefficient may be written

\[ D = \frac{(-1)^{-\alpha} e^{m(z-z_1)}}{Pe^{m(z+z_1)} \frac{\gamma-1}{\gamma}} \]

(10) \[ D = \frac{(-1)^{-\alpha}}{P} e^{-2iz_1\sqrt{k_0^2-a^2}} \]

A computer program was written to calculate the complex parameters of the profile and to find \( V \) and \( D \). Incorporated in the program was a subroutine for calculating the hypergeometric series. At the edges of the plateau where \( z = \pm z_1 \), the argument of \( u_1, u_3, u_5 \), and their derivatives was always \( \xi = 0.5 \). Sufficient accuracy from the hypergeometric series was usually obtained with about 10 terms.

Brekhovskikh has derived an expression for the transmission and reflection coefficients for the symmetrical Epstein layer utilizing analytic continuation. For this profile, which has no homogeneous plateau, the coefficients are expressed in terms of gamma functions. Rather than calculate the reflection and transmission coefficients in this way, the methods outlined here for matching boundary conditions where the plateau thickness is zero should suffice. As an example, the permittivity profile

\[ \epsilon_r(z) = 1 - \frac{2.444 e^{100z}}{(1 + e^{100z})^2} \]
was chosen. A calculation for this lossless profile gave

\[ V = 0.06416 \ e^{-93.75^\circ} \]
\[ D = 0.998 \ e^{-179.0^\circ} \]

An additional calculation was performed using the numerical difference equation method described previously. The same values were obtained for the coefficients \( V \) and \( D \).

Propagation in a lossy plasma profile with a homogeneous plateau was also investigated. The plateau had permittivity \( \varepsilon_r = 1.0 - 2.53(1.0 - i0.283) \) and thickness 0.14 inches. The inhomogeneous boundary layers on each side of the plateau had spatial dependence \( \frac{e^{100z}}{(1+e^{100z})^2} \). For a 10 GHz plane wave incident at 41° and polarized perpendicular to the plane of incidence, the reflection and transmission coefficients were found to be

\[ V = 0.8293 \ e^{-i134.8^\circ} \]
\[ D = 0.3566 \ e^{i125.2^\circ} \].
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