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VIA QUEUING MODELS AND GRAPH THEORY

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for the Degree Doctor of Philosophy in the
Graduate School of The Ohio State University

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* * * * * *
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CHAPTER I
INTRODUCTION

The scientific study of traffic congestion, whether intended to describe or to ameliorate, has been a natural consequence of man's enforced interest in his increasingly overcrowded world. Unlike most other congestion situations, vehicular traffic presents more than purely mathematical difficulties since the basic parameters are not agreed upon. Vehicular traffic situations represent an often bewildering mixture of psychological factors which are little understood in isolation and much less understood in combination. The most fully developed mathematical theory of congestion is the queuing theory, which deals with accumulation at a fixed point, caused by the need for service and changing with the passage of time. The history of queuing theory falls neatly into two periods, separated by the very important paper of D. G. Kendall [11]. In the first period, beginning with the work of Erlang [5] early in the century and lasting until 1951, the principal investigators were concerned with the problems of telephone traffic. During the past few years the increasing urgency of vehicular traffic congestion problems from both the technical and social points of view has focused the attention of many investigators on this area of research. Only the simplest facts are known, far less than are required to devise anything like optimal systems.

One very intricate problem causing vehicular traffic congestion on the road is that of merging. One of the earliest problems which
attracted interest in this area is the following: A car arrives at a stop sign at time $t = 0$, and the driver observes cars on a main intersecting road arriving at random times. The waiting driver is delayed for a period random in nature until a suitable opportunity presents itself for crossing or merging with the traffic. The statistics of the delay of the stopped car are required.

The merging situation is more general than that of crossing by virtue of the simple fact that the driver who wishes to merge can himself partly control the variables which characterize his position with respect to the traffic flow. Once it is realized that there is a question of policy involved, the merging problem becomes quite difficult to formulate. It is necessary to determine the extent of the information about the traffic flow, the driver employs and it is necessary to determine the specific goal or criterion used by the driver in initiating the merge. Is the object of the driver to merge as safely as possible, or as quickly as possible, or as downstream as possible? In real life, all of these considerations play a part in a driver's decisions. These considerations complicate matters beyond any exact formulation of a mathematical model.

Once the information used by the driver and goals are specified, and his policy obtained, one should be able to obtain a statistical description of the result of many merges choosing individual policies. Basically the problem reduces to the study of the queue formation by virtue of a merging criterion. Kendall [12] and Lindley [13] suggested the use of imbedded Markov chains to analyze such queuing problems.
The Concept of Imbedded Chains

A queue or waiting line is formed when customers arrive at a counter offering certain facilities and demand service. As examples, one may consider subscribers' calls arriving at a telephone exchange, patients waiting in a doctor's reception room, machines waiting to be serviced by a repairman, or drivers waiting at a traffic intersection. A queuing system is completely described by (1) the input, (2) the queue-discipline, and (3) the service mechanism. These are defined as follows:

The input describes the way customers arrive and join the system. The number of customers may be finite or infinite, and they may arrive individually or in groups. Let the successive customers arrive at the instants \( t_0, t_1, t_2, \ldots \); the interarrival times are then \( u_r = t_r - t_{r-1} \) \((r = 1,2,\ldots)\). It is assumed that \( u_1, u_2, \ldots \), are random variables which are mutually independent and have the same distribution function \( A(u); (0 \leq u < \infty) \).

The queue discipline is the rule determining the formation of the queue. The simplest discipline is "first come, first served," according to which customers are served in the order of their arrival. The service mechanism describes the arrangements for serving the customers. If the customers are drivers waiting to move at traffic lights or waiting to move from a minor road to a major road, service is restricted both by the need to wait until a 'free period' occurs, and by the need to wait until the previous customers in the queue have been served. The time which elapses while a particular customer is being
served is called his service time. It is assumed that the service
times \( v_1, v_2, v_3, \ldots \), of the successive customers are random variables,
which are mutually independent and also independent of \( u_1, u_2, u_3, \ldots \);
and that they all have the same distribution function \( B(v) (0 \leq v < \infty) \),
a distribution that generally differs from \( A(u) \).

When the input and service time distributions assume general
forms, the process \( n(t) \), the length of queue at time \( t \), is no longer
Markovian; i.e., the specifications of the queue length at time \( \tau \) are
not adequate to predict its length at a time \( \tau + t \). The correct speci-
fications of the queuing system are then given by the vector process
\( [n(t), u(t), v(t)] \), where \( u(t) \) is the time which has elapsed since the
last arrival, and \( v(t) \) is the expanded service time of the customer be-
ing served at time \( t \). One can study this vector process, though the
theory will be rather complicated. However, it is still possible to do
something about the process \( n(t) \) by making use of its "regeneration
points" [18].

A set \( S \) of time-instants is said to be a set of regeneration
points for a stochastic process \( X(t) \) if and only if, for all \( t > t_0 \),
\[
distr [X(t) \mid X(t_0)] = distr [X(t) \mid X(\tau) \text{ for all } \tau \leq t_0],
\]
where \( t_0 \) is known to belong to set \( S \). This condition implies that the
development of the process during \( t > t_0 \) is independent of the history
of the process during \( (0,t_0) \). For a Markov process, the whole range of
\( t \) values is a set of regeneration points. Suppose that there exists a
denumerable \( S \)-set of regeneration points \( \{t_n, n = 0,1,2,\ldots\} \) such that
\( t_0 < t_1 < t_2 < \ldots \); and put \( X_n = X(t_n) \); then it follows from the above
definition that the sequence of random variables \( \{X_n\} \) forms a Markov chain. This chain is said to be imbedded in the given process \( X(t) \). In the queuing system with Poisson input and general service time distribution, the instants at which the customers leave the system are points of regeneration.

The concept of regeneration points is due to Palm [17], and the technique of imbedded Markov chains is due to Kendall [12]. Lindley [13] carried out the imbedded Markov chain analysis of waiting time in a queuing system.

The problem of merging becomes even more acute in the case of the freeway on-ramp situation, where the 'capacity' of the ramp is finite. This situation is shown in Figure 1. In most operational problems involving flow, the first step is to ensure that mean capacities can handle average flow so that persistent bottlenecks do not occur. But even if mean capacities can handle average flows, transient "traffic jams" can occur because the actual flow fluctuates, being sometimes larger than its mean value. The purpose of this treatise is to study and describe how the effects of such fluctuations can be analyzed and how to remove the bottlenecks.

In Chapter II, the concept of imbedded markov chain is utilized to specify the design of an on-ramp dynamic storage facility \( S \), Figure 2, to accommodate the overflow from the finite space on-ramp. The mathematical models of merging discussed in Sections 2.2, 2.3, and 2.4 are:
Fig. 1. Freeway on-ramp situation.

Fig. 2. Freeway on-ramp dynamic storage facility.
a) Single Service Model (SSM) -- Only one vehicle is allowed to merge at the appearance of a 'gap', irrespective of the size of the 'gap'.

b) Variable Bulk Service Model (VBSM) -- The number of vehicles merging into the freeway from the on-ramp is proportional to the gap size.

To utilize the storage facility efficiently and to avoid the ambiguity of decisions on the part of the driver, a logical control system is designed to do the following:

a) Divert the traffic, wishing to join the ramp, to the storage facility if the need arises.

b) Allow the vehicles from the storage to join the ramp.

The logic block for on-ramp traffic control is developed in Chapter III.

The transient behavior of the queues poses quite interesting nonlinear stochastic problems. Cosgriff [3] developed procedures for the study of nonlinear stochastic systems. Hemami and Cosgriff [9] used graph techniques together with the nonharmonic series representation of stationary Gaussian signals to analyze a class of cascade of linear time-invariant, nonlinear zero-memory and linear time-invariant systems. In Chapter IV their work is extended to analyze a wider class of nonlinear problems, i.e., a cascade of zero-memory nonlinear, linear time-invariant and a zero-memory nonlinear systems. This analysis will be of considerable help in studying transient behavior of the queues.
2.1 Introduction

In Chapter I the problem of merging was introduced. In all freeway on-ramp situations a driver must select the right moment for merging into the freeway traffic flow. His decision must be based upon some criterion that will insure a safe merge, and this criterion invariably causes a queue formation on the on-ramp. Kendall[11,12] and Lindley[13] studied some basic problems in the theory of queues and proposed the use of imbedded Markov chains for their analysis. Following their classical treatment Oliver[15,16] and many other authors[6,8,21] have studied various criteria of merge and obtained the statistics of the queue that is formed on the ramp; however, they assume that there is an unlimited amount of waiting space on the ramp. Finch[7] has discussed a more practical queuing problem of finite waiting space, but the service distribution assumed in the analysis has a serious limitation, namely, that the waiting driver does not wait for a suitable 'gap' on the freeway. In this chapter this limitation has been removed and a more practical situation that arises for the case of a finite waiting space on the ramp and a simple merging criterion is examined.

The merge is allowed at the availability of a 'gap', where a gap is defined to be the headway between adjacent vehicles on the freeway greater than or equal to a real number $x$. Two mathematical models are
discussed in Sections 2.2, 2.3, and 2.4: (a) Single Service Model (SSM)—only one vehicle is allowed to merge at the appearance of a gap, irrespective of the size of the gap; (b) Variable Bulk Service Model (VBSM)—number of vehicles merging into the freeway is proportional to the gap size.

The size of storage facility needed to accommodate the overflow from the ramp in both cases is discussed in Section 2.5.

2.2 Single Service Model (SSM)

In this section on queue statistics the mean queue length \( \bar{n} \) on the on-ramp is derived by using the concept of embedded Markov chains and regeneration points. The capacity of the ramp is assumed to be finite, \( N \). It is assumed that the arrivals on the ramp are Poisson distributed while the intervehicle headways on the freeway are arbitrary but identically and independently distributed random variables. The queue discipline is "first come--first served" and the service distribution depends on the availability of gaps in the major stream.

Only one vehicle is allowed to merge at the appearance of a gap. It is assumed that when a driver arrives to find the waiting room full, he departs never to return (or if he does return, he does so later without upsetting the normal traffic pattern). In order to describe the queue statistics, it is further assumed that when a vehicle leaves the ramp and joins the freeway, it does so at the instant that the beginning of gap crosses in front of the ramp. If there is a second vehicle in the queue on the ramp, it must wait for the arrival of the next gap.
Fig. 3. Queuing for gaps.

0 = Beginning of gaps

t > 0

t = 0

Direction of flow
If no vehicles are in the queue on the ramp, the first vehicle to arrive must also wait for the beginning of a gap. Figure 3 illustrates the subset of vehicles on the freeway that signal the beginning of gaps.

The development to obtain an expression for the mean queue length, $\bar{n}$, on the ramp is as follows.

Suppose that the vehicles arrive in a Poisson process at a rate $\lambda$, and that the service time $v$ has the distribution $dB(v)$ ($0 < v < \infty$). Let $t_r$ denote the instants at which the $r^{th}$ vehicle on the ramp leaves the ramp ($r = 0, 1, 2, \ldots, t_0=0$). Then the instants $(t_r+0)$ are obviously points of regeneration of the queue length process $n(t)$. Let $n_r = n(t_r+0)$, then $\{n_r\}$ is a time-homogeneous Markov Chain; let its transition probabilities be denoted by:

$$P_{ij} = \text{Prob. (number of vehicles in queue} = j \text{ at a regeneration point} | \text{number of vehicles} = i \text{ at preceding regeneration point)}$$

Let $k_r$ denote the probability of $r$ arrivals on the ramp during a service period of arbitrary duration; i.e.,

$$k_r = \int_0^\infty e^{-\lambda v} \frac{(\lambda v)^r}{r!} dB(v) \quad (r \geq 0)$$

The transition probabilities of the process $n(t)$ can thus be written as:

$$P_{0j} = k_j \quad (2.2)$$

$$P_{ij} = k_{j-i+1} \quad i > 0, j < N \quad (2.3)$$
The steady-state probability, \( p_j \), that there are \( j \) vehicles on the ramp is easily obtained from the transition probabilities.

\[
P_j = \sum_{i=0}^{j+1} \pi_i p_{ij}
\]

The probability generating function for the \( p_j \)'s is defined by

\[
P(z) = \sum_{j=0}^{N} p_j z^j
\]

and the probability generating function of the distribution \( \{k_j\} \) is defined by

\[
K(z) = \sum_{j=0}^{N} k_j z^j
\]

From Eq. (2.2) to Eq. (2.7) it follows that

\[
P(z) = \sum_{i=1}^{N+1} \sum_{j=i-1}^{N} p_i k_{j-i+1} z^j + \sum_{j=0}^{N} p_0 k_j z^j
\]

Substituting Eq. (2.7) in (2.8) gives

\[
P(z) = \sum_{i=1}^{N+1} \sum_{j=i-1}^{N} p_i k_{j-i+1} z^j + p_0 K(z)
\]

Let \( \ell = j-i+1 \),

\[
P(z) = \sum_{i=1}^{N+1} \sum_{\ell=0}^{N-i+1} p_i k_\ell z^{\ell+i-1} + p_0 K(z)
\]
Simplification yields

\[ P(z) = \frac{p_0 K(z)(1-z)}{K(z) - z} + \frac{K(z)p_{N+1}z^{N+1}}{K(z) - z} + \frac{z^{N+1}}{K(z) - z} \left[ \sum_{i=1}^{N} p_i z^{i-1} \sum_{k=N-i+2}^{N} k_i z^k \right] \]  \hspace{1cm} (2.11)

The second term of \( P(z) \) in Eq. (2.11) is zero since \( p_{N+1} = 0 \) and the double summation in the last term has \( N+1 \) as the minimum power of \( z \), which is multiplied by a power series in \( z \). Thus the last term contributes \( z^{N+2} \) and higher order terms and, therefore, can be neglected since \( P(z) \) by definition does not contain such terms. Thus

\[ P(z) = p_0 \frac{K(z)(1-z)}{K(z) - z} \]  \hspace{1cm} (2.12)

This agrees with a result due to Kendall[11].

For Poisson flow on the freeway with mean \( \alpha \), the service time distribution can be expressed as [16].

\[ B(v) = I = \frac{X}{x} \left( -\alpha \right)^i \frac{e^{-\alpha x}(v - ix)^i}{i!} \]  \hspace{1cm} (2.13)

where \( I = \frac{X}{x} \) is the integral part of \((vx^{-1})\), \( v \) is the service time, and \( x \) is the length of the gap.

Substituting Eqs. (2.1) and (2.13) in Eq. (2.7) gives

\[ K(z) = \sum_{j=0}^{N} z^j \left[ \int_0^{\infty} \sum_{i=0}^{j} \frac{(-\alpha)^i e^{-\alpha x(v - ix)} v^i}{i!} dv \right] \]  \hspace{1cm} (2.14)
Simplification yields\(^1\)

\[
K(z) = [\lambda(1 - zN \log_e(1 + \frac{1}{N})) + \alpha e^{-(\alpha + \lambda)x} (1 + \frac{1}{N})^{\lambda x N}]^{-1}
\]

\[
= [\lambda(1 - zN \log_e(1 + \frac{1}{N})) + \alpha e^{-(\alpha + \lambda)x} e^{\lambda x N \log_e(1 + \frac{1}{N})}]^{-1}
\]

Let \(N \log_e(1 + \frac{1}{N}) = m\)
\(\alpha e^{-(\alpha + \lambda)x} = b\)
\(\lambda x = c\)

Equation (2.16) can then be written as:

\[
K(z) = [\lambda(1 - m z) + b e^{c m z}]^{-1}
\]

(2.17)

Substituting Eq. (2.17) in Eq. (2.12) yields:

\[
P(z) = p_0(1 - z)\frac{1 + a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots}{1 + b_1 z + b_2 z^2 + \ldots + b_n z^n + \ldots}
\]

(2.18)

Let \(c m = t\),

\[
P(z) = p_0(1 - z)\frac{1 + a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots}{1 + b_1 z + b_2 z^2 + \ldots + b_n z^n + \ldots}
\]

(2.19)

where \(a_n = 0\) \(n \geq 1\)
\(b_1 = -(\lambda + b)\)
\(b_2 = (\lambda m - b t)\)
\(b_n = -\frac{b}{(n-1)!} t^{n-1}\) for \(n \geq 3\)

(2.20)

\(^1\)See Appendix A.
Equation (2.19) can be rewritten as:

\[ P(z) = p_0(1-z)[1 + c_1 z + c_2 z^2 + \ldots + c_n z^n + \ldots] \quad (2.21) \]

where

\[ c_1 = a_1 - b_1 \]
\[ c_2 = a_2 - (b_1 c_1 + b_2) \]
\[ c_3 = a_3 - (b_1 c_2 + b_2 c_1 + b_3) \]
\[ \vdots \]
\[ c_n = a_n - (b_1 c_{n-1} + b_2 c_{n-2} + \ldots + b_{n-1} c_1 + b_n) \quad (2.22) \]

The probability \( p_n \), that there are \( n \) cars on the ramp is easily obtained from the probability generating function \( P(z) \). It is the coefficient of \( z^n \) in \( P(z) \) and is given by

\[ P_n = p_0(c_n - c_{n-1}) \quad (2.23) \]

The above equation determines the probability of \( n \) cars on the ramp in terms of \( p_0 \), the probability of an empty queue. The probability \( p_0 \) can be obtained from the usual normalization technique, i.e.,

\[ \sum_{n=0}^{N} p_n = 1 \quad (2.24) \]

Thus

\[ p_0\left[ \sum_{n=0}^{N} (c_n - c_{n-1}) \right] = 1 \quad (2.25) \]

This yields

\[ p_0 = [c_N]^{-1} \quad (2.26) \]

and therefore,

\[ P_n = (c_n - c_{n-1}) c_n^{-1} \quad (2.27) \]

where \( c_n \) is given by Eq. (2.22).
The expected value of the number of cars on the ramp, i.e., mean queue length \( \bar{n} \), is therefore

\[
\bar{n} = \sum_{n=0}^{N} n p_n = c_n^{-1} \sum_{n=0}^{N} n(c_n - c_{n-1})
\]  

(2.28)

Another expression can be used to determine the approximate mean queue length \( \bar{n} \) by approximating the probability generating function of the distribution \( \{k_j\} \) by

\[
K(z) = \sum_{j=0}^{\infty} k_j z^j
\]  

(2.29)

Thus \( K(z) \) can be determined from Eq. (2.17) by letting \( N \to \infty \).

\[
K(z) = \left[ 1 + d(1-z) e^{-cz} \right]^{-1}
\]  

(2.30)

where \( d = \lambda/b \).  

(2.31)

Substituting Eq. (2.30) into Eq. (2.12) and simplifying yields

\[
P(z) = p_o (1 - dz e^{-cz})^{-1}
\]  

(2.32)

Thus \( p_n \), the probability of \( n \) cars on the ramp, is the coefficient of \( z^n \) in the above expression. Thus

\[
p_n = p_o \sum_{k=0}^{n-1} (-c)^k \frac{d^{n-k}(n-k)!}{k!} \quad (n \geq 1)
\]  

(2.33)

The probability \( p_o \) of an empty queue is again obtained by normalization. On simplification it yields

\[
p_o = \left[ 1 + \sum_{k=0}^{N-1} \sum_{r=1}^{N-k} \frac{(-rc)^k}{k!} d^r \right]^{-1}
\]  

(2.34)
Thus the probability of \( n \) cars on the ramp is

\[
P_n = \frac{\sum_{k=0}^{n-1} \frac{(-c)^k}{k!} d^{(n-k)(n-k)} k!}{1 + \sum_{k=0}^{N-1} \sum_{r=1}^{N-k} \frac{(-rc)^k}{k!} d^r}
\]

(2.35)

Therefore the expected number of vehicles on the ramp, i.e., mean queue length \( \bar{n} \), is

\[
\bar{n}' = \sum_{n=0}^{N} n p_n
\]

\[
\bar{n} = \frac{\sum_{k=0}^{N-1} \frac{(-c)^k}{k!} \sum_{r=1}^{N-k} (r + k)d^r r^k}{1 + \sum_{k=0}^{N-1} \sum_{r=1}^{N-k} \frac{(-rc)^k}{k!} d^r}
\]

(2.36)

If the restriction on the waiting space on the ramp is relaxed, i.e., \( N \to \infty \), the probability of an empty queue, \( p_0' \), can be obtained from Eq. (2.34) as

\[
p_0' = \frac{1}{1 + \sum_{r=1}^{\infty} \sum_{k=0}^{(-rc)^k}} = (1 - de^{-c})
\]

(2.37)

Also the mean queue length \( \bar{n}' \) becomes

\[
\bar{n}' = \lim_{N \to \infty} \sum_{n=0}^{N} n p_n = p_0' \frac{(1-c)d e^{c}}{1 - de^{-c} - q^2}
\]

(2.38)
It is noted that \( \bar{n}' \) can also be obtained from the classical technique, i.e.,

\[
\bar{n}' = \left. \frac{d}{dz} P(z) \right|_{z=1}
\]

For \( x = 0 \), Eq. (2.38) becomes

\[
\bar{n}' = \frac{d}{1-d} = \frac{\lambda}{\alpha - \lambda}
\]

which is a standard result in the theory of queues\(^4\).

The probability of an empty queue \( p_0 \), \( p_0' \) and the mean queue length \( \bar{n} \) and \( \bar{n}' \) are tabulated for \( N = 10, 20, 30 \) in Tables 1, 2, and 3 for \( \lambda \) varying from 0.01 sec\(^{-1} \) to 0.05 sec\(^{-1} \) and \( \alpha \) varying from 0.1 sec\(^{-1} \) to 0.5 sec\(^{-1} \). The probability of an empty queue for \( N = 10, x = 5 \) sec is plotted in Figure 4, and the mean queue length, \( \bar{n} \), on the ramp with finite waiting space (\( N = 10 \)) is plotted in Figure 5.

The queue length for finite as well as infinite waiting space on the ramp is a convex function of the arrival rate, \( \alpha \), on the freeway, since \( \frac{d^2 \bar{n}}{d\alpha^2} > 0 \) and \( \frac{d^2 \bar{n}'}{d\alpha^2} > 0 \). Since \( \bar{n}(\alpha) \) and \( \bar{n}'(\alpha) \) are convex functions, they have unique minima, which occur at \( x\alpha = 1 \). This result is supported by the reasoning that the gap flow rate, \( \alpha e^{-\alpha x} \), has a maximum for \( x\alpha = 1 \), and thus with maximum availability of gaps, the queue length is minimized. It is also observed that \( \frac{\partial \bar{n}}{\partial \alpha} \) (or \( \frac{\partial \bar{n}'}{\partial \alpha} \)) is not always positive, so \( \bar{n} \) (or \( \bar{n}' \)) is not a monotonically increasing function of \( \alpha \).
### TABLE 1

Mean Queue Length and Probability of an Empty Queue for $N = 10$

<table>
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<tr>
<th>$N$</th>
<th>$\alpha$</th>
<th>$\lambda$</th>
<th>$p_0$</th>
<th>$p'_0$</th>
<th>$\bar{n}$</th>
<th>$\bar{n}'$</th>
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<tr>
<td>10</td>
<td>0.10</td>
<td>0.01</td>
<td>0.827</td>
<td>0.835</td>
<td>0.209</td>
<td>0.187</td>
</tr>
<tr>
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<td>0.636</td>
<td>0.670</td>
<td>0.573</td>
<td>0.442</td>
</tr>
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</tr>
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</tr>
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</tr>
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</tr>
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</tr>
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TABLE 2

Mean Queue Length and Probability of an Empty Queue for $N = 20$

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<th>$P_0'$</th>
<th>$\bar{n}$</th>
<th>$\bar{n}'$</th>
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<td>0.835</td>
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TABLE 3
Mean Queue Length and Probability of an
Empty Queue for N = 30

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<th>p₀</th>
<th>p₀'</th>
<th>⃗n</th>
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<td>0.835</td>
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<td>0.670</td>
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</tr>
<tr>
<td>30</td>
<td>0.40</td>
<td>0.03</td>
<td>0.356</td>
<td>0.445</td>
<td>1.8</td>
<td>1.05</td>
</tr>
<tr>
<td>30</td>
<td>0.40</td>
<td>0.04</td>
<td>0.101</td>
<td>0.261</td>
<td>7.9</td>
<td>2.26</td>
</tr>
<tr>
<td>30</td>
<td>0.40</td>
<td>0.05</td>
<td>--</td>
<td>0.076</td>
<td>24.7</td>
<td>9.07</td>
</tr>
<tr>
<td>30</td>
<td>0.50</td>
<td>0.01</td>
<td>0.743</td>
<td>0.756</td>
<td>0.34</td>
<td>0.30</td>
</tr>
<tr>
<td>30</td>
<td>0.50</td>
<td>0.02</td>
<td>0.461</td>
<td>0.512</td>
<td>1.16</td>
<td>0.85</td>
</tr>
<tr>
<td>30</td>
<td>0.50</td>
<td>0.03</td>
<td>0.151</td>
<td>0.269</td>
<td>5.13</td>
<td>2.3</td>
</tr>
<tr>
<td>30</td>
<td>0.50</td>
<td>0.04</td>
<td>--</td>
<td>0.025</td>
<td>24.8</td>
<td>3.06</td>
</tr>
<tr>
<td>30</td>
<td>0.50</td>
<td>0.05</td>
<td>--</td>
<td>--</td>
<td>28.2</td>
<td>--</td>
</tr>
</tbody>
</table>
Fig. 4. Probability of an empty queue on the ramp.
Fig. 5. Steady-state average queue length on the ramp with finite waiting space, $N = 10$. 
2.3 Variable Bulk Service Model (VBSM)

In Section 2.2 it was assumed that only one vehicle from the ramp could merge with the freeway traffic flow whenever the gap was available. This merging mechanism, though quite satisfactory for high flow rates, is not very realistic. In this section the above model is modified by allowing the merging number of vehicles to vary according to the size of the gap. It is called variable bulk service model (VBSM). Most of the work that has been reported in literature on bulk queues is related to a fixed batch size. Jaiswal[10] and Bhat[1] have investigated the bulk service queuing problem with variable 'capacity'. In this section the variable bulk service model is studied at regeneration points for a finite-capacity, $N$, ramp. The probability generating function $P(z)$ of steady-state probabilities $p_j$'s will be determined in terms of the probability generating function of the distribution $\{k_j\}$. From $P(z)$, the probability $p_n$ is determined and finally the mean queue length $\bar{n}$ is obtained. The development is as follows.

Let $k_j$ denote the probability of $j$ arrivals during a service period of arbitrary duration and let $p_{ij,r}$ denote the probability that the number of vehicles $= j$ at a regeneration point/number of vehicles $= i$ at the preceding regeneration point and $r$ vehicles can merge in the freeway traffic at the availability of a gap. The transition probabilities are:

$$P_{0j} = k_j$$

(2.39)

$$P_{ij,r} = k_j \quad 0 \leq i < r$$

(2.40)

$$= k_{j-i+r} \quad j-i+r \geq 0; \quad i > 0; \quad j \leq N$$

(2.41)
Thus

\[ p_{ij} = \sum_r p_{ij,r} p(r) \quad (2.42) \]

where \( p(r) \) is the probability that \( r \) vehicles can merge at the availability of a gap. This depends on the length of the gap. If the gap length is \( t \) and \( x < t < 2x \), only one vehicle will be allowed to merge with the freeway traffic; if \( 2x < t < 3x \), two vehicles will be allowed, and so on.

The density function of gap sizes \( g(t) \) is plotted in Figure 6.

\[ g(t) = a e^{-\lambda(t-x)} \quad t \geq x \quad (2.43) \]

Thus

\[ p(r) = e^{-rax} (e^{ax} - 1) \quad (2.44) \]

The steady state, equilibrium, probabilities \( p_j \), that there are \( j \) vehicles on the ramp is

\[ p_j = \sum_i p_{ij} p_i \quad (2.45) \]

The probability generating function of \( p_j \)'s is defined by

\[ P(z) = \sum_{j=0}^{N} p_j z^j \quad (2.46) \]

and the probability generating function of the distribution \( \{k_j\} \) is defined by

\[ K(z) = \sum_{j=0}^{N} k_j z^j \quad (2.47) \]

Substituting Eq. (2.39), (2.40), (2.41), and (2.45) in (2.46) yields

\[ P(z) = \sum_{j=0}^{N} z^j \left[ \sum_{r=1}^{\infty} \left( \sum_{i=0}^{j+r-1} k_{j-i+r} p_i \right) p(r) \right] \quad (2.48) \]
Fig. 6. Density function of gap sizes.
Substituting Eq. (2.44) in the above equation, after considerable ma-
nipulation,\(^1\) yields

\[
P(z) = K(z) \sum_{i=0}^{\infty} p_i e^{-i \alpha x} + \frac{e^{\alpha x} - 1}{ze^{\alpha x} - 1} P(z) K(z)\]

\[
- \frac{e^{\alpha x} - 1}{ze^{\alpha x} - 1} K(z) \sum_{i=0}^{\infty} p_i e^{-i \alpha x} \]

\[
- (e^{\alpha x} - 1) \sum_{r=1}^{N} e^{-r \alpha x} \sum_{i=r}^{N+r} \sum_{l+i-r=N+1}^{\infty} p_i k_{ij} z^{l+i-r} \ (2.49)\]

The last term of \(P(z)\) in Eq. (2.49) does not contribute to the desired
probability generating function since it contributes \(z^{N+1}\) and higher
order terms. Since

\[
\sum_{i} p_i e^{-i \alpha x} = P(e^{-\alpha x}) \ (2.50)\]

the probability generating function \(P(z)\) for the VBSM is obtained from
Eqs. (2.49) and (2.50).

\[
P(z)\bigg|_{\text{VBSM}} = \frac{P(e^{-\alpha x}) K(z) (1 - z)}{(1 - e^{-\alpha x}) K(z) - (z - e^{-\alpha x})} \ (2.51)\]

Remark: For large \(\alpha x\), \(P(e^{-\alpha x}) \to p_0\) and \(e^{-\alpha x} \to 0\). Thus

\[
P(z)\bigg|_{\text{VBSM}} = p_0 \frac{K(z) (1 - z)}{K(z) - z} = P(z)\bigg|_{\text{SSM}} \ (2.12)\]

\(^1\)See Appendix B.
This was expected, since for high flow rates the availability of larger gaps is very small and so the result should remain unchanged. This indicates that there is hardly any improvement over SSM for high flow rates.

Now the statistics of the queue length for VBSM are investigated. Substituting Eq. (2.30) in Eq. (2.51), one can obtain:

$$P(z) = P(e^{-ax})[1 - d(z - e^{-ax})e^{-cz}]^{-1} \quad (2.52)$$

First the case of on-ramp of infinite waiting space is considered. The probability generating function of $p_j$'s is defined as:

$$P(z) = \sum_{j=0}^{\infty} p_j z^j \quad (2.53)$$

Since $P(1) = 1 \quad (2.54)$

one can thus obtain

$$P(e^{-ax}) = [1 - d(1 - e^{-ax})e^{-c}]$$

$$= 1 - \lambda/a(e^{ax} - 1) \quad (2.55)$$

$$P(z) = P(e^{-ax})[1 + d(z - e^{-ax})e^{-cz} + d^2(z - e^{-ax})^2e^{-2cz} +$$

$$+ \ldots + d^n(z - e^{-ax})^ne^{-ncz} + \ldots] \quad (2.56)$$

The probability of an empty queue is the constant term in Eq. (2.56).

Thus

$$P_0 = P(e^{-ax})[1 - de^{-ax} + d^2e^{-2ax} \ldots]$$

$$P_0 = P(e^{-ax}) \cdot \frac{1}{1 + \frac{\lambda}{a} e^{-ax}}$$
The mean queue length on the ramp can be easily computed by differentiating $P(z)|_{\text{VBSM}}$ and setting $z = 1$.

$$
\pi' \big|_{\text{VBSM}} = \frac{d}{dz} P(z) \big|_{z=1} = \frac{\lambda}{a} \left(1 - \frac{\lambda x}{a} + \frac{\lambda x}{a} \right) \left(1 - \frac{\lambda x}{a} + \frac{1}{a} \right)
$$

$\pi'(a)$ is a convex function, thus it has a unique minimum and it occurs for $x a < 1$.

Finite waiting space on the ramp. Now the case of a finite-capacity ramp is considered for VBSM. Here again the mean queue length $\pi$ is derived. Rewriting Eq. (2.52) gives

$$
\frac{P(z)}{P(e^{-ax})} = \left[1 - d(z - e^{-ax}) e^{-cz}\right]^{-1}
$$

Let $e^{-ax} = \mu$

Then

$$
\frac{P(z)}{P(e^{-ax})} = \left[1 - d(z - \mu) e^{-cz}\right]^{-1}
$$

$$
= \sum_{q=0}^\infty d^q(z - \mu)^q e^{-qcz}
$$

$$
= \sum_{q=0}^\infty d^q \sum_{k=0}^q (-1)^{q-k} \binom{q}{k} \mu^{q-k} z^k \sum_{m=0}^\infty (-1)^m \frac{(ae)^m}{m!} z^m
$$
\[ \frac{P(z)}{P(e^{-ax})} = \sum_{q=0}^{\infty} \sum_{n=0}^{\infty} z^n \frac{(-1)^q z^k}{m!} \frac{\sum_{m+k=n}^{\infty} (-1)^q (q)_k (q)_k (q)_k (q)_k (q)_k (q)_k}{m!} \] (2.61)

\[ P(z) = P(e^{-ax}) \sum_{n=0}^{\infty} z^n \frac{(-1)^q z^k}{m!} \frac{\sum_{m+k=n}^{\infty} (-1)^q (q)_k (q)_k (q)_k (q)_k (q)_k (q)_k}{m!} \] (2.62)

Since \( P(z) = \sum_{j=0}^{N} p(z)^j \), \( p_n \), the probability that there are \( n \) cars on the ramp, is obtained from Eq. (2.62) as the coefficient of \( z^n \).

\[ p_n = (-1)^n P(e^{-ax}) \sum_{q=0}^{\infty} (-1)^q z^k \frac{\sum_{m+k=n}^{\infty} (-1)^q (q)_k (q)_k (q)_k (q)_k (q)_k (q)_k}{m!} \] (2.63)

Since \( 0 \leq k = n-m \leq q \)

\( n-q \leq m \leq n \)

\( \max(0, n-q) \leq m \leq n \)

\[ p_n = \sum_{q=0}^{\infty} \frac{(-1)^n \mu^n}{m!} \sum_{q=0}^{\infty} (-1)^q (\mu v)^q \sum_{m=0}^{\infty} \frac{(vq)^m}{m!} \] (2.64)

or

\[ p_n = \sum_{q=0}^{\infty} \frac{(-1)^n \mu^n P(e^{-ax})}{m!} \sum_{m=0}^{\infty} \frac{(vq)^m}{m!} \sum_{n-m \leq q < \infty} \frac{p^q (q)_q^m}{(n-m)^q} \] (2.65)

where \( p = (-\mu v) \). It can be simplified further as:

\[ \text{See Appendix C.} \]
\[ p_n = p(e^{-ax}) \frac{d^n}{(1 + du)^n} \frac{n}{x} \sum_{m=0}^{\infty} (-1)^m \frac{(uc)^m}{m!} \frac{(1 + du)^m}{m!} \]

\[ \sum_{i=1}^{m+1} a_{k,m} \frac{(\xi + n - m - 1)!}{\xi - 1! (1 + du)^k} \]  \hspace{1cm} (2.66)

where

\[ a_{k,m} = (-1)^{m-2+i} \sum_{i=0}^{\xi-1} (-1)^i \frac{(\xi-1)!}{i!} (\xi - i)^m \]  \hspace{1cm} (2.67)

Since

\[ \sum_{n=0}^{N} p_n = 1 \]  \hspace{1cm} (2.68)

one can obtain from Eq. (2.66) to Eq. (2.68)

\[ P(e^{-ax}) = \left[ \sum_{n=0}^{\infty} \frac{d^n}{(1+du)^n} \sum_{m=0}^{n} \frac{(u)^m}{m!} \frac{(1+du)^m}{m!} \sum_{k=1}^{\infty} a_{k,m} \frac{(\xi+n-m-1)!}{(\xi-1)! (1+du)^k} \right]^{-1} \]  \hspace{1cm} (2.69)

Thus the expected number of cars on the ramp, i.e., mean queue length \( \bar{n} \), is obtained as

\[ \bar{n}_{VBSM} = \sum_{n=0}^{N} n p_n \]

\[ = \sum_{n=0}^{N} \frac{n}{(1+du)^n} \sum_{m=0}^{n} \frac{(u)^m}{m!} \frac{(1+du)^m}{m!} \sum_{k=1}^{\infty} a_{k,m} \frac{(\xi+n-m-1)!}{(\xi-1)! (1+du)^k} \]  \hspace{1cm} (2.70)
2.4 Comparison of SSM and VBSM

It was pointed out in Sections 2.2 and 2.3 that the queue length \( \bar{N} \) (or \( \bar{N}' \)) is a convex function as a function of arrival rate on the freeway for both the SSM and the VBSM. However, for VBSM the unique minimum occurs for \( xa < 1 \) as compared to \( xa = 1 \) for SSM. This suggests that the queue length for the VBSM is smaller than for the SSM. Let \( M \) denote the merit of VBSM over SSM:

\[
M = \left. \bar{N}' \right|_{\text{SSM}} - \left. \bar{N}' \right|_{\text{VBSM}}
\]  

(2.71)

Substituting Eq. (2.38) and Eq. (2.58) in (2.71) yields

\[
M = \frac{\lambda (\frac{1}{a} e^{ax} - x)}{(1 - \frac{\lambda}{a} e^{ax})(1 - \frac{\lambda}{a} e^{ax} + \frac{\lambda}{a})}
\]

(2.72)

\( M \) is a positive quantity for \( ax > 0 \). Thus the queue length is reduced in variable bulk service model, which was anticipated.

2.5 Design of Storage Facility

In Sections 2.2 and 2.3 the expected value of the queue length \( \bar{N} \) for the SSM and the VBSM was obtained for the case of a finite capacity, \( N, \) ramp. Quite often more explicit information about the behavior of the queue over fairly short times is required. This leads to a non-equilibrium theory of queues which is usually much more difficult than the equilibrium theory discussed in previous sections. One interesting and very practical example where non-equilibrium theory is needed occurs when the rate of arrival is suddenly increased for some period. This sudden increase of arrival rate commences the 'rush-hour'. The effect
of 'rush-hour' over the system behavior has to be considered in designing any storage facility. An elaborate account of 'rush-hour' is available in Cox and Smith[4]. The exact calculations are prohibitive. In this section an approximate effect of 'rush-hour' is considered in specifying the design of an on-ramp dynamic storage facility.

Let \( \bar{n} \) = number of vehicles on the ramp at \( t = 0 \), and at \( t = 0^+ \) the rush-hour starts; i.e., the arrival rate \( \lambda \) changes to a higher value such that the traffic intensity \( \rho (\rho = \lambda/\alpha e^{\alpha t}) \) becomes more than 1. Let \( A_t \) be the expected number of arrivals in time \( t \) after the start of 'rush-hour' and \( L_t \) be the expected number of vehicles merging with the freeway traffic in the same time; then if \( A_t - L_t > 0 \), the queue grows indefinitely. Let \( N_t \) = expected number of vehicles on the ramp at time \( t \).

\[
N_t = \bar{n} + A_t - L_t
\]

Thus the size of storage facility, \( S \), needed to accommodate the overflow from the finite capacity ramp is

\[
S = N_t - N = (\bar{n} + A_t - L_t) - N
\]

where \( N \) is the capacity of the ramp, and \( \bar{n} \) is obtained from Eq. (2.36) for the SSM or Eq. (2.70) for the VBSM. The 'rush-hour' behavior of the queue on the ramp is plotted in Figure 7. The capacity of storage facility needed to accommodate 30 minutes of 'rush-hour' for a particular traffic intensity \( \rho = 1.46 \) is given in Table 4.
Fig. 7. Rush hour behavior of the queue on the ramp.

\begin{align*}
x &= 5 \text{ sec} \\
\alpha &= 0.5 \text{ sec}^{-1} \\
\lambda_{0-} &= 0.4 \text{ sec}^{-1} \\
\lambda_{0+} &= 0.6 \text{ sec}^{-1} \\
\rho_{0+} &= \frac{\lambda}{\alpha} e^{\alpha x} = 1.46
\end{align*}
TABLE 4
Capacity of Storage Facility Required to Accommodate 30 Minutes of Rush Hour

<table>
<thead>
<tr>
<th>Capacity of the Ramp, N</th>
<th>Capacity of Storage Required, S</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>30</td>
</tr>
<tr>
<td>20</td>
<td>30</td>
</tr>
<tr>
<td>30</td>
<td>30</td>
</tr>
</tbody>
</table>
CHAPTER III
SYNTHESIS OF A LOGICAL TRAFFIC CONTROL SYSTEM

In Chapter II the statistics of the queue length for a finite capacity on-ramp were obtained for single service and variable bulk service queuing models. The capacity of dynamic storage facility to accommodate the overflow from the on-ramp was also determined. Figure 2 describes the freeway on-ramp situation with the necessary dynamic storage facility. It is obvious that such a storage scheme will only add to confusion if traffic is not controlled at the entrance of the ramp and the exit from the storage. In this chapter a logical control system is designed to utilize the storage facility efficiently and to avoid the ambiguity of decisions on the part of the driver. A logic block is developed to do the following: (a) divert the traffic, wishing to join the ramp, to the storage facility if the need arises; (b) allow the vehicles from the storage to join the ramp.

Figure 2 can be represented by the following block diagram (Figure 8) where \( f_1, f'_1, f_2, \) and \( f'_2 \) are flow rates on the freeways. \( F_I \) and \( F_{II} \) denote the freeways and \( R \) and \( S \) denote the ramp and storage respectively. It is evident that Figure 8 needs to be modified to avoid collisions at the ramp between the vehicles released from \( S \) and the vehicles on \( F_I \) wishing to join \( R \). Essentially some logical design to channelize the traffic flow properly is required. A modified situation is shown in Figure 9.
Fig. 8. Freeway on-ramp situation with on-ramp storage facility.

Fig. 9. Freeway on-ramp situation with on-ramp storage and channelizing gates.
G₁ and G₂ are gates required to channelize the flow.

Let

\[ N = \text{capacity of the ramp} \]
\[ \bar{n} = \text{mean queue length on ramp} \]
\[ S = \bar{n} - N = \text{size of storage facility needed to accommodate the overflow} \]

For proper channelization the conditions for gate G₁ become:

G₁ open if \( n \leq N \)

G₂ closed if \( 0 < m < S \), and \( n > N \)

This simple modification, of course, neglects the convenience of vehicles in the storage facility. This factor will be taken into account when a logic circuit is developed to operate G₁ and G₂.

Let two forward-backward counters C₁ and C₂ keep a count of vehicles on the ramp and in the storage, respectively, and install two traffic lights L₁ and L₂ at the entrance of the ramp and storage and at the exit of storage, respectively. The following notations are used:

L₁ green - vehicle joins the ramp
L₁ red - vehicle joins the storage
L₂ green - vehicle merges with the ramp from the storage
L₂ red - vehicle stays in storage

Thus Figure 9 can be represented as a system with the arrival of vehicles as independent variables and the output of lights L₁ and L₂ as dependent variables. It is described in Figure 10. Different arrival situations that may arise and the desired output for the traffic lights are tabulated in Table 5.
Fig. 10. System representation of on-ramp traffic control.
### TABLE 5

**Desired Output of Logic Block**

<table>
<thead>
<tr>
<th>No.</th>
<th>C₁</th>
<th>C₂</th>
<th>L₁</th>
<th>L₂</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>&lt;N</td>
<td>0</td>
<td>G</td>
<td>x</td>
<td>G = Green</td>
</tr>
<tr>
<td>2</td>
<td>&lt;N</td>
<td>0 &lt; C₂ ≤ S</td>
<td>R</td>
<td>G</td>
<td>R = Red</td>
</tr>
<tr>
<td>3</td>
<td>N</td>
<td>0</td>
<td>R</td>
<td>x</td>
<td>x = Don't care</td>
</tr>
<tr>
<td>4</td>
<td>N</td>
<td>0 &lt; C₂ ≤ S</td>
<td>R</td>
<td>R</td>
<td></td>
</tr>
</tbody>
</table>

### TABLE 6

**Truth Table of Logic Block**

<table>
<thead>
<tr>
<th>No.</th>
<th>C₁</th>
<th>C₂</th>
<th>L₁</th>
<th>L₂</th>
<th>Remarks for L₁, L₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>x</td>
<td>1 = Green</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0 = Red</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>x</td>
<td>x = Don't care</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>
Denote that in Table 5

\[ C_1 < N \text{ by } a = 0, \]
\[ C_1 = N \text{ by } b = 1, \]
and
\[ C_2 = 0 \text{ by } A = 0, \]
\[ C_2 \neq 0 \text{ by } B = 1. \]

Thus the counters can be represented by variables \( x(a,b) \) and \( y(A,B) \) as shown in Table 6.

From Table 6, one can write

\[ L_1 = \overline{x} \overline{y} \] (3.1)

and
\[ L_2 = \overline{x} y \] (3.2)

Thus Eqs. (3.1) and (3.2) can be represented as shown in Figure 11. It is noted that \( x \) and \( y \) are activated by another logic block which is dependent on counter outputs. This is described in Tables 7 and 8. From Table 7 it is noticed that one needs a logic circuit to excite \( x \) with a logic 1 if \( n = N \) and with logic 0 if \( n < N \). This is demonstrated by the following example.

Let \( N = 30; \) \( (30)_{10} = 11110 \). The counter \( C_1 \) is shown in Figure 12. \( x \) can be excited by an AND circuit as shown in Figure 13. From Table 8, it is clear that one needs a logic 0 for \( C_2 = 0 \) and logic 1 otherwise for the excitation of \( y \). It is explained below for \( S = 10 \). The counter \( C_2 \) has the form shown in Figure 14. Since \( (10)_{10} = 1010 \) and it is required to have a logic 0 for \( C_2 = 0 \) and logic 1 otherwise, one can achieve this with an AND circuit and an inverter as shown in Figure 15. Hence for the above example for \( N = 30 \) and \( S = 10 \), the on-ramp traffic control is achieved by the circuit as shown in Figure 16.
Fig. 11. Logic circuit for operating $L_1, L_2$. 
### TABLE 7

Output of Counter $C_1$

<table>
<thead>
<tr>
<th>${n}$</th>
<th>$x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&lt;N$</td>
<td>0</td>
</tr>
<tr>
<td>$N$</td>
<td>1</td>
</tr>
</tbody>
</table>

### TABLE 8

Output of Counter $C_2$

<table>
<thead>
<tr>
<th>${m}$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\leq S$</td>
<td>1</td>
</tr>
</tbody>
</table>
Fig. 12. Counter C₁.

Fig. 13. Logic circuit for excitation of x.

\[ x = \text{logic 1 for } N = 30 \]
Fig. 14. Counter C₂.

Fig. 15. Logic circuit for excitation of y.
Fig. 16. On-ramp traffic control system for $N = 30$ and $S = 10$. 
CHAPTER IV

GRAPH THEORY IN TRAFFIC FLOW PROBLEMS

4.1 Introduction

In Chapter II some queuing models for freeway on-ramp situations were discussed. The analysis was based on steady-state conditions. Quite often more explicit information about the behavior of the queue over fairly short times is required. This necessitates the study of transient behavior of the queues. Recently the literature on transient behavior of queues has been growing very rapidly. A comprehensive bibliography is available in Prabhu[18], Saaty[19], and Tackas[20]. The transient behavior of the queues poses quite interesting nonlinear stochastic problems. To facilitate the study of the transient behavior, a knowledge of the probability density function p(n) of the stochastic process n(t), the queue length, and p[n(t)/n(0)], the conditional probability density function are often adequate. For the present, the objective shall be centered on computation of p(n), the absolute probability density function of the queue length. For this purpose a model should be chosen to simulate traffic flow distribution on a freeway (or a ramp). A cascade of nonlinear zero-memory, linear time-invariant and nonlinear zero memory system as shown in Figure 17 is chosen since it can be used to generate an output whose distribution resembles a typical traffic flow distribution on a freeway (or a ramp). This is explained as follows.
Fig. 17. Block diagram of the cascade system to simulate traffic flow distribution.
If the input, \( x(t) \), to the system shown in Figure 17 is a stationary Gaussian signal with a continuous power density spectrum and zero mean, and if the first nonlinear zero-memory system is an ideal relay, its output, \( y(t) \), resembles a telegraph signal. This signal \( y(t) \) can be differentiated to obtain positive and negative pulses, \( z(t) \), at the zero crossings of the Gaussian signal. A second nonlinear zero-memory system, such as a rectifier, can be used to convert the negative pulses to positive pulses. These sequences of positive pulses, \( w(t) \), correspond to the vehicles or elements of a traffic queue. Thus one can conclude that with the adjustment of the cascade system shown in Figure 17, one may be able to simulate a variety of queuing problems. The probability density function of \( w(t) \) can be generated from the knowledge of its moments. Hemami and Cosgriff[9] obtained moments of probability density function of a functional of a Gaussian signal directly using graph techniques and non-harmonic series representation of stationary Gaussian signals. In this chapter their work is extended to obtain the moments of probability density function of a functional of a Gaussian signal for a wider class of nonlinear stochastic problem as shown in Figure 17.

In Section 4.2 representation of random stationary Gaussian signals is described, which is used in Section 4.3 to analyze the system shown in Figure 17. Graph theory is applied in the analysis to obtain autocorrelation and other moments of \( w(t) \) which can subsequently be used to generate the probability density function of \( w(t) \). The computation of moments requires generation of a class of 'irregular graphs',
counting and classifying them according to 'distinct perfect matchings'.
Irregular graphs and 'perfect matchings' are introduced in Section 4.4 and
an algorithm is developed to count and determine the 'distinct'
perfect matchings of an irregular graph. Rather than discussing the
general method of how nonharmonic series and graph theory can be used
to obtain moments of \( w(t) \), the rest of the chapter, Sections 4.5 and
4.6, is devoted to a particular example of evaluating autocorrelation
function of \( w(t) \) for specific linear and nonlinear blocks of Figure 17.
By detailed discussion of the example, the approach, the procedure and
means of solving such problems are thus elaborated.

4.2 Representation of Random
Stationary Gaussian Signals

As suggested by Cosgriff[3], a stationary Gaussian signal \( x(t) \)
with a continuous power density spectrum \( \Phi_{xx}(j\omega) \) and zero mean is ap­
proximated by a series

\[
x_{\Delta}(t) = \sum_{\lambda_k \in S_1} a_k e^{j\lambda_k t}
\]

(4.1)

where \( k \) ranges over all positive and negative integers (zero excluded),
and for positive \( k \), the elements \( \lambda_k \) are linearly independent. For \( x(t) \)
to be real

\[
\lambda_{-k} = \lambda_k \quad (4.2)
\]

\[
a_{-k} = a_k^* \quad (4.3)
\]

The aggregate of all positive and negative \( \lambda_k \) is defined as set \( S_1 \).
The elements \( \lambda_k \) fall in an interval defined as follows:
\[(k-1)\Delta < \lambda_k < k \Delta\]  

where \(\Delta\) is a positive increment and, for an exact representation of \(x(t)\), tends to zero. The constant amplitude \(a_k\) is related to the power density spectrum \(\phi_{xx}(j\omega)\) as follows:

\[|a_k|^2 = \frac{\Delta}{2\pi} \phi_{xx}(j\lambda_k)\]  

\[\sum_k |a_k|^2 = \frac{\Delta}{2\pi} \phi_{xx}(j\lambda_k)\]  

As \(\Delta\) tends to zero:

\[\sum_k |a_k|^2 = \int_{-\infty}^{\infty} \phi_{xx}(j\omega) \frac{d\omega}{2\pi}\]  

It can be shown[3] that based on the above representation of \(x(t)\) the \(n^{th}\) Hermite polynomial of \(x(t)\) has the following form:

\[H_n[x(t)] = \sum_{\lambda_k \in S_1} \cdots \sum_{\lambda_k \in S_1} a_k, \ldots, a_k e^{j(\lambda_k1+\ldots+\lambda_kn)t}\]  

where \(\mid_r\) indicates that the summations are to be carried out such that no two \(\lambda_k\) in the exponent sum to zero, i.e., \(\lambda_k \neq -\lambda_k\). All elements \(\lambda_k1+\ldots+\lambda_kn\) with the above restriction are said to belong to set \(S_n\). Thus

\[H_n[x(t)] = \sum_{\lambda_k \in S_1} \cdots \sum_{\lambda_k \in S_1} a_k, \ldots, a_k e^{j(\lambda_k1+\ldots+\lambda_kn)t}\]  

Set \(S_0\) corresponding to zero frequency component, and thus any term which has a component in \(S_0\) has a time average.

Let a functional of \(x(t)\) be defined as
\[
L_n[x(t)] = \lambda_{k_1} + \ldots + \lambda_{k_n} s_a \sum_{s} a_{k_1}, \ldots, a_{k_n} \cdot H(j(\lambda_{k_1}, \lambda_{k_2}, \ldots, \lambda_{k_n}))
\]

\[
= \sum_{s} a_{k_1}, \ldots, a_{k_n} \cdot H(j(\lambda_{k_1}, \lambda_{k_2}, \ldots, \lambda_{k_n}))
\]

where \( H(j(\lambda_{k_1}, \ldots, \lambda_{k_n})) \) is an arbitrary bounded function of its \( n \) variables. These functionals are orthogonal over \(-\infty < t < \infty\) since multiplication of \( L_n[x(t)] \) and \( L_m[x(t)] \) does not contain any term belonging to set \( S_0 \). The functional \( L_n[x(t)] \) is represented by a column of \( n \) points. The multiplication of \( L_n[x(t)] \) and \( L_m[x(t)] \) is represented by constructing an array—two columns of \( n \) and \( m \) points, respectively, next to each other. The product of \( L_n[x(t)] \) and \( L_m[x(t)] \) has a time average only if it has a component in \( S_0 \). For example, if \( n = 3 \) and \( m = 5 \), the product can be represented as

\[
L_3[x(t)]L_5[x(t)] = \begin{array}{c}
  \vdots \\
  \vdots \\
  \vdots \\
  \vdots \\
  \vdots
\end{array}
\]

It has components in \( S_8, S_6, S_4 \), and \( S_2 \) but no component in \( S_0 \), thus the time average of Eq. (4.11) is zero. The use of these orthogonal functionals in computing the autocorrelation function of \( w(t) \) of Figure 17 is described in the following section.

In Section 4.1 it was explained that a cascade of nonlinear zero-memory, linear time-invariant and nonlinear zero-memory systems as shown in Figure 17 can be used to generate an output, \( w(t) \), which resembles a typical traffic flow distribution. In this and the following sections the analysis will be restricted to the computation of the autocorrelation function of \( w(t) \) for a stationary Gaussian input, \( x(t) \), with zero mean. The output \( w(t) \) is expanded into orthogonal functionals \( L_n[x(t)] \) as

\[
w(t) = \sum_{n=1}^{\infty} c_n L_n[x(t)]
\]

(4.12)

The autocorrelation function of \( w(t) \) thus reduces to

\[
w(t) w(t+\tau) = \sum_{n=1}^{\infty} c_n^2 \phi_{L_n} L_n(\tau)
\]

(4.13)

where \( \phi_{L_n} L_n(\tau) \) is the autocorrelation function of \( L_n[x(t)] \), since two functionals of different \( S_i \) are orthogonal over \(-\infty < t < \infty\). The output, \( w(t) \), is first derived in terms of its set frequencies. The analysis is as follows: \( x(t) \) is assumed to be a stationary Gaussian signal with a continuous power density spectrum and zero mean, and is represented by nonharmonic series representation as explained in Section 4.2. Let the nonlinear zero-memory system I be such that its output, \( y(t) \), is the sum of several Hermite polynomials of the input signal. Thus \( y(t) \) can be expressed as:

\[
y(t) = \sum_{i=1}^{n} H_i(x)
\]

(4.14)
Substituting Eq. (4.9) in (4.14) and letting \( n = 3 \),

\[
y(t) = \sum_{s_1} a_{k_1} e^{j\lambda_{k_1} t} + \sum_{s_2} a_{k_2} a_{k_3} e^{j(\lambda_{k_2} + \lambda_{k_3}) t} \\
+ \sum_{s_3} a_{k_4} a_{k_5} a_{k_6} e^{j(\lambda_{k_4} + \lambda_{k_5} + \lambda_{k_6}) t}
\]  

(4.15)

The output, \( z(t) \), of the linear time-invariant system is given by:

\[
z(t) = \sum_{s_1} a_{k_1} H(j\lambda_{k_1}) e^{j\lambda_{k_1} t} + \sum_{s_2} a_{k_2} a_{k_3} H(j(\lambda_{k_2} + \lambda_{k_3})) e^{j(\lambda_{k_2} + \lambda_{k_3}) t} \\
+ \sum_{s_3} a_{k_4} a_{k_5} a_{k_6} H(j(\lambda_{k_4} + \lambda_{k_5} + \lambda_{k_6})) e^{j(\lambda_{k_4} + \lambda_{k_5} + \lambda_{k_6}) t}
\]  

(4.16)

where \( H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt \)  

(4.17)

and \( h(t) \) is the impulse response of the linear time-invariant system.

The second nonlinear zero-memory system is assumed to be analytic and single-valued. In particular, let

\[
w(t) = z^2(t)
\]  

(4.18)

Thus the output, \( w(t) \), in terms of its set frequencies is given by

\[
w(t) = \left[ \sum_{s_1} a_{k_1} H(j\lambda_{k_1}) e^{j\lambda_{k_1} t} + \sum_{s_2} a_{k_2} a_{k_3} H(j(\lambda_{k_2} + \lambda_{k_3})) e^{j(\lambda_{k_2} + \lambda_{k_3}) t} \\
+ \sum_{s_3} a_{k_4} a_{k_5} a_{k_6} H(j(\lambda_{k_4} + \lambda_{k_5} + \lambda_{k_6})) e^{j(\lambda_{k_4} + \lambda_{k_5} + \lambda_{k_6}) t} \right] x
\]

\[
\left[ \sum_{s_1} a_{k_7} H(j\lambda_{k_7}) e^{j\lambda_{k_7} t} + \sum_{s_2} a_{k_8} a_{k_9} H(j(\lambda_{k_8} + \lambda_{k_9})) e^{j(\lambda_{k_8} + \lambda_{k_9}) t} \\
+ \sum_{s_3} a_{k_{10}} a_{k_{11}} a_{k_{12}} H(j(\lambda_{k_{10}} + \lambda_{k_{11}} + \lambda_{k_{12}})) e^{j(\lambda_{k_{10}} + \lambda_{k_{11}} + \lambda_{k_{12}}) t} \right]
\]  

(4.19)
To obtain the autocorrelation of \( w(t) \), one has to determine the terms of \( \overline{w(t)w(t+\tau)} \) that fall in set \( S_0 \). To compute the contribution in set \( S_0 \) directly, the orthogonal functionals described in Section 4.2 are employed. It is evident from Eq. (4.19) that to obtain autocorrelation function of \( w(t) \), the time average of the product of several functionals must be derived. The time average of \( w(t)w(t+\tau) \) correspond to those terms of product of the orthogonal functionals which belong to set \( S_0 \). The functional \( \int_s x(t) \) is represented by a column of \( m \) points and the multiplication of \( \int_s x(t) \), \( \int_n x(t) \), \( \int_p x(t) \) is represented by writing the columns of \( m, n, \ldots, p \) points next to each other. In order to obtain the terms belonging to set \( S_0 \), all the different ways in which the points of the array can be connected by pairs must be counted. Of course, points of the same column cannot be connected because of the definition of the sets. Thus it is clear that one is concerned with only those products of \( \int_m x(t), \int_n x(t), \ldots, \int_p x(t) \) for which \( m + n + \ldots + p \) is even, since the zero frequency component, corresponding to \( S_0 \), cannot be obtained if \( m + n + \ldots + p \) is odd. Given \( K \) such columns of points, the ordering of columns is immaterial so far as the time average is concerned.

Consider a typical multiple sum of \( w(t)w(t+\tau) \) where \( w(t) \) is given by Eq. (4.19).
The array for the product of orthogonal functionals appearing in Eq. (4.23) is constructed as shown in Figure 18. For a particular pairing of the above array as shown in Figure 19, a simpler sum results:

\[ \sum_{S_2} \sum_{S_3} \sum_{S_3} a_{k_2} a_{k_3} a_{k_6} a_{k_9} a_{k_{16}} a_{k_{17}} a_{k_{18}} a_{k_{22}} a_{k_{23}} a_{k_{24}} \]

\[ H(J(\lambda_{k_2}+\lambda_{k_3})) H(J(\lambda_{k_6}+\lambda_{k_9})) H(J(\lambda_{k_{16}}+\lambda_{k_{17}}+\lambda_{k_{18}})) x \]

\[ J(\lambda_{k_2}+\lambda_{k_3}) t \quad J(\lambda_{k_6}+\lambda_{k_9}) t \]

\[ J(\lambda_{k_{22}}+\lambda_{k_{23}}+\lambda_{k_{24}}) \epsilon \]

\[ \epsilon \]

\[ j(\lambda_{k_{16}}+\lambda_{k_{17}}+\lambda_{k_{18}})(t+t) \quad j(\lambda_{k_{22}}+\lambda_{k_{23}}+\lambda_{k_{24}})(t+t) \]  

(4.23)

In the limit as \( \Delta \to 0 \), Eq. (4.24) becomes an integral:

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_{xx}(J\omega_2) \phi_{xx}(J\omega_3) \phi_{xx}(J\omega_{16}) \phi_{xx}(J\omega_{17}) \phi_{xx}(J\omega_{18}) |H(J(\omega_2+\omega_3))|^2 x \]

\[ |H(J(\omega_{16}+\omega_{17}+\omega_{18}))|^2 \frac{d\omega_2}{2\pi} \frac{d\omega_3}{2\pi} \frac{d\omega_{16}}{2\pi} \frac{d\omega_{17}}{2\pi} \frac{d\omega_{18}}{2\pi} \]  

(4.25)

There are many more possibilities of pairing the array shown in Figure 18. Each pairing will correspond to an integral as shown in Eq. (4.25). Some of these integrals are identical and thus the computation is reduced considerably. The number of distinct integrals correspond to the number of distinct pairings of an array which is explained in the following section. If \( x_1 \) is the number of columns of length \( m \), \( x_2 \) is the number of columns of length \( n \), and finally \( x_p \) is the number of columns of length \( p \), then the total number of columns \( J \) is given by
Fig. 18. An array corresponding to Eq. 4.23.

Fig. 19. A paired array.
\[
\prod_{i=1}^{p} x_i = J
\]  
\[\text{(4.26)}\]

and

\[m + n + \ldots + p = 2r \quad m,n,p,\ldots,r = 1,2,3,\ldots \]  
\[\text{(4.27)}\]

Then the number \(N\),

\[
N = \frac{J!}{x_1! x_2! \ldots x_p!}
\]  
\[\text{(4.28)}\]

corresponds to various permutations of the columns of the array which contribute to the time average of \(w(t) w(t+r)\). When the number of points in each column of the array is not the same, it is called an irregular graph. Without loss of generality one can arrange the columns of the array in an ascending order. A typical array pertaining to an irregular graph is shown in Figure 20.

4.4 Matchings of a Graph

Before considering the problem of counting, a few relevant concepts from the theory of graphs[2] are described. A graph can be defined as follows. A graph consists of a non-empty set \(V\) of vertices, a (possibly empty) set \(E\) of edges disjointed from \(V\), and a mapping of \(E\) into \(V\) and \(V\). The elements of \(V\) and \(E\) are called the vertices and edges of the graph, respectively. Let \(G = (V,E)\) be any graph having no loops. A loop is an edge that is incident with only one vertex. A set of edges \(M \subseteq E\) is called a matching in \(G\), as shown in Figure 21, if no two edges of \(M\) are adjacent where two edges having at least one common end point are called adjacent. Thus each vertex of \(G\) is incident with at most one edge of \(M\). A vertex is said to be either covered or exposed, relative to \(M\), depending on whether or not it meets an edge of \(M\),
Fig. 20. An array pertaining to an irregular graph.
Fig. 21. Matching of a graph.

Non-matching

Matching

All vertices are covered.

Two vertices are exposed.

Fig. 22. Covered and exposed vertices.
as shown in Figure 22. If every vertex is covered, the matching is said to be perfect. Note that no perfect matching can exist if \(|V|\), i.e., number of vertices is odd. Thus the problem of counting the number of ways of pairwise connections of an array is essentially an example of perfect matchings of a graph with the restriction that the points in the same column of an array cannot be connected to each other.

To obtain various moments of \(w(t)\), the output of the cascade system shown in Figure 17, one has to obtain the perfect matchings of an array of the type shown in Figure 20. This array can be represented by a graph \(G = (V,E)\), where \(V = (V_1, V_2, V_3, V_t)\), i.e., the \(i^{th}\) column has \(V_i\) points, and

\[
\sum_{i=1}^{t} V_i = 2n \quad (n = 1,2,3,...)
\]

(4.29)

Let \(M(V_1, V_2, ..., V_t)\) denote the number of perfect matchings of the above graph \(G\). The drawing of each matching is cumbersome and impractical. To avoid this tedious enumeration of matchings the following theorem is developed.

**Theorem 4.1.** Let \(G = (V,E)\) be a graph where \(V = (V_1, V_2, ..., V_t)\) and \(\sum_{i=1}^{t} V_i = 2n, n = 1,2,3,...\); and let \(M(V_1, V_2, ..., V_t)\) = number of perfect matchings of this graph. Let \(P\) represent the distinct partitions of \((2n - 2V_1)\) into \((t-1)\) columns \((V_2', V_3', ..., V_t')\) such that

\[
\sum_{i=2}^{t} V_i' = (2n - 2V_1)
\]

and 

\[\max(0, (V_1 - V_1')) \leq V_i' \leq V_i\]

If \(C(V_2', V_3', ..., V_t'; V_1, V_2, ..., V_t)\) denotes the number of ways of
obtaining the partitioning of \((2n - 2V_1)\) into \((V_2', V_3', \ldots, V_t')\). Then

\[
M(V_1, V_2, \ldots, V_t) = \sum_{p} C(V_2', V_3', \ldots, V_t'; V_1, V_2, \ldots, V_t) \times M(V_2', V_3', \ldots, V_t')
\] (4.30)

**Proof:** Matching \(M_1 \otimes M_2\), i.e., matching \(M_1\) is equivalent to \(M_2\) if and only if the first column of the array \((V_1, V_2, \ldots, V_t)\) determines the same partition, \((V_2', V_3', \ldots, V_t')\) in \(P\). Therefore

\[
M(V_1, V_2, \ldots, V_t) = \sum_{i} |[M_i]|
\] (4.31)

where \([M_i]\) denotes the number of perfect matchings corresponding to a partition \(i\) in \(P\). \([M_i]\) is determined by:

1. Choice of partition \((V_2', \ldots, V_t')\) in \(P\).
2. Number of ways of obtaining \((V_2', V_3', \ldots, V_t')\) from \((V_1, V_2, \ldots, V_t)\), i.e., \(C(V_2', V_3', \ldots, V_t'; V_1, V_2, \ldots, V_t)\)
3. Number of matchings of \((V_2', V_3', \ldots, V_t')\), i.e., \(M(V_2', \ldots, V_t')\).

Thus

\[
|[M_i]| = C(V_2', V_3', \ldots, V_t'; V_1, \ldots, V_t) \times M(V_2', \ldots, V_t')
\] (4.32)

Hence

\[
M(V_1, V_2, \ldots, V_t) = \sum_{p} C(V_2', V_3', \ldots, V_t'; V_1, V_2, \ldots, V_t) \times M(V_2', \ldots, V_t')
\]

Q.E.D.

The above theorem can be applied recursively to reduce \(M(V_1, V_2, \ldots, V_t)\) to \(M(V_1 V_4)\). Thus the problem of counting reduces to:
(1) Determination of distinct partitions, $P$, of $(2n - 2V_1)$ into $(V'_2, V'_3, \ldots, V'_t)$ as discussed in Appendix D.

(2) Determination of $C(V'_2, V'_3, \ldots, V'_t, V_1, V_2, \ldots, V_t)$.

(3) Evaluation of $M(V'_i, V'_j)$.

Evaluation of $M(V'_i, V'_j)$ is very simple indeed.

\[ M(V'_i, V'_j) = \begin{cases} 
0 & \text{if } V'_i \neq V'_j \\
V'_i! & \text{otherwise}
\end{cases} \quad (4.33) \]

The value of $C(V'_2, \ldots, V'_t; V_1, \ldots, V_t)$ can be obtained using simple combinational relations as follows: Consider $G = (V, E)$ and a particular partition $P$ corresponding to $(V'_2, \ldots, V'_t)$. Given $(V'_2, \ldots, V'_t)$, the matchings of $V_1$ is uniquely determined up to the equivalence discussed before. Let $K$ be the number of columns into which $V_1$ can be matched to obtain the given partition.

Let

\[
\begin{align*}
m_1 &= \text{number of columns having } V'_p \text{ points} \\
m_2 &= \text{number of columns having } V'_q \text{ points} \\
&\vdots \\
m_s &= \text{number of columns having } V'_s \text{ points}
\end{align*}
\]

such that

\[ \sum_{i=1}^{s} m_i = K \quad (4.35) \]

and

\[ K \leq (t - 1) \quad (4.36) \]

Given partition $(V'_2, V'_3, \ldots, V'_t)$ the number of ways of obtaining this partition from the array is

\[ K! \quad (4.37) \]

\[ \frac{1}{m_1! \, m_2! \, \ldots \, m_s!} \]
Having chosen the partition, the points within a column can be permuted.

Thus

$$C(v_2^i, \ldots, v_t^i; v_1, \ldots, v_t) = \frac{K!}{m_1! m_2! m_s!} \prod_{i=1}^{m_1} v_a^{(i)} v_p^{(i)} \prod_{i=1}^{m_2} v_q^{(i)} v_q^{(i)} \prod_{i=1}^{m_s} v_f^{(i)} v_f^{(i)} v_1$$

where

$$\frac{v_a^{(i)}}{v_p^{(i)}} = \frac{v_a^{(i)}}{(v_a^{(i)} - v_p^{(i)})!} v_p^{(i)}$$

4.5 An Example.

The procedure outlined in the preceding sections is now illustrated by deriving the autocorrelation function of $w(t)$. The autocorrelation function $R_{ww}(\tau)$ of $w(t)$ is defined as:

$$R_{ww}(\tau) = \overline{w(t)w(t+\tau)}$$

Substituting Eq. (4.19) in Eq. (4.39) gives

$$R_{ww}(\tau) = \text{time average of } \left[ \sum_{i=1}^{S_1} a_{k_1}^H j^{(\lambda_{k_1})t} + \right.$$}

$$\left. \sum_{i=1}^{S_2} a_{k_2}^H (j^{(\lambda_{k_2}^{+\lambda_{k_3}})})^{j^{(\lambda_{k_2}^{+\lambda_{k_3}})}} + \right.$$

$$\left. \sum_{i=1}^{S_3} a_{k_4}^H (j^{(\lambda_{k_4}^{+\lambda_{k_5}^{+\lambda_{k_6}}})})^{j^{(\lambda_{k_4}^{+\lambda_{k_5}^{+\lambda_{k_6}}})}} \right] x$$

$$\left[ \sum_{i=1}^{S_1} a_{k_7}^H (j^{(\lambda_{k_7}^{+\lambda_{k_9}})})^{j^{(\lambda_{k_7}^{+\lambda_{k_9}})}} + \sum_{i=1}^{S_2} a_{k_8}^H (j^{(\lambda_{k_8}^{+\lambda_{k_9}})})^{j^{(\lambda_{k_8}^{+\lambda_{k_9}})}} \right] x$$

$$\left[ \sum_{i=1}^{S_3} a_{k_{10}}^H (j^{(\lambda_{k_{10}^{+\lambda_{k_{11}^{+\lambda_{k_{12}}}}})})^{j^{(\lambda_{k_{10}^{+\lambda_{k_{11}^{+\lambda_{k_{12}}}}}})}} \right] x$$

$$\left[ \sum_{i=1}^{S_1} a_{k_{13}}^H (j^{(\lambda_{k_{13}}^{+\lambda_{k_{13}}^{+\lambda_{k_{13}}}})})^{j^{(\lambda_{k_{13}}^{+\lambda_{k_{13}}^{+\lambda_{k_{13}}}})}} \right] x$$
The time average of Eq. (4.40) is given by terms belonging to set \( S_0 \). Thus only those terms for which \( S_i + S_j + S_k + S_l \) are even, where \( i, j, k, l = 1, 2, 3 \), will contribute to the time average. Therefore only \( S_i + S_j + S_k + S_l = 4, 6, 8, 10, \) and 12 should be considered. The distinct ways of obtaining the above sum and the number of terms belonging to each class, which are obtained from Eq. (4.28) are tabulated in Table 9.

It is now required to determine the perfect matchings of the configurations listed in Table 9. As discussed in the previous section the number of perfect matchings is:

\[
M(V_1V_2V_3V_4) = \sum_p C(V_2^p V_3^p V_4^p; V_1V_2V_3V_4)M(V_2^p V_3^p V_4^p) \tag{4.41}
\]

Thus to compute the perfect matchings of the arrays shown in Table 9, one needs the following:

1. a. Distinct partitioning of \((2n-2V_1)\) to \((V_2^p V_3^p V_4^p)\).
   b. Distinct partitioning of \((2n-2V_1-2V_2^a)\) to \((V_3^p V_4^p)\).
TABLE 9

Distinct Configurations Corresponding to
\[ S_i + S_j + S_k + S_l = 4, 6, 8, 10, 12 \]

<table>
<thead>
<tr>
<th>No.</th>
<th>( S_i + S_j + S_k + S_l )</th>
<th>Configuration</th>
<th>( N = \text{No. of terms belonging to each configuration} )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>12</td>
<td>( \cdot \cdot \cdot \cdot \cdot )</td>
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(2) Determination of the number of ways of obtaining (1a) and (1b), i.e., \( C(V_3V_1V_0; V_2V_4V_5) \) and \( C(V_3V_1V_0, V_2V_4V_5) \).

(3) Evaluation of the number of perfect matchings of \( y(V_3, V_4) \), i.e., \( M(V_3V_4) \).

The above calculations are shown in Table 10. The paired arrays for distinct classes (partitions) are also drawn. As discussed before each paired array has an integral representation. The evaluation of such integrals is discussed in the following section.

The number of perfect matchings \( M(V_1V_2V_3V_4) \) for different arrays shown in Table 9 are thus determined. There are \( N \) terms belonging to each configuration as shown in Table 9, which contribute to the time average of \( R_{ww}(\tau) \). Therefore the perfect matchings of each distinct paired array is multiplied by corresponding \( N \). Thus the problem has been reduced to evaluating integrals corresponding to each distinct paired array and the autocorrelation function \( R_{ww}(\tau) \) thus becomes:

\[
R_{ww}(\tau) = \sum_{l} M_{l}(V_1V_2V_3V_4) N_{l} I_{l}(\tau)
\]

(4.6.2)

where \( M_{l}(V_1V_2V_3V_4) \) denotes the number of perfect matchings of paired array (Table 10), \( N_{l} \) corresponds to the number of terms belonging to the \( l^{th} \) configuration (Table 9), and \( I_{l}(\tau) \) is the value of the integral associated with the \( l^{th} \) paired array.

The evaluation of the necessary integrals is discussed in the following section.
<table>
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<th>Configuration of the array ((v_1, v_2, v_3, v_4)) ((v_2 = v_4))</th>
<th>Distinct Partition of ((v_1, v_2, v_3, v_4)) ((v_2 = v_4))</th>
<th>Distinct Partition of ((v_1, v_2, v_3, v_4)) ((v_2 = v_4))</th>
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<th>(M(v_1, v_2, v_3, v_4))</th>
<th>Paired Array</th>
<th>(H_4(v_1, v_2, v_3, v_4))</th>
<th>(N(v_1, v_2, v_3, v_4))</th>
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</tbody>
</table>
4.6 Integrals of Quadratic Forms

In Section 4.3 it was pointed out that an integral is associated with each distinct paired array. In this section the integral representation for a particular paired array of Table 9 is developed and the forms of integrals involved for other distinct arrays occurring in Table 9 are listed in Table 11. The evaluation of such integrals is facilitated by reducing the integrand to a quadratic form. Some properties of the integrals of quadratic forms are discussed. A sample calculation for evaluating one such integral is shown and finally the autocorrelation function, $R_{ww}(\tau)$, is plotted in Figure 24.

Consider a typical array configuration (1133), i.e.,

from Table 9. Corresponding to this array, Eq. (4.40) yields the following multiple sum.

$$I_\xi(\tau) = \text{average of } S_1 S_1 S_3 S_3 \sum_{k_1} a_k H(j\lambda_{k_1} e^{j\lambda_{k_1} \tau}) \sum_{k_2} a_k H(j\lambda_{k_7} e^{j\lambda_{k_7} \tau}) x$$

$$a_{k_16} a_{k_17} a_{k_18} H(j(\lambda_{k_16} + \lambda_{k_17} + \lambda_{k_18})) e^{j(\lambda_{k_16} + \lambda_{k_17} + \lambda_{k_18}) (t+\tau)}$$

$$a_{k_22} a_{k_23} a_{k_24} H(j(\lambda_{k_22} + \lambda_{k_23} + \lambda_{k_24})) e^{j(\lambda_{k_22} + \lambda_{k_23} + \lambda_{k_24}) (t+\tau)}$$

One paired array of the above configuration shown in Figure 23 results from

$$\lambda_{k_1} + \lambda_{k_16} = 0$$

$$\lambda_{k_7} + \lambda_{k_22} = 0$$

$$\lambda_{k_17} + \lambda_{k_23} = 0$$

$$\lambda_{k_18} + \lambda_{k_24} = 0$$

(4.43)
Fig. 23. A paired array for configuration (1133).
Thus Eq. (4.43) becomes

\[ I_\lambda(\tau) = \sum_{s_1} \sum_{s_1} \sum_{s_1} \sum_{s_1} |a_{k_1}|^2 |a_{k_7}|^2 |a_{k_17}|^2 |a_{k_18}|^2 H(J\lambda_{k_1}) H(J\lambda_{k_7}) \times \]

\[ H(J(-\lambda_{k_1} - \lambda_{k_17} - \lambda_{k_18})) H(J(-\lambda_{k_7} - \lambda_{k_17} - \lambda_{k_18})) \epsilon^{-J\lambda_{k_1} \tau} \epsilon^{-J\lambda_{k_7} \tau} \]

\[ (4.45) \]

On substituting Eq. (4.7) in the above equation and renumbering the subscripts it leads to

\[ I_\lambda(\tau) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} \int_{-\infty}^{\frac{1}{\lambda}} \phi_{xx}(j\omega_1) H(j\omega_1) H(j\omega_2) H(j(\omega_3 + \omega_4 - \omega_1)) \times \]

\[ H(-j(\omega_2 + \omega_3 + \omega_4)) \epsilon^{-j\omega_1 \tau} \epsilon^{-j\omega_2 \tau} \]

\[ \epsilon \omega_1 \omega_2 \omega_3 \omega_4 \]

\[ (4.46) \]

Consider

\[ \phi_{xx}(j\omega) = \frac{1}{1 + \omega^2} \]

\[ (4.47) \]

and

\[ H(j\omega) = \frac{1}{1 + j\omega} \]

\[ (4.48) \]

Approximating Eq. (4.47) by \( \epsilon^{-\omega^2} \) and Eq. (4.48) by \( \epsilon^{-j\omega - \omega^2/2} \) and substituting in Eq. (4.46) it yields:

\[ I_\lambda(\tau) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} \int_{-\infty}^{\frac{1}{\lambda}} \epsilon^{[-2\omega_1^2 + 2\omega_1^2 + 2\omega_2^2 + 2\omega_3^2 - \omega_1 \omega_3 - \omega_1 \omega_4 + 2\omega_3 \omega_4 + \omega_2 \omega_3 + \omega_2 \omega_4]} \times \]

\[ + j\omega_1 \tau + j\omega_2 \tau] \]

\[ \epsilon \omega_1 \omega_2 \omega_3 \omega_4 \]

\[ (4.49) \]

The exponent in the above equation can be reduced to a quadratic form,

\[ 2\omega_1^2 + 2\omega_2^2 + 2\omega_3^2 - \omega_1 \omega_3 - \omega_1 \omega_4 + \omega_2 \omega_3 + \omega_2 \omega_4 + 2\omega_3 \omega_4 + C \]

by changing the variables as:
\[ \omega_1 = \omega_1' + \alpha \]
\[ \omega_2 = \omega_2' + \beta \]
\[ \omega_3 = \omega_3' + \gamma \]
\[ \omega_4 = \omega_4' + \delta \]

This leads to:
\[ \omega_1'(4\alpha - \gamma - \delta + j\tau) = 0 \quad (4.51) \]
\[ \omega_2'(4\beta + \gamma + \delta + j\tau) = 0 \quad (4.52) \]
\[ \omega_3'(4\gamma - \alpha + \beta + 2\delta) = 0 \quad (4.53) \]
\[ \omega_4'(4\delta - \alpha + \beta + 2\gamma) = 0 \quad (4.54) \]

and
\[ C = 2\alpha^2 + 2\beta^2 + 2\gamma^2 + 2\delta^2 - \alpha\beta - \alpha\delta + \beta\gamma + \beta\delta \]
\[ + 2\gamma\delta + j\tau\alpha + j\tau\beta \quad (4.55) \]

Since \( \omega_1', \omega_2', \omega_3', \omega_4' \neq 0 \), the values of \( \alpha, \beta, \gamma, \delta \) are obtained from simultaneous equations (4.51) to (4.54). This gives
\[ \alpha = \beta = -j\tau/4, \text{ and } \gamma = \delta = 0 \quad (4.56) \]

Substituting the values of \( \alpha, \beta, \gamma, \delta \) from Eq. (4.56) in Eq. (4.55) gives \( C = \tau^2/4 \). Thus Eq. (4.49) becomes
\[ I_k(\tau) = \frac{e^{-\tau^2/4}}{(2\pi)^{4\cdot1/4}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\omega_1^2/2 - \omega_2^2/2 - \omega_3^2/2 - \omega_4^2/2 - \omega_1\omega_3 - \omega_1\omega_4 + \omega_2\omega_3 + \omega_2\omega_4 + \omega_3\omega_4} \]
\[ \omega_1' \omega_2' \omega_3' \omega_4' \quad (4.57) \]

The above integral can be easily evaluated from the theory of integrals of quadratic forms[14] as explained below.

Let \( Y = (y_1, y_2, \ldots, y_n) \) be an \( n \)-dimensional vector and \( P_n \) be a positive definite \( nxn \) symmetric matrix independent of \( Y_n \). It is required to compute the integral:
over all of n-dimensional Euclidean space. This n-fold integral can be compactly written as:

\[ v = \int_{-\infty}^{\infty} I_n(y_n) \, dy_n \]  

(4.59)

where \( dy_n = dy_1 \, dy_2 \ldots \, dy_n \).

Since \( P_n \) is a positive definite matrix, there exists a non-singular square matrix \( Q_n \), such that \( P_n = Q_n^T Q_n \). Let

\[ z_n = Q_n^{-1} y_n \]  

(4.60)

Then Eq. (4.59) becomes:

\[ v = \int_{-\infty}^{\infty} \varepsilon^{\frac{-1}{2}} z_n^T z_n \, J \, dz_n \]  

(4.61)

where \( J = \left| \frac{\partial(y_n)}{\partial(z_n^T)} \right| = \left| \text{det. } Q_n \right| \)  

(4.62)

is the Jacobian of Transformation, Eq. (4.60). Thus

\[ v = \left| \text{det. } Q_n \right| \int_{-\infty}^{\infty} \varepsilon^{\frac{-1}{2}} 2z_2 \]  

\[ = (2\pi)^{n/2} \left| \text{det. } Q_n \right| \]  

(4.63)

But \( (\text{det. } Q_n)^2 = \text{det. } P_n \)  

(4.64)

Thus

\[ \int_{-\infty}^{\infty} \varepsilon^{\frac{-1}{2}} y_n^T P_n^{-1} y_n \, dy_n = (2\pi)^{n/2} \sqrt{\text{det. } P_n} \]  

(4.65)

Eq. (4.57) can be rewritten in the form of Eq. (4.59) as follows:

\[ I_k(t) = \frac{\varepsilon^{\frac{-1}{2}/4}}{(2\pi)^{1/4}} \int_{-\infty}^{\infty} \varepsilon^{\frac{-1}{2} \omega_n^T P_n^{-1} \omega_n} \, d\omega_1 \, d\omega_2 \, d\omega_3 \, d\omega_4 \]  

(4.67)
where

$$[p_n^{-1}] = \begin{bmatrix} 4 & 0 & -1 & -1 \\ 0 & 4 & 0 & 1 \\ -1 & 0 & 4 & 2 \\ -1 & 1 & 2 & 4 \end{bmatrix}$$  \hspace{1cm} (4.68)$$

$$|P_n| = \text{det. } P_n = 1/161$$  \hspace{1cm} (4.69)$$

Thus from Eq. (4.66) to Eq. (4.69) it can be shown that

$$I_4(\tau) = \frac{1}{(2\pi)^2} \frac{1}{\sqrt{161}} e^{-\tau^2/4}$$  \hspace{1cm} (4.70)$$

The integrals associated with each of the distinct paired arrays of Table 10 are tabulated in Table 11, and the autocorrelation function $R_{ww}(\tau)$ is plotted in Figure 24. Higher order moments of $w(t)$ can be similarly obtained and thus the probability density function of output $w(t)$ can be generated. This analysis, therefore, will be of considerable help in studying the transient behavior of traffic flow.
<table>
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<tr>
<th>Configuration</th>
<th>Paired Array</th>
<th>Form of the Integral</th>
</tr>
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<td>[ \frac{1}{(2\pi)^3} \int \int \int_{-\infty}^{\infty} \pi \phi_{xx}(j\omega_1)</td>
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TABLE II, contd.

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</table>
| ![Configuration 1](image1.png) | ![Paired Array 1](image2.png) | \[
\frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_{xx}(j\omega_1) H(j(\omega_1+\omega_2)) H(j(\omega_3-\omega_1)) H(j(\omega_4-\omega_2)) \\
\times H(-j(\omega_3+\omega_4)) e^{-j(\omega_2+\omega_3)\tau} d\omega_1 d\omega_2 d\omega_3 d\omega_4
\] |
| ![Configuration 2](image3.png) | ![Paired Array 2](image4.png) | \[
\frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_{xx}(j\omega_1) |H(j\omega_1)|^2 |H(j(\omega_2+\omega_3+\omega_4)|^2 d\omega_1 d\omega_2 d\omega_3 d\omega_4
\] |
| ![Configuration 3](image5.png) | ![Paired Array 3](image6.png) | \[
\frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_{xx}(j\omega_1) H(j\omega_1) H(j\omega_2) H(j(\omega_3-\omega_1)) H(-j(\omega_2+\omega_3+\omega_4)) \\
\times H(-j(\omega_1+\omega_2)) e^{-j(\omega_1+\omega_2)\tau} d\omega_1 d\omega_2 d\omega_3 d\omega_4
\] |
| ![Configuration 4](image7.png) | ![Paired Array 4](image8.png) | \[
\frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_{xx}(j\omega_1) H(j\omega_1) H(j(\omega_2-\omega_1)) H(j(\omega_3+\omega_4)) \\
\times H(-j(\omega_2+\omega_3+\omega_4)) e^{-j\omega_2\tau} d\omega_1 d\omega_2 d\omega_3 d\omega_4
\] |
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<th>Form of the Integral</th>
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<td>$H(-j(\omega_1+\omega_2)) \epsilon \omega_2^\tau$</td>
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<td>$\frac{1}{(2\pi)^5} \int_{-\infty}^{\infty} \pi \phi_{xx}(j\omega_1) H(j\omega_1) H(j(\omega_2+\omega_3-\omega_1)) H(j(\omega_4+\omega_5-\omega_2))$</td>
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<td>$H(-j(\omega_3+\omega_4+\omega_5)) \epsilon \omega_2^\tau$</td>
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### TABLE 11, contd.

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</table>
|               | ![Diagram](image1.png) | \[
\frac{1}{(2\pi)^5} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_{xx}(j\omega_i) H(j(\omega_1+\omega_2)) H(j(\omega_3-\omega_1)) H(j(\omega_4+\omega_5-\omega_2)) \\
\times e^{-j(\omega_2+\omega_3)\tau} H(-j(\omega_3+\omega_4+\omega_5)) e^{j(\omega_2+\omega_3+\omega_4+\omega_5)\tau} d\omega_i 
\]

|               | ![Diagram](image2.png) | \[
\frac{1}{(2\pi)^5} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_{xx}(j\omega_i) H(j(\omega_1+\omega_2)) H(j(\omega_3+\omega_4)) H(j(\omega_5-\omega_1-\omega_3)) \\
\times e^{-j(\omega_1+\omega_2+\omega_3+\omega_4+\omega_5)\tau} H(-j(\omega_1+\omega_2+\omega_3+\omega_4+\omega_5)) e^{j(\omega_1+\omega_2+\omega_3+\omega_4+\omega_5)\tau} d\omega_i 
\]

|               | ![Diagram](image3.png) | \[
\frac{1}{(2\pi)^6} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_{xx}(j\omega_i) |H(j(\omega_1+\omega_2+\omega_3)|^2 |H(j(\omega_4+\omega_5+\omega_6)|^2 d\omega_i 
\]

|               | ![Diagram](image4.png) | \[
\frac{1}{(2\pi)^6} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_{xx}(j\omega_i) H(j(\omega_1+\omega_2+\omega_3)) H(j(\omega_4-\omega_1-\omega_2)) H(j(\omega_5+\omega_6-\omega_3)) \\
\times e^{-j(\omega_3+\omega_4+\omega_5+\omega_6)\tau} H(-j(\omega_1+\omega_2+\omega_3+\omega_4+\omega_5+\omega_6)) e^{j(\omega_3+\omega_4+\omega_5+\omega_6)\tau} d\omega_i 
\]
### TABLE II, contd.

<table>
<thead>
<tr>
<th>Configuration</th>
<th>Paired Array</th>
<th>Form of the Integral</th>
</tr>
</thead>
</table>
|               | ![Diagram](image) | \[
\frac{1}{(2\pi)^6} \int_{-\infty}^{\infty} \prod_{i=1}^{6} \Phi_{XX}(j\omega) H(j(\omega_1+\omega_2+\omega_3)) H(j(\omega_4+\omega_5-\omega_1)) H(j(\omega_6-\omega_2-\omega_4))
\]

\[
-\frac{1}{(2\pi)^6} \int_{-\infty}^{\infty} H(-j(\omega_3+\omega_5+\omega_6)) e^{-i\omega_1 \cdot \omega_1}
\]
Fig. 24. Autocorrelation function of $w(t)$. 
CHAPTER V
CONCLUSIONS

This dissertation has been devoted to the study of some traffic intersection problems. In particular the freeway on-ramp situation is discussed via two queuing models: Single Service and Variable Bulk Service. The concept of imbedded Markov chain is used to obtain the statistics of the queue length on the finite capacity on-ramp. It is shown that for both the single service and the variable bulk service model, the mean queue length, $\bar{n}$, on the ramp is a convex function of the mean arrival rate, $\alpha$, on the freeway and thus has a unique minimum. This occurs when the 'gap' flow rate is the maximum. The effect of the 'rush-hour' is incorporated to obtain the extent of overflow from the finite capacity on-ramp. An on-ramp dynamic storage facility is suggested to accommodate this overflow and its size is determined in Section 2.5. To utilize this storage facility effectively and efficiently, a logical traffic control system is developed in Chapter III. This accomplishes the following: (1) divert the traffic wishing to join the ramp to the storage facility if the need arises; (2) allow the vehicles from the storage to join the ramp at proper instants.

Chapter IV is devoted to the study of transient behavior of the queues. In particular the objective has been centered on computation of $p(n)$, the absolute probability density function of the queue length. For this purpose a cascade of nonlinear zero-memory, linear time
invariant, and nonlinear zero-memory system has been used to simulate a
typical traffic flow distribution on a freeway (or a ramp). Graph
theory is applied in the analysis to obtain autocorrelation and other
moments of this distribution which can subsequently be used to generate
the desired probability density function. The computation of moments
requires the generation of a class of 'irregular graphs', and the count-
ing and classifying of them according to distinct perfect matchings of
the graph. Irregular graphs and their perfect matchings are discussed
in Section 4.4 and an algorithm has been developed to count and deter-
mine the distinct perfect matchings of such an irregular graph. This
analysis will be of considerable help in studying the transient behav-
ior of the queues and thus in evaluating the problems of traffic con-
gestion.
The detailed steps leading to the probability generating function $K(z)$ of the distribution $\{k_j\}$ are as follows:

\[
K(z) = \frac{N}{\sum_{j=0}^{\infty} [\sum_{i=0}^{x} (-a)^i \frac{e^{-aix(v-ix)i}}{i!} \ dv] z^j}
\]

\[
= \frac{\infty}{\sum_{j=0}^{\infty} (\sum_{i=0}^{j} (-a)^i \frac{e^{-aix(v-ix)i}}{i!}) \ dv}
\]

For large values of $N$,

\[
1 + \frac{y}{1!} + \frac{y^2}{2!} + \ldots + \frac{y^N}{N!} = (1 + \frac{1}{N})^N
\]

Therefore (A-1) becomes

\[
K(z) = \frac{\infty}{\sum_{j=0}^{\infty} (\sum_{i=0}^{j} (-a)^i \frac{e^{-aix(v-ix)i}}{i!}) \ dv}
\]

Since

\[
(1 + \frac{1}{N})^N = e
\]

\[
K(z) = \frac{\infty}{\sum_{j=0}^{\infty} (\sum_{i=0}^{j} (-a)^i \frac{e^{-aix(v-ix)i}}{i!}) \ dv}
\]

Let $\lambda(1 - zN \text{ log}_e(1 + \frac{1}{N})) = \beta$

\[
K(z) = \frac{\infty}{\sum_{j=0}^{\infty} [\sum_{i=0}^{x} (-a)^i \frac{e^{-aix(v-ix)i}}{i!} \ dv]
\]

\[
= \frac{\infty}{\sum_{j=0}^{\infty} [\sum_{i=0}^{x} (-a)^i \frac{e^{-aix(v-ix)i}}{i!} \ dv]
\]

\[
= \frac{\infty}{\sum_{j=0}^{\infty} [\sum_{i=0}^{x} (-a)^i \frac{e^{-aix(v-ix)i}}{i!} \ dv]
\]

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\[
\begin{align*}
\frac{2x}{\pi} - & \frac{\varepsilon}{\pi} \left[ \sum_{i=0}^{\infty} (-\alpha)^i \frac{e^{-\alpha i(x-ix)i}}{i!} \right] dv \\
\frac{3x}{\pi} - & \frac{\varepsilon}{\pi} \left[ \sum_{i=0}^{\infty} (-\alpha)^i \frac{e^{-\alpha i(x-ix)i}}{i!} \right] dv \\
\frac{(n+1)x}{\pi} - & \frac{\varepsilon}{\pi} \left[ \sum_{i=0}^{\infty} (-\alpha)^i \frac{e^{-\alpha i(x-ix)i}}{i!} \right] dv + \cdots \quad (A-5)
\end{align*}
\]

For \( nx \leq v \leq (n+1)x \), \( \left[ \frac{v}{x} \right] = n \). Thus

\[
K(z) = \frac{2x}{\pi} - \frac{\varepsilon}{\pi} \frac{e^{-\alpha nx}}{n!} dv \\
\frac{3x}{\pi} - \frac{\varepsilon}{\pi} \frac{e^{-\alpha nx}}{n!} dv \\
\frac{(n+1)x}{\pi} - \frac{\varepsilon}{\pi} \frac{e^{-\alpha nx}}{n!} dv + \cdots \quad (A-6)
\]

\[
= \sum_{n=0}^{\infty} \frac{\varepsilon^{-\alpha nx}}{n!} \left[ \sum_{i=0}^{\infty} (-\alpha)^i \frac{e^{-\alpha i(x-ix)i}}{i!} \right] dv \\
= \sum_{n=0}^{\infty} \frac{\varepsilon^{-\alpha nx}}{n!} \left[ \sum_{i=0}^{\infty} (-\alpha)^i \frac{e^{-\alpha i(x-ix)i}}{i!} \right] dv + \cdots \quad (A-7)
\]

Let \( v-nx = T \),

\[
K(z) = \sum_{n=0}^{\infty} \frac{\varepsilon^{-\alpha nx}}{n!} \left[ \sum_{i=0}^{\infty} (-\alpha)^i \frac{e^{-\alpha i(x-ix)i}}{i!} \right] dv \\
= \sum_{n=0}^{\infty} \frac{\varepsilon^{-\alpha nx}}{n!} \left[ \sum_{i=0}^{\infty} (-\alpha)^i \frac{e^{-\alpha i(x-ix)i}}{i!} \right] dv + \cdots \quad (A-8)
\]

Let \( A_n = \int_{0}^{\infty} e^{-\beta T} T^n dT \) \quad (A-9)
\[ A_n = \frac{\Gamma(n+1)}{\beta^{n+1}} = \frac{n!}{\beta^{n+1}} \]  \hspace{1cm} (A-10)

Thus \[ K(z) = \sum_{n=0}^{\infty} (-a)^n \frac{e^{-(a+\beta)nx}}{n!} \frac{n!}{\beta^{n+1}} \]

\[ = \frac{1}{\beta} \sum_{n=0}^{\infty} \left( \frac{-a e^{-\beta x}}{\beta} \right)^n \]

\[ = \frac{1}{\beta + a e^{-ax} e^{-\beta x}} \]  \hspace{1cm} (A-11)

Since \( \beta = \lambda(1 - zN \log_e(1 + \frac{1}{N})) \)

\[ K(z) = \left[ \lambda(1 - zN \log_e(1 + \frac{1}{N})) + a e^{-(a+\lambda)x} \lambda zN \log_e(1+\frac{1}{N}) \right]^{-1} \]  \hspace{1cm} (A-12)
The probability generating function $P(z)$ for the variable bulk service model is obtained as follows:

$$P(z) = \sum_{r=0}^{N} \left[ \sum_{i=0}^{r} k_{j}p_{i} + \sum_{i=r}^{j+r} k_{j-i+r}p_{i} \right] p(r)z^{j} \quad (B-1)$$

Substituting for $p(r)$ from Eq. (2.14) it becomes

$$P(z) = (e^{ax-1}) \sum_{r=1}^{\infty} e^{-rax} \left[ \sum_{i=0}^{r} k_{j}p_{i} + \sum_{i=r}^{j+r} k_{j-i+r}p_{i} \right]z^{j}$$

$$= (e^{ax-1}) \sum_{r=1}^{\infty} e^{-rax} \left[ \sum_{i=0}^{r} k_{j}p_{i} \right]z^{j}$$

$$+ (e^{ax-1}) \sum_{r=1}^{\infty} e^{-rax} \sum_{i=r}^{j+r} k_{j-i+r}p_{i}z^{j} \quad (B-2)$$

$$= (e^{ax-1}) \sum_{i=0}^{\infty} p_{i} \sum_{r=i+1}^{\infty} e^{-rax} \left[ \sum_{j=0}^{\infty} k_{j}z^{j} \right]$$

$$+ (e^{ax-1}) \sum_{r=1}^{\infty} e^{-rax} \sum_{i=r}^{j+r} k_{j-i+r}p_{i}z^{j} \quad (B-3)$$

Since \( \sum_{r=i+1}^{\infty} e^{-rax} = \frac{e^{-i\alpha x}}{e^{ax} - 1} \quad (B-4) \)

Eq. (B-3) becomes

$$P(z) = (e^{-i\alpha x})k(z) + (e^{ax-1}) \sum_{r=1}^{\infty} e^{-rax} \sum_{i=r}^{j+r} k_{j-i+r}p_{i}z^{j} \quad (B-5)$$
Let
\[ P(z) = A(z) + B(z) \quad \text{(B-6)} \]
where
\[ A(z) = (\sum_{i=0}^{\infty} p_i e^{-i\alpha x}) K(z) \quad \text{(B-7)} \]
and
\[ B(z) = (e^{\alpha x-1}) \sum_{r=1}^{\infty} e^{-\alpha x} \sum_{j=0}^{N} \sum_{i=0}^{r} k_{j-i+r} p_i z^j \]
\[ = (e^{\alpha x-1}) \sum_{r=1}^{\infty} e^{-\alpha x} N+r \sum_{i=0}^{N} p_i \sum_{j=1-r}^{N} k_{j-i+r} z^j \quad \text{(B-8)} \]

Let \( j - i + r = \ell \),
\[ B(z) = (e^{\alpha x-1}) \sum_{r=1}^{\infty} e^{-\alpha x} N+r \sum_{i=0}^{N} k_{\ell} z^{\ell+i-r} \]
\[ = (e^{\alpha x-1}) \sum_{r=1}^{\infty} e^{-\alpha x} \sum_{i=0}^{N} p_i \sum_{\ell=0}^{\infty} k_{\ell} z^{\ell} \]
\[ = (e^{\alpha x-1}) \sum_{r=1}^{\infty} e^{-\alpha x} z^{-r} \sum_{i=0}^{N} \sum_{\ell=0}^{N-i+r} k_{\ell} z^\ell \]
\[ = (e^{\alpha x-1}) \sum_{r=1}^{\infty} e^{-\alpha x} z^{-r} (\sum_{i=0}^{N} p_i z^i - \sum_{i=0}^{r-1} p_i z^i) \times \]
\[ (K(z) - \sum_{\ell=N-i+r+1}^{N} k_{\ell} z^{\ell}) \]
\[ = (e^{\alpha x-1}) \sum_{r=1}^{\infty} e^{-\alpha x} z^{-r} ((P(z) - \sum_{i=0}^{r-1} p_i z^i) K(z)) \]
\[ - (e^{\alpha x-1}) \sum_{r=1}^{\infty} e^{-\alpha x} z^{-r} \sum_{i=0}^{N} p_i z^i \sum_{\ell=N-i+r+1}^{N} k_{\ell} z^\ell \]
\[(\text{01-2})\]

\[x_{\text{wx}} x_{\text{wx}} \in \mathcal{L}(T^{-\infty}) - x_{\text{wx}} x_{\text{wx}} \frac{0^{\infty}}{z \in \mathcal{L}(T^{-\infty})} - \frac{T^{-\infty}}{z \in \mathcal{L}(T^{-\infty})} \]

\[(\text{6-8})\]

\[x_{\text{wx}} x_{\text{wx}} \in \mathcal{L}(T^{-\infty}) - x_{\text{wx}} x_{\text{wx}} \frac{0^{\infty}}{z \in \mathcal{L}(T^{-\infty})} - \frac{T^{-\infty}}{z \in \mathcal{L}(T^{-\infty})} \]

\[(\text{88})\]

\[x_{\text{wx}} x_{\text{wx}} \in \mathcal{L}(T^{-\infty}) - x_{\text{wx}} x_{\text{wx}} \frac{0^{\infty}}{z \in \mathcal{L}(T^{-\infty})} - \frac{T^{-\infty}}{z \in \mathcal{L}(T^{-\infty})} \]
APPENDIX C

The probability, \( p_n \), that there are \( n \) vehicles on the ramp for variable bulk service model is the coefficient of \( z^n \) in \( P(z) \). The detailed steps leading to \( p_n \) (Eq. 2.66) are as follows:

\[
p_n = (-1)^n \mu^{-n} p(e^{-\alpha x}) \sum_{m=0}^{n} \frac{(\mu c)^m}{m!} \sum_{n-m \leq k < \infty} q^k \binom{n-m}{q} q^m
\]

where \( p = -d\mu \). Let

\[
g(p) = \sum_{n-m \leq q < \infty} q^m \binom{n-m}{q} p^q
\]

\[
e = \sum_{n-m \leq q < \infty} q^m q(q-1)(q-2) \ldots (q+1-(n-m)) p^q
\]

Let \( q - (n-m) = k \)

\[
g(p) = \sum_{0 \leq k < \infty} (k+1)(k+2) \ldots (k+(n-m)) \binom{n-m}{k} p^k
\]

or

\[
\int g(p) \frac{dp}{p^{n-m}} = \sum_{0 \leq k < \infty} (k+2)(k+3) \ldots (k+n-m) \binom{n-m}{k+1} p^{k+1} + c
\]

Integrating it \( (n-m-1) \) times,

\[
\int \ldots \int g(p) \frac{dp}{p^{n-m}} = \sum_{0 \leq k < \infty} (k+n-m)^{k+n-m} + \ldots
\]

Denote \((k+n-m)\) by \( S \), Eq. (C-5) becomes

\[
\int \ldots \int g(p) \frac{dp}{p^{n-m}} = \sum_{n-m \leq S < \infty} S^n p^S + \ldots
\]
or
\[
\int \frac{1}{p} \int \frac{1}{p} \cdots \int \frac{1}{p} \int \frac{1}{p} \cdots \int \frac{g(p)}{p^{n-m}} (dp)^n = \sum_{n-m \leq s \leq m} \frac{1}{p^s} + \cdots
\]
\[
= \frac{p^{n-m}}{1-p} + \cdots = \frac{1}{1-p} - (1 + p + p^2 + \cdots + p^{n-m-1}) + \cdots \quad (C-7)
\]

Only the first term on the right-hand side of Eq. (C-7) will determine \(g(p)\) since the other terms will be reduced to zero after indicated differentiations. Thus, consider

\[
\int \frac{1}{p} \int \frac{1}{p} \cdots \int \frac{1}{p} \int \frac{1}{p} \cdots \int \frac{g(p)}{p^{n-m}} (dp)^n = \frac{1}{1-p} \quad (C-8)
\]

Let
\[
\int \frac{1}{p} \int \frac{1}{p} \cdots \int \frac{1}{p} G(p)(dp)^m = \frac{1}{1-p} \quad (C-9)
\]

where
\[
G(p) = \int \int \cdots \int \frac{g(p)}{p^{n-m}} (dp)^{n-m} \quad (C-10)
\]

Define
\[
H_0 = G(p) \quad (C-11)
\]

then
\[
H_{m-k} = \int \frac{1}{p} \int \frac{1}{p} \cdots \int \frac{1}{p} G(p)(dp)^{m-k} = \int \frac{1}{p} H_{m-k-1} \ dp \quad (C-12)
\]

\[
H_{m-k-1} = p \frac{d}{dp} H_{m-k} \quad (C-13)
\]

\[
H_m = \frac{1}{1-p} = \frac{a_{1,0}}{1-p} \quad (C-14)
\]
From (C-13) and (C-14), one can obtain

\[ H_{m-1} = p \frac{d}{dp} H_m = \frac{a_{1,0} p}{(1-p)^2} = -\frac{a_{1,0}}{1-p} + \frac{a_{1,0}}{(1-p)^2} \]

\[ = \frac{a_{1,1}}{1-p} + \frac{a_{2,1}}{(1-p)^2} \]

(C-15)

By induction suppose

\[ H_{m-k} = \frac{a_{1,k}}{1-p} + \frac{a_{2,k}}{(1-p)^2} + \ldots + \frac{a_{k+1,k}}{(1-p)^{k+1}} \]  

(C-16)

Then

\[ H_{m-k-1} = p \frac{d}{dp} H_{m-k} \]

\[ = \frac{a_{1,k} p}{(1-p)^2} + \frac{2a_{2,k} p}{(1-p)^3} + \ldots + \frac{(k+1)a_{k+1,k} p}{(1-p)^{k+2}} \]

\[ = -\frac{a_{1,k}}{1-p} + \frac{(a_{1,k} - 2a_{2,k})}{(1-p)^2} + \frac{(2a_{2,k} - 3a_{3,k})}{(1-p)^3} + \ldots \]

\[ + \frac{(ka_{k,k} - (k+1)a_{k+1,k})}{(1-p)^{k+1}} + \ldots \]

\[ + \frac{(ka_{k,k} - (k+1)a_{k+1,k})}{(1-p)^{k+1}} + \frac{(k+1)a_{k+1,k}}{(1-p)^{k+2}} \]

(C-17)

\[ = \frac{a_{1,k+1}}{1-p} + \frac{a_{2,k+1}}{(1-p)^2} + \ldots + \frac{a_{k+2,k+1}}{(1-p)^{k+2}} \]

(C-18)
From (C-17) and (C-18),
\[ a_{1,k+1} = - a_{1,k} \]
\[ a_{2,k+1} = a_{1,k} - 2a_{2,k} \]
\[ \ldots \]
\[ a_{k,k+1} = (k-1)a_{k-1,k} - a_{k,k} \]
\[ \ldots \]
\[ a_{k+2,k+1} = (k+1)a_{k+1,k} \]
\[ \text{(C-19)} \]
where
\[ a_{1,0} = 1 \]
\[ a_{k,0} = 0 \quad k > 2 \]
\[ \text{(C-20)} \]
and \[ a_{1,m} = (-1)^m \]

It is required to compute
\[ H_0 = \sum_{k=1}^{m+1} \frac{a_{2,m}}{(1-p)^k} \]
\[ \text{(C-21)} \]
From (C-19) it is obvious that
\[ a_{k,m} = (k-1)a_{k-1,m-1} - a_{k,m-1} \]
\[ \text{(C-22)} \]
Replacing \( m \) by \( m-1, m-2, \ldots \), in (C-22) one can obtain
\[ a_{k,m-1} = (k-1)a_{k-1,m-2} - a_{k,m-2} \]
\[ \text{(C-23)} \]
\[ a_{k,m-2} = (k-1)a_{k-1,m-3} - a_{k,m-3} \]
\[ \ldots \]
\[ a_{k,1} = (k-1)a_{k-1,0} - a_{k,0} \]
\[ \text{(C-24)} \]
\[ \text{(C-25)} \]
Multiplying (C-22), (C-23), (C-24), and (C-25) by \( 1, (-k), (-k)^2 \), and \( (-k)^m \) respectively, and adding, it yields:
\[ a_{\ell,m} = (\ell - 1)[a_{\ell-1,m-1} + (-\ell)a_{\ell-1,m-2} + (-\ell)^2a_{\ell-1,m-3} + \ldots + (-\ell)^{m-1}a_{\ell-1,0}] \]  

(C-26)

Since \( a_{\ell,0} = 0 \) for \( \ell \geq 2 \), it can now be proved by induction that

\[ a_{\ell,m} = (-1)^{m-\ell+1} \left\{ \ell^m - \binom{\ell-1}{1} (\ell-1)^m + \binom{\ell-1}{2} (\ell-2)^m + \ldots + (-1)^{\ell-1} \binom{\ell-1}{\ell-1} 1^m \right\} \]

(C-27)

**Proof:** Since \( a_{1,m} = (-1)^m \), suppose it is true for \( a_{\ell-1,m} \) for \( \ell \geq 2 \) and every \( m \). From (C-26),

\[ a_{\ell,m} = (\ell - 1) \sum_{j=0}^{m-1} (-\ell)^{m-1-j} a_{\ell-1,j} \]

Substituting for \( a_{\ell-1,j} \) for \( j = 0,1, (m-1) \) from the induction hypothesis yields:

\[ a_{\ell,m} = (\ell - 1) \sum_{j=0}^{m-1} (-\ell)^{m-1-j} (-1)^j \ell^2 \sum_{i=0}^{\ell-2} (-1)^i \binom{\ell-2}{i} (\ell-i-1)^j \]

\[ = (-1)^{m-\ell+1}(\ell - 1) \sum_{i=0}^{\ell-2} (-1)^i \binom{\ell-2}{i} m-1-j(\ell-1-i)^j \]

Since

\[ (\ell - 1) \binom{\ell-2}{i} = \binom{\ell-1}{i} (\ell-i-1) \]

\[ a_{\ell,m} = (-1)^{m-\ell+1} \sum_{i=0}^{\ell-2} (-1)^i \binom{\ell-1}{i+1} (\ell^m - (\ell-i-1)^m) \]
Thus the induction proof of (C-27) is complete.

Recall (C-10) and (C-11),
\[
H_0 = G(p) = \int \cdots \int g(p) (dp)^{n-m}
\]
\[
H_0 = \frac{m+1}{l+1} \sum_{l=1}^{m+1} \frac{a_{l,m}}{(1-p)^{l+1}}
\]

Differentiating \( H_0 \) \((n-m)\) times, one can obtain,
\[
g(p) = \frac{m+1}{l+1} \sum_{l=1}^{m+1} \frac{a_{l,m}}{(1-p)^{l+1}} \frac{(l+n-m-1)!}{(l-1)!} \frac{1}{(1-p)^{l+n-m}}
\]

Thus
\[
g(p) = \left( \frac{p}{1-p} \right)^{n-m} \frac{(p^{m+1} a_{l,m}) (l+n-m-1)!}{l+1 (l-1)! (1-p)^{l+n-m}}
\]

and hence substituting (C-2) and (C-29) in (C-1), it yields:
\[
P_n = P(e^{-\alpha x})(-1)^n \mu^{-n} \sum_{m=0}^{n} \frac{(\nu c)^m}{m!} \left( \frac{p}{1-p} \right)^{n-m} \left( \frac{m+1}{l+1} \sum_{l=1}^{m+1} \frac{a_{l,m}}{(1-p)^{l+1}} \right) \frac{(l+n-m-1)!}{(l-1)!} \frac{1}{(1-p)^{l+n-m}}
\]
APPENDIX D

Let $G = (V,E)$ be a graph where $V = (v_1, v_2, \ldots, v_t)$ and $\sum_{i=1}^{t} v_i = 2n$, $n = 1, 2, 3, \ldots$. Let $P$ represent the distinct partitions of $(2n - 2v_1)$ into $(t-1)$ columns $(v_2', v_3', \ldots, v_t')$ such that

$$\sum_{i=2}^{t} v_i' = (2n - 2v_1)$$

and

$$\max(0, (v_i - v_1)) \leq v_i' \leq v_i$$

Without loss of generality one can write $V = (v_1, v_2, \ldots, v_t)$ such that $v_1 \leq v_2 \leq v_3 \ldots \leq v_t$. To determine $P$, one needs to find all the distinct ways of mapping $v_j$, $j = 1, 2, \ldots, t-1$, into the rest of the columns $v_k$, $k = j+1, j+2, \ldots, t$, of the graph. The following steps, then, determine the distinct partitions $P$.

1. Set $j$ and exhaust $k$. Let $v_k$ represent the last $v_j$ mapped. If two columns have the same length, consider any one of them for mapping.

2. Increment $j$ until $v_j > v_k$ and repeat step (1).

3. Split $v_j$ into its possible distinct combinations and map each onto the rest of the $v_k$ columns.

The above procedure is explained by an example. Let $V = (2233)$. To determine $P$ the following steps are necessary.

(a) Map 2 of the first column onto 2 of the second column which gives $P_1 = (033)$. 

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(b) Map 2 of the first column to 2 of the third column. This gives $P_2 = (213)$ since mapping 2 of the first column to 2 of the last column will yield the same as $P_2$. This should be omitted.

(c) Split 2 of the first column into $(1,1)$ and map this onto the second and third columns. It yields $P_3 = (123)$. Mapping $(1,1)$ onto the second and fourth columns gives the same as $P_3$ so it should be omitted.

(d) Map $(1,1)$ onto the third and fourth columns giving $P_4 = (222)$. Thus the partitioning of $(2233)$ is

$$P = \{P_r\}, \quad r = 1,2,3,4.$$
REFERENCES


