MEAN VALUE PROPERTIES OF
GENERALIZED MATRIX FUNCTIONS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By


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The Ohio State University
1966

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ACKNOWLEDGMENTS

I am grateful to Professor Herbert J. Ryser of Syracuse University, my former faculty adviser, who first introduced me to matrix theory. His ready willingness to concern himself with the problems of an old student will not be forgotten.

Professor Marvin D. Marcus of The University of California, Santa Barbara, has contributed his ideas to virtually all the principal results in this paper. I am most pleased to acknowledge his participation.

Finally, I thank Lt. Col. John V. Armitage, USAF, Director of the Applied Mathematics Research Laboratory, through whom I received the support of the Aerospace Research Laboratories.
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PUBLICATIONS

A Search for Analogues of the Mathieu Groups (with E. T. Parker), Mathematical Tables and Other Aids to Computation, 12(1958), 38-43.

Permanents of Incidence Matrices, Mathematical Tables and Other Aids to Computation, 14(1960), 262-266.

Reduction of Two-Point Boundary Value Problems in a Vector Space to Initial Value Problems by Projection (with Karl G. Cuderley), Numerische Mathematik, 8(1966), 270-289.

FIELDS OF STUDY

Major Field: Mathematics

Studies in Algebra. Professors Herbert J. Ryser, Marshall Hall, Jr., Erwin Kleinfeld, and Henry B. Mann

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I. INTRODUCTION

Only in recent years have the problems and methods of combinatorial mathematics begun to take the form of a coherent mathematical discipline. Certainly, the most important of the unifying forces in combinatorics has been the role of matrix theory, for one of the most useful tools in the study of combinatorial designs is the incidence matrix. Let $S$ denote a set of $n$ elements $x_1, \ldots, x_n$ and let $T_1, \ldots, T_m$ denote a collection of $m$ subsets of $S$. Form the $m$ by $n$ $(0,1)$-matrix $A = [a_{ij}]$ by setting $a_{ij} = 1$ or $0$ according as $x_j$ is an element of the set $T_i$ or not. $A$ is called the incidence matrix for the subsets $T_1, \ldots, T_m$ of the set $S$, and it describes completely this combinatorial configuration. The study of such designs is often facilitated by determining which if any of its properties are left invariant by certain permutations of the subscripts $(1, \ldots, m)$ of $T_1, \ldots, T_m$ and $(1, \ldots, n)$ of $x_1, \ldots, x_n$. Such a rearrangement of the elements $x_j$ in the sets $T_i$ is equivalent to permutations of the rows and columns of the incidence matrix $A$. Hence, we are interested in matrix functions which exhibit symmetry properties similar to those we study in the combinatorial designs. Of these matrix functions, the permanent is a conspicuous example.

Suppose that $A = [a_{ij}]$ is an $n$-square matrix with entries in a field $R$. Then if $S_n$ is the full symmetric group of degree $n$

$$\text{per } A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$
defines the permanent of the matrix $A$. per $A$ is linear in the rows (columns) of $A$ and is clearly invariant under all permutations of the rows (columns) of $A$. Moreover, if $A$ is an incidence matrix, per $A$ counts the number of systems of distinct representatives for the combinatorial arrangement determining the matrix $A$. Another example is the familiar determinant function

$$\det A = \sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}$$

which is skew-symmetric and alternating as well as linear in the rows (columns) of $A$. $\varepsilon(\sigma)$ represents the sign function on $S_n$; i.e., $\varepsilon(\sigma)$ is +1 for $\sigma$ an even permutation and -1 for $\sigma$ odd. Combinatorial aspects of the permanent and determinant are discussed at length in the book of H. Ryser [25]\(^1\). The permanent as a matrix function is treated in the recent survey of M. Marcus and H. Minc [18] which contains an extensive bibliography and list of references. These matrix functions are examples of a very general class of matrix functions first studied systematically by I. Schur [27].

Let $A = [a_{ij}]$ be any $n$-square matrix with entries in a field $R$, $H$ any subgroup of $S_n$, and $\lambda: H \to R$ any function from the group $H$ to the field $R$. The matrix function

$$d^{H}_{\lambda}(A) = \sum_{\sigma \in H} \lambda(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}$$

is called the generalized matrix function associated with $H$ and $\lambda$. If $H$ is $S_n$ and $\lambda$ is identically 1 then $d^{S}_{\lambda}(A)$ is the permanent function. If $\lambda = \varepsilon$ then $d^{S}_{\varepsilon}(A) = \det A$. We shall drop the superscript $H$ in the nota-

\(^1\) Numbers in brackets correspond to citations in REFERENCES.
tion $d_{\lambda}^H(A)$ and reserve this position for exponents. The dependence of $d_{\lambda}(A)$ on the subgroup is always implicitly assumed. Schur was interested in comparing such functions over the class of positive semi-definite hermitian matrices. By relying heavily on exceedingly intricate methods of group representations, he proved the matrix inequality

$$\det A \leq d_{\chi}(A)$$

(1.1)

where $\chi$ is a character of degree 1 of the subgroup H of $S_n$. New techniques for proving Schur's inequality have been recently introduced by M. Marcus and collaborators [10, 19, 21]. However, these methods were adopted initially to attack a different problem, the van der Waerden conjecture, a matrix problem with definite combinatorial interest.

Let $A$ denote a matrix with non-negative entries whose row and column sums are all unity. Such a matrix is called doubly stochastic. In 1926 B. van der Waerden [29] posed the problem of determining the minimal permanent over all $n$-square doubly stochastic matrices $A$ and conjectured that

$$\text{per } A \geq \frac{n!}{n^n}$$

with equality if and only if $A = J_n$, the $n$-square matrix with every entry equal to $1/n$. Using methods of multilinear algebra, M. Marcus and M. Newman [20] in 1961 succeeded in resolving the van der Waerden conjecture in the affirmative in case $A$ is positive semi-definite hermitian. The conjecture remains undecided for arbitrary doubly stochastic $A$. Since this initial success, methods of multilinear algebra have been brought to bear on many of the questions arising from the paper of
Schur. Indeed, it is inequalities for generalized matrix functions with which we shall be concerned here.

In §II and §III we present sufficient material from multilinear algebra and the theory of symmetry classes of tensors to enable the reader to familiarize himself with current literature. Few of these results are new, but the presentation in readable, coherent form is due to M. Marcus [12]. §IV contains some results from current periodical literature necessary to maintain continuity and provide motivation for our principal results which appear in §V.

The first of our results estimates the arithmetic mean of the generalized matrix functions of the \( \binom{n}{m} \) principal \( m \)-square submatrices of an \( n \)-square positive semi-definite hermitian matrix \( A \). A lower bound on this sum is obtained in terms of the \( m \)-th elementary symmetric function of the row sums of \( A \) when \( \lambda \) is identically 1. A companion result in terms of the homogeneous product sums is obtained for the permanent function. Next we imbed the inequality (1.1) of Schur in a continuous hierarchy of matrix inequalities. We make use of the classical power means and the spectral theorem in an appropriate symmetry class of tensors. The identification of matrix functions with power means enables us to use reversal techniques to produce complementary matrix inequalities. Examples are provided which produce new inequalities for the permanent and determinant. Finally, in §VI we discuss some recent results which promise to be very effective in producing new matrix inequalities. We also pose some questions.
II. MULTILINEAR FUNCTIONS AND TENSORS

Let $V_1, \ldots, V_m$ and $U$ be vector spaces over the same field $R$ and let $V_1 \times \cdots \times V_m$ denote the cartesian product of $V_1, \ldots, V_m$. A function $\phi: V_1 \times \cdots \times V_m \to U$ is called $m$-linear or simply multilinear if $\phi = \phi(v_1, \ldots, v_m)$ is linear in each vector variable $v_i \in V_i$, $i = 1, \ldots, m$. The totality of such $m$-linear functions $\phi$ we denote by $M(V_1, \ldots, V_m; U)$. If $U = R$ we call $\phi$ an $m$-linear functional. For $c$ and $d$ in $R$ and $\phi$ and $\theta$ in $M(V_1, \ldots, V_m; U)$ define $c\phi + d\theta$ to be the $m$-linear function whose value at $(v_1, \ldots, v_m)$ is $c\phi(v_1, \ldots, v_m) + d\theta(v_1, \ldots, v_m)$. Under this definition $M(V_1, \ldots, V_m; U)$ is itself a vector space over $R$. Before discussing a basis for $M(V_1, \ldots, V_m; U)$ we introduce some notation.

Let $n_1, \ldots, n_m$ be positive integers. Denote by $\Gamma^{n_1, \ldots, n_m}$ the totality of sequences of positive integers $\omega = (\omega_1, \ldots, \omega_m)$ satisfying $1 \leq \omega_k \leq n_k$, $k = 1, \ldots, m$. If $1 \leq r \leq n$ then $\Gamma_{r, n}$ is the set $n, \ldots, n$ in which $n$ is repeated $r$ times. When the context permits we abbreviate $\Gamma^{n_1, \ldots, n_m}$ or $\Gamma_{r, n}$ to $\Gamma$. We order the sequences $\alpha = (\alpha_1, \ldots, \alpha_m)$ and $\beta = (\beta_1, \ldots, \beta_m)$ in $\Gamma^{n_1, \ldots, n_m}$: $\alpha$ precedes $\beta$ in the lexicographic ordering if the first non-zero difference $\beta_i - \alpha_i$ is positive, $i = 1, \ldots, m$, and we write $\alpha < \beta$.

**THEOREM 2.1.** Let $\dim U = n$ and $\dim V_i = n_i$, $i = 1, \ldots, m$. Then $M = M(V_1, \ldots, V_m; U)$ is a vector space over $R$ of dimension $n_1 \cdots n_m$.

**PROOF.** It is routine to verify that $M$ is a vector space under the
above mentioned operations of addition and scalar multiplication. We con-
struct a basis for $M$. For each $t$, $t = 1, \ldots, m$, let $\{e_{t1}, \ldots, e_{tn_t}\}$ be
a basis of $V_t$ and let $\{u_1, \ldots, u_n\}$ be a basis of $U$. For each $\alpha \in \Gamma$, $n_1, \ldots, n_m = \Gamma$ and each $j$, $j = 1, \ldots, n$, set $\phi_{\alpha,j}(e_{1\alpha_1}, \ldots, e_{m\alpha_m}) = \delta_{\alpha,\beta}u_j$ for $\beta = (\beta_1, \ldots, \beta_m) \in \Gamma$ where $\delta_{\alpha,\beta}$ is the Kronecker $\delta$ on $\Gamma$. Extend $\phi_{\alpha,j}$ to all of $V_1 \times \cdots \times V_m$ by multilinear extension, i.e.,

\[
\phi_{\alpha,j}(x_1, \ldots, x_m) = \sum_{\beta \in \Gamma} \prod_{t=1}^m c_{t\beta_t} \phi_{\alpha,j}(e_{1\beta_1}, \ldots, e_{m\beta_m})
\]

where $x_t = \sum_{j=1}^{n_t} c_{tj}e_{tj}, t = 1, \ldots, m$. Clearly $\phi_{\alpha,j} \in M$. Suppose that for
each $\beta \in \Gamma$, $\phi(e_{1\beta_1}, \ldots, e_{m\beta_m}) = \sum_{j=1}^n c_{\beta_j}u_j$. We assert

\[
\phi = \sum_{\beta \in \Gamma} \sum_{j=1}^n c_{\beta_j} \phi_{\beta,j}.
\]

Applying the right hand side of (2.2) to the $m$-tuple of vectors

$(e_{1\alpha_1}, \ldots, e_{m\alpha_m})$ we obtain

\[
\sum_{\beta \in \Gamma} \sum_{j=1}^n c_{\beta_j} \phi_{\beta,j}(e_{1\alpha_1}, \ldots, e_{m\alpha_m}) = \sum_{\beta \in \Gamma} \sum_{j=1}^n c_{\beta_j} \delta_{\alpha,\beta}u_j
\]

\[
= \sum_{\beta \in \Gamma} \delta_{\alpha,\beta} \sum_{j=1}^n c_{\beta_j}u_j
\]

\[
= \sum_{\beta \in \Gamma} \delta_{\alpha,\beta} \phi(e_{1\beta_1}, \ldots, e_{m\beta_m})
\]

\[
= \phi(e_{1\alpha_1}, \ldots, e_{m\alpha_m}).
\]

Thus for every $\alpha \in \Gamma$, $\phi$ agrees with the right-hand side of (2.2) on

$(e_{1\alpha_1}, \ldots, e_{m\alpha_m})$; hence, by multilinearity on all of $V_1 \times \cdots \times V_m$. If

$\phi = 0$ in (2.2), we evaluate its right-hand side on all $m$-tuples of vec-
tors $(e_{1\omega_1}, \ldots, e_{m\omega_m})$, $\omega \in \Gamma$, and obtain $\sum_{j=1}^n c_{\omega_j}u_j = 0$. But this implies

$c_{\omega_j} = 0$ for $\omega \in \Gamma$, $j = 1, \ldots, n$, so that the $\phi_{\alpha,j}, \alpha \in \Gamma, j = 1, \ldots, n$, are linearly independent. There are exactly $n! n_{t=1}^m n_t$ such functions.
Let $E_i = \{e_{i1}, \ldots, e_{in_i}\}, i = 1, \ldots, m$, and $F = \{u_1, \ldots, u_n\}$ be bases of $V_i, i = 1, \ldots, m$, and $U$, respectively. Then the functions $\phi_{\alpha,j}$ defined by (2.1) for $\alpha \in \Gamma, j = 1, \ldots, n$, and ordered lexicographically in the $(m+1)$-tuple $(\alpha, j) = (\alpha_1, \ldots, \alpha_m, j)$ are called dual to the bases $E_1, \ldots, E_m, F$. We have shown that the $\phi_{\alpha,j}$ constitute a basis of $M$.

Denote by $V_i^*, i = 1, \ldots, m$, the dual space of $V_i$ and let $\phi_i \in V_i^*$. The function $\phi: V_1 \times \cdots \times V_m \to R$ defined by

$$
\phi_{V_1, \ldots, V_m} = \prod_{i=1}^m \phi_i(v_i)
$$

is the product of the linear functionals $\phi_i$ and is written $\phi = \prod_{i=1}^m \phi_i$.

**THEOREM 2.2.** The product $\phi = \prod_{i=1}^m \phi_i$ is an $m$-linear functional. If $\{h_{i1}, \ldots, h_{in_i}\}$ is a basis of $V_i^*, i = 1, \ldots, m$, then the $m$-linear functionals $\phi_{\alpha} = \prod_{t=1}^m h_{t\alpha_t}, \alpha \in \Gamma, n_1, \ldots, n_m$, form a basis of $M(V_1, \ldots, V_m: R)$.

**PROOF.** It is routine to verify that $\phi$ is $m$-linear. We proceed with the proof of the second assertion. Let $E_i = \{e_{i1}, \ldots, e_{in_i}\}$ be a basis of $V_i$ dual to $\{h_{i1}, \ldots, h_{in_i}\}, i = 1, \ldots, m$. We use the multiplicative identity 1 of $R$ as a basis of $R$ and compute

$$
\phi_{\alpha}(e_{1\beta_1}, \ldots, e_{m\beta_m}) = \prod_{t=1}^m h_{t\alpha_t}(e_{t\beta_t}) = \prod_{t=1}^m \delta_{\alpha_t, \beta_t} = \delta_{\alpha, \beta}, \alpha, \beta \in \Gamma.
$$

Thus the $\phi_{\alpha}, \alpha \in \Gamma$, are dual to the bases $E_i$ of $V_i$ and (1) of $R, i = 1, \ldots, m$, and hence form a basis of $M(V_1, \ldots, V_m: R)$ by Theorem 2.1.

Consider now the vector space $M(V_1, \ldots, V_m: R)$. We call the dual space of $M(V_1, \ldots, V_m: R)$ the tensor product of $V_1, \ldots, V_m$ and denote...
it by $V_1 \otimes \cdots \otimes V_m$ or $\bigotimes_{i=1}^m V_i$. Given $x_i \in V_i$, $i = 1, \ldots, m$, we define the tensor product of vectors $x_i$ to be the element $f \in \bigotimes_{i=1}^m V_i$ whose value on any $\phi \in M(V_1, \ldots, V_m : R)$ is $\phi(x_1, \ldots, x_m) = f(\phi)$ and write $x_1 \otimes \cdots \otimes x_m$ for $f$. We call the elements of $\bigotimes_{i=1}^m V_i$ tensors. If for a tensor $g$ there exist $x_i \in V_i$, $i = 1, \ldots, m$, so that $g = x_1 \otimes \cdots \otimes x_m$ then $g$ is a decomposable tensor.

**Theorem 2.3.** If $\dim V_i = n_i$, $i = 1, \ldots, m$, then $\dim V_1 \otimes \cdots \otimes V_m = \prod_{i=1}^m n_i$. If $E_i = \{e_{i1}, \ldots, e_{in_i}\}$ is a basis of $V_i$, $i = 1, \ldots, m$, then the decomposable tensors $e_{i\alpha_1} \otimes \cdots \otimes e_{m\alpha_m}$, $\alpha \in \Gamma_1, \ldots, n_m$, form a basis of $V_1 \otimes \cdots \otimes V_m$. This basis is dual to the basis of $M(V_1, \ldots, V_m : R)$ which is in turn dual to the bases $E_i$ of $V_i$ and $\{1\}$ of $R$. The decomposable tensor $x_1 \otimes \cdots \otimes x_m$ is linear in each $x_t$, $t = 1, \ldots, m$. If $x_i = \sum_{j=1}^{n_i} c_{ij} e_{ij}$, $i = 1, \ldots, m$, then

$$x_1 \otimes \cdots \otimes x_m = \sum_{\alpha_1, \ldots, \alpha_m} c_{1\alpha_1} e_{1\alpha_1} \otimes \cdots \otimes c_{m\alpha_m} e_{m\alpha_m}.$$ 

**Proof.** By Theorem 2.1 we have $\dim \bigotimes_{i=1}^m V_i = \dim M(V_1, \ldots, V_m : R) = \prod_{i=1}^m n_i$. The functionals $\phi_\alpha \in M$ defined by

$$\phi_\alpha(e_{1\beta_1} \otimes \cdots \otimes e_{m\beta_m}) = \delta_{\alpha, \beta}, \alpha, \beta \in \Gamma$$

and by multilinear extension in (2.1) form a basis of $M$. Alternatively we write

$$e_{1\beta_1} \otimes \cdots \otimes e_{m\beta_m} (\phi_\alpha) = \delta_{\alpha, \beta}$$

so that the tensors $e_{1\beta_1} \otimes \cdots \otimes e_{m\beta_m}$, $\beta \in \Gamma$, form precisely what was defined to be a dual basis to the basis of $m$-linear functionals $\phi_\alpha$, $\alpha \in \Gamma$. Let $\phi \in M(V_1, \ldots, V_m : R)$. Then for $(x_1, \ldots, x_m) \in V_1 \times \cdots \times V_m$
Thus $x_1 \otimes \cdots \otimes x_m$ is linear in each $x_t$, $t = 1, \ldots, m$. We can now compute

$$x_1 \otimes \cdots \otimes x_m = \sum_{\alpha_1 = 1}^{n_1} c_{\alpha_1} e_{\alpha_1} \otimes \cdots \otimes \sum_{\alpha_m = 1}^{n_m} c_{\alpha_m} e_{\alpha_m}$$

proving Theorem 2.3.

For $\alpha \in \Gamma$, $n_1, \ldots, n_m = \Gamma$ we write $e_\alpha = e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_m}$. If $x_t = \sum_{j=1}^{n_t} c_{\tau_t} e_{\tau_t}$, $t = 1, \ldots, m$, we set $c_\alpha = \prod_{t=1}^{m} c_{\tau_t}$ and write

$$x_1 \otimes \cdots \otimes x_m = \sum_{\alpha \in \Gamma} c_\alpha e_\alpha.$$

There is available an alternative definition of the tensor product $V_1 \otimes \cdots \otimes V_m$. Let $V_1, \ldots, V_m$ be vector spaces over the field $R$. Let $P$ be a vector space over $R$ with the following property. There exists an $f \in M(V_1, \ldots, V_m; P)$ such that if $U$ is any vector space over $R$ and $\phi \in M(V_1, \ldots, V_m; U)$ then there exists a linear map $h: P \to U$ satisfying $\phi = hf$. The pair $(P, f)$ is said to satisfy the universal factorization property. We say that a pair $(P, f)$ which satisfies the universal factorization property and also has the property that the linear closure
of the range of \( f \) coincides with \( P \), i.e., \( \langle \text{rng } f \rangle = P \), is called a tensor product of \( V_1, \ldots, V_m \). We illustrate the mappings involved in a diagram.

\[
\begin{array}{c}
V_1 \times \cdots \times V_m \\
\downarrow \phi \\
U
\end{array}
\xleftarrow{f} P
\xleftarrow{h} Q
\xleftarrow{g} \begin{array}{c}
V_1 \times \cdots \times V_m \\
\downarrow \phi \\
U
\end{array}
\xleftarrow{k} P

That under this definition the tensor product of \( V_1, \ldots, V_m \) is essentially unique is given by the following result.

**Theorem 2.4.** If \((P, f)\) and \((Q, g)\) are each tensor products of \( V_1, \ldots, V_m \) then there exists a regular linear map \( T: P \rightarrow Q \) from \( P \) onto \( Q \) for which \( g = Tf \). Moreover, \( T \) is unique.

**Proof.** The diagrams

\[
\begin{array}{c}
V_1 \times \cdots \times V_m \\
\downarrow g \\
Q
\end{array}
\xrightarrow{f} P
\xleftarrow{h} Q
\xleftarrow{g} \begin{array}{c}
V_1 \times \cdots \times V_m \\
\downarrow g \\
Q
\end{array}
\xrightarrow{f} P

indicate the mappings involved and the method of proof. There exist linear maps \( h: P \rightarrow Q \) and \( k: Q \rightarrow P \) such that \( g = hf \) and \( f = kg \). Call \( \theta \) the linear map \( kh: P \rightarrow P \) and let \( \nu = f(v_1, \ldots, v_m) \in P \). Then \( \theta(\nu) = \nu \) for any \( \nu \in \text{rng } f \). But \( \langle \text{rng } f \rangle = P \). Thus \( \theta = I_P \), the identity map from \( P \) to itself. Similarly \( hk = I_Q \). Thus \( h = k^{-1} \). Setting \( T = h \) we have \( g = Tf \) as required. We assert \( T \) is unique. Let \( T_1: P \rightarrow Q \) be linear from \( P \) to \( Q \) which also satisfies \( g = T_1f \). Then \( T \) and \( T_1 \) agree on a basis of \( P \) since they agree on \( \text{rng } f \) and \( \langle \text{rng } f \rangle = P \). Since \( T \) and \( T_1 \) agree on a basis, they coincide.
The next result relates the two definitions of tensor product of vector spaces.

**THEOREM 2.5.** Let $V_1, \ldots, V_m$ be vector spaces over the field $R$.

Let $P = V_1 \otimes \cdots \otimes V_m$ and let $f$ be the decomposable tensor $x_1 \otimes \cdots \otimes x_m$. Then $(P, f)$ is a tensor product of $V_1, \ldots, V_m$.

**PROOF.** By Theorem 2.3, $f$ is $m$-linear and $\langle \text{rng } f \rangle = P$. Let $E_1 = \{e_{i1}, \ldots, e_{in_1}\}$ be a basis of $V_i$, $i = 1, \ldots, m$. Let $U$ be any vector space over $R$ and let $\phi \in M(V_1, \ldots, V_m; U)$. Set $u_\alpha = \phi(e_{1\alpha}, \ldots, e_{m\alpha})$, define $h(e_\alpha) = u_\alpha$, $\alpha \in \Gamma_{n_1, \ldots, n_m}$, and extend $h$ to all of $P$ by linearity.

One can verify easily that $(P, f)$ satisfies the universal factorization property.

We have now shown that the tensor products of vector spaces obtained through the two definitions are essentially the same. We are free to make use of either or both in the proofs of results which follow.
III. SYMMETRY CLASSES AND STAR PRODUCTS

Let $S_m$ denote the symmetric group of degree $m$ and let $H$ of order $p$ be a subgroup of $S_m$. Let $\chi: H \to \mathbb{R}$ denote a character of degree $1$ of $H$ to the field $\mathbb{R}$ which we now assume to have characteristic greater than $p$. For $m$ and $n$ positive integers define a binary relation on the sequence set $\Gamma_{m,n}$ by: $\alpha \sim \beta$ if there exists a permutation $\sigma \in H$ for which $\alpha^\sigma = (\alpha_\sigma(1), \ldots, \alpha_\sigma(m)) = (\beta_1, \ldots, \beta_m) = \beta$. It is routine to verify that $\sim$ is an equivalence relation on $\Gamma = \Gamma_{m,n}$. As such it partitions $\Gamma$ into mutually exclusive and exhaustive equivalence classes which we denote by $\Gamma_{m,n}^H(\omega) = \Gamma(\omega)$, $\omega \in \Gamma$. We write $\Delta_{m,n}^H = \Delta$ for a system of distinct representatives for $\sim$ so chosen that each $\omega \in \Delta$ is the first sequence in $\Gamma(\omega)$ in the lexicographic ordering. Denote by $H_\alpha$ the subgroup of $H$ fixing $\alpha$, $\alpha \in \Gamma$, and by $\nu(\alpha)$ its order. Finally we set $\Delta_{m,n}^H = \Delta$ equal to the subset of $\Delta$ consisting of all sequences $\omega$ for which $\sum_{\sigma \in H} \chi(\sigma) \neq 0$. We prove the following combinatorial result. It first appeared in [19].

**LEMMA 3.1.** In the notation just introduced, $\gamma^\sigma$ coincides with each sequence in $\Gamma(\gamma)$ exactly $\nu(\gamma)$ times as $\sigma$ runs over $H$. If $f: \Gamma \to V$ is any function on the sequence set $\Gamma$ to an additive abelian group $V$ then

$$
\sum_{\omega \in \Gamma} f(\omega) = \sum_{\gamma \in \Delta} (1/\nu(\gamma)) \sum_{\sigma \in H} f(\gamma^\sigma).
$$

If $\omega \in \Gamma$ and $\omega \sim \alpha \in \Delta$ then

$$
\sum_{\sigma \in H} \chi(\sigma) = \begin{cases} 
\nu(\omega) & \text{if } \alpha \in \Delta, \\
0 & \text{if } \alpha \notin \Delta.
\end{cases}
$$
PROOF. The sequence $\gamma^\sigma$ runs over $\Gamma(\gamma)$ as $\sigma$ runs over $H$. Let $\omega \in \Gamma(\gamma)$ and $\gamma^\phi = \omega$, $\phi \in H$. Then $\gamma^\sigma = \gamma$ if and only if $\gamma^\sigma \phi = \omega$. The correspondence $\sigma \leftrightarrow \sigma \phi$ is 1-1 so that $\gamma^\sigma$ coincides with $\omega$ precisely as often as does $\gamma$; namely, $\nu(\gamma)$ times. To prove (3.1) we compute

$$\sum_{\omega \in \Gamma} f(\omega) = \sum_{\gamma \in \Delta} \sum_{\omega \in \Gamma(\gamma)} f(\omega) = \sum_{\gamma \in \Delta} \left(1/\nu(\gamma)\right) \sum_{\sigma \in H} f(\gamma^\sigma).$$

Finally, $\omega \sim \alpha \in \Delta$ implies there is a $\phi \in H$ for which $\omega = \alpha^\phi$. Thus $\omega^\sigma = \omega$ if and only if $\alpha^\phi \sigma^\phi^{-1} = \alpha$; i.e., $\sigma \in H_\omega$ if and only if $\phi \sigma^\phi^{-1} \in H_\alpha$. Hence

$$\sum_{\omega \in H_\alpha} \chi(\omega) = \sum_{\omega \in H_\alpha} \chi(\phi^{-1} \theta \phi) = \sum_{\omega \in H_\alpha} \chi(\theta).$$

Let 1 denote the identity of $R$. Since $H_\alpha$ is a group $\sum_{\omega \in H_\alpha} \chi(\omega)$ is either 0 or $\nu(\alpha) \cdot 1 = \nu(\alpha)$ according as $\alpha \in \Delta$ or not. Thus (3.2) is proved.

Let $V$ be a vector space over the field $R$. Let $Q = \bigoplus_{i=1}^m V$ and $f(v_1, \ldots, v_m) = v_1 \otimes \cdots \otimes v_m$. For $\sigma \in S_m$ let $\theta = \sigma^{-1}$ and set $g(v_1, \ldots, v_m) = v_{\theta(1)} \otimes \cdots \otimes v_{\theta(m)}$. By Theorem 2.5 the pair $(Q, f)$ is a tensor product of $V$ taken with itself $m$ times. Thus in the diagram

$$\begin{array}{ccc}
V \times \cdots \times V & \xrightarrow{f} & Q \\
\downarrow{g} & & \downarrow{P(\sigma)} \\
Q & & Q
\end{array}$$

$P(\sigma) : Q \to Q$ is a linear map and satisfies

(3.3) \hspace{1cm} P(\sigma) v_1 \otimes \cdots \otimes v_m = v_{\theta(1)} \otimes \cdots \otimes v_{\theta(m)}.$
$P(\sigma)$ is called a permutation operator. If $\lambda : S_m \to R$ is any function from $S_m$ to $R$ then the linear map $T : Q \to Q$ defined by

$$T = \sum_{\sigma \in S_m} \lambda(\sigma)P(\sigma)$$

is called a symmetry operator. The range of $T$ is called a symmetry class of tensors. We call the tensor $T(v_1 \star \cdots \star v_m)$ the star product of the vectors $v_1, \ldots, v_m$ and denote it by $v_1 \star \cdots \star v_m$. For $v_1, \ldots, v_n$ in $V$ and $\omega \in \Gamma_{m,n}$ we write $v_\omega^* = v_{\omega_1}^* \star \cdots \star v_{\omega_m}^*$. If $H$ is a subgroup of $S_m$ and $\phi \in M(V, \ldots, V : U)$ where $U$ is any vector space over $R$, then $\phi$ is said to be symmetric with respect to $H$ and $\lambda$ if

$$\phi(v_{\sigma(1)}, \ldots, v_{\sigma(m)}) = \lambda(\sigma)\phi(v_1, \ldots, v_m)$$

for all $\sigma \in H$ and all $v_i \in V$, $i = 1, \ldots, m$. The totality of all $\phi \in M(V, \ldots, V : U)$ symmetric with respect to $H$ and $\lambda$ we denote by $M_m(V, U, H, \lambda)$. If $P$ is any vector space over $R$ and $f$ any $m$-linear function in $M(V, \ldots, V : P)$ such that for any $\phi \in M_m(V, U, H, \lambda)$ there exists a linear map $h : P \to U$ satisfying $\phi = hf$ then the pair $(P, f)$ is said to have the universal factorization property with respect to $H$ and $\lambda$. The by now familiar diagram illustrates.

$$\begin{array}{ccc}
V \times \cdots \times V & \xrightarrow{f} & P \\
\downarrow \phi & & \downarrow h \\
& Q & \\
\end{array}$$

If the pair $(P, f)$ satisfies the universal factorization property with respect to $H$ and $\lambda$ and in addition $\langle mg f \rangle = P$, then $(P, f)$ will be called an $m$-th star product of $V$ with respect to $H$ and $\lambda$. The following result is central to our theory.
THEOREM 3.1. Let $V$ be a vector space over $R$ and let $H$ be a subgroup of $S_m$. Let $x$ be a character of degree 1 of $H$ and assume that $R$ has characteristic greater than $p$, the order of $H$. Let

$$T = (1/p)\sum_{\sigma \in H} x(\sigma)P(\sigma)$$

and denote rng $T$ in the map $T: \bigotimes_{i=1}^m V \rightarrow \bigotimes_{i=1}^m V$ by $V^m_x(H)$. Then if $e_1, \ldots, e_n$ is a basis of $V$, the tensors $e^*_\omega$, $\omega \in \Delta$, form a basis of $V^m_x(H)$. The function $f(v_1, \ldots, v_m) = v_1 \ast \cdots \ast v_m$ is linear in each $v_t$, $t = 1, \ldots, m$, and symmetric with respect to $H$ and $x$; the pair $(V^m_x(H), f)$ is an $m$-th star product of $V$ with respect to $H$ and $x$.

PROOF. Since $v_1 \otimes \cdots \otimes v_m$ is linear in each $v_t$ and since $T$ is linear, $v_1 \ast \cdots \ast v_m$ is linear in each $v_t$. We assert $T$ is idempotent; i.e., $T^2 = T$. We have for $\phi$ in $H$

$$P(\phi)T = P(\phi)(1/p)\sum_{\sigma \in H} x(\sigma)P(\sigma)$$

$$= (1/p)\sum_{\sigma \in H} x(\sigma)P(\phi)P(\sigma)$$

$$= (1/p)\sum_{\sigma \in H} x(\sigma)P(\phi\sigma)$$

$$= (1/p)\sum_{\theta \in H} x(\phi^{-1}\theta)P(\theta)$$

$$= x(\phi^{-1})T.$$ 

Similarly $TP(\phi) = x(\phi^{-1})T$. Hence, $x(\phi)P(\phi)T = T$. Summing this expression over $H$ we obtain $T^2 = T$. We also have from these results that

$${v}_1(1) \ast \cdots \ast {v}_m(m) = TP(\phi^{-1})(v_1 \otimes \cdots \otimes v_m)$$

$$= x(\phi)T(v_1 \otimes \cdots \otimes v_m).$$
so that $f(v_1, \ldots, v_m) = v_1 \ast \cdots \ast v_m$ is symmetric with respect to $\mathcal{H}$ and $x$. Next let $v_i \in V$, $v_i = \sum_{j=1}^{n} c_{ij} e_j$, $i = 1, \ldots, m$. Then

$$v_1 \ast \cdots \ast v_m = T(v_1 \boxtimes \cdots \boxtimes v_m)$$

$$= T\left(\sum_{\omega \in \Gamma} c_{\omega} e_{\omega}\right)$$

$$= \sum_{\omega \in \Gamma} c_{\omega} e_{\omega}^*$$

$$= \sum_{\omega \in \Delta} \left(1/\nu(\omega)\right) \sum_{\sigma \in \mathcal{H}} c_{\omega \sigma} e_{\omega \sigma}$$

$$= \sum_{\omega \in \Delta} \left(1/\nu(\omega)\right) (\sum_{\sigma \in \mathcal{H}} \chi(\sigma) c_{\omega \sigma}) e_{\omega}^*;$$

the last two relations hold by Lemma 3.1. Now suppose $\omega \in \Delta$ but $\omega \notin \Delta$ so that $\sum_{\sigma \in \mathcal{H}} \chi(\sigma) = 0$. Then for any $\sigma \in \mathcal{H}$, $e_{\omega \sigma}^* = e_{\omega}^* = \chi(\sigma) e_{\omega}^*$ since $\omega^\sigma = \omega$. Summing over $H_\omega$ we obtain

$$v(\omega) e_{\omega}^* = (\sum_{\sigma \in \mathcal{H}} \chi(\sigma)) e_{\omega}^*$$

$$= 0.$$

Hence $e_{\omega}^* = 0$ if $\omega \in \Delta$, $\omega \notin \Delta$, so that

$$v_1 \ast \cdots \ast v_m = \sum_{\omega \in \Delta} \left(1/\nu(\omega)\right) (\sum_{\sigma \in \mathcal{H}} \chi(\sigma) c_{\omega \sigma}) e_{\omega}^*;$$

This result holds whether or not $e_1, \ldots, e_n$ is a basis of $V$. Since all tensors in $V_{\chi}(\mathcal{H})$ are linear combinations of star products, it follows that the $e_{\omega}^*$, $\omega \in \Delta$, span $V_{\chi}(\mathcal{H})$. We assert they are linearly independent. Suppose that for some $d_\omega \in \mathbb{R}$, $\omega \in \Delta$, $\sum_{\omega \in \Delta} d_{\omega} e_{\omega}^* = 0$; let
Let $h_1, \ldots, h_n$ be a basis of $V^*$ dual to the basis $e_1, \ldots, e_n$ of $V$. Let $\gamma \in \Delta$. Then with $\theta = \sigma^{-1}$

$$
e^*(h_{\gamma_1} \cdots h_{\gamma_m}) = (1/p) \sum_{\sigma \in H} \chi(\sigma) e_{\theta}^*(h_{\gamma_1} \cdots h_{\gamma_m}) = (1/p) \sum_{\sigma \in H} \chi(\sigma) \delta_{\omega, \gamma}.$$

Now both $\omega$ and $\gamma$ are in $\Delta$; hence, $\omega \sim \gamma$ if and only if $\omega = \gamma$. Thus $\delta_{\omega, \gamma} = 1$ if and only if $\omega = \gamma^\sigma$ and $\sigma \in H$. Then using (3.2) we obtain

$$e^*(h_{\gamma_1} \cdots h_{\gamma_m}) = (\delta_{\omega, \gamma}/p) \sum_{\sigma \in H} \chi(\sigma)$$

$$= \delta_{\omega, \gamma} v(\omega)/p.$$

Applying this result to the equation $\sum_{\omega \in \Delta} d e^* = 0$ we obtain $0 = \sum_{\omega \in \Delta} d e^*(h_{\gamma_1} \cdots h_{\gamma_m}) = v(\gamma) d\gamma/p$ so that the $e^*$ are linearly independent. Next we note that $\langle \text{mg f} \rangle$ contains a basis of star products; hence $\langle \text{mg f} \rangle = V^m_H(\chi)$. Finally we assert that $(V^m_H(\chi), f)$ satisfies the universal factorization property with respect to $H$ and $\chi$. Let $U$ be any vector space over $\mathbb{R}$ and let $\phi$ be an $m$-linear function symmetric with respect to $H$ and $\chi$ whose range is contained in $U$. We first show that the values $\phi(v_1, \ldots, v_m)$ are determined by the fixed values $\phi(e_{\omega_1}, \ldots, e_{\omega_m}), \omega \in \Delta$. For $\sigma \in H$ we have $\phi(v_{\sigma(1)}, \ldots, v_{\sigma(m)}) = \chi(\sigma) \phi(v_1, \ldots, v_m)$ or

$$\phi(v_1, \ldots, v_m) = \chi(\sigma^{-1}) \chi(\sigma^{-1}) v_1 \otimes \cdots \otimes v_m(\phi).$$

Summing over all $\sigma \in H$ and dividing by $p \cdot 1 = p$ we have

$$\phi(v_1, \ldots, v_m) = T(v_1 \otimes \cdots \otimes v_m)(\phi).$$
But by (3.6) for some $d_\omega$, $\omega \in \Delta$,

$$v_1 \ast \cdots \ast v_m = \sum_{\omega \in \Delta} d_\omega e_\omega^*$$

Thus

$$\phi(v_1, \ldots, v_m) = \sum_{\omega \in \Delta} d_\omega \phi(e_\omega, \ldots, e_\omega)$$

But $e_\omega^* = \phi(e_\omega, \ldots, e_\omega)$ so that

$$\phi(v_1, \ldots, v_m) = \sum_{\omega \in \Delta} d_\omega \phi(e_\omega, \ldots, e_\omega)$$

Now consider the diagram

$$\begin{array}{ccc}
V \times \cdots \times V & \xrightarrow{f} & V^m_X(H) \\
\downarrow \phi & & \downarrow h \\
U & & U
\end{array}$$

Define $h$ on the basis $e_\omega^*$, $\omega \in \Delta$, of $V^m_X(H)$ by

$$h(e_\omega^*) = \phi(e_\omega^*, \ldots, e_\omega^*), \omega \in \Delta;$$

then we assert $\phi = hf$. For (i) $hf \in M(V, \ldots, V: U)$; (ii) $hf$ is symmetric with respect to $H$ and $\chi$; (iii) $hf$ agrees with $\phi$ for all $(e_\omega, \ldots, e_\omega)$, $\omega \in \Delta$, and is completely determined by its values on these $m$-tuples of vectors. This completes the proof of Theorem 3.1.

If the pair $(P, f)$ is an $m$-th star product of $V$ with respect to $H$ and $\chi$ then the linear map $h$ in the diagram

$$\begin{array}{ccc}
V \times \cdots \times V & \xrightarrow{f} & P \\
\downarrow \phi & & \downarrow h \\
Q & & Q
\end{array}$$
is uniquely determined. For since \(<\text{rng } f> = P\) we can select a basis of P of vectors of the form \(f(v_1, \ldots, v_m)\). From the equation

\[ \phi(v_1, \ldots, v_m) = h(f(v_1, \ldots, v_m)) \]

we conclude that h is uniquely determined.

We observe here that the matrix function \(d_\lambda\) associated with \(H\) and \(\lambda\) appears in (3.6) of Theorem 3.1. If \(v_i = \sum_{j=1}^{n} c_{ij} e_j, i = 1, \ldots, m,\) then

\[ v_1 \ast \cdots \ast v_m = \sum_{\omega \in \Delta} (1/n(\omega)) \sum_{\sigma \in H} \chi(\sigma) \prod_{t=1}^{m} c_{t\omega(\sigma(t))} e^*_\omega. \]

For \(\alpha = (\alpha_1, \ldots, \alpha_r) \in \Gamma_{r,m}\) and \(\beta = (\beta_1, \ldots, \beta_s) \in \Gamma_{s,n}\) we denote the matrix obtained from the matrix \(C = [c_{ij}]\) by \(C[\alpha|\beta]\). We then write

(3.7) \[ v_1 \ast \cdots \ast v_m = \sum_{\omega \in \Delta} (1/n(\omega)) d_\lambda(C[1, \ldots, m|\omega]) e^*_\omega. \]

We shall work out concurrently two examples of symmetry classes of tensors. One example gives rise to the determinant function and the other to the permanent function, the object of our initial investigations in the theory of symmetry classes. Let \(V\) be a vector space over a field \(R\) whose characteristic is assumed here to exceed \(m!\). Assume \(\dim V = n\) and that \(m \leq n\). Let \(H = S_m\) and let \(\chi(\sigma), \sigma \in S_m, \) be \(\varepsilon(\sigma),\) the alternating character on \(S_m\). The equivalence relation in \(\Gamma_{m,n}\) is given by: \(\alpha \sim \beta\) if and only if \(\alpha^\sigma = \beta\) for some \(\sigma \in H\). Thus \(\Gamma(\omega)\) is the set of all sequences of the form \(\omega^\sigma, \sigma \in S_m\). Clearly the first sequence in \(\Gamma(\omega)\) in the lexicographic order is that with entries in non-decreasing order. Thus \(\Delta\) contains \(G_{m,n}\), the set of all sequences in \(\Gamma_{m,n}\).
whose entries are in non-decreasing order. Next we evaluate 
\[ \sum_{\sigma \in \mathcal{H}_\gamma} \varepsilon(\sigma) \] for \( \gamma \in \Delta \) and \( \mathcal{H} = \mathcal{S}_m \). Denote by \( m_j(\gamma) \) the number of times

the integer \( j \) occurs in the sequence \( \gamma \). Let \( m_j(\gamma) \) be strictly greater than 1 for \( k \) distinct integers \( j \) in \( \gamma \) and label these multiplicities \( m_1, \ldots, m_k \). Then \( \mathcal{H}_\gamma = (\mathcal{S}_m)_\gamma \) consists of products of permutations \( \sigma_i \) where each \( \sigma_i \) permutes among themselves the elements of the \( i \)-th block of \( m_i \) integers and fixes the rest, \( i = 1, \ldots, k \). Thus each such set of \( \sigma_i \) is itself a symmetric group of degree \( m_i \) which we denote by \( \mathcal{S}^i_m \). We now calculate

\[ (3.8) \quad \sum_{\sigma \in (\mathcal{S}_m)_\gamma} \varepsilon(\sigma) = \sum_{\sigma \in (\mathcal{S}_m)_\gamma} \varepsilon(\sigma_1 \ldots \sigma_k) \]

\[ = \sum_{\sigma \in (\mathcal{S}_m)_\gamma} \varepsilon(\sigma_1) \ldots (\sigma_k) \]

\[ = (\sum_{\sigma \in \mathcal{S}^1_m} \varepsilon(\sigma_1)) \ldots (\sum_{\sigma \in \mathcal{S}^k_m} \varepsilon(\sigma_k)). \]

Since each \( \mathcal{S}^i_m \) contains as many even as odd permutations and since \( \varepsilon(\sigma) \) is +1 or -1 according as \( \sigma \) is even or odd, each term in the preceding product is zero unless each \( \mathcal{S}^i_m \) consists of the identity permutation alone. Hence \[ \sum_{\sigma \in (\mathcal{S}_m)_\gamma} \varepsilon(\sigma) = 0 \] unless \( m_1 = \ldots = m_k = 1 \). Thus for the

character \( \chi = \varepsilon \), \( \Lambda = Q_m, \) the set of all \( \binom{n}{m} \) strictly increasing sequences in \( \mathcal{P}_{m,n} \). The relations \( m_1 = \ldots = m_k = 1 \) are not possible unless \( m < n \). If \( m > n \) then \( \Lambda \) is empty, and the symmetry class is 0. For \( \mathcal{H} = \mathcal{S}_m \) and \( \chi = \varepsilon \) the star products satisfy

\[ \star_{\sigma(1)} * \cdots * \star_{\sigma(m)} = \varepsilon(\sigma) v_1 * \cdots * v_m, \]

a property called skew symmetry. \( \star_{\chi}^m(H) \) in this case is the \( m \)-th Grassman space and is denoted by \( \Lambda^m V \). We write \( v_1 \Lambda \cdots \Lambda v_m \) for the star
product of $v_1, \ldots, v_m$ and call it the Grassman product of $v_1, \ldots, v_m$.

We summarize our results.

**Theorem 3.2.** Let $V$ denote a vector space of dimension $n$ over a field $R$ of characteristic greater than $m!$ where $m < n$. Then

$$\text{dim } \Lambda^m V = \binom{n}{m}. \quad (3.9)$$

If $e_1, \ldots, e_n$ is a basis of $V$ then the Grassman products

$$e_{\sigma(1)} \wedge \cdots \wedge e_{\sigma(m)} = \varepsilon(\sigma) v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(m)}. \quad (3.10)$$

If $v_i = \sum_{j=1}^{n} a_{ij} e_j, i = 1, \ldots, m$, and $m = n$ then with $A = [a_{ij}]$

$$v_1 \wedge \cdots \wedge v_n = (\text{det } A) e_1 \wedge \cdots \wedge e_n. \quad (3.11)$$

**Proof.** All but (3.11) has been proved. For the proof of (3.11) consider the general formula (3.6). We have

$$v_1 \wedge \cdots \wedge v_n = \sum_{j=1}^{n} a_{1j} e_j \wedge \cdots \wedge \sum_{j=1}^{n} a_{nj} e_j$$

$$= \sum_{\omega \in Q_{n,n}} (1/\nu(\omega)) \sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{t=1}^{n} a_{t \sigma(t)} e_{\sigma(1)} \wedge \cdots \wedge e_{\sigma(n)}.$$

Now $Q_{n,n}$ consists of the single sequence $(1, \ldots, n)$. Thus $(S_m)_\omega$ consists of the identity permutation only so that $\nu(\omega) = 1$, and

$$v_1 \wedge \cdots \wedge v_n = (\sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{t=1}^{n} a_{t \sigma(t)}) e_1 \wedge \cdots \wedge e_n$$

$$= (\text{det } A) e_1 \wedge \cdots \wedge e_n.$$
Let us briefly duplicate the preceding argument in case the character \( \chi \) is identically 1. In this case

\[
\sum_{\sigma \in H} \chi(\sigma) = \sum_{\sigma \in (S_m)} 1
\]

\[
= (\sum_{\gamma \in S_1} 1) \cdots (\sum_{\gamma \in S_k} 1)
\]

\[
= m_1! \cdots m_k!
\]

and for no \( \gamma \in \Delta \) is \( \sum_{\sigma \in H} \chi(\sigma) = 0 \). Hence \( \Delta = G_{m,n} \) and \( \nu(\gamma) = m_1! \cdots m_k! \). The star products satisfy for \( H = S_m \) and \( \chi = 1 \)

\[
\nu_\sigma(1) \ast \cdots \ast \nu_\sigma(m) = v_1 \ast \cdots \ast v_m,
\]

a property known as complete symmetry. We write \( v_1 \cdots v_m \) for the star product of \( v_1, \ldots, v_m \) and call it the symmetric product of \( v_1, \ldots, v_m \). The corresponding symmetry class is called the \( m \)-th symmetric space and is denoted by \( V^{(m)} \).

**Theorem 3.3.** Let \( V \) be a vector space of dimension \( n \) over a field \( R \) of characteristic greater than \( m! \). Then

\[
\dim v^{(m)} = \binom{n + m - 1}{m}.
\]

If \( e_1, \ldots, e_n \) is a basis of \( V \) then the symmetric products \( e_{\omega_1} \cdots e_{\omega_m} \) form a basis of \( V^{(m)} \). For \( v_i \in V, i = 1, \ldots, m, \) and \( \sigma \in S_m \)

\[
v_\sigma(1) \ast \cdots \ast v_\sigma(m) = v_1 \ast \cdots \ast v_m.
\]

Let \( v_i = \sum_{j=1}^n a_{ij} e_j, i = 1, \ldots, m, m = n, \) and \( \mu(\omega) = \pi_{i=1}^n m_1(\omega)! \). Then

\[
v_1 \ast \cdots \ast v_n = \sum_{\omega \in G_{n,n}} (1/\mu(\omega)) \text{per} A[1, \ldots, n|\omega] e_{\omega_1} \cdots e_{\omega_n}.
\]
PROOF. To obtain the dimension of \( V^{(m)} \) we calculate the number of distinct sequences in \( A = G_{m,n} \). Consider the correspondence

\[
\gamma \leftrightarrow \gamma^*
\]

where \( \gamma = (\gamma_1, \ldots, \gamma_m) \in G_{m,n} \) and \( \gamma^* = (\gamma_1 + 0, \ldots, \gamma_m + m - 1) \in Q_{m,n+m-1} \). The correspondence is 1-1 from \( G_{m,n} \) onto \( Q_{m,n+m-1} \). Hence \( G_{m,n} \) contains exactly \( \binom{n+m-1}{m} \) sequences proving (3.12). The remainder of the result is clear.

We next define an important subclass of the class of all linear transformations from a symmetry class \( V_x(H) \) to itself. Let \( T: V \rightarrow V \) be a linear map from \( V \) to itself and consider the diagram

\[
\begin{array}{ccc}
V \times \cdots \times V & \xrightarrow{f} & V_x^m(H) \\
\downarrow{\phi} & & \downarrow{h} \\
V_x^m(H) & \xrightarrow{h} & V_x^m(H) \\
\end{array}
\]

where \( f(v_1, \ldots, v_m) = v_1 * \cdots * v_m \) and

\[
\phi(v_1, \ldots, v_m) = f(Tv_1, \ldots, Tv_m)
\]

\[
= Tv_1 * \cdots * Tv_m.
\]

Clearly \( \phi \) is symmetric with respect to \( H \) and \( x \), and thus there exists a unique linear \( h: V_x^m(H) \rightarrow V_x^m(H) \) such that \( \phi = hf \), i.e.,

\[
h(v_1 * \cdots * v_m) = Tv_1 * \cdots * Tv_m
\]

The linear map \( h \) is called the induced transformation and we write \( h = K(T) \).
THEOREM 3.4. If $S: V \rightarrow V$ and $T: V \rightarrow V$ are linear maps then

\[(3.15) \quad K(ST) = K(S)K(T).\]

If $T$ is regular so is $K(T)$ and

\[(3.16) \quad K^{-1}(T) = K(T^{-1}).\]

If $T$ has proper values $\lambda_1, \ldots, \lambda_n$ and corresponding proper vectors $u_1, \ldots, u_n$ then $K(T)$ has proper values $\lambda_\omega = \prod_{t=1}^{n} \lambda_t^{m_t(\omega)}$ and corresponding proper vectors $u^*_{\omega}$, $\omega \in \Delta$.

PROOF. To prove (3.15) write

\[
K(ST)v_1 \ast \cdots \ast v_m = STv_1 \ast \cdots \ast STv_m
= S(Tv_1) \ast \cdots \ast S(Tv_m)
= K(S)Tv_1 \ast \cdots \ast Tv_m
= K(S)K(T)v_1 \ast \cdots \ast v_m.
\]

Denoting the identity map from $V$ to itself by $I_V$ we have

\[
K(I_V)v_1 \ast \cdots \ast v_m = I_Vv_1 \ast \cdots \ast I_Vv_m
= v_1 \ast \cdots \ast v_m
= I^m_{V^m(\Delta)}v_1 \ast \cdots \ast v_m
\]

so $K(I_V) = I_{V^m(\Delta)}$. Thus from (3.15)

\[
K(I_V) = K(TT^{-1})
= K(T)K(T^{-1})
\]
so (3.16) follows. Finally let \( \omega = (\omega_1, \ldots, \omega_m) \in \Delta \). Then \( T_{\omega_i} = \lambda_{\omega_i} u_{\omega_i} \) so that

\[
K(T)u^* = Tu_{\omega_1} \ast \cdots \ast Tu_{\omega_m}
\]

\[
= \lambda_{\omega_1} u_{\omega_1} \ast \cdots \ast \lambda_{\omega_m} u_{\omega_m}
\]

\[
= (\prod_{i=1}^{m} \lambda_{\omega_i}) u^*
\]

\[
= (\prod_{i=1}^{n} \lambda_t(\omega)) u^*
\]

\[
= \lambda_{\omega} u^*.
\]

There is an interesting and important relationship between the associated matrix function and the matrix representation of the induced map \( K(T) \).

**Theorem 3.5.** Let \( E = \{e_1, \ldots, e_n\} \) be a basis of \( V \) and let \( E^* = \{e^*_\alpha, \alpha \in \Delta\} \) be the corresponding induced basis of \( V^* \) ordered lexicographically. Let \( A = [a_{ij}] \) denote the matrix representation of the map \( T \) relative to \( E \) and let \( A^T \) denote the matrix transpose of \( A \). Then the \( \alpha, \beta \) entry in the matrix representation of the induced map \( K(T) \) relative to the basis \( E^* \) is

\[
(3.17) \quad (1/\nu(\alpha)) d_\chi(A^T[\beta|\alpha]).
\]

**Proof.** Let \( C = A^T[\beta|1, \ldots, n] \). Then

\[
K(T)e^*_\beta = K(T)e_{\beta_1} \ast \cdots \ast e_{\beta_m}
\]

\[
= Te_{\beta_1} \ast \cdots \ast Te_{\beta_m}
\]
by (3.7) proving (3.17).

We call the matrix representation of the induced transformation $K(T)$ from Theorem 3.5 the associated matrix of $A$ and write $K(A)$, an $N$-square matrix where $N$ is the number of sequences in $\bar{\rightarrow}$. If the field $\mathbb{R}$ contains the square roots of the positive integral multiples of its identity $1$, we may then write

$$K(T)e_\beta/\sqrt{\nu(\beta)} = \sum_{\omega \in \bar{\rightarrow}} (1/\sqrt{\nu(\omega)\nu(\beta)})d_{\chi}(A^T[\beta|\omega])e_\omega/\sqrt{\nu(\omega)}.$$ 

If $E_k = \{e_\alpha/\sqrt{\nu(\alpha)}$, $\alpha \in \bar{\rightarrow}\}$ then the matrix representation of $K(T)$ relative to $E_k$ is the $N$-square matrix whose $\alpha, \beta$ entry is

$$(3.18) \quad d_{\chi}(A^T[\beta|\alpha])/\sqrt{\nu(\alpha)\nu(\beta)}.$$

Generally we prefer to use the basis $E_k$ rather than $E_k$ whenever possible since, as we shall see in the next section, $E_k$ is an orthonormal basis when $E$ is. Before proceeding with a discussion of inner products, we summarize some elementary results.

**THEOREM 3.6.** Let $\mathbb{R}$ be any field. Then if $I_n$ and $I_N$ denote, respectively, the $n$-square and $N$-square identity matrices

$$(3.19) \quad K(I_n) = I_N;$$
(3.20) \[ K(AB) = K(A)K(B) \]

(3.21) \[ K(A^{-1}) = K^{-1}(A). \]

If \( R \) is the complex field then

(3.22) \[ K(A^*) = K^*(A) \]

where \( A^* \) denotes the complex conjugate - transpose \( \overline{A^T} \) of \( A \). If \( A \) is normal, hermitian, unitary, or, for \( m \) odd, skew - hermitian then \( K(A) \) has the same property.

**PROOF.** (3.19) follows from the proof of (3.16) in Theorem 3.4 and from basic properties of matrix representations of linear mappings. (3.20) follows from (3.15) and (3.21) from (3.16). To prove (3.22) we compute for \( \alpha, \beta \in \mathbb{A} \)

\[
d_x ((A^*)^T [\beta | \alpha]) = d_x (\overline{A} [\beta | \alpha])
\]

\[
= \sum_{\sigma \in H} \chi(\sigma) \prod_{i=1}^{m} \bar{a}_{\beta_i} \alpha_{\bar{\sigma}(i)}
\]

\[
= \sum_{\sigma \in H} \bar{\chi}(\sigma) \prod_{i=1}^{m} a_{\beta_i} \alpha_{\sigma(i)}
\]

\[
= \sum_{\sigma \in H} \chi(\sigma^{-1}) \prod_{i=1}^{m} a_{\beta_i} \alpha_{\sigma(i)}
\]

\[
= \sum_{\sigma \in H} \chi(\sigma) \prod_{i=1}^{m} a_{\beta} \alpha_{\sigma(i)}
\]

\[
= d_x (A^T [\alpha | \beta]).
\]

The rest of the assertions now follow easily. Note that if \( A \) is skew - hermitian then \( K^*(A) = K(A^*) = K(-A) = -K(A) \); the last equality holds if \( m \) is odd.
In case $V^m(H)$ is the $m$-th Grassman space, $K(A)$ is called the $m$-th compound matrix of $A$ and is denoted usually by $C_m(A)$. If $\chi$ is identically 1 and $H = S_m$ then $V^m_\chi(H) = V^m$, the $m$-th symmetric space, and $K(A)$ is denoted by $P_m(A)$. $P_m(A)$ is called the $m$-th power matrix of $A$. We write $C_m(T)$ and $P_m(T)$ for the corresponding induced transformations.
IV. LENGTHS OF STAR PRODUCTS

Let \( V \) denote an \( n \)-dimensional unitary vector space with inner product \((x, y)\). Denote by \( \mathbb{M}^n V \) the tensor product of \( V \) taken with itself \( m \) times. Then it is well-known that every inner product on \( V \) induces a corresponding inner product on \( \mathbb{M}^n V \) [24]. If \( x_1 \otimes \cdots \otimes x_m \) and \( y_1 \otimes \cdots \otimes y_m \) are decomposable tensors in \( \mathbb{M}^n V \) then their induced inner product is given by

\[
(x_1 \otimes \cdots \otimes x_m, y_1 \otimes \cdots \otimes y_m) = \prod_{i=1}^{m} (x_i, y_i)
\]

If \( e_1, \ldots, e_n \) is an orthonormal basis of \( V \) then the basis \( e_\alpha = e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_m} \), \( \alpha \in \Gamma_m, n \), of decomposable tensors is an orthonormal basis of \( \mathbb{M}^n V \). We shall be interested in the inner product (4.1) on the symmetry class of tensors \( V_x^m(H) \), a subspace of \( \mathbb{M}^n V \). The following theorem summarizes some important results [19].

**Theorem 4.1.** Let \( V \) denote an \( n \)-dimensional unitary space with inner product \((x, y)\). Let \( H \) of order \( p \) be any subgroup of \( S_m \) and let \( x \) be a character of degree 1 of \( H \). Denote by \( T = (1/p) \sum_{\sigma \in H} x(\sigma) P(\sigma) \) the symmetry operator determined by \( H \) and \( x \). Then \( T \) is idempotent; moreover, \( T \) is hermitian with respect to the inner product (4.1). Let \( x_1, \ldots, x_m \) and \( y_1, \ldots, y_m \) be two sets of \( m \) vectors each in \( V \). Then the induced inner product satisfies

\[
(x_1 \ast \cdots \ast x_m, y_1 \ast \cdots \ast y_m) = (1/p) d^x [(x_i, y_j)] .
\]
Let \( e_1, \ldots, e_n \) be an orthonormal basis of \( V \). For \( \omega \in \tilde{A} \) let \( \nu(\omega) \) be the order of the subgroup \( H_\omega \) of \( H \) fixing \( \omega \). Then the star products 
\[ \sqrt{p/\nu(\omega)} e_\omega^x, \omega \in \tilde{A}, \]
constitute an orthonormal basis of \( V^m_{\chi(H)} \).

**Proof.** We showed in proving Theorem 3.2 that \( T^2 = T \). Next let \( \theta = \sigma^{-1} \) and compute

\[
(Tx_1 \otimes \cdots \otimes x_m, y_1 \otimes \cdots \otimes y_m)
\]

\[
= \left( (1/p) \sum_{\sigma \in H} x(\sigma)x_\theta(1) \otimes \cdots \otimes x_\theta(m), y_1 \otimes \cdots \otimes y_m \right)
\]

\[
= \left( (1/p) \sum_{\sigma \in H} x(\sigma)\prod_{i=1}^m (x_\theta(i), y_1) \right)
\]

\[
= \left( (1/p) \sum_{\sigma \in H} x(\sigma)\prod_{i=1}^m (x_i, y_\sigma(i)) \right)
\]

\[
= \left( (1/p) \sum_{\sigma \in H} x(\sigma)\prod_{i=1}^m (y_\sigma(i)^*, x_i) \right)
\]

\[
= \left( (1/p) \sum_{\sigma \in H} x(\sigma)\prod_{i=1}^m (y_\sigma(i), x_i) \right)
\]

\[
= \left( (1/p) \sum_{\sigma \in H} x(\sigma)\prod_{i=1}^m (y_\theta(i)^*, x_i) \right)
\]

\[
= \left( (1/p) \sum_{\sigma \in H} x(\sigma)\prod_{i=1}^m (y_\theta(i), x_i) \right)
\]

\[
= \left( (1/p) \sum_{\sigma \in H} x(\sigma)_\theta(1) \otimes \cdots \otimes x_\theta(m), x_1 \otimes \cdots \otimes x_m \right)
\]

\[
= \left( x_1 \otimes \cdots \otimes x_m, Ty_1 \otimes \cdots \otimes y_m \right)
\]

so that \( T \) is hermitian with respect to the induced inner product (4.1). Since

\[
(Tx_1 \otimes \cdots \otimes x_m, y_1 \otimes \cdots \otimes y_m) = (Tx_1 \otimes \cdots \otimes x_m, Ty_1 \otimes \cdots \otimes y_m),
\]

(4.2) follows. If \( e_1, \ldots, e_n \) is an orthonormal basis of \( V \) then the linear functionals defined by \( h_j(e_i) = (e_i, e_j) = \delta_{ij}, i, j = 1, \ldots, n, \) and linear extension constitute a basis of \( V^* \) dual to the basis \( e_1, \ldots, e_n \).
of V. For any dual basis we proved in the course of proving Theorem 3.2 that $e^*_{\omega}(h_\gamma) = e^*_{\omega}(h_\gamma \cdots h_\gamma_m) = \delta_{\omega,\gamma}(\omega)/p$, $\omega, \gamma \in \Delta$. But with $\theta = \omega^{-1}$

\[ e^*_{\omega}(h_\gamma \cdots h_\gamma_m) = (1/p) \sum_{\sigma \in H} x(\sigma) e_{\omega}(h_\gamma \cdots h_\gamma_m) \]

\[ = (1/p) \sum_{\sigma \in H} x(\sigma) \prod_{i=1}^m h_{\gamma_i} (e_{\omega}(i)) \]

\[ = (1/p) \sum_{\sigma \in H} x(\sigma) \prod_{i=1}^m (e_{\omega}(i), e_{\gamma_i}) \]

\[ = (e^*_{\omega}, e^*_\gamma) \]

which completes the proof.

Let us now apply the Cauchy - Schwarz inequality to the inner product (4.2). For $x_1, \ldots, x_m$ and $y_1, \ldots, y_m$ any vectors in $V$

\[ (1/p) |d^\pi (x_i, y_j)| = |(x_1 * \cdots * x_m, y_1 * \cdots * y_m)| \]

\[ \leq ||x_1 * \cdots * x_m|| ||y_1 * \cdots * y_m|| \]

\[ = ((1/p) d^\pi [(x_1, x_j)])^{1/2} ((1/p) d^\pi [(y_1, y_j)])^{1/2} \]

or

\[ (4.3) \quad |d^\pi [(x_i, y_j)]| \leq (d^\pi [(x_i, x_j)])^{1/2} (d^\pi [(y_i, y_j)])^{1/2}. \]

We may call the non-negative real number $||x_1 * \cdots * x_m||$ the length of the star product $x_1 * \cdots * x_m$ since the inner product (4.2) determines a metric on $V_m^\pi(H)$. The inequality (4.3) is the starting point for the development of many matrix inequalities. Before proceeding, however, we face the question of determining the cases of equality in
(4.3). Necessary and sufficient conditions for equality are in general unknown, but the question can be resolved in many interesting and useful situations.

Equality, we know, holds in the Cauchy-Schwarz inequality in case the two vectors under comparison are linearly dependent or in case one or both is the zero vector. The following lemma helps settle the cases of equality in one of our principal results. Having first appeared in [19], it will be supplemented later in this section with even more recent criteria. We omit the proof.

**Lemma 4.1.** If $x_i$ and $y_i$ are in $V$, $i = 1, \ldots, m$, $H$ is any subgroup of $S_m$, and $\chi = 1$ then

(i) $x_1 \ast \cdots \ast x_m = 0$ if and only if some $x_i = 0$;

(ii) if $x_1 \ast \cdots \ast x_m = y_1 \ast \cdots \ast y_m \neq 0$ then there exists a $\sigma \in S_m$ and constants $d_i \neq 0$, $i = 1, \ldots, m$, such that $x_i = d_i y_{\sigma(i)}$, $i = 1, \ldots, m$. Moreover, $\prod_{i=1}^{m} d_i = 1$.

The following theorem is an important matrix theoretic interpretation of the inequality (4.3). It was first proved in the case $\chi = 1$ and appeared in [20]. The result is proved for any character $\chi$ in [19].

**Theorem 4.2.** Let $A$ be an $m$ by $n$ matrix and $B$ an $n$ by $m$ matrix with entries in the field of complex numbers. Then

\[ |d_\chi(AB)| \leq d_\chi^{1/2}(AA^*) d_\chi^{1/2}(B^*B). \]

In case $\chi = 1$, equality holds in (4.4) if and only if (i) $A$ has a zero row or $B$ has a zero column, or (ii) $A = DPB^*$, where $D$ is a diagonal matrix and $P$ is a permutation matrix.

**Proof.** Let $e_1, \ldots, e_n$ be an orthonormal basis of the unitary space
V. Let $x_i = \sum_{k=1}^{n} a_{ik} e_k$ and $y_j = \sum_{k=1}^{n} b_{kj} e_k$ where $A = [a_{ik}]$ and $B = [b_{kj}]$. Then (4.4) follows from (4.3). The case of equality for $\chi = 1$ follows from Lemma 4.1.

If in (4.4) we set $B = I_m$ and $m = n$, we obtain

$$(4.5) \quad |d_\chi(A)| \leq d_\chi^{1/2}(AA^*).$$

Let $K$ denote a positive semi-definite hermitian matrix. Then there is a triangular matrix $A = [a_{ij}]$ satisfying $AA^* = K$ [17]. But then $d_\chi(A) = \prod_{i=1}^{n} a_{ii} = \det A = \det^{1/2} K$ so that (4.5) yields the classical inequality of Issai Schur [27]

$$(4.6) \quad \det K \leq d_\chi(K).$$

As pointed out in [14], one can obtain directly from (4.6) several classical matrix inequalities. For example, let $q$ be any integer between $1$ and $n$. Let $H$ be the direct product of the symmetric group of degree $q$ on the integers $1, \ldots, q$ and the symmetric group of degree $n - q$ on the integers $q + 1, \ldots, n$. Set $\chi = \varepsilon$, the alternating character on $H$. Then (4.6) becomes

$$(4.7) \quad \det K \leq \det K[1, \ldots, q|1, \ldots, q] \det K[q+1, \ldots, n|q+1, \ldots, n],$$

Fischer's generalization of Hadamard's inequality.

The determination of the cases of equality in the inequality (4.6) is the subject of an intensive investigation by S. Williamson [30]. A research announcement [31] summarizes his findings. We state those of his results which apply to the inequality (4.6) as a lemma and omit the proof.
LEMMA 4.2. Let $A = [a_{ij}]$ denote an $m$-square positive definite hermitian matrix. Denote by $H$ a subgroup of the symmetric group $S_m$ of degree $m$, $\chi$ a character of degree 1 of $H$, and $d_\chi(A)$ the matrix function of $A$ associated with $H$ and $\chi$. Let $S_m(A)$ be the subgroup of $S_m$ generated by all transpositions $(i \, j)$ for which $a_{ij} \neq 0$. Then $\det A = d_\chi(A)$ if and only if $S_m(A)$ is a subgroup of $H$ and $\chi$ restricted to $S_m(A)$ is $\varepsilon$, the alternating character. Furthermore, let $A = PP^*$ where $P$ is $m$-square and triangular. Then $\det A = d_\chi(A)$ if and only if $S_m(P)$ is a subgroup of $H$ and $\chi$ restricted to $S_m(P)$ is $\varepsilon$.

It is instructive to interpret Lemma 4.2 for $\chi = 1$ in the light of what we know from Lemma 4.1. Lemma 4.1 asserts that equality holds in the inequality (4.6) for $K = AA^*$ positive definite if and only if there exists a $\sigma \in S_m$ and constants $d_i \neq 0$ such that $\sum_{k=1}^m a_{ik}e_k = x_i = d_i\sigma(i) = d_i e_{\sigma(i)}$, $i = 1, \ldots, m$. But then $a_{ii} = (x_i, e_i) = d_i(e_{\sigma(i)}, e_i) \neq 0$ so that $\sigma(i) = i$ whence $K$ is diagonal. On the other hand by Lemma 4.2 equality in (4.6) holds for $K$ positive definite if and only if $S_m(K)$ is contained in $H$ and $\chi$ restricted to $S_m(K)$ is $\varepsilon$. But $\chi = 1$ on $H$ implies every permutation in $S_m(K)$ is even so that $S_m(K)$ must consist of the identity permutation alone. Hence $K$ is diagonal.

Our final preliminary topic concerns relations which hold between a pair of orthonormal bases in an $n$-dimensional unitary space $V$ and the corresponding pair of induced orthonormal bases in an arbitrary symmetry class $V_m^\chi(H)$. Let $T: V \to V$ be a normal transformation from $V$ to itself. Then by definition $TT^* = T^*T$ so that by (3.15) the induced map $K(T): V_m^\chi(H) \to V_m^\chi(H)$ is also normal. Suppose $T$ has proper values $\lambda_1, \ldots, \lambda_n$ and corresponding orthonormal proper vectors $u_1, \ldots, u_n$. Then
for any unit vector \( e \) of \( V \) one may write using the spectral theorem

\[(4.8)\]
\[
(\text{T}e,e) = \sum_{i=1}^{n} |(e_i,u_i)|^2 \lambda_i
\]
with
\[
\sum_{i=1}^{n} |(e_i,u_i)|^2 = 1.
\]

Now let \( e_1, \ldots, e_n \) be an orthonormal basis of \( V \). Then for \( \omega \in \tilde{\Delta} \)
\[
\sqrt{p/\nu}(\omega)e_\omega^* \text{ is a unit vector in } V_\chi(H).
\]
By Theorem 3.4 the proper values of \( \text{K}(\text{T}) \) are
\[
\lambda_\omega = \prod_{t=1}^{n} \lambda_t^{m_t(\omega)}
\]
and corresponding orthonormal proper vectors are
\[
\sqrt{p/\nu}(\omega)u_\omega^*, \omega \in \tilde{\Delta}.
\]
Then by the spectral theorem for \( V_\chi(H) \) we obtain for every \( \omega \in \tilde{\Delta} \)

\[(4.9)\]
\[
(\text{K}(\text{T})e_\omega^*, e_\omega^*) = \sum_{\gamma \in \tilde{\Delta}} \prod_{t=1}^{n} \lambda_t^{m_t(\omega)} |(e_\omega^*, \sqrt{p/\nu}(\gamma)u_\gamma^*)|^2
\]
with
\[
\sum_{\gamma \in \tilde{\Delta}} |(\sqrt{p/\nu}(\omega)e_\omega^*, \sqrt{p/\nu}(\gamma)u_\gamma^*)|^2 = 1.
\]

The following theorem provides a useful and important identity. It first appeared in [11].

**THEOREM 4.3.** Let \( e_1, \ldots, e_n \) and \( v_1, \ldots, v_n \) be two orthonormal bases in \( V \). Then for each \( \omega \in \tilde{\Delta} \) and for each \( t, t = 1, \ldots, n \),

\[(4.10)\]
\[
\sum_{i=1}^{n} m_i(\omega) |(e_i,v_t)|^2 = \sum_{\gamma \in \tilde{\Delta}} m_t(\gamma) \left| (\sqrt{p/\nu}(\omega)e_\omega^*, \sqrt{p/\nu}(\gamma)v_\gamma^*) \right|^2.
\]

**PROOF.** Our proof follows that for the case \( \omega = (\omega_1, \ldots, \omega_m) = (1, \ldots, m) \) given in [19]. Let \( T \) denote a normal transformation on \( V \) having fixed proper vectors \( v_1, \ldots, v_n \) with corresponding proper values \( \lambda_1, \ldots, \lambda_n \). Denote by \( A = [a_{ij}] \) the \( m \times m \) matrix \( [(T_{e_i}v_j)] \). Set

\[
c_{\omega,\gamma} = |(\sqrt{p/\nu}(\omega)e_\omega^*, \sqrt{p/\nu}(\gamma)v_\gamma^*)|^2.
\]
Then by (4.2) and (4.9) we obtain the following identity each side of which we regard as a function of the n complex variables \( \lambda_1, \ldots, \lambda_n \):

\[
(4.11) \quad d_{\chi}(A) = -\nu(\omega)\sum_{\gamma \in \mathbb{H}} \prod_{t=1}^{n} \lambda_t^{m_t(\gamma)} c_{\omega, \gamma}.
\]

We differentiate each side of (4.11) with respect to \( \lambda_t, t = 1, \ldots, n \), and evaluate the resulting identity at the "point" \( (\lambda_1, \ldots, \lambda_n) = (1, \ldots, 1) = \pi \). We first write

\[
d_{\chi}(A) = \sum_{\sigma \in \mathbb{H}} x(\sigma) \prod_{i=1}^{m} a_{i\sigma(i)}
= a_{rs}\sum_{\sigma(r)=s} x(\sigma) \prod_{i=1, i\neq r}^{m} a_{i\sigma(i)} + \sum_{\sigma(r)\neq s} x(\sigma) \prod_{i=1}^{m} a_{i\sigma(i)}
\]

and note that \( \sum_{\sigma(r)\neq s} x(\sigma) \prod_{i=1}^{m} a_{i\sigma(i)} \) is independent of \( a_{rs} \). Thus

\[
(4.12) \quad \frac{\partial d_{\chi}(A)}{\partial a_{rs}} = \sum_{\sigma(r)\neq s} x(\sigma) \prod_{i=1, i\neq r}^{m} a_{i\sigma(i)}.
\]

Regarding \( d_{\chi}(A) \) as a composite function of \( \lambda_1, \ldots, \lambda_n \) we obtain

\[
(4.13) \quad \frac{\partial d_{\chi}(A)}{\partial \lambda_t} = \sum_{r,s=1}^{m} \left( \frac{\partial d_{\chi}(A)}{\partial a_{rs}} \right) (\partial a_{rs} / \partial \lambda_t).
\]

Since \( T = I_V \) and, for \( \sigma \in \mathbb{H}, a_{i\sigma(i)} = (Te_{\omega_i}, e_{\omega_{\sigma(i)}}) = (e_{\omega_i}, e_{\omega_{\sigma(i)}}) = \delta_{i\sigma(i)} \) when \( (\lambda_1, \ldots, \lambda_n) = \pi \),

\[
\frac{\partial d_{\chi}(A)}{\partial a_{rs}} \bigg|_{\pi} = 0
\]

for \( r \neq s \). For \( r = s \)

\[
\frac{\partial d_{\chi}(A)}{\partial a_{rr}} \bigg|_{\pi} = \sum_{\sigma(r)=r} x(\sigma) \prod_{i=1, i\neq r}^{m} \delta_{i\sigma(i)}
= \sum_{\sigma \in \mathbb{H}} x(\sigma)
= \nu(\omega).
\]
Thus, using (4.8), (4.13) for \((\lambda_1, \ldots, \lambda_n) = (1, \ldots, 1) = \pi\) becomes

\[
\frac{\partial d\chi(A)}{\partial \lambda_t} \bigg|_{\pi} = \nu(\omega) \frac{\partial}{\partial \lambda_t} \left( \sum_{i=1}^{m} a_{i} \right) \bigg|_{\pi} = \nu(\omega) \frac{\partial}{\partial \lambda_t} \left( \sum_{i=1}^{m} (Te_i, e_i) \right) \bigg|_{\pi} = \nu(\omega) \frac{\partial}{\partial \lambda_t} \left( \sum_{i=1}^{m} m_i(\omega)(Te_i, e_i) \right) \bigg|_{\pi} = \nu(\omega) \frac{\partial}{\partial \lambda_t} \left( \sum_{i=1}^{m} m_i(\omega) \sum_{s=1}^{n} \lambda_s |(e_i, v_s)|^2 \right) \bigg|_{\pi} = \nu(\omega) \frac{\partial}{\partial \lambda_t} \left( \sum_{i=1}^{m} m_i(\omega) |(e_i, v_t)|^2 \right).
\]

On the other hand

\[
\frac{\partial d\chi(A)}{\partial \lambda_t} \bigg|_{\pi} = \nu(\omega) \sum_{\gamma \in \mathcal{A}} c_{\omega, \gamma} m_t(\gamma) \lambda_t \prod_{s=1, s \neq t}^{n} \lambda_s \bigg|_{\pi} = \nu(\omega) \sum_{\gamma \in \mathcal{A}} m_t(\gamma) \left( \sqrt{p/\sqrt{\nu(\omega)}} e^{*}_{\omega}, \sqrt{p/\sqrt{\nu(\gamma)}} v^{*}_{\gamma} \right)^2 \bigg|_{\pi} \]

completing the proof.
V. PRINCIPAL RESULTS

Let $A$ denote an $m$-square positive semi-definite hermitian matrix having row sums $r_1, \ldots, r_m$ satisfying $\sum_{i=1}^{m} r_i = r \neq 0$. Let $d_{\chi}(A)$ be the matrix function associated with the character $\chi = 1$ and subgroup $H$ of order $h$ of $S_m$. In [10] M. Marcus proved that

\begin{equation}
\sum_{i=1}^{m} |r_i|^2
\end{equation}

with equality holding in (5.1) if and only if $A$ either has a zero row or has rank 1. If we restrict the entries of $A$ to be non-negative reals, set $r_1 = \cdots = r_m = 1$, and take $H = S_m$ then it is clear that (5.1) is the resolution of the van der Waerden conjecture in the case $A$ is positive semi-definite hermitian. Our first result is a generalization of (5.1). An $m$-square matrix $B$ is a principal submatrix of an $n$-square matrix $A$ in case $B = A[\omega|\omega]$, $\omega \in Q_{m,n}$. We estimate the arithmetic mean of the associated matrix functions of the $m$-square principal submatrices of an $n$-square positive semi-definite hermitian matrix in the case $\chi = 1$.

**Theorem 5.1.** Let $A$ denote an $n$-square positive semi-definite hermitian matrix with row sums $r_1, \ldots, r_n$ satisfying $\sum_{i=1}^{n} r_i = r \neq 0$. Let $p_m(|r_1|, \ldots, |r_n|) = \sum_{\omega \in Q_{m,n}} |r_\omega|/(r_m^m)$, the $m$-th weighted elementary symmetric function of $|r_1|, \ldots, |r_n|$. Then for any subgroup of order $h$ of $S_m$ and $\chi = 1$

\begin{equation}
\sum_{\omega \in Q_{m,n}} d_{\chi}(A[\omega|\omega])/(r_m^n) \geq (h/r_m) p_m^\omega(|r_1|, \ldots, |r_n|).
\end{equation}
Equality holds in (5.2) if and only if either (i) the number \( k \) of non-zero rows of \( A \) is less than \( m \), or (ii) \( A \) has rank 1 and either \( m = n \), \( m = k \), or all the non-zero entries of \( A \) have equal modulus.

**PROOF.** \( A \) is positive semi-definite hermitian and, hence, is a Gram matrix based on a set of vectors \( x_1, \ldots, x_n \); i.e., \( A = [a_{ij}] \) satisfies \( a_{ij} = (x_i, x_j) \). Let \( \omega = (\omega_1, \ldots, \omega_m) \in \mathbb{Q}_{m,n} \) and consider the star products \( x_\omega = x_{\omega_1} \star \cdots \star x_{\omega_m} \in V^m(\mathbb{H}) \). By (4.2) these satisfy

\[
(5.3) \quad ||x_\omega||^2 = (x_{\omega_1} \star \cdots \star x_{\omega_m}, x_{\omega_1} \star \cdots \star x_{\omega_m}) \\
= (1/h) d_x([x_{\omega_1}, x_{\omega_1}]) \\
= (1/h) d_x(A[\omega|\omega]).
\]

Let \( u = x_1 + \cdots + x_n \). Then \( ||u||^2 = (u, u) = \sum_{i,j=1}^{n} (x_i, x_j) = \sum_{i=1}^{n} r_i = r \neq 0 \). Now \( r \neq 0 \) implies \( r > 0 \); hence, \( u \neq 0 \). Next let \( v = u \star \cdots \star u \in V^m(\mathbb{H}) \) which by Lemma 4.1 is non-zero. Then

\[
||v||^2 = ||u \star \cdots \star u||^2 \\
= (1/h) d_x([(u, u)]) \\
= (1/h) d_x([r]) \\
= r^m
\]

so that the tensor \( w = (1/r^m)^{1/2} v \) is a star product of unit length in \( V^m(\mathbb{H}) \). Hence by (5.3) and the Cauchy-Schwarz inequality

\[
(1/h) d_x(A[\omega|\omega]) = (1/h) d_x([x_{\omega_1}, x_{\omega_1}]) \\
= ||x_\omega||^2
\]
\[ \geq |(x^\omega, w)|^2 \]
\[ = |(x^\omega, u \ast \cdots \ast u)|^2 / r^m \]
\[ = (1/n^2 r^m) |\chi_d([x^\omega])|^2 \]
\[ = (1/n^2 r^m) |\chi_d([x^\omega, x_1 + \cdots + x_n])|^2 \]
\[ = (1/n^2 r^m) h^m \prod_{i=1}^m |r_{\omega_i}|^2 \]
\[ = |r_\omega|^2 / r^m. \]

Summing over all \( \omega \in Q_{m,n} \) we obtain
\[(5.4) \quad \sum_{\omega \in Q_{m,n}} d\chi(A[\omega|\omega]) \geq h^{\sum_{\omega \in Q_{m,n}} |r_\omega|^2 / r^m}. \]

Now the real function \( f(t) = t^2 \) is everywhere strictly convex. Hence
\[(5.5) \quad \sum_{\omega \in Q_{m,n}} |r_\omega|^2 \geq (\sum_{\omega \in Q_{m,n}} |r_\omega|)^2 / (\sum_{\omega \in Q_{m,n}} |r_\omega|^2). \]
\[ = \frac{\mu^2}{m}(|r_1|, \ldots, |r_n|)_m. \]

Combining (5.4) and (5.5) we obtain (5.2). In order that equality hold in (5.4) it is necessary and sufficient that for every \( \omega \in Q_{m,n}, x^\omega \) and \( u \ast \cdots \ast u \) be linearly dependent in \( V_{\chi^m}(\mathbb{D}) \). Since \( u \neq 0 \), it follows by Lemma 4.1 that for every \( \omega = (\omega_1, \ldots, \omega_m) \in Q_{m,n} \) either (a) \( x_{\omega_i} = 0 \) for some \( i, i = 1, \ldots, m \), or (b) \( x_{\omega_i} = c_\omega u \neq 0 \) for all \( i, i = 1, \ldots, m \); i.e., \( A[\omega|\omega] \) has rank 1. If (a) holds for every \( \omega \in Q_{m,n} \) then clearly \( k < m \). On the other hand if \( k < m \) then each \( A[\omega|\omega] \) has a zero row so that both sides of (5.2) are zero. Now suppose \( k \geq m \). Then \( A \) must have rank 1; otherwise \( A \) has an \( m \)-square principal submatrix without a zero row whose rank exceeds 1 violating (b) [17]. Hence let \( A = [c_{ij}] \). Then
\[ d(\Lambda[\omega]) = \sum_{\omega \in Q_{m,n}} d(\Lambda[\omega]) = h \sum_{\omega \in Q_{m,n}} |c_{\omega}|^2. \]

But \( r_i = c_i \sum_{j=1}^n \bar{c}_j \), \( i = 1, \ldots, n \), so

\[ \prod_{i=1}^m |r_{\omega_i}|^2 = |\sum_{j=1}^n c_j|^2 \prod_{i=1}^m |c_{\omega_i}|^2 \]

and \( r^m = (\sum_{i=1}^n r_i)^m = (\sum_{i=1}^n c_i \sum_{j=1}^n \bar{c}_j)^m = |\sum_{i=1}^n c_i|^2 m. \)

Hence

\[ \frac{h}{r^m} \sum_{\omega \in Q_{m,n}} |r_{\omega}|^2 = \frac{h}{r^m} \sum_{\omega \in Q_{m,n}} \prod_{i=1}^m |r_{\omega_i}|^2. \]

\[ = h \sum_{\omega \in Q_{m,n}} \prod_{i=1}^m |c_{\omega_i}|^2 \]

\[ = h^\sum_{\omega \in Q_{m,n}} |c_{\omega}|^2. \]

Thus equality holds in (5.4) for any \( A \) of rank 1. But equality holds in (5.5) if and only if all summands \( |c_{\omega}| \) are equal; i.e., \( m = n, m = k \), or all the non-zero \( |c_i| \) are equal.

If we now set \( m = n \) in (5.2) we obtain (5.1). In addition we have

COROLLARY 5.1. Let \( A \) have non-negative real entries and let the row sums satisfy \( r_1 = \cdots = r_n = 1 \); i.e., let \( A \) be doubly stochastic. Then

(5.6) \[ \sum_{\omega \in Q_{m,n}} \text{per } A[\omega] \geq \frac{(n)^m}{m!}. \]

Equality holds in (5.6) if and only if \( A \) is the matrix all of whose entries are \( 1/n \).

PROOF. The inequality (5.6) follows directly from (5.2) upon setting \( H = S_m \). The specification of the case of equality follows from the
observation that the only doubly stochastic matrix of rank 1 is that with every entry equal to $1/n$.

It is noteworthy that (5.6) fails when $A$ is symmetric but indefinite. For example, let $J$ denote the n-square matrix each of whose entries is equal to 1. Set $A = (1/(n-1))J - (1/(n-1))I_n$. Then with $m = 1$ we have by (5.6) that $0 = \text{tr} A \geq n$, a contradiction.

A companion result to that of Theorem 5.1 estimates the arithmetic mean of the permanents of all $m$-square matrices $A[\omega|\omega]$, $\omega \in G_{m,n}$, obtained from an $n$-square positive semi-definite hermitian matrix $A$. We obtain a lower bound in terms of the weighted completely symmetric functions [7] of the row sums of $A$.

**THEOREM 5.2.** Let $A$ be the $n$-square matrix of Theorem 5.1. Let

$$q_m(|r_1|, \ldots, |r_n|) = \sum_{\omega \in G_{m,n}} |r_\omega|/(n + m - 1)$$

the $m$-th weighted completely symmetric function of $|r_1|, \ldots, |r_n|$. Then

$$(5.7) \sum_{\omega \in G_{m,n}} \text{per} A[\omega|\omega]/(n + m - 1) \geq m!q_m^2(|r_1|, \ldots, |r_n|)/r^m.$$ 

The inequality in (5.7) is strict unless $A$ has rank 1. If $A$ has rank 1 then equality holds in (5.7) if and only if all the non-zero entries of $A$ have equal modulus.

**PROOF.** The proof follows that of Theorem 5.1. For $\omega = (\omega_1, \ldots, \omega_m) \in G_{m,n}$ we use the Cauchy-Schwarz inequality to estimate the lengths of all the symmetric products $x_{\omega_1} \cdots x_{\omega_m}$ in $V^{(m)}$. Lemma 4.1 again enables us to decide the cases of equality. We omit the details.

We continue our study of associated matrix functions with mean values as our dominant theme. We shall show how the inequality (4.6) of Schur can be imbedded in a continuous chain of matrix inequalities. Our
principal tools will be the classical inequalities for power means and the spectral theorem as it applies to positive hermitian transformations on a unitary vector space. We begin by recalling the definition of power means.

For any non-negative real values \( (x) = [x_1, \ldots, x_N] \) and positive real weights \( (q) = [q_1, \ldots, q_N] \), \( \sum_{i=1}^{N} q_i = 1 \), and for any real \( t \) define the \( t \)-th power mean or \( t \)-norm of the values \( (x) \) with weights \( (q) \) by

\[
M_t(x|q) = \begin{cases} 
(\sum_{i=1}^{N} q_i x_i)^{1/t} & \text{if } t \neq 0 \\
\prod_{i=1}^{N} x_i & \text{if } t = 0 \\
0 & \text{if } t < 0, \text{ and some } x_i = 0.
\end{cases}
\]

In case \( t = -1, 1, \) or \( 2 \) the \( t \)-norms are, respectively, the harmonic mean, the arithmetic mean, and the root-mean-square. If \( t = 0 \), \( M_t(x|q) \) is the geometric mean. An extensive literature surrounds power means and their varied applications [1, 7]. We summarize relevant properties in the following theorem and omit the proof.

**THEOREM 5.3.** For fixed \( (x) \) and \( (q) \), \( M_t(x|q) \) is continuous, monotone non-decreasing for all real \( t \). If no \( x_i \), \( i = 1, \ldots, N \), is zero, \( M_t(x|q) \) is strictly increasing for all \( t \) unless all the \( x_i \) are equal. \( M_t(x|q) \) is strictly increasing for positive \( t \) unless all the \( x_i \) are equal. The function \( t \log M_t(x|q) \) is a convex function of \( t \) for positive \( x_i \) or if \( t \) is positive and is strictly convex unless all the \( x_i \) are equal. Moreover

\[
\lim_{t \to \infty} M_t(x|q) = \max(x)
\]

and
\[(5.9) \quad \lim_{t \to -\infty} M_t(x|q) = \min(x).\]

We return to the \(n\)-dimensional unitary space \(V\) with orthonormal basis \(e_1, \ldots, e_n\) and again denote by \(T: V \to V\) a normal transformation from \(V\) to itself. By using the spectral theorem we can define in the usual way [6] the map \(T^t: V \to V\) for any non-negative real \(t\), and if \(T\) is regular, \(T^t\) is defined for all \(t\). \(T^t\) is again normal. If \(T\) is hermitian and positive so is \(T^t\). Let \(V^m(x,H)\) be the symmetry class associated with the subgroup \(H\) of order \(h\) of \(S_m\) and character \(x\) of degree 1 of \(H\). Then we know that the induced map \(K(T^t): V^m(x,H) \to V^m(x,H)\) is normal. If \(T\) has proper values \(\lambda_1, \ldots, \lambda_n\) and corresponding orthonormal proper vectors \(u_1, \ldots, u_n\) then \(K(T^t)\) has proper values \(\lambda^t_\omega = \sum_{j=1}^{n} \lambda_j^{tm_j(\omega)}\) and corresponding orthonormal proper vectors \(\sqrt{h/\nu(\omega)}u^*_{\omega}, \omega \in \hat{\Lambda}\). If \(e^\hat{\Lambda}_\omega, \omega \in \hat{\Lambda}\), is the orthogonal basis of \(V^m(x,H)\) induced by the orthonormal basis \(e_1, \ldots, e_n\) of \(V\) then by (4.9) the complex numbers 

\[\left(\sqrt{h/\nu(\omega)}K(T)e^*_{\omega}, \sqrt{h/\nu(\omega)}e^*_{\omega}\right)\] 

are convex combinations of the proper values \(\lambda_\gamma\) of \(K(T), \gamma \in \hat{\Lambda}\). We specialize these remarks to positive semi-definite hermitian matrices.

Denote by \(C^n\) the vector space of \(n\)-tuples of complex numbers. For \(x = (x_1, \ldots, x_n)\) and \(y = (y_1, \ldots, y_n)\) in \(C^n\), \((x,y) = \sum_{i=1}^{n} x_i \bar{y}_i\) is the \textit{standard inner product}, and the space \(C^n\) together with the inner product \((x,y)\) is an \(n\)-dimensional unitary vector space. For \(i = 1, \ldots, n\) set \(e_i = (\delta_{i1}, \ldots, \delta_{in})\), an orthonormal basis of \(C^n\). Let \(T: C^n \to C^n\) be a positive hermitian map defined by \(T = A^T\) where \(A = [a_{ij}]\) is an \(n\)-square positive semi-definite hermitian matrix and \(A^T\) is its transpose. Suppose \(T\) has as non-negative real proper values \(\lambda_1, \ldots, \lambda_n\) arranged in
descending order and corresponding orthonormal proper vectors \( u_1, \ldots, u_n \). Then by (4.2)

\[
\begin{align*}
h(K(T)e^*_\omega, e^*_\omega) &= d_x\left([((T e^*_\omega, e^*_\omega)]\right) \\
&= d_x\left([(A^T e^*_\omega, e^*_\omega)]\right) \\
&= d_x([a^*_\omega, a^*_\omega]) \\
&= d_x(A[\omega|\omega])
\end{align*}
\]

for any \( \omega \in \mathbb{A} \). By (4.9) we have

\[
(5.10) \quad d_x(A[\omega|\omega]) = \nu(\omega) \sum_{\gamma \in \mathbb{A}} \prod_{j=1}^{n} \lambda_j^{m_j}(\gamma) |(\sqrt[\gamma^*]{\nu(\omega)} e^*_\omega, \sqrt[\gamma^*]{\nu(\gamma)} u^*_\gamma)|^2;
\]

associated matrix functions are weighted arithmetic means of the non-negative proper values \( \lambda_\gamma \) of \( K(A) \). M. Marcus has used this observation [11,14] to obtain new matrix inequalities which include as special cases that of E. Fischer (4.7) and those of K. Fan [3]. We may now assert the following.

**THEOREM 5.4.** Let \( A = [a_{ij}] \) denote an \( n \)-square positive semi-definite hermitian matrix. Then the function

\[
g(t) = \begin{cases} 
  \det A & \text{if } t = 0 \\
  d^{1/t}_x(A^t) & \text{if } t \neq 0 \\
  0 & \text{if } t < 0, A \text{ singular}
\end{cases}
\]

is continuous, monotone non-decreasing for all real \( t \). If \( A \) is regular then \( g(t) \) is strictly increasing for all \( t \) unless (i) the subgroup \( S_n(A) \) of \( S_n \) generated by all transpositions \( (i \ j) \) for which \( a_{ij} \neq 0 \) is
a subgroup of $H$ and $x$ restricted to $S_n(A)$ is $\epsilon$, the alternating character. The function $\log d_x(A_t)$ is a convex function of $t$ and in case $A$ is regular is strictly convex unless (i) holds. If $\Lambda_{\text{max}}$ and $\Lambda_{\text{min}}$ denote, respectively, the maximal and minimal proper values of $K(A)$ then for all $t$, $g(t)$ satisfies the inequalities

$$
(5.11) \quad \Lambda_{\text{min}} \leq g(t) \leq \Lambda_{\text{max}}. 
$$

**PROOF.** We apply (5.10) to the positive transformation $T^t: \mathbb{C}^n \rightarrow \mathbb{C}^n$ where $T = A^t$. Set

$$
c_{\omega, \gamma} = |(\sqrt[n]{v(\omega)}e_\omega^*, \sqrt[n]{v(\gamma)}u_\gamma^*)|^2. $$

For $\omega \in \tilde{A}$ let $\Delta(\omega)$ denote the set of all sequences $\gamma \in \tilde{A}$ for which $c_{\omega, \gamma} > 0$. If $m = n$ and $\omega = (1, \ldots, n)$ then $v(\omega) = 1$ and (5.10) becomes

$$
(5.12) \quad d^{1/t}(A_t) = \left( \sum_{\gamma \in \Delta(l, \ldots, n)} c_{(l, \ldots, n), \gamma} \lambda_\gamma^t \right)^{1/t},
$$

the $t$-th power mean of the proper values $\lambda_\gamma$, $\gamma \in \Delta(l, \ldots, n)$, of $K(A)$ with weights $c_{(l, \ldots, n), \gamma}$. By Theorem 5.3 $g(t)$ is continuous and monotone non-decreasing for all real $t \neq 0$. We assert

$$
\det A = \prod_{\gamma \in \Delta(l, \ldots, n)} \lambda_\gamma^{c_{(l, \ldots, n), \gamma}}. 
$$

By (4.10) we have for $j = 1, \ldots, n$

$$
\sum_{\gamma \in \Delta(l, \ldots, n)} m_j(\gamma)c_{(l, \ldots, n), \gamma} = \sum_{\gamma \in \Delta} m_j(\gamma)c_{(l, \ldots, n), \gamma} = \sum_{i=1}^n m_i(l, \ldots, n)|e_i^*u_j|^2 = \sum_{i=1}^n |(e_i, u_j)|^2 = 1.
$$
Hence
\[ \prod_{\gamma \in \Delta(1,\ldots,n)} \lambda_{\gamma}^{c_{\gamma}(1,\ldots,n)}, \gamma = \prod_{\gamma \in \Delta(1,\ldots,n)} \prod_{j=1}^{n} \lambda_{j}^{m_{j}(\gamma)c_{\gamma}(1,\ldots,n)}, \gamma \]
\[ = \prod_{j=1}^{n} \prod_{\gamma \in \Delta(1,\ldots,n)} \lambda_{j}^{m_{j}(\gamma)c_{\gamma}(1,\ldots,n)}, \gamma \]
\[ = \prod_{j=1}^{n} \sum_{\gamma \in \Delta(1,\ldots,n)} \lambda_{j}^{m_{j}(\gamma)c_{\gamma}(1,\ldots,n)}, \gamma \]
\[ = \prod_{j=1}^{n} \lambda_{j} \]
\[ = \det A. \]

Thus \( g(t) \) is everywhere continuous. Now \( g(t) \) is by Theorem 5.3 either everywhere constant or strictly increasing. Hence \( g(t) \) is constant everywhere if and only if \( g(0) = g(1) \); i.e., \( \det A = \chi(A) \). But if \( A \) is regular, by Lemma 4.2 equality can occur if and only if (i) holds. The rest of the theorem follows by direct application of the previous observation and by Theorem 5.3.

We can, as usual, say considerably more if we know that \( \chi = 1 \). The following assertion is only a slight generalization of a result of M. Marcus and M. Newman [21] but places their hitherto somewhat isolated theorem in its proper context. We state our result as a corollary.

**COROLLARY 5.1.** Let \( A \) be as in the previous theorem and assume \( \chi = 1 \). If \( A \) is regular then \( g(t) \) is strictly increasing for all \( t \) unless \( A \) is diagonal. If \( A \) is singular, \( g(t) \) is strictly increasing for positive \( t \) unless \( A \) has a zero row. If the proper vector corresponding to the maximal proper value \( \lambda_{1} \) of \( A \) has no zero co-ordinates then

\[ \lim_{t \to \infty} g(t) = \lambda_{1}^{n}. \]
If the proper vector corresponding to the minimal proper value \( \lambda_n \) of \( A \) has no zero co-ordinates then

\( \lim_{t \to -\infty} g(t) = \lambda_n \).

**PROOF.** The first part of the assertion is clear by the previous theorem and by Lemma 4.1. We next note that the sequence set \( \tilde{A} = A \) for \( \chi = 1 \) and any subgroup \( H \) of \( S_n \) coincides with \( A \) since for \( \omega \in \Lambda \),

\[
\sum_{\sigma \in H} \chi(\sigma) = \sum_{\sigma \in H} \omega \neq 0 \text{ where } H_\omega \text{ is the subgroup of } H \text{ fixing } \omega. \text{ Then,}
\]
too, \( A \) contains \( G_{n,n} \), the set of non-decreasing sequences. Thus both

(1, ..., 1) and (n, ..., n) are in \( \tilde{A} \) so here \( K(A) \) has \( \lambda_1 \) and \( \lambda_n \) as its maximal and minimal proper values, respectively. If \( u = (u_{11}, ..., u_{nn}) \) and \( u_n = (u_{1n}, ..., u_{nn}) \) are the proper vectors of \( A^T \) corresponding to \( \lambda_1 \) and \( \lambda_n \), respectively, then \((\tilde{u}_{11}, ..., \tilde{u}_{nn})\) and \((\tilde{u}_{1n}, ..., \tilde{u}_{nn})\) are their counterparts for \( A \). We show that \((1, ..., 1) \in \tilde{A}(1, ..., n)\). Since \( \nu(1, ..., 1) = \sum_{\sigma \in H} (1, ..., 1) = \chi = h \) we have

\[
c(1, ..., n), (1, ..., 1) = |(\sqrt{h}e_1 \ast \ldots \ast e_n, u_1 \ast \ldots \ast u_1)|^2
= |d_\chi((e_1, u_1))|^2/h
= h^2 \left| \prod_{i=1}^{n} a_{ii} \right|^2/h
= h \prod_{i=1}^{n} |\tilde{u}_{1i}|^2
\neq 0.
\]

Similarly, \( c(1, ..., n), (n, ..., n) \neq 0 \), and the proof is complete.

It is clear from (5.10) and the proof of Theorem 5.4 that we may state a slightly more general result in terms of a principal submatrix
A[\omega|\omega], \omega \in Q_{m,n}, of the positive semi-definite hermitian matrix A.

There is no known analogue, however, of Lemma 4.1 or Lemma 4.2 to aid in the determination of the cases of equality. In the following result as in the sequel we distinguish between the matrices $A^t[\omega|\omega]$ and $(A[\omega|\omega])^t$.

**THEOREM 5.5.** Let A denote an n-square positive semi-definite hermitian matrix having proper values $\lambda_1, \ldots, \lambda_n$ and corresponding orthonormal proper vectors $u_j = (u_{1j}, \ldots, u_{nj}), j = 1, \ldots, n$. Let $\omega = (\omega_1, \ldots, \omega_m) \in Q_{m,n}$. Then the function

$$g_\omega(t) = \begin{cases} \frac{1}{t} \log d_\chi(A^t[\omega|\omega]) & \text{if } t \neq 0 \\ \prod_{j=1}^{n} \left( \sum_{i=1}^{m} |u_{\omega_j}|^2 \right) & \text{if } t = 0 \end{cases}$$

is continuous, monotone non-decreasing for all real $t$ if A is regular and for all positive $t$ if A is singular. The function $\log d_\chi(A^t[\omega|\omega])$ is a convex function of $t$. If $\Lambda^{\min}$ and $\Lambda^{\max}$ are respectively, the minimal and maximal proper values of the associated matrix $K(A)$ then

$$\Lambda^{\min} \leq g_\omega(t) \leq \Lambda^{\max}.$$  

**PROOF.** The proof follows from (5.10) and the proof of the corresponding assertions in Theorem 5.4.

As an example of the preceding result let $V^m_{\chi}(\Omega) = \Lambda^m V$, the m-th Grassman space. Then $d_\chi(A[\omega|\omega]) = \det A[\omega|\omega]$, the determinant function. Here $K(A)$ is denoted by $C_m(A)$, the m-th compound of the matrix A whose proper values are $\lambda_\gamma, \gamma \in Q_{m,n}$. Hence $C_m(A)$ has minimal proper value $\prod_{i=1}^{n} \lambda_{n-i+1}$ and maximal proper value $\prod_{i=1}^{m} \lambda_i$ if $\lambda_1 \geq \cdots \geq \lambda_n$ are the proper values of A. Thus $g_\omega(t) = \det^{1/t}(A^t[\omega|\omega])$ satisfies for all $t$
(5.16) \[ \prod_{i=1}^{m} \lambda_{n-i+1} \leq \det^{1/t}(A^t|\omega|) \leq \prod_{i=1}^{m} \lambda_i. \]

When \( t = 1 \), (5.16) is an inequality of M. Marcus and J. McGregor [16].

In the remainder of this chapter we shall give some applications of Theorems 5.4-5.5. We know from these results that inequalities for the power means \( M_t(x|q) \) provide inequalities for associated matrix functions. The cases of equality are troublesome, however, in that when equality is specified in inequalities involving power means, it is usually specified in terms of the \( x_i \) in \( (x) = [x_1, \ldots, x_N] \). In our applications the \( x_i \) are the proper values of the associated matrices \( K(A) \) about which we generally possess little information. Then, too, we are given the weights \( (q) = [q_1, \ldots, q_N] \) as non-negative rather than strictly positive numbers. Thus the only cases of equality we can so far decide are those provided by Theorem 5.4 and Corollary 5.1. Such is the case in the following result.

**Corollary 5.2.** Let \( t = \sum_{i=1}^{k} p_i s_i \) be a convex combination of \( k \) real numbers \( s_i \). Then if \( A \) is regular or if all the \( s_i \) are non-negative

(5.17) \[ d(\chi(A^t|\omega|)) \leq \prod_{i=1}^{k} d(\chi(A_i|\omega|)) \]

for any \( \omega \in Q_{m,n}^+ \).

**Proof.** Since \( \log d(\chi(A^t|\omega|)) \) is convex in \( t \)

\[ \log d(\chi(A^t|\omega|)) \leq \sum_{i=1}^{k} p_i \log d(\chi(A_i^s|\omega|)) \]

\[ = \sum_{i=1}^{k} \log d(\chi(A_i^s|\omega|)) \]

\[ = \log \prod_{i=1}^{k} d(\chi(A_i^s|\omega|)) \]

and the result follows.
Much attention has been devoted in the recent literature to the techniques of developing new inequalities from established ones. For example, consider the Cauchy-Schwarz inequality in the unitary vector space $V$. For all $u, v \in V$ we have

$$|(u, v)| \leq ||u|| \cdot ||v||.$$  

It is clear that no universal constant $C$ exists for which

$$\text{(5.18)} \quad ||u|| \cdot ||v|| \leq C|(u, v)|$$

for all $u$ and $v$ in $V$. For certain restricted $u$ and $v$, however, a $C$ may be found for which (5.18) holds. An inequality of the kind (5.18) is called a complementary inequality, and the process of determining a complementary inequality is called reversal. Naturally, the most interesting complementary inequalities are those which specify when precisely the inequality becomes equality; e.g., given $u$ and $v$ in some subset of $V$, determine the smallest $C$ for which (5.18) holds. Theorem 5.3 provides a continuous chain of inequalities for the power means $M^r(x|q)$ from which we obtained a continuous chain of matrix inequalities. We state without proof a result of G. Cargo and O. Shisha [2] which yields a continuous chain of inequalities complementary to those of Theorem 5.3. See also [23].

**Theorem 5.6.** Let $r$, $s$, $a$, and $b$ be given real numbers where $r < s$ and $0 < a < b$. Let $\gamma = b/a$ and let $I = \{x_1, \ldots, x_N \mid a \leq x_k \leq b, \ k = 1, \ldots, N\}$ denote the $N$-dimensional cube. Then for fixed weights $q_i > 0, \sum_{i=1}^{N} q_i = 1$,

$$\text{(5.19)} \quad M_s(x|q)/M_r(x|q) \leq T_1$$
where

\[
\begin{align*}
\{(r(Y^S - Y^r)) &/ [(s - r)(Y^r - 1)]\}^{1/s} \\
\times \{[s(Y^r - Y^S)]/[r - s)(Y^S - 1)]\}^{-1/r} & \quad \text{if } rs \neq 0 \\
\end{align*}
\]

\[(5.20) \quad \tau_1 = \begin{cases} 
(r^S/(Y^S - 1))/[e \log r^S/(Y^S - 1)]^{1/s} & \quad \text{if } r = 0 \\
(r^r/(Y^r - 1))/[e \log r^r/(Y^r - 1)]^{-1/r} & \quad \text{if } s = 0.
\end{cases}
\]

Let

\[
\begin{align*}
\{r/(Y^r - 1) - s/(Y^S - 1)\} &/ (s - r) \quad \text{if } rs \neq 0 \\
0 = \begin{cases} 
1/(s \log Y^S) - 1/(Y^S - 1) & \quad \text{if } r = 0 \\
1/(r \log Y^r) - 1/(Y^r - 1) & \quad \text{if } s = 0.
\end{cases}
\end{align*}
\]

Then \(0 < \theta < 1\). Equality in (5.20) holds for a point \([x_1, \ldots, x_N] \in I\) if and only if there exists a subsequence \((k_1, \ldots, k_p)\) of \((1, \ldots, N)\) such that \(\sum_{j=1}^{p} q_{kj} = \theta, x_{kj} = b\) for \(j = 1, \ldots, p\), and \(x_k = a\) for all \(k \neq k_j, j = 1, \ldots, p\).

The preceding result gives an upper bound for the ratio of two power means. The following theorem provides a bound on the difference of two such means. This form of reversal yields a complementary inequality with an additive constant rather than a multiplicative one such as in (5.18) or (5.19). The proof of this and similar results can be found in [28].

THEOREM 5.7. Let \(r, s, a, b, \gamma, I, (x),\) and \((q)\) be as in Theorem 5.6, but assume \(s > 1\). Denote by \(0\) the unique solution of
\[ [y(y^s - 1) + 1]^{1/s-1}(y^s - 1)/s - [y(y^r - 1) + 1]^{1/r-1}(y^r - 1)/r = 0 \]
\[ 0 < y < 1 \]

if \( r \neq 0 \) or

\[ [y(y^s - 1) + 1]^{1/s-1}(y^s - 1)/s - y \log y = 0 \]
\[ 0 < y < 1 \]

if \( r = 0 \). Then

(5.21) \[ M_y(x|q) - M_x(x|q) \leq \alpha \]

where

\[ T_2 = \begin{cases} 
[y(y^s - 1) + 1]^{1/s-1}[y(y^r - 1) + 1]^{1/r} & \text{if } r \neq 0 \\
[y(y^s - 1) + 1]^{1/s-1}y & \text{if } r = 0.
\end{cases} \]

Equality in (5.21) holds for a point \([x_1, \ldots, x_N]\) if and only if there exists a subsequence \((k_1, \ldots, k_p)\) of \((1, \ldots, N)\) such that \(\sum_{j=1}^{p} q_{k_j} = \theta, \quad x_{k_j} = b \quad \text{for } j = 1, \ldots, p, \quad \text{and } x_k = a \quad \text{for all } k \neq k_j, \quad j = 1, \ldots, p.\)

It is clear that Theorems 5.4-5.5 together with the two preceding results will produce inequalities for associated matrix functions. The following theorem is an example of the kind of result we may expect.

**THEOREM 5.8.** Let \( A \) denote an \( n \)-square positive definite hermitian matrix not a scalar matrix. Let \( \lambda_1 \) and \( \lambda_n \) denote, respectively, the maximal and minimal proper values of \( A \) and let \( \kappa = \kappa(A) = \lambda_1/\lambda_n \), the condition number relative to the spectral norm of the matrix \( A \). Set \( \gamma = \kappa^n \). Then for \( H \) a subgroup of \( S_n \) and \( \chi \) a character of degree 1 of \( H \)

(5.23) \[ d_{x}^{1/s}(A^s)/d_{x}^{1/r}(A^r) \leq T_1 \]
for \( r < s \), and if \( s \geq 1 \)

\[(5.24) \quad d_x^{1/s}(\lambda^s) - d_x^{1/r}(\lambda^r) \leq \lambda_n^{nt_2} \]

PROOF. The proof follows from Theorem 5.4 and Theorems 5.6-5.7 upon setting \( a = \lambda_n^n \) and \( b = \lambda_1^n \).

If we set \( r = 0 \) and \( s = 1 \) we obtain

\[(5.25) \quad d_x(A) \leq \left( \kappa^n/(\kappa^n-1) \det A \right)/\left( e \log \kappa^n/(\kappa^n-1) \right) \]

from (5.23) and

\[(5.26) \quad d_x(A) \leq \det A + \lambda_n^n \left[ ((\kappa^n - 1)/(n \log \kappa)) \log[(\kappa^n - 1)/(en \log \kappa)] + 1 \right] \]

from (5.24). These are complements of the inequality (4.6) of Schur.

Suppose we take \( H = \{I\} \), the identity permutation alone. Then \( V_x^m(H) = N^mV \), the tensor product of \( V \) taken with itself \( m \) times. If \( m = n \) then \( d_x(A) = \prod_{i=1}^n a_{ii} \), the product of the diagonal entries of \( A \), and Schur's inequality, as Schur himself points out [27], becomes

\[(5.27) \quad \det A \leq \prod_{i=1}^n a_{ii} \]

the classical Hadamard inequality. In this case (5.26) yields

\[(5.28) \quad \prod_{i=1}^n a_{ii} - \det A \leq \lambda_n^n \left[ ((\kappa^n - 1)/(n \log \kappa)) \log[(\kappa^n - 1)/(en \log \kappa)] + 1 \right] \]

Recently M. Marcus and G. Soules [22], utilizing inequalities derived from a generalized Laplace expansion theorem, obtained the estimate
(5.29) \[ \prod_{i=1}^{n} a_{ii} - \det A \leq \lambda_1^{n-2} \sum_{1 \leq i < j \leq n} |a_{ij}|^2 \]

together with the cases of equality.

As a final example, we can reverse the inequalities (5.16) derived from Theorem 5.5. Let \( \gamma = \prod_{i=1}^{m} \lambda_i / \lambda_{n-i+1} \) and let \( \omega \in Q_{m,n} \). Then if \( A \) is any positive definite hermitian matrix not a scalar matrix with proper values \( \lambda_1 \geq \cdots \geq \lambda_n \) we have for \( r < s \)

(5.30) \[ \det^{1/s}(A^{s}[\omega|\omega]) / \det^{1/r}(A^{r}[\omega|\omega]) \leq T_1 \]
and if \( s > 1 \)

(5.31) \[ \det^{1/s}(A^{s}[\omega|\omega]) - \det^{1/r}(A^{r}[\omega|\omega]) \leq T_2 \prod_{i=1}^{m} \lambda_{n-i+1} \cdot \]

If in (5.30) we set \( r = -1 \) and \( s = 1 \), we obtain

(5.32) \[ 1 \leq \det(A[\omega|\omega]) \det(A^{-1}[\omega|\omega]) \leq (\gamma^{1/2} + \gamma^{-1/2})^2 / 4 , \]
a result of M. Marcus and N. Khan [15].
VI. SOME COMMENTS AND QUESTIONS

The field of matrix theory and matrix inequalities is most active at the present time. Stimulated by a paper of W. Frank [4], M. Marcus and H. Minc have shown that generalized matrix functions associated with a group $H$ and a character $\chi$ of degree 1 on $H$ are monotone matrix functions in the sense of C. Loewner [1]; i.e., if $A \geq B$, meaning that the matrix $A - B$ is positive semi-definite hermitian when $A$ and $B$ are, then $d_\chi(A) \geq d_\chi(B)$. Moreover, it is true that the associated induced matrices satisfy $K(A) \geq K(B)$. For the sake of completeness we include brief proofs of these results, for they imply our inequalities given by Theorems 5.1 - 5.2 as well as an inequality for the $m$-th power matrix $P_m(A)$ announced recently by D. Sasser and M. Slater [26]. Our proof rests on the following

**LEMMA 6.1.** Let $A$ and $B$ denote $n$-square positive semi-definite hermitian matrices and suppose that $A$ is regular. Then the proper values of $A^{-1}B$ are real and non-negative. Moreover, $A \geq B$ if and only if the proper values of $A^{-1}B$ are at most equal to 1.

**PROOF.** Let $C$ denote the unique positive definite square root of $A$. Since $A \geq B$ if and only if $((A - B)x,x) \geq 0$ for all $x$, it is easy to see that the partial ordering $A \geq B$ is preserved under a congruence transformation. Hence $I_n = C^{-1}AC^{-1} = (C^{-1})^*AC^{-1} \geq (C^{-1})^*BC^{-1} = C^{-1}BC^{-1} = \ldots$

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1Private communication.
The proper values of $A^{-1}B$ coincide with those of $CA^{-1}BC^{-1}$, a matrix hermitely congruent to $B$. Thus $A^{-1}B$ has real, non-negative proper values all less than or equal to 1. The reverse argument proves the converse.

The result of Marcus and Minc now follows readily.

**Theorem 6.1.** If $A \preceq B$ then $K(A) \preceq K(B)$.

**Proof.** Assume first that $A$ is regular. Then $A^{-1}B$ has all of its proper values between 0 and 1. But by Theorem 3.4 so does $K(A^{-1}B) = K^{-1}(A)K(B)$. Hence $K(A) \succeq K(B)$. A routine argument based on continuity extends the inequality to possibly singular $A$.

**Corollary 6.1.** Let $H$ be any permutation group of order $h$ and degree $m$ and let $\chi$ denote a character of degree 1 on $H$. If $\omega = (\omega_1, \ldots, \omega_m) \in \Delta^H_{m,n}$ then $d_\chi(A[\omega|\omega]) \geq d_\chi(B[\omega|\omega])$ when $A \succeq B$.

**Proof.** Let $e_1, \ldots, e_n$ denote the standard basis of $\mathbb{C}^n$, the unitary space of $n$-tuples of complex numbers. Then using standard inner products with $S = A^T$ and $T = B^T$

$$d_\chi(A[\omega|\omega]) = d_\chi(((Se_{\omega_i}, e_{\omega_j}))$$

$$= h(K(S)e^*_\omega, e^*_\omega)$$

$$\geq h(K(T)e^*_\omega, e^*_\omega)$$

$$= d_\chi(((Te_{\omega_i}, e_{\omega_j}))$$

$$= d_\chi(B[\omega|\omega]).$$
The following theorem also communicated to us by M. Marcus now relates the foregoing discussion to two of our principal results.

**THEOREM 6.2.** Let \( A \) denote an \( n \)-square positive semi-definite hermitian matrix whose row sums \( r_i \) satisfy \( \sum_{i=1}^{n} r_i = r > 0 \). Then

\[
(6.1) \quad A \preceq \begin{bmatrix} r_i^{-1} & r_j \end{bmatrix} \frac{r}{r}.
\]

**PROOF.** Again, we can assume by continuity that \( A \) is regular. Let \( A = PP^* \) where \( P \) is now regular. Define \( Q \) to be that matrix each of whose column vectors is the sum of the \( n \) row vectors of the matrix \( P \). Denote by \( J_n \) the \( n \)-square positive semi-definite matrix of rank 1 each of whose entries is \( 1/n \). We observe that \( Q^*Q = r_nJ_n = QQ^* \) and that \( PQ \) has \( r_i \) as its \( i,j \) entry. Also note that \( I_n \succeq J_n \). For a unitary matrix \( U \) exists for which \( U^*J_nU = E_{11} \), the \( n \)-square matrix with 1 in the 1,1 position and 0 elsewhere, and clearly \( I_n \succeq E_{11} \). Hence \( r_nA = r_PP^* \preceq r_nP^*_n \geq r_nP^*_n = PQQ^*P = PQ(PQ)^* = n[R^{-1}_i r_j] \).

If in Corollary 6.1 we set \( B = [r_i^{-1} r_j/r] \) and \( \chi = 1 \), apply the result to each sequence \( \omega \in Q_{m,n} \), and sum we obtain the inequality (5.4) from which (5.1) follows. The inequality (5.7) is proved similarly.

The preceding developments promise to be a fruitful source of matrix inequalities. Another observation that should prove useful is that contained in equation (5.10); i.e., \( d^\chi(A) \) is a convex combination of the proper values of the associated matrix \( K(A) \) when \( A \) is a normal matrix. For example, suppose \( A \) is a circulant. Then the unitary matrix \( U = [\exp(-2\pi ij\sqrt{-1}/n)]/\sqrt{n} \) diagonalizes \( A \) [17, p. 66], so that a suitable set of weights in (5.10) would be \( |d^\chi(U[1,\ldots,n \mid \gamma])|^2/\nu(\gamma), \gamma \in \Delta \). It
would be helpful to know which of these constants are strictly positive. Along the same lines, one can ask for the maximal and minimal proper values of \( K(A) \) in terms of \( H, \chi \), and the proper values of \( \Lambda \). What are the maximal and minimal sequences lexicographically in \( \Lambda_{m,n}^H \)? These are difficult questions, but progress is being made. The rank and determinant of \( K(A) \) have been calculated as well as the dimension of \( V_x^m(H) \) [13].

Finally, we mention some conjectures which have arisen through the study of symmetry classes. One due to M. Marcus asserts that for \( A = [a_{ij}] \) an \( \text{n-square positive semi-definite hermitian matrix} \)

\[
d_x(A) \leq \text{per } A
\]

for any subgroup \( H \) of \( S_n \) and character \( \chi \) of degree 1 on \( H \). It has already been shown [9] that a "Hadamard" inequality for the permanent

\[
\prod_{i=1}^{n} a_{ii} \leq \text{per } A
\]

is valid. Moreover, E. Lieb [8] recently proved that for \( q = 1, \ldots, n \)

\[
\text{per } A[1, \ldots, q|1, \ldots, q] \text{per } A[q+1, \ldots, n|q+1, \ldots, n] \leq \text{per } A,
\]

the analogue for the permanent of Fischer's inequality (4.7). There is also some evidence to support the existence of the analogue for the permanent function of the Minkowski determinant inequality [1]

\[
\text{per}^{1/n}(A + B) \leq \text{per}^{1/n}(A) + \text{per}^{1/n}(B),
\]

but no proof has yet appeared.
REFERENCES


13. __________, Symmetry classes and combinatorial identities, Notices Amer. Math. Soc., 13(1966), p. 120.