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DIFFICULTIES ENCOUNTERED BY ELEMENTARY ALGEBRA STUDENTS IN SOLVING EQUATIONS IN ONE UNKNOWN--A DIAGNOSIS OF ERRORS AND A COMPARISON AFTER FORTY YEARS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

Harold A. Leonard, B.S., M.A.

* * * * * * *

The Ohio State University

1966

Approved by

[Signature]

Adviser

School of Education
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CHAPTER I

INTRODUCTION

Algebra: a branch of mathematics in which arithmetic relations are generalized and explored by using letter symbols to represent numbers, variable quantities, or other mathematical entities (as vectors and matrices), the letter symbols being combined especially in forming equations, in accordance with assigned rules.\footnote{Webster's Third New Unabridged Dictionary.}

Background of the study

The definition of algebra quoted above adequately describes the powerful investigative tool which forms a part of what Harold Fawcett has called "one face of mathematics," referring to algebra's utility in man's endeavors to understand and manipulate his environment. However, the bare definition carries no hint of the power of "an algebra" to excite men's minds as an art form, of a testament to the marvel of the physical universe and the wonder of creation. The exposition of this "second face" of mathematics has been increasingly observable in the evolving mathematics curriculum of the 20th century. A principal reason for this change in emphasis has been the belief that the interest generated by the presentation of algebra through an exposition of its "two faces"...
would lead to greater student motivation and an accompanying clearer student understanding of the properties of a number field. One of the expected consequences of such an approach is an improvement of algebraic skills, as measured by student performance in standardized tests.

The record of mathematics education is replete with evidence that educators have been dissatisfied with the teaching of algebra. W. D. Reeve, in 1926, reported:

> It is a well-known fact that the percentage of failures in algebra and geometry is too high in many schools. For example, it is not uncommon to find as high as 25 per cent or 30 per cent of a normal group of pupils failing in freshman algebra, - a situation which cannot be justified from any modern point of view. (7,2)

J. P. Everett, in 1927, called attention to a comment on the teaching of algebra in mid-nineteenth century:

> Imperfect preparation in algebra is so common as to compel the conviction, that insufficient attention is given to this study in our Preparatory Schools, or that inadequate textbooks are used. . . Nor is acquaintance with the processes of the science sufficient, without a thorough knowledge of the principles upon which these processes are based. (4,4)

Apparently, students have performed poorly on tests in the past and, judging from the concern evinced recently, continue to do so today. It has become both fashionable and an element of the conventional wisdom to assume that drill in the fundamental manipulative processes of algebra unaccompanied by an exposition of the subject matter in a manner that stresses understanding of basic
concepts will not be effective in improving algebraic competence. The Commission on Mathematics of the College Entrance Examination Board, in 1959, recommended that algebra be given increased attention but that the emphasis be placed upon an understanding of the properties of a number field. Although the authors believed that the teaching of manipulative skills had an important place in the development of a mathematics program, they insisted that such development be placed second to the development of the fundamental ideas and concepts (2, 20-21).

This position affirmed the evolutionary change in emphasis from the kind of approach to the teaching of algebra typified by Hall and Knight's Algebra (15) to the concept-discovery approach of what has come to be known as "modern mathematics."

Though it might be still too early to criticize the results of many of the newer approaches and text books used in algebra classes across the nation, the current state of teacher re-education being perhaps too potent a variable for significant analysis of the effects of new programs, it is nevertheless painfully obvious to the researcher in his present setting that junior high school algebra students are still performing poorly in uniform tests. The New York State Education Department reported that 38.2 percent of the students taking the June 1965 Ninth Year Mathematics (Elementary Algebra) Regents Examination throughout the State of New York failed to pass! The examination appeared to be an uncomplicated test of minimum competence in 9th grade algebra. This situation
was somewhat frustrating to Long Island educators whose results were not significantly above the state average despite a nearly uniform attempt to offer conceptually-oriented courses claiming superiority to the traditional, manipulative-oriented courses. The demonstrated student incompetence in attempting to solve relatively simple equations in one unknown on the Regents Examination was particularly galling.

In the preface to their recent algebra text, Johnson, Lendsey and Slesnick wrote:

In addition to its concern with solving equations, Modern Algebra supplies a language and a pattern of reasoning to the rest of mathematics. It is the aim of this book to show some of the facets of the new and quite exciting branch of 20th century mathematics. (16, iii)

One hears the cry today that something seems to be wrong with the teaching of algebra. Can it be claimed that as a result of the changing emphasis in the teaching of algebra, the student becomes aware of the true meaning and structure of algebra when results of uniform testing expose his poor skills in solving simple equations in one unknown?

In 1926, W. D. Reeve published a set of algebra examinations which he designed to test broad areas of student competence in algebraic skills. One of these tests consists of a set of thirty-one equations in one unknown. This test was given to 1,204 ninth grade algebra students. In 1928, J. P. Everett, analyzing Reeve's results as well as results of a battery of tests which he himself constructed, concluded that the algebra students appeared to possess
the necessary manipulative skills. These skills he described as being "... self-contained in the sense that their performance itself indicated the essential nature of the mental processes which are involved. ..." (4, 9)

Why then did the students perform so poorly? Everett postulated the existence of a set of skills he called "associative," describing these skills as follows:

Back of every act that belongs to the level of skilled occupations or professions there seems to belong some subtle factor of appreciation or a recognition of significance that is not put into operation by any clearly observable external stimuli and that is not exercised in accordance with, or accompanied by manipulative expressions that can be perceived. (4, 9)

It appeared to Everett that the mere act of solving an equation correctly "conveys no intimations whatsoever of whether one knows that x is a number, or that the results obtained have any significance in relation to the original equation" (4, 10). In his thesis, Everett listed and described forty-four categories of associative skills and recommended that the teaching of these skills be given a high priority in the mathematics curriculum. Success in algebra, and possibly in many an unrelated field of life's endeavors might well depend

... not in any act or in the use of special instruments or formulas, but in the sensitiveness to some element of the situation that results in the right action at the appropriate time. In algebra success often depends only in a secondary sense upon ability to perform a mathematical operation; the real test of efficiency lies in whether or not the mind has grasped some subtle understanding of relationships that sets the correct manipulative machinery in motion at the right time. (4, 12)
Statement of the problem

In the years following the works of Reeve and Everett quoted above, and particularly in the past dozen years, the mathematics curriculum has reflected an increasing concern for the presentation of algebra in a manner that stresses not only the basic skills which are manipulative in nature, but also a deeper understanding of basic notions, structure and patterns, skills which might be termed associative in the sense that Everett employed the term.

If the new emphases are indeed improving the teaching of algebra, we should expect significantly better results when today's students attempt tests which were prepared by educators in the past. If the results of such tests should be available to the researcher, and if these tests were to be administered to students today, data might be obtained which could be useful for the purpose of comparison of student performance.

To limit the size of the undertaking and to allow for depth in analysis, the researcher decided to limit his investigation to a narrow area of algebraic skills. The skills chosen to be tested were those required to solve equations in one unknown.

The researcher asks:

Question 1

If a test in solving equations in one unknown, attempted by a number of ninth grade algebra students approximately forty years ago, were administered to a comparable number of elementary algebra students in 1966, would there be any significant difference in student performance?
In an attempt to obtain some background information which might be of value in analyzing and interpreting student performance on the test, the writer decided to examine some textbooks which were in use during the first quarter of the Twentieth Century and to compare the presentation of the topic of solution of equations found in them with that found in modern texts. This information was obtained for the purpose of exploring the following question:

Question 2

To what extent might differences in student performance be attributable to changing emphases in presentation as reflected in the textbooks generally in use by students of the 1920's and today?

The researcher was able to examine the test papers submitted by the 1966 sample. He observed, analyzed, and categorized student errors for the purpose of exploring the following questions:

Question 3

What are some of the significant areas of student difficulty in attempting to solve equations of the type appearing on the 1966 test?

Question 4

To what extent are the difficulties experienced by students indications of inadequate manipulative and associative skills?
CHAPTER II

RESEARCH DESIGN AND INSTRUMENTATION

Design of the study

The researcher arranged for a test, consisting in part, of selected equations taken from a set prepared by Reeve in the early 1920's and administered to 1,204 ninth grade algebra students, to be given to 1,226 elementary algebra students of eight junior and senior high schools on Long Island, New York.

Seven different textbooks are presently in use for ninth grade algebra in these eight schools. The participating students were "Regents Track;" no general, shop, business, or consumer mathematics students were involved. All participating schools present courses in algebra which follow the Ninth Year Mathematics Syllabus of the New York State Education Department. Such courses normally terminate in a "Regents Examination." (The Ninth Year Mathematics Regents Examination was reinstated for a three-year experimental period in 1964 after a long period of disuse.)

The researcher attempted to answer question 1 by comparing results of items answered by students in the two population samples. By employing a standard test of statistical significance (t-test) for the differences noted, he was in a position to make a number of
assertions concerning significance of differences in student performance at the 1 per cent level of confidence.

An examination of two textbooks which are representative of a traditional approach to the teaching of algebra and comparison with a 'modern' approach as reflected in the seven texts used by the students in the 1966 sample assisted the researcher in his exploration of question 2.

By observing the kinds of errors students made in their attempts at solutions, the researcher was able to identify certain areas of weakness (Q.3), and to diagnose student difficulties (Q.4).

In structuring his thesis, the researcher intends first to present the data, develop his arguments, and submit the results of his investigation of the above four questions in sequence; then, he will summarize his findings and restate his conclusions somewhat more formally in the final chapter. At the conclusion, the writer will also list nine recommendations for overcoming a number of the difficulties encountered by the students as demonstrated by their performance on the 1966 test.

Selection of a test for comparison purposes

The researcher surveyed the literature of the early years of the Twentieth Century in search of a suitable instrument for his study. Only two tests were found which appeared to be serviceable. One was a test constructed by William D. Reeve and administered to 1,204 ninth grade algebra students in the early 1920's, and the other
was the "Hotz First Year Algebra Scale in Equation and Formula."

There were two references in the literature to the use of the latter examination. Schreiber, in 1925, reported using the Hotz test in a study involving 160 students in eight classes (26, 65). The same test was used on a much larger scale in the Pacific Northwest, with 4,195 students involved (25, 418).

There were two objections to using the Hotz test as a vehicle for this study. First, the range of topical coverage appeared too broad. (The twenty-five equations on the test ranged from linear to radical equations.) Second, the statistics available did not permit a breakdown of student achievement on an item-by-item basis. The Reeve equation test, covering what might be termed introductory skills for equation solving, appeared to be more useful in that the topic was manageable and Reeve's results were provided in an appendix to his study (7). These results were expressed as per cents correct, on an item-by-item basis, enhancing the test's serviceability for purposes of comparison of student achievement. Consequently, the researcher decided to make use of the Reeve test. A description of this test and some details of its administration follow.

**The Reeve Equation Test - Test 1**

In a book published in 1926, William D. Reeve reported on a study he had done of teaching problems in high school mathematics (7). The book was a reprint of a Ph.D. thesis which he had submitted in 1924 while pursuing his doctorate at the University of Minnesota. In the period of time from 1915 to 1921 Reeve was employed as a teacher
of mathematics at the University of Minnesota High School, and from 1921 to 1923 he served as principal of that school. 2

Within the body of the thesis, Reeve displayed a set of tests which he constructed to assist him in diagnosing student difficulties in algebra and geometry. Among these tests (the first of his so-called composite tests) was one consisting of thirty-one equations in one unknown (7, 32). This test will be referred to by the researcher as Test 1.

Test 1 was administered to 1,204 ninth grade algebra students in a number of schools in the Minneapolis area (7, 33). Reeve reported that he did the experimental work with the assistance of a number of mathematics teachers in Minneapolis and St. Paul who were interested in the experiment. Reeve wrote:

The teachers who assisted in the study are all in schools which are accredited by the North Central Association of Colleges and Secondary Schools. (7, 4)

The test was administered immediately after thirteen lessons were given in the topic of solution of equations of the type tested. Reeve allotted forty-five minutes for completion of the thirty-one items (7, 29).

Reeve constructed a record sheet which he used to record the student results. On this sheet, incorrect responses were marked

2Biographical information was obtained from Who's Who in American Education, 16th ed., 1953-4, p. 1041.
with the symbol "x." If no response was made to an item, the symbol "o" was entered.

In an appendix to the thesis, Reeve included a table containing the results of Test 1, represented as per cents of correct responses for each item (7, 104).

John P. Everett, in a thesis which was published in 1927, made use of some material from Reeve's study. He analyzed student difficulties in four of the examples of Test 1 (4, 16-18). Everett identified some of the manipulative skills needed by the student to solve the equations, and on the basis of his analysis, concluded that though the students appeared to possess the necessary manipulative skills, they often lacked insights into the problem of knowing when to perform the appropriate operation.

Everett termed the abilities needed to attain these necessary insights "associative skills." The identification of these skills constituted the body of Everett's thesis.

The Equation Test--1966

The test administered to the 1966 group consisted of thirty items, of which only twenty-six were used for comparison purposes. The Reeve Test 1, described in the previous section, contained a number of equations which seemed to involve duplication of processes

3 The researcher had the privilege of interviewing Dr. Everett at a meeting of the National Council of Teachers of Mathematics in New York, in April of 1966. Understandably, at 90 years of age, he could recall very few of the details of Reeve's study. However, he did recall that he considered his thesis to "follow-up" the work done by Reeve.
already tested, and which also involved a large number of processes per item. These items, five in all, the researcher omitted.

The items numbered 7, 28, 29, and 30 on the 1966 test were equations added by the researcher. The added equations offered an opportunity to test certain equation solving skills in addition to those required for correct performance on the Reeve Test 1. (A copy of the test may be found in Appendix D.)

In the following chapter, the details of administration and scoring of this test will be described.
CHAPTER III

ADMINISTRATION OF THE TEST AND THE
RECORDING OF RESULTS

To prepare for administration of the test described in the previous chapter, the researcher, in January, 1966, conducted a pilot study with 143 students of algebra in his own school, Smithtown High School, a school which did not otherwise participate in the study. The pilot study had as its purpose the determination of suitable procedures for administering the test and compiling data.

On March 10, 1966, letters were sent to mathematics department chairmen and supervisors of eight schools on Long Island describing the researcher's project and inviting participation (Appendix A). In every case, the recipients of the letter indicated an interest in the project and a willingness to participate. On April 26 the participating chairmen and supervisors met with the researcher to plan for the administration of the test. Due to the large number of classroom teachers involved—(26), it was deemed advisable for each chairman to accept responsibility for briefing his teachers and arranging for proper administration of the test.

Each classroom teacher was responsible for completing the heading and listing the names of participating students on the Class Tally Sheet (Appendix E). Teachers were apprised of the importance
of following closely the instructions for administering the test. They were to make sure that students were aware that they were to show all work. No scrap paper was permitted. Pencil was to be used, and erasing was allowed. Thirty-five minutes were allotted for the examination.

Responsibility for the scoring rested with the researcher. A staff consisting of thirty-five Smithtown Central High School seniors was given the necessary preparation to assist in the task of scoring. Since the researcher was not only concerned with obtaining per cents of students obtaining correct answers, but also in tabulating, categorizing and analyzing student errors, he devised a scoring method whereby the incorrect responses to items were entered for each equation. The entry 'N' was made in the case where no numerical solution was submitted by the student despite some indication that a solution was attempted. The entry 'T' indicated that there was no evidence on the paper that the student attempted a solution.

Error tally sheets for each item were prepared (Appendix F). A tally was made of all incorrect answers submitted. These tallies were categorized on the Error-Tally sheets according to school and textbook in use. In cases where two different textbooks were being used in a school, two entry categories were established for tallying purposes.

When the scoring was completed, the researcher had available the following data:

1. The per cent of correct responses for each item on the test.
2. The number of students who failed to supply answers for each item. ("T" and "N")

3. The total number of non-equivalent incorrect responses submitted for each item.

4. The frequencies of particular incorrect responses.

The individual scores of students were not deemed significant for the purposes of this study. These statistics, however, are available to the participating schools on request.

Also available to the researcher were the test papers themselves. With them, one is able to observe student procedures both correct and incorrect. These observations were used to form conclusions concerning student abilities which will be discussed in a subsequent chapter.

In this chapter, the details of administration and scoring were outlined. We will now examine the results.
CHAPTER IV

COMPARISON OF STUDENT PERFORMANCE

Preliminary assumptions

The equation test administered to 1,226 students in 1966 consisted of twenty-six of the thirty-one items of the Reeve Test 1, and four additional equations making a total of thirty test items. The equations of Test 1 which were not included in the 1966 test were Test 1 items number 7, 23, 29, 30, and 31. In this chapter, comparisons will be made of achievement on those items of the two tests which were attempted by students in both samples.

Since an account of the conditions and details of administration of the Reeve Test 1 is not available to the researcher, he poses a set of assumptions in light of which he will make his analysis and comparison of results.

Assumption 1:

The Test 1 population sample and the 1966 population sample consist of students of comparable ability to learn mathematics at the level of elementary algebra.

In the eight schools which participated in the 1966 project, a sizeable fraction of ninth grade students, those with low ability in mathematics, are enrolled in courses designated General, Consumer, or Business Mathematics. These students did not take the equation test.
Davis, in 1925, describing the high drop-out rates for schools of his day, wrote:

It is clear, first, that enormous losses of pupils from the schools take place particularly in grades seven and eight; second, that of those who finish the eighth grade only about one-half to two-thirds enter high school. (3, 59)

Therefore, it is conceivable that some measure of attrition effectively removed a fraction of the low achievers in mathematics from the ninth grade school population in Reeve's day.

Assumption 2:

The subject matter of the test and the items themselves were in the syllabi of the courses for both student samples.

The textbooks used in the participating schools in 1966 all gave prominence to the topic of equation solving. Equations of the kind on Test 1 can be found in all seven of the texts in use by the 1966 group.

Reeve claimed that the equations of Test 1 were representative of current usage in his day (7, 32).

Assumption 3:

Time pressures for students taking both tests were not of a degree to affect the validity of the inferences drawn from a comparison of the results.

The students taking the equation test in 1966 were allotted thirty-five minutes for the test. The researcher, in a pilot project, observed that the mode time for completion of a test nearly identical with the final version was eighteen minutes. The test was designed
so that the last three items were not from the Reeve Test 1, and consequently these items were not involved in the comparison of achievement.

Reeve allotted forty-five minutes for his test; he did not report that students experienced time pressures. The researcher did not make use of the last three items on Test 1.

Assumption 4:

The students in both samples were adequately prepared for the subject matter of the test.

The 1966 group was tested toward the conclusion of the year's work, but before intensive review prior to final examinations. (The New York State Regents Examination in Ninth Year Mathematics, in 1966, took place on June 17th.) The teachers of these students had available, in the texts in use, cumulative review exercises containing equations of the type found on the test. Also, equations of this kind might well have been confronted by the student when working with other topics during the year.

Reeve's group was tested immediately after thirteen lessons in equation solving. Conceivably, with the subject fresh in their minds, the performance of students on the test might well be biased to some extent in favor of Reeve's group.

Significance of differences in achievement

Table 1 contains the results, expressed as per cents of correct responses, for items of Test 1 which were attempted by students in the 1966 sample.
<table>
<thead>
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<th>1966 Test</th>
<th>Reeve Test 1</th>
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<tbody>
<tr>
<td>1</td>
<td>(x + 5 = 9)</td>
<td>99.2</td>
<td>97.5</td>
<td>+ 1.7</td>
</tr>
<tr>
<td>2</td>
<td>(2z = 10)</td>
<td>98.8</td>
<td>97.3</td>
<td>+ 1.5</td>
</tr>
<tr>
<td>3</td>
<td>(4x + 5 = 17)</td>
<td>98.4</td>
<td>95.7</td>
<td>+ 2.7</td>
</tr>
<tr>
<td>4</td>
<td>(x - 3 = 4)</td>
<td>97.1</td>
<td>93.4</td>
<td>+ 3.7</td>
</tr>
<tr>
<td>5</td>
<td>(2y - 3 = 9)</td>
<td>95.9</td>
<td>91.1</td>
<td>+ 4.8</td>
</tr>
<tr>
<td>6</td>
<td>(8x = 5x + 12)</td>
<td>92.6</td>
<td>89.3</td>
<td>+ 3.3</td>
</tr>
<tr>
<td>8</td>
<td>(1/5 x = 6)</td>
<td>87.8</td>
<td>82.8</td>
<td>+ 5.0</td>
</tr>
<tr>
<td>9</td>
<td>(3.4y - 1.2y + 4.8y = 70)</td>
<td>70.5</td>
<td>80.0</td>
<td>- 9.5</td>
</tr>
<tr>
<td>10</td>
<td>(6x + 3 = 2x + 35)</td>
<td>88.2</td>
<td>80.0</td>
<td>+ 8.2</td>
</tr>
<tr>
<td>11</td>
<td>(3/2 x = 12/2)</td>
<td>88.3</td>
<td>80.0</td>
<td>+ 8.3</td>
</tr>
<tr>
<td>12</td>
<td>(3/4 y = 6)</td>
<td>86.8</td>
<td>76.5</td>
<td>+10.3</td>
</tr>
<tr>
<td>13</td>
<td>(13x + 12 - 3x - 2 = 20)</td>
<td>88.5</td>
<td>73.8</td>
<td>+14.7</td>
</tr>
<tr>
<td>14</td>
<td>(1/3x + 1/2 x = 30)</td>
<td>69.9</td>
<td>70.7</td>
<td>- 0.8</td>
</tr>
<tr>
<td>15</td>
<td>(2x + 6 1/2 = 12)</td>
<td>66.6</td>
<td>65.4</td>
<td>+ 1.2</td>
</tr>
<tr>
<td>16</td>
<td>(5x - 8 1/2 = 9.5)</td>
<td>73.6</td>
<td>66.1</td>
<td>+ 7.5</td>
</tr>
<tr>
<td>17</td>
<td>(8x + 30 - 4x = 60 + 3x)</td>
<td>85.2</td>
<td>67.4</td>
<td>+17.8</td>
</tr>
<tr>
<td>18</td>
<td>(6y + 4 = 52 - 2y)</td>
<td>81.2</td>
<td>63.9</td>
<td>+17.3</td>
</tr>
<tr>
<td>19</td>
<td>(1/4 x - 1/8 x = 2)</td>
<td>71.3</td>
<td>64.9</td>
<td>+ 6.4</td>
</tr>
<tr>
<td>20</td>
<td>(1/3 y + 2 = 5)</td>
<td>80.1</td>
<td>64.3</td>
<td>+15.8</td>
</tr>
<tr>
<td>21</td>
<td>(4.3x + 0.24 = 8.84)</td>
<td>66.5</td>
<td>64.0</td>
<td>+ 2.5</td>
</tr>
<tr>
<td>22</td>
<td>(2/9x + 1/6x = 1/18 x + 1/3)</td>
<td>65.0</td>
<td>55.2</td>
<td>+ 9.8</td>
</tr>
<tr>
<td>23</td>
<td>(1/3 z - 5 = 9)</td>
<td>70.8</td>
<td>53.7</td>
<td>+17.1</td>
</tr>
</tbody>
</table>
TABLE 1 (Contd.)

<table>
<thead>
<tr>
<th>Item No.</th>
<th>Equation</th>
<th>1966 Test</th>
<th>Reeve Test 1</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>24</td>
<td>12 - 8x = 3 - 2x</td>
<td>64.4</td>
<td>46.9</td>
<td>+ 17.5</td>
</tr>
<tr>
<td>25</td>
<td>0.4x - 5 = 3.8</td>
<td>56.2</td>
<td>45.3</td>
<td>+ 10.9</td>
</tr>
<tr>
<td>26</td>
<td>1/2 x = 7/4 - 1/3 x</td>
<td>55.1</td>
<td>44.7</td>
<td>+ 10.4</td>
</tr>
<tr>
<td>27</td>
<td>8y - 14 - 3y + 9 = 0</td>
<td>76.4</td>
<td>39.6</td>
<td>+ 36.8</td>
</tr>
<tr>
<td>Composite Means</td>
<td></td>
<td>79.8</td>
<td>71.1</td>
<td>+ 8.7</td>
</tr>
</tbody>
</table>

(Item numbers correspond to question numbers on 1966 test.)

Note: Items 7, 28, 29 and 30 were not used for comparison. Student performance on these items will be reported separately in Chapter VII.

To obtain a statistical measure of the significance of the differences in achievement observed for the twenty-six items taken as a whole, a t-test was employed. As a result of this test, a t value was obtained which would allow the researcher to reject the following hypothesis at the one per cent confidence level. (See Table 2.)

Hypothesis T:

There is no significant difference in achievement, by students in the two population samples, in attempting to solve the twenty-six equations from the Reeve Test 1.

In order to compare skills in performing operations necessary for correct solution of the equations, the researcher established eleven sub-sets of equations, described in Table 3.
### Table 2

**Results of Tests for Significance of Differences**

<table>
<thead>
<tr>
<th>Hyp.</th>
<th>No. of Items</th>
<th>d</th>
<th>(d-d_0)^2</th>
<th>d.f.</th>
<th>t</th>
<th>s/.05</th>
<th>s/.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>26</td>
<td>8.650</td>
<td>1935.44</td>
<td>25</td>
<td>5.103</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>A</td>
<td>8</td>
<td>14.450</td>
<td>793.56</td>
<td>7</td>
<td>3.839</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>B</td>
<td>12</td>
<td>9.375</td>
<td>525.46</td>
<td>11</td>
<td>4.699</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>C</td>
<td>9</td>
<td>9.144</td>
<td>235.24</td>
<td>8</td>
<td>5.057</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>D</td>
<td>21</td>
<td>8.648</td>
<td>1726.79</td>
<td>20</td>
<td>4.266</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>E</td>
<td>9</td>
<td>9.144</td>
<td>235.24</td>
<td>8</td>
<td>5.057</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>F</td>
<td>4</td>
<td>2.850</td>
<td>239.07</td>
<td>3</td>
<td>0.6387</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>G</td>
<td>2</td>
<td>4.350</td>
<td>19.85</td>
<td>1</td>
<td>1.379</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>H</td>
<td>7</td>
<td>12.043</td>
<td>189.50</td>
<td>6</td>
<td>5.6672</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>I</td>
<td>7</td>
<td>10.743</td>
<td>1307.20</td>
<td>6</td>
<td>1.877</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>J</td>
<td>4</td>
<td>9.150</td>
<td>137.21</td>
<td>3</td>
<td>2.7066</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>K</td>
<td>11</td>
<td>11.691</td>
<td>1392.55</td>
<td>10</td>
<td>3.286</td>
<td>yes</td>
<td>yes</td>
</tr>
</tbody>
</table>

Conclusions regarding significance of t-values obtained were made with the aid of a table in Snedecor's "Statistical Methods." (10, 65)
### TABLE 3

**SUBSETS OF EQUATIONS REQUIRING PARTICULAR SKILLS FOR SOLUTION**

<table>
<thead>
<tr>
<th>Sub-set</th>
<th>Types of Equations or Operational Skills</th>
<th>Item Numbers</th>
<th>No. of Items</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Adding a positive-signed number to both sides of the equation</td>
<td>4, 5, 16, 18, 23, 24, 25, and 27</td>
<td>8</td>
</tr>
<tr>
<td>B</td>
<td>Adding of a negative-signed number to both sides of an equation, or subtracting a positive signed number from both sides of an equation</td>
<td>1, 3, 6, 10, 13, 15, 17, 18, 20, 21, 22, and 24</td>
<td>12</td>
</tr>
<tr>
<td>C</td>
<td>Multiplying both sides of an equation by a number</td>
<td>8, 11, 12, 14, 19, 20, 22, 23, and 26</td>
<td>9</td>
</tr>
<tr>
<td>D</td>
<td>Dividing both sides of an equation by a number</td>
<td>All items except 1, 4, 8, 20 and 23</td>
<td>21</td>
</tr>
<tr>
<td>E</td>
<td>Working with fractions</td>
<td>8, 11, 12, 14, 19, 20, 22, 23 and 26</td>
<td>9</td>
</tr>
<tr>
<td>F</td>
<td>Working with decimals</td>
<td>9, 16, 21, 25</td>
<td>4</td>
</tr>
<tr>
<td>G</td>
<td>Working with mixed numbers</td>
<td>15, 16</td>
<td>2</td>
</tr>
<tr>
<td>H</td>
<td>Working with unknowns on both sides of the equation</td>
<td>6, 10, 17, 18, 22, 24 and 26</td>
<td>7</td>
</tr>
<tr>
<td>I</td>
<td>Combining signed terms on one side of an equation</td>
<td>9, 13, 14, 17, 19, 22 and 27</td>
<td>7</td>
</tr>
<tr>
<td>J</td>
<td>Equations which have non-integral solutions</td>
<td>15, 16, 24 and 26</td>
<td>4</td>
</tr>
<tr>
<td>K</td>
<td>Equations requiring more than two operations for correct solution</td>
<td>9, 10, 13, 14, 17, 18, 19, 22, 24, 26 and 27</td>
<td>11</td>
</tr>
</tbody>
</table>

*It should be noted that a student using a multiplicative inverse approach would consider that he was making use of the multiplicative property in cases such as 7x=70 and 1/5x=6. However, in this study, these cases are studied separately, in the traditional manner, so that comparisons of achievement might be made in a meaningful way for equations of this type.*
Differences in achievement were calculated and t-values obtained to test for statistical significance of the differences. The resulting t-values obtained may be found in Table 2.

As a result of applying the t-test to the differences obtained for each sub-set, the following null hypotheses were rejected at the one per cent confidence level:

There is no significant difference in student performance in solving equations in one unknown which involve--

A: adding a positive-signed number to both sides of an equation,
B: adding a negative-signed number to both sides of an equation, or subtracting a positive-signed number from both sides of an equation,
C: multiplying both sides of an equation by a number,
D: dividing both sides of an equation by a number,
E: working with fractions,
H: working with unknowns on both sides of an equation,
K: more than two operations for a correct solution.

As a result of applying the t-test, the following null hypotheses could not be rejected at the five per cent confidence level:

There is no significant difference in student performance in solving equations in one unknown which involve--

F: working with decimals,
G: working with mixed numbers,
I: combining signed terms on one side of an equation,
J: non-integral solutions.
Analysis of student performance

The researcher is able to assert that the 1966 population sample out-performed the earlier group on the twenty-six comparison items, by a decisive margin. This assertion supports the hypothesis that elementary algebra students today are more skilled at solving equations of the type found on Reeve's Test 1 than were their counterparts in the 1920's.

We may assert from our data that the students in 1966 did significantly better than the students in the 1920's working with what have traditionally been termed the four basic axioms of algebra (Hyp. A, B, C, and D).

Students in the 1966 sample demonstrated significantly better performance in working with the fractions appearing in the equations (Hyp. E).

A significant difference in performance of the two samples occurred in situations involving unknowns on both sides of the equation (Hyp. H). Comparing results of items in this category such as numbers 13, 17, 18, and 24, among others, would seem to indicate that this category represented a particularly weak area for the students of Reeve's day. For the seven examples constituting subset H, the 1966 group achieved a 76.0 per cent mean for correct responses, while the earlier group achieved a 63.9 per cent score, a difference of 12.1 per cent.

Another area of significant difference appeared to be in situations requiring more than two operations for a correct solution (Hyp. K). These items might well be classed as the more difficult
examples of the set being compared. The Reeve group exhibited particularly poor results with equations in this category. Note particularly examples 17, 18, 24, and 27. In example 27, we note that approximately four out of ten students were able to supply correct answers in the Reeve group, while in the 1966 sample more than seven out of ten correct responses were recorded. An analysis of difficulties peculiar to this type of equation will be discussed in greater detail in a subsequent chapter. However, it might be appropriate to note that the "0" on the right side of the equation might well cause problems for a student whose conception of the nature of an additive identity is weak.

In comparing student achievement in situations involving decimals, mixed numbers, combining signed terms on one side of an equation, and for equations having non-integral solutions, we were unable to reject hypotheses that no significant difference in performance occurred (Hyp. F, G, I, and J). However, since the mean differences, expressed as per cents of correct responses, were non-negative for all four cases, it cannot be asserted that the Reeve group out-performed the 1966 sample in any of the four categories. The relatively poor performance displayed by students in both population samples might indicate that these four categories represent areas of continuing difficulty.

For a final observation, the researcher calls attention to the fact that in no case did fewer students in the 1966 sample submit correct answers than incorrect answers to any of the items of the
set used for purposes of comparison. In the Reeve sample, there were more incorrect than correct responses to four items, numbers 24, 25, 26, and 27.

In this chapter, the researcher presented data which allowed a comparison of achievement of the two groups. He then provided an answer to Question 1 on the basis of an analysis of this data. In the following chapter, the researcher will present the results of his investigation of the changing mode of topical presentation as reflected in textbooks.
CHAPTER V

TEXTUAL PRESENTATION OF THE TOPIC

In this chapter, the researcher will describe the mode of presentation of the topic of solution of equations in one unknown as found in two algebra texts which were widely used in the early decades of the Twentieth Century. These books are: Elementary Algebra, by Hall and Knight (15), and Algebra, by Longley and Marsh (18). He will then compare the approach found in the above texts with that found in the seven books in use by students of the 1966 population sample. From a comparison of approaches, the writer hoped to secure information which would assist him to explore question 2.

Two traditional approaches

Elementary Algebra by Hall and Knight

Hall and Knight's Elementary Algebra had its first printing in 1885 (15). Up until 1908, it had appeared in eight editions and had been reprinted fifteen times. It is no exaggeration to call this text one of the standard algebra books of its day.

The approach to algebra in this book might be characterized as "how to do it." (One conjures up an image of a young man in a log cabin who uses this book to teach himself algebra to pass some qualifying examination.)
The presentation of topics followed a pattern of definition, rule, example, and drill. No attempt was made to present the material in an interesting manner; no effort was made to give the student an intuitive grasp prior to formal instruction in manipulative procedures.

Hall and Knight defined an equation as being a "statement that two algebraical expressions are equal" (15, 52). The parts separated by the "=" sign were called members. Conditional equations were defined as follows:

If two expressions are only equal for a particular value or values of the symbols, the equation is called an "equation of condition," or more usually an "equation" simply. (15, 52)

The "particular values" were called "roots," and they were said to "satisfy" an equation.

The student was informed that--

The process of solving a simple equation depends only on the following axioms:

1. If to equals we add equals the sums are equal.
2. If from equals we take equals the remainders are equal.
3. If equals are multiplied by equals the products are equal.
4. If equals are divided by equals the quotients are equal. (15, 52)

(n.b. There is no zero restriction mentioned.)

After a number of examples, followed by a number of drill exercises, the process of solving was summarized in a "general rule."
RULE:
First, if necessary, clear of fractions; then transpose all the terms containing the unknown quantity to one side of the equation, and the known quantities to the other. Collect the terms on each side; divide both sides by the coefficient of the unknown quantity and the value required is obtained. (15, 54)

"Beginners" were advised to "verify" their answers by substituting. A verification procedure was demonstrated, and in all subsequent drill exercises involving equations the students were instructed to "solve and verify."

Algebra by Longley and Marsh

The Hall and Knight text provides a fair indication of the mode of presentation of algebra in the years prior to the 1920's. During the third decade of the Twentieth Century, an algebra text appeared which, in significant ways, differed in spirit and approach from the earlier work.

Longley and Marsh's algebra textbook was first published in 1926. The degree of acceptance and success of this book may be gauged by the fact that the text underwent three revisions and six reprintings from 1926 to 1933 (18).

An important departure from the definition, rule, example, and drill procedure of previous algebra texts was the authors' use of the balance analogy in an attempt to give the student an intuitive grasp of the equation concept (18, 14). Here were authors who felt the need to motivate students; to make the study of algebra more interesting.
In another departure from tradition, Longley and Marsh informed students that some simple equations could, and should, be solved by inspection. A set of exercises were included in which the student was to attempt to "see" the solution before any rules for formal, manipulative solution were introduced.

So that teachers would not get the mistaken impression that increase in understanding through better motivation and an intuitive approach was gained at the expense of manipulative skills, the authors, in the preface affirm:

> The intention of modern instruction in algebra is that the pupil shall acquire absolute mastery of the technique that has been agreed upon as essential. (18, v-vi)

With the exception of the departures mentioned above, the Longley and Marsh book presents the topic of solving equations in a manner little different from Hall and Knight's text. The definitions of equation and root are the same; roots are again said to "satisfy" equations.

Solving an equation is said to be the process in which we "find the value (of the unknown) for which the equation will be true" (18, 14). This is accomplished upon correct application of the following principles:

The roots of an equation are not change if--

1. The same number is added to each side.
2. The same number is subtracted from each side.
3. Both sides are multiplied by the same number (provided that number is not 0 or an expression containing the unknown).

4. Both sides are divided by the same number. (provided that number is not 0 or an expression containing the unknown). (18, 121)

No explanation is given for the restrictions involving zero; the student is informed that "whenever both sides of an equation are multiplied by an expression containing the unknown quantity, new roots may be introduced." (18, 121)

The student is provided with a procedure for formal solution before examples are given of solution methods.

RULE: To solve a linear equation—

1. Remove all parentheses.

2. Transpose all terms containing the unknown letter to one side of the equation (generally the left), and all other terms to the other side.

3. Simplify both sides of the equation by combining similar terms.

4. Divide both sides of the last equation by the coefficient of the unknown letter. (18, 123)

The process of checking is demonstrated and, in drill exercises, pupils are instructed to check.

The following sequence might be a fair description of the presentation of the topic in this text:

1. An intuitive approach to the topic is attempted through the use of a "balance analogy" and by encouraging students to solve simple equations by inspection prior to the presentation of formal solving procedures.

2. Terms such as equation, solution, and root are defined.
3. A presentation of a set of principles governing operations with equations is given.

4. A procedure is suggested for solving linear equations. (Linear equations are defined in the text as equations which, when reduced to simplest form, contain only the first power of the unknown.) (18, 120)

5. Examples illustrating the use of the solution procedures are given.

6. A set of exercises is included for drill.

Modern approaches

In this section, the researcher will describe various approaches to the topic of solution of equations in one unknown. As resource material he will examine the seven textbooks used by the students who participated in the project in 1966. Assuming that these books are representative samples of textbooks in use today, the researcher will attempt to arrive at a synthesis which would be a fair characterization of a "modern" approach to the teaching of equation solving.

The texts examined will be referred to here by letter designation. The reader may refer to Appendix C for a listing of textbooks, authors, publishers, and dates of publication.

How are equations defined?

<table>
<thead>
<tr>
<th>Text</th>
<th>Definition</th>
<th>Found on Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>We generally use the term equations for sentences in which it is indicated that two expressions are equal.</td>
<td>32</td>
</tr>
<tr>
<td>B</td>
<td>An equation is a statement that two numbers or algebraic expressions are equal.</td>
<td>39</td>
</tr>
</tbody>
</table>
Any (algebraic) sentence using the symbol = is called an equation.

You are familiar with two types of open sentences. Sentences in the form $3x-5=7$ and $x=x+1$ are called equations. Such sentences are formed when two expressions are joined by an equal sign.

If a sentence states that two expressions represent the same number, it is an equation.

An equation is a mathematical statement that two quantities are equal.

Equations are mathematical sentences. They may be true sentences, false sentences, or sentences which are neither true nor false.

On the following page, is a checklist which may be used to relate certain mathematical language to the textbooks which make use of it.
Reference is made to right and left member of an equation.  

The term solution set is used.  

The term replacement set is used.  

The term root is used for the solution of an equation with one unknown.  

A root is said to satisfy an equation.  

An equation is referred to as an open sentence.  

Operational rules are called axioms.  

Operational rules are called principles.  

Operational rules are derived by theorem.  

The expression equivalent equations is used.  

Additive and multiplicative inverses are terms employed.  

Inverse operations are referred to.  

The word undo is used to express the result of use of an inverse operation.  

The word opposite is used to explain the nature of an additive inverse.  

Also of interest to the reader may be the checklist below which describes the method of topical introduction employed by the authors.  

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>Attempt is made at an intuitive approach by use of a balance analogy.</td>
<td>*</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>Attempt is made at an intuitive approach by having students test elements from a given replacement set.</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td></td>
</tr>
</tbody>
</table>
Attempt is made at an intuitive approach by having students guess solutions.

Attempt is made to have students discover operational principles before formal introduction.

Examples are given to illustrate the procedures used in applying axioms, principles, or theorems.

A specific list of procedures or rules for solving equations is NOT given.

Includes at least 100 exercises consisting of equations of the type found in the Reeve Test 1.

Includes at least 200 exercises consisting of equations of the type found in the Reeve Test 1.

Includes over 500 exercises consisting of equations of the type found in the Reeve Test 1.

Describes a procedure for checking solutions.

Instructs students to check in exercises involving solution of equations.

Textbook F

Of the seven textbooks examined, Textbook F comes closest to the traditional approach as characterized by the Longley and Marsh algebra text. The following parallels in presentation are noted:

1. Similar definitions of an equation.
2. The use of the "balance analogy."
3. The use of the expression "satisfies an equation" when referring to the term "root."
4. Students are instructed to check answers.
Significant differences:

1. The use of the concept of "inverse operations" in describing solution procedures (undoing principle).

2. No stated set of rules or procedures for solving equations either at the beginning or at the end of the topical development.

3. The relatively large number of exercises consisting of examples of the type found on the Reeve Test 1.

The Other Six Textbooks

The other six textbooks more nearly represent what might be termed a modern approach to the presentation of the topic. This approach might be synthesized from information gathered in examining the texts as follows:

1. An equation is characterized as an open sentence which contains a symbol or symbols representing one or more unknown numbers. Such a statement is neither true nor false.

2. Replacement of the symbol or symbols from a set of numbers, called a replacement set, allows for a determination of whether the resulting statement is true or false.

3. The set of all replacements which make an open sentence true is called the solution set.

4. A number, which is an element of the solution set is called a solution, truth element, or root.

5. An intuitive approach is used as students are:

   a. given a finite replacement set and asked to check for truth value for each of the resulting statements.

   b. asked to "guess" at solutions before formal solution procedures are introduced.
c. encouraged to discover principles which might be useful for solution by a formal process (Texts D and G).

6. Formal solution is presented, based on axioms, postulated principles, and, in Text E, principles logically deduced.

7. There is no listing of formal procedure or rules for solving equations either at the beginning or at the end of the development of the topic.

8. Examples are supplied to illustrate solution procedures.

9. The process of checking a solution is not generally introduced, nor are students generally instructed to check solutions to exercises (except for Texts B, G, and F).

It could be noted that approaches to the presentation of the topic such as those used in Texts B and G might be described as "transitional" in that some accommodation appears to have been made with tradition with respect to language (e.g., the use of the expression "satisfy an equation" when referring to roots) and the inclusion of checking procedure. These accommodations, however, appear to be minor, and examination of the above checklist would indicate that the approach to the topic in these two texts is much closer to that of the other four texts in this group than it is to that of Text F.

Comparison of approaches

Examining textbooks for the purpose of determining what is being taught assumes that

1. Textbooks provide good indications of the curriculum.

2. Teachers use textbooks in a manner intended by the authors (or the school supervisors, for that matter).
Are we assuming too much?

Shibli, in 1932, wrote:

In American schools, the textbook in geometry is of vital importance because the teacher and pupil depend entirely on the text they use. The curriculum is the textbook; and any developments in the syllabus of geometry would, therefore, be indicated by the geometry texts. (9, 120)

Assuming that Shibli's remarks were applicable to algebra texts as well, and also that his observation was correct for his day, how valid would the assertion be if made today? Though it cannot be said that Shibli made his observation at a time when the curriculum was static, we recognize that in recent years there has been an out-pouring of new texts and text series at a previously unheard-of rate. Pressures for curricular revision and the products of a number of nationally-known writing groups have given impetus to a wave of 'new' algebra texts and 'new' geometry books. (Publishers today would probably be reluctant to issue a mathematics text which did not, in its title, suggest that it was 'new' or 'modern.')

A major effort has been made in the past decade to re-educate mathematics teachers so that they might adjust their teaching methods to newer ideas.

While some teachers enthusiastically follow the spirit of a new approach, and others remain inflexible, it seems reasonable to expect that most teachers, after a period of adjustment, will teach in a manner which approximates the presentation in the textbook. Consequently, we should not be surprised to find that, at a time of much curricular experimentation, the approach to the teaching of algebra
used in the classroom is somewhat behind the development of the subject matter one sees in some of the newer texts. However, as experience in teaching the new programs increases, the gap should be narrowed.

Let us assume for the purposes of this study that the texts examined approximate the way algebra is taught today, and in the case of the old books, the way algebra was taught during the first quarter of the Twentieth Century. What significant changes in emphasis can we detect, as we examine these books, which might have been a factor in the better performance of the 1966 group in this study.

First, let us observe that both the Longley and Marsh book and the texts used by the 1966 group represent different stages in an evolutionary trend away from a purely manipulative approach in the teaching of algebra such as the one employed in the Hall and Knight book. Newer texts place much emphasis on intuitive processes and an understanding of basic concepts while, at the same time, the authors profess not to neglect manipulative skills. The words "interesting" and "meaningful" are often used to describe the mode of presentation. Benefits to students studying algebra with a "modern" approach are described by the authors of Text D as follows:

By stressing individual inquiry and participation, and by leading the student through a carefully planned series of discoveries, teacher and text can provide important benefits for the student. These benefits include an interest in mathematics; an understanding of the foundations of mathematics; and manipulative skills based on understanding. (21, vii)
Emphasis on guiding students to discovery of concepts as an important goal of its text program, is affirmed by the authors of Text G.

... this book is intended to help the teacher to present algebra to the student as an ever unfolding structure; to guide the student to the discovery of mathematical principles. (19, v)

If we accept the premise that the authors of these quotes are reflecting the views of progressive mathematics educators and, further, if the present state of the curriculum is the result of an evolutionary process as described above, then even a fair measure of success in implementing a modern mathematics program should result in better performance of students today as compared with their counterparts of a generation ago. This view appears to be confirmed by the test results in this study. However, an assertion that the changing mode of topical presentation was to some degree responsible for the resulting differences in student performance would have to be based on the establishment of a cause-effect relationship which, the researcher realizes, is beyond his power to achieve. Consequently, he cannot make a categorical response to Question 2.
CHAPTER VI

DIAGNOSIS OF STUDENT DIFFICULTIES

The two previous chapters were devoted to comparisons of pupil achievement and textbook presentation of the topic of solution of equations in one unknown. We now turn our attention to the diagnostic phase of the study as we investigate questions 3 and 4.

In attempting to diagnose student difficulties, the researcher used the procedure of examining test papers and tabulating the type and frequency of errors students made. A similar procedure was recommended by Everett (4).

The analysis will be presented in this chapter as follows: first, errors in the twenty-six comparison items made by a significant number of students will be described; second, a report on student performance in the four additional items of the 1966 test will be given; finally, a summary of student difficulties will conclude this chapter.

Student errors in the comparison items

Failure to answer

The greatest indication of student difficulty with the test was the high incidence of failure to arrive at a solution for certain items.
Though Reeve allowed for a separate notation of "no answer" in his tallying procedures, his thesis did not contain data to show the number of students failing to answer, or attempting to answer each item. There is some indication, however, that Reeve did keep such a record, since he reported that, in the case of Item 7 (our Item 13) "the third largest source of error was failure to attempt the exercise" (7, 35). Reeve used the symbol '0' on his record sheet to indicate 'no answer.' He tallied answers given under "attempts." There is no indication in the thesis that Reeve differentiated between 'no answer' and 'no attempt' in his recording procedure.

Concerning his equation 19 (our Item 20), Reeve wrote:

The third cause of failure was that the pupils simply abandoned the work at some point in the solution, evidently because they were not sure what they should do next. (7, 36)

This observation seems to indicate that Reeve had available test papers on which students were required to show their work.

In the 1966 test, the students were required to show all work; they were not permitted to use scrap paper. Two 'no answer' categories were established:

'T,' no evidence of an attempt to solve,

'N,' evidence present of an attempt to solve, but no answer offered.

Table 4 shows the incidence of 'no answer' for the comparison items on the test when five per cent or more of the students are involved.
<table>
<thead>
<tr>
<th>Item No.</th>
<th>Per Cent Correct</th>
<th>Equation</th>
<th>T+N Per Cent</th>
<th>T Per Cent</th>
<th>N Per Cent</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>70.5</td>
<td>3 \cdot 4y - 1.2y + 4.8y = 70</td>
<td>86 7.0</td>
<td>43 3.5</td>
<td>43 3.5</td>
</tr>
<tr>
<td>14</td>
<td>69.9</td>
<td>1/3 x + 1/2 x = 30</td>
<td>89 7.3</td>
<td>45 3.7</td>
<td>44 3.6</td>
</tr>
<tr>
<td>15</td>
<td>66.6</td>
<td>2x + 6 \frac{1}{2} = 12</td>
<td>61 5.0</td>
<td>27 2.2</td>
<td>32 2.6</td>
</tr>
<tr>
<td>16</td>
<td>73.6</td>
<td>5x - 8 \frac{1}{2} = 9.5</td>
<td>80 6.5</td>
<td>59 4.8</td>
<td>21 1.7</td>
</tr>
<tr>
<td>17</td>
<td>85.2</td>
<td>8x + 30 - 4x = 60 + 3x</td>
<td>66 5.4</td>
<td>37 3.0</td>
<td>29 2.4</td>
</tr>
<tr>
<td>19</td>
<td>71.3</td>
<td>1/4 x - 1/8 x = 2</td>
<td>104 8.5</td>
<td>71 5.8</td>
<td>33 2.7</td>
</tr>
<tr>
<td>21</td>
<td>66.5</td>
<td>4.3x + 0.24 = 8.84</td>
<td>99 8.1</td>
<td>64 5.2</td>
<td>35 2.9</td>
</tr>
<tr>
<td>22</td>
<td>65.0</td>
<td>2/9 x + 1/6 x = 1/18 x + 1/3</td>
<td>193 15.7</td>
<td>132 10.8</td>
<td>61 5.0</td>
</tr>
<tr>
<td>23</td>
<td>70.8</td>
<td>1/3 z - 5 = 9</td>
<td>89 7.3</td>
<td>72 5.9</td>
<td>17 1.4</td>
</tr>
<tr>
<td>24</td>
<td>64.4</td>
<td>12 - 8x = 3 - 2x</td>
<td>115 9.4</td>
<td>65 5.3</td>
<td>50 4.1</td>
</tr>
<tr>
<td>25</td>
<td>56.2</td>
<td>0.4x - 5 = 3.8</td>
<td>159 13.0</td>
<td>131 10.7</td>
<td>28 2.3</td>
</tr>
<tr>
<td>26</td>
<td>55.1</td>
<td>1/2 x = 7/4 - 1/3 x</td>
<td>243 19.8</td>
<td>172 14.0</td>
<td>71 5.8</td>
</tr>
<tr>
<td>27</td>
<td>76.4</td>
<td>8y - 14 - 3y + 9 = 0</td>
<td>150 12.2</td>
<td>124 10.1</td>
<td>26 2.1</td>
</tr>
</tbody>
</table>
As one examines Table 4, certain questions arise:

1. Could the relatively large incidence of 'T' in Items 21 and 25 have been caused by unfamiliarity with the symbols 0.24 and 0.4x? No comparable difficulty exists in Item 9, which also contains decimals. Less than five per cent of the students failed to attempt Item 9 (actually 3.5 per cent).

2. Could the large incidence of 'T' in Items 22 and 26 have been caused by unfamiliarity with equations of this type, containing fractional coefficients? Was the relatively large incidence of 'N' for these items due to lack of skill in working with fractions?

3. Item 23 differs from Item 5 principally in that the coefficient of the symbol representing the unknown number is a fraction. No students failed to attempt Item 5; why then, did 72 students fail to attempt equation 23?

4. Though, by comparison with the Reeve population sample, almost twice as many students were able to solve Item 27 correctly, why did one student out of ten in the 1966 group fail to attempt to solve the equation? Could the '0' on the right side of the equation be a source of difficulty?

5. Was unfamiliarity with the presence of mixed numbers in an equation the reason why so many students did not supply answers to Items 15 and 16? The items themselves appear to be relatively uncomplicated, given the necessary skills with mixed numbers. Note the resemblance of Item 15 to Item 3 (98.4 per cent correct), and Item 16 to Item 5 (95.9 per cent correct).
6. Despite the greater number of correct answers in Item 19 over Item 14 (the reverse of what one might expect and what the Reeve group achieved), why did more students fail to supply an answer to Item 19 than Item 14?

Work with decimals

A second significant area of student difficulty appears to be in work with decimals. Four test items involved the use of decimals: Items 9, 16, 21, and 25. In Item 16, a mixed number also appears. In comparing achievement of the two groups, we were unable to assert significantly different performance at the 5 per cent confidence level.

Table 5 shows student achievement for these four examples. The column headed E contains the number of non-equivalent incorrect answers, exclusive of "T" or "N," submitted.

The large variety of erroneous answers is startling. Reeve noted that when the equations began to involve more terms and operations, or to become more complex in any other way, a large number of incorrect solutions appeared. It is remarkable how pupils can find so many incorrect answers to an equation. (7, 35)

Item 9: 3.4y - 1.2y + 4.8y = 70

The manipulative skills needed to solve the equation in Item 9 are as follows:

1. Combine terms on the left side.
2. Divide by 7, or multiply by 1/7.
### TABLE 5
STUDENT ACHIEVEMENT ON THE ITEMS INVOLVING DECIMALS

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>70.5</td>
<td>80.0</td>
<td>$3.4y - 1.2y + 4.8y = 70$</td>
<td>43</td>
<td>43</td>
<td>86</td>
<td>125</td>
</tr>
<tr>
<td>16</td>
<td>73.6</td>
<td>66.1</td>
<td>$5x - 8 1/2 = 9.5$</td>
<td>59</td>
<td>21</td>
<td>80</td>
<td>81</td>
</tr>
<tr>
<td>21</td>
<td>66.5</td>
<td>64.0</td>
<td>$4.3x + 0.24 = 8.84$</td>
<td>64</td>
<td>35</td>
<td>99</td>
<td>117</td>
</tr>
<tr>
<td>25</td>
<td>56.2</td>
<td>45.3</td>
<td>$0.4x - 5 = 3.8$</td>
<td>132</td>
<td>28</td>
<td>160</td>
<td>121</td>
</tr>
</tbody>
</table>
The following insights are needed:

1. Realization that the terms on the left are combinable and should be combined.

2. Realization that the division axiom, or principle, should be used, or that a multiplication by a multiplicative inverse is in order.

The equivalent equations which should have been obtained were \(7y = 70\), and \(y = 10\).

Almost three out of ten students answered incorrectly. Eighty-six did not supply an answer; thirty-five students wrote the following: \(7.0y = 70; \ y = 1,\) indicating some decimal confusion.

The difficulties rising from this item were compounded by the necessity to perform two operations. This appeared to be a problem for a large number of students. The difficulty seemed to be in faulty execution, that is, in the exercise of manipulative skills.

Item 21: \(4.3x + 0.24 = 8.84\)

Examination of the student work done in answering Item 21 gives further evidence of lack of skill in working with decimals. In this item, the correct answer is 2. Seventy-one students (5.8 per cent) wrote: \(4.3x = 8.60; \ x = 2.2,\) while 39 students wrote: \(4.3x = 8.60; \ x = 20.\) Answers of .02, .2, and even 200 appeared.

Item 25: \(0.4x - 5 = 3.8\)

In Item 25, one hundred twelve students (9.1 per cent) offered \(0.4x = 8.8; \ x = 2.2.\) Twenty-six students decided to multiply both sides of the equation as follows: \(0.4x - 5 = 3.8\)

\[
\begin{align*}
4x - 5 & = 38 \\
4x & = 43 \\
x & = 10 \frac{3}{4}
\end{align*}
\]
Item 16: $5x - 8\frac{1}{2} = 9.5$

The decimal in Item 16 did not appear to cause difficulties. The students appeared to recognize the equivalence of .5 and $\frac{1}{2}$.

Examination of the work done in Items 9, 21 and 25 would seem to indicate that a significant fraction of the students have difficulty with decimals, particularly with division of decimals.

Judgment in choosing a procedure for solution

The difficulties encountered by the students in working with decimals appeared to involve manipulative skills. A significant area of weakness involving poor insight, or perhaps poor judgment, appeared to be causative in the use of incorrect axioms or principles. This was noticeable in cases where a "-" sign appeared in the equation. Consider Items 4, 5, 16, and 23.

Item 4: $x - 3 = 4$

In Item 4, of thirty-five incorrect solutions, twenty-four involved subtraction: $x - 3 = 4; x = 1$.

Item 5: $2y - 3 = 9$

In Item 5, twenty-seven students wrote:

$2y - 3 = 9$
$2y = 6$
$y = 3$
Item 16:  \(5x - 8\frac{1}{2} = 9.5\)

In Item 16, although the decimal did not cause difficulties, seventy-six students (6.2 per cent) obtained the answer 1/5, in this manner:

\[
\begin{align*}
5x - 8\frac{1}{2} &= 9.5 \\
5x &= 1 \\
x &= 1/5
\end{align*}
\]

Item 23:  \(\frac{1}{3}z - 5 = 9\)

In Item 23, seventy-seven students wrote:

\[
\begin{align*}
\frac{1}{3}z - 5 &= 9 \\
\frac{1}{3}z &= 4 \\
z &= 12
\end{align*}
\]

Item 25:  \(0.4x - 5 = 3.8\)

Item 25 is interesting in that the above difficulty did not seem to exist. Were students inhibited from subtraction by the signed number problem they would face?

No comparable problem in judgment was found in cases where a subtraction was required, as in Items 1, 3, 15, and 20.

Item 8:  \(\frac{1}{5}x = 6\)

Another case of poor judgment occurred when a division operation was chosen in place of a multiplication. In Item 8, forty-seven students wrote: \(\frac{1}{5}x = 6; x = 1 \frac{1}{5}\).

Item 12:  \(\frac{3}{4}y = 6\)

In Item 12, thirty-four made the same kind of error, answering \(4\frac{1}{2}\).
Item 14: \( \frac{1}{3}x + \frac{3}{5}x = 30 \)

In Item 14, after improperly combining \( \frac{1}{3}x \) and \( \frac{3}{5}x \) to obtain \( \frac{1}{5}x \), forty-six students wrote: \( \frac{1}{5}x = 30; x = 6 \).

Item 19: \( \frac{1}{4}x - \frac{1}{8}x = 2 \)

In Item 19, after a great variety of correct and incorrect combinations for the two terms on the left side of the equation, the same kind of error was observed.

Item 23: \( \frac{1}{3}z - 5 = 9 \)

Again in Item 23, for which we have already shown a tendency to use subtraction in place of addition, we note that seventy-seven students wrote: \( \frac{1}{3}z - 5 = 9 \)
\[ \frac{1}{3}z = 14 \]
\[ z = 4 \frac{2}{3} \]

Examination of the papers did not disclose a comparable confusion leading to a multiplication instead of a division. We ask:

To what extent is the subtraction-addition confusion due to poor skills with signed numbers? To what extent is the division-multiplication confusion due to poor skills with fractions?

(It is the opinion of the researcher that the Reeve Test I does not provide adequate material to test skills in working with signed numbers. In a subsequent section we will have an opportunity to observe the degree of success the 1966 group revealed in examples which tested their skill with signed numbers.)

Fractions

We have already made mention of some of the difficulties experienced by the students in dealing with fractions. (Note the
comments concerning Items 14 and 19 above.) In an earlier chapter, we asserted with confidence that the 1966 group significantly outperformed the Reeve group in solving equations which involve fractions.

Item 11: \( \frac{3}{2}x = \frac{12}{2} \)

We have already noted the division-multiplication confusion in equations such as in Item 8. Item 11 was solved correctly by 88.3 per cent of the students, but thirty-four students committed the following atrocity:

\[
\begin{align*}
\frac{3}{2}x &= \frac{12}{2} \\
x &= \frac{36}{4} \\
x &= 9
\end{align*}
\]

Item 12: \( \frac{3}{4}y = 6 \)

Again in Item 12 we note thirty-four cases of

\[
\begin{align*}
\frac{3}{4}y &= 6 \\
y &= \frac{18}{4} \\
y &= 4\frac{1}{2}
\end{align*}
\]

Mixed numbers

Item 15: \( 2x + 6\frac{1}{2} = 12 \)

Another area of student difficulty was exposed by Item 15. Approximately two out of three students were able to solve this equation. There were ninety different incorrect answers given; there was no significant accumulation of any particular incorrect response. The most common error was: (34 students)

\[
\begin{align*}
2x + 6\frac{1}{2} &= 12 \\
2x &= 6\frac{1}{2} \\
x &= 3\frac{1}{4}
\end{align*}
\]
Quite a large number of students appeared to have difficulty proceeding correctly after writing $2x = 5\frac{1}{2}$. Answers of $5\frac{1}{2}$, $\frac{5.5}{2}$, or $\frac{11}{2}$ were not accepted. (For all items on the test, equivalent forms of the correct answer were acceptable only if the answers were expressed as integers, decimals, mixed numbers, or as fractions with integral numerators and denominators. If the eighteen students who offered the three non-acceptable forms above were to have their answers marked correct, then the 1966 result for this item would have been raised from 66.6 per cent to 68.0 per cent.)

It appears clear that the difficulties experienced by students with Item 15 were primarily caused by the presence of the mixed number.

In this section, the researcher described pupil performance on the twenty-six comparison items of the 1966 test. In the following section, performance on the four additional items of that test will be examined.

**Student errors in the four non-comparison items**

Examination of student performance in attempting to solve the four non-comparison equations on the 1966 test gave the researcher an opportunity to test some algebraic skills which he felt were not adequately tested in Reeve's Test 1. The non-comparison items were used to test the following skills:

1. Addition of signed numbers with a negative result.
2. Division of signed numbers with a negative result.
3. Use of parentheses.

4. Solution of an equation which is in the form of a proportion.

Table 6 shows pupil achievement in the four non-comparison items of the 1966 test.

**TABLE 6**

1966 STUDENT ACHIEVEMENT IN THE NON-COMPARISON ITEMS

<table>
<thead>
<tr>
<th>Item No.</th>
<th>Equation</th>
<th>Per Cent Corr.</th>
<th>E*</th>
<th>T</th>
<th>N</th>
<th>T+N</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>18x + 3x - 16x + 48 = 18</td>
<td>68.9</td>
<td>60</td>
<td>27</td>
<td>28</td>
<td>55</td>
</tr>
<tr>
<td>28</td>
<td>1/3 (x + 4) = 6</td>
<td>53.2</td>
<td>105</td>
<td>136</td>
<td>58</td>
<td>195</td>
</tr>
<tr>
<td>29</td>
<td>x - 3 (2 - 5x) = 32</td>
<td>31.0</td>
<td>160</td>
<td>210</td>
<td>120</td>
<td>330</td>
</tr>
<tr>
<td>30</td>
<td>( \frac{5}{2-x} = \frac{3}{4+x} )</td>
<td>23.4</td>
<td>105</td>
<td>344</td>
<td>156</td>
<td>500</td>
</tr>
</tbody>
</table>

*The column headed E contains the number of non-equivalent incorrect responses, exclusive of T and N.*

Item 7: 18x + 3x - 16x + 48 = 18

A correct sequence of equivalent equations might be:

18x + 3x - 16x + 48 = 18  
5x + 48 = 18  
5x = -30  
x = -6

Examination of student papers indicated that nearly all students who attempted this item sought to combine terms on the left side of the equation and followed this with a division or a multiplication by an inverse. (Only one student offered 18/53 as an answer, the result of a careless combination of unlike terms.)
Two hundred fourteen students, representing 17.5 per cent of the 1966 sample, responded 6. Two types of erroneous procedure were observed:

(a) \[18x + 3x - 16x + 48 = 18\]
\[5x + 48 = 18\]
\[5x = 30\]
\[x = 6\]

(b) \[18x + 3x - 16x + 48 = 18\]
\[5x + 48 = 18\]
\[5x = -30\]
\[x = 6\]

In School 1, where students were taught to use an inverse approach (Text A), of twenty-six students who answered 6, nineteen had sequence "a" and seven had sequence "b." In School 7, students using Text F employed a division approach. Of thirty-three students who answered 6 in this group, twenty-nine had sequence "a" while only four had "b."

In Table 7, the students who answered 6 are categorized according to text used. The sub-script \(i\) indicates that a multiplicative inverse approach is employed in the text, while the sub-script \(d\) indicates that a division approach is used in the text.

**Table 7**

**Distribution of Students Answering 6 for Item 7**

<table>
<thead>
<tr>
<th>Text</th>
<th>(A_i)</th>
<th>(B_d)</th>
<th>(C_d)</th>
<th>(D_i)</th>
<th>(E_i)</th>
<th>(F_d)</th>
<th>(G_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of Students answering 6</td>
<td>51</td>
<td>21</td>
<td>31</td>
<td>41</td>
<td>11</td>
<td>33</td>
<td>26</td>
</tr>
<tr>
<td>Per cent of students using text</td>
<td>18</td>
<td>17</td>
<td>15</td>
<td>12</td>
<td>13</td>
<td>42</td>
<td>25</td>
</tr>
</tbody>
</table>
The data does not indicate bias in the direction of either approach. (It should be noted that the poor showing of students using Text F is consistent with the poor showing of that group on the whole test. This should not be interpreted as indicating, necessarily, that either the text or its mode of presentation is inferior. Quite possibly, the group using Text F was a homogeneous group of low achievers. It has become a policy of a number of mathematics departments in the area of the study to assign newer texts and texts with newer approaches to the better students, and to assign texts with a more traditional approach to poorer students, in the belief that these books would be more suitable for them.)

The results of test Item 7 would seem to indicate that student difficulty with signed number work is common for all groups making up the 1966 population sample and, possibly, difficulty of this kind is common among students in algebra classes of today.

Item 30: \[
\frac{5}{2-x} = \frac{3}{4+x}
\]

This item served two purposes. It provided a check on the processes students use to solve equations which are in the form of proportions. Also, it allowed the researcher to check, once more, the student's ability to deal with a situation where a negative answer results.

A surprising number of students, 500 (over two out of five), did not arrive at an answer; 344 of them (over 28 per cent) did not even attempt to solve. However, an unknown factor here might well be time pressures, this being the last item on the test. Unfamiliarity
with this type of equation might also have contributed significantly
to the rather large group who failed to attempt it.

Table 8 shows the number of students who did not attempt Item
30, categorized according to textbook in use.

**Table 8**

"NO ATTEMPTS" EXPRESSED AS
A PER CENT OF TEXT GROUP

<table>
<thead>
<tr>
<th>Text</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>T as per cent of text group</td>
<td>31</td>
<td>28</td>
<td>32</td>
<td>21</td>
<td>50</td>
<td>50</td>
<td>11</td>
</tr>
</tbody>
</table>

Examination of the texts shows that the two texts with the
largest number of exercises of this type are F and G, in that order.
Text D contains relatively few equations of this kind, but it does
contain one on page 181 which is very much like the test item.
Text E contains only one exercise of the type under consideration,
and that one near the close of the book.

The skills required to solve correctly include the intuitive
knowledge which would permit the student to apply manipulative
skills in order to arrive at the equivalent equation: \(20 + 5x = 6 - 3x\).
At this point, the student is faced with an equation similar to that
of Item 18, which 81.2 per cent of the students solved correctly. A
correct solution might be obtained by the following sequence of equiva-

te equations:

\[
20 + 5x = 6 - 3x \\
8x = -14 \\
x = -\frac{7}{4}
\]
An examination of the results indicates that 120 students, almost one out of ten, answered \( \frac{7}{4} \); that is, they were able to arrive at the first equivalent equation, but were denied a correct result by sign difficulties of the type already described in the case of Item 7.

In obtaining the first equation equivalent to the given one, the students appeared to be using three processes:

(a) "cross-multiplication" (the proportion property),

(b) use of a common multiplier such as \((2-x)(4+x)\),

and (c) a common denominator approach in the following manner:

\[
\frac{4+x}{4+x} \cdot \frac{5}{2-x} = \frac{3}{4+x} \cdot \frac{2-x}{2-x}
\]

\[(4+x)5 = 3(2-x)\]

Usually, all three approaches were to be found in all textbook groups.

Examination of the results for Item 30 would indicate that students might profit from greater experience with examples of this type since such proportions are not uncommon in mathematics.

Items 28 and 29

These items were included in the test to offer the researcher an opportunity to gauge the ability of today's students to solve equations involving parentheses. The poor results obtained indicate that the researcher's expectations exceeded the performance on the test. Yet, the record of students' efforts to solve this type of
problem provides a mine of information which can be valuable in an effort to diagnose difficulties.

Let us first consider Item 29.

Item 29: \( x - 3(2 - 5x) = 32 \)

A list of manipulative skills might be as follows:

1. "Clear" of parentheses.
2. Combine like terms.
3. Add six.
4. Divide by 16, or multiply by \( \frac{1}{16} \).

A correct sequence of equivalent equations would be:

\[
\begin{align*}
x - 3(2-5x) &= 32 \\
x - 6 + 15x &= 32 \\
16x - 6 &= 32 \\
16x &= 38 \\
x &= 2 \frac{3}{8}
\end{align*}
\]

Once more the large number of students not attempting this item is baffling. Again, time pressure may have contributed significantly to this result.

Sixty-two students, slightly over 5% per cent, contributed the following sequence of equations:

\[
\begin{align*}
x - 3(2-5x) &= 32 \\
x - 6 - 15x &= 32 \\
14x - 6 &= 32 \\
14x &= 38 \\
x &= 2 \frac{5}{7}
\end{align*}
\]

Yet, interestingly, only thirteen students obtained \(-2 \frac{5}{7}\).

Calculation difficulties were also noted here. Fifty-eight students, almost one out of twenty, concluded: \( 16x = 38; x = 2; \) and twenty-five students, that \( x = 3 \).
Thirty-eight answered $2\frac{1}{2}$ after being faced with the 'necessity' to reduce the fraction $\frac{38}{16}$.

Thirty students wrote:

\[ 16x - 6 = 32 \]
\[ 16x = 26 \]
\[ x = 1\ \frac{5}{8} \]

With so many opportunities to 'go wrong,' the fund of incorrect answers contained 160 non-equivalent varieties, an astounding figure considering the fact that slightly fewer than three out of four students even supplied an answer! (One wonders what Reeve's group of 1,204 students would have done with this equation, after 'suitable preparation.' More to the point, one wonders what a comparable sample of algebra students would do with this equation forty years hence.)

Item 28: \( \frac{1}{3}(x + 4) = 6 \)

This equation was a very useful diagnostic tool for identifying student difficulties. Somewhat over half the students in the group were able to answer correctly.

Two correct approaches employed were these:

(a) Multiplication by 3 to obtain

\[ \frac{1}{3}(x + 4) = 6 \]
\[ x + 4 = 18 \]
\[ x = 14 \]

(b) Distribution of the multiplication by \( \frac{1}{3} \) to obtain

\[ \frac{x}{3} + \frac{4}{3} = \frac{18}{3} \]
\[ \frac{x}{3} = \frac{14}{3} \]
\[ x = 14 \]
A number of students arrived at \( x + 4 = 2 \) by an incorrect choice of procedures, and then only eleven "correctly" obtained a (-2); forty-six, however, obtained (2).

Thirty-seven students ignored the parentheses, writing:

\[
\frac{1}{3} (x + 4) = 6 \\
\frac{1}{3}x = 2 \\
x = 6
\]

(with five answering 2/3)

Another kind of error noted might have been due to an incorrect understanding of the factor concept. A total of sixty-five students (over 5 per cent) multiplied both factors on the left side of the equation, obtaining one of the following sequences:

(a) \[
\frac{1}{3} (x+4) = 6 \\
3x+12 = 18 \\
3x = 6 \\
x = 2
\]

(b) \[
\frac{1}{3} (x+4) = 6 \\
3x+12 = 18 \\
3x = 30 \\
x = 10
\]

(c) \[
\frac{1}{3} (x+4) = 6 \\
3x+12 = 18 \\
3x = 14 \\
x = 4 \quad \frac{2}{3}
\]

(d) \[
\frac{1}{3} (x+4) = 6 \\
3x+12 = 18 \\
3x = 22 \\
x = 7 \quad \frac{1}{3}
\]

Again, there were a startling number of types of incorrect response.

An interesting sidelight to this item arose in the opportunity given to students to solve by an intuitive approach. Such an approach might involve the student asking himself: "One-third of what number equals six?" He might conclude that \( x + 4 \) must represent eighteen, and consequently, that \( x \) must represent fourteen.

On a number of papers, the given equation was immediately followed below by the equation \( x + 4 = 18 \), and then by \( x = 14 \). (A
few papers had \( \frac{1}{3} (18) = 6; x = 14 \). There was no indication on these papers that any intermediate manipulative processes had taken place. Might these students have obtained correct answers by the intuitive process described above? It is conceivable that a student of today might employ such reasoning. Let us ask: is it conceivable that a "Hall and Knight student" would attempt to solve this way? Would a "Longley and Marsh student" employ such a method? Or, to put the question another way: would a student who has been brought to understand that the symbol \((x + 4)\) represents a number, as a result of an approach which stresses the meaning and understanding of algebraic symbols, be more likely to solve an equation in the intuitive manner described than one who is taught algebra in a fashion which stresses "absolute mastery of the technique" of manipulating symbols and applying rules?

In this chapter, the researcher has identified some areas of student difficulties, as observed in the 1966 test papers:

1. Failure to attempt an item.
2. Abandoning an equation after making an attempt to solve.
3. Poor judgment in choice of axiom or principle.
4. Difficulties in work with parentheses.
5. Difficulties in work with signed numbers.
6. Difficulties in work with decimals.
7. Difficulties in work with fractions.
8. Difficulties in work with mixed numbers.
CHAPTER VII

FUNDAMENTAL SKILLS FOR EQUATION SOLVING

In the previous chapter, the researcher identified certain areas of weakness as observed on the 1966 test papers. In this chapter, he will comment on the skills displayed by the students as they attempted to solve the equations.

Everett, in his thesis "The Fundamental Skills of Algebra" (4), placed student errors into two general categories: associative and manipulative. He defined the associative skills needed for solving equations to be those abilities involved which are "outside, and distinct from, the ability to add, subtract, multiply, or divide" (4, 17). The four operational or manipulative skills are applied in situations where integers, fractions, decimals, and mixed numbers are involved. They are required for successful operation with arithmetic number symbols, and with algebraic number symbols involving letters which represent numbers.

In this study, the researcher identifies two basic categories of non-manipulative skills. They are: Skills of Interpretation and Skills in Exercising Judgment. These categories are by no means clearly separable. For example, it is conceivable that poor exercise of judgment in a situation might well be caused by a misinterpretation of the symbols.
Skills of interpretation

What does a student understand the statement "4x + 5 = 17" to mean? Conceivably, a student could solve this equation by purely manipulative means, following a set of instructions well learned and skillfully executed. But, does he know that an equation is a statement about numbers? Does he understand that in the equation above, x represents a number? Is he aware that the equation informs him that seventeen is five more than the number represented by 4x?

Our student might well have been introduced to the equation concept in a manner which would allow for an affirmative answer to each question. Nevertheless, when asked to solve the equation, does he bring these understandings to bear?

Correct interpretation can be a key which springs the lock of understanding, allowing for a fruitful use of intuitive powers. A student solving the above equation in a manner which makes use of interpretive skills would ask himself what number 4x must represent if 4x + 5 = 17. He should conclude that 4x must represent twelve since 12 + 5 = 17. In the same manner, he should "see" that x must represent three.

Compare this approach with the one used by a student who "recognizes" the form of the equation and proceeds to "get rid" of the 5 in order to have only terms with the unknown "letter" on the left side of the equation. Certainly, an approach which instructs the student to operate on "letters" will not serve to enlighten him as to the meaning of the process of solving equations; one which
instructs him to "transpose" will probably vitiate any effort on the part of the teacher to develop an understanding of the importance of necessary inference in mathematics.

Observation of the pupil's work will not always provide the researcher with insights into the mental activity which resulted in the written record. Consider a sequence of equivalent equations such as: $4x + 5 = 17; 4x = 12; x = 3$, where no indications of manipulative procedure are visible on the paper. One cannot assume that an intuitive approach was used since many bright students are able to "manipulate" mentally in simple examples. (The reader may remember that in the previous chapter, it was noted that many students wrote the sequence: $\frac{1}{3} (x + 4) = 6; x + 4 = 18$ without indicating a multiplication by three in the work shown on the paper.)

What might the researcher conclude if he were to observe the following in Item 8: $8x = 5x + 12; 12 = 3x; x = 4$, again without written evidence of manipulation? (Such a sequence was not observed on any paper.) Might he make some assertion as to the manner of solution if he were to see $8x - 4 = 6x; 4 = 2x; x = 2$?

Examination of the test papers did not provide evidence that a significant number of students were making use of interpretive skills to solve equations. They may possess these skills, but their approach to solution appeared to be generally manipulative.

**Skills in exercising judgment**

The exercise of good judgment requires a base of experience and an ability to discern those elements in a situation which call to
mind previously learned notions. The latter may well be related to intelligence. These skills are used in solving equations, first, to recognize the configuration, then to recall a suitable axiom or principle, and finally, to choose a correct procedure for solution. Failure to identify the configuration might cause a student to give up entirely, leaving no sign of an attempt to solve.

A more pragmatic approach, employing interpretive skills and a spirit of experimentation might overcome, to some extent, non-recognition of a configuration. Item 30, for example, might have appeared strange to a student who had little opportunity to deal with an equation of this kind. However, it is a number statement which informs us that the symbols $\frac{5}{2-x}$ and $\frac{3}{4+x}$ represent the same rational number.

The large number of "no attempts" on the test might indicate that, in general, the students were unprepared (or unwilling) to attack unfamiliar equation configurations.

In the twenty-six items of the comparison set of equations, and in Item 7, the students generally appeared to exercise good judgment in their choice of procedures. Only in Items 22 and 16 of the comparison set did a significant number of students make poor choices of procedures to the extent that they were unable to submit an answer (N). It should be noted that both of these items involved relatively difficult equations with three or more situations in each where procedural choices were called for.
For three non-comparison items, however (items 28, 29, 30), there appeared to be considerable difficulty in exercise of judgment. Though large fractions of the sample failed to attempt these items, a large fraction of those who did indicated by their ineptness an unfamiliarity with equations of this type.

The researcher appreciates that poor judgment is often the result of a lack of understanding of fundamental concepts. This could be the result of poor teaching, or a consequence of a presentation of the subject matter which does not allow for sufficient practice in applications of the concepts taught. (Actually, it could be a combination of both.) If the latter more nearly describes the true situation, then unfamiliarity with the equations might well explain the poor exercise of judgment observed in items 28, 29, and 30.

Manipulative skills

Everett described the manipulative skills required to solve simple equations as those involving the four basic operations. (Operations with parentheses might be classed as multiplicative; in this way the skills involved would fit into the above category.) For the purposes of this study, it will be convenient to consider the existence of two interdependent categories of manipulative skills: Arithmetic Computation Skills, and Algebraic Manipulative Skills.

Arithmetic Computation Skills

The results of the 1966 test showed considerable student difficulty with a number of arithmetic processes, particularly in those involving decimals, fractions, and mixed numbers.
The algebra student who brings to the subject a poor set of computational skills labors under a serious handicap in his efforts to study algebra. The teacher of algebra all too often deplores the level of arithmetic competence displayed by a large fraction of his charges, but he does not feel called upon to do much about it. Yet, here might be an area where pupil performance in algebra can be upgraded substantially by added attention to maintaining previously learned arithmetic skills, and, if necessary, reteaching some basic arithmetic concepts. Diagnostic tests should be helpful in determining pupil difficulties in arithmetic; remedial work might be done on an individual, small group, or class basis. Since algebra texts do not as a rule provide sufficient materials for arithmetic review, these might have to be introduced by the teacher.

Algebraic Manipulative Skills

For the task of solving equations in one unknown of the type appearing on the 1966 test, the student was expected to be able to apply the four basic operations in situations involving algebraic expressions.

The student of algebra today usually is introduced to concepts underlying the development of a number of these skills in previous grades. He may even bring to the algebra class considerable skill in solving simple equations. However, previous experiences with algebraic concepts and skills are usually considered introductory in nature, in that an elementary algebra teacher presents a course which, if textbooks are reliable indicators, takes only minor cognizance (if
any at all) of the student's previous experiences. Thus, it is conceivable that a student could have an approach to algebra in an earlier grade which is radically different from the one he experiences in the ninth grade. This experience might not be uncommon in the light of the numerous recent adoptions of new mathematics programs. (One wonders to what extent such a situation would contribute to confusion leading to poor performance on standardized tests.)

The performance of the 1966 sample in situations requiring the exercise of algebraic skills indicated that either some concepts underlying the effective use of these skills are inadequately developed, or that insufficient opportunity is given for the acquisition of operational proficiency. This was particularly noticeable in situations involving signed numbers (results for Item 7), parentheses (results for Items 28 and 29), and algebraic fractions.

In this chapter, the researcher identified various associative and manipulative skills required for successful solution of equations in one unknown of the type appearing on the 1966 test. His observation of student performance leads him to conclude that the results of that test were affected to a considerable extent by student difficulty in situations requiring the use of associative and manipulative skills.
Summary, conclusions, and recommendations

Where is science taking us? In the direction of our dreams. Mathematics, the language of science, is the language of dreamers who plan to achieve their dream.

Man has reached a point in his evolutionary development at which he no longer feels bound by the shackles of "the possible." It is an age when the sky no longer limits; when dreams are molded into achievement as the dreamer marshalls the tremendous storehouse of knowledge accumulated by those with vision in every age, and applies it to the problems of the day, confident that solution is sure; today's dream will be tomorrow's reality.

In such an age, education becomes more than just a way of preparing the young to function in society. Education, if it is to preserve the direction and increase the rate of progress, must also build understanding and appreciation, understanding of the complex world in which we live, and appreciation of the great forces which are the instruments of progress. Therefore, it follows that every individual who claims to be educated must possess some ability, commensurate with his role in a progressive society, to speak and understand the language of these forces.

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If we accept that mathematics is the language of science, then we are bound to shape a curriculum which strives to teach this language; to provide the background of information which will enable the student to appreciate the place of mathematics in his society; to build an appreciation of the beauty of mathematics and of its power to inspire men to dream and create; and to encourage the able among them to make active use of the language in making valuable contributions to their fellow men.

At every stage in the evolutionary development of the curriculum, the words of educational leaders attest to the view that the teaching of mathematics should be done in a manner reflecting the contemporary understanding of the power of mathematics to shape their society. The apparent burst of concern for revising the curriculum and improving the teaching of mathematics in recent years testifies to the degree of current recognition of that power, reinforced, almost daily, by reports of technological achievements which were obviously brought about through the application of mathematical knowledge.

It is understandable that there might be concern that in the process of shaping the curriculum to reflect the "second face" of mathematics, with great emphasis on its sociological and conceptual aspects, the task of developing and maintaining computational and manipulative skills would, in some degree, be assigned a lower priority and lead to a deterioration of the quality of student performance in tasks requiring these basic skills. In an attempt to test the validity of the concern over the ability of today's student to employ manipulative skills, the researcher decided to compare the
performance of a group of 1,226 elementary algebra students, in 1966, with that of 1,204 ninth grade algebra students of approximately forty years ago, on a test which was designed to assess ability to solve algebraic equations in one unknown.

The question was posed:

**Question 1**

If a test in solving equations in one unknown, attempted by a number of ninth grade algebra students approximately forty years ago, were administered to a comparable number of elementary algebra students in 1966, would there be any significant difference in student performance?

The results showed that the 1966 group out-performed the earlier group in solving the twenty-six equations attempted by the two samples. Significantly better performance was recorded by the 1966 group in situations involving use of the four fundamental operations with signed numbers, in work with fractions, in solving more difficult equations involving work with unknowns on both sides of the equation, and in working with equations where more than two operations were required for a correct solution.

The results are particularly significant in that the comparison group of the 1920's studied algebra at a time when emphasis was on developing computational skills in a manner that stressed the application of rules.

Granted that the scope of the subject area being tested was rather narrow, the skills involved are basic for subsequent work in mathematics. It appears reasonable to assume that the superior level of skill demonstrated by the 1966 group would give them an advantage.
over the older group in situations involving the application of these skills which might arise at a later point in the development of an elementary algebra course.

The notion that an approach to the teaching of algebra which merely emphasized the manipulative aspects would lead to unsatisfactory results is certainly not a new one. Writing at about the time of Reeve's study, E. L. Thorndike wrote:

It may be possible for a pupil to learn to manipulate coefficients, exponents, radicals, signs, parentheses, numerators, and denominators, by habituation to a fixed rule . . . with very little genuine understanding of what he is doing or why he does it. It is, however, extremely hard for a pupil to learn to operate algebraically in this way. (12,447)

The changing emphasis in the teaching of algebra over the past half-century attests to the fact that mathematics educators came to realize the reasonableness of Thorndike's observation. Evidence of a change in emphasis may be found in the textbooks used by students of both eras.

We asked:

**Question 2**

To what extent might differences in student performance be attributable to changing emphases in presentation as reflected in the textbooks generally in use by students of the 1920's and today?

Examination of the textbooks used by students over the past fifty years indicates a progressively increasing concern for: development of a topic in a manner which is calculated to interest the
student and motivate him to learn, encouraging the student to use his intuitive faculties, building an appreciation of the significance of mathematics, and encouraging the student to discover fundamental mathematical principles. The authors assert that the importance of drill for the purpose of maintaining learned skills has not been overlooked. Consequently, the researcher, aware that a firm response to Question 2 based on the identification of a causal relationship cannot be made, nevertheless asks the reader to consider the reasonableness of his suspicion that the changing emphases in the teaching of algebra as reflected in the textbooks examined in this study might have been a factor in the improved performance of the 1966 sample.

It should be noted, however, that current instructional practices in algebra are probably behind the approaches found in some of the newer texts, the amount of lag depending, to a great extent, on the degree of personal commitment on the part of the classroom teacher to act as an agent of the text authors in a fair attempt to implement their program.

Question 3

What are some of the significant areas of student difficulty in attempting to solve equations of the type appearing on the 1966 equation test?

1. Examination of the test results and the students' papers indicates that the largest area of student weakness is in failure to arrive at a solution. At a time when stress in algebra instruction is purported to be on giving the student a better understanding of the equation concept and, consequently, a stronger base for experimentation
and use of intuitive powers, it seems incongruous that the degree of
timidity could be so great. The researcher suspects that unfamiliarity
with certain equation forms caused the failure of previously learned
concepts to transfer.

Recommendation #1

The students should be given more opportunity
to apply their knowledge of basic principles
in a larger variety of situations. This might
be accomplished by increasing the number of
drill exercises and introducing more problem
situations where they would have an opportunity
to solve equations in a meaningful setting.

Recommendation #2

An attitude of experimentation should be
fostered which should result in student
acceptance of an unusual or unfamiliar form
of an equation as a challenge to his ingenuity
and ability to apply his acquired skills.

2. The nature and extent of student errors in work with
decimals and mixed numbers indicates that previous work in these
areas has not led to a satisfactory degree of retention of the con­
cepts and accompanying skills.

Recommendation #3

A greater number of situations involving
decimals and mixed numbers should be employed
in an algebra course in order to reinforce
concepts and processes taught in previous
grades. This might be accomplished through
specific drill and by the inclusion of more
problem situations in which decimals and mixed
numbers arise.

3. Operation with signed numbers appears to be difficult for
an appreciable fraction of the students. This was particularly
noticeable in addition when the result of adding two signed numbers with opposite sign results in a negative number, and also when signed numbers are divided.

Recommendation #4

More opportunity for work in fundamental operations with signed numbers is needed, since subsequent work in mathematics is seriously impeded by a lack of proficiency in working with signed numbers.

4. Examination of student papers indicates a considerable amount of difficulty in working with fractions. In the use of all four fundamental operations, too many students demonstrated a poor degree of retention of concepts and skills developed in lower grades. Conceivably, insufficient effort might have been given in the seventh and eighth grades to maintain these skills.

Recommendation #5

Specific provision should be made to reinforce manipulative skills with fractions through the use of drills and in problem situations in which fractions arise.

5. The poor results observed for Items 28 and 29 were to a large extent due to the presence of the parentheses. Examination of a number of texts used by the students indicates that, though the concepts of the distributive property and the inclusion function of parentheses appear to be adequately presented, insufficient opportunities exist for transfer in equation solving situations. This might explain why so many students did not attempt to solve these items.
Recommendation #6

Since the presence of parentheses in algebraic equations is a common phenomenon, more opportunity should be given students to develop their skills in solving such equations.

Question 4

To what extent are the difficulties experienced by students indications of inadequate manipulative and associative skills?

Two kinds of associative, or non-manipulative categories of skills were discussed: Interpretive Skills, and Skills in Exercising Judgment.

There was no evidence on the test that students were employing interpretive skills to any significant extent. Their approach to solving equations appeared to be primarily manipulative, judging from the processes of solution observed on the test papers.

For the most part, student difficulty in exercising judgment did not appear to be extensive in the comparison items. On the whole, their difficulties in choosing correct procedures were considerably less of a problem than correctly executing the attempted procedures chosen.

The above statement, however, does not take into account the relatively large number of students who did not attempt to solve. The difficulty might have been caused by recall problems due to unfamiliarity with the form of the equation.

Examination of the test papers exposed significant weakness in the exercise of arithmetic skills.
Recommendation #7

The maintenance and reinforcement of basic arithmetic skills should have a high priority in an elementary algebra program. Supplementary materials should be introduced if, in the opinion of the teacher, the text in use does not provide for adequate arithmetic review.

The relatively large number of "no-attempts" on the comparison items and the poor results on Items 28, 29, and 30, the researcher believes, were caused mainly by unfamiliarity with some types of equations, particularly those involving parentheses and fractions.

Recommendation #8

More time should be devoted to teaching students to solve equations, particularly equations involving fractions, decimals, mixed numbers, and parentheses.

Four out of seven texts in use by schools in the 1966 sample did not require students to check their equation solutions. In the opinion of the researcher, encouraging the checking habit could lead to improved performance. For a number of items on the test, a check of an incorrect answer could have led a student to reconsider his process of solution.

Recommendation #9

Teachers of algebra should require their students to check their equation solutions even if the text in use does not require this.

The researcher concludes:

There has been a change in emphasis in the teaching of algebra, over the past fifty years, from a primarily manipulative approach to one which seeks to motivate the learner and give him some basis for
understanding the beauty and power of mathematics as it affects the world about him.

As a consequence of this change in emphasis, less attention is given in the instructional program to the development and maintenance of manipulative skills. Despite this de-emphasis, a group of 1,226 students of elementary algebra in 1966 performed significantly better than did a comparable group of 1,204 students of forty years ago in a test of their skills in equation solving.

Examination of the test papers of the 1966 group, and analysis of student difficulties as reflected in the errors recorded, indicate that there are certain areas of weakness which suggest that more attention to the development and maintenance of basic skills, both algebraic and arithmetic, might be needed in order to meet the high expectations of a modern mathematics program.

**Significance of the study**

The contrast between the expectations of the various "new mathematics" programs, recently introduced into a number of North Shore Long Island schools, and the poor results obtained on the New York State Ninth Year Mathematics Regents Examination in 1965 by students in these schools, has caused much concern among parents, teachers and administrators, who ask whether, in adopting a modern mathematics program, they might have attempted to do too much, too soon.

Sponsors of newer approaches to the teaching of mathematics assert that an important goal of their programs is the teaching of
manipulative skills through understanding. However, there are some who are critical of the approach, and are pointing to the results of standardized tests as indicating a deterioration of mathematical skills. Quite vocal are some who assert that children do not appear to have the degree of computational skill possessed by the students of the previous generation, a generation in which algebra was studied through rule, example, and drill.

Educators generally appreciate that these new programs are somewhat experimental in nature, and that the recently introduced programs are in a "shakedown" stage; teachers and pupils need time to adjust to the new language, new emphases and new approaches. Unfortunately, the State Examination does not measure the increased student interest in studying mathematics brought about by these new programs; nor does it allow for the occasional clumsy efforts of teachers who, with interest and sincerity, are attempting to change the teaching habits of many years.

If their original decision to adopt a new program in mathematics was correct, then educators should allow for time and experience to do their work in narrowing the gap between results and expectations.

The results of this study do not lend support to the position that the "basic" approach of a generation ago was superior in imparting algebraic skills; our students today, students who, so to speak, are in the midst of the controversy, out-performed the comparison group in the use of skills which are fundamental to successful algebraic performance.
However, some added attention to the preservation and maintenance of skills previously learned, and some provision made for increased drill and review of fundamental algebraic operations to supplement the conceptual approach of a modern mathematics program, might go a long way toward the attainment of respectable results on standardized tests and, in effect, ease the civic pressures from quarters which are clamoring for a return to the "good old days" with their "good old ways" of teaching algebra.

The diagnostic procedures employed by the researcher in this study followed, to some degree, those suggested by John Phelps Everett in 1927. The technique of examining and categorizing student errors can be a valuable tool for the researcher and the classroom teacher as an instrument for diagnosing pupil difficulties with mathematics at all levels of instruction. The data obtained from the project and the test papers themselves represent a potentially rich source of information for a researcher who might wish to investigate any phase of the study which is of interest to him, or to explore any avenue of inquiry emerging from the study.

Suggestions for further research

The researcher suggests investigation of the following questions:

1. Would a similar comparative study of pupil use of algebraic skills required for other topical areas such as factoring, or problem solving, reveal significant differences in performance which would tend to corroborate the findings of this study?
2. Would the results of a test, consisting of the twenty-six comparison items of Test 1, when administered to a present-day group of students in a different geographical area, reveal similar differences in student performance?

3. To what extent does a textbook presentation of an elementary algebra course reflect the instructional methods in a class using that text?

4. To what extent is diagnostic testing used today by teachers in their instructional practices?

5. How effective are current practices of developing skill in the use of signed numbers?

6. What effect does emphasis on teaching algebraic concepts in the lower grades have on pupil performance on standardized tests given in the ninth grade?
March 10, 1966

Dear Colleague,

The purpose of this letter is to solicit your help in a study which might provide answers to some of the vexing problems we face in the teaching of algebra. Student performance on standardized tests and on uniform examinations such as the Ninth Year Mathematics Regents Examination, you will surely agree, leaves much to be desired.

In my study, I plan to observe the procedures used by students as they attempt to solve a selected set of equations in one unknown. By analyzing the results and cataloging student errors, I expect to be able to diagnose the students' difficulties. The test they will take was prepared and administered to elementary algebra students approximately forty years ago. At that time, an analysis of student difficulties was made which led to the conclusion that the students appeared to possess sufficient manipulative skill in their algebraic operations; the kind of errors that they made seemed to indicate a lack of understanding of some of the basic concepts of algebra.

We expect that the evolutionary trend towards a down-grading of emphasis on manipulative processes and an increase in emphasis on teaching basic concepts which has characterized the newer approaches to the teaching of mathematics today would lead to improved student performance. By means of this test, I will be able to compare the performance of a comparable number of ninth grade students in seven schools with that of the group which attempted the test in the mid-1920's.

I am at this time submitting a copy of the proposal for this study for your examination. If you agree to participate, I would welcome your assistance in planning for the administration of the test. I plan to arrange for a meeting of department chairmen of participating schools in the near future at which time we could discuss the project and set up procedures which would ensure that valid inferences might be drawn and significant results derived from the study. I expect that the results will be made available to participating schools.

Should you require additional information, please feel free to contact me. Thank you for your consideration.

Cordially,

Harold Leonard

"Education is the difference between civilization and chaos"
APPENDIX B
<table>
<thead>
<tr>
<th>School Designation</th>
<th>Name of School</th>
<th>Address</th>
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Total: 1226
## TEXTBOOKS USED BY STUDENTS IN THE 1966 GROUP

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<tr>
<th>Letter Designation</th>
<th>Authors, Text, and Publisher</th>
<th>Schools Using Texts</th>
<th>Number of Students</th>
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</table>
The following are a set of equations which you are to attempt to solve. In each case, solve for x, y or z. **ALL WORK MUST BE SHOWN**, and no scrap paper will be allowed. Erasing is permitted. Enter your answer to each question in the space provided at the lower right hand corner of each question box. Checking is not required.

1. \( x + 5 = 9 \)
2. \( 2z = 10 \)
3. \( 4x + 5 = 17 \)
4. \( x - 3 = 4 \)
5. \( 2y - 3 = 9 \)
6. \( 8x = 5x + 12 \)
7. \( 18x + 3x - 16x + 48 = 18 \)
8. \( \frac{1}{2}x = 6 \)
9. \( 3.4y - 1.2y + 4.8y = 70 \)
18x + 3x - 16x + 48 = 18

\( \frac{1}{2}x = 6 \)

3.4y - 1.2y + 4.8y = 70

6x + 3 = 2x + 35

\( \frac{3}{2}x = \frac{12}{2} \)

\( \frac{3}{4}y = 6 \)

13x + 12 - 3x - 2 = 20

\( \frac{1}{3}x + \frac{1}{2}x = 30 \)

2x + 6\frac{1}{2} = 12
<p>| | | |</p>
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<tr>
<td>(16)</td>
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<tr>
<td>$5x - 8 \frac{1}{2} = 9.5$</td>
<td>$8x + 30 - 4x = 60 + 3x$</td>
<td>$6y + 4 = 52 - 2y$</td>
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<td>(19)</td>
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<tr>
<td>$\frac{1}{4}x - \frac{1}{8}x = 2$</td>
<td>$\frac{1}{3}y + 2 = 5$</td>
<td>$4.3x + 0.24 = 8.84$</td>
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<tr>
<td>(22)</td>
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<tr>
<td>$\frac{2}{3}x + \frac{1}{6}x = \frac{1}{18}x + \frac{1}{3}$</td>
<td>$\frac{1}{3}x - 5 = 9$</td>
<td>$12 - 8x = 3 - 2x$</td>
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</table>
(22) \[ \frac{2}{3}x + \frac{1}{6}x = \frac{1}{18}x + \frac{1}{3} \]

(23) \[ \frac{1}{3}x - 5 = 9 \]

(24) \[ 12 - 8x = 3 - 2x \]

(25) \[ 0.4x - 5 = 3.8 \]

(26) \[ \frac{1}{2}x = \frac{7}{4} - \frac{1}{3}x \]

(27) \[ 8y - 14 - 3y + 9 = 0 \]

(28) \[ \frac{1}{2}(x + 4) = 6 \]

(29) \[ x - 3(2 - 5x) = 32 \]

(30) \[ \frac{5}{2 - x} = \frac{3}{4 + x} \]
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Example 5
Answer 6

\[2y - 3 = 9\]

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BIBLIOGRAPHY

Books


97
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