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THE APPROXIMATE ANALYSIS OF THIN SHELLS
BY THE FINITE ELEMENT METHOD

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

by

Cornelis Visser, M.Sc.

* * * * * *

The Ohio State University
1966

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INTRODUCTION

During the last two decades structural engineers were faced with the design and analysis of highly complex structures. This was particularly true in the aircraft industry, where structural engineers were challenged by the problem of determining stresses, deformations and vibration frequencies of low aspect ratio wings, complex airframe configurations, etc. The elementary methods of analysis, which had been in use until then, were inadequate for the analysis of these structures.

One of the methods, which were developed by aeronautical structural engineers to handle the analysis of these complex structures, is the finite element method. A deciding factor in the development of the method was the availability of large automatic digital computers in the aircraft industry. A large electronic computer is necessary to perform the vast amount of routine computations which arise in the application of the finite element method.

The finite element method can easily be formulated in matrix notation, which considerably simplifies the task of programming the method.

In the last few years the finite element method has been extended to problems in other fields, such as civil
engineering, two- and three-dimensional elasticity problems, plate bending problems and shells of revolution.

In this study the method is extended to thin shells of arbitrary curvature. Since it is desirable that the method can be used for shells with irregular boundaries, cutouts, etc., a curved triangular shell element is developed. The development of the triangular shell element is achieved through a suitable definition of the boundary curves of the element on the middle surface of the shell.

Three displacement functions are needed; namely, two membrane displacement functions and a normal displacement function. Displacement functions are chosen that are continuous across adjacent element boundaries and are suitable for use with the Ritz method.

The convergence of the approximate solutions obtained with the above element and displacement functions is shown from the point of view of the Ritz method. It is demonstrated that the assumed displacement functions are relatively complete which is a sufficient condition for the convergence of the potential energy of the assemblage of elements to that of the exact solution of the problem. The proof of convergence is extended to the displacements themselves.

The expressions for the stiffness matrix for the triangular curved shell element are derived as well as the corresponding load vector and the consistent mass matrix.
The mathematical conditions of relative completeness are applied to the generalized displacement functions of several elements which are used in plane elasticity and plate bending problems. It is found that these mathematical conditions are equivalent to definite physical conditions.
CHAPTER I

THE FINITE ELEMENT DISPLACEMENT METHOD

The chapter begins by giving a brief introduction to the finite element method as it was originally developed. The method is then demonstrated on a very simple example. The Ritz method is reviewed and the connection which exists between it and the finite element displacement method is discussed. The convergence of the displacement method is next approached from the point of view of the Ritz method and is demonstrated on the simple example. There is a brief review of the literature on the displacement method. The advantages and disadvantages of the method are stated and finally the scope of the study is outlined.

1.1 The finite element method

In engineering analysis the real physical system is replaced by a mathematical model. This model is an idealization which is obtained from the real system by making certain simplifying assumptions. These assumptions are of a physical nature and do not involve mathematical approximations. For example, in the usual structural analysis of a truss the real structure is replaced by an assemblage of one-dimensional structural members, interconnected by pins at a number of
joints. To this model the methods of structural analysis are applied to obtain the forces in the members caused by the applied loads. The results are then assumed to be applicable not only to the mathematical model but also to the more complex physical system. No mathematical approximation is required in the structural analysis.

The finite element method, as originally developed in the aircraft industry, is very similar to the methods used in the structural analysis of trusses and rigid frames. The real structure or continuum is replaced by an assemblage of a finite number of individual elements. This concept makes it possible to apply the finite element method to a great variety of problems such as two- or three-dimensional elasticity problems, plates, shells, etc. The elements can have a variety of shapes to suit the particular problem. The elements are connected at a finite number of joints, called the nodes. The nature of these connections depends on the type of problem. For example, if bending stresses are involved, as in the case of plate bending problems, these connections should be able to transmit moments as well as forces. The assemblage of elements, by which the continuum is replaced, is the mathematical model to which the techniques of structural analysis are applied. In general these techniques, whether they are applied to a truss, a framed structure, or a model representing a continuum, require that the stiffness
properties of the individual structural elements be known. The stiffness coefficients representing these properties define the relationship between the forces or moments and the corresponding displacements at the nodes. In the case of a simple one-dimensional truss member these stiffness coefficients can be easily and directly obtained by the use of Hooke's law. The stiffness coefficients for more complex elements such as the triangular curved shell element can only be determined by approximate methods. The reasons for this are stated below.

The two basic methods of structural analysis which are used for trusses and rigid frame structures and which can also be used with the finite element method are these:

1) The force method, in which the nodal forces are taken as the unknowns, and

2) The displacement method in which the nodal displacements are taken as the unknowns.

It has been found (ref. 10) that in general the displacement method results in a simpler formulation for complex structures. In this study we will confine ourselves to the displacement method.

With the displacement method the stiffness coefficients of an element are determined by assuming displacement functions which are continuous on the element and depend linearly
on the displacements of the nodes. Castigliano's first theorem can be used to compute the stiffness coefficients.

At this stage it may be instructive to illustrate the displacement method on a very simple problem. Consider a one-dimensional member with crosssectional area \( A \) and of length \( l \), loaded by a force \( F \) at one end and fixed at the other end (see Figure 1). The mathematical model on which the analysis is to be performed consists of two elements, each of length \( l/2 \) (see Figure 2). The analysis can be performed in a number of steps:

![Fig. 1.---One-dimensional member.](image)

![Fig. 2.---Mathematical model of the one-dimensional bar.](image)
1) We consider a typical element with end points 1 and j, which have coordinates $x_1$ and $x_j$. We assume that the displacements are given by:

$$\hat{u} = a_0 + a_1 x^2 \tag{1-1}$$

where $a_0$ and $a_1$ are unknown coefficients. This displacement function will not yield the exact solution since the corresponding strain is not constant. It has been chosen to illustrate the method.

2) The coefficients $a_0$ and $a_1$ can be expressed in terms of nodal displacements $\hat{u}_1$ and $\hat{u}_j$. To accomplish this we compute the nodal displacements from equation (1-1) and obtain

$$\hat{u}(x_1) = \hat{u}_1 = a_0 + a_1 x_1^2 \tag{1-2}$$

$$\hat{u}(x_j) = \hat{u}_j = a_0 + a_1 x_j^2$$

Upon solving for $a_0$ and $a_1$, we find

$$a_0 = \frac{x_1^2 \hat{u}_j - x_j^2 \hat{u}_1}{x_1^2 - x_j^2} \tag{1-3}$$

$$a_1 = \frac{\hat{u}_1 - \hat{u}_j}{x_1^2 - x_j^2}$$

3) Equation (1-1) can now be put in the form
\[ \hat{u} = \frac{x_i^2 - x_j^2}{2} \hat{u}_i + \frac{x_j^2 - x_i^2}{2} \hat{u}_j \]  

(1-4)

giving the displacement function as a linear function of the nodal displacements.

4) The strain energy stored in the element is given by

\[ U_s = \frac{1}{2} \int_{x_i}^{x_j} \bar{\sigma} \varepsilon \, dx = \frac{E}{2} \int_{x_i}^{x_j} \varepsilon^2 \, dx \]  

(1-5)

where \( \bar{\sigma} \) is the stress and \( \varepsilon \) the corresponding strain at a typical point of the element and \( E \) the modulus of elasticity of the material of the bar.

5) The nodal forces can be obtained by Castigliano's first theorem which in this case gives for the typical element

\[ F_i = \frac{\partial U_s}{\partial \hat{u}_i} \]  

(1-6)

\[ F_j = \frac{\partial U_s}{\partial \hat{u}_j} \]

where \( F_i \) is the force applied at the \( i^{th} \) node in the direction of the displacement \( \hat{u}_i \) and similar for \( F_j \). The forces \( F_i \) and \( F_j \) can be considered generalized faces conjugate to the generalized displacements \( \hat{u}_i \) and \( \hat{u}_j \). Note that since \( \hat{u} \) is linear in \( \hat{u}_i \) and \( \hat{u}_j \) the function \( U_s \) is quadratic.
in \( \hat{u}_1 \) and \( \hat{u}_j \). The equations (1-6) will thus become linear algebraic equations in the \( \hat{u}_1 \) and \( \hat{u}_j \).

In the case of the illustrative problem we obtain for element 1

\[
\varepsilon^{(1)} = \frac{d\hat{u}}{dx} = \frac{2x}{x_2^2} \hat{u}_2
\]  

(1-7)

\[
U_s^{(1)} = \frac{A\varepsilon^{(1)}}{2} \int_0^{b_2} (\varepsilon^{(1)})^2 \, dx = \frac{4}{3} \frac{EA}{\ell} \hat{u}_2^2
\]  

(1-8)

since the first node is fixed and thus \( \hat{u}_1 = 0 \). Thus,

\[
F_2^{(1)} = \frac{\partial U_s^{(1)}}{\partial \hat{u}_2} = \frac{8}{3} \frac{EA}{\ell} \hat{u}_2.
\]  

(1-9)

The stiffness coefficient for the first element is \( \frac{8}{3} \frac{EA}{\ell} \).

Similarly we find for the second element

\[
U_s^{(2)} = \frac{28}{27} \frac{EA}{\ell} (\hat{u}_2 - \hat{u}_3)^2
\]  

(1-10)

\[
F_2^{(2)} = \frac{\partial U_s^{(2)}}{\partial \hat{u}_2} = \frac{56}{27} \frac{EA}{\ell} (\hat{u}_2 - \hat{u}_3).
\]  

(1-11)

\[
F_3^{(2)} = \frac{\partial U_s^{(2)}}{\partial \hat{u}_3} = \frac{56}{27} \frac{EA}{\ell} (\hat{u}_2 - \hat{u}_3).
\]  

(1-12)

In this case the stiffness coefficient is \( \frac{56}{27} \frac{EA}{\ell} \).

6) We can now solve the structural system consisting of the two elements by one of the standard methods of structural analysis. Using the method of joints we obtain
\[ \sum F_2 = 0 = \frac{8}{3} \frac{EA}{l} \hat{u}_2 + \frac{56}{27} \frac{EA}{l} (\hat{u}_2 - \hat{u}_3). \quad (1-13) \]

\[ \sum F_3 = F = -\frac{56}{27} \frac{EA}{l} (\hat{u}_2 - \hat{u}_3). \quad (1-14) \]

Upon solving the above algebraic equations we find

\[ \hat{u}_2 = \frac{3}{6} \frac{F \ell}{EA} \quad (1-15) \]

\[ \hat{u}_3 = \frac{6}{7} \frac{F \ell}{EA}. \quad (1-16) \]

The above example is very simple and the exact nodal forces could easily have been obtained. For more complex problems this is not the case.

Consider for example a shell problem. The shell can, somehow, be divided into triangular elements interconnected at the node points. To determine the exact stiffness coefficients, i.e., the exact nodal forces due to a virtual unit displacement of one of the nodal points while the others are kept fixed in place, is a very difficult problem. In addition, the use of the exact stiffness coefficients obtained in this manner would result in a very crude approximation of the real behavior of the shell. The elements would deform independently of each other, except at the nodes; gaps and overlaps would occur along their common boundaries and high stress concentrations would develop at the nodal points due to the nodal forces. Such gaps, overlaps, and stress concentrations would not exist in the real shell structure and the
stresses and displacements of the idealization would be entirely different from those occurring in the real structure. In order to obtain a reasonable approximation of the actual behavior of the shell, the assemblage of triangular shell elements should deform in a similar manner as the real structure. The deformations should be continuous within the elements and across element boundaries, but their derivatives may be discontinuous at the element boundaries. With the use of such displacements the finite element method is actually a special form of the Ritz method. In the next sections we discuss the Ritz method and illustrate its use on the simple example of this section.

1.2 The Ritz method

The Ritz method is one of the direct methods of the calculus of variations. It is an approximate method for the solution of extremum problems.

The problems of elasticity can be expressed mathematically in terms of certain boundary value problems, involving differential equations, as well as in terms of certain integrals, which are related to the work performed on the elastic body in the process of deformation. These two mathematical representations are completely equivalent. In fact, the differential equations of the boundary value problem are in general identical with the Euler-Lagrange equations of the corresponding extremum problem of the calculus of variations.
Mathematically the problem of solving a boundary value problem is equivalent to solving the corresponding extremum problem.

1.2.1 The Ritz method as an approximate mathematical method

In order to gain insight into the Ritz method we review here the general problem of the minimum of the integral of the one-dimensional problem (ref. 1)

\[ I(u) = \int_{x_0}^{x_1} F(x,u,u')dx \]  

(1-17)

under the conditions

\[ u(x_0) = u_0 \]

\[ u(x_1) = u_1 \]  

(1-18)

where \( F(x,u,u') \) is a continuous function.

We suppose that \( u^*(x) \) is the exact solution to this problem and that \( I(u^*) = m \) is the minimum value of the functional.\(^1\)

The Ritz method treats the minimal problem directly by inserting a trial family of functions with undetermined parameters in the expression of the functional. Let this family of functions be given by

\(^1\)The term functional is used in the calculus of variations to describe functions defined by integrals whose arguments themselves are functions.
\[ u_n = \varphi_n(x, a_1, a_2, \ldots, a_n). \quad (1-19) \]

\((n = 1, 2, \ldots)\)

where the \(a\)'s are undetermined parameters and each family \(u_k\) includes in it families with subscripts less than \(k\).

Furthermore let the functions \(u_n\) and their derivatives satisfy the conditions of equation (1-18) and be piecewise continuous on the interval \((x_0, x_1)\). The function \(F(x, u_n, u'_n)\) is then at least piecewise continuous on \((x_0, x_1)\) and is integrable over this interval since it has at most a finite number of discontinuities. If we substitute equation (1-19) into equation (1-17) we obtain a function in the \(n\) variables \(a_1, a_2, \ldots, a_n\); namely, \(I = I(a_1, a_2, \ldots, a_n)\). The minimum value of this function can be found by satisfying the system of equations

\[
\frac{\partial I}{\partial a_k} = 0, (k = 1, 2, \ldots, n). \quad (1-20)
\]

This system of equations can be solved to determine the values of the parameters \(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n\). The required approximate solution to the problem is then given by

\[ \bar{u}_n = \varphi_n(x, \bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n). \quad (1-21) \]

Since each successive family contains all the functions of the preceding, it is clear that the successive minima are
non-increasing; i.e.,

\[ I(\overline{u}_1) \geq I(\overline{u}_2) \geq \ldots. \]

A theorem from calculus states that since the non-increasing infinite sequence \( \{I(\overline{u}_n)\} \) is bounded below it is convergent. However, this does not guarantee that it will converge to the minimum \( I(u^*) = m \). In order to ensure convergence of the sequence \( \{I(\overline{u}_n)\} \) to \( I(u^*) \) we must restrict the family of functions of equation (1-19) to those functions \( u_n(x) \) of equation (1-19) which are relatively complete of order one. The set of functions \( u_n(x) \) is relatively complete of order one if for every function \( u(x) \) which satisfies the conditions of equation (1-18) and belongs to class \( C_1 \) and for every \( \xi > 0 \) there exists an \( n \) and a function of the \( n \)th family

\[ u^*_n(x) = \phi_n(x, a_1^*, a_2^*, \ldots, a_n^*). \]

such that

\[ \left| u^*_n - u \right| < \xi \quad (1-21) \]

\[ \left| \frac{du^*_n}{dx} - \frac{du}{dx} \right| < \xi \quad (1-22) \]

\[ \overline{u}_n \]

The order of relative completeness of the function \( u_n \) must be the same as the order of the highest derivative occurring in the functional.

\( C_k \) is the class of all functions which are continuous together with their \( k \)th partial derivatives. In the following sections functions used in connection with the Ritz method are always assumed to be of class \( C_1 \).
for all \( x \) in the interval \( (x_1, x_j) \) where \( \frac{du_n^*}{dx} \) exist. This condition is sufficient for the sequence to the minimal. To show this, we apply it to the exact solution of the problem \( u^* \). The condition of relative completeness now ensures that there exists a function \( u_n^*(x) \) from a certain \( n^{th} \) family

\[
u_n^*(x) = \phi_n(x, a_1^*, a_2^*, ..., a_n^*).
\]

such that

\[
\left| u_n^* - u^* \right| < \varepsilon
\]

\[
\left| \frac{du_n^*}{dx} - \frac{du^*}{dx} \right| < \varepsilon
\]

As a consequence of the continuity of \( F \) we have that the difference

\[
\left| F(x,u_n^*, \frac{du_n^*}{dx}) - F(x, u^*, \frac{du^*}{dx}) \right|
\]

will be arbitrarily small in the interval \( (x_1, x_j) \) and is continuous except perhaps at a finite number of discreet points.

The difference of the integrals of these functions will also be arbitrarily small; i.e.,

\[
\left| I(u_n^*) - I(u^*) \right| < \varepsilon'
\]

where
The function $u^*_n$ is one of the functions of the $n^{th}$ family (1-23). The function $\tilde{u}_n$ gives the integral its least value in comparison with all the functions of this family. Therefore, we must have $I(\tilde{u}_n) \leq I(u^*_n)$ or $I(u^*_n) \leq I(\tilde{u}_n) \leq I(u^*_n) \leq I(u^*) + \epsilon'$. Since $\epsilon'$ is arbitrarily small we have that $\lim I(\tilde{u}_n) = I(u^*_n) = m$. Note that the convergence of $I(u_n)$ to $I(u^*_n)$ does not imply that $u_n$ will converge to be $u^*_n$.

In a later chapter we shall apply the same ideas as outlined above to obtain an approximate solution to the shell problem. The integrals involved there are surface integrals and the requirement of relative completeness includes certain partial derivatives. Furthermore, the proof of convergence will be extended to the function $u_n$ itself.

1.3.2. The Ritz method applied to the problems of the theory of elasticity

In the introduction of section 1.3 we have stated that the problems of elasticity can be expressed mathematically in terms of certain extremum problems. Several theorems govern the extremum problems of the small displacement theory of elasticity. One of these theorems is important for the development of the theory in the following sections.

Theorem of minimum potential energy (ref. 6). Of all displacements satisfying the given geometric boundary conditions those which satisfy the equilibrium equations make the potential energy an absolute minimum.
This theorem is only valid for strains which are small in comparison with unity and for small displacements; i.e., displacements which do not affect the action of the applied loads. The potential energy of a body $V$, which is subjected to certain body and surface forces is defined by the formula

$$
\Pi = U_s - \int_S [T_x u + T_y v + T_z w] \, ds \\
- \int_V [F_x u + F_y v + F_z w] \, dV \quad (1-24)
$$

where $U_s$ is the strain energy; $T_x$, $T_y$ and $T_z$ are the components of the prescribed surface forces; $S_T$ is the part of the boundary surface $S$ where the surface forces are prescribed; $u$, $v$, and $w$ are the components of admissible displacements satisfying the compatibility equations and the displacement boundary conditions; and $F_x$, $F_y$ and $F_z$ are the components of the body forces acting on $V$.

The strain energy $U_s$, the surface forces represented by $T_x$, $T_y$ and $T_z$ and the body forces represented by $F_x$, $F_y$ and $F_z$ are all assumed to be conservative.

The Ritz method can be applied to the problems of the theory of elasticity, plates and shells by assuming a piece-wise continuously differentiable family of displacement functions with undetermined parameters, substituting these into the expression for the potential energy equation (1-24) and
minimizing the resulting expression with respect to the undetermined parameters. If the assumed displacement functions are linear in the parameters the resulting equations will be linear algebraic equations with the parameters as the unknowns. The displacement functions for one-dimensional problems are of the type shown in equation (1-19) while for two- or three-dimensional problems they are similar functions of two or three variables. It can be shown (ref. 2) that these families of functions must satisfy the geometric boundary conditions of the problem but need not satisfy the natural or additional boundary conditions.\(^4\)

The maximum displacements obtained by the Ritz method for a certain elasticity problem are usually less than or equal to the maximum exact displacements for that problem. The reason for this is that by assuming the displacement to be of a certain form we introduce additional constraints on the body or structure, which have the effect of decreasing the displacements. Another characteristic of the Ritz method is that the stresses obtained are usually less accurate in comparison with those of the exact solution than the displacements. This is due to the fact that the stresses are

\(^4\)The geometric boundary conditions are also referred to as the principal or essential boundary conditions.

\(^5\)In the case of the problems of the theory of elasticity, plates and shells the geometric boundary conditions refer to the conditions which prescribe the displacements or their first derivatives on the boundaries. The natural boundary conditions refer to the conditions which prescribe the stresses on the boundaries.
obtained by differentiating the displacements and the dis-
placements obtained by the Ritz method are better approxima-
tions than their slopes. The most difficult step in the
Ritz method, when it is applied to elasticity problems, is
the selection of the families of displacement functions, which
are relatively complete and satisfy the geometric boundary
conditions. This is particularly true if the body or struc-
ture has holes, cut outs or complicated boundaries.

The next section shows that under certain conditions
the finite element method is a special form of the Ritz
method.

1.3-1. The finite element method
as a special form of the
Ritz method

In the preceding section we have shown that the Ritz
method can be used to obtain approximate solutions to the
problems of the theory of elasticity. The family of dis-
placement functions with undetermined parameters must be
piecewise continuous together with their first derivatives and satisfy the boundary conditions of the problem.

6It is interesting to note that several authors (ref. 22,20) have stated that the Ritz method requires that the displacement functions must be compatible; i.e., the func-
tions and certain of their derivatives must be continuous across the element boundaries. This condition caused serious difficulties in plate bending problems since it requires that the slopes normal to the boundaries be continuous. Appar-
ently, this is not a mathematical requirement, although physically it seems desirable that the elements fit together after deformation.
The displacement functions of the finite element method generally satisfy these conditions; i.e., they usually are continuous on the elements and across element boundaries, but their derivatives often have discontinuities at the boundaries. The parameters of the family displacement functions can be chosen to be the displacements of the nodal points. Since the element displacement functions are linear functions of the nodal displacements the equations resulting from minimizing the potential energy are linear algebraic equations with the nodal displacements as the unknowns. These equations are completely equivalent to those obtained by the finite element method described in section 1.2. To illustrate this, we apply the Ritz method to the example of section 1.2. The displacement functions of equation (1-3) are continuous functions. The potential energy for the first element is given by (see section 1.2)

\[ II^{(1)} = U_s^{(1)} = \frac{4}{3} \frac{EA}{L} \hat{u}_2^2 \]  

(1-25)

and for the second element by

\[ II^{(2)} = U_s^{(2)} - F\hat{u}_3 = \frac{28}{27} \frac{EA}{L} (\hat{u}_2 - \hat{u}_3)^2 - F\hat{u}_3. \]  

(1-26)

The total potential energy for the entire bar is represented by

\[ II = II^{(1)} + II^{(2)} \]

\[ = \frac{4}{3} \frac{EA}{L} \hat{u}_2^2 + \frac{28}{27} \frac{EA}{L} (\hat{u}_2 - \hat{u}_3)^2 - F\hat{u}_3. \]  

(1-27)
This function can be minimized by taking the partial derivatives of $II$ with respect to $\hat{u}_2$ and $\hat{u}_3$ and setting them equal to zero. In this manner we obtain

$$\frac{\delta II}{\delta \hat{u}_2} = 0 = \frac{8}{3}\frac{EA}{L} \hat{u}_2 + \frac{56}{27}\frac{EA}{L} (\hat{u}_2 - \hat{u}_3)$$  \hspace{1cm} (1-28)

$$\frac{\delta II}{\delta \hat{u}_3} = 0 = -\frac{56}{27}\frac{EA}{L} (\hat{u}_2 - \hat{u}_3) - F.$$  \hspace{1cm} (1-29)

These equations are identical with equations (1-13) and (1-14), respectively.

The convergence of the solution can also be investigated by applying the concept of relative completeness to the element displacement functions. To illustrate the method by which this can be accomplished we apply it to the one-dimensional example problem of section 1.2. Suppose an arbitrary continuously differentiable function $u = u(x)$ is defined on the bar. Consider the displacement function $\hat{u}$ of equation (1-4) defined on the typical element with end points $i$ and $j$ of section 1.2. In this particular case the conditions of relative completeness imply that the displacement function $\hat{u}$ should be capable of approximating the function $u$ and its first derivative such that the error can be made arbitrarily small. It is intuitively obvious that the approximation of $u$ can be achieved by choosing the nodal displacements (which are equivalent to the undetermined parameters of the Ritz method) equal to the values of $u$ at the points $i$ and $j$; i.e.,
\( \hat{u}_1 = u(x_1) \) and \( \hat{u}_j = u(x_j) \). We can show this mathematically by expanding the nodal displacements in a Taylor series about an interior point of the element and substituting these series into the expression of \( \hat{u} \).

The Taylor series of \( u = u(x) \) about a point \( x_1 \) is given by

\[
u(x) = u(x_1) + (x-x_1)u'(x_1) + \frac{(x-x_1)^2}{2!} u''(x_1) + \cdots + \frac{(x-x_1)^n}{n!} u^{(n)}(x_1) + R_{n+1}
\]

(1-30)

Where

\[
R_{n+1} = \frac{1}{(n+1)!} u^{(n+1)}(t) (x-x_1)^{n+1}
\]

is the remainder and \( t \) lies between \( x_1 \) and \( x \).

The remainder can also be indicated by the symbol \( O(h^{n+1}) \). The symbol \( O(h^{n+1}) \) stands for "a quantity that is, in absolute value, at most equal to a constant multiple of the absolute value of \( h^{n+1} \)," where \( h \) is the length of the element; i.e., \( h = |x_1-x_j| \). To obtain the Taylor expansion of the nodal displacement \( \hat{u}_1 = u(x_1) \) we replace \( x \) with \( x_1 \) and \( x_1 \) with \( x \) and find from equation (1-30)

\[
\hat{u}_1 = u(x_1) = u(x) - (x-x_1)u'(x) + \frac{(x-x_1)^2}{2!} u''(x) + \cdots + (-1)^n \frac{(x-x_1)^n}{n!} u^{(n)}(x) + O(h^{n+1}).
\]

(1-31)
The displacement function is given by (see section 1.1)

\[ \hat{u}(x) = \frac{x^2 - x_j^2}{x_1^2 - x_j^2} \hat{u}_1 + \frac{x_1^2 - x^2}{x_1^2 - x_j^2} \hat{u}_j \]  

(1-3)

and its derivative by

\[ \hat{u}' = \frac{2x}{x_1^2 - x_j^2} \hat{u}_1 - \frac{2x}{x_1^2 - x_j^2} \hat{u}_j. \]  

(1-32)

Equation (1-31) can be written in the form

\[ \hat{u}_1 = u(x) + 0(h) \]  

(1-33)

On substituting this equation and a similar expression for \( \hat{u}_j \) into equation (1-3) we find

\[ \hat{u} = \frac{x^2 - x_j^2}{x_1^2 - x_j^2} [u(x) + O(h)] + \]

\[ \frac{x_1^2 - x^2}{x_1^2 - x_j^2} [u(x) + O(h)]. \]

Since \( \frac{x^2 - x_j^2}{x_1^2 - x_j^2} \) and \( \frac{x_1^2 - x^2}{x_1^2 - x_j^2} \) are bounded on the element this last expression can be replaced by

\[ \hat{u} = u(x) + O(h) \]  

(1-34)

If now the size of the element decreases so that \( h \to 0 \) then \( \hat{u} \to u(x) \).
Similarly we can write (1-31) in the form
\[ \hat{u}_i = u(x) - (x-x_i)u'(x) + O(h^2) \] (1-35)

Substitution of this equation and a similar one for \( \hat{u}_j \) into equation (1-32) yields
\[ \hat{u}' = \frac{2x}{x_i^2-x_j^2} [u(x)-(x-x_i)u'(x) + O(h^2)] \]
\[ - \frac{2x}{x_i^2-x_j^2} [u(x)-(x-x_j)u'(x) + O(h^2)] \] (1-36)

Since \( \frac{2x}{x_i^2-x_j^2} = \frac{2x}{(x_i+x_j)(x_i-x_j)} \) is bounded on the element and is at most a multiple of \( 1/h \) we can write (1-36) in the form
\[ \hat{u}' = \frac{2x}{x_i+x_j} u'(x) + O(h) \]

Again when \( h \to 0 \) we have that \( x_i \to x_j \) and \( x \to x_i \) so that \( \hat{u}' \to u'(x) \).

Note that the displacements \( \hat{u} \) are continuous across the element boundaries but that their derivatives are discontinuous at the boundaries.

It should be emphasized that good results cannot be obtained if the element displacement functions are not capable of giving a close approximation to the exact displacements of the problem. The conditions of relative completeness are essentially mathematical expressions of this fact; however,
its fulfillment does not necessarily guarantee quick convergence.

Physical insight can be gained into the conditions that should be imposed on the displacement functions by generalizing these functions and applying the above procedures to these functions. To illustrate this we apply it to the displacement function equation (1-3) of section 1.1. In this particular case the generalized displacement function becomes

\[ \hat{u} = f_1(x) \hat{u}_1 + f_2(x) \hat{u}_j \]  

and its derivative

\[ \hat{u}' = f'_1(x) \hat{u}_1 + f'_2(x) \hat{u}_j. \]

Following the procedure used above we substitute equation (1-33) and a similar expression for \( \hat{u}_j \) into equation (1-37) and find

\[ \hat{u} = f_1(x) [u + 0(h)] + \\
+ f_2(x) [u + 0(h)] \]

The function \( \hat{u} \) will converge to \( u \) when \( h \to 0 \) if

\[ f_1(x) + f_2(x) \to 1 \]

and the functions \( f_1 \) and \( f_2 \) are bounded on the element.
Similarly by substitution of equation (1-35) and a similar expression for \( \hat{u}_j \) into equation (1-38) we obtain for the derivative of \( \hat{u} \)

\[
\hat{u}' = f'_1(x) \left[ u-(x-x_1)u'(x) + O(h^2) \right] \\
+ f'_2(x) \left[ u-(x-x_j)u'(x) + O(h^2) \right]
\]

Now if \( h \to 0 \) we have that \( \hat{u}' \to u' \) when

\[
f'_1(x) + f'_2(x) \to 0 \tag{1-40}
\]

\[
x_1f'_1(x) + x_jf'_2(x) \to 1 \tag{1-41}
\]

A quicker rate of convergence can be expected if the functions \( f_1(x) \) and \( f_2(x) \) satisfy the conditions

\[
f_1(x) + f_2(x) = 1 \tag{1-42}
\]

\[
x_1f'_1(x) + x_jf'_2(x) = 1 \tag{1-43}
\]

for any size of the elements.

In that case equation (1-40) becomes

\[
f'_1(x) + f'_2(x) = 0.
\]

Substituting equation (1-44) into equation (1-43) gives

\[
(x_1 - x_j)f'_1 = 1
\]

so that \( f_1(x) = \frac{x}{x_1-x_j} + \text{constant} \).
Physically these results mean that the displacement function \( \hat{u} \) should be able to represent a condition of constant strain. In fact, if the functions \( f_1(x) \) and \( f_2(x) \) are chosen as linear functions of \( x \) the displacement method will, in the particular case of this example, yield the exact solution to the problem.

In Appendix A the requirements on the displacement function of several elements of plane elasticity and plate bending problems have been investigated using the approach outlined above.

1.3. Matrix formulation of the displacement method

The number of computations which result from the application of the finite element displacement method to complex problems is very large. However, they can be conveniently carried out on an electronic digital computer. The most practical way of organizing these computations is by the use of matrix algebra. It is, therefore, desirable to express the equations arising in the displacement method in matrix form.

The procedure to be followed can be outlined in a number of steps. In the following pages we illustrate the results obtained in each step on the simple one-dimensional example of section 1.1.
1) The element displacement functions are expressed in matrix form by

\[
\begin{align*}
\{ d \} &= B \{ a \} \\
\text{(1-44)}
\end{align*}
\]

where the components of the vector \( \{ d \} \) represent the displacement components; the components of the vector \( \{ a \} \) are parameters, the number of which correspond to the total number of the nodal displacement components; the coefficients of the matrix \( B \) are functions in the variables of the coordinate system used.

The matrix equation (1-43) corresponds to equation (1-1) of the example of section 1.1. In this particular case the vector \( \{ d \} \) has only one component namely \( \hat{u} \). The vector \( \{ a \} \) is given by

\[
\{ a \}^T = \{ a_0 a_1 \}
\]

and the matrix \( B \) by

\[
B = [1 \ x^2]
\]

2) The vector \( \{ a \} \) can be expressed in terms of the nodal displacements. This can be accomplished by substituting the coordinates of the nodal points into the functions which form the coefficients of the \( B \) matrix and solving the resulting equations. This results in
\{ g \} = M \{ a \} \quad (1-45)

where the components of the vector \{ g \} are the nodal displacement components. The matrix M is a square matrix. The functions of equation (1-44) must be chosen such that the vectors formed by the rows of the matrix B evaluated at the nodal points are linearly independent.

This implies that the inverse of M exists.

From equation (1-44) we obtain

\[ \{ a \} = M^{-1} \{ g \} \quad (1-46) \]

where \( M^{-1} \) is the inverse of M.

The equations (1-2) of the example problem correspond to the matrix equation (1-45) and the M matrix is in this case given by

\[ M = \begin{bmatrix} 1 & x_i^2 \\ 1 & x_j^2 \end{bmatrix} \]

The vector \{ g \} is represented by

\[ \{ g \}^T = \{ \hat{u}_i \hat{u}_j \} \]

The inverse of the matrix M becomes

\[ M^{-1} = \frac{1}{x_i - x_j} \begin{bmatrix} -x_j^2 & x_i^2 \\ 1 & -1 \end{bmatrix} \]
and equations (1-3) are equivalent to the matrix equation (1-46).

3) The displacement functions can be expressed in terms of the nodal displacements by substituting equation (1-46) into equation (1-44)

\[
\begin{align*}
\{d\} &= BM^{-1} \{g\} \\
&= D \{g\} \\
\end{align*}
\]

(1-47)

This matrix equation is equivalent to equation (1-4) of the example problem. The matrix \(D\) is represented by

\[
D = \frac{1}{x_1^2-x_2^2} [(x_2^2x_1^2)(x_1^2-x_2^2)]
\]

4) The strains corresponding to the assumed displacements can be expressed as

\[
\{e\} = C \{d\}
\]

(1-48)

where the components of the vector \(\{e\}\) consist of the components of strain appropriate to the problem at hand; the coefficients of the matrix \(C\) are differential operators.

In the case of the example problem the vector \(\{e\}\) has only one component namely \(\varepsilon\) and the matrix \(C\) has only one term namely \(\frac{d}{dx}\).
5) The relationship between stress and strain is expressed by the generalized Hooke's law

\[
\{ \mathbf{G} \} = \mathbf{E} \begin{bmatrix} e \end{bmatrix} \quad (1-49)
\]

where \( \{ \mathbf{G} \} \) is the stress vector and \( \mathbf{E} \) is a symmetric matrix the coefficients of which represent the material properties of the problem at hand.

6) The strain energy which is stored in the element can be expressed as

\[
U_s = \frac{1}{2} \int_{V_{el}} \begin{bmatrix} G \end{bmatrix}^T \begin{bmatrix} e \end{bmatrix} \, dV_{el} \quad (1-50)
\]

Substituting equation (1-49) into equation (1-50) we obtain

\[
U_s = \frac{1}{2} \int_{V_{el}} \begin{bmatrix} e \end{bmatrix}^T \mathbf{E} \begin{bmatrix} e \end{bmatrix} \, dV_{el} \quad (1-51)
\]

The strain energy can now be expressed in terms of the nodal displacement components by substituting equation (1-48) into equation (1-51).

\[
U_s = \frac{1}{2} \int_{V_{el}} \begin{bmatrix} d \end{bmatrix}^T \mathbf{C}^T \mathbf{E} \mathbf{C} \begin{bmatrix} d \end{bmatrix} \, dV_{el}. \quad (1-52)
\]

By substituting equation (1-47) into equation (1-52) we finally obtain

\[
U_s = \frac{1}{2} \int_{V_{el}} \mathbf{d}^T [\mathbf{D}^T \mathbf{C} \mathbf{E} \mathbf{C} \mathbf{D}] \, dV_{el} \begin{bmatrix} g \end{bmatrix} . \quad (1-53)
\]
The nodal displacement components represented by the vector \( \{ g \} \) have been removed from the integral since they are independent of the coordinates. The matrix

\[
K = \int_{V_{el}} [B^T C^T E D] dV_{el}
\]

is called the element stiffness matrix. The coefficients of the element stiffness matrix \( K \) have a definite physical meaning. The coefficient \( K_{mn} \) represents the generalized force at the nodal point \( m \) due to a unit generalized displacement at the nodal point \( n \). By virtue of the fact that \( E \) is a symmetric matrix we conclude that \( K \) is also symmetric.

Equation (1-52) is equivalent to equation (1-8) for the first element and equation (1-10) for the second element. The stiffness matrix of the first element has only one coefficient namely \( \frac{8}{3} \frac{EA}{L} \), while the stiffness matrix of the second element consists of two coefficients both equal to \( \frac{56}{27} \frac{EA}{L} \).

7) The potential energy is of the form given in equation (1-24). If we assume that no body forces are present we can express the contribution of the applied loads to the potential energy as follows:

\[
P = - \int_{S_T} \left\{ p \right\}^T \left\{ d \right\} ds \quad \text{(1-55)}
\]

where the components of the vector \( \{ p \} \) consist of the components of the applied loads.
The work $P$ can be expressed in terms of the nodal displacements by substituting equation (1-47) into equation (1-55)

$$P = -\int_{S_T} \left[ \{ p \}^T D \{ g \} \right] \, ds$$

$$= -\{ g \}^T \{ q \} .$$

(1-56)

The vector

$$q = \int_{S_T} [D \{ p \}] \, ds$$

is called the load vector. The potential energy of the element is then given by

$$II^{el} = \frac{1}{2} \{ g \}^T K \{ g \} - \{ q \}^T \{ g \}$$

(1-57)

This equation is equivalent to equation (1-25) for the first element of the example problem. The load vector in this case is the zero vector. The equivalent equation for the second element is given by equation (1-26). The load vector in that case has two components namely zero and the force $F$.

8) The total potential energy of the assemblage of elements can be obtained by adding the potential energy of the individual elements

$$II = \sum II^{el} .$$

(1-58)

This equation corresponds to equation (1-27) obtained for the example problem.
9) Minimization of II with respect to the nodal displacements furnishes the condition that

$$\sum [K \{g\} - \{q\}] = 0 \quad (1-59)$$

This equation can be written in the form

$$K_{\text{tot}} \{g_{\text{tot}}\} - \{q_{\text{tot}}\} = 0 \quad (1-60)$$

where $K_{\text{tot}}$ represents the total stiffness matrix for the assemblage of elements; the vector $\{g_{\text{tot}}\}$ has as its components the nodal displacement components of the entire assemblage of elements; and the vector $\{q_{\text{tot}}\}$ is the load vector for the entire assemblage of elements.

The matrix equation (1-60) is equivalent to the system of equations (1-29) and (1-30) of the example problem. The coefficients of $K_{\text{tot}}$ have the same physical meaning as those of the elements stiffness matrix namely the coefficient $k_{mn}$ is the generalized force at the nodal point $m$ due to a unit generalized displacement of the nodal point $n$. In this case, however, the coefficient $k_{mn}$ is different from zero only if $m = n$ or if the node $m$ belongs to the same element as $n$. It follows from this that the matrix $K_{\text{tot}}$ is sparcely populated since coupling only occurs between nodal points which are adjacent to each other. This fact can be used to advantage in the solution of equation (1-60) if the number of the components of the vector $\{g\}$ is very large.
1.5 Advantages and disadvantages of the displacement method

1.5-1 Advantages

The advantages of the displacement method of analysis are:

1) The method can be applied to structures having complex geometry. For example, shell problems with irregular boundaries, holes, etc., can be closely approximated by the use of suitable elements. It should be noted that triangular elements are, in this respect, superior to rectangular elements.

2) The material characteristics can be taken into account in the development of the element stiffness matrices. By assigning individual elastic properties to each element, the method will handle problems with non-homogeneous materials. The effects of stiffening members and material distribution through the structure can also be taken into account.

3) The redundancy of the structure does not affect the procedures of the displacement method and can thus be easily incorporated in the analysis.

4) The method can be formulated in matrix notation and can be programmed on an electronic digital computer.

5) If displacement functions are used which satisfy the geometric boundary conditions exactly the maximum displacement obtained with the displacement method will usually
be smaller than the maximum displacement of the exact solution. A refinement of the mesh will show monotonic convergence of the potential energy from above. If the displacement functions are relatively complete the potential energy will converge to that of the exact solution.

1.5-2 Disadvantages

1) The real boundaries of the structure are replaced by the boundaries of the elements. The nodal points can be located on the real boundaries but the boundaries in between the nodal point are those of the elements. In certain cases the element boundaries will be the same as the real boundaries. The approximation can be improved by locating a large number of nodal points on the real boundary.

2) The prescribed displacements are replaced by the displacements of the elements. The nodal points can be given the prescribed displacements of the problem but the displacements of the in between points will only be approximate. In certain cases it may be possible to represent the prescribed displacements exactly. This depends on the type of problem and the displacement functions.

3) The solutions obtained by the displacement method do not in general satisfy the equilibrium equations. The overall equilibrium of the applied loads and the generalized nodal forces acting on each element and on the entire structure can be obtained by including rigid body modes in the
element displacement functions (ref. 19). However, even this is sometimes difficult to achieve as is the case with shell element displacement functions.

4) At this time no known relation exists between the accuracy of the solution and the number of elements.

1.6 Review of the literature

The concept of replacing a continuum by an assemblage of elements is not new. Engineers have, for a long time, used it in the analysis of framed structures, beam grid structures, etc. It is quite natural to apply this concept to that type of structure. Levy (ref. 8), however, was the first to replace a low aspect-ratio wing by an assemblage of elements. Turner et al. (ref. 9) refined and extended the method. Clough (ref. 11) and Wilson (ref. 13) applied the method to two-dimensional elasticity problems. The applicability of the displacement method to plate bending problems was demonstrated by Tocher (ref. 14), Clough and Tocher (ref. 15), Melosh (ref. 16) and Zienkiewicz and Cheung (ref. 17). Clough and Rashid (ref. 24) extended the displacement method to axi-symmetrical solids, while Melosh (ref. 22) published a number of stiffness matrices for the analysis of three-dimensional solids.

These papers all attempted to demonstrate numerically that the answers approached the exact solution in the limit. No theoretical treatment of the convergence of the displace-
ment method was published until 1963. Besseling (ref. 34), Melosh (ref. 19) and DeVeubeke (ref. 20) established the relationship between the displacement method and the Ritz method. Melosh (ref. 19) considered the convergence of the potential energy and McLay (ref. 25) demonstrated convergence of the displacements on a two-dimensional elasticity problem using rectangular elements. Bazeley et al. (ref. 18) and Clough and Tocher (ref. 15) obtained stiffness matrices for a triangular plate element, based on compatible displacement functions.

Adini (ref. 26), Greene et al. (ref. 27) and Melosh (ref. 30) applied the displacement method to thin shell structures using flat elements. Grafton and Strome (ref. 28), Percy et al. (ref. 29) and Klein (ref. 31) obtained solutions for shells of revolution using curved elements. Borner et al. (ref. 32) indicated a method to obtain the stiffness matrix for a compatible rectangular shell element.

1.7 Scope of the study

The study is limited to thin shells and considers only small deformations; i.e., strains which are small compared to unity.

The purpose of the study is to--

1) Define a triangular shell element on a shell of arbitrary curvature,
2) Define a suitable set of compatible displacement functions on this element,

3) Develop the element stiffness matrix for this element,

4) Prove the convergence of the solutions obtained with this element in the limit, and

5) Develop a consistent mass matrix for this element.

The equations in the following chapters have been arranged in such a way that they are convenient for use in a computer program.
CHAPTER II

THE CURVED TRIANGULAR SHELL ELEMENT AND THE CORRESPONDING DISPLACEMENT FUNCTIONS

The analysis of thin shells of arbitrary curvature by the finite element method can be achieved through the development of a curved triangular shell element and the corresponding displacement functions. Three displacement functions are required namely two membrane displacement functions, denoted by \( u \) and \( v \), and a normal displacement function, denoted by \( w \).

The chapter begins by reviewing the basic concepts of the differential geometry of surfaces which are needed in the following sections. The boundaries of the shell element are next defined. Then the displacement functions are defined and are shown to be continuous functions, suitable for use with the Ritz method.

2.1 Some basic concepts of the differential geometry of surfaces

Let the \((X,Y,Z)\) coordinate system, shown in Figure 3, be a fixed righthanded rectangular coordinate system. Let the coordinates of the surface \( S \) be given by
Fig. 3.—Curvilinear coordinates of a surface.
\[ X = f_1(\alpha, \beta) \]
\[ Y = f_2(\alpha, \beta) \quad (2-1) \]
\[ Z = f_3(\alpha, \beta) \]

where \( \alpha \) and \( \beta \) are two independent parameters and \( f_1, f_2 \) and \( f_3 \) are continuous, single-valued functions of these parameters.

If we let \( \beta \) take different constant values and let \( \alpha \) vary, we obtain a family of curves lying on the surface. These curves are called the \( \alpha \) coordinate lines. Similarly, if we let \( \alpha \) take different constant values and let \( \beta \) vary, we find a second family of curves, the \( \beta \) coordinate lines.

The vector equation
\[ \vec{r} = \vec{r}(\alpha, \beta) \quad (2-2) \]
is equivalent to the three scalar equations (2-1).

The partial derivatives of \( \vec{r} \) with respect to \( \alpha \) and \( \beta \)
\[ \vec{r}_\alpha = \frac{\partial \vec{r}}{\partial \alpha} \]
\[ \vec{r}_\beta = \frac{\partial \vec{r}}{\partial \beta} \quad (2-3) \]
are, at each point of the surface, tangent to the \( \alpha \) and \( \beta \) lines respectively.

The total differential \( d\vec{r} \) represents an increase of the vector \( \vec{r} \) on the surface and is given by
\[ d\vec{r} = \vec{r}_\alpha \, d\alpha + \vec{r}_\beta \, d\beta \quad (2-4) \]
The square of the length of this vector is
\[ ds^2 = dr^2 \cdot dr^2 \]  \hspace{1cm} (2-5)
\[ = (\vec{r}_{,\alpha} \cdot \vec{r}_{,\alpha})d\alpha^2 + 2(\vec{r}_{,\alpha} \cdot \vec{r}_{,\beta})d\alpha d\beta + (\vec{r}_{,\beta} \cdot \vec{r}_{,\beta})d\beta^2. \]

In the case the coordinate lines \( \alpha \) and \( \beta \) are orthogonal the scalar product \((\vec{r}_{,\alpha} \cdot \vec{r}_{,\beta})\) vanishes and equation (2-5) assumes the form
\[ ds^2 = (\vec{r}_{,\alpha} \cdot \vec{r}_{,\alpha})d\alpha^2 + (\vec{r}_{,\beta} \cdot \vec{r}_{,\beta})d\beta^2 \]  \hspace{1cm} (2-6)

If \( d\beta = 0 \) we have
\[ dx = A_\alpha \, d\alpha \]  \hspace{1cm} (2-7)
and similarly if \( d\alpha = 0 \)
\[ dy = A_\beta \, d\beta \]

where \( dx \) and \( dy \) are increases of arc length along the coordinate lines \( \alpha \) and \( \beta \), due to the increases of the curvilinear coordinates \( \alpha \) and \( \beta \) respectively.

The quantities
\[ A_\alpha = (\vec{r}_{,\alpha} \cdot \vec{r}_{,\alpha})^{1/2} \]  \hspace{1cm} (2-8)
and
\[ A_\beta = (\vec{r}_{,\beta} \cdot \vec{r}_{,\beta})^{1/2} \]
are called the Lame' parameters. Equations (2-3) can be written in the form

\[ \mathbf{r}_{,\alpha} = A_\alpha \mathbf{e}_\alpha \]  
\[ \mathbf{r}_{,\beta} = A_\beta \mathbf{e}_\beta \]

where \( \mathbf{e}_\alpha \) and \( \mathbf{e}_\beta \) are unit vectors tangent to the \( \alpha \) and \( \beta \) lines respectively (see Fig. 3).

The unit vector \( \mathbf{e}_n \) normal to the middle surface of the shell is given by

\[ \mathbf{e}_n = \mathbf{e}_\alpha \times \mathbf{e}_\beta \]  \hspace{1cm} (2-9)

The system \( \mathbf{e}_\alpha \), \( \mathbf{e}_\beta \), \( \mathbf{e}_n \) is assumed to be right handed and is assumed to be directed to the convex side of the surface (or to the side of the centers of negative curvature if the signs of the radii of curvature are not equal).

The area of a surface element of dimensions \( dx \) by \( dy \) is

\[ dA = A_\alpha A_\beta d\alpha d\beta \]

2.2 Geometry of the curved triangular shell element

Let the vertices of a typical curved triangular element be located on the middle surface\(^7\) of the shell by the coordinates \((\alpha_i, \beta_i), (\alpha_j, \beta_j)\) and \((\alpha_k, \beta_k)\) (see Fig. 4). The

\(^7\)The middle surface of a shell is defined as the locus of points which lie at equal distances from the two curved surfaces which form the boundaries of the shell body.
Fig. 4.—A typical triangular shell element.
boundary curves connecting the points i, j and k are defined by the equation \( \beta = m\alpha + n \). For example, the curve C connecting the points i and j is represented by the equation

\[
\beta = \frac{\beta_{ij}}{\alpha_{ij}} \alpha + \frac{\alpha'_{ij} \beta_i - \alpha_i \beta_{ij}}{\alpha_{ij}}
\]  

(2-10)

where \( \alpha'_{ij} = \alpha_i - \alpha_j \)

and \( \beta_{ij} = \beta_i - \beta_j \)

It should be noted that the boundary curves defined above are curves on the surface making constant angles with the parametric lines. In the case of a flat surface and a cartesian coordinate system these boundary curves become straight lines.

2.3 The displacement functions for the curved triangular shell element

The displacements of a shell are most conveniently represented by its components in the directions of \( \vec{e}_\alpha \), \( \vec{e}_\beta \), and \( \vec{e}_n \). The components in the directions of \( \vec{e}_\alpha \) and \( \vec{e}_\beta \) are called the membrane displacements and are denoted by \( \hat{u} \) and \( \hat{v} \) respectively. The component in the direction of \( \vec{e}_n \) is called the normal displacement and is denoted by \( \hat{w} \).

The membrane displacement functions for the triangular shell element can be obtained from those derived for the triangular element of plane elasticity analysis. Similarly, the normal displacement functions can be adapted from the displacement function developed for the triangular element of plate bending analysis.
2.3-1 The membrane displacement functions

The boundaries of the curved triangular shell element were defined in section 2.2 and it was found that the properties of these boundary curves are very similar to those of a straight line in the plane. The similarity of these properties suggest the adoption of displacement functions similar to those of plane elasticity for the membrane displacements of the shell problem.

The membrane displacement functions are thus defined as follows (see Appendix B)

\[
\hat{u} = L_1 \hat{u}_1 + L_j \hat{u}_j + L_k \hat{u}_k \\
\hat{v} = L_1 \hat{v}_1 + L_j \hat{v}_j + L_k \hat{v}_k
\]  

(2-14)

where

\[
L_1 = \frac{1}{2\Delta} (a_1 + b_1 \alpha + c_1 \beta) \\
L_j = \frac{1}{2\Delta} (a_j + b_j \alpha + c_j \beta) \\
L_k = \frac{1}{2\Delta} (a_k + b_k \alpha + c_k \beta)
\]  

(2-15)

\[
a_r = \begin{vmatrix}
\alpha_s & \alpha_t \\
\beta_s & \beta_t
\end{vmatrix}
\]

(2-16)

\[
b_r = \beta_s - \beta_t = \beta_{st}
\]

\[
c_r = \alpha_s - \alpha_t = \alpha_{st}
\]
and \( r, s \) and \( t \) are cyclic combinations of \( i, j \) and \( k \);
\( \hat{u}_i \) and \( \hat{v}_i \) are the displacements of the \( i^{th} \) nodal point in the
\( \alpha \) and \( \beta \) directions respectively.

The membrane displacement functions \( \hat{u} \) and \( \hat{v} \), as defined above, have properties very similar to the displacement functions of the flat triangular element of plane elasticity (see Appendix B, section B-1). They are continuous on the element and the displacements on the boundaries depend linearly on the displacements of their end points. However, their derivatives are not continuous across the boundaries. In this case no rigid body modes are included in the displacement functions. Rigid body modes lead to zero strains, while the strains due to \( \hat{u} \) and \( \hat{v} \) of equations (2-14) are generally not zero on the shell element (see Equations (C-1), Appendix C).

2.3-2 The normal displacement function

The normal displacement function for the curved triangular shell element can be obtained from the displacement function developed for the triangular plate element (see Appendix B, section B-2).

The complete displacement function is

\[
2 \Delta = \begin{bmatrix}
1 & \alpha_i & \beta_i \\
1 & \alpha_j & \beta_j \\
1 & \alpha_k & \beta_k
\end{bmatrix}
\]
\[ \hat{w} = L_1 \hat{w}_1 + L_j \hat{w}_j + L_k \hat{w}_k \]
\[ + (\beta_{ij} \psi_{ij} + \beta_{ik} \psi_{ik}) \theta_{\alpha_i}^* \]
\[ + (\alpha_{ij} \psi_{ij} + \alpha_{ik} \psi_{ik}) \theta_{\beta_i}^* \]
\[ + (\beta_{jk} \psi_{jk} + \beta_{ji} \psi_{ji}) \theta_{\alpha_j}^* \]
\[ + (\alpha_{jk} \psi_{jk} + \alpha_{ji} \psi_{ji}) \theta_{\beta_j}^* \]
\[ + (\beta_{kl} \psi_{kl} + \beta_{kj} \psi_{kj}) \theta_{\alpha_k}^* \]
\[ + (\alpha_{kl} \psi_{kl} + \alpha_{kj} \psi_{kj}) \theta_{\beta_k}^* \] 

(2-18)

where

\[ \psi_{ij} = L_1^2 L_j + \frac{1}{2} L_1 L_j L_k \]

(2-19)

\[ \psi_{ik} = L_1^2 L_k + \frac{1}{2} L_1 L_j L_k \]

The other \( \psi \)'s can be obtained from those shown in equation (2-19) by a cyclic interchange of the \( i, j \) and \( k \).

\[ \theta_{\alpha_i}^* = \theta_{\alpha_i} + \frac{1}{2\Delta} (c_i \hat{w}_i + c_j \hat{w}_j + c_k \hat{w}_k) + \frac{\hat{v}_l}{R_{\beta l}}. \]

(2-20)

\[ \theta_{\beta_i}^* = \theta_{\beta_i} + \frac{1}{2\Delta} (b_i \hat{w}_i + b_j \hat{w}_j + b_k \hat{w}_k) + \frac{\hat{u}_l}{R_{\alpha l}}. \]

(2-21)

\[ R_{\alpha l} = R_{\alpha}(\alpha_l, \beta_l) \]

(2-21)

\[ R_{\beta l} = R_{\beta}(\alpha_l, \beta_l) \].
are the radii of curvature in the $\alpha$- and $\beta$- directions; $\theta_{\alpha l} = -\frac{1}{A_\alpha} \frac{\partial \hat{W}(\alpha_\ell\beta_\ell)}{\partial \beta} + \frac{\hat{V}_l}{R_{\alpha l}}$ is the slope about the $\beta$ line at the $l^{th}$ node ($l=i,j,k$), and

$$\theta_{\beta l} = -\frac{1}{A_\beta} \frac{\partial \hat{W}(\alpha_\ell\beta_\ell)}{\partial \alpha} + \frac{\hat{U}_l}{R_{\beta l}}$$

is the slope about the $\beta$ line at the $l^{th}$ node.

These functions have the property that on the boundary curves $\hat{w}$ is a linear combination of the displacements and rotations of the end points. For example on boundary curve $i-j$ we have

$$L_k = \psi_{ki} = \psi_{ik} = 0$$

so that

$$\hat{w} = L_i \hat{w}_i + L_j \hat{w}_j$$

$$+ \beta_{ij} L_i^2 L_j \theta^{*}_{\alpha i} + \alpha_{ij} L_i^2 L_j \theta^{*}_{\beta i} + \beta_{ji} L_j^2 L_i \theta^{*}_{\alpha j} + \alpha_{ji} L_j^2 L_i \theta^{*}_{\beta j}. \tag{2-22}$$

On substituting equations (2-20) into equation (2-22) we find that the normal displacements on the curve $i-j$ depend only on the displacements and rotations of the nodes $i$ and $j$. The normal displacements are thus continuous across the boundaries. The slopes and changes of curvature are in general not continuous across the boundary curves. This condition is similar to the discontinuity at the boundaries of the derivatives of the membrane displacement functions of section 2.3.1. Similar conditions are frequently encountered in finite element applications of other problems.
CHAPTER III

THE STRAIN ENERGY FOR A THIN SHELL

In order to apply the Ritz method to a shell structure it is necessary to develop the expression for the strain energy. For the purpose of this study it will be sufficient to consider an orthotropic shell. The stress-strain relations for an orthotropic material are usually expressed in terms of the stresses and strains in the directions of the axes of orthotropy. In the case of an orthotropic shell these axes of orthotropy are represented by two mutually orthogonal families of curves, lying on the middle surface of the shell.

The most convenient expressions for the stresses and strains in the theory of thin shells are related to the directions of the principal lines of curvature of the middle surface. It is, therefore, necessary to determine the angle between a line of curvature and an axis of orthotropy at an arbitrary point of the middle surface.

3.1 The angle between a principal line of curvature and an axis of orthotropy

Let the curve $C_1$, which lies on the middle surface of the shell, be one of the axes of orthotropy and let the curve
Let $ds_1$ be a line element along $C_1$. The unit tangent vector at some point $P$ of $C_1$ is given by

$$
\vec{T}_1 = \frac{dr}{ds_1} = r, \alpha \frac{d\alpha}{ds_1} + r, \beta \frac{d\beta}{ds_1}
$$

(3-2)

By the chain rule we find

$$
\frac{d\beta}{ds_1} = \frac{d\beta}{d\alpha} \frac{d\alpha}{ds_1} = g'(\alpha) \frac{d\alpha}{ds_1}
$$

(3-3)

so that

$$
\vec{T}_1 = \frac{d\alpha}{ds_1} \left( A_\alpha \vec{e}_\alpha + g'(\alpha) A_\beta \vec{e}_\beta \right)
$$

(3-4)

The unit tangent vector to the line $\beta = \text{constant}$ passing through $P$ is $\vec{e}_\alpha$ and the angle $\theta$ between $\vec{T}_1$ and $\vec{e}_\alpha$ can be determined from the relations

$$
\cos \theta = A_\alpha \frac{d\alpha}{ds_1}
$$

(3-5)

and

$$
\sin \theta = g'(\alpha) A_\beta \frac{d\alpha}{ds_1}
$$

(3-6)

A more convenient form is

$$
\tan \theta = \frac{A_\beta}{A_\alpha} g'(\alpha) = \frac{dy}{dx}
$$

(3-7)

where $dx$ and $dy$ are given by equation (2-7).
Fig. 5.—The axes of orthotrophy.
3.2 The strain energy for a thin shell

Let the stresses in the directions of the principal lines of curvature at an arbitrary point of the shell be denoted by

\[ \sigma_{xx}, \sigma_{yy}, \sigma_{xy}, \sigma_{xz}, \text{etc.} \]

and the stresses in the directions of the axes of orthotropy at the same point by

\[ \sigma_{11}, \sigma_{22}, \sigma_{12}, \sigma_{13}, \text{etc.} \]

The stresses in the directions of the axes of orthotropy can be expressed in terms of the stresses in the directions of the principal lines of curvature. This can be achieved by an orthogonal transformation of the coordinates (ref. 6).

The resulting relationship is given in matrix form by

\[ \begin{bmatrix} \sigma_1 \end{bmatrix} = T_1 \begin{bmatrix} \sigma_2 \end{bmatrix} \]  \hspace{1cm} (3-8)

where

\[ \begin{bmatrix} \sigma_1 \end{bmatrix}^T = \begin{bmatrix} \sigma_{11} & \sigma_{22} & \sigma_{33} & \sigma_{12} & \sigma_{13} & \sigma_{23} \end{bmatrix} \]  \hspace{1cm} (3-9)

\[ \begin{bmatrix} \sigma_2 \end{bmatrix}^T = \begin{bmatrix} \sigma_{xx} & \sigma_{yy} & \sigma_{zz} & \sigma_{xy} & \sigma_{xz} & \sigma_{yz} \end{bmatrix} \]  \hspace{1cm} (3-10)
\[
T_1 = \begin{bmatrix}
\cos^2 \theta & \sin^2 \theta & 0 & \sin 2\theta & 0 & 0 \\
\sin^2 \theta & \cos^2 \theta & 0 & -\sin 2\theta & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
-\frac{1}{2} \sin 2\theta & \frac{1}{2} \sin 2\theta & 0 & \cos 2\theta & 0 & 0 \\
0 & 0 & 0 & 0 & \cos \theta & \sin \theta \\
0 & 0 & 0 & 0 & -\sin \theta & \cos \theta
\end{bmatrix}
\]

(2-11)

The strains in the directions of the axes of orthotropy can also be expressed in terms of the strains in the directions of the principal lines of curvature by

\[
\{\xi_1\} = T_2 \, \{\xi_2\}
\]

(3-12)

where

\[
\{\xi_1\}^T = \{\xi_{11} \, \xi_{22} \, \xi_{33} \, \xi_{12} \, \xi_{13} \, \xi_{23}\}
\]

(3-13)

\[
\{\xi_2\}^T = \{\xi_{xx} \, \xi_{yy} \, \xi_{zz} \, \xi_{xy} \, \xi_{xz} \, \xi_{yz}\}
\]

(3-14)

\[
T_2 = \begin{bmatrix}
\cos^2 \theta & \sin^2 \theta & 0 & \frac{1}{2} \sin 2\theta & 0 & 0 \\
\sin^2 \theta & \cos^2 \theta & 0 & -\frac{1}{2} \sin 2\theta & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
-\sin 2\theta & \sin 2\theta & 0 & \cos 2\theta & 0 & 0 \\
0 & 0 & 0 & 0 & \cos \theta & \sin \theta \\
0 & 0 & 0 & 0 & -\sin \theta & \cos \theta
\end{bmatrix}
\]

(3-15)
The stress-strain relations for an orthotropic material are given by (ref. 6)

\[
\{\sigma_1\} = A_1 \{\varepsilon_1\}
\]  (3-16)

where the \(A_1\) matrix is given by (ref. 6)

\[
A_1 = \begin{bmatrix}
a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\
a_{12} & a_{22} & a_{23} & 0 & 0 & 0 \\
a_{13} & a_{23} & a_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & a_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & a_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & a_{66}
\end{bmatrix}
\]  (3-17)

The elements of this matrix represent the elastic properties of the material.

The theory of thin shells is based on similar assumptions which Kirchhoff made to simplify the theory of thin plates. These assumptions, as stated by Novozhilov (ref. 4), are

1) The straight fibers of a plate which are perpendicular to the middle surface before deformation remain so after deformation and do not change their length.

2) The normal stresses acting on planes parallel to the
middle surface may be neglected in comparison with the other stresses.

The last assumption is equivalent to neglecting \( \sigma_{33} \) with respect to the other stresses, while the first assumption implies that the strains \( \varepsilon_{13} \) and \( \varepsilon_{23} \) are zero.

From equation (3-16) we find by neglecting the stress \( \sigma_{33} \)

\[
\varepsilon_{33} = \frac{a_{23} \varepsilon_{22} - a_{13} \varepsilon_{11}}{a_{33}}
\]  
(3-18)

The equation (3-16) can then be rewritten in the form

\[
\{ \sigma \} = A \{ \varepsilon \}
\]  
(3-19)

where

\[
\{ \sigma \}^T = \{ \sigma_{11} \ \sigma_{22} \ \sigma_{12} \}
\]  
(3-20)

\[
\{ \varepsilon \}^T = \{ \varepsilon_{11} \ \varepsilon_{22} \ \varepsilon_{12} \}
\]  
(3-21)

\[
A = \frac{1}{a_{33}} \begin{bmatrix}
(a_{11}a_{33}-a_{13}^2) & (a_{12}a_{33}-a_{13}a_{23}) & 0 \\
(a_{12}a_{33}-a_{13}a_{23}) & (a_{22}a_{33}-a_{23}^2) & 0 \\
0 & 0 & a_{33}a_{44}
\end{bmatrix}
\]  
(3-22)

The matrix equation (3-12) can be simplified by using the first assumption. Substituting \( \varepsilon_{13} = \varepsilon_{23} = 0 \) into (3-12) results in

\[
\{ \varepsilon \} = T \{ \varepsilon_3 \}
\]  
(3-23)
where
\[
\{\varepsilon\}_T = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{12} & \varepsilon_{22} \end{bmatrix} \quad (3-24)
\]
\[
\{\varepsilon\}_3^T = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{yy} & \varepsilon_{xy} \end{bmatrix} \quad (3-25)
\]

\[
T = \begin{bmatrix} \cos^2\theta & \sin^2\theta & \frac{1}{2} \sin2\theta \\ \sin^2\theta & \cos^2\theta & \frac{1}{2} \sin2\theta \\ -\sin2\theta & \sin2\theta & \cos2\theta \end{bmatrix} \quad (3-26)
\]

The stresses in the directions of the axes of orthotropy can now be expressed in terms of the strains in the directions of the principal lines of curvature by substituting equation (3-23) into equation (3-19)

\[
\{G\} = A T \{\varepsilon\}_3 \quad (3-27)
\]

From the theory of elasticity we know that the strain energy is given by the volume integral

\[
U_s = \frac{1}{2} \int_V \{G\}^T \{\varepsilon\} dV. \quad (3-28)
\]

where \( V \) is the volume of shell.

Substituting equation (3-23) and (3-27) into equation (3-28) we obtain

\[
U_s = \frac{1}{2} \int_V \left[ \{\varepsilon\}_3^T T^T \Phi T \{\varepsilon\}_3 \right] dV. \quad (3-29)
\]

In the case of a shell the volume element is (ref. 4)

\[
dV = A_\alpha A_\beta \left( 1 + \frac{Z}{R_\alpha} \right) \left( 1 + \frac{Z}{R_\beta} \right) d\alpha d\beta dz \quad (3-30)
\]
and the strains at an arbitrary point of the shell in terms of the strains of the middle surface are (see Appendix C):

\[ \varepsilon_{xx} = \frac{1}{1 + \frac{2}{R \alpha}} \left( \varepsilon_{\alpha\alpha} + z \kappa_{\alpha} \right) \]  

(3-31)

\[ \varepsilon_{yy} = \frac{1}{1 + \frac{2}{R \beta}} \left( \varepsilon_{\beta\beta} + z \kappa_{\beta} \right) \]  

(3-32)

\[ \varepsilon_{xy} = \frac{1}{(1 + \frac{2}{R \alpha})(1 + \frac{2}{R \beta})} \left[ \left(1 + \frac{z^2}{R \alpha R \beta}\right) \omega + 2 \left(1 + \frac{1}{R \alpha} + \frac{1}{R \beta} + \frac{z^2}{2 R \alpha R \beta}\right) z \tau' \right] \]  

(3-33)

where \( \varepsilon_{\alpha\alpha}, \varepsilon_{\beta\beta} \) and \( \omega \) are the strains of the middle surface corresponding to \( \varepsilon_{xx}, \varepsilon_{yy} \) and \( \varepsilon_{xy} \), respectively, \( \kappa_{\alpha}, \kappa_{\beta} \) and \( \tau' \) characterize the change of curvature and torsion of the middle surface (see ref. 4, chapter 1, sec. 4). The expressions for \( \varepsilon_{\alpha\alpha}, \varepsilon_{\beta\beta}, \omega, \kappa_{\alpha}, \kappa_{\beta} \) and \( \tau \) in terms of the displacements \( u, v \) and \( w \) are shown in Appendix C.

The theory of thin shells deals with shells for which the thickness of the shell \( t \) is small in comparison with the radius of curvature. The quantity \( t/R \) may thus be neglected with respect to unity and terms containing \( t/R \) may be omitted.

In order to simplify the expression for the strain energy the integrand of equation (3-29) can be expanded into a power series in \( z \) where terms up to and including \( z^2 \) are retained.
On substituting equations (3-31), (3-32) and (3-33)
into equation (3-29) we obtain in this manner

$$U_s = \frac{1}{2} \int \int \left[ \int_{-t/2}^{t/2} \left( Q_0 + Q_1 z + Q_2 z^2 \right) \, d\alpha \, d\beta \right] A_\alpha A_\beta \, d\alpha \, d\beta . \tag{3-34}$$

where

$$Q_0 = \varepsilon_{\alpha \alpha} \left( b_{11} + \varepsilon_{\alpha \alpha} \varepsilon_{\beta \beta} (b_{12} + b_{21}) + \varepsilon_{\alpha \alpha} \omega (b_{13} + b_{31}) \right) + \varepsilon_{\beta \beta} b_{22} + \varepsilon_{\beta \beta} \omega (b_{23} + b_{32}) + \omega^2 b_{33} \tag{3-35}$$

$$Q_1 = \frac{2}{R_\alpha} \left( \varepsilon_{\alpha \alpha} R_\alpha - \varepsilon_{\alpha \alpha} ^2 \right) b_{11} + \left[ R_\alpha R_\beta \varepsilon_{\alpha \alpha} \varepsilon_{\beta \beta} + R_\alpha R_\beta \varepsilon_{\beta \beta} \varepsilon_{\alpha \alpha} \varepsilon_{\beta \beta} (R_\alpha + R_\beta) \right] \frac{b_{12} + b_{21}}{R_\alpha R_\beta} \tag{3-36}$$

$$+ \left( \varepsilon_{\beta \beta} \omega + 2 \varepsilon_{\alpha \alpha} \omega - \frac{2 \varepsilon_{\alpha \alpha} \omega}{R_\alpha} - \frac{\varepsilon_{\alpha \alpha} \omega}{R_\beta} \right) (b_{13} + b_{31}) + \frac{2}{R_\beta} (\varepsilon_{\beta \beta} \omega + 2 \varepsilon_{\alpha \alpha} \omega - \frac{2 \varepsilon_{\beta \beta} \omega}{R_\beta} + \varepsilon_{\beta \beta} \omega) (b_{23} + b_{32}) + \left( 4 \omega \tau - 2 \omega^2 \frac{R_\alpha + R_\beta}{R_\alpha R_\beta} \right) b_{33} .$$

$$Q_2 = \left( \varepsilon_{\alpha \alpha} R_\alpha^2 - 4 \varepsilon_{\alpha \alpha} \varepsilon_{\alpha \alpha} R_\alpha + 3 \varepsilon_{\alpha \alpha}^2 \right) \frac{b_{11}}{R_\alpha^2} + \left\{ \varepsilon_{\alpha \alpha} R_\alpha R_\beta \left( R_\alpha + R_\beta \right) + \varepsilon_{\alpha \alpha} \varepsilon_{\beta \beta} \left( R_\alpha^2 + R_\beta^2 + R_\alpha R_\beta \right) \right\} - \varepsilon_{\alpha \alpha} \varepsilon_{\beta \beta} R_\alpha R_\beta \left( R_\alpha + R_\beta \right) + \varepsilon_{\alpha \alpha} \varepsilon_{\beta \beta} \left( R_\alpha^2 + R_\beta^2 + R_\alpha R_\beta \right) \tag{3-37}$$
\[-\epsilon_{\beta\gamma} k_\alpha r_\alpha r_\beta (r_\alpha + r_\beta) \left\{ \frac{b_{12} + b_{21}}{R_\alpha^2 R_\beta^2} \right\} + \left\{ -\epsilon_{\alpha\gamma} \tau (r_\alpha + 3 r_\beta) \right\} \]

\[+ \epsilon_{\alpha\gamma} \omega \frac{R_\alpha^2 + r_\alpha r_\beta + r_\beta^2}{r_\alpha r_\beta} + 2 \tau k_\beta r_\alpha r_\beta - \omega k_\beta (r_\alpha + 2 r_\beta) \left\{ \frac{b_{13} + b_{31}}{R_\alpha r_\beta} \right\} \]

\[+ \left\{ r_\alpha^2 r_\beta - 4 \epsilon_{\beta\gamma} k_\beta r_\beta + 3 k_\beta^2 \right\} \frac{b_{22}}{R_\beta^2} + \left\{ -\epsilon_{\beta\gamma} \tau (3 r_\alpha + r_\beta) \right\} \]

\[+ \epsilon_{\beta\gamma} \omega \frac{3 r_\alpha + r_\alpha r_\beta + r_\beta^2}{r_\alpha r_\beta} + 2 \tau k_\beta r_\alpha r_\beta - \omega k_\beta (2 r_\alpha + r_\beta) \frac{b_{23} + b_{32}}{r_\alpha r_\beta} \]

\[+ \left\{ 4 \tau^2 r_\alpha r_\beta - 6 \tau \omega (r_\alpha + r_\beta) \right\} \]

\[+ \omega^2 \frac{3 r_\alpha + 2 r_\alpha r_\beta}{r_\alpha r_\beta} \left\{ \frac{b_{33}}{r_\alpha r_\beta} \right\} . \]

and the $b_{ij}$ are the coefficients of the matrix $B = T^T A T$.

The integral

\[ \int_{-t/2}^{t/2} Q_\perp z \, d\,z = 0 \]

After carrying out the integration with respect to $z$ there remains the surface integral.
The terms \( Q_0 \) and \( Q_2 \) contain the strains \( \varepsilon_{\alpha\alpha} \), \( \varepsilon_{\beta\beta} \) and \( \omega \), which are dimensionless quantities, and \( \kappa_{\alpha} \), \( \kappa_{\beta} \) and \( \tau \), which are of dimension \( L^{-1} \). In order to make all terms dimensionally similar we introduce the parameters

\[
\varepsilon'_{\alpha\alpha} = \frac{t}{2} \kappa_{\alpha} \tag{3-39}
\]

\[
\varepsilon'_{\beta\beta} = \frac{t}{2} \kappa_{\beta} \tag{3-40}
\]

\[
\omega' = t \tau \tag{3-41}
\]

The \( \varepsilon'_{\alpha\alpha} \) and \( \varepsilon'_{\beta\beta} \) represent the strains in the extreme fibers of the shell arising from the bending and \( \omega' \) is the shear strain in the extreme fibers caused by the torsion.

Introducing \( \varepsilon'_{\alpha\alpha} \), \( \varepsilon'_{\beta\beta} \) and \( \omega' \) into the expression for \( Q_2 \) and neglecting terms containing \( \frac{t}{R} \) or \( \left( \frac{t}{R} \right)^2 \) we obtain

\[
U_s = \frac{1}{2} \int_5 \left[ \varepsilon_{\alpha\alpha}^2 b_{11} + \varepsilon_{\alpha\alpha}' \varepsilon_{\beta\beta}' (b_{12} + b_{21}) + \varepsilon_{\beta\beta}^2 b_{22} + \varepsilon_{\alpha\alpha} \omega (b_{13} + b_{31}) \\
+ \varepsilon_{\beta\beta} \omega (b_{23} + b_{32}) + \omega^2 b_{33} \right] A_{\alpha} A_{\beta} \, d\alpha \, d\beta 
+ \frac{t}{6} \int_5 \left[ (\varepsilon_{\alpha\alpha}')^2 b_{11} + \varepsilon_{\alpha\alpha}' \varepsilon_{\beta\beta}' (b_{12} + b_{21}) + (\varepsilon_{\beta\beta}')^2 b_{22} + \varepsilon_{\alpha\alpha}' \omega' (b_{13} + b_{31}) \\
+ \varepsilon_{\beta\beta}' \omega' (b_{23} + b_{32}) + (\omega')^2 b_{33} \right] A_{\alpha} A_{\beta} \, d\alpha \, d\beta . \tag{3-42}
\]
The $\varepsilon'_{\alpha\alpha}$, $\varepsilon'_{\beta\beta}$ and $\omega'$ can now be eliminated from this equation by substituting equations (3-39), (3-40) and (3-41) back into equation (3-42).

Carrying out this substitution and expressing the result in matrix notation we obtain

$$U_S = U_m + U_b$$ (3-43)

where

$$U_m = \frac{t}{2} \int \int_s \{ e \}^T T^T A T \{ e \} A_{\alpha\beta} d\alpha d\beta$$ (3-44)

$$U_b = \frac{t^3}{24} \int \int_s \{ k \}^T T^T A T \{ k \} A_{\alpha\beta} d\alpha d\beta$$ (3-45)

$$\{ e \}^T = \{ \varepsilon_{\alpha\alpha}, \varepsilon_{\beta\beta}, \omega \}$$ (3-46)

$$\{ k \}^T = \{ k_{\alpha}, k_{\beta}, 2\tau \}$$ (3-47)

In these equations $U_m$ represents the strain energy due to the membrane stresses and $U_b$ represents the strain energy due to bending and torsion.

For an isotropic material we have $\theta = 0$ so that $T$ is the identity matrix and the $A$ matrix becomes

$$A = \frac{\overline{E}}{1 - \mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1 - \mu}{2} \end{bmatrix}$$ (3-48)
where $E$ is the modulus of elasticity of the material and $\mu$ is Poison's ratio.

The expression for the strain energy reduces in this case to

$$U_s = \frac{t}{2(1-\mu^2)} \int \left[ (\varepsilon_{\alpha\beta}^\alpha + \varepsilon_{\beta\alpha}^\beta)^2 - 2(1-\mu)(\varepsilon_{\alpha\beta}^\alpha \varepsilon_{\beta\alpha}^\beta - \frac{\omega^2}{4}) \right] A_{\alpha} A_{\beta} \, d\alpha \, d\beta$$

$$+ \frac{t^3 E}{24(1-\mu^2)} \int \left[ (\kappa_{\alpha}^\alpha + \kappa_{\beta}^\beta)^2 - 2(1-\mu)(\kappa_{\alpha}^\alpha \kappa_{\beta}^\beta - \tau^2) \right] A_{\alpha} A_{\beta} \, d\alpha \, d\beta. \tag{3-49}$$

3.3 The forces and moments in terms of the strains of the middle surface

The internal forces and moments acting in the shell can be expressed in terms of the strains of the middle surface. Consider the first variation of the strain energy.

This variation may be obtained from

$$U_s = \frac{1}{2} \int_V \{ \mathbf{G}_3 \}^T \{ \varepsilon_3 \} \, dV. \tag{3-50}$$

and is given by

$$\delta (U_s) = \frac{1}{2} \int_V \{ \mathbf{G}_3 \}^T \delta \{ \varepsilon_3 \} \, dV. \tag{3-51}$$

where $\delta (\quad)$ is the symbol of the variational increment.

The variation of the vector $\{ \varepsilon_3 \}$ can be obtained from equations (3-31), (3-32) and (3-33)
\[ \delta(\varepsilon_{xx}) = \frac{1}{1 + z/R_\alpha} \left[ \delta(\varepsilon_{\alpha\alpha}) + z \delta(\kappa_{\alpha}) \right] \]  

\[ \delta(\varepsilon_{yy}) = \frac{1}{1 + z/R_\beta} \left[ \delta(\varepsilon_{\beta\beta}) + z \delta(\kappa_{\beta}) \right] \]  

\[ \delta(\varepsilon_{xy}) = \frac{1}{(1 + z/R_\alpha)(1 + z/R_\beta)} \left[ (1 + \frac{z^2}{R_\alpha R_\beta}) \delta(\omega) \right. \]
\[ + 2 \left\{ 1 + \left( \frac{1}{R_\alpha} + \frac{1}{R_\beta} \right) \frac{z}{2} \right\} \delta(\tau) \]  

The membrane forces and moments per unit length of shell measured along the and lines are shown in Appendix C. The expressions for the membrane forces and moments can be substituted into equation (3-51). In this manner we find

\[ \delta(U_S) = \iint_s \left[ T_\alpha \delta(\varepsilon_{\alpha\alpha}) + T_\beta \delta(\varepsilon_{\beta\beta}) + S \delta(\omega) + M_\alpha \delta(\kappa_\alpha) \right. \]
\[ + M_\beta \delta(\kappa_\beta) + 2 H \delta(\omega) \left] \right| A_\alpha A_\beta d\alpha d\beta. \]  

This equation can be rewritten in matrix form by introducing

\[ \{G\}^T = \{ T_\alpha \ T_\beta \ S \} \]  

\[ \{H\}^T = \{ M_\alpha \ M_\beta \ H \} \]  

Equation (3-55) then becomes

\[ (U_S) = \iint_s \left[ \{G\}^T \delta(\{\varepsilon\}) + \{H\}^T \delta(\{\kappa\}) \right] A_\alpha A_\beta d\alpha d\beta. \]  

(3-58)
The first variation of the strain energy can also be obtained from equation (3-45)

\[
(U_s) = t \iint_{\mathbb{S}} \left[ \{e\}^T \delta \{e\} \right] A_\alpha A_\beta d\alpha d\beta \\
+ \frac{t^3}{12} \iint_{\mathbb{S}} \left[ \{k\}^T \delta \{k\} \right] A_\alpha A_\beta d\alpha d\beta.
\] (3-59)

By comparing the coefficients for the variational changes of equations (3-58) and (4-59) we find

\[
\{G\}^T = t \{e\}^T T^T A T
\] (3-60)

\[
\{H\}^T = \frac{t^3}{12} \{k\}^T T^T A T
\] (3-61)

In the case of an isotropic material \(T\) is the unit matrix, \(A\) is given in equation (3-48) and the equations (3-60) and (3-61) result in

\[
T_\alpha = \frac{\bar{E} t}{1 - \mu^2} \left( \varepsilon_{\alpha\alpha} + \mu \varepsilon_{\beta\beta} \right).
\] (3-62)

\[
T_\beta = \frac{\bar{E} t}{1 - \mu^2} \left( \varepsilon_{\beta\beta} + \mu \varepsilon_{\alpha\alpha} \right).
\] (3-63)

\[
S = \frac{\bar{E} t}{2(1+\mu)} \omega.
\] (3-64)

\[
M_\alpha = \frac{\bar{E} t^3}{12(1-\mu^2)} \left( \kappa_\alpha + \mu \kappa_\beta \right).
\] (3-65)

\[
M_\beta = \frac{\bar{E} t^3}{12(1-\mu^2)} \left( \kappa_\beta + \mu \kappa_\alpha \right).
\] (3-66)

\[
H = \frac{\bar{E} t^3}{12(1+\mu)} \tau.
\] (3-67)
CHAPTER IV

THE STIFFNESS MATRIX OF THE CURVED TRIANGULAR SHELL ELEMENT AND THE CORRESPONDING LOAD VECTOR

In this chapter the stiffness matrix of the curved triangular element defined in Chapter II is derived. Since each nodal point has five degrees of freedom the stiffness matrix is a 15 x 15 matrix. The corresponding load vector is next derived and has fifteen components.

The notation used in this chapter is that of Chapters I, II, and III.

4.1 The stiffness matrix

The derivation of the stiffness matrix of the curved triangular shell element presented here follows essentially the steps outlined in section 1.4, Chapter I.

The expressions for the strains of the middle surface in terms of the displacement components, u, v and w are shown in Appendix C. These equations can be written in matrix form

\[ \{ e \} = C \{ d \} \]  \hspace{1cm} (4-1)

where \( \{ e \} \) and \( \{ d \} \) are the vectors defined by equations (3-46) and (3-47).
The displacement functions \( \hat{u}, \hat{v} \) and \( \hat{w} \) of Chapter II can also be represented in matrix form

\[
\{d\} = D_1 \{f\}
\]  

where the components of the vector \( \{f\} \) are

\[
\hat{u}_s, \hat{v}_s, \hat{w}_s, \theta^*_s, \theta^*_s \quad (s = i, j, k)
\]  

Substitution of equation (4-5) into equation (4-1) results in

\[
\{e\} = CD_1 \{f\}
\]  

The vector \( \{f\} \) can be expressed in terms of the nodal displacements by

\[
\{f\} = R \{g\}
\]  

where the components of the vector \( \{g\} \) are the nodal displacements

\[
\hat{u}_s, \hat{v}_s, \hat{w}_s, \theta^*_s, \theta^*_s \quad (s = i, j, k).
\]  

The strains of the middle surface can now be expressed
in terms of the nodal displacements by substituting equation (4-8) into equation (4-7)

\[ \begin{bmatrix} e \\ g \end{bmatrix} = C D_1 R \begin{bmatrix} e \\ g \end{bmatrix} = N \begin{bmatrix} e \\ g \end{bmatrix} \] (4-9)

The \( N \) matrix is shown in Appendix D.

The expression for the strain energy of a thin shell is shown in equations (3-43), (3-44) and (3-45).

The strain energy due to the membrane stresses can be expressed in terms of the nodal displacements by substituting equation (4-9) into equation (3-44)

\[ U_m = \frac{1}{2} \begin{bmatrix} e \end{bmatrix}^T K_m \begin{bmatrix} e \end{bmatrix} \] (4-10)

where

\[ K_m = t \int_{S_{el}} (\mathbf{TN})^T \mathbf{A}(\mathbf{TN}) A_{\alpha} A_{\beta} d\alpha d\beta. \] (4-11)

and \( S_{el} \) is the surface area of the element.

The bending stresses can be treated in the same manner.

The \( \{ k \} \) vector of equation (3-47) can also be expressed in terms of the displacement vector \( \{ d \} \) (see Appendix C).

\[ \{ k \} = G \{ d \} \] (4-12)

where
The vector can be expressed in terms of the nodal displacements by substituting equations (4-5) and (4-8) into equation (4-12)

\[ \{ k \} = GD_1R \{ g \} = P \{ g \} \quad (4-14) \]

The P matrix is shown in Appendix D.

The strain energy due to bending and torsion equation (3-45) can now be represented by

\[ U_b = \frac{1}{2} \begin{bmatrix} g \end{bmatrix}^T K_b \begin{bmatrix} g \end{bmatrix} \quad (4-15) \]

where

\[ K_b = \frac{t^3}{12} \int_{S_{el}} (TP)^T A(TP) A_\alpha A_\beta d\alpha d\beta. \quad (4-16) \]

The stiffness matrix of the curved triangular shell element is now given by

\[ K = K_m + K_b \quad (4-17) \]
The integrands of equations (4-11) and (4-17) are very complex functions. The evaluation of the integrals is, therefore, most conveniently carried out numerically on an electronic computer. Appendix D shows all the necessary information to accomplish this.

4.2 The load vector

The definition of the potential energy is given in section 1.3.2. The contribution of the surface loads is given by (see equation (1-24))

\[- \int_{S_T} [T_x u + T_y v + T_z w] \, ds \quad (4-18)\]

The contribution of the applied loads on the shell element is thus

\[- \int_{S_{el}} (p_\alpha \, \hat{u} + p_\beta \, \hat{v} + p_n \, \hat{w}) \, A_{\alpha} A_{\beta} \, d\alpha \, d\beta \quad (4-19)\]

Where \( p_\alpha \), \( p_\beta \) and \( p_n \) are the components of the applied surface loads in the directions of \( \hat{e}_\alpha \), \( \hat{e}_\beta \) and \( \hat{e}_n \).

The equation can be brought in vector form by

\[- \int_{S_{el}} \{p\}^T \{d\} \, A_{\alpha} A_{\beta} \, d\alpha \, d\beta \quad (4-20)\]

where

\[\{p\}^T = \{p_\alpha \, p_\beta \, p_n\}\]  

(4-21)
By substituting equations (4-5) and (4-8) into equation (4-20) we obtain an expression of the potential energy of the applied loads in terms of the nodal displacements

$$-\iint_{S_{el}} \{p\}^T S \{g\} A_\alpha A_\beta d\alpha d\beta = -\{q\}^T \{g\}$$  \hspace{1cm} (4-22)

where

$$S = D_1 R$$  \hspace{1cm} (4-23)

and the load vector \(\{q\}\) is given by

$$\{q\}^T = \iint_{S_{el}} \{p\}^T S A_\alpha A_\beta d\alpha d\beta$$  \hspace{1cm} (4-24)

$$= \iint_{S_{el}} \{q_1\}^T A_\alpha A_\beta d\alpha d\beta$$  \hspace{1cm} (4-25)

where

$$\{q_1\}^T = \{p\}^T S.$$  \hspace{1cm} (4-26)

The vector \(\{q_1\}^T\) and the matrix \(S\) are shown in Appendix D.

Again it will be necessary to determine the components of the load vector \(\{q\}\) by numerical integration.
CHAPTER V

CONVERGENCE OF THE APPROXIMATE SOLUTIONS

In this chapter the convergence of the approximate solutions obtained with the displacement method described in the previous chapter is approached from the point of view of the Ritz method. Sufficient conditions on the displacements functions for the convergence of the potential energy are stated in the first section. The next two sections show that the assumed displacement functions satisfy these conditions. In the last two sections the proof of convergence is extended to the displacements themselves.

5.1 The conditions of relative completeness

We have previously demonstrated that the membrane displacement functions $\hat{u}$ and $\hat{v}$, and the normal displacement function $\hat{w}$ are continuous on the curved triangular shell element and across the element boundary curves and that their derivatives have discontinuities at the boundaries. Physically this means that the strains, stresses and moments are discontinuous on the boundaries of the elements. The displacement functions, therefore, can be used with the Ritz method, where the nodal displacements and rotations are the undetermined parameters.
The potential energy of the shell in terms of $\hat{u}$, $\hat{v}$ and $\hat{w}$ is given by the total potential energy of all the elements (see equations (1-58) and (1-59)).

Sufficient conditions for the convergence of the potential energy to that of the exact solution are that the displacement functions $\hat{u}$, $\hat{v}$ and $\hat{w}$ must be relatively complete. The proof of this is almost identical with that given for the one-dimensional case of Chapter I, section 1.3.1 and will be omitted here. The proof depends on the fact that a function which is continuous on a region, except at certain points which lie on a finite number of smooth curves, is integrable.

The conditions of relative completeness for the displacement function $u$ can be stated as follows: Let the middle surface $S$ of the shell structure be partitioned into triangular elements (see section 2.2) and let the boundary of this surface $S$ be defined by some of the boundaries of some of the elements. Furthermore, let $u$ be an arbitrary continuously differentiable function on $S$ which satisfies certain boundary conditions. The function $\hat{u}$ is relatively complete of order one if for every $\varepsilon > 0$ there exist a partition with $n$ nodal points and nodal displacements $\hat{u}_r$ ($r = 1,2,\ldots,n$) such that on each triangular element and its boundaries we have

\[
\left| \hat{u} - u \right| < \varepsilon \quad (5-1)
\]

\[
\left| \frac{\partial \hat{u}}{\partial a} - \frac{\partial u}{\partial a} \right| < \varepsilon \quad (5-2)
\]
On the boundaries of the elements the partial derivatives of \( \hat{u} \) of equations (5-2) and (5-3) represent the one sided derivatives.

The conditions of relative completeness for the \( \hat{v} \) function may be obtained from the ones shown above by replacing \( \hat{u} \) by \( \hat{v} \) and \( u \) by \( v \). The conditions of equations (5-2) and (5-3) mean that the slopes must be smooth functions on the elements; i.e., they must not oscillate violently on the element. Physically the fulfillment of these equations mean that the strains will vary smoothly on the element.

Similarly the conditions of relative completeness for the normal displacement function \( \hat{w} \) can be expressed as follows: Let \( w \) be an arbitrary function defined on the region \( S \), which is continuous together with its first and second order partial derivatives and satisfies certain boundary conditions. Again let the region \( S \) be partitioned as defined above. The function \( \hat{w} \) is relatively complete of order two if for every \( \varepsilon > 0 \) there exists a partition with \( n \) nodal points, nodal displacements \( \hat{w}_r \) and nodal rotations \( \theta_{ar} \) and \( \theta_{br} \) (\( r = 1, 2, \ldots, n \)) such that on each element we have

\[
\left| \frac{\partial \hat{w}}{\partial \beta} - \frac{\partial u}{\partial \beta} \right| < \varepsilon \quad (5-3)
\]

\[
\left| \frac{\partial \hat{w}}{\partial \alpha} - \frac{\partial w}{\partial \alpha} \right| < \varepsilon \quad (5-4)
\]

\[
\left| \frac{\partial \hat{w}}{\partial \alpha} - \frac{\partial w}{\partial \alpha} \right| < \varepsilon \quad (5-5)
\]
On the boundaries of the elements the partial derivatives of $\hat{w}$ of equations (5-7), (5-8) and (5-9) represent the one sided derivatives. Equations (5-7), (5-8) and (5-9) mean that the change in curvature must be smooth on the element. This in turn guarantees that the moments will vary smoothly on the element.

It should be noted that the Ritz method requires that the trial functions satisfy the geometric boundary conditions. This means that the boundary conditions on the functions $u$, $v$ and $w$ referred to above must be restricted to those boundary conditions which can be satisfied exactly by the corresponding displacement functions $\hat{u}$, $\hat{v}$ and $\hat{w}$. The proof of convergence of this and the following sections fails in the case the boundary conditions cannot be satisfied exactly by the functions $\hat{u}$, $\hat{v}$ and $\hat{w}$. However, the approximate solution may still converge to the exact solution since in the limit the boundary conditions may be satisfied.
5.2 Proof of the relative completeness of the membrane displacement functions

The conditions of relative completeness for the displacement function $u$ are shown in the previous section (equations (5-1), (5-2), and (5-3)). The function $u(\alpha, \beta)$ is an arbitrary continuously differentiable function on $S$.\footnote{It should be noted that by assuming that the first derivatives of the function $u$ are continuous on $S$, we exclude the case in which the loading in the direction of $\mathbf{e}$ consists of concentrated loads, since such loads would lead to singularities in the membrane displacements at the point of application of such loads. However, this restriction is not of great importance since we can always represent a concentrated load by a distributed load over a small area.} The assumed displacement function on a typical element with nodal points, $i$, $j$ and $k$ is (see equation (2-20))

$$\hat{u}(\alpha, \beta) = L_i \hat{u}_i + L_j \hat{u}_j + L_k \hat{u}_k \quad (5-10)$$

We can consider the values of the functions $\hat{u}$ and $u$ at each point of the element as coordinates in the direction of the unit normal vector $\mathbf{e}_n$. The functions $\hat{u}$ and $u$ can thus be thought of as surfaces lying above or below the middle surface of the shell. The condition of equation (5-1) means geometrically that it must be possible to make the distance (measured along $\mathbf{e}_n$) between the surfaces represented by the functions $\hat{u}$ and $u$ arbitrarily small. It is easy to see that this can be accomplished by a suitable choice of the size and shape of the elements and by choosing the nodal displacements such that
\[ \hat{u}_r = u(\alpha_r, \beta_r) \quad (r = i, j, k) \quad (5-11) \]

We can show this mathematically by expanding the nodal displacements of equation (5-11) into a Taylor series about an interior point of the element

\[ \hat{u}_r = u(\alpha_r, \beta_r) = u(\alpha, \beta) - (\alpha - \alpha_r) \frac{\partial u(\alpha, \beta)}{\partial \alpha} - (\beta - \beta_r) \frac{\partial u(\alpha, \beta)}{\partial \beta} \]

\[ + \frac{1}{2} \left[ (\alpha - \alpha_r) \frac{\partial^2 u(\alpha, \beta)}{\partial \alpha^2} + (\beta - \beta_r) \frac{\partial^2 u(\alpha, \beta)}{\partial \beta^2} \right] u(\alpha, \beta) + \ldots \quad (5-12) \]

This equation can be written in the form

\[ \hat{u}_r = u(\alpha, \beta) + 0(h) \quad (5-13) \]

where \( h \) is the maximum difference of the coordinates of the nodal points, i.e.,

\[ h = \text{Max} \left( \left| \alpha_r - \alpha_s \right|, \left| \beta_r - \beta_s \right| \right); \quad s = i, j, k \quad (5-14) \]

Substituting equation (5-13) into equation (5-10) we have

\[ \hat{u} = L_i [u(\alpha, \beta) + 0(h)] + L_j [u(\alpha, \beta) + 0(h)] + L_k [u(\alpha, \beta) + 0(h)] \quad (5-15) \]

Now \( 0 \leq L_\ell \leq 1, \quad \ell = i, j, k \) on the element and the boundary curves. In addition we have that

\[ L_i + L_j + L_k = 1 \quad (5-16) \]
So that we can write equation (5-15) in the form

\[ \hat{u} = u(\alpha, \beta) + o(h) \]  \hspace{1cm} (5-16)

If we now refine the partition such that \( h \to 0 \) we have

\[ \hat{u} \to u(\alpha, \beta) \] \hspace{1cm} (5-17)

The refinement of the partition can be accomplished in a practical manner as follows: Subdivide each triangular element by joining the midpoints of the boundary curves by curves having the equation \( \beta = m\alpha + n \). In this way the elements will be reduced in size gradually and uniformly.

The partial derivative of \( \hat{u} \) with respect to \( \alpha \) is given by

\[ \frac{\partial \hat{u}}{\partial \alpha} = \frac{1}{2\Delta} \left( b_i \hat{u}_i + b_j \hat{u}_j + b_k \hat{u}_k \right) \] \hspace{1cm} (5-18)

The Taylor expansion of equation (5-12) can be written in the form

\[ \hat{\alpha}_r = u(\alpha, \beta) - (\alpha - \alpha_r) \frac{\partial u(\alpha, \beta)}{\partial \alpha} - (\beta - \beta_r) \frac{\partial u(\alpha, \beta)}{\partial \beta} + O(h^2). \] \hspace{1cm} (5-19)

Substituting equation (5-19) into equation (5-18) gives

\[ \frac{\partial \hat{u}}{\partial \alpha} = \frac{b_i}{2\Delta} \left[ u(\alpha, \beta) - (\alpha - \alpha_i) \frac{\partial u(\alpha, \beta)}{\partial \alpha} - (\beta - \beta_i) \frac{\partial u(\alpha, \beta)}{\partial \beta} + O(h^2) \right] + \]
\[ + \frac{b_i}{2\Delta} \left[ u(\alpha, \beta) - (\alpha - \alpha_i) \frac{\partial u(\alpha, \beta)}{\partial \alpha} - (\beta - \beta_i) \frac{\partial u(\alpha, \beta)}{\partial \beta} + O(h^2) \right] \]

\[ + \frac{b_k}{2\Delta} \left[ u(\alpha, \beta) - (\alpha - \alpha_k) \frac{\partial u(\alpha, \beta)}{\partial \alpha} - (\beta - \beta_k) \frac{\partial u(\alpha, \beta)}{\partial \beta} + O(h^2) \right] \]  \hspace{1cm} (5-20)

Computations will show that

\[ b_i + b_j + b_k = 0 \]  \hspace{1cm} (5-21)

\[ \frac{1}{2\Delta} \left( \alpha_i b_i + \alpha_j b_j + \alpha_k b_k \right) = 1 \]  \hspace{1cm} (5-22)

and

\[ \frac{1}{2\Delta} \left( \beta_i b_i + \beta_j b_j + \beta_k b_k \right) = 0 \]  \hspace{1cm} (5-23)

Substituting equations (5-21), (5-22) and (5-23) into equation (5-20) we have

\[ \frac{\partial \hat{u}}{\partial \alpha} = -\frac{\partial u(\alpha, \beta)}{\partial \alpha} + O(h) \]  \hspace{1cm} (5-24)

When the size of the element is reduced such that

\[ h \to 0 \] we have

\[ \frac{\partial \hat{u}}{\partial \alpha} \to -\frac{\partial u(\alpha, \beta)}{\partial \alpha} \]  \hspace{1cm} (5-25)

Similarly we can show when \( h \to 0 \)

\[ \frac{\partial \hat{u}}{\partial \beta} \to -\frac{\partial u(\alpha, \beta)}{\partial \beta} \]  \hspace{1cm} (5-26)
It should be noted that for certain choices of elements one or more of the $b_l$ ($l = 1, j, k$) can be very large. This is the case when long narrow elements are used. It can be seen from equation (5-20) that the use of such long narrow elements will give a slower rate of convergence of the first partial derivatives of $\hat{u}$. Similarly, if the refinement of the partition results in long narrow elements slower convergence of the partial derivatives of $u$ can be expected.

5.3 Proof of the relative completeness of the normal displacement function

The conditions of relative completeness for the normal displacement function $w$ are stated in section 5.1 (equations (5-4) through (5-9)). The function $w(\alpha, \beta)$ is an arbitrary continuously differentiable function on $S$. The assumed displacement function $\hat{w}$ (see equation (2-31) of section 2.3.2) can be written in the following equivalent form

\[
\hat{w} = - F_{\alpha i} \frac{\partial \hat{w}(\alpha_i, \beta_i)}{\partial \beta} - F_{\beta i} \frac{\partial \hat{w}(\alpha_i, \beta_i)}{\partial \alpha} \\
- F_{\alpha j} \frac{\partial \hat{w}(\alpha_j, \beta_j)}{\partial \beta} - F_{\beta j} \frac{\partial \hat{w}(\alpha_j, \beta_j)}{\partial \alpha} \\
- F_{\alpha k} \frac{\partial \hat{w}(\alpha_k, \beta_k)}{\partial \beta} - F_{\beta k} \frac{\partial \hat{w}(\alpha_k, \beta_k)}{\partial \alpha} \\
+ \left[ L_i + \psi_{ij} + \psi_{ik} - \psi_{ji} - \psi_{ki} \right] \hat{w}_i \\
+ \left[ L_j + \psi_{jk} + \psi_{ij} - \psi_{kj} - \psi_{ij} \right] \hat{w}_j \\
+ \left[ L_k + \psi_{ki} + \psi_{kj} - \psi_{ik} - \psi_{jk} \right] \hat{w}_k.
\] (5-27)
where \( F_{\alpha r} \) and \( F_{\beta r} \) are given by

\[
F_{\alpha r} = \beta_{rs} \psi_{rs} + \beta_{rt} \psi_{rt} \quad (5-28)
\]

\[
F_{\beta r} = \alpha_{rs} \psi_{rs} + \alpha_{rt} \psi_{rt} \quad r, s, t = 1, j, k \quad (5-29)
\]

The functions \( L_1, L_j \) and \( L_k \) are defined on the closed regions formed by the triangular elements. On these regions we have

\[
0 \leq L_l \leq 1 \quad , \quad l = 1, j, k \quad (5-30)
\]

From the definition of the \( \psi \) functions (equation (2-33), section 2.3.2) we see that, on using the relations developed above we have

\[
\begin{align*}
|\psi_{1j}| &= \left| L_1^2 L_j + \frac{1}{2} L_1 L_j L_k \right| \\
&\leq \left| L_1^2 L_j \right| + \frac{1}{2} \left| L_1 L_j L_k \right| \leq \frac{1}{2}. \quad (5-31)
\end{align*}
\]

In a similar manner we can show that all the functions as well as their first and second partial derivatives are bounded on the regions formed by the triangular shell elements. In order to prove that the normal displacement function \( w \) is relatively complete we set the nodal displacements and rotations equal to those of the arbitrary function \( w(\alpha, \beta) \) at the nodal points, i.e., \( \hat{w}_l = w(\alpha, \beta) \),

\[
\frac{\partial \hat{w}}{\partial \alpha} = \frac{\partial w(\alpha, \beta)}{\partial \alpha}, \quad \text{etc.}, \quad \text{for} \quad l = 1, j, k.
\]
Hence

\[ w = - F_{\alpha i} \frac{\partial w(\alpha_i, \beta_i)}{\partial \beta} - F_{\beta i} \frac{\partial w(\alpha_i, \beta_i)}{\partial \alpha} \]

\[ - F_{\alpha j} \frac{\partial w(\alpha_j, \beta_i)}{\partial \beta} - F_{\beta j} \frac{\partial w(\alpha_j, \beta_i)}{\partial \alpha} \]

\[ - F_{\alpha k} \frac{\partial w(\alpha_k, \beta_k)}{\partial \beta} - F_{\beta k} \frac{\partial w(\alpha_k, \beta_k)}{\partial \alpha} \]

\[ + \left( L_i + \psi_{ij} + \psi_{ik} - \psi_{ji} - \psi_{ki} \right) w(\alpha_i, \beta_i) \]

\[ + \left( L_j + \psi_{jk} + \psi_{ji} - \psi_{kj} - \psi_{ij} \right) w(\alpha_j, \beta_j) \]

\[ + \left( L_k + \psi_{ki} + \psi_{kj} - \psi_{ik} - \psi_{jk} \right) w(\alpha_k, \beta_k). \quad (5-32) \]

Following the procedure used in the proof of section 5.2 we now expand the \( w(\alpha'_l, \beta'_l) \), \( \frac{\partial w}{\partial \alpha} (\alpha'_l, \beta'_l) \) and \( \frac{\partial w}{\partial \beta} (\alpha'_l, \beta'_l) \) for \( l = i, j, k \) into Taylor series about an arbitrary point \( (\alpha, \beta) \) of the element.

\[ w(\alpha'_l, \beta'_l) = w(\alpha, \beta) - (\alpha - \alpha'_l) \frac{\partial w(\alpha, \beta)}{\partial \alpha} - (\beta - \beta'_l) \frac{\partial w(\alpha, \beta)}{\partial \beta} \]

\[ + \frac{1}{2!} \left[ (\alpha - \alpha'_l) \frac{\partial}{\partial \alpha} + (\beta - \beta'_l) \frac{\partial}{\partial \beta} \right]^2 w(\alpha, \beta) + \cdots \]

and

\[ \frac{\partial w(\alpha'_l, \beta'_l)}{\partial \alpha} = \frac{\partial w(\alpha, \beta)}{\partial \alpha} - (\beta - \beta'_l) \frac{\partial^2 w(\alpha, \beta)}{\partial \alpha \partial \beta} \]

\[ - (\alpha - \alpha'_l) \frac{\partial^2 w(\alpha, \beta)}{\partial \alpha^2} + \cdots \]

Equation (5-36) can be written in the form

\[ w(\alpha'_l, \beta'_l) = w(\alpha, \beta) + O(h) \quad (5-35) \]
Noting that the $\psi$ functions are bounded on the elements, that the $\beta_{l,m}$ and $\alpha_{l,m}$ $(l,m = 1,J,k)$ are smaller than or equal to $h$ and that \( \frac{\partial w(\alpha_l, \beta_l)}{\partial \alpha} \) and \( \frac{\partial w(\alpha_l, \beta_l)}{\partial \beta} \) $(l = 1,J,k)$ are constants we can write for equation (5-32)

\[
\hat{w}(\alpha, \beta) = (L_i + L_j + L_k) w(\alpha, \beta) + O(h)
\]

\[
+ (L_i + \psi_{ij} + \psi_{ik} - \psi_{ji} - \psi_{ki}) O(h)
\]

\[
+ (L_j + \psi_{jk} + \psi_{ji} - \psi_{kj} - \psi_{ij}) O(h)
\]

(5-36)

\[
+ (L_k + \psi_{ki} + \psi_{kj} - \psi_{ik} - \psi_{jk}) O(h).
\]

or

\[
\hat{w}(\alpha, \beta) = w(\alpha, \beta) + O(h)
\]

(5-37)

when $h \rightarrow 0$ we have that

\[
\hat{w}(\alpha, \beta) \rightarrow w(\alpha, \beta)
\]

(5-38)

In a similar way we can show that the first and second derivatives of $\hat{w}(\alpha, \beta)$ converge to those of the arbitrary function $w(\alpha, \beta)$. Since the proof involves a great amount of algebraic operations we show it here in abbreviated form for the second derivative with respect to $\alpha$ only.

From equation (5-27) we obtain

\[
\frac{\partial^2 \hat{w}}{\partial \alpha^2} = - \left( \beta_{ij} \frac{\partial^2 \psi_{ij}}{\partial \alpha^2} + \beta_{ik} \frac{\partial^2 \psi_{ik}}{\partial \alpha^2} \right) \frac{\partial w(\alpha_l, \beta_l)}{\partial \beta}
\]

\[
- \left( \alpha_{ij} \frac{\partial^2 \psi_{ij}}{\partial \alpha^2} + \alpha_{ik} \frac{\partial^2 \psi_{ik}}{\partial \alpha^2} \right) \frac{\partial w(\alpha_l, \beta_l)}{\partial \alpha}
\]
- \left( \beta_{jk} \frac{\partial^2 \psi_{jk}}{\partial \alpha^2} + \beta_{ij} \frac{\partial^2 \psi_{ji}}{\partial \alpha^2} \right) \partial w(\alpha_j, \beta_{j})

- \left( \alpha_{jk} \frac{\partial^2 \psi_{jk}}{\partial \alpha^2} + \alpha_{ij} \frac{\partial^2 \psi_{ji}}{\partial \alpha^2} \right) \partial w(\alpha_j, \beta_{j})

- \left( \beta_{ki} \frac{\partial^2 \psi_{ki}}{\partial \alpha^2} + \beta_{kj} \frac{\partial^2 \psi_{kj}}{\partial \alpha^2} \right) \partial w(\alpha_k, \beta_{k})

- \left( \alpha_{ki} \frac{\partial^2 \psi_{ki}}{\partial \alpha^2} + \alpha_{kj} \frac{\partial^2 \psi_{kj}}{\partial \alpha^2} \right) \partial w(\alpha_k, \beta_{k})

+ \left( \frac{\partial^2 \psi_{ii}}{\partial \alpha^2} + \frac{\partial^2 \psi_{ik}}{\partial \alpha^2} - \frac{\partial^2 \psi_{ji}}{\partial \alpha^2} - \frac{\partial^2 \psi_{ki}}{\partial \alpha^2} \right) \partial \hat{w}_i

+ \left( \frac{\partial^2 \psi_{jk}}{\partial \alpha^2} + \frac{\partial^2 \psi_{ji}}{\partial \alpha^2} - \frac{\partial^2 \psi_{kj}}{\partial \alpha^2} - \frac{\partial^2 \psi_{ik}}{\partial \alpha^2} \right) \partial \hat{w}_j

+ \left( \frac{\partial^2 \psi_{ki}}{\partial \alpha^2} + \frac{\partial^2 \psi_{kj}}{\partial \alpha^2} - \frac{\partial^2 \psi_{ik}}{\partial \alpha^2} - \frac{\partial^2 \psi_{jk}}{\partial \alpha^2} \right) \partial \hat{w}_k \quad (5-39)

We now write equations (5-33) and (5-34) in the form

\hat{w}_f = \psi(\alpha_f, \beta_f) = \psi(\alpha_f, \beta_f) - (\alpha_f - \alpha_f) \frac{\partial \psi(\alpha_f, \beta_f)}{\partial \alpha} - (\beta_f - \beta_f) \frac{\partial \psi(\alpha_f, \beta_f)}{\partial \beta}

\quad \quad \quad + \frac{1}{2} \left[ (\alpha_f - \alpha_f) \frac{\partial}{\partial \alpha} + (\beta_f - \beta_f) \frac{\partial}{\partial \beta} \right]^2 \psi(\alpha_f, \beta_f) + O(h^3). \quad (5-40)

\frac{\partial \psi(\alpha_f, \beta_f)}{\partial \alpha} = \frac{\partial \psi(\alpha_f, \beta_f)}{\partial \alpha} - (\beta_f - \beta_f) \frac{\partial^2 \psi(\alpha_f, \beta_f)}{\partial \alpha \partial \beta} - (\alpha_f - \alpha_f) \frac{\partial^2 \psi(\alpha_f, \beta_f)}{\partial \alpha^2} + O(h^3). \quad (5-41)
\[
\frac{\partial^2 w(\alpha_2, \beta_2)}{\partial \beta^2} = \frac{\partial w(\alpha, \beta)}{\partial \beta} - (\alpha - \alpha_2) \frac{\partial^2 w(\alpha, \beta)}{\partial \alpha \partial \beta} - (\beta - \beta_2) \frac{\partial^2 w(\alpha, \beta)}{\partial \beta^2} + O(h^3). \tag{5-42}
\]

for \( l = 1, j, k. \)

Substituting equations (5-40), (5-41) and (5-42) into equation (5-39) gives

\[
\frac{\partial^2 \hat{w}}{\partial \alpha^2} = \left[ \beta_{ij} \frac{\partial^2 \psi_{ij}}{\partial \alpha^2} + \beta_{ik} \frac{\partial^2 \psi_{ik}}{\partial \alpha^2} \right] \left[ \frac{\partial w}{\partial \beta} - (\beta - \beta_i) \frac{\partial^2 w}{\partial \beta^2} \right] - (\alpha - \alpha_i) \frac{\partial^2 w}{\partial \alpha \partial \beta} + O(h^2). \right]
\]

\[
- \left[ \alpha_{ij} \frac{\partial^2 \psi_{ij}}{\partial \alpha^2} + \alpha_{ik} \frac{\partial^2 \psi_{ik}}{\partial \alpha^2} \right] \left[ \frac{\partial w}{\partial \beta} - (\beta - \beta_i) \frac{\partial^2 w}{\partial \beta^2} \right] - (\alpha - \alpha_j) \frac{\partial^2 w}{\partial \alpha \partial \beta} + O(h^2). \right]
\]

\[
- \left[ \beta_{jk} \frac{\partial^2 \psi_{jk}}{\partial \alpha^2} + \beta_{ji} \frac{\partial^2 \psi_{ji}}{\partial \alpha^2} \right] \left[ \frac{\partial w}{\partial \beta} - (\beta - \beta_j) \frac{\partial^2 w}{\partial \beta^2} \right] - (\alpha - \alpha_j) \frac{\partial^2 w}{\partial \alpha \partial \beta} + O(h^2). \right]
\]

\[
- \left[ \alpha_{jk} \frac{\partial^2 \psi_{jk}}{\partial \alpha^2} + \alpha_{ji} \frac{\partial^2 \psi_{ji}}{\partial \alpha^2} \right] \left[ \frac{\partial w}{\partial \beta} - (\beta - \beta_j) \frac{\partial^2 w}{\partial \beta^2} \right] - (\alpha - \alpha_k) \frac{\partial^2 w}{\partial \alpha \partial \beta} + O(h^2). \right]
Collecting terms and simplifying the above equation leads to

\[
\frac{\partial^2 \hat{\omega}}{\partial \alpha^2} = -\left[ \frac{\alpha_{ij}^2}{2} \frac{\partial^2 \hat{\psi}_{ij}}{\partial \alpha^2} \left( \psi_{ij} + \psi_{ji} \right) + \frac{\alpha_{ik}^2}{2} \frac{\partial^2 \hat{\psi}_{ik}}{\partial \alpha^2} \left( \psi_{ik} + \psi_{ki} \right) \right] \frac{\partial^2 \hat{w}}{\partial \alpha^2} + \frac{\alpha_{jk}^2}{2} \frac{\partial^2 \hat{\psi}_{jk}}{\partial \alpha^2} \left( \psi_{jk} + \psi_{kj} \right) \right] \frac{\partial^2 \hat{w}}{\partial \alpha^2} + \frac{\beta_{ij}^2}{2} \frac{\partial^2 \hat{\psi}_{ij}}{\partial \alpha^2} \left( \psi_{ij} + \psi_{ji} \right) + \frac{\beta_{ik}^2}{2} \frac{\partial^2 \hat{\psi}_{ik}}{\partial \alpha^2} \left( \psi_{ik} + \psi_{ki} \right) + \frac{\beta_{jk}^2}{2} \frac{\partial^2 \hat{\psi}_{jk}}{\partial \alpha^2} \left( \psi_{jk} + \psi_{kj} \right) \right] \frac{\partial^2 \hat{w}}{\partial \alpha^2} + O(\alpha^4) \]
Observing that

\[ \psi_{rs} + \psi_{sr} = L_p L_s \quad (r, s = i, j, k) \]  

and carrying out the indicated differentiation we obtain from equation (5-44)

\[
\frac{\partial^2 \hat{W}}{\partial \alpha^2} = -\frac{1}{4\Delta} \left[ \alpha_{ij}^2 b_i b_j + \alpha_{ik}^2 b_i b_k + \alpha_{jk}^2 b_j b_k \right] \frac{\partial^2 w}{\partial \alpha^2} \\
- \frac{1}{4\Delta} \left[ \beta_{ij}^2 b_i b_j + \beta_{ik}^2 b_i b_k + \beta_{jk}^2 b_j b_k \right] \frac{\partial^2 w}{\partial \beta^2} \\
- \frac{1}{4\Delta} \left[ \alpha_{ij} \beta_{ij} b_i b_j + \alpha_{ik} \beta_{ik} b_i b_k + \alpha_{jk} \beta_{jk} b_j b_k \right] \frac{\partial^2 w}{\partial \alpha^2 \partial \beta} + O(h^3). 
\]  

(5-46)

On substituting the values of \( b_1, b_j \) and \( b_k \) into equation (5-46) we finally obtain

\[
\frac{\partial^2 \hat{W}}{\partial \alpha^2} = \frac{\partial^2 w(\alpha, \beta)}{\partial \alpha^2} + O(h^3). 
\]  

(5-47)

When \( h \to 0 \) we have \( \frac{\partial^2 \hat{W}}{\partial \alpha^2} \to \frac{\partial^2 w(\alpha, \beta)}{\partial \alpha^2} \).

The shape of the elements and the manner in which the refinement of the partition is accomplished will influence the rate of convergence. The reasons for this are identical with the ones shown in section 5.2.
5.4 Proof of convergence of the normal displacements to the exact solution

We have seen in the preceding sections that the assumed displacement functions \( \hat{u}, \hat{v}, \) and \( \hat{w} \) are relatively complete and that this is a sufficient condition for the convergence of the potential energy to that of the exact solution. However, convergence of the potential energy does not imply convergence of the displacements. Fortunately, the proof can be extended to the normal displacements. The proof given here is essentially due to Trefftz (ref. 37) who applied it to plate bending problems.

In Chapter I we defined the potential energy. For the shell the potential energy is

\[
II = U_s - \int_S \left( p_\alpha u + p_\beta v + p_n w \right) A_\alpha A_\beta \, d\alpha \, d\beta \tag{5-48}
\]

where we have omitted the body forces.

Clayperon's Theorem (ref. 6) states that the strain energy of a body which is in equilibrium under a given system of forces is equal to one-half the work that would be done by these forces acting through the displacements from the unstressed state to the state of equilibrium. It follows from this that the minimum potential energy is given by

\[
II_{\text{min}} = -\frac{1}{2} \int_S \left( p_\alpha u + p_\beta v + p_n w \right) A_\alpha A_\beta \, d\alpha \, d\beta . \tag{5-49}
\]
We now consider a particular shell and add to the original loading \( (p_\alpha, p_\beta, p_n) \) a concentrated normal load acting at an arbitrary point \((\alpha, \beta)\) of the shell. The shell problem consisting of original loading and the concentrated normal load \( \gamma \) is called the "associated problem."

If we place the concentrated load \( \gamma \) on the shell by itself the work done by it will be \( \frac{1}{2} \gamma^2 G(\alpha, \beta) \), where \( G(\alpha, \beta) \) is the normal deflection of the point \((\alpha, \beta)\) due to a unit load at that point. In view of the Kirchhoff assumption \( \sigma_{zz} = 0 \) (see section 3.2) this deflection will be finite.

The loads \( p_\alpha \), \( p_\beta \), and \( p_n \) are now placed on the shell in addition to the concentrated load \( \gamma \). The work done by the loads \( p_\alpha \), \( p_\beta \), and \( p_n \) is equal to \( \frac{1}{2} \iint_{S} (p_\alpha u + p_\beta v + p_n w) A_\alpha A_\beta \, d\alpha \, d\beta \) while the concentrated load \( \gamma \) does work equal to \( \gamma w(\alpha, \beta) \).

The total potential energy due to \( p_\alpha \), \( p_\beta \), \( p_n \) and \( \gamma \) is now

\[
II_A(\alpha, \beta, \gamma) = -\gamma w(\alpha, \beta) - \frac{1}{2} \gamma^2 G(\alpha, \beta) + \frac{1}{2} \iint_{S} (p_\alpha u + p_\beta v + p_n w) A_\alpha A_\beta \, d\alpha \, d\beta.
\]

The deflection at point \((\alpha, \beta)\) may now be expressed in terms of the potential energy \( II_A \) of the "associated problem" by first letting \( \gamma = 1 \) and then \( \gamma = -1 \). We then obtain

\[
II_A(\alpha, \beta, 1) = -w(\alpha, \beta) - \frac{1}{2} G(\alpha, \beta) + \frac{1}{2} \iint_{S} (p_\alpha u + p_\beta v + p_n w) A_\alpha A_\beta \, d\alpha \, d\beta.
\]
and

$$II_A(\alpha, \beta, -1) = w(\alpha, \beta) - \frac{1}{2} G(\alpha, \beta)$$

$$- \frac{1}{2} \int \int_{S} p_{\alpha} u + p_{\beta} v + p_{n} w A_{\alpha} A_{\beta} d\alpha d\beta \quad (5-52)$$

We now subtract the equation (5-51) from equation (5-52) and obtain

$$w(\alpha, \beta) = \frac{1}{2} [II_A(\alpha, \beta, 1) - II_A(\alpha, \beta, -1)] \quad (5-53)$$

The normal displacement $w(\alpha, \beta)$ at an arbitrary point $(\alpha, \beta)$ due to the loading $p_{\alpha}, p_{\beta}$, and $p_{n}$ has thus been expressed in terms of the potential energy of two of the associated problems. In the above development the displacements $u, v$ and $w$ represented the exact displacements of the particular shell problem considered.

We now repeat the above development using the assumed displacement functions. Again we obtain the potential energy of the problem, namely (see section 1.4)

$$II_n = \frac{1}{2} \{ g \}^T K \{ g \} - \{ g \}^T \{ q \} \quad (5-54)$$

where the subscript $n$ refers to the particular partition of the shell with $n$ nodal points.

If we minimize the potential energy equation (5-54) with respect to the nodal displacements we obtain

$$K \{ g \} = \{ q \} \quad (5-55)$$
We now multiply both sides of this equation by $\{g\}^T$. This results in

$$\{g\}^T K \{g\} = \{g\}^T \{q\} \quad (5-56)$$

On substituting this equation into equation (5-54) we find

$$II_{n_{\text{min}}} = -\frac{1}{2} \{g\}^T \{q\} \quad (5-57)$$

The minimum potential energy, obtained by applying the Ritz method and using the assumed displacement functions $\hat{u}$, $\hat{v}$ and $\hat{w}$ is thus equal to one-half of the negative of the work done by the applied forces in moving through the displacements $\hat{u}$, $\hat{v}$ and $\hat{w}$.

We now consider the "associated problem" and obtain for it the minimum potential energy by the Ritz method. Using equation (5-57) we find that in this case the minimum potential energy obtained by the Ritz method is equal to

$$II_{n_{A}}(\alpha, \beta, \gamma) = -\gamma w_n(\alpha, \beta) - \frac{1}{2} \{q\}^T \{g\} - \frac{1}{2} \{q_{\gamma}\}^T \{g\} \quad (5-58)$$

where $\{q_{\gamma}\}$ is the load vector which results if we use as the loading the concentrated load $\gamma$ at the point $(\alpha, \beta)$ on the middle surface of the shell. We now apply the same sequence of loading as in the case of the exact displacements $u$, $v$ and $w$ and determine the potential energy with $\gamma = 1$ and $\gamma = -1$. This results in an
equation for \( \hat{w}_n(\alpha,\beta) \) similar to equation (5-53) namely

\[
\hat{w}_n(\alpha,\beta) = \frac{1}{2} \left[ II_{nA}(\alpha,\beta,-1) - II_{nA}(\alpha,\beta,+1) \right] \tag{5-59}
\]

We have thus obtained an expression for the approximate normal displacement \( \hat{w}_n \) in terms of the minimum potential energy of the approximate solution, which is in every respect similar to the expression for the exact displacement \( w \) shown in equation (5-53). The relative completeness of the assumed displacement functions \( \hat{u}, \hat{v} \) and \( \hat{w} \) guarantees the convergency of the minimum potential energy obtained by the Ritz method to the potential energy of the exact solution; i.e., when \( h \to 0 \) we have

\[
II_{nA}(\alpha,\beta,-1) \to II_A(\alpha,\beta,-1) \tag{5-60}
\]

and

\[
II_{nA}(\alpha,\beta,+1) \to II_A(\alpha,\beta,+1) \tag{5-61}
\]

It now follows from equations (5-53), (5-59), (5-60) and (5-61) that when \( h \to 0 \)

\[
\hat{w}_n(\alpha,\beta) \to w(\alpha,\beta) \tag{5-62}
\]

Furthermore, the convergence of the minimum potential energy \( II_{nA}(\alpha,\beta,-1) \) to \( II_A(\alpha,\beta,-1) \) is monotone, since the approximation improves or at worst, remains unchanged with each refinement of the partition; i.e., with each component added to the generalized displacement vector \( \{g\}_j \). In addi-
tion the potential energy is a continuous function of the variables \( \alpha \) and \( \beta \). Dini's Theorem (ref. 38) asserts that a convergent sequence of continuous functions will converge uniformly if it approximates a continuous function monotonically. Hence \( II_{nA}(\alpha, \beta, -1) \) and \( II_{nA}(\alpha, \beta, +1) \) converge uniformly to \( II_A(\alpha, \beta, -1) \) and \( II_A(\alpha, \beta, +1) \) respectively. Furthermore, it follows that \( \hat{w}_n(\alpha, \beta) \) converges uniformly to \( w(\alpha, \beta) \).

The convergence of \( \hat{w} \) to \( w \) implies that the derivatives of \( \hat{w} \) converge to the derivatives of \( w \). The discontinuities in the moments at the boundaries of the elements will thus approach zero in the limit. However, the convergence of \( \hat{w} \) to \( w \) does not imply that if for a certain finite partition \( \hat{w} \) gives a good approximation to \( w \) that the derivatives of \( \hat{w} \) give good approximations to the derivatives of \( w \). It is the relative completeness of \( \hat{w} \) which ensures that a good approximation of \( w \) implies a good approximation of the first two derivatives of \( w \). Physically this means that if a good approximation to the displacement \( w \) is obtained we can expect a good approximation to the moments and stresses. To obtain a good approximation in regions where high stress gradients exist a fine partition will be required.

5.5 Proof of convergence of the "average" membrane displacements to the exact solution

The proof of the convergence of the normal displacement function \( \hat{w} \) cannot be used in the same way for the membrane
displacements $\hat{u}$ and $\hat{v}$. In the proof of the previous section we obtained the deformation $G(\alpha, \beta)$ due to a unit normal concentrated load at point $(\alpha, \beta)$ of the middle surface. If we place a unit concentrated load tangent to the middle surface of the shell we obtain an "infinite" deformation at the point of application of the load. Hence the associated problem does not have a solution and the proof cannot be used in this form. However, we can consider the "average" deflection over a small portion $S_1$ of the surface. Suppose that $\gamma$ is a load per unit area which is applied on $S_1$ in the direction of the $\hat{u}$ displacements, i.e., tangential to the shell middle surface and parallel to an $\alpha$-line. Let us denote the displacements due to $\gamma$ on $S_1$ by $u^*, v^*$ and $w^*$. The potential energy of the associated problem becomes in this case

$$II_A(\alpha, \beta, \gamma) = -\gamma \iint_{S_1} u(\alpha, \beta) A_\alpha A_\beta \, d\alpha \, d\beta - \frac{1}{2} \gamma^2 \iint_{S_1} u^* A_\alpha A_\beta \, d\alpha \, d\beta$$

$$- \frac{1}{2} \iint_{S_1} (p_\alpha u + p_\beta v + p_n w) A_\alpha A_\beta \, d\alpha \, d\beta \quad (5-63)$$

We now let $\gamma = -1$ and $\gamma = 1$ and obtain in the same way as before

$$\iint_{S_1} u(\alpha, \beta) A_\alpha A_\beta \, d\alpha \, d\beta = \frac{1}{2} [II_A(S_1, -1) - II_A(S_1, +1)] \quad (5-64)$$

The minimum potential energy of the associated problem obtained by the Ritz method gives

$$II_{nA}(S_1, \gamma) = -\gamma \iint_{S_1} \hat{u}_n(\alpha, \beta) \, dS - \frac{1}{2} \begin{bmatrix} q \end{bmatrix}^T \{g\} - \frac{1}{2} \begin{bmatrix} q \end{bmatrix}^T \{g\}$$

$$\quad (5-65)$$
and hence

\[ \int_{S_1} \hat{u}_n(\alpha, \beta) \ A_\alpha \ A_\beta \ d\alpha \ d\beta = \frac{1}{2} [ \text{II}_{nA}(S_1, -1) - \text{II}_{nA}(S_1, +1)] \quad (5-66) \]

The relative completeness of the displacement functions \( \hat{u}, \hat{v}, \) and \( \hat{w} \) leads to the convergence of

\[ \int_{S_1} \hat{u}_n(\alpha, \beta) \ A_\alpha \ A_\beta \ d\alpha \ d\beta \text{ to } \int_{S_1} u(\alpha, \beta) \ A_\alpha \ A_\beta \ d\alpha \ d\beta \]

in a similar manner as shown in section 5.4. It should be noted that \( S_1 \) can be taken as very small but not infinitesimal. For practical engineering purposes this means that the approximate solution is convergent to the exact solution.
6.1 Introduction

The finite element displacement method of analysis can be used to advantage to determine the natural frequencies of a shell structure. In the previous chapters we have seen that in the case the assumed displacement functions $\hat{u}$, $\hat{v}$ and $\hat{w}$ are continuous and the geometric boundary conditions are satisfied exactly the displacement method is a special form of the Ritz method. This means that if the above conditions are satisfied the frequencies obtained by the displacement method will usually not be less than the exact natural frequencies of the shell. The reason for this is that the deviation of the assumed displacements from the exact displacements has the same effect as certain constraints imposed on the system. This increases the stiffness of the shell structure (see also section 1.3.2). In order that the entire analysis be consistent the same displacements $\hat{u}$, $\hat{v}$ and $\hat{w}$ which are used for the determination of the stiffness matrix and the load vector must be utilized for the derivation of the mass matrix. The mass matrix obtained in this manner is called the consistent mass matrix (ref. 23).
The derivation of the consistent mass matrix follows the same procedure as is shown for the load vector in section 4.2. In the case of the mass matrix, the loading consists of the inertia forces which can be obtained by D'Alembert's principle. The determination of the approximate natural frequencies requires the solution of an eigenvalue problem. Since, in this case, the system of equations is very large, it is convenient to obtain the solution by an iterative procedure on an electronic computer. It is for this reason that in the following section the derivation is carried out in matrix notation.

6.2 Derivation of the consistent element mass matrix

In the preceding chapters we have defined the potential energy of the shell as $\Pi = \text{Strain Energy}$—the work done by the external forces $p_\alpha$, $p_\beta$, and $p_n$ in moving through the displacements $u$, $v$ and $w$. We have further shown that the strain energy of a triangular shell element expressed in terms of the assumed displacement functions $\hat{u}$, $\hat{v}$ and $\hat{w}$ of Chapter II is represented by (see equation (4.10))

$$U_s = \frac{1}{2} \begin{bmatrix} g \end{bmatrix}^T K \begin{bmatrix} g \end{bmatrix} \quad (6-1)$$

and the work done by the external forces $p_\alpha$, $p_\beta$, and $p_n$ by (see equation (4.22))

$$- \begin{bmatrix} g \end{bmatrix}^T \begin{bmatrix} q \end{bmatrix} \quad (6-2)$$
So that the potential energy for the triangular shell element is given by

\[ II = \frac{1}{2} \{ g \}^T K \{ g \} - \{ g \}^T \{ q \}. \] (6-3)

For the determination of the consistent mass matrix we assume that the inertia forces act on the middle surface of the shell and that no other forces are present. From D'Alembert's principle we find that these forces are given by

\[ p_\alpha = -m t \ddot{u} \] (6-4)
\[ p_\beta = -m t \ddot{v} \] (6-5)
\[ p_n = -m t \ddot{w} \] (6-6)

or in matrix form

\[ \{ p \} = -m t \{ \ddot{d} \} = mtS \{ \ddot{g} \} \] (6-7)

where \( m \) is the mass density of the shell; \( S \) is the matrix of equation (4-23) and is shown in Appendix D; \( t \) is the thickness of the shell; \( \ddot{g} \) denotes the derivative of \( g \) with respect to time twice; and thus \( \{ \ddot{d} \}^T = \{ \ddot{u} \ \ddot{v} \ \ddot{w} \} \), etc.

The natural modes of vibration can be obtained in the classical manner by assuming the displacements to be sinusoidal functions of time with frequency \( \omega \) so that

\[ \{ \ddot{g} \} = -\omega^2 \{ g \} \] (6-8)
On substituting equation (6-8) into equation (6-7), we have

\[
\begin{bmatrix} p \end{bmatrix} = \text{mt}^2 \omega^2 S \begin{bmatrix} g \end{bmatrix} \quad (6-9)
\]

The inertia forces which form the components of the vector \( \{ p \} \) depend linearly on the nodal displacements which are the components of the vector \( \{ g \} \). The work done by these inertia forces \( \{ p \} \) in moving through the displacements \( \{ g \} \) is equal to

\[
\begin{bmatrix} g \end{bmatrix}^T \begin{bmatrix} q \end{bmatrix} = \frac{1}{2} \iint_{\mathcal{S}} \begin{bmatrix} p \end{bmatrix}^T S \begin{bmatrix} g \end{bmatrix} A_\alpha A_\beta d\alpha d\beta. \quad (6-10)
\]

We now substitute equation (6-9) into equation (6-10) and obtain

\[
\begin{bmatrix} g \end{bmatrix}^T \begin{bmatrix} q \end{bmatrix} = \frac{1}{2} \iint_{\mathcal{S}} \begin{bmatrix} g \end{bmatrix}^T S^T \text{mt} S \begin{bmatrix} g \end{bmatrix} A_\alpha A_\beta d\alpha d\beta
= \frac{1}{2} \begin{bmatrix} g \end{bmatrix}^T \omega^2 M \begin{bmatrix} g \end{bmatrix}. \quad (6-11)
\]

where

\[
M = \iint_{\mathcal{S}} S^T S \text{mt} A_\alpha A_\beta d\alpha d\beta \quad (6-12)
\]

The M matrix is called the element mass matrix.

The expression for the potential energy now becomes

\[
\Pi = \frac{1}{2} \begin{bmatrix} g \end{bmatrix}^T K \begin{bmatrix} g \end{bmatrix} - \frac{1}{2} \begin{bmatrix} g \end{bmatrix}^T \omega^2 M \begin{bmatrix} g \end{bmatrix}. \quad (6-13)
\]
Minimization of the potential energy with respect to the nodal displacements which form the components of the vector \( \{ g \} \) gives

\[
[K - \omega^2 M] \{ g \} = 0
\]  \hspace{1cm} (6-14)

The natural frequencies \( \omega \) can be determined from the eigenvalue problem

\[
\text{det.} (K - \omega^2 M) = 0
\]  \hspace{1cm} (6-15)

From equation (6-12) we obtain in the case of constant thickness \( t \) and uniform mass density \( m \)

\[
M = mt \int_S M_1 A_\alpha A_\beta d\alpha d\beta
\]  \hspace{1cm} (6-16)

where \( M_1 = S^T S \)

The matrix \( M_1 \) is a 15 x 15 matrix and is shown in Appendix D.

In order to obtain a physical interpretation of the elements of the mass matrix we obtain from equation (6-10)

\[
\{ q \} = \omega^2 M \{ g \} = -M \{ \ddot{g} \}
\]  \hspace{1cm} (6-18)

The components of the vector \( \{ q \} \) represent the forces at the node points due to the inertia forces acting on the element caused by the accelerations of the nodal points. The coefficients of the mass matrix \( M \) have a clear physical meaning. For example, inspection of the components of the
vector \( \{ \mathbf{g} \} \) (see section 4.1) reveals that the third component is the normal displacement \( w_i \) at the \( i^{th} \) node and the ninth component is the rotation \( \Theta_{\alpha_j} \) about the tangent to the \( \alpha \)-line at the node \( j \). It follows that the coefficient \( m_{39} \) of the matrix \( M \) represents the mass inertia force at the \( i^{th} \) node acting in the normal direction due to a unit angular acceleration about the tangent to the \( \alpha \)-line at the \( j^{th} \) node.
CHAPTER VII

THE COMPUTER PROGRAM

The chapter gives a brief description of a computer program that is being written as a part of a research project and is almost complete. The program does not include the mass matrix or the solution of the eigenvalue problem to obtain the natural frequencies. The properties of the surfaces which have been incorporated in the program are also described.

7.1 Main features of the computer program

The program is divided into three segments. Schematic flow charts for the segments are shown in Figures 6, 7 and 8.

The first segment computes the stiffness matrix for each element and forms the total stiffness matrix for the entire assemblage of elements. The double integrals involved are evaluated numerically. The method used is Gaussian quadrature (ref. 36). This means that the value of the integrand has to be computed at a number of points of the element. Gaussian quadrature uses unequal spacing of these points which results in greater accuracy with a certain number of points than when equal spacing is used with the same number of points.
READ INPUT DATA
k IS NO. OF ELEMENTS
l IS NO. OF POINTS FOR NUMERICAL INTEGRATION.

l = 1

EVALUATE CONSTANTS FOR $i^{th}$ ELEMENT.

n = 1

CALCULATE COORDINATES AT POINTS OF ELEMENTS WHERE THE INTEGRANDS ARE TO BE EVALUATED.

FORM $[T]$  
FORM $[N]$  
FORM $[P]$  

$K_m = N^T T^T A T N.$

$K_b = P^T T^T A T P$

SELECT COEFFICIENT C FOR NUMERICAL INTEGRATION

$K = C \left[ h K_m + \frac{h^3}{12} K_b \right].$

Fig. 6.—Flow chart of segment 1.
CALL &EGME.NT 2. FOR THE TOTAL STIFFNESS MATRIX $K_{tot}$.  
DELETE THE COEFFICIENTS OF $K_{tot}$ CORRESPONDING TO THE PRESCRIBED DISPLACEMENTS.  
CALL SEGMENT 2.  

**SEGMENT 2.**  
READ LOADS FOR EACH ELEMENT.  
FORM THE LOAD VECTOR $\{q\}$ BY NUMERICAL INTEGRATION OVER EACH ELEMENT.  
MERGE THE ELEMENT LOAD VECTORS TO FORM THE TOTAL LOAD VECTOR $\{q_{tot}\}$.  
CALL SEGMENT 3.  
CHANGE $\{q_{tot}\}$ TO TAKE THE PRESCRIBED DISPLACEMENTS INTO ACCOUNT.  

*Fig. 7.*—Flow chart of segment 2.
Form the vector \( \{ \mathbf{z} \} \) such that
\[
\mathbf{X} \mathbf{z} = \{ \mathbf{q} \}_{\text{tot}}
\]

Solve for \( \{ \mathbf{q} \} \) from
\[
\mathbf{X} \mathbf{q} = \{ \mathbf{z} \}
\]

Print out solution vector \( \{ \mathbf{q} \} \)

Compute moments and forces at interior points of the elements

Stop.

Fig. 8.--Flow chart of segment 3.
The process of forming the total stiffness matrix $K_{\text{tot}}$ of the entire assemblage of elements is called the merge.

The coefficients of the total stiffness matrix $K_{\text{tot}}$ corresponding to nodal displacements prescribed as boundary conditions can be deleted from the stiffness matrix.

The total stiffness matrix $K_{\text{tot}}$ is sparcely populated (see Chapter I, section 1.3). By a suitable numbering of the nodal points the matrix $K_{\text{tot}}$ will only have non-zero coefficients in a band parallel to and centered about the main diagonal of the stiffness matrix. This fact can be used in the computer program to reduce the amount of storage required for the stiffness matrix $K_{\text{tot}}$. A shell structure with cutouts may require a very large number of elements. Since each nodal point in turn results in five rows and columns in the stiffness matrix very large stiffness matrices may occur. A reduction in storage requirements is thus of great practical importance.

The second segment computes the load vector $\{q\}$ for each element and forms the total load vector $\{q_{\text{tot}}\}$ for the entire assemblage of elements. Numerical integration is used to evaluate the double integrals involved.

The total load vector $\{q_{\text{tot}}\}$ is adjusted to take into account the prescribed nodal displacements. This adjustment is required since the coefficients of $K_{\text{tot}}$ corresponding to nodal displacements prescribed as boundary conditions are
deleted from \( K_{\text{tot}} \) in segment one. It is accomplished by subtracting the product of the prescribed nodal displacements and the corresponding coefficients of the stiffness matrix \( K_{\text{tot}} \) from the corresponding components of the vector \( \mathbf{q}_{\text{tot}} \).

The last segment computes the solution of the equation

\[
K_{\text{tot}} \{ \mathbf{e}_{\text{tot}} \} = \{ \mathbf{q}_{\text{tot}} \}
\]  

(7-1)

Since the matrix \( K_{\text{tot}} \) can be very large it is important to use a method of solution which makes use of the particular properties of this matrix. Such a method has been developed by Gatewood and Ohanian (ref. 35).

Since the method of solution of equation (7-1) is a very important part of the computer program, particularly where very large matrices are involved, we give here a brief review of the method as developed by the above authors.

The method is a modified form of Cholesky's method. The inversion of the matrix \( K_{\text{tot}} \) is avoided in this method since for large matrices this often leads to considerable round off errors.

The matrix \( K_{\text{tot}} \) is partitioned into \( mxm \) submatrices \( K_{i,j} \) with the properties

1) \( K_{i,i} \) is a non singular square matrix

\[
i = 1,2, \ldots, m
\]  

(7-2)

2) \( K_{i,i+j} = 0 \), for \( j > n, n < m \)

\[
i = 1,2, \ldots, m
\]
The above equations mean that all submatrices in $K_{tot}$ are zero outside a diagonal band of width $2n+1$ centered about the main diagonal or $K_{tot}$. The vectors $\{g_{tot}\}$ and $\{q_{tot}\}$ are also divided into subvectors $\{g_i\}$ and $\{q_i\}$, $i = 1, 2, \ldots, m$.

The matrix $K_{tot}$ can be written as

$$K_{tot} = [K_{1,i-n} \ldots K_{1,i-1} K_{1,i} K_{1,i+1} \ldots K_{1,i+n}]^\top$$  (7-3)

The matrix $K_{tot}$ can be expressed as the product of a lower triangular matrix $X$ and an upper triangular matrix $Y$ with

$$X = [X_{1,i-n} \ldots X_{1,i}]^\top$$  (7-4)

$$Y = [Y_{1,i} Y_{1,i+1} \ldots Y_{1,i+n}]^\top$$  (7-5)

If we equate $K_{tot}$ with the product of $X$ and $Y$; i.e.,

$$K_{tot} = X Y$$  (7-6)

we obtain the recursion formulas

$$X_{1,i-j} = K_{1,i-j} - \sum_{k=1}^{n-j} X_{1,i-j-k} Y_{1-j-k,i-j}$$  (7-7)

$j = n, n-1, \ldots, 1, 0; \ j < i; \ k < (i-j)$

$$Y_{1,i+j} = X_{1,i}^{-1} [K_{1,i+j} - \sum_{k=1}^{n-j} X_{1,i-k} Y_{1-k,i+j}]$$  (7-8)

$j = 1, 2, \ldots, n; \ j \leq (m-1); \ k < i$

$i = 1, 2, \ldots, m$
We now define the vector \( \{ Z \} \) by

\[
\{ Z \} = Y \{ g_{\text{tot}} \}
\]  

(7-9)

From equations (7-1), (7-6) and (7-9) we obtain

\[
K_{\text{tot}} \{ g_{\text{tot}} \} = XY \{ g_{\text{tot}} \} = X \{ Z \} = \{ q_{\text{tot}} \}
\]

or

\[
X \{ Z \} = \{ q_{\text{tot}} \}
\]  

(7-10)

Let \( \{ Z \} \) be divided into subvectors \( \{ Z_i \} \),

\( i = 1, 2, \ldots, m \) in the same way as \( \{ g_{\text{tot}} \} \) and \( \{ q_{\text{tot}} \} \).

From equations (7-4) and (7-10) we obtain the recursion formula

\[
\{ Z_1 \} = X_{1,1} \{ q_1 \} - \sum_{k=1}^{n} X_{1,1-k} \{ Z_{1-k} \}
\]  

(7-11)

\( k < 1 \); \( l = 1, 2, \ldots, m \).

This last recursion for \( \{ Z_1 \} \) can be obtained simultaneously with the recursions for \( X_{1,1-j} \) and \( Y_{1,1+j} \).

Substitution of equations (7-4) and (7-5) into equation (7-9) results in

\[
\{ g_i \} = \{ Z_1 \} - \sum_{k=1}^{n} Y_{1,1+k} \{ g_{1+k} \}
\]  

(7-12)

\( k \leq m-1 \); \( i = m, m-1, \ldots, 1 \).

This yields \( \{ g_i \} \) for \( i = m, m-1, \ldots, 1 \). The remaining \( \{ g_i \} \) can be obtained from equations (7-1) and (7-3).
\[ \left\{ g_{1-n} \right\} = K^{-1}_{i,i-n} \left[ q_i - \sum_{k=1}^{2n} K_{i,i-n} \left\{ g_{i-n-k} \right\} \right]. \quad (7-13) \]

7.2 Properties of the middle surface of the shell

The theory of the preceding chapters is valid for any thin shell. The properties of the middle surface of the shell are represented by the parameters \( A_\alpha, A_\beta, R_\alpha, R_\beta \) and their derivatives. It is desirable that the program be set up in such a way that several different types of shells can be handled by it. This can be accomplished by incorporating the above expressions for the parameters in subroutines, any number of which can be added to the program. Three such subroutines have been incorporated in the present program; namely,

1) The plane with a cartesian coordinate system
2) The sphere
3) The circular cylinder

The parameters in the first subroutine are

\[ A_\alpha = A_\beta = 1 \]
\[ R_\alpha = R_\beta = \infty \]

It should be noted that with these parameters the program is capable of obtaining solutions for plane elasticity problems as well as plate bending problems. Both problems are included in the shell problem and can be handled as special cases.
Fig. 9.—Curvilinear coordinates for the spherical surface.

\[ A_\alpha = R \cos \beta ; \quad A_\beta = R_\alpha = R_\beta = R \]

Fig. 10.—Curvilinear coordinates for the circular cylindrical surface.

\[ A_\alpha = 1, \quad R_\alpha = \infty, \quad A_\beta = R_\beta = R \]
The parameters for the sphere are incorporated in the second subroutine, they are

\[ A_\alpha = R \cos \]
\[ A_\beta = R_\alpha = R_\beta = R \]

The coordinate system is shown in Figure 9.

Some difficulties arise with this coordinate system. For example, for a complete sphere the coordinates of a point i located on the reference line \( \alpha = 0 \) can have the coordinates \((o, \beta_i)\) or \((2\pi, \beta_i)\). As a consequence, if we connect point i and some point j with a curve \( \beta = m\alpha + n \), a certain curve is obtained with the coordinates \((o, \beta_i)\). With the coordinates \((2\pi, \beta_i)\), however, an entirely different curve is obtained. This difficulty is peculiar to a complete surface. It can be overcome by giving each points on the reference line \( \alpha = 0 \) two different numbers, say \( r \) and \( s \), and represent the coordinates of \( r \) by \((o, \beta_r)\) and of \( s \) by \((2\pi, \beta_r)\). The displacements of \( r \) and \( s \) are then set equal to each other.

The parameters of the circular cylinder are incorporated in the third subroutine, they are

\[ A_\alpha = 1 \]
\[ A_\beta = R \]
\[ R_\alpha = \infty \]
\[ R_\beta = R \]

The coordinate system is shown in Figure 10. The complete cylindrical surface has the same difficulties as the sphere.
CHAPTER VIII

SUMMARY AND RECOMMENDATIONS FOR FURTHER STUDY

8.1 Summary

The work of the preceding chapters can be summarized as follows:

1. Displacement functions for a curved triangular thin shell element of arbitrary curvature have been obtained by a suitable definition of the element boundary curves.

2. It has been proved that these displacement functions are relatively complete which is a sufficient condition for the convergence of the potential energy to that of the exact solution.

3. The proof of convergence has been extended to the normal displacements and the "average" membrane displacements.

4. The stiffness matrix for the triangular shell element has been derived in a form suitable for use in a computer program.

5. The consistent mass matrix for the determination of the natural frequencies of a shell has been derived.

6. It has been shown that the conditions of relative completeness have a definite physical meaning for some ele-
ment displacement functions. For plane elasticity problems these conditions are equivalent to the requirement that the displacement functions must be capable of representing a condition of constant strain. For plate bending problems the conditions of relative completeness are equivalent to the requirement that the displacement functions must be capable of representing a condition of constant curvature.

8.2 Recommendations for further study

It is recommended that the following studies be undertaken:

1. A study to determine the maximum errors that can be expected as related to element size and shape. The importance of element shape has been pointed out in Chapter V. A study of this type may lead to practical rules for the determination of size and shape of the elements.

2. Development of stiffness and mass matrices for a curved triangular shell element using the force method and stress functions which satisfy the Cauchy equilibrium equations.

3. An investigation of the possibility of extending the finite element shell analysis to large deflections. The step by step analysis proposed by Wilson (ref. 13) leads to
difficulties since no mathematical expressions for the deformed shell surface are obtained at the end of each step.

4. An extension of the finite element displacement method to the analysis of plastic shell structures.

5. An extension of the present study to elastic stability of shells.
APPENDIX A

The condition of relative completeness guarantees the convergence of the potential energy to that of the exact solution. The mathematical condition of relative completeness corresponds to a certain physical condition. This last condition varies with the type of problem considered. It is the purpose of this Appendix to investigate this condition for several problems.

A.1 The rectangular element of plane stress analysis

The displacement functions \( \hat{u} \) and \( \hat{v} \) have in this case the general form (see Figure 11)

\[
\hat{u} = f_1(x,y)\hat{u}_1 + f_j(x,y)\hat{u}_j + f_k(x,y)\hat{u}_k + f_\ell(x,y)\hat{u}_\ell
\]

\[
\hat{v} = f_1(x,y)\hat{v}_1 + f_j(x,y)\hat{v}_j + f_k(x,y)\hat{v}_k + f_\ell(x,y)\hat{v}_\ell
\] (A-1)

Since the functions \( \hat{u} \) and \( \hat{v} \) are of the same form we need only consider the function \( \hat{u} \). The highest derivative of \( \hat{u}(x,y) \) occurring in the expression for the potential energy is of order one. The function \( \hat{u}(x,y) \) must, therefore, also be relatively complete of order one.

In order that the function \( \hat{u}(x,y) \) be relatively complete of order one we must have that for a given function
Fig. 11.—The rectangular element.

$u(x,y)$ defined on a region which is partitioned in rectangular elements

$$
\left| \hat{u} - u \right| < \varepsilon
$$

$$
\left| \frac{\partial \hat{u}}{\partial x} - \frac{\partial u}{\partial x} \right| < \varepsilon
$$

$$
\left| \frac{\partial \hat{u}}{\partial y} - \frac{\partial u}{\partial y} \right| < \varepsilon
$$

(A-2)
On the boundaries of the elements the partial derivatives of \( \hat{u} \) in equation (A-2) must be replaced by the one sided derivatives of \( \hat{u} \).

To satisfy the first of equation (A-2) we select the values of the undetermined parameters \( \hat{u}_i, \hat{u}_j, \hat{u}_k \) and \( \hat{u}_\ell \) so that they are equal to the values of the function \( u \) at the points \( i, j, k \) and \( \ell \), respectively; i.e., \( \hat{u}_i = u(x_i, y_i) \), etc.

Expansion of \( u(x_i, y_i) \) in the Taylor series about an interior point of the element gives

\[
\hat{u}_i = u(x_i, y_i) = u(x, y) - (x-x_i) \frac{\partial u(x,y)}{\partial x} (y-y_i) \frac{\partial u(x,y)}{\partial y} + \frac{1}{2} [(x-x_i) \frac{\partial}{\partial x} + (y-y_i) \frac{\partial}{\partial y}]^2 u(x,y) + ... \quad (A-3)
\]

This equation may be written in the form

\[
u(x_i, y_i) = u + O(h) \quad (A-4)
\]

where \( h \) is the maximum dimension of the element.

Substitution of equation (A-4) and similar equations for \( u(x_j, y_j), u(x_k, y_k) \) and \( u(x_\ell, y_\ell) \) into the first of equation (A-1) leads to

\[
\hat{u} = (f_1 + f_j + f_k + f_\ell)u + O(h) \quad (A-5)
\]

In the limit when \( h \to 0 \) we have that the first of equation (A-2) is satisfied if

\[
\lim_{h \to 0} (f_1 + f_j + f_k + f_\ell) \to 1. \quad (A-6)
\]
and the functions \( f_r (i,j,k,l) \) are bounded on the element.

This condition is certainly satisfied if

\[
f_i + f_j + f_k + f_l = 1 \quad (A-7)
\]

regardless of the size of the element.

The condition of equation (A-7) leads to simpler displacement functions than equation (A-6). This simplification also will improve the convergence.

The partial derivative of \( \hat{u} \) with respect to \( x \) is

\[
\frac{\partial \hat{u}}{\partial x} = \frac{\partial f_i}{\partial x} \hat{u}_i + \frac{\partial f_j}{\partial x} \hat{u}_j + \frac{\partial f_k}{\partial x} \hat{u}_k + \frac{\partial f_l}{\partial x} \hat{u}_l. \quad (A-8)
\]

Equation (A-3) can be written in the form

\[
u(x_1, y_1) = u - (x-x_i) \frac{\partial u}{\partial x} - (y-y_i) \frac{\partial u}{\partial y} + O(h^2) \quad (A-9)
\]

Substitution of equation (A-9) and similar expressions for \( u(x_j, y_j), u(x_k, y_k) \) and \( u(x_l, y_l) \) into equation (A-8) gives

\[
\frac{\partial \hat{u}}{\partial x} = \frac{\partial}{\partial x} (f_i + f_j + f_k + f_l) \left[ u - x \frac{\partial u}{\partial x} - y \frac{\partial u}{\partial y} \right] +
\]

\[
\frac{\partial}{\partial x} \left( x_i f_i + x_j f_j + x_k f_k + x_l f_l \right) \frac{\partial u}{\partial x} +
\]

\[
\frac{\partial}{\partial y} \left( y_i f_i + y_j f_j + y_k f_k + y_l f_l \right) \frac{\partial u}{\partial y} + O(h^2). \quad (A-10)
\]

The second condition of equation (A-2) will be satisfied if

\[
\lim_{h \to 0} \frac{\partial}{\partial x} (x_i f_i + x_j f_j + x_k f_k + x_l f_l) = 1
\]
and \( \lim_{h \to 0} \frac{\partial}{\partial y} [y_i f_i + y_j f_j + y_k f_k + y_\ell f_\ell] = 0 \)

and the partial derivatives of the \( f \) functions are bounded on the element.

These last conditions are certainly satisfied if

\[
\frac{\partial}{\partial x} [x_i f_i + x_j f_j + x_k f_k + x_\ell f_\ell] = 1 \quad (A-11)
\]

and

\[
\frac{\partial}{\partial x} [y_i f_i + y_j f_j + y_k f_k + y_\ell f_\ell] = 0 \quad (A-12)
\]

In a similar manner we find for the last condition of equation (A-2)

\[
\frac{\partial}{\partial y} [x_i f_i + x_j f_j + x_k f_k + x_\ell f_\ell] = 0 \quad (A-13)
\]

\[
\frac{\partial}{\partial y} [y_i f_i + y_j f_j + y_k f_k + y_\ell f_\ell] = 1 \quad (A-14)
\]

The question now arises what the physical meaning of these equations is. In order to investigate this question we write these equations in the form

\[
\frac{\partial}{\partial x} [x_i (f_i + f_j) + x_k (f_k + f_\ell)] = 1 \quad (A-15)
\]

\[
\frac{\partial}{\partial x} [y_i (f_i + f_j) + y_k (f_k + f_\ell)] = 0 \quad (A-16)
\]

\[
\frac{\partial}{\partial y} [x_i (f_i + f_j) + x_k (f_k + f_\ell)] = 0 \quad (A-17)
\]

\[
\frac{\partial}{\partial y} [y_i (f_i + f_\ell) + y_k (f_k + f_j)] = 1 \quad (A-18)
\]
Integration of (A-15) and (A-16) gives

\[ x_i (f_i + f_j) + x_k (f_k + f_\ell) = x + g_1(y) \quad (A-19) \]

and

\[ y_i (f_i + f_\ell) + y_k (f_k + f_j) = g_2(y) \quad (A-20) \]

Substituting equations (A-19) and (A-20) into equations (A-17) and (A-18) yields

\[ g'_1(y) = 0 \text{ and hence } g_1(y) = C_1 \]

and

\[ g'_2(y) = 1 \text{ so that } g_2(y) = y + C_2 \]

where \( C_1 \) and \( C_2 \) are constants.

The following relations are thus obtained

\[ x_i (f_i + f_j) + x_k (f_k + f_\ell) = x + C_1 \]

\[ y_i (f_i + f_\ell) + y_k (f_k + f_j) = y + C_2 \quad (A-21) \]

\[ f_i + f_j + f_k + f_\ell = 1 \]

Combining these last equations leads to

\[ f_k + f_\ell = \frac{1}{x_{1k}} (x_1 - x - C_1) \quad (A-22) \]

\[ f_i + f_\ell = \frac{1}{y_{1k}} (y_i - y - C_2) \]

The left hand side of the first equation of (A-22) represents the \( \hat{u} \) displacement when the nodal displacements are taken to be
\[ \hat{u}_1 = \hat{u}_j = 0 \]
\[ \hat{u}_k = \hat{u}_l = 1 \]

Equations (A-22) thus represents the condition that the displacement functions must be capable of representing a condition of constant strain.

Similar conditions can be found for the displacement functions of the triangular element. In fact, the displacement functions for the triangular element shown in Appendix B, section B.1 can be derived from the conditions of relative completeness.

A.2 The element of beam bending analysis

Beam elements have been used extensively in the analysis of wing structures. In this section a beam element of constant depth is considered (see Figure 12)

Fig. 12.—The beam element of constant depth.
The displacement function can be taken to be

$$\hat{w} = f_i(x) \hat{w}_i + f_j(x) \hat{w}_j + f_{xi}(x) \theta_{xi} + f_{xj}(x) \theta_{xj}$$

(A-23)

where \( \hat{w}_i \) and \( \hat{w}_j \) are the nodal displacements normal to the
x-axis. \( \theta_{xi} \) and \( \theta_{xj} \) are the nodal slopes at the
points i and j.

The functions \( f_i \) and \( f_j \) are dimensionless while the
functions \( f_{xi} \) and \( f_{xj} \) have the dimension of length.

Let \( \ell \) be the length of the beam and let \( w \) be an
arbitrary continuous function on the interval \( [0, \ell] \).

Since the highest derivative of \( \hat{w}(x) \) occurring in the
potential energy is of order two the function \( \hat{w}(x) \) must be
relatively complete of order two. The conditions of relative
completeness are in this case

$$\left| \hat{w} - w \right| < \varepsilon$$

$$\left| \frac{\partial \hat{w}}{\partial x} - \frac{\partial w}{\partial x} \right| < \varepsilon$$

(A-24)

$$\left| \frac{\partial^2 \hat{w}}{\partial x^2} - \frac{\partial^2 w}{\partial x^2} \right| < \varepsilon$$

On the boundaries of the elements the partial derivatives of
\( \hat{w} \) in equation (A-24) must be replaced by the one sided
derivatives of \( \hat{w} \).
Proceeding in the same way as shown in the previous section we assume \( \hat{w}_i = w(x_i, y_i) \), etc., and expand these in a Taylor series about an interior point of the element. This leads to

\[
\hat{w} = f_i(x)[w + O(h)] + f_j(x)[w + O(h)] +
\]

\[
+ f_{xi}(x) \theta_{xi} + f_{xj}(x) \theta_{xj}. \tag{A-25}
\]

where \( h \) is the length of the element.

The first condition of equation (A-23) is satisfied if \( f_i + f_j \rightarrow 1 \) as \( h \rightarrow 0 \) and \( f_{xi} \) and \( f_{xj} \) are both smaller than or equal to \( h \) on the element.

Better convergence can be expected if

\[
f_i + f_j = 1 \tag{A-26}
\]

regardless of the size of the element.

Similarly we have for the derivative of \( w \)

\[
\hat{w}' = f'_i(x) \hat{w}_i + f'_j(x) \hat{w}_j + f'_{xi}(x) \theta_{xi} + f'_{xj}(x) \theta_{xj}
\]

\[
= f'_i(x) [w-(x-x_i)w' + O(h^2)]
\]

\[
+ f'_j(x) [w-(x-x_j)w' + O(h^2)]
\]

\[
+ f'_{xi}(x) [w' + O(h)]
\]

\[
+ f'_{xj}(x) [w' + O(h)]
\]
The second condition of equation (A-24) leads to

\[ f_i' + f_j' \to 0 \quad (A-27) \]

\[ xf_i' + xf_j' + f_i' + f_j' \to 0 \]

\[ i \quad j \quad xi \quad xj \]

when \( h \to 0 \).

This condition is also satisfied if

\[ f_i' + f_j' = 0 \]

\[ x_i f_i' + x_j f_j' + f_i' + f_j' = 1 \quad (A-28) \]

for any size of the element.

The first equation of (A-28) is satisfied if equation (A-26) is. In that case the second equation of (A-28) may be written in the form

\[ x_{ij} f_i' + f_i' + f_j' = 1 \quad (A-29) \]

The third equation of (A-24) finally leads to the condition

\[ \frac{x_i^2}{2} f_i'' + \frac{x_j^2}{2} f_j'' + x_i f_i' x_i + x_j f_j' x_j = 1 \quad (A-30) \]

Substitution of equations (A-26) and (A-29) into equation (A-30) gives

\[ f_i'' + f_j'' = \frac{2}{x_{ij}} \text{ (a constant)} \quad (A-31) \]

This equation can be obtained from equation (A-23) by assuming \( \hat{\omega}_i = \hat{\omega}_j = 0 \) and \( \theta_{x_i} = - \theta_{x_j} = 1 \). The physical
meaning of equation (A-31) is that the displacement function of equation (A-23) must admit a condition of constant curvature.

Functions $f$ and $f_x$ which satisfy the above requirements can be obtained from the theory of beam deflections. For example for $x_i = 0$, $f_j$ and $f_{xj}$ are given by

$$f_j = -\frac{2}{\ell^3} x^3 + \frac{3}{\ell^2} x^2$$

$$f_{xj} = -\frac{1}{\ell^2} x^3 + \frac{1}{\ell} x^2$$

and their shapes are shown in Figures 13, and 14.

---

**Fig. 13.**—Displacements for $\hat{w}_j = 1$.

**Fig. 14.**—Displacement for $\theta_{xj} = -1$. 
A.3 The rectangular element of plate bending analysis

The normal displacement function has in this case the form (see Figure 11)

\[ \hat{w} = f_1(x,y)\hat{w}_1 + f_j(x,y)\hat{w}_j + f_k(x,y)\hat{w}_k + f_\ell(x,y)\hat{w}_\ell \]

\[ + f_{x1}(x,y)\theta_{x1} + f_{xj}(x,y)\theta_{xj} + f_{xk}(x,y)\theta_{xk} + f_{x\ell}(x,y)\theta_{x\ell} \]  \hspace{1cm} (A-32)

\[ + f_{y1}(x,y)\theta_{y1} + f_{yj}(x,y)\theta_{yj} + f_{yk}(x,y)\theta_{yk} + f_{y\ell}(x,y)\theta_{y\ell}. \]

where \( \hat{w}_r \) \( (r=1, j, k, \ell) \) are the normal nodal displacements;
\( \theta_{xr} \) \( (r=1, j, k, \ell) \) are the nodal rotations about the \( x \)-axis;
and \( \theta_{yr} \) \( (r=1, j, k, \ell) \) are the nodal rotations about the \( y \)-axis.

The conditions of relative completeness in this case lead to

\[ f_1 + f_j + f_k + f_\ell = 1 \] \hspace{1cm} (A-33)

\[ x_{ik} \frac{\partial}{\partial x}(f_1 + f_j) + x_{jk} \frac{\partial}{\partial y}(f_1 + f_j + f_k + f_\ell) = 1 \] \hspace{1cm} (A-34)

\[ y_{ik} \frac{\partial}{\partial x}(f_1 + f_j) + y_{jk} \frac{\partial}{\partial y}(f_1 + f_j + f_k + f_\ell) = 0 \] \hspace{1cm} (A-35)

\[ \frac{x_{1}^2 - x_{k}^2}{2} \frac{\partial^2}{\partial x^2}(f_1 + f_j) + x_{1} \frac{\partial^2}{\partial x^2}(f_{y1} + f_{yj}) + x_{k} \frac{\partial^2}{\partial x^2}(f_{yk} + f_{y\ell}) = 1 \] \hspace{1cm} (A-36)

and similar equations for the other derivatives.

On substituting equations (A-33) and (A-34) into equation (A-36) we arrive at
\[ \frac{\partial^2}{\partial x^2} (f_{y1} + f_{yj}) - \frac{\partial^2}{\partial y^2} (f_{yk} + f_{y\ell}) = \frac{2}{x_{1k}} \text{ (a constant)} \quad (A-37) \]

It is seen that this equation is similar to equation (A-31) and also represents a condition of constant curvature.

The same procedure leads to very complicated expressions when it is applied to the triangular element of plate bending analysis and the physical meaning of these expressions is not as easily established as in the above case. However, the condition of constant curvature has been incorporated in the displacement functions of the triangular element (see Appendix B) and as a result these functions are relatively complete (see Chapter V).
APPENDIX B

B.1 The triangular element of plane stress analysis

The displacement functions for the triangular element were introduced by Turner et al. (ref. 9).

The assumed displacements are linear and contain rigid body modes

\[ \hat{u} = a_1 + a_2 x + a_3 y \]
\[ \hat{v} = a_4 + a_5 x + a_6 y \]  \hspace{1cm} (B-1)

The generalized coordinates are chosen to be the nodal displacements shown in Figure 15.

Fig. 15.— The triangular element.
Substitution of the coordinates of point $i$ into equation (B-1) gives

$$\hat{u}_i = a_1 + a_2x_i + a_3y_i$$  \hspace{1cm} (B-2)

$$\hat{v}_i = a_4 + a_5x_i + a_6y_i$$

where $\hat{u}_i$ is the displacement of the $i^{th}$ node in the $x$-direction and;

$\hat{v}_i$ is the displacement of the $i^{th}$ node in the $y$-direction.

Two similar sets of equations for the points $i$ and $j$ are

$$\hat{u}_j = a_1 + a_2x_j + a_3y_j$$ \hspace{1cm} (B-3)

$$\hat{v}_j = a_4 + a_5x_j + a_6y_j$$

$$\hat{u}_k = a_1 + a_2x_k + a_3y_k$$ \hspace{1cm} (B-4)

$$\hat{v}_k = a_4 + a_5x_k + a_6y_k$$

The $a$'s can be expressed in terms of the nodal displacements by solving the equations (B-2), (B-3) and (B-4). Substitution of the $a$'s into equation (B-1) then yields the equations

$$\hat{u} = L_i\hat{u}_i + L_j\hat{u}_j + L_k\hat{u}_k$$

$$\hat{v} = L_i\hat{v}_i + L_j\hat{v}_j + L_k\hat{v}_k$$ \hspace{1cm} (B-5)
where

\[ L_1 = \frac{1}{2\Delta} (\overline{a}_1 + \overline{b}_1 x + \overline{c}_1 y) \]  
\[ L_j = \frac{1}{2\Delta} (\overline{a}_j + \overline{b}_j x + \overline{c}_j y) \]  
\[ L_k = \frac{1}{2\Delta} (\overline{a}_k + \overline{b}_k x + \overline{c}_k y) \]  

\[ \overline{a}_r = \begin{vmatrix} x_s & x_t \\ y_s & y_t \end{vmatrix} \]  
\[ \overline{b}_r = y_x - y_t = y_{st} \]  
\[ \overline{c}_r = x_s - x_t = x_{st} \]  

(B-6)

and the \( r, s, \) and \( t \) are cyclic combinations of the \( i, j \) and \( k \).

It is easy to show that at the \( i \)th node \( L_1 = 1 \) and

\[ L_j = L_k = 0 \]  

so that

\[ \hat{u}(x_i, y_i) = \hat{u}_i \]  
\[ \hat{v}(x_i, y_i) = \hat{v}_i \]  

Similar conditions exist at the \( j \)th and \( k \)th nodes.

The area of the triangle is given by

\[ \frac{1}{2} \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix} = \overline{a}_1 + \overline{a}_j + \overline{a}_k \]  

(B-8)
The L-functions (equation (B-6) are dimensionless and represent the so-called area coordinates of a typical point P inside the triangle. For instance

\[ L_1 = \frac{A_1}{\Delta} \]

where \( A_1 \) is the area if the subtriangle opposite point \( i \) (see Figure 17).

It is easy to show that

\[ L_1 + L_j + L_k = 1 \] (B-9)

When the point P lies on the boundary j-k we have that \( L_1 = 0 \) and the displacements of P become

\[ \hat{u} = L_j \hat{u}_j + L_k \hat{u}_k \]

\[ \hat{v} = L_j \hat{v}_j + L_k \hat{v}_k \] (B-10)

These equations express the fact that the displacements on the boundary j-k vary linearly and depend uniquely on the displacements of the nodal points j and k. The elements will thus fit together after deformation and the displacement functions are continuous.

The functions represented by equation (B-5) are also relatively complete. The proof of this is exactly the same as that shown in Chapter V.

The L-functions are so-called pyramid functions. The function \( L_1 \) is illustrated in Figure 16.
Fig. 16. — Displacements $\hat{u}$ for $\hat{u}_1=1$, $\hat{u}_j=\hat{u}_k=0$.

The stresses corresponding to the displacements of equation (B-5) are constants and are

$$
\sigma_{xx} = \frac{b_1 \hat{u}_1 + b_j \hat{u}_j + b_k \hat{u}_k}{2\Delta}
$$

$$
\sigma_{yy} = \frac{c_1 \hat{v}_1 + c_j \hat{v}_j + c_k \hat{v}_k}{2\Delta}
$$

$$
\sigma_{xy} = \frac{c_1 \hat{u}_1 + c_j \hat{u}_j + c_k \hat{u}_k + b_1 \hat{v}_1 + b_j \hat{v}_j + b_k \hat{v}_k}{2\Delta}.
$$

(B-11)

These equations show that the stresses are not continuous on the element boundaries.
B.2 The triangular element of plate bending analysis

The triangular element has been used for some time (ref. 14) in the analysis of plate bending problems. The displacement functions used in this reference, however, did not give very good results. It was only recently (refs. 15, 18) that displacement functions were developed which yielded good results.

The normal displacement function considered here is one of the functions developed in ref. 18. In the following treatment we briefly review the development of this function.

The nodal displacements, which are the undetermined parameters of the Ritz method, are in this case the three normal displacements and the six rotations in the x- and y-directions of the nodal points. These are denoted by

\[ \mathbf{w}_r, \varphi_{xr}, \text{ and } \varphi_{yr} \quad (r=1,j,k) \]  

where

\[ \varphi_{xr} = - \frac{\partial \mathbf{w}_r}{\partial y} \bigg|_r \]  

\[ \varphi_{yr} = - \frac{\partial \mathbf{w}_r}{\partial x} \bigg|_r \]  

In order to ensure equilibrium of the applied forces and the nodal forces of each element and of the entire plate, it is necessary to include rigid body nodes in the displacement function (ref. 19). The total displacement is

\[ \mathbf{w} = \mathbf{w}^* + \mathbf{w}^R \]
where $w^R$ is the rigid body translation and is given by

$$w^R = L_1 \hat{w}_1 + L_j \hat{w}_j + L_k \hat{w}_k; \quad (B-15)$$

and $w^*$ is the normal displacement relative to the rigid body translation.

In view of the properties of the $L$ functions we have that $w^R(x_s, y_s) = \hat{w}_s \ (s=i,j,k)$. It follows that $w^*$ must have zero values at the nodes. Since $\hat{w}$ must be a linear function of the nodal displacements it is clear that $w^*$ can be defined in terms of the six slopes imposed at the nodes

$$w^* = F_{x_1} w_{x_1}^* + F_{y_1} w_{y_1}^* + F_{x_2} w_{x_2}^* + F_{y_2} w_{y_2}^* + F_{x_3} w_{x_3}^* + F_{y_3} w_{y_3}^* \quad (B-16)$$

where

$$\theta^*_x = \theta_x - \frac{\partial w^R}{\partial y}$$

$$\theta^*_y = \theta_y - \frac{\partial w^R}{\partial x}$$

$$\frac{\partial w^R}{\partial x} = \frac{(b_i w_i + b_j w_j + b_k w_k)}{2\Delta} \quad (B-17)$$

$$\frac{\partial w^R}{\partial y} = \frac{(c_i w_i + c_j w_j + c_k w_k)}{2\Delta}$$

and no summation is implied where double subscripts are used.

The $F$-functions must satisfy certain conditions. For example since $w^*$ must be zero at all nodes, it follows that

$$F_{x_r} = F_{y_r} = 0 \ (r = i,j,k) \text{ at all nodes.}$$
Furthermore \( \frac{\partial w^*}{\partial y} \) must be equal to \( \theta^* \) at the \( r^{th} \) node, but must be zero at the other nodes. Hence we must have that

\[
\frac{\partial F_{xr}}{\partial y} = -1 \text{ at node } r \text{ and }
\]

\[
\frac{\partial F_{xr}}{\partial y} = 0 \text{ at the two remaining nodes}
\]

while

\[
\frac{\partial F_{xr}}{\partial x} = 0 \text{ at all nodes.}
\]

A polynomial expression which satisfies these conditions is

\[
F_{xi} = y_{ij} L_i^2 L_j + y_{ik} L_i L_k \quad (B-18)
\]

Similarly

\[
F_{yi} = x_{ji} L_i^2 L_j + x_{ki} L_i L_k \quad (B-19)
\]

In Appendix A we have shown that, in the case of bending problems, the condition of relative completeness corresponds to a condition of constant curvature.\(^9\) The functions of equations (B-18) and (B-19), however, do not satisfy this requirement. To correct this, the function \( L_i^2 L_j L_k \) can be added in any desired proportion, since this function has zero slopes and deflections at all nodes. This results in

\(^9\)The authors of (ref. 18) did not arrive at this requirement by means of the conditions of relative completeness but used physical reasoning.
\[ F_{xi} = y_{ij}(L_1^2L_j + \alpha L_1L_jL_k) + y_{ik}(L_1^2L_k + \alpha L_1L_jL_k) \]  
\[ F_{yi} = x_{ji}(L_1^2L_j + \alpha L_1L_jL_k) + x_{ki}(L_1^2L_k + \alpha L_1L_jL_k) \]  
\[ (B-20) \]

The other functions are defined similarly.

The condition of constant curvature requires that \( w^* \) be a simple quadratic

\[ w^* = A_1L_jL_k + A_jL_iL_i + A_kL_iL_j \]  
\[ (B-21) \]

where the A's can take up any prescribed values.

Differentiating \((B-21)\) with respect to \( y \) gives

\[ \frac{\partial w^*}{\partial y} = (A_1c_jL_k + A_jc_kL_j + \]

\[ A_jc_kL_j + A_kc_jL_k + \]

\[ A_kc_iL_i + A_jc_jL_i)/2\Delta \]

If we evaluate this derivative at the \( i^{th} \) nodal point we have

\[ \theta^*_{xi} = -\frac{\partial w^*(x_i,y_i)}{\partial y} = - (A_jc_k + A_kc_j)/2\Delta \]

\[ = -(A_jx_{ji} + A_kx_{ik})/2\Delta \]  
\[ (B-22) \]

since at the \( i^{th} \) node \( L_i = 1 \) and \( L_j = L_k = 0 \).

Similarly we have

\[ \theta^*_{yi} = (a_jy_{ij} + A_ky_{ki})/2\Delta \]  
\[ (B-23) \]

With similar expression for the other \( \theta \)'s.
Substituting these into (B-16) gives

\[ 2 \Delta W^* = A_i (-x_{ji} F_{xj} + y_{ij} F_{yj} - x_{ik} F_{xk} + y_{ki} F_{yk}) + \]

\[ A_j (-x_{kj} F_{xk} + y_{jk} F_{yk} - x_{ji} F_{xj} + y_{ij} F_{yj}) + \]  

\[ A_k (-x_{ik} F_{xi} + y_{ki} F_{yi} - x_{kj} F_{xj} + y_{jk} F_{yj}). \]  

The coefficient of equation (B-20) can be evaluated by substituting equation (B-20) into equation (B-24). In this way it is found that the constant curvature condition is possible with the assumed displacement functions only if \( \alpha = \frac{1}{2} \).  

The function \( \hat{w} \) is not compatible; i.e., the slopes are not continuous across the element boundaries. The authors of ref. 18 considered the possibility of making the \( w \) function compatible by adding suitable functions to the \( F_x \) and \( F_y \) functions of equation (B-24). The resulting expression for \( \hat{w} \) is very complex and since compatibility is not a requirement for the Ritz method the additional functions have been omitted here.

The equation (B-19) becomes now

\[ F_{xi} = y_{ij} \psi_{ij} + y_{ik} \psi_{ik} \]  

\[ F_{yi} = x_{ji} \psi_{ij} + x_{ki} \psi_{ik} \]  

where

\[ \psi_{ij} = L_i^2 L_j + \frac{1}{2} L_i L_j L_k \]  

\[ \psi_{ik} = L_i^2 L_k + \frac{1}{2} L_i L_j L_k \]  

(B-26)
The other functions may be obtained by a cyclic rotation of the indices and are shown in Chapter II.

Figure 17 shows the displacement function for $\hat{w}_1 = 1$ and all other nodal displacements and rotations equal to zero.

Figure 18 shows the displacement along curve i-j.

Figures 19 and 20 show the displacements with $\theta_{x1} = 1$ with all other nodal displacements and rotations equal to zero.

These curves are very similar to those obtained in Appendix A, Figures 13 and 14.
Fig. 17.—Level curves of $\hat{w}$ for $\hat{w}_1 = 1$, $\hat{w}_j = \hat{w}_k = 0$. 
Fig. 18.—Displacements along $i-j$ for $w_i=1$.

Fig. 19.—Displacements along $i-j$ for $\theta_{x1}=1$.

Fig. 20.—Displacements along $i-k$ for $\theta_{x1}=1$. 

APPENDIX C

In this Appendix the equations of the theory of thin shells, which are relevant to the theory of this study, are summarized. The equations shown are essentially as shown in ref. 4, and are based on a principal coordinate system on the surface.

The strains of the middle surface are given in terms of the displacements by

\[ \epsilon_{\alpha \alpha} = \frac{1}{A_\alpha} \frac{\partial u}{\partial \alpha} + \frac{1}{A_\alpha A_\beta} \frac{\partial A_\alpha}{\partial \beta} v + \frac{w}{R_\alpha}. \]  \hspace{1cm} (C-1)

\[ \omega = \frac{A_\beta}{A_\alpha} \frac{\partial}{\partial \alpha} \left( \frac{v}{A_\beta} \right) + \frac{A_\alpha}{A_\beta} \frac{\partial}{\partial \beta} \left( \frac{u}{A_\alpha} \right). \]  \hspace{1cm} (C-2)

\[ \epsilon_{\beta \beta} = \frac{1}{A_\beta} \frac{\partial v}{\partial \beta} + \frac{1}{A_\alpha A_\beta} \frac{\partial A_\beta}{\partial \alpha} u - \frac{w}{R_\beta}. \]  \hspace{1cm} (C-3)

The parameters \( \kappa_\alpha, \kappa_\beta \) and \( \tau \) are given by

\[ \kappa_\alpha = -\frac{1}{A_\alpha} \frac{\partial}{\partial \alpha} \left( \frac{1}{A_\alpha} \frac{\partial w}{\partial \alpha} - \frac{u}{R_\alpha} \right) \]

\[ \hspace{2cm} - \frac{1}{A_\alpha A_\beta} \frac{\partial A_\alpha}{\partial \beta} \left( \frac{1}{A_\beta} \frac{\partial w}{\partial \beta} - \frac{v}{R_\beta} \right) \hspace{1cm} (C-4) \]

\[ \kappa_\beta = -\frac{1}{A_\beta} \frac{\partial}{\partial \beta} \left( \frac{1}{A_\beta} \frac{\partial w}{\partial \beta} - \frac{v}{R_\beta} \right) \]

\[ \hspace{2cm} - \frac{1}{A_\alpha A_\beta} \frac{\partial A_\beta}{\partial \alpha} \left( \frac{1}{A_\alpha} \frac{\partial w}{\partial \alpha} - \frac{u}{R_\alpha} \right). \hspace{1cm} (C-5) \]
The forces per unit length of shell measured along the $\alpha$- and $\beta$- lines are (see Figures 22 and 23)

\[ T_\alpha = \int_{-t/2}^{t/2} \sigma_{x0}(1 + z/R_\beta) \, dz \]

\[ T_\beta = \int_{-t/2}^{t/2} \sigma_{y0}(1 + z/R_\alpha) \, dz \]

\[ T_{\alpha\beta} = \int_{-t/2}^{t/2} \sigma_{x0}(1 + z/R_\beta) \, dz \]

\[ N_\alpha = \int_{-t/2}^{t/2} \sigma_{z0}(1 + z/R_\beta) \, dz \]

\[ N_\beta = \int_{-t/2}^{t/2} \sigma_{z0}(1 + z/R_\alpha) \, dz \]

\[ S = T_{\alpha\beta} - \frac{M_{\beta\alpha}}{R_\beta} = T_{\beta\alpha} - \frac{M_{\alpha\beta}}{R_\alpha} \]
The moments per unit length of shell measured along the $\alpha$- and $\beta$-lines are (see Figure 23)

\[
M_\alpha = \int_{-t/2}^{t/2} \sigma_{\alpha\alpha} z (1 + z/R_\beta) \, dz.
\]  \hspace{1cm} (C-13)

\[
M_{\alpha\beta} = \int_{-t/2}^{t/2} \sigma_{\alpha\beta} z (1 + z/R_\beta) \, dz.
\]  \hspace{1cm} (C-14)

\[
M_\beta = \int_{-t/2}^{t/2} \sigma_{\beta\beta} z (1 + z/R_\alpha) \, dz.
\]  \hspace{1cm} (C-15)

\[
M_{\beta\alpha} = \int_{-t/2}^{t/2} \sigma_{\beta\alpha} z (1 + z/R_\alpha) \, dz.
\]  \hspace{1cm} (C-16)

\[
H = \frac{1}{2} \left( M_{\alpha\beta} + M_{\beta\alpha} \right).
\]  \hspace{1cm} (C-17)

The stresses $\sigma_{\alpha\alpha}$, $\sigma_{\beta\beta}$ and $\sigma_{\alpha\beta}$ are shown in Figure 21.

Fig. 21.—Stresses in the shell.
Fig. 22.--Forces in the shell figure.

Fig. 23.--Moments in the shell figure.
**APPENDIX D**

\[ D_1 = \begin{bmatrix}
L_i & 0 & 0 & 0 & 0 & L_j & 0 & 0 & 0 & L_k & 0 & 0 & 0 \\
0 & L_i & 0 & 0 & 0 & L_j & 0 & 0 & 0 & L_k & 0 & 0 & 0 \\
0 & 0 & L_i & F_{ai} & F_{ai} & 0 & 0 & L_j & F_{aj} & F_{aj} & 0 & 0 & L_k & F_{ak} & F_{ak} \\
\end{bmatrix} \]

\[ R = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & R_{bi} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & R_{ci} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & R_{aj} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix} \]
The $N$ matrix is

\[
n_{1, r} = \frac{b_3}{2\Delta A_\alpha} - \frac{F_{\alpha\beta}}{R_\alpha R_\alpha s}.
\]

\[
n_{1, r+1} = \frac{L_s}{A_\alpha A_\beta} \frac{\partial A_\alpha}{\partial \beta} - \frac{F_{\alpha\beta s}}{R_\alpha R_\beta s}.
\]

\[
n_{1, r+2} = \frac{1}{R_\alpha} \left[ L_s + \frac{1}{2\Delta} \left\{ c_s (F_{\alpha \beta} + F_{\alpha j} + F_{\alpha k}) + b_s (F_{\beta i} + F_{\beta j} + F_{\beta k}) \right\} \right].
\]

\[
n_{1, r+3} = \frac{F_{\alpha\beta s}}{R_\alpha}.
\]

\[
n_{1, r+4} = \frac{F_{\alpha\beta s}}{R_\alpha}.
\]

\[
n_{2, r} = \frac{L_s}{A_\alpha A_\beta} \frac{\partial A_\beta}{\partial \alpha} - \frac{F_{\alpha\beta}}{R_\beta R_\alpha s}.
\]

\[
n_{2, r+1} = \frac{c_s}{2\Delta A_\beta} - \frac{F_{\alpha\beta s}}{R_\beta R_\beta s}.
\]

\[
n_{2, r+2} = \frac{1}{R_\beta} \left[ L_s + \frac{1}{2\Delta} \left\{ c_s (F_{\alpha \beta} + F_{\alpha j} + F_{\alpha k}) + b_s (F_{\beta i} + F_{\beta j} + F_{\beta k}) \right\} \right].
\]

\[
n_{2, r+3} = \frac{F_{\alpha\beta s}}{R_\beta}.
\]

\[
n_{2, r+4} = \frac{F_{\alpha\beta s}}{R_\beta}.
\]

\[
n_{3, r} = \frac{1}{A_\beta} \left( \frac{c_s}{2\Delta} - \frac{L_s}{A_\alpha} \frac{\partial A_\alpha}{\partial \beta} \right).
\]

\[
n_{3, r+1} = \frac{1}{A_\alpha} \left( \frac{b_s}{2\Delta} - \frac{L_s}{A_\beta} \frac{\partial A_\beta}{\partial \alpha} \right).
\]

\[
n_{3, r+2} = 0.
\]

\[
n_{3, r+3} = 0.
\]

\[
n_{3, r+4} = 0.
\]

For $s = 1, 2, 3$; $r = 5s - 4$; $i = 1, j = 2, k = 3$. 
The $P$ matrix is

\[
P_{1,r} = \frac{1}{A_\alpha} \left[ \frac{1}{R_\alpha} \left( \frac{b_\alpha}{2A_\alpha} - \frac{L_\alpha}{A_\alpha} \frac{\partial R_\alpha}{\partial \alpha} \right) - \frac{1}{R_{\alpha s}} \left( - \frac{1}{A_\alpha} \frac{\partial^2 F_{\beta s}}{\partial \alpha^2} + \frac{1}{A_\alpha^2} \frac{\partial A_\alpha}{\partial \alpha} \frac{\partial F_{\beta s}}{\partial \beta} - \frac{1}{A_\beta} \frac{\partial A_\beta}{\partial \beta} \frac{\partial F_{\alpha s}}{\partial \alpha} \right) \right] \]

\[
P_{1,r+1} = \frac{1}{A_\alpha} \left[ \frac{L_\alpha}{A_\alpha R_\beta} - \frac{1}{R_{\beta s}} \left( - \frac{1}{A_\alpha} \frac{\partial^2 F_{\beta s}}{\partial \alpha^2} + \frac{1}{A_\alpha^2} \frac{\partial A_\alpha}{\partial \alpha} \frac{\partial F_{\beta s}}{\partial \alpha} \right) \right]
\]

\[
P_{1,r+2} = \frac{1}{A_\alpha} \left( \frac{b_\alpha}{2A_\alpha} \frac{\partial A_\alpha}{\partial \alpha} - \frac{c_\alpha}{2A_\alpha^2} \frac{\partial A_\alpha}{\partial \beta} \right)
\]

\[
+ \frac{c_\alpha}{2A_\alpha} \left( - \frac{1}{A_\alpha} \frac{\partial^2}{\partial \alpha^2} + \frac{1}{A_\alpha^2} \frac{\partial A_\alpha}{\partial \alpha} \frac{\partial}{\partial \alpha} - \frac{1}{A_\beta} \frac{\partial A_\beta}{\partial \beta} \frac{\partial}{\partial \alpha} \right) \left( F_{\alpha i} + F_{\alpha j} + F_{\alpha k} \right)
\]

\[
+ \frac{b_\alpha}{2A_\alpha} \left( - \frac{1}{A_\alpha} \frac{\partial^2}{\partial \alpha^2} + \frac{1}{A_\alpha^2} \frac{\partial A_\alpha}{\partial \alpha} \frac{\partial}{\partial \alpha} - \frac{1}{A_\beta} \frac{\partial A_\beta}{\partial \beta} \frac{\partial}{\partial \alpha} \right) \left( F_{\beta i} + F_{\beta j} + F_{\beta k} \right)
\]

\[
P_{1,r+3} = \frac{1}{A_\alpha} \left( - \frac{1}{A_\alpha} \frac{\partial^2}{\partial \alpha^2} + \frac{1}{A_\alpha^2} \frac{\partial A_\alpha}{\partial \alpha} \frac{\partial}{\partial \alpha} - \frac{1}{A_\beta} \frac{\partial A_\beta}{\partial \beta} \frac{\partial}{\partial \alpha} \right) F_{\alpha s}
\]

\[
P_{1,r+4} = \frac{1}{A_\alpha} \left( - \frac{1}{A_\alpha} \frac{\partial^2}{\partial \alpha^2} + \frac{1}{A_\alpha^2} \frac{\partial A_\alpha}{\partial \alpha} \frac{\partial}{\partial \alpha} - \frac{1}{A_\beta} \frac{\partial A_\beta}{\partial \beta} \frac{\partial}{\partial \alpha} \right) F_{\beta s}
\]

\[
P_{2,r} = \frac{1}{A_\beta} \left[ \frac{L_\beta}{A_\alpha R_\alpha} - \frac{1}{R_{\alpha s}} \left( - \frac{1}{A_\alpha} \frac{\partial^2}{\partial \alpha^2} + \frac{1}{A_\alpha^2} \frac{\partial A_\alpha}{\partial \alpha} \frac{\partial}{\partial \alpha} \right) \left( F_{\alpha i} + F_{\alpha j} + F_{\alpha k} \right) \right]
\]

\[
+ \frac{c_\alpha}{2A_\alpha} \left( - \frac{1}{A_\alpha} \frac{\partial^2}{\partial \alpha^2} + \frac{1}{A_\alpha^2} \frac{\partial A_\alpha}{\partial \alpha} \frac{\partial}{\partial \alpha} - \frac{1}{A_\beta} \frac{\partial A_\beta}{\partial \beta} \frac{\partial}{\partial \alpha} \right) \left( F_{\beta i} + F_{\beta j} + F_{\beta k} \right)
\]

\[
P_{2,r+1} = \frac{1}{A_\beta} \left[ \frac{1}{R_\beta} \left( \frac{c_\beta}{2A_\beta} - \frac{L_\beta}{A_\beta} \frac{\partial R_\beta}{\partial \beta} \right) - \frac{1}{R_{\beta s}} \left( - \frac{1}{A_\beta} \frac{\partial^2}{\partial \beta^2} + \frac{1}{A_\beta^2} \frac{\partial A_\beta}{\partial \beta} \frac{\partial}{\partial \beta} \right) \left( F_{\alpha i} + F_{\alpha j} + F_{\alpha k} \right) \right]
\]

\[
+ \frac{c_\beta}{2A_\beta} \left( - \frac{1}{A_\beta} \frac{\partial^2}{\partial \beta^2} + \frac{1}{A_\beta^2} \frac{\partial A_\beta}{\partial \beta} \frac{\partial}{\partial \beta} + \frac{1}{A_\beta^2} \frac{\partial A_\beta}{\partial \beta} \frac{\partial}{\partial \beta} \right) \left( F_{\alpha i} + F_{\alpha j} + F_{\alpha k} \right) \right]
\]

\[
P_{2,r+2} = \frac{1}{2A_\beta^2} \left( - \frac{b_\beta}{A_\beta} \frac{\partial A_\beta}{\partial \alpha} + \frac{c_\beta}{A_\beta^2} \frac{\partial A_\beta}{\partial \beta} \right)
\]

\[
+ \frac{c_\beta}{2A_\beta} \left( - \frac{1}{A_\beta} \frac{\partial^2}{\partial \beta^2} + \frac{1}{A_\beta^2} \frac{\partial A_\beta}{\partial \beta} \frac{\partial}{\partial \beta} \right) \left( F_{\alpha i} + F_{\alpha j} + F_{\alpha k} \right)
\]

\[
+ \frac{b_\beta}{2A_\beta} \left( - \frac{1}{A_\beta} \frac{\partial^2}{\partial \beta^2} + \frac{1}{A_\beta^2} \frac{\partial A_\beta}{\partial \beta} \frac{\partial}{\partial \beta} \right) \left( F_{\beta i} + F_{\beta j} + F_{\beta k} \right).
\]
\[ p_{z, r+3} = \frac{1}{A_\beta} \left( -\frac{1}{A_\beta} \frac{\partial^2}{\partial \beta^2} - \frac{1}{A_\beta^2} \frac{\partial^2 A_A}{\partial \alpha \partial \beta} + \frac{1}{A_\beta} \frac{\partial^2 A_A}{\partial \beta \partial \beta} \right) F_{\alpha \beta} \]

\[ p_{z, r+4} = \frac{1}{A_\beta} \left( -\frac{1}{A_\beta} \frac{\partial^2}{\partial \beta^2} - \frac{1}{A_\beta^2} \frac{\partial^2 A_A}{\partial \alpha \partial \beta} + \frac{1}{A_\beta} \frac{\partial^2 A_A}{\partial \beta \partial \beta} \right) F_{\rho \beta} \]

\[ p_{z, r} = \frac{1}{A_\beta} \left[ \frac{1}{R_\alpha} \left( \frac{b_A}{2 \Delta} - \frac{L_s}{A_\alpha} \frac{\partial A_A}{\partial \beta} \right) - \frac{1}{A_\beta} R_{\alpha \delta} \left( -\frac{\partial^2}{\partial \alpha \partial \delta} + \frac{1}{A_\alpha} \frac{\partial^2 A_A}{\partial \beta \partial \delta} \frac{\partial A_A}{\partial \alpha} \right) \right] F_{\rho \beta} \]

\[ p_{z, r+1} = \frac{1}{A_\alpha} \left[ \frac{1}{R_\alpha} \left( \frac{b_A}{2 \Delta} - \frac{b_A}{A_\alpha} \frac{\partial A_A}{\partial \beta} \right) - \frac{1}{A_\beta} R_{\rho \beta} \left( -\frac{\partial^2}{\partial \alpha \partial \beta} + \frac{1}{A_\alpha} \frac{\partial^2 A_A}{\partial \beta \partial \delta} \frac{\partial A_A}{\partial \alpha} \right) \right] F_{\alpha \delta} \]

\[ p_{z, r+2} = \frac{1}{A_\alpha A_\beta} \left[ \frac{1}{2 \Delta} \left( \frac{b_A}{A_\alpha} \frac{\partial A_A}{\partial \beta} + \frac{b_A}{A_\beta} \frac{\partial A_A}{\partial \alpha} \right) \right. \]

\[ + \frac{c_s}{2 \Delta} \left( -\frac{\partial^2}{\partial \alpha \partial \beta} + \frac{1}{A_\alpha} \frac{\partial A_A}{\partial \beta} \frac{\partial A_A}{\partial \alpha} + \frac{1}{A_\beta} \frac{\partial A_A}{\partial \alpha} \frac{\partial A_A}{\partial \beta} \right) \left( F_{\alpha i} + F_{\alpha j} + F_{\alpha k} \right) \]

\[ + \frac{b_A}{2 \Delta} \left( -\frac{\partial^2}{\partial \alpha \partial \beta} + \frac{1}{A_\alpha} \frac{\partial A_A}{\partial \beta} \frac{\partial A_A}{\partial \alpha} + \frac{1}{A_\beta} \frac{\partial A_A}{\partial \alpha} \frac{\partial A_A}{\partial \beta} \right) \left( F_{\beta i} + F_{\beta j} + F_{\beta k} \right) \]

\[ p_{z, r+3} = \frac{1}{A_\alpha A_\beta} \left( -\frac{\partial^2}{\partial \alpha \partial \beta} + \frac{1}{A_\alpha} \frac{\partial A_A}{\partial \beta} \frac{\partial A_A}{\partial \alpha} + \frac{1}{A_\beta} \frac{\partial A_A}{\partial \alpha} \frac{\partial A_A}{\partial \beta} \right) F_{\alpha \beta} \]

\[ p_{z, r+4} = \frac{1}{A_\alpha A_\beta} \left( -\frac{\partial^2}{\partial \alpha \partial \beta} + \frac{1}{A_\alpha} \frac{\partial A_A}{\partial \beta} \frac{\partial A_A}{\partial \alpha} + \frac{1}{A_\beta} \frac{\partial A_A}{\partial \alpha} \frac{\partial A_A}{\partial \beta} \right) F_{\rho \beta} \]

For \( s = 1, 2, 3 \); \( r = 55-4 \); \( i \approx 1, j \approx 2, k \approx 3 \).
The $S$ matrix is

$$S = \begin{bmatrix}
L_i & 0 & 0 & 0 & 0 & | & L_j & 0 \\
0 & L_i & 0 & 0 & 0 & | & 0 & L_j \\
-\frac{F_{\alpha i}}{R_{\alpha i}} & -\frac{F_{\alpha i}}{R_{\beta i}} & S_{3,3} & F_{\alpha i} & F_{\beta i} & | & -\frac{F_{\beta i}}{R_{\alpha j}} & -\frac{F_{\alpha i}}{R_{\beta j}} \\
0 & 0 & 0 & | & L_k & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & | & 0 & L_k & 0 & 0 & 0 \\
S_{3,8} & F_{\alpha i} & F_{\beta i} & -\frac{F_{\alpha k}}{R_{\alpha k}} & -\frac{F_{\alpha k}}{R_{\beta k}} & S_{3,13} & F_{\alpha k} & F_{\beta k}
\end{bmatrix}$$

$$S_{3,3} = L_i + \frac{c_i}{2\Delta} (F_{\alpha i} + F_{\alpha j} + F_{\alpha k}) + \frac{b_i}{2\Delta} (F_{\beta i} + F_{\beta j} + F_{\beta k}).$$

$$S_{3,8} = L_j + \frac{c_j}{2\Delta} (F_{\alpha i} + F_{\alpha j} + F_{\alpha k}) + \frac{b_j}{2\Delta} (F_{\beta i} + F_{\beta j} + F_{\beta k}).$$

$$S_{3,13} = L_k + \frac{c_k}{2\Delta} (F_{\alpha i} + F_{\alpha j} + F_{\alpha k}) + \frac{b_k}{2\Delta} (F_{\beta i} + F_{\beta j} + F_{\beta k}).$$

The vector $\{q_{\gamma_i}\}$ is given by

$$\{q_{\gamma_i}\}^T = \left\{ \left( q_{\alpha L_i} - q_n \frac{F_{\alpha i}}{R_{\alpha i}} \right) \left( q_{\beta L_j} - q_n \frac{F_{\beta i}}{R_{\beta i}} \right) \left( q_{\gamma S_{3,3}} \right) \right. \left. \left( q_{\gamma F_{\alpha i}} \right) \left( q_{\gamma F_{\beta i}} \right) \left( q_{\gamma L_j} - q_n \frac{F_{\beta j}}{R_{\alpha j}} \right) \left( q_{\gamma L_i} - q_n \frac{F_{\alpha j}}{R_{\beta j}} \right) \left( q_{\gamma L_k} - q_n \frac{F_{\alpha k}}{R_{\alpha k}} \right) \left( q_{\gamma S_{3,13}} \right) \left( q_{\gamma F_{\alpha k}} \right) \left( q_{\gamma F_{\beta k}} \right) \right\}. $$
The coefficients of the $M_t$ matrix are:

\[
m_{t,r} = L_l L_s + \frac{F_{\beta l} F_{\beta s}}{R_{\beta l} R_{\beta s}}
\]

\[
m_{t,r+1} = \frac{F_{\beta l}}{R_{\beta l}} \frac{F_{\alpha s}}{R_{\alpha s}}
\]

\[
m_{t,r+2} = -\frac{F_{\beta l}}{R_{\beta l}} S_{3,r+2}
\]

\[
m_{t,r+3} = -\frac{F_{\beta l}}{R_{\beta l}} F_{\alpha s}
\]

\[
m_{t,r+4} = -\frac{F_{\beta l}}{R_{\beta l}}_F_{\beta s}
\]

\[
m_{t+1,r} = \frac{F_{\alpha l}}{R_{\alpha l}} \frac{F_{\beta s}}{R_{\beta s}}
\]

\[
m_{t+1,r+1} = -L_l L_s + \frac{F_{\alpha l} F_{\alpha s}}{R_{\alpha l} R_{\alpha s}}
\]

\[
m_{t+1,r+2} = -\frac{F_{\alpha l}}{R_{\alpha l}} S_{3,r+2}
\]

\[
m_{t+1,r+3} = -\frac{F_{\alpha l}}{R_{\alpha l}} F_{\alpha s}
\]

\[
m_{t+1,r+4} = -\frac{F_{\alpha l}}{R_{\alpha l}} F_{\beta s}
\]

\[
m_{t+2,r} = -\frac{F_{\alpha l}}{R_{\alpha l}} S_{3,t+2}
\]

\[
m_{t+2,r+1} = -\frac{F_{\alpha l}}{R_{\alpha l}} S_{3,t+2}
\]

\[
m_{t+2,r+2} = S_{3,r+2} S_{3,t+2}
\]

\[
m_{t+2,r+3} = F_{\alpha l} S_{3,t+2}
\]

\[
m_{t+2,r+4} = F_{\beta l} S_{3,t+2}
\]
\[
\begin{align*}
m_{t+3,r} &= F_{\alpha l} \frac{F_{\beta s}}{R_{\alpha s}} \\
m_{t+3,r+1} &= F_{\alpha l} \frac{F_{\alpha s}}{R_{\beta s}} \\
m_{t+3,r+2} &= F_{\alpha l} S_{3,r+2} \\
m_{t+3,r+3} &= F_{\alpha l} F_{\alpha s} \\
m_{t+3,r+4} &= F_{\alpha l} F_{\beta s} \\
m_{t+4,r} &= -F_{\beta l} \frac{F_{\beta s}}{R_{\alpha s}} \\
m_{t+4,r+1} &= -F_{\beta l} \frac{F_{\alpha s}}{R_{\beta s}} \\
m_{t+4,r+2} &= F_{\beta l} S_{3,r+2} \\
m_{t+4,r+3} &= F_{\beta l} F_{\alpha s} \\
m_{t+4,r+4} &= F_{\beta l} F_{\beta s}
\end{align*}
\]

for \( s = 1, 2, 3 \); \( r = 5s-4 \)
\( l = 1, 2, 3 \); \( t = 5l-4 \)
\( i \approx 1 \), \( j \approx 2 \), \( k \approx 3 \).
\[ \psi_{ik} = L_i^2 L_k + \frac{1}{2} L_i L_j L_k \]

\[ \frac{\partial \psi_{ik}}{\partial \alpha} = \frac{1}{2 \Delta} \left[ b_i L_k \left( 2 L_i + \frac{1}{2} L_j \right) + \frac{1}{2} b_j L_i L_k + b_k L_i \left( L_i + \frac{1}{2} L_j \right) \right] \]

\[ \frac{\partial \psi_{ik}}{\partial \beta} = \frac{1}{2 \Delta} \left[ c_i L_k \left( 2 L_i + \frac{1}{2} L_j \right) + \frac{1}{2} c_j L_i L_k + c_k L_i \left( L_i + \frac{1}{2} L_j \right) \right] \]

\[ \frac{\partial^2 \psi_{ik}}{\partial \alpha^2} = \frac{1}{4 \Delta^2} \left[ b_i b_j L_k + b_i b_k \left( 4 L_i + L_j \right) + b_j b_k L_i + 2 b_i^2 L_k \right] \]

\[ \frac{\partial^2 \psi_{ik}}{\partial \beta^2} = \frac{1}{4 \Delta^2} \left[ c_i c_j L_k + c_i c_k \left( 4 L_i + L_j \right) + c_j c_k L_i + 2 c_i^2 L_k \right] \]

\[ \frac{\partial^2 \psi_{ik}}{\partial \alpha \partial \beta} = \frac{1}{4 \Delta^2} \left[ L_i \left( 2 b_i c_k + \frac{1}{2} b_j c_k + 2 b_k c_i + \frac{1}{2} b_k c_j \right) + L_j \left( \frac{1}{2} b_k c_i + \frac{1}{2} b_i c_k \right) + L_k \left( \frac{1}{2} b_i c_j + \frac{1}{2} b_j c_i + 2 b_i c_i \right) \right] \]

The corresponding expressions for \[ \psi_{j1} \] and \[ \psi_{kj} \] can be obtained from those shown for \[ \psi_{1k} \] by a cyclic interchange of the \( i, j \) and \( k \).
\[ \psi_{ij} = L_i^2 L_j + \frac{1}{2} L_i L_j L_k \]

\[ \frac{\partial \psi_{ij}}{\partial \alpha} = \frac{1}{2\Delta} \left[ b_i L_j (2L_i + \frac{1}{2} L_k) + b_j L_i (L_i + \frac{1}{2} L_k) + \frac{1}{2} b_k L_i L_j \right] \]

\[ \frac{\partial \psi_{ij}}{\partial \beta} = \frac{1}{2\Delta} \left[ c_i L_j (2L_i + \frac{1}{2} L_k) + c_j L_i (L_i + \frac{1}{2} L_k) + \frac{1}{2} c_k L_i L_j \right] \]

\[ \frac{\partial^2 \psi_{ij}}{\partial \alpha^2} = \frac{1}{4\Delta^2} \left[ b_i b_j (4L_i + L_k) + b_i b_j L_j + b_j b_k L_i + 2b_i L_j \right] \]

\[ \frac{\partial^2 \psi_{ij}}{\partial \beta^2} = \frac{1}{4\Delta^2} \left[ c_i c_j (4L_i + L_k) + c_i c_j L_j + c_j c_k L_i + 2c_i^2 L_j \right] \]

\[ \frac{\partial^2 \psi_{ij}}{\partial \alpha \partial \beta} = \frac{1}{4\Delta^2} \left[ L_i \left( 2b_i c_j + 2b_j c_i + \frac{1}{2} b_j c_k + \frac{1}{2} b_k c_j \right) + L_j \left( 2b_i c_i + \frac{1}{2} b_i c_k + \frac{1}{2} b_k c_i \right) + L_k \left( \frac{1}{2} b_i c_j + \frac{1}{2} b_j c_i \right) \right]. \]

The corresponding expressions for \( \psi_{jk} \) on \( \psi_{ki} \) can be obtained from those shown for \( \psi_{ij} \) by a cyclic interchange of the \( i, j \) and \( k \).
REFERENCES


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