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SYNTHESIS OF A CLASS OF RADAR AMBIGUITY FUNCTIONS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

Morgan Stuart Waugh, B.E.E.

* * * * * * * *

The Ohio State University
1966

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ACKNOWLEDGMENTS

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VITA

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INTRODUCTION

Radar is perhaps the most widely used and often the only available technique for measuring a limited number of physical characteristics of inaccessible objects. Of particular interest are target distance, velocity, shape, aspect, and rotation. Radar allows the measurement of these properties within limitations imposed by design, operation, and environment.

Regardless of the form of implementation of the radar system, when white Gaussian noise corrupts the received signal and maximum likelihood detection is assumed, the probability of target detection can be shown to be dependent on the ratio of returned signal energy to ambient noise power \((E/N_0)\) [1]. The greater the returned signal energy and the smaller the ambient noise level the more probable will be the detection of those objects present in the radar beam.

Resolution, ambiguity, and accuracy have been suggested as criteria of a radar's ability to determine particular properties of targets after they have been detected [2]. These properties of a radar depend on the probing signal transmitted as well as on the \((E/N_0)\) ratio available at the receiver terminals [2-4].

As nearly as possible, the following definitions will be used.

Resolution. the ability of a radar to determine correctly:
(1) the number of targets present within a given volume of space, or
(2) the slight differences in observable parameters that may be used to distinguish between several targets.
Ambiguity. the uncertainty involved in correctly separating true target returns from incidental signals arising from the choice of transmitted signals and from the effects of environmental noise. As an example, range measured by the transmission of a series of pulses may be ambiguous.

Accuracy. the ability of a given radar to measure correctly absolute values of specific properties of one or several targets.

This paper is concerned with the choice of transmitted waveforms which enhance the measurement capabilities of a radar. It will generally be assumed that the target detection problem has been solved even if by application of the "brute force" method of increasing peak and average signalling energies. In the case where more than a single target is present, the assumption is that one or more but not necessarily all targets have been detected.

Two analytic techniques have been used extensively in studying the problem of probing signal selection. Slepian [5] has extended the mathematical theory of estimation to the determination of amplitude, range (signal propagation delay time), and velocity (doppler frequency shifts) of a received radar target echo embedded in white Gaussian noise. This approach offers important quantitative information about a particular signal's usefulness for one type of measurement or another. The complexity of the mathematical results, however, generally make it extremely difficult to visualize what modifications of the probing signals would produce desirable changes in system performance.

On the other hand, the radar signal time-frequency (t-f) correlation or ambiguity function introduced by Ville [6] and later by Woodward [2]
allows an interesting interpretation of the relationship between signal waveform, target range and velocity resolution, and measurement accuracy and ambiguity. This correlation function may be thought of as a surface over the time-frequency plane with the shape of the surface being indicative of the associated waveform's usefulness as a radar signal. Unfortunately the synthesis problem of determining an exact radar signal from a desirably shaped ambiguity surface has not been solved.

The central results of this paper are concerned with obtaining descriptions of a class of ambiguity surfaces optimal in the sense that, for a given target range, the radar receiver will have a minimized response to those target echoes possessing a doppler frequency shift (velocity) lying within a selected band of frequencies. The transmitted radar pulse required for this condition has an amplitude modulated envelope whose shape is related to that of a truncated zero order prolate spheroidal wave function. Likewise, one may specify an ambiguity surface optimal in the sense that, for a given target velocity, the receiver will have a minimized response to those targets lying outside a selected range region. One realization of this surface requires a rectangular envelope radar pulse with a nonlinear frequency modulation characteristic related to the prolate spheroidal wave functions. This result is closely related to the linearly frequency modulated signals of pulse compression radars [4, sec.10.9].

A second realization of the ambiguity surface with optimum range resolution characteristics involves simultaneous pulse amplitude and frequency modulation. A simple linear frequency modulation is required while the amplitude modulation is related to the prolate spheroidal wave
functions. If one assumes that the matched filter receiver for the nonlinearly frequency modulated radar signal can be built, use of this signal is shown to have a distinct advantage over use of the simultaneously amplitude and frequency modulated pulse. The advantage is simply that the rectangular envelope nonlinearly frequency modulated pulse allows more energy to be transmitted per pulse. Target detection probability is best with non-amplitude modulated radar signals.

Urkowitz et al. [7], have extended the concept of the t-f correlation or ambiguity function to the measurement of other target parameters. These extensions are not considered here since target range and velocity determination as well as the number of targets present are generally of first interest.

In Chapters I and II we elaborate on the mathematical properties of ambiguity functions. Next we take up brief explanations of ambiguity surface synthesis techniques recently developed and then proceed to the specification of the optimal surfaces and signals mentioned above.
A. Signal representation

The mathematical analysis of radar systems is considerably simplified if the analytic signal representation of modulated waveforms is used. [2,8] Important features of the analytic signal are that its Fourier transform contains only positive frequency components and that the instantaneous magnitude and phase of an analytic signal uniquely defines the envelope and phase of its associated real (physical) signal.

As an example, consider the real signal $u(t)$ possessing the Fourier transform $U(\omega)$ having negligible components outside the band $(-\omega_m, \omega_m)$. The amplitude modulated signal $s(t)$,

$$s(t) = u(t) \cos(\omega_0 t + \phi),$$

possesses both positive and negative frequency Fourier transform components centered at $+\omega_0$ and $-\omega_0$, respectively, as in Fig. 1. The Fourier spectrum of the analytic signal representation of the modulated carrier $s(t)$ contains only positive frequency components.

If $f_a(t)$ is the analytic signal and $u(t)$ is as in (1.1),

$$f_a(t) = u(t) e^{j(\omega_0 t + \phi)}$$

(1.1)
In order for \( f_a(t) \) to remain strictly analytic one must require \( U(\omega) = 0, |\omega| > \omega_m, \) and \( \omega_0 > \omega_m. \) When these conditions are met, every linear operation on \( f_a(t) \) will result in an analytic signal whose real part corresponds to the physical signal obtained when the same linear operator is applied to the real signal \( s(t). \)

The complex representation of the double sideband phase and amplitude modulated signal

\[
s(t) = u(t) \cos (\omega_0 t + \phi(t))
\]

may be written from

\[
s(t) = \Re\{u(t) e^{j(\omega_0 t + \phi(t))}\}
\]

as

\[
f(t) = u(t) e^{j(\omega_0 t + \phi(t))}.
\]

It is well known \([9,10]\) that \( f(t) \) may be thought of as an analytic signal only in a practical sense, for the Fourier spectrum of the exponential \( e^{j\phi(t)} \) contains spectral components over the whole region \( (-\infty < \omega < \infty) \) even when \( \phi(t) \) is band limited. In order to preserve the simplifying notions of envelope and phase representation by the analytic signal it is necessary to assume (and require) that the significant spectral components of \( e^{j\phi(t)} \) are limited to a bandwidth \( 2\omega_m << \omega_0. \) The Fourier transform of \( f(t) \) \( \mathcal{F} \) \( F(\omega) \) then effectively contains only positive frequency components and linear operations on \( f(t) \) also yield analytic signals. A signal which is analytic may be envelope detected, since

\[
|f(t)| = u(t).
\]
In the following, the energy of a signal is often referred to and unless otherwise specified it will be implied that the energy of the analytic signal is meant. It is easily shown [1, chpt. 6] that

\[ E_a = 2E_f \]

\[ E_f = \int |f(t)|^2 = \int |f(t)|^2 dt \]

where

- \( E_f \) is the energy of the physical signal,
- \( E_a \) is the energy of the analytic representation of \( f(t) \).
\( u(t) \overset{F}{\rightarrow} U(\omega) \)

Fourier spectra of the real low-pass modulation.

\( S(\omega) \overset{F}{\leftrightarrow} s(t) = u(t)\cos(\omega_0 t + \phi) \)

Fourier spectra of real amplitude modulated carrier \( \omega_0 \).

\[ F_a(\omega) = 2S(\omega) \quad \omega > 0 \]

Fourier spectra of analytic signal representation of \( s(t) \).

Figure 1. The relationship of the spectra of a real and its associated analytic signal.
B. The ambiguity function as a matched filter response

We introduce the following definitions:

- $t_1$, $r_1$: expected delay and doppler frequency of a target return.
- $t_2$, $r_2$: actual delay and doppler frequency of a target return.
- $t$: real time, measured with respect to the transmitted pulse leading edge.
- $t_0$: sample time at which a filter matched to the expected signal produces a maximum instantaneous $(E/N_0)$ ratio.

$$S(x) = f(x, t_2)e^{j[(\omega_0 + r_2)(t_0 - t - r_1)x] + n(x)}$$

$S(x)$ = received signal.

$n(x)$ = noise component of the received signal.

A linear filter response $g(t)$ to an arbitrary input $S(t)$ may be obtained by the general convolution integral. If the filter is of the matched variety [11] associated with the deterministic portion of (1.8), its impulse response for parameters $t_1$, $r_1$ may be written as in (1.9).

$$h(x) = f^*(t_0 - t_1 - x)e^{-j[(\omega_0 + r_1)(t_0 - t_1 - x) + \psi]}$$

So

$$g(t) = \int_{-\infty}^{\infty} f(x, t_2)e^{j[(\omega_0 + r_2)(x-t_2) + \psi]} h(t-x) \, dx$$

$$g(t) = e^{-j[(\omega_0 + r_2)t_2 + (\omega_0 + r_1)(t_0 - t_1)]} \cdot \int_{-\infty}^{\infty} f(x-t_2) f^*(x+t_0-t_1)e^{j(r_2-r_1)x} \, dx$$

(1.10)
A linear change of variables is helpful. Let \( y = x - t_2 \),

\[
x - t_1 + t_2 - t_2 = y + (t_2 - t_1) = y + \tau
\]

and let \( r_1 - r_2 = 2\pi v \).

Thus

\[
g(t) = e^{-j[(\omega_0 + r_1)(t_0 - t + \tau) - 2\pi vt_2]} \int_{-\infty}^{\infty} f(y)f^*(y + \tau + t_0 - t)e^{-j2\pi vy} dy.
\]

In the derivation of (1.11) it is assumed that the target radial velocity component \( v_r \) is small in comparison to the propagation velocity of electromagnetic waves and that the significant spectral components of the response \( f(t) \) are contained in a bandwidth of \( 2\pi W \) radians, small in comparison to the carrier frequency \( \omega_0 \). Given these conditions and the following inequality \([12]\),

\[
\frac{2v_r}{c} (WT) \ll 1
\]

with \( T = \) duration of a probing pulse, \( c = \) velocity of electromagnetic energy propagation, one may assume that the effect of target motion on the radar echo signal frequency is a linear, doppler-frequency shift. Several authors have considered the effects of dispersive doppler shifts on the design of matched filters and on the accuracy of radar measurements of high velocity and highly accelerating targets \([13,14]\).

When \( 2\pi W \) is small in comparison to \( \omega_0 \), the modulus of (1.11) is a slowly varying quantity which may be envelope detected. Sampling the matched filter at time \( t = t_0 \) one obtains from (1.11) by ignoring the carrier component, the radar ambiguity function.
The integral of (1.13) has a value dependent on the delay and doppler deviations \( \tau = t_2 - t_1 \) and \( 2\pi v = r_1 - r_2 \), respectively. Equation (1.13) has an alternate representation in the frequency domain that can be developed by insertion of the Fourier transforms of \( f(y) \) and \( f^*(y+\tau) \) into that expression obtaining (1.14).

\[
a(\tau, v) = \int_{-\infty}^{\infty} F(f) F(f + v) e^{-j2\pi f} df
\]

The ambiguity function may be represented by a surface over the two-dimensional \( \tau-v \) plane. This surface has several interesting properties which will be described later. Points on the surface represent the instantaneous sampled value \( (t = t_o) \) of the envelope of the matched filter output. If instead of sampling at \( t = t_o \), the envelope or absolute value of \( g(t,\tau,v) \), (1.11), is continuously recorded; the record will have the appearance of a constant doppler \( (v = v_1) \) vertical cut through the ambiguity surface of (1.13) parallel to the delay \( (v = 0) \) axis.

Figure 2 demonstrates a typical normalized ambiguity surface and indicates the waveforms which might be recorded at the filter output at (a) \( t = t_0 \) and (b) over the range \( t_1 < t < t_2 \), when the target return deviates from the expected delay by \( \tau_1 \) and from the expected doppler shift by \( v_1 \). The continuous form of the matched filter output is often required as this additional information may be useful in resolving multiple targets or estimating more difficult to determine target characteristics such as size and shape.
Figure 2. A normalized ambiguity surface and its relation to a received signal's matched filter response.
C. The ambiguity function and target resolution

The response of a matched filter to a time-frequency shifted replica of the expected signal is characterized by a time-frequency correlation function \( a(\tau, \nu) \). The previous derivation of \( a(\tau, \nu) \) was described in terms of matched filter response because the signal of issue was assumed embedded in white Gaussian noise and because a detection philosophy required the availability of the maximum instantaneous signal-to-noise energy ratio at the receiver output. It is interesting that the ambiguity function is also a natural result of the application of a resolution philosophy to the general radar problem. For instance, one can seek some criteria that may be used in the construction of signals which are resolvable from their own time-frequency translates.

Let \( s_i(t) \) and \( s_j(t) \) be the signals to be resolved and as a measure of resolution choose the distance \( D \) as

\[
D = \int |s_i(t) - s_j(t)|^2 dt \geq 0
\]

\[
D = \int |s_i(t)|^2 dt + \int |s_j(t)|^2 dt - 2\text{Re} \int s_i(t)^* s_j(t) dt.
\]

(1.1)

Let \( s_j(t) \) have a delay \( \tau \) and frequency shift \( \nu \) with respect to \( s_i(t) \) so that

\[
s_j(t) = s_i(t+\tau) e^{j2\pi\nu(t+\tau)} = f_i(t+\tau) e^{j(\omega_0+2\pi\nu)(t+\tau)}
\]

(1.1)

\[
\int |s_i(t)|^2 dt = \int |s_j(t)|^2 dt = E_f
\]

(1.1)

where \( E_f = \) energy of the signal \( f_i(t) \).

---

1Unless otherwise specified, integrals without limits are taken over \((-\infty, \infty)\) of the integration variable.
\[
D = 2E_f - 2\text{Re} \int f(t)f(t+\tau)e^{-(j\omega_0+2\pi\nu)\tau} e^{-j2\pi\nu t} dt
\]
\[
= 2E_f(1 - \text{Re} \left[ \frac{e^{-j\omega_0 + 2\pi\nu t}}{E_f} \int f(t)f(t+\tau)e^{-j2\pi\nu t} dt \right])
\]

Note from (1.18), or Property (10) to follow, \( a(0,0) = E_f \) so that

\[
D = 2E_f(1 - \text{Re} \left[ e^{-j\omega_0 + 2\pi\nu t} \frac{a(\tau,\nu)}{a(0,0)} \right]). \tag{1.19}
\]

Equation (1.19) shows that it is only the term \( \frac{a(\tau,\nu)}{a(0,0)} \) which is important in determining the distance between \( s_i(t) \) and \( s_j(t) \). For good resolution of the translated signals one would wish \( a(\tau,\nu) \) to decrease rapidly as \( (\tau,\nu) \) deviate from \( (0,0) \) or at least \( a(\tau,\nu) \) should be sharply suppressed in those areas of the \( \tau-\nu \) plane where resolution of the two signals is particularly desired. Thus to achieve the desired resolution one should pick \( f(t) \) so that the ambiguity surface defined by (1.20) has a desirable shape.

\[
\frac{a(\tau,\nu)}{a(0,0)} = \frac{1}{E_f} \int f(t)f^*(t+\tau)e^{-j2\pi\nu t} dt \tag{1.20}
\]

In addition to playing a central role in discussions of detection (1.13) and resolution (1.20) of radar targets, the ambiguity surface may be used to advantage in the study of signal selection for communication systems [15].
D. Summary of fundamental ambiguity function relations

Not every real or complex function of two variables \((\tau, \nu)\) may be written as an integral of the type (1.20) and thus be a member of the class of ambiguity functions. Although a great deal of effort has been directed toward discovering necessary and sufficient conditions such that a complex function \(a(\tau, \nu)\) be representable as the integral (1.20) over a complex waveform \(f(t)\), complete understanding has not been attained [16-18, 26]. In other words, the general synthesis problem remains unsolved.

Siebert [16], using a slightly different mathematical definition but equivalent in interpretation to (1.20), compiled a list of mathematical properties of the ambiguity function. Stutt [17,18] and others [19,20] have contributed to this knowledge. The more basic of these results have been translated into the notation of this paper and are listed with a reference to a primary source and comments where appropriate. The first six properties and other relations are well explained by Siebert [16].

1. A function \(a(\tau, \nu)\) is an ambiguity function if and only if it can be represented by an integral of the type of (1.20) or an equivalent

This condition requires that the function \(p(x, \tau)\)

\[
p(x, \tau) = \int_{-\infty}^{\infty} a(t, \nu) e^{j2\pi \nu x} \, dv
\]  

(1.21)

be a product of the form \(p(x, \tau) = f(x) f^*(x + \tau)\). Those surfaces \(a(\tau, \nu)\), with associated \(p(x, \tau)\)'s which factor in the above manner, are
therefore true ambiguity surfaces.

\[ \int f(x) f^*(x + \tau) e^{-j2\pi vx} \, dx = a(\tau, v) \quad (1.22) \]

2. Given that \( a_1(\tau, v) \) and \( a_2(\tau, v) \) are ambiguity functions, then

\[ a_0 = a_1 + a_2 \quad (1.23) \]

is also an ambiguity function if and only if \( a_1 = Ca_2 \), where \( C \) is an arbitrary complex (or real) constant.

3. If \( a(\tau, v) \) is an ambiguity function

\[ a(\tau, v) = a^*(-\tau, -v). \quad (1.24) \]

4. The generation of \( a(\tau, v) \) from the complex signal \( f(t) \) may be denoted as

\[ f(t) \xrightarrow{a} a(\tau, v) \]

then

\[ f(bt) \xrightarrow{1/b} \frac{a}{b}[a(b\tau, v/b)]. \quad (1.25) \]

5. If \( f(t) \) generates \( a(\tau, v) \), then

\[ g(t) = f(t) e^{i(kt^2/2)} e^{j(kt^2/2)} a(\tau, v + \frac{kt}{2\pi}) \quad (1.26) \]

and in the frequency domain definition (1.14) if the complex spectrum \( F(\omega) \) of a signal \( f(t) \) generates an ambiguity function say, \( b(\tau, v) \),

\[ F(\omega) \xrightarrow{a} b(\tau, v) \]

then

\[ G(\omega) = F(\omega) e^{i(k\omega^2/2)} e^{j(k/2)(2\pi v)^2} b(\tau + 2\pi kv, v). \quad (1.27) \]
Note that a multiplicative quadratic phase modulation of \( f(t) \), or a quadratic phase distortion of \( F(\omega) \) results in an ambiguity function which is a distorted version of that associated with \( f(t) \) or \( F(\omega) \). The distortion is a skewing of the ambiguity surfaces which leaves only the cuts \( a(0,v) \) in (1.26) and \( a(\tau,0) \) in (1.27) unchanged.

6. The convolution of ambiguity functions is also an ambiguity function. Thus if

\[
F_1(\omega) \xrightarrow{a} b_1(\tau,v) \quad \text{and} \quad F_2(\omega) \xrightarrow{a} b_2(\tau,v) ,
\]

then

\[
b_0(\tau,v) = \int b_1(\xi,v) b_2(\tau-\xi,v) d\xi \tag{1.28}
\]

\[
= \int F_1^*(f) F_1(f+v) e^{-j2\pi f \xi} df \int F_2(g) F_2(g+v) e^{-j2\pi (\tau-\xi) g} dg d\xi
\]

\[
= \int F_1^*(f) F_2^*(g) F_1(f+v) F_2(g+v) e^{-j2\pi (f-g) \xi} d\xi e^{-j2\pi \tau g} df dg
\]

\[
b_0(\tau,v) = \int F_1^*(f) F_2^*(f) F_1(f+v) F_2(f+v) e^{-j2\pi \tau f} df . \tag{1.29}
\]

Next let \( F_0(f) = F_1(f) F_2(f) \) which shows that the form of (1.29) verifies that the convolution of ambiguity functions is also an ambiguity function.

In the time domain, if

\[
f_1(t) \xrightarrow{a} a_1(\tau,v)
\]

\[
f_2(t) \xrightarrow{a} a_2(\tau,v)
\]

then

\[
f_0(t) = f_1(t) f_2(t)
\]

\[
f_0(t) \xrightarrow{a} a_0(\tau,v)
\]

then by a proof similar to (1.29) based on the time domain definition of
an ambiguity function

\[
a_0(\tau, \nu) = \int a_1(\tau, \xi) a_2(\tau, \nu - \xi) \, d\xi.
\] (1.30)

7. A useful set of Fourier transforms may be derived from the ambiguity function definitions [20].

\[
a(\tau, \nu) = \int f(t) f^*(t+\tau) e^{-j2\pi \nu t} \, dt
\]

\[
a(0, \nu) = \int |f(t)|^2 e^{-j2\pi \nu t} \, dt
\]

so

\[
a(0, \nu) \mathcal{F} |f(t)|^2.
\] (1.31)

By setting \( t = 0 \) in the inverse transform,

\[
|f(0)|^2 = \int a(0, \nu) e^{j2\pi \nu t} \, dv |_{t=0}.
\]

Also

\[
a(\tau, \nu) = \int \mathcal{F}^* (f) \mathcal{F}(f+\nu) e^{-j2\pi \nu t} \, df
\]

\[
a(\tau, 0) = \int |\mathcal{F}(f)|^2 e^{-j2\pi \nu t} \, df
\]

\[
a(\tau, 0) \mathcal{F} \mathcal{F} |\mathcal{F}(f)|^2.
\] (1.32)

Likewise

\[
|\mathcal{F}(0)|^2 = \int a(\tau, 0) e^{j2\pi \nu t} \, dt \bigg|_{t=0}.
\]

8. Denoting \( A(\tau, \nu) = |a(\tau, \nu)|^2 \), necessary but not sufficient conditions are known such that \( A(\tau, \nu) \) be the squared magnitude of a valid ambiguity function [16]. One such condition is that \( A(\tau, \nu) \) be self-reciprocal with respect to a double Fourier transformation.

Thus

\[
A(\sigma, \xi) = \int A(\tau, \nu) e^{-j2\pi(\tau \xi - \nu \sigma)} \, d\tau \, d\nu.
\] (1.33)
9. Interesting Fourier transforms [21] may be derived from (1.33) which show the relationship between line integrals along constant $\tau$ and $\nu$ contours of $A(\tau,\nu)$ to the axial distributions $A(0,\nu)$ and $A(\tau,0)$ respectively. Specifically,

$$fA(\tau,\nu)d\nu = fA(0,\nu) e^{-j2\pi\nu\tau}d\nu,$$

$$fA(\tau,\nu)d\tau = fA(\tau,0) e^{-j2\pi\nu\tau}d\tau.$$  \hspace{1cm} (1.34)

10. If $\phi(\tau,\nu)$ is any quadratically integrable function possessing the two dimensional Fourier transform $\phi(\sigma,\xi)$,

$$\phi(\sigma,\xi) = f\phi(\tau,\nu) e^{-j2\pi(\tau\xi-\nu\sigma)}d\tau d\nu,$$

then

$$fA(\tau,\nu) \phi(\tau,\nu)d\tau d\nu = fA(\tau,\nu) \phi(\tau,\nu)d\tau d\nu.$$  \hspace{1cm} (1.36)

Price and Hofstetter [22] have found the last relation useful in the study of bounds on the synthesis of ambiguity functions with sharp central peaks of ambiguity volume and "nearly" clear surrounding areas. Such ambiguity surfaces may be useful since their shape implies an increased ability to resolve closely spaced targets. The concept of "cleared areas" will be explained more fully in a following chapter.

11. If $f_i(t)$ is a complex signal and a member of a set $(i = 0,1,...n)$ of equal energy signals, such that

$$\text{energy of } f_i(t) = \left| f_i(t) \right|^2 = \int |f_i(t)|^2 dt = E_a,$$

if for every $i$ $f_i(t) \rightarrow a_i(\tau,\nu)$,
then \[ I_i = \int \int |a_i(\tau,\nu)|^2 \, d\tau \, d\nu \]

\[ = \int \int A_i(\tau,\nu) \, d\tau \, d\nu \]

\[ = \frac{E_a^2}{n} \quad \text{all } i = 0, 1, \ldots, n \quad (1.37) \]

The invariant nature of the integral \( I \) with changes in waveshape, but fixed signal energy, has been advanced as a kind of uncertainty relation limiting the radar analyst's attempts to enhance target resolution [2, 16]. In fact, the crux of the radar signal design problem lies in distributing the invariant volume \( I = E_a^2 \) under the ambiguity surface about the \( \tau-\nu \) plane in such a way as to achieve optimum system performance under various criteria of operation.

From property 8, (1.33) repeated here,

\[ A(\sigma, \xi) = \int \int A(\tau, \nu) \, e^{-j2\pi(\tau\xi - \nu\sigma)} \, d\tau \, d\nu, \]

it follows \[ A(0, 0) = \int \int A(\tau, \nu) \, d\tau \, d\nu = E_a^2 \quad (1.38) \]

\[ A(0, 0) = a(0, 0) \, a^*(0, 0) = E_a^2 \]

\[ a(0, 0) = E_a \quad (1.39) \]

From the defining relation (1.22), it is easy to see that \( a(0, 0) \) must be real. Although it is not generally the practice here, a common simplification of ambiguity function discussions sets the signal energy \( E_a \equiv 1 \) so that both \( a(0, 0) = 1 \) and \( A(0, 0) = 1 \).
E. Examples of ambiguity functions

The preceding sections have defined and demonstrated some mathematical properties of the ambiguity functions. In this section, a few examples are developed to demonstrate physical realizations of these surfaces. Reference [23] contains an extensive collection of ambiguity surfaces associated with signals meaningful in modern radar practice.

A. Rectangular Pulse (-T/2 ≤ t ≤ T/2):

Let \( f(t) = \sqrt{E/T} \left[ u(t + T/2) - u(t - T/2) \right] e^{j\omega_0 t} \). (1.40)

\[
a_R(\tau, \nu) = \begin{cases} 
\frac{E}{T} e^{-j\omega_0 \tau} \int_{-T/2}^{T/2} e^{-j2\pi\nu t} dt & \text{for} \quad \tau > 0 \\
\frac{E}{T} e^{-j\omega_0 \tau} \int_{-T/2}^{-\tau} e^{-j2\pi\nu t} dt & \text{for} \quad \tau < 0 \\
0 & \text{for} \quad |\tau| > T
\end{cases}
\]

\[
a_R(\tau, \nu) = \begin{cases} 
E e^{-j(\omega_0 + \nu \tau)} \sin \pi \nu T \frac{1 - |\tau|}{T} & \text{for} \quad |\tau| < T \\
0 & \text{elsewhere}
\end{cases}
\]

Figure 3 displays \( \left| \frac{a_R(\tau, \nu)}{a_R(0, 0)} \right| \).

B. Linearly Frequency Modulated Rectangular Pulse (-T/2 ≤ t ≤ T/2):

This is the well known "chirp" radar signal [24].
The instantaneous frequency of the pulse is
\[ \xi_i(t) = \frac{1}{2\pi} \frac{d\phi(t)}{dt} = \frac{1}{2\pi} \left[ \omega_0 + \Delta \omega \frac{t}{T} \right]. \]  
(1.42)

Thus
\[ \xi_i(\frac{T}{2}) = \frac{1}{2\pi} \left( \omega_0 - \frac{\Delta \omega}{2} \right) = \xi_0 - \frac{\Delta \xi}{2} \]
and
\[ \xi_i(\frac{T}{2}) = \frac{1}{2\pi} \left( \omega_0 + \frac{\Delta \omega}{2} \right) = \xi_0 + \frac{\Delta \xi}{2}. \]

By substitution of (1.41) into (1.20) and completing the integration,

\[ a_c(\tau, \nu) = E e^{-j((\omega_0 + \pi \nu)\tau - \frac{\nu \Delta \xi \tau^2}{T}) \sin(\pi \nu T + \pi \Delta \xi \tau)(1 - \frac{\vert \tau \vert}{T})} \]
\[ (\pi \nu T + \pi \Delta \xi \tau) \]
\[ \vert \tau \vert < T. \]  
(1.43)

Figure 4 shows \( a_c(\tau, \nu)/a_c(0,0) \). It can be seen that the result is a skewed version of the result of Example A, Figure 3, as predicted by Property 5.

C. Impulse Train of Finite Duration:

Let
\[ f_I(t) = \sqrt{\frac{E}{2N+1}} \sum_{k=-N}^{N} \delta(t - kT) \]  
(1.44)

with the energy of
\[ f_I(t), \quad ||f_I(t)||^2 = E. \]
(1.45)

\[ a_I(\tau, \nu) = \frac{E}{2N+1} \sum_{-NT}^{NT} \sum_{k=-N}^{N} \delta(t-kT) \sum_{j=-N}^{N} \delta(t+jT)e^{-j2\pi \nu t} \]
\[ dt \]  
(1.46)
When \( \tau = 0 \),

\[
\alpha_I(0,v) = \frac{E}{2N+1} \sum_{k=-N}^{N} \int_{-NT}^{NT} \delta(t-kT) e^{-j2\pi vt} \, dt
\]

\[
\alpha_I(0,v) = \frac{E}{2N+1} \sum_{k=-N}^{N} e^{-j2\pi vkt}
\]  \hspace{1cm} (1.47)

\[
\alpha_I(0,v) = \frac{E}{2N+1} \frac{\sin(2N+1)\nu T}{\sin \pi \nu T}
\]  \hspace{1cm} (1.48)

A well known identity has been used in passing from (1.47) to (1.48) [25].

It is easy to see from (1.46) that as \( \tau \) increases, \( \sum_{j=-N}^{N} \delta(t+\tau-jT) \) is a 2NT long train of impulses which sweeps across the train of impulses \( \sum_{k=-N}^{N} \delta(t-kT) \), the integrand having a value only when \( \tau \) is an integer multiple of \( T \).

\[
\alpha_I(\ell T,v) = \frac{E}{2N+1} \sum_{k=-N}^{N-\ell} e^{-j2\pi vkt} = \frac{E}{2N+1} [\left( \sum_{k=-N}^{0} + \sum_{k=0}^{N-\ell} \right) e^{-j2\pi vkt} - 1]
\]  \hspace{1cm} (1.49)

Likewise,

\[
\alpha_I(-\ell T,v) = \frac{E}{2N+1} \sum_{k=-N+\ell}^{N} e^{-j2\pi vkt} = \frac{E}{2N+1} [\left( \sum_{k=-N+\ell}^{0} + \sum_{k=0}^{N-\ell} \right) e^{-j2\pi vkt} - 1]
\]  \hspace{1cm} (1.50)

\[
\alpha_I(\ell T,v) = \frac{E}{2N+1} \left[ \sum_{k=0}^{k=N} e^{j2\pi vkt} + \sum_{\ell=0}^{N-\ell} e^{-j2\pi vkt} - 1 \right]
\]  \hspace{1cm} (1.51)

\[
\alpha_I(-\ell T,v) = \frac{E}{2N+1} \left[ \sum_{k=0}^{0} e^{j2\pi vkt} + \sum_{\ell=0}^{N-\ell} e^{-j2\pi vkt} - 1 \right]
\]  \hspace{1cm} (1.52)

Obviously,

\[
\alpha_I^* (\ell T,v) = \alpha_I(-\ell T,v).
\]  \hspace{1cm} (1.53)
These last expressions may be written as result (1.54) for $\tau > 0$ [25, p. 78]. Equation (1.53) may be used in (1.54) to obtain $a_\tau$ for $\tau < 0$.

$$a_\tau(\ell T, \nu) = \frac{E}{2N+1} \left[ e^{j\pi\nu T} \frac{\sin((N+1)\pi \nu T)}{\sin \pi \nu T} + e^{-j(N-\ell)\pi \nu T} \frac{\sin(N-\ell+1)\pi \nu T}{\sin \pi \nu T} - 1 \right]$$

integers ($0 \leq \ell \leq 2N$)

(1.54)

Figure 5 displays $|a_\tau(\tau, \nu)/a_\tau(0,0)|$ for a choice of $N = 5$. As time domain impulses have been assumed, $a_\tau(\tau, \nu)$ is made up of a series of infinitely thin $\nu$ directed sheets of ambiguity spaced at integer multiples of $T$, $|\tau| = |\ell T| \leq 2NT$. In a case where the ideal impulses are broadened into short rectangular pulses, the resulting ambiguity function will look much like Figure 5 in doppler directed distribution; but the thin ambiguity sheets of the ideal case will take on a finite thickness in the $\tau$ direction.

If the ideal impulse train grows very long ($N \to \infty$), the surface (1.54) approaches the so-called "bed of nails" ambiguity function which possesses impulses distributed at the lattice points of a rectangular grid. The impulses are spaced at $\Delta \tau = T$ and $\Delta \nu = 1/T$ intervals.
Figure 3. Ambiguity surface of a rectangular pulse. Constant doppler frequency profiles spaced at $2/\pi T$ cps.

Figure 4. Ambiguity surface of the linearly frequency modulated, rectangular pulse of $TW = 80/\pi$. Constant doppler frequency profiles spaced at $8/\pi T$. 
Figure 5. Ambiguity surface of a sequence of five impulses. Constant delay profiles spaced at T seconds.
CHAPTER II
AMBIGUITY FUNCTION SYNTHESIS TECHNIQUES

The ambiguity function was introduced in the preceding chapter and its fundamental importance in the design and investigation of the measurement properties of radar signals was discussed. Some additional aspects of ambiguity functions and an approach to their synthesis will be examined in this chapter.

A. The ideal ambiguity function

The simple examples of Chapter I demonstrate the appearance of several ambiguity surfaces. An ideal surface is rather difficult to define as its properties such as shape and extent should be matched to the radar's operating environment of noise and number and distribution of targets. Rihaczek has pointed out that optimum ambiguity functions may be optimum only in the sense of being the best possible solution for a particular situation [19].

Suppose one were to require simultaneously a very sharp matched filter response to a specific doppler frequency shifted and time delayed target echo. The associated ambiguity surface would necessarily have a very narrow origin-centered peak containing a volume $V_0$. This surface would indicate that the signal and its matched filter are capable of resolving closely spaced targets and would allow accurate measurements of target range and velocity. In order to prevent the radar receiver matched filter from responding to target echoes elsewhere in the $\tau$-$\nu$
plane, the ambiguity surface must fall rapidly to zero as the points \((\tau, v)\) deviate from the point \((0, 0)\). This surface would then have a very small amplitude in the non-central region of the ambiguity plane and is said to have low sidelobes. The last concept is obtained from a simple analogy with beam shaping problems of antenna theory. The analogy will be of use in a later chapter. Figure 6 depicts a "near ideal" or "general-purpose" ambiguity function with the properties just described.

It is well known [2, chpt. 7] that the requirements of resolution, accuracy, and ambiguity control are incompatible. Woodward's invariant volume principle implies that when the sharp central peak necessary for good target resolution contains only a small portion \(V_0\) of the total volume, the remainder of the invariant ambiguity volume must be distributed about the rest of the time-frequency plane in some manner that could possibly allow ambiguous responses of the matched filter to remote targets. This concept is discussed quantitatively in the following section.

B. Cleared area concept

Take \(A\) as an area in the ambiguity plane containing a centered spike of volume \(V_0\) surrounded by a convex subregion \(C(A)\) of \(A\). For good target resolution we wish to clear \(C(A)\) of all ambiguity volume other than \(V_0\). Price and Hofstetter [22] have shown that (2.1) is a fundamental inequality relating the volume \(V(A)\) over \(C(A)\) to the central spike volume \(V_0\).

\[
V(A) \geq \frac{1}{4} C(A) V_0
\]

(2.1)

where \(V(A) = \) lower bound on the total ambiguity volume above \(C(A)\) including \(V_0\).
\( C(A) \) = subregion of A desired cleared of ambiguity.

From (2.1) one may conclude that subregions \( C(A) \) of A may not be cleared of ambiguity volume in excess of \( V_0 \) if \( C(A) > 4 \). For smaller areas it is at least theoretically possible to realize a near ideal ambiguity surface with a centered impulse \( V_0 \) surrounded by a volume cleared area \( C(A) \). Outside this area the ambiguity distribution is not controlled except by the volume invariance principle. The ideal situation is demonstrated by the "bed of nails" ambiguity distribution [26] associated with an unrealizable radar signal of an infinitely long series of impulses, as when \( N \to \infty \) in Example I-C. By inspection of Figure 7 one sees that each nail is surrounded by a clear area \( C(A) \) of size \( C(A) = 4(T \times 1/T) = 4 \).

Again following Price and Hofstetter [22], if one allows a volume \( \alpha \) (ambiguity/unit area) to be spread over the area \( C(A) \), the inequality (2.1) is modified as in (2.2).

\[
V(A) \geq (1/4) C(A) V_0
\]

\[
V_0 + \alpha C(A) \geq (1/4) C(A) V_0 \quad (2.2)
\]

\[
C(A) \leq V_0/(V_0/4 - \alpha) \quad (2.3)
\]

With the premise of the origin centered volume impulse \( V_0 \), the ambiguity distribution over an area as large as \( C(A) = 20 \) may be suppressed to an approximate level \( \alpha = V_0/5 \). Note (2.2) implies that \( C(A) \) may be as large as desired if one does not attempt to suppress the volume distribution below \( \alpha = V_0/4 \).
Practical radar signals do not allow the realization of the ideal impulse ambiguity function. Instead, one must be satisfied with an ambiguity surface whose central peak has a finite base area and whose shape and non-central volume distribution is controlled to meet as best possible the various resolution and accuracy requirements. Rihaczek [19] has discussed the philosophy of ambiguity surface selection for various operating conditions. If we assume that the specifications for a desired surface have been translated into a mathematical expression, the next task is to find a signal which produces the specified ambiguity distribution.

C. The constructive check

Since Property 1 of the ambiguity function given by (1.22) is of a rather special form it is obvious that not every function of two variables \((\tau,v)\) belongs to the class of true ambiguity functions. In fact, one would not often be able to write a valid mathematical description of an ambiguity function unless a great deal of a priori information concerning its shape was available. It appears that the synthesis of a completely known ambiguity surface is not generally the problem which faces the radar signal designer.

Assume that a potential ambiguity function \(a_p(\tau,v)\) is available and has properties making its synthesis desirable, the following procedure may be used to determine if the proposed function may be realized exactly or not. If the checking method proves the function to be a valid ambiguity function it is shown that the required radar signal is generated by one of the steps in the test.
Figure 6. A general-purpose ambiguity function.

Figure 7. "Bed of nails" ambiguity function associated with an infinitely long series of transmitted impulses.
From Property 1, Chapter I, it is known that the factoring of the inverse Fourier transform $\mathcal{F}^{-1}[a_p(\tau,\nu)] = g(\tau,\tau) = f(t)f^*(t+\tau)$ is a necessary and sufficient condition that $a_p$ be a valid ambiguity function. Thus it is only necessary to demonstrate that such a factoring does or does not exist and thus that $a_p(\tau,\nu)$ is or is not a valid expression. A constructive check could be as follows.

Let $a_p(\tau,\nu)$ be integrable square in $\nu$ so that it possesses a Fourier transform which may in general not be factorable by inspection, then
\[
\int a_p(\tau,\nu) e^{i2\pi\nu t} d\nu = g(t,\tau). \tag{2.4}
\]
Of course, if $g(t,\tau)$ factors immediately into the form $f(t)f^*(t+\tau)$ then one has proof that a particular $a_p(\tau,\nu)$ is a valid ambiguity function and also one has the required waveform $f(t)$ directly.

Temporarily assuming $a_p(\tau,\nu)$ is valid but that the transform (2.4) is not factorable by inspection one has (2.5) by setting $t = 0$ in (2.4) and taking the conjugate
\[
\int a_p^*(\tau,\nu) d\nu = f^*(0) f(\tau). \tag{2.5}
\]
At most $f(0) = c e^{i\lambda}$ \quad \tag{2.6}
so that from (2.5)
\[
\int a_p^*(0,\nu) d\nu = |f(0)|^2 = c^2. \tag{2.7}
\]
From (2.5) it is apparent that a change in the dummy variable $\tau$ to the variable $t$ yields
\[
\int a_p^*(t,\nu) d\nu = c e^{-i\lambda} f(t) \tag{2.8}
\]

\[1\text{Integrals written without limits will be taken over } (-\infty, \infty).\]
and \[ \int a_p(t+\tau,\nu) \, d\nu = c e^{i\lambda} f^\#(t+\tau). \quad (2.9) \]

Multiplying (2.8) and (2.9) together and with \( c^2 \) obtainable from (2.7)

\[
(\int a^\#_p(t,\nu) \, d\nu) (\int a_p(t+\tau,\nu) \, d\nu) = c^2 f(t) f^\#(t+\tau). \quad (2.10)
\]

The manipulations leading to \( g(t,\tau) \) in (2.4) were performed without any assumptions about the validity of \( a_p(\tau,\nu) \). One now can compare the result (2.4) with (2.10) which was based on a premise of the validity of \( a_p(\tau,\nu) \).

\[
g(t,\tau) \neq 1/c^2 (\int a^\#_p(t,\nu) \, d\nu) (\int a_p(t+\tau,\nu) \, d\nu) \quad (2.11)
\]

If the equality holds in (2.11) one can conclude that the factoring of \( g(t,\tau) \) into the desired form \( f(t)f^\#(t+\tau) \) is indeed possible and is sufficient proof of the validity of \( a_p(\tau,\nu) \) as an ambiguity function. Note that not only does the suggested method allow a check of \( a_p(\tau,\nu) \) but also yields a positive result. Equality in (2.11) indicates that (2.8) constructs the desired signal to within a complex constant. Further, every quantity needed for the checking process is available from the proper simplification of (2.4).

It is interesting that the arbitrary phase constant \( e^{i\lambda} \) has no effect on the shape of the realized ambiguity surface. Thus if

\[
h(t) = e^{-i\lambda} f(t), \quad (2.12)
\]

\[
a_h(\tau,\nu) = \int h(t) \, h^\#(t+\tau) \, e^{-j2\pi \nu t} \, dt
\]

\[
= \int e^{-i\lambda} f(t) \, e^{i\lambda} f^\#(t+\tau) \, e^{-j2\pi \nu t} \, dt
\]

so \( a_h(\tau,\nu) = a_f(\tau,\nu) \). \quad (2.13)
D. Sussman-Wilcox method

For reasons previously set forth it is quite likely that the inequality sign will hold in (2.11) indicating that a potential surface \( a_p(\tau, \nu) \) is not exact in the sense of Property 1, Chapter I. It has been proposed [27] that such a surface be approximated by a linear combination of valid ambiguity surfaces \( \{ K_{ij}(\tau, \nu) \} \). The "induced basis" ambiguity surfaces are time-frequency auto and cross-correlations of a complete orthonormal set of basis waveforms \( \{ \phi_i(t) \} \). It may be proved that the completeness of the set \( \{ \phi_i \} \) implies the completeness of the set \( \{ K_{ij} \} \). The waveform \( v(t) \) necessary to generate the approximation \( a_p(\tau, \nu) \) is given by a linear combination of the \( \{ \phi_i \} \).

\[
v(t) = \sum_i a_i \phi_i(t) \tag{2.14}
\]

In essence, the mathematical problem is to determine the \( a_i \)'s so that with a controllably small error \( \varepsilon \)

\[
a_p(\tau, \nu) = \sum_i B_{ij} K_{ij}(\tau, \nu) + \varepsilon(\tau, \nu) \tag{2.15}
\]

with

\[
B_{ij} = \iint a_p(\tau, \nu) K_{ij}(\tau, \nu) \, d\tau \, d\nu \tag{2.16}
\]

Then

\[
[a] = \begin{bmatrix}
a_1 \\
a_2 \\
\vdots \\
\end{bmatrix}
\]

\[
[b] = [a] [a^*]^T. \tag{2.17}
\]

The cross and auto-ambiguity functions over \( \{ \phi_i(t) \} \) are defined in a slightly different but equivalent form of (1.22) as
\[ K_{ij}(\tau, v) = \int \phi_i(t - \tau/2) \phi_j^*(t + \tau/2) e^{-j2\pi \tau \nu} dt \] (2.18)

so that \[ K_{ij}(\tau, v) = K_{ji}^*(-\tau, -v). \] (2.19)

Using a least mean square error criterion, Sussman has shown that the problem of finding the expansion coefficients \( a_i \) may be solved by applying a variational approach to a matrix algebra formulation of the minimized error criterion. A matrix eigenvalue problem results which is simple in form if not in solution. If the desired surface has the property \( a_p(\tau, v) = a_p^*(-\tau, -v) \), it may be synthesized in terms of auto-ambiguity functions alone so that only terms of the type \( K_{ij}(\tau, v) \) are necessary in (2.15). The results for this case are the same as those obtained by an approach of Wilcox [28].

The realization error \( \xi \) in the synthesis of \( a_p(\tau, v) \) is

\[ \xi = \iint |a_p(\tau, v)|^2 \, d\tau \, dv - E \] (2.20)

where \[ E = \sum_i |a_i|^2. \] (2.21)

When one is allowed the complete set \( \{\phi_i(t)\} \), \( \xi \) may be reduced as desired; however, when the series (2.14) for \( v(t) \) is truncated to \( N \) terms the realization error for \( a_p(\tau, v) \) is highly dependent on the choice of basis waveforms. For instance in synthesizing those surfaces possessing the symmetry property \( a_p(\tau, v) = a_p^*(-\tau, -v) \), one should pick \( \{\phi_i(t)\} \) to insure that each member of the set \( \{K_{ij}\} \) also possesses this symmetry.

As in other aspects of ambiguity function synthesis, there are definite difficulties involved with the Sussman-Wilcox approach. Fore-
most is the lack of control over and visualization of the modulation, duration, and bandwidth requirements of the realized generating signal \( v(t) \) for a specific surface \( a_p(\tau,v) \). Secondly, even though the integrated error \( \xi = \int \int |e(\tau,v)|^2 d\tau \, dv \) is made reasonably small by sensible choices of \( \{\phi_i(t)\} \) and by large \( N \)'s, one should investigate the realized ambiguity surface \( F(\tau,v) \)

\[
F(\tau,v) = \sum_{i=1}^{N} B_{ii} K_{i}(\tau,v) \quad (2.22)
\]

to see in what way this error arises. If \( e(\tau,v) \) is uniformly distributed over the extent of \( a_p(\tau,v) \) then the realization \( F(\tau,v) \) will most probably be acceptable. On the other hand, a system of tall, narrow error spikes distributed about the \( \tau,v \) plane might result in a small value of \( \xi \), but be unacceptable for some operational situation. In the latter case, the alternative is to pick a new set of basis waveforms \( \{\psi_i(t)\} \), form a new set of "induced basis" surfaces \( \{L_{ij}(\tau,v)\} \), and a new expansion of \( a_p(\tau,v) \) in terms of these surfaces. The error involved in this new expansion must be checked and additional modifications made to the basis waveforms as necessary to suppress unacceptable errors.

E. The ambiguity magnitude function synthesis problem

Underlying the preceding synthesis technique is the assumption that one can describe suitably shaped ambiguity surfaces by appropriate expressions in \( (\tau,v) \). In particular one can write

\[
a_p(\tau,v) = |a_p(\tau,v)| e^{j\gamma(\tau,v)} \quad (2.23)
\]
so that the complete specification of \( a_p \) requires both a magnitude and a phase relation.

In incoherently detecting radar schemes, only the magnitude function \( |a_p| \) is of interest. A synthesis method avoiding the necessity of phase specification would appear, in principle, to be more useful than one which requires both \( |a_p| \) and \( \gamma \). In fact, little is known about the synthesis of arbitrary magnitude functions and the results in this area have been particularly discouraging. Recently, additional properties of the magnitude function have become available [16,27] which show that a proposed (desired) surface must be constrained in the following ways if it is to be synthesized by any method.

1. An extension of Woodward's volume invariance result (Chapter I, Property 10) to the \( p \)th ordered norm defined by the left side of (2.24) shows that a valid ambiguity surface \( A(\tau,\nu) = |a|^2 \) satisfies the relation

\[
\int \int |a(\tau,\nu)|^{2p} d\tau d\nu \leq \left\{ [p]^{-1} (1 + [p]^{-1})^{[p]-p} \right\} a(0,0)^{2p}
\]

(2.24)

for all \( p > 1 \) and \([p]\) = largest integer \(< p\).

For the special case of integer \( p > 1 \), (2.24) simplifies to

\[
\int \int |a(\tau,\nu)|^{2p} d\tau d\nu \leq 1/p \ a(0,0)^{2p}.
\]

(2.25)

2. Stutt [18] has derived relations between the magnitude and phase components of an ambiguity surface.

If

\[
A(\tau,\nu) = a(\tau,\nu) a^*(\tau,\nu)
\]

(2.26)

\[
a(\tau,\nu) = |a(\tau,\nu)| e^{i \gamma(\tau,\nu)}
\]

(2.27)
then the quantities \( A(\tau, v) \frac{\partial \gamma(\tau, v)}{\partial \tau} \) and \( A(\tau, v) \frac{\partial \gamma(\tau, v)}{\partial v} \) are self-reciprocal with respect to a double Fourier transform.

\[
\int \int A(\tau, v) e^{j2\pi(\tau y - \nu x)} d\tau \, dv = A(x, y) \frac{\partial \gamma(x, y)}{\partial x} \quad (2.28)
\]

\[
\int \int A(\tau, v) \frac{\partial \gamma(\tau, v)}{\partial v} e^{j2\pi(\tau y - \nu x)} d\tau \, dv = A(x, y) \frac{\partial \gamma(x, y)}{\partial y} \quad (2.29)
\]

The phase functions are not unique for if \( \gamma_1 \) and \( A_1 \) satisfy (2.28) and (2.29) so also does \( A_1 \) and \( \gamma_2 = \gamma_1 + \alpha \tau + \beta \nu \) [18]. The class of phase functions is limited since \( \gamma(\tau, v) = -\gamma(-\tau, -\nu) \) is a necessary requirement.

F. Summary

This chapter has pointed out that the constructive check approach to synthesis of ambiguity functions is useful when an exact ambiguity expression is available. The successful application of the check to an ambiguity expression will yield the required radar signal.

The Sussman-Wilcox approach to ambiguity surface synthesis allows an approximate realization of a desired distribution of ambiguity volume over the \( \tau-v \) plane. In this sense, it is the only formal approach to the synthesis problem yet devised. From an engineering viewpoint the theory is not particularly useful for it fails to allow the designer reasonable control of the amplitude or angle modulation of the generating signal \( v(t) = \sum_i a_i \phi_i(t) \), \( |t| \leq T/2 \). Thus an ambiguity surface of desirable shape and a convenient set of basis waveforms \( \{\phi_i(t)\} \) may require a generating signal \( v(t) \) which is very unsuitable as a radar probing signal.

In the following discussions it will be shown that useful synthesis methods exist which satisfactorily relate certain properties of the
ambiguity function surface to specific properties of the required generating signal. In such a synthesis procedure one must accept at the outset the fact that generally it will not be possible to control the shape of the ambiguity surface at every point of the $\tau$-$\nu$ plane.
CHAPTER III

ALTERNATE AMBIGUITY FUNCTION SYNTHESIS METHODS

An alternate approach to the synthesis problem is available if one is willing to trade pointwise approximation of the magnitude function in most areas of the τ-v plane for the ability to relate the generating signal and its spectrum to specific properties of |a(τ,ν)|.

A. The uncertainty ellipse

Helstrom [3, pp. 18,280] has shown by means of a Taylor series expansion of a possibly complex signal f(t) that, for small (τ,ν), the contours of |af(τ,ν)| associated with f(t) take on approximately elliptical shapes. The ellipse size and orientation above the τ-v plane is determined by a nominal duration T, bandwidth W, and phase modulation α of f(t) = u(t)e^{jφ(t)} and its Fourier transform F(ω) in (3.1) - (3.3).

It is assumed that the time and positive frequency origins are taken at the centers of gravity of f(t) and F(ω) so that,

\[ T^2 = \frac{\int_{-\infty}^{\infty} t^2 |f(t)|^2 dt}{\int_{-\infty}^{\infty} |f(t)|^2 dt} \quad (3.1) \]

\[ W^2 = \frac{\int_{-\infty}^{\infty} \omega^2 |F(\omega)|^2 d\omega}{\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega} \quad (3.2) \]
\[
\alpha = \frac{\int_{-\infty}^{\infty} t \phi'(t) |f(t)|^2 dt}{\int_{-\infty}^{\infty} |f(t)|^2 dt}
\] 

(3.3)

Helstrom's so-called "uncertainty ellipse" may be obtained by passing a plane parallel to the \(\tau - \nu\) plane through the central response peak at the \(|a(\tau, \nu)|^2 = 1/2\) level. The resulting intersection of the surfaces is the half-power -3db elliptic contour shown in Fig. 8 and approximately given by

\[
W^2 \tau^2 + 2a \tau \nu + T^2 \nu^2 = 1/4
\] 

(3.4)

When there is no phase (frequency) modulation, the major and minor axes of the ellipse are oriented along the \(\tau, \nu\) axes. For all but large values of \(a\), the effect of phase (frequency) modulation is a rigid rotation of the ellipse in the \(\tau, \nu\) plane. Under these conditions the ellipse area is constant, \(A = 2\pi\). One might establish the rough rule that any pair of target returns falling within the uncertainty ellipse will be unresolvable by the radar. By increasing \(T\) or \(W\) one can increase the \(\nu\) or \(\tau\) directed resolution respectively by narrowing the distribution of the ambiguity central peak along those axes. Since the area of the ellipse is essentially constant, increasing \(T\) or \(W\) means that the ellipse must expand along its major axis and the resolution of multiple targets oriented along the ellipse major axis will be significantly degraded. For large \(a\) such as might be produced by quadratic or higher order frequency modulation, the ellipse area \(A\) may be decreased thus indicating that increased complexity of the generating signal leads to greater possible resolution of multiple target echoes.
Figure 8. Contour of $|a(\tau,\nu)|^2 = 3/4$ defined by Helstrom's uncertainty ellipse $W^2 \tau^2 + 2\alpha \tau \nu + T^2 \nu^2 = 1/4$. 
The uncertainty ellipse is an approximate representation of the ambiguity distribution in the central peak. Those surfaces associated with generating signals of like parameters \( T, W, \) and \( \alpha \) will have essentially the same uncertainty ellipse representation of their central peak -3 db contour. However, these ambiguity surfaces may be quite different elsewhere in the \( \tau, v \) plane. Clearly, if a satisfactory shape and orientation of an uncertainty ellipse is the only criteria for some radar signal design problem, one should choose the most convenient signal that will give the desired \( T, W, \) and \( \alpha \). These concepts are clearly explained by Rihaczek [19].

B. Axial ambiguity distributions

A somewhat different approach to the synthesis problem involves more direct relations between generating signal waveshapes and their spectra and control of a specific sector of the ambiguity surface. As an example, when either the range or velocity of a target is approximately known and the other is to be determined, Fig. 8 indicates that the synthesis of a suitable ambiguity function involves control of the surface in the immediate vicinity of the known \( \tau \) or known \( v \). Since design delay or center frequency of a matched filter can be chosen at will, the approach is no less general if the known \( \tau \) or \( v \) is set equal to zero. Thus when \( \tau = 0 \) is a known target parameter it is of interest to control the ambiguity distribution about the \( v \) axis, that is, \( a(0,v) \). On the other hand, when \( v = 0 \) is known the quantity of interest is \( a(\tau,0) \) as may be inferred from the qualitative discussion of Helstrom's uncertainty ellipse.
Property 7 states the two transform pairs \( u^2(t) \xrightarrow{\mathcal{F}} a(0, \nu) \) and \( U^2(f) \xrightarrow{\mathcal{F}} a(\tau, 0) \). This pair of relations may be used to advantage in an ambiguity function synthesis technique.

Radar system models

Figure 9 illustrates a simplified radar signal processing system in which the total range of expected target return doppler frequency shifts has been divided into \( n \), \( W \) cps wide doppler cells by \( n \) local oscillators and ideal bandpass filters. The signal in each doppler cell is applied to an associated bank of \( k \) filters matched to delayed versions of the transmitted waveshape. In the usual case, the \( k \) "range" filters are adjusted to respond to signals in \( k \) overlapping range cells which include all target ranges expected. Each filter should be of essentially constant frequency response over the width of the preceding "doppler" filter. The bandpass width \( W \) cps when too small results in serious signal distortion and thus degradation of the matched receiver response. Choosing \( W \) small also increases the number of matched filters and the complexity of the system required to process signals throughout a specified doppler interval. Wider passbands decrease the doppler resolution of multiple targets but tends to preserve signal shape and thus range resolution. Accurate range determination depends on the control of the shape of \( a(\tau, 0) \) of the individual filter responses and on their spacing in delay (range). Thus, from a practical viewpoint, a synthesis problem of some importance is one based on the proper choice and realization of \( a(\tau, 0) \) by signals of passband width \( W \) cps.
Figure 9. Simplified pulse radar range tracker with matched filter receiver for positive doppler shifts.
Figure 10. Simplified noncoherent pulsed doppler radar signal processor.
In Fig. 10 a complementary situation is illustrated in which a gating process is used to define \( n \) range cells by dividing the signal received between pulse transmissions into \( n \), \( \Delta t \) length segments. Each signal segment is then examined by \( k \) filters matched to the possible doppler shifts of a target return. The frequency response or, equivalently, the impulse response of each filter is chosen to allow optimum resolution and suppressed ambiguity (sidelobes) in the doppler domain. Thus in the specification of the required signal and its matched filter it is the axial distribution \( a(0,v) \) of \( a(t,v) \) that is to be controlled. The system of Fig. 10 is of practical importance as it is a simplified example of the doppler tracking portion of widely used noncoherent pulsed doppler radars [23, p. 383].

Axial distribution sidelobes

The central peak of an axial ambiguity distribution generally is surrounded by a series of local maxima referred to as "sidelobes". The control of the height of these sidelobes is important and as a first step it is of interest to investigate just how they arise.

In a study of asymptotic forms of Fourier integrals, Erdelyi [29, p. 46] has shown for \( f(t) \) a \( N \)-times differentiable function over \(|t| \leq \frac{T}{2}\) that

\[
\lim_{\omega \to \infty} \int_{-T/2}^{T/2} g(t) e^{-j\omega t} dt = 0(\omega^{-N}) + B_N(\omega) - A_N(\omega). \tag{3.6}
\]

Where

\[
0(\omega^{-N}) = (j\omega)^{-N} \int_{-T/2}^{T/2} g(N)(t) e^{-j\omega t} dt
\]

(which is of order \( \omega^{-N} \))
\[ B_N(\omega) = \sum_{n=0}^{N-1} (-j)^n \frac{g(n)(T/2)\omega^{-n-1}}{\omega} e^{-j\omega T/2} \]
\[ A_N(\omega) = \sum_{n=0}^{N-1} (-j)^n \frac{g(n)(-T/2)\omega^{-n-1}}{\omega} e^{j\omega T/2} \]

and \[ g(n)(t) = \frac{d^n}{dt^n} (g(t)). \]

When \( g(t) \) is an even function of \( t \)
\[ g(n)(T/2) = g(n)(-T/2) \quad n = 0, 2, \ldots \text{ even} \]
\[ g(n)(T/2) = -g(n)(-T/2) \quad n = 1, 3, \ldots \text{ odd} \] (3.7)

By the proper simplifications of (3.6) it is not difficult to show that
\[
\lim_{\omega \to +\infty} \int_{-T/2}^{T/2} g(t)e^{-j\omega t}dt = 0(\omega^{-N}) + 2 \sum_{n=0}^{N-1} j^n g(n)(\frac{T}{2})\omega^{-n-1} \sin \frac{\omega T}{2} 
\]
\[ + 2 \sum_{n=0}^{N-1} j^n g(n)(\frac{T}{2})\omega^{-n-1} \cos \frac{\omega T}{2} \]

(3.8)

An equivalent (imaginary) result is obtained if \( g(t) \) is an odd function of \( t \). The sidelobe structure of the asymptotic form of the Fourier integral is evident in (3.8) since the sine and cosine terms add together to give a sinusoidal function with peak value determined by the values of \( g(n)(\frac{T}{2}) \). In principle these terms may be suppressed by requiring \( g(n)(\frac{T}{2}) = 0, n = 0, 1, 2, \ldots, \ell < N - 1 \). If we let \( g(t) = u^2(t) \), it follows that the sidelobe structure of the associated ambiguity surface cut \( a_g(0, v) \) may be suppressed by requiring \( (d^n/dt^n)(u^2(t)) \bigg|_{t=\frac{T}{2}} = 0, \) for all \( n = 0, 1, \ldots, \ell \leq N - 1 \). Thus the sidelobes of the ambiguity distributions
(or antenna fields) are generated by the Fourier transform operation on finite duration integrands that possesses a non-zero end condition and possibly non-zero derivatives at the integrand end points.

Sidelobe suppression by zero removal

In his now classic paper, Taylor [30, pp. 17-19] showed that end point discontinuities of an antenna aperture current distribution were necessary to obtain a narrow main beam of the field pattern. The analogous situation in signal design indicates that an even pulse should have leading and trailing edge discontinuities if its associated doppler axis ambiguity distribution is to have a narrow central response peak. The results of (3.8) and those of Taylor's paper reaffirm the difficulty of simultaneously obtaining a narrow central response and suppressed sidelobes.

If one accepts whatever central response broadening that occurs, an interesting method of sidelobe suppression based on (3.8) involves finding pulse shapes $u(t)$ for which $u^2(t)$ has no end-discontinuities and possesses a number of zero derivatives at the pulse edges.

Campbell et al. [31, p. 44] have stated that the removal of $k$ zeroes of the spectrum of a pulse increases the order of its differentiability by $k$. Under these authors' definitions, the derivatives created are zero at the pulse end points. Consider the even function $u(t)$ with real Fourier spectrum $U(\omega)$. The removal of $k$ zeroes of $U(\omega)$ may be effected by passing $u(t)$ through cascaded filters whose transfer functions have zero-cancelling poles at the desired frequencies. If $\tilde{U}(\omega)$ is the filter modified version of $U(\omega)$, its inverse Fourier transform is
at least \( k \) times differentiable pulses \( \overline{u}(t) \). If \( \overline{u}(t) \) is always positive and \( k \) times differentiable, then \( \overline{u}^2(t) \) is at least \( k \) times differentiable. By virtue of (3.8) and the construction of \( \overline{u}^2(t) \) such that
\[
\left. \frac{d^n}{dt^n} (\overline{u}^2(t)) \right|_{t=\pm T/2} = 0 \quad n = 0, 1, \ldots, k,
\]
one can see that for moderate choices of \( k \) the Fourier transform \( a_2(0, \nu) \) of \( \overline{u}^2(t) \) should possess a very small sidelobe structure.

An example of this method of sidelobe suppression has not been included since the ideas presented here leave several questions unanswered. For instance, as we know that the pulse shape \( u(t) \) is to be modified by the zero removal process, what is a wise initial choice of \( u(t) \) and which and how many zeroes of \( U(\omega) \) should be removed? Importantly, one also fails to obtain any idea about the ambiguity distribution in the region between the central peak and that of large frequency \( \nu \).
The zero removal process is quite interesting, however, and offers a good area for further study. In fact, zero manipulations appear to be a fundamental concept in a unified theory of modulation that ultimately must apply to radar problems [46].

The next section treats an alternate method of sidelobe control which has an advantage over the preceding postulated method in that a simple development yields a problem solution which is optimal in a sense to be defined. Again the pair \( u^2(t) \leftrightarrow a(0, \nu) \) will be considered.

C. Taylor ambiguity distributions

If the synthesis of an ambiguity function is to be based on the realization of properly shaped axial distributions, an ideal form appears to be one whose main response peak half-power width is minimized for a
fixed bound on the maximum height of those surrounding secondary response peaks which may occur. Such a situation is reminiscent of the problem of synthesis of antenna patterns possessing a minimum half-power main beam-width for a specified sidelobe level. Indeed the analogy is a powerful one for the antenna far electric field pattern is, to a good approximation, the Fourier transform of the antenna aperture current distribution just as a \((t,0)\) is the Fourier transform of \(U^2(f)\) or as a \((0,v)\) is the Fourier transform of \(u^2(t)\). See Appendix A.

In the sense mentioned, the optimum realizable antenna pattern (or the analogous axial ambiguity distribution) is the Taylor approximation to the non-realizable Dolph-Chebyshev pattern [32, p. 1083]. The Taylor pattern possesses \(n - 1\) equal height sidelobes to either side of the central response peak. These sidelobes merge at the \(n\)th pattern zero with a region containing an infinite number of smaller sidelobes whose height decreases as \(\sin (kv)/(kv)\). The Taylor distribution main response peak is broader by a factor \(\sigma\) than is that of the Dolph-Chebyshev pattern of an equal sidelobe level \(R\).

The Taylor antenna field pattern requires a finite length continuous aperture current distribution. Taylor axial ambiguity distributions require continuous but time-limited signal envelopes \(u^2_{\text{Ty}}(t)\) or continuous bandlimited spectral envelopes \(U^2_{\text{Ty}}(f)\). Examples of Taylor pulses and the associated doppler axis ambiguity distributions \(a_{\text{Ty}}(0,v|\bar{n},A)\) are presented in Fig. 11 for an arbitrary choice of parameters \(\bar{n} = 4, R = 20\) and 30 decibels. The general pulse shape is given by (3.9) where the coefficients \(a_m\) are determined following a standard method [30, pp. 16-28].
The ambiguity distribution and the broadening factor $\sigma$ are defined in (3.10) and (3.11).

\[ u_{Ty}(t) = a_0 + 2 \sum_{m=1}^{\infty} a_m \cos(2\pi mt/T) \]  
\[ (3.9) \]

\[ a_{Ty}(0,\nu|\bar{n},A) = \sum_{n=1}^{\bar{n}-1} \frac{\pi}{n} \left( 1 - \frac{(\nu T/\sigma)^2}{A^2 + (n - 1/2)^2} \right) \frac{[(\bar{n} - 1)!]^2}{(\bar{n} - 1 + \nu T)!((\bar{n} - 1 - \nu T)!} \]  
\[ (3.10) \]

\[ \sigma = \frac{\bar{n}}{[(\bar{n} - 1/2)^2 + A^2]^{1/2}} \]  
\[ (3.11) \]

Where \[ A = (1/\pi)(\cosh^{-1}R) \] .  
\[ (3.12) \]

Focusing on the doppler axis distribution (3.10), one may see that a disadvantage of the Taylor distribution is that the sidelobe structure of $a_{Ty}(0,\nu)$ of an individual filter does not decrease rapidly with changes of $\nu$ away from the central peak ($\nu=0$). Instead, there are $\bar{n} - 1$ equal height sidelobes along the doppler frequency axis to each side of the central peak. In order that few real targets escape detection, it is necessary for the doppler filters of Fig. 10 to be spaced in frequency so that their central response peaks overlap significantly. A result is that the sidelobe structure of a filter with Taylor response distribution generally extends for a significant distance along the doppler axis and overlaps the response distributions of several consecutive filters.
Often a very strong target echo will cause several doppler filters to respond simultaneously. Such a situation is depicted in Fig. 12 where a single target echo with doppler frequency \( v = 1/2T \) cps caused significant response of two adjacent filters \( F_0 \) and \( F_\perp \) and a slight response of a third filter \( F_2 \), somewhat removed in center frequency from \( F_0 \) and \( F_\perp \). The response of the third filter is a result of the hypothetical target having a doppler frequency (velocity) coinciding with one of that filter's \( a_T(0,v) \) sidelobe peaks.

The simultaneous response of these filters is ambiguous as it may not be clear to the radar operator whether one or more targets are actually present. Intuitively, one method of minimizing filter responses to targets distant in doppler frequency is to choose a transmitted signal and matched receiver so that \( a(0,v) \) is compressed or concentrated into a central \( v \) region. In following sections we investigate this method of controlling the \( a(0,v) \) distribution sidelobe level and note that the formulation of the problem gives some interesting bounds on how ambiguity may be distributed along the \( v \) axis. By direct analogy, the results hold for developing the proper spectral envelope \( U^2(f) \) which optimally concentrates the axial distribution \( a(\tau,0) \).

D. Axial ambiguity distribution concentration and sidelobe suppression

In order to establish a formal approach to the synthesis of axial distributions, the following simple method of finding distributions with suppressed sidelobes will be examined.
Figure 11. Taylor pulse square-envelope and associated doppler axis ambiguity distribution, $\bar{n} = 4.$
Figure 12. Superposed doppler axis ambiguity distributions for five adjacent Taylor filters and their relative responses to a signal of 0 dbm strength and $v = 1/2T$ cps doppler frequency.
Let $\Omega$ be an arbitrary doppler frequency and $T$ be the duration of the radar pulse to be transmitted. Form the ratio $\beta$

$$\beta = \frac{\int_{-\Omega}^{\Omega} |a(0,v)|^2 dv}{\int_{-\infty}^{\infty} |a(0,v)|^2 dv}.$$  \hfill (3.13)

Thus $\beta$ represents the proportion of ambiguity $|a(\tau,v)|^2$ contained in an incremental slice of width $\Delta \tau$ oriented along the $v(\tau=0)$ axis between the doppler frequencies $\pm \Omega$ as in Fig. 13. Substitution of the Fourier transform (1.31) into (3.13), inversion of the order of integration and completion of the integration on $v$ in both the numerator and denominator terms gives (3.15);

$$\beta = \frac{\int_{-T/2}^{T/2} \int_{-T/2}^{T/2} u^2(t) e^{-j2\pi v t} \frac{1}{2\pi u^2(s) \sin 2\pi \Omega(t-s)} dt ds}{\int_{-T/2}^{T/2} \int_{-T/2}^{T/2} u^2(t) e^{-j2\pi v t} \delta(t-s) dt ds}.$$  \hfill (3.14)

With the substitution $g(t) = u^2(t)$, and $\delta(t-s)$ the delta function

$$\delta(t-s) = \begin{cases} 1 & t = s \\ 0 & \text{otherwise} \end{cases}$$
Figure 13. Doppler axis ambiguity distribution (-Ω, Ω).
\[ \beta = \frac{\int_{-T/2}^{T/2} g(t) g(s) \frac{\sin 2\pi \Omega(t-s)}{\pi(t-s)} \, dt \, ds}{\int_{-T/2}^{T/2} g^2(t) \, dt} \]  

(3.15)

It is possible to minimize the distributed ambiguity in the vicinity of the \( v(t=0) \) axis outside \(|v| < \Omega\) by selecting the proper \( g(t) = u^2(t) \) to maximize \( \beta \). The form of \( \beta \) indicates that it is only the shape of \( g(t) \) which affects the possible values of \( \beta \). The amplitude of \( g(t) \) is unimportant since every constant multiplicative factor of \( g(t) \) may be extracted from the integrals and cancelled. The solutions of (3.15) for \( \beta_{\text{max}} \) depend on a function \( c \) of the constants \( \Omega \) and \( T \). For a particular \( c \), that \( g(t) = g(c,t) = u^2(c,t) \) which maximizes \( \beta \) may be amplitude scaled so as to obtain a constant signalling energy \( E \).

\[ E = \int_{-T/2}^{T/2} |f(c,t)|^2 \, dt = \int_{-T/2}^{T/2} k^2u^2(c,t) \, dt \]  

(3.16)

\[ = k^2 M_T^2(c), \]  

(3.17)

where \( M_T^2(c) = \int_{-T/2}^{T/2} u^2(c,t) \, dt \)  

(3.18)

and \( k = \sqrt{E/M_T(c)} \).  

(3.19)

For the maximized \( \beta \) condition, the transmitted signal is

\[ f(c,t) = \frac{\sqrt{E}}{M_T(c)} u(c,t) e^{j(\omega_0 t + \phi(t))}, \]  

(3.20)

where \( \phi(t) \) is an arbitrary phase modulation yet to be specified and
having no effect on the realized $|a_c(0,v)|$. Property 10 of Chapter I shows that $a_c(0,0) = E$ when the signal energy is adjusted to be $E$.

Regardless of the value of parameter $c$, the shape of $f(c,t)$, or the ambiguity distribution sidelobe level, when the signals are all scaled to have equal energy they will all allow matched filter detection of a given target with equal probability.

The maximizing problem solution

In this and a following section, the mathematical aspects of the $\beta$ maximization problem are further developed. The physical implications of these results are discussed later.

It is fortunate that the $\beta$ maximization problem (3.13) reduces to the form (3.15), for the solution of this problem is well known [33, pp. 54-59] and may be obtained by solving the integral equation (3.21).

$$\lambda g(t) = \int_{-T/2}^{T/2} g(s) \frac{\sin 2\pi \Omega (t-s)}{\pi(t-s)} \, ds , \quad |t| < \frac{T}{2} \quad (3.21)$$

The solution for (3.21) is

$$g(t) = g(c; t) = S_{\infty}^{(1)} (c, \frac{2t}{T}), \quad |t| \leq T/2. \quad (3.22)$$

Take

$$S_{\infty}^{(1)} (c, \frac{2t}{T}) \Delta S_0 (c, \frac{2t}{T})$$

= the angular prolate spheroidal wave function of the first kind, zero order, here abbreviated APSWF.

$$c = \pi \Omega T \quad (\Omega, \text{doppler frequency in cps};$$

$$T, \text{pulse duration in seconds}).$$
There are other solutions of (3.21), namely, $S_{\ell}(c, \frac{2t}{T})$, $\ell = 1, 2, \ldots, n$. None of these solutions result in as large $\beta$ values as does the zero order solution $\beta$. The zero-order integral equation solution (3.22) exists only for eigenvalues $\lambda_0 = \lambda_0(c)$ which as indicated depends on the parameter $c$. Recent interest in the spheroidal wave functions has resulted in some extensions of available tables of the eigenvalues $\lambda_0(c)$, some values of which are given in Table 1. The spheroidal wave functions themselves are not well tabulated. For digital computer calculations the APSWFs may be obtained on the range $|\frac{2t}{T}| \leq 1$ by a series expansion in spherical Bessel functions as shown in Appendix B. Figure 14 displays $u(t) = \sqrt{S_0(c,2t/T)}$ of several $c$ dependent spheroidal wave functions normalized in the manner of Flammer [34, p. 213] so that $S_0(c,0) = 1$, all $c$.

Insert the $\beta_{\text{max}}$ solution (3.22) into (3.15) through (3.20):

$$E = \int_{-T/2}^{T/2} k^2 S_0(c, \frac{2t}{T}) dt = k^2 M_T^2(c), \quad (3.23)$$

where $M_T^2(c) = \int_{-T/2}^{T/2} S_0(c, \frac{2t}{T}) dt$. \quad (3.24)

See Table 1 for values of $M_T^2(c)$.\quad (3.25)

$$k = \sqrt{\frac{E}{M_T(c)}}$$

$$f(c, t) = \frac{\sqrt{E}}{M_T(c)} S_0(c, \frac{2t}{T}) \exp(i\omega_0 t + \phi(t)), |t| \leq \frac{T}{2} \quad (3.26)$$

The arbitrary phase function $\phi(t)$ has no effect on the realization of $|a_c(0, v)|$.\quad (3.26)
A simply obtained but important result is immediate from substitution of (3.21) into (3.15).

\[ \beta_{\text{max}} = \frac{\int_{-T/2}^{T/2} g(t) \int_{-T/2}^{T/2} \frac{g(s) \sin 2\pi \Omega(t-s)}{\pi(t-s)} ds dt}{\int_{-T/2}^{T/2} g^2(t) dt} \]

\[ \beta_{\text{max}} = \frac{\int_{-T/2}^{T/2} \lambda_0(c) g^2(t) dt}{\int_{-T/2}^{T/2} g^2(t) dt} = \lambda_0(c) . \]  

(3.27)

E. Ambiguity functions for prolate spheroidal waveshapes

The complete ambiguity function for the APSWF pulse shapes cannot be obtained without a selection of a specific phase modulation \( \phi(t) \), however, the general result after suppressing the carrier component is

\[ a_0(\tau, \nu) = \frac{E}{M_T^2(c)} \int_{-T/2}^{T/2} \frac{\sqrt{S_0(c, \frac{2t}{T}) \sqrt{S_0(c, \frac{2}{T} (t+\tau))}}}{\nu} e^{j[\phi(t) - \phi(t+\tau)]} e^{-j2\pi \nu \tau} dt . \]  

(3.28)

Figure 15 presents examples of \( |a_0(\tau, \nu)| \) for \( c = 4 \) and \( c = 6 \) and a linear phase term \( \phi(t) = bt \).
TABLE 1

Eigenvalues $\lambda_0(c)$ of the Zero Order Spheroidal Wave Function Solution of the Integral Equation (3.21) and the Normalizing Coefficient $M_T^2(c)/T$

<table>
<thead>
<tr>
<th>c</th>
<th>$\lambda_0(c)$</th>
<th>$M_T^2(c)/T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.30969</td>
<td>0.98637</td>
</tr>
<tr>
<td>1.0</td>
<td>0.57258</td>
<td>0.94837</td>
</tr>
<tr>
<td>2.0</td>
<td>0.88056</td>
<td>0.83162</td>
</tr>
<tr>
<td>3.0</td>
<td>0.97583</td>
<td>0.71480</td>
</tr>
<tr>
<td>4.0</td>
<td>0.99589</td>
<td>0.62537</td>
</tr>
<tr>
<td>5.0</td>
<td>0.99935</td>
<td>0.56032</td>
</tr>
<tr>
<td>6.0</td>
<td>0.99990</td>
<td>0.51164</td>
</tr>
<tr>
<td>7.0</td>
<td>0.99999</td>
<td>0.47370</td>
</tr>
<tr>
<td>8.0</td>
<td>1.00000</td>
<td>0.44311</td>
</tr>
</tbody>
</table>
Figure 14. Spheroidal wave function pulse envelopes

\[ u(t) = \sqrt{S_0} \phi(\alpha, 2t/T) \]

- \( \alpha = 0(2)8 \).
Figure 15. Ambiguity surfaces for prolate spheroidal wave-shaped pulses, linear phase case. Constant doppler frequency profiles spaced at Δv cps.
Of more immediate interest is the doppler axis distribution \( a_c(0, v) \). If we set \( \tau = 0 \) in (3.28),

\[
\begin{align*}
  a_c(0, v) &= \frac{E}{M_T(c)} \int_{-T/2}^{T/2} S_0(c, \frac{2t}{T}) e^{-j2\pi vt} dt. \\
&= \frac{E}{M_T(c)} 2S_0(c, 0) e^{-j2\pi vt} dt. \\
&= \frac{E}{M_T(c)} T^2 2t -j2\pi vt \\
&= \frac{E}{M_T(c)} 2S_0(c, 0) e^{-j2\pi vt} dt. \\
\end{align*}
\]

Equations (3.28) and (3.29) correctly reduce to \( a_c(0, 0) = E \). Slepian and Pollak [33, p. 57] show that the zero order APSWF is Fourier transformable over the interval \( |x| < 1 \) with the result:

\[
2R_{oo}^{(1)}(c, l) S_0(c, z) = \int_{-1}^{1} S_0(c, x) e^{jczx} dx ,
\]

where

\[
R_{oo}^{(1)}(c, l) = \text{radial prolate spheroidal wave function of zero order, first kind, parameter } c, \text{ argument } |x| = 1.
\]

The evenness of \( S_0(c, x) \) allows rewriting (3.30) with a negative exponential so that the integral becomes the usual forward Fourier transform integral. When \( x = 2t/T \), then \( 2cz/T = 2\pi \nu \) and

\[
2R_{oo}^{(1)}(c, l) S_0(c, \frac{\pi \nu T}{c}) = \frac{2}{T} \int_{-T/2}^{T/2} S_0(c, \frac{2t}{T}) e^{-j2\pi \nu t} dt. 
\]

For \( \nu = 0 \),

\[
\begin{align*}
TR_{oo}^{(1)}(c, l) S_0(c, 0) &= \int_{-T/2}^{T/2} S_0(c, \frac{2t}{T}) dt = M_T^2(c) . \\
TR_{oo}^{(1)}(c, l) S_0(c, 0) &= \frac{T/2}{-T/2} S_0(c, \frac{2t}{T}) dt = M_T^2(c) . \\
\end{align*}
\]

Recall that \( S_0(c, 0) = 1 \) by the Flammer normalization. When (3.29), (3.31), and (3.32) are combined and \( c = \pi \Omega T \);

\[
a_c(0, v) = \frac{E}{TR_{oo}^{(1)}(c, l) S_0(c, 0)} [TR_{oo}^{(1)}(c, l)] S_0(c, \frac{\pi \nu T}{c})
\]
It is found that the doppler axis ambiguity distribution (3.33) is given directly by the APSWF, \( S_0(c,x) \). For \(|x| < 1\) the values of the axial distribution are obtainable from the Legendre expansion previously mentioned. In the range \(|x| > 1\), \( S_0(c,x) \) may be found by numerical evaluation of an appropriate modification of (3.29) as well as by the expansion in spherical Bessel functions. Some sample doppler axis ambiguity distributions are shown in Fig. 16. The distributions are even functions of doppler frequency and thus are shown only for positive \( v \) for each of several values of the parameter \( c \).

Reference [34] shows that \( R_{oo}^{(1)}(c,x) \), the radial prolate spheroidal wave function, may be written

\[
S_{oo}^{(1)}(c,x) = k_0(c) R_{oo}^{(1)}(c,x) \quad |x| \geq 1 ,
\]

where \( k_0(c) \) is a \( c \)-dependent constant referred to as a joining factor.

An interesting asymptotic form of \( R_{oo}^{(1)}(c,x) \) is [34, p. 31]

\[
R_{oo}^{(1)}(c,x) \approx \frac{1}{cx} \cos \left( cx - \frac{\pi}{2} \right) = \frac{\sin cx}{cx}
\]

so that

\[
S_0(c,x) \approx k_0(c) \frac{\sin cx}{cx} \quad \text{as } x \to \infty .
\]
It is not surprising that the asymptotic form $a_c(0,v) \approx k_c(c) \frac{\sin \frac{\pi v}{\Omega}}{\frac{\pi v}{\Omega}}$ as $v \to \infty$ exhibits a definite sidelobe structure which falls off at -20db/frequency decade. Erdelyi's asymptotic expansion of the Fourier integral (3.8) combined with the known fact that $S_0(c,x)$ possesses end-point discontinuities at $x = \pm 1$ predicts an identical result.

Properties of $|a_c(0,v)|$

Calculation of a number of optimal ambiguity distributions like those displayed in Fig. 16 enables one to determine the maximum sidelobe levels of $|a_c(0,v)|$ as a function of $c = \pi \Omega T$. For small values of $c$, Fig. 14 shows that the APSWF pulse envelope is nearly rectangular. For that case the sidelobe level is roughly -13.2 db as is the sidelobe level of the axial ambiguity distribution $|a_R(0,v)|$ associated with an exactly rectangular pulse. As the APSWF pulse shape changes with increasing $c$, the concentration $\beta = \lambda_0(c)$ of $|a_c(0,v)|$ increases as is shown in Figs. 17 and 18. The sharply decreasing sidelobe level $R(c)$ of $|a_c(0,v)|$ for increasing $c$ is also demonstrated in Fig. 17 and is the result of the monotonically increasing concentration of ambiguity residing in the $\Delta t$ wide strip along the $\nu$ axis between doppler frequencies $\pm \Omega = c/\pi T$.

The sidelobes of $|a_c(0,v)| = E S_0(c,\frac{\nu}{\Omega})$ are always outside the $\pm \Omega$ region since $S_0(c,x)$ has no zeroes in $|x| \leq 1$ for any $c$. For this reason increasing $c$, and thus $\lambda_0(c)$, means that the sidelobe structure outside $\pm \Omega$ is suppressed while nearly the complete main lobe of $|a_c(0,v)|$ is contained in $\pm \Omega$ as can be noted from the examples of $c = 2, 4,$ and 6 in Fig. 16. Figure 18 contains a more easily read scale by which to examine
the manner in which $\lambda_o(c)$ approaches unity. A very complete study of asymptotic forms of the spheroidal wave functions and of $\lambda_o(c)$ have recently been published by Slepian [36, 37].

In Figs. 19 and 20, $\lambda_o(c)$ has been compared with the ambiguity concentration $\beta(\Omega)$ given in (3.37) for the previously discussed Taylor pulse and the truncated Gaussian pulse of (3.38).

\[
\beta(\Omega) = \frac{\int_{-\Omega}^{\Omega} |a(0,\nu)|^2 d\nu}{\int_{-\infty}^{\infty} |a(0,\nu)|^2 d\nu} (3.37)
\]

It is evident that $\lambda_o(c)$ is an upper bound for the $\beta(\Omega)$ curves for any Gaussian, Taylor, or other pulse of duration $T$. For example, taking $\Omega = \Omega_1$ form $c_1 = \pi \Omega_1 T$. The associated APSWF pulse of envelope

\[
u_G^2(k,t) = e^{-k(\frac{2t}{T})^2} |\frac{2t}{T}| \leq 1 (3.38)
\]

gives the axial ambiguity distribution

\[
a_o(0,\nu|c_1) = E S_o(c_1, \frac{\nu}{\Omega_1}) . (3.40)
\]

As has been shown, $\beta_o(\Omega_1) = \lambda_o(c_1) > \beta(\Omega_1)$ for any other ambiguity distribution and thus for any other pulse shape. Since $\Omega_1$ was chosen arbitrarily, $\lambda_o(c)$ is everywhere an upper bound to all ambiguity concentrations $\beta(\Omega)$. 
Figure 16. The normalized doppler axis ambiguity distribution $|a_c(0,v)|$ for APSWF pulses of parameter $c$, $v > 0$. 
Figure 17. Doppler axis ambiguity function characteristics of the APSWF amplitude modulated pulses.

Figure 18. $\lambda_0(c)$. 
Figure 19. Doppler axis ambiguity concentration for Taylor pulses, n=4.

Figure 20. Doppler axis ambiguity concentration for truncated Gaussian pulses.
Figure 21. Doppler axis ambiguity concentration for the APSWF pulse shapes.
for pulses of duration $T$. Thus $\lambda_0(c)$ does not represent one $\beta(\Omega)$ curve for a specific pulse shape-ambiguity distribution pair. Rather, the APSWF pulse shapes change with changing $c$ and $\lambda_0(c = \pi \Omega T)$ represents the largest possible value of $\beta(\Omega)$ for a particular doppler frequency realizable by any pulse. To further illustrate this point, Fig. 21 shows $\lambda_0(c)$ and the doppler axis ambiguity concentrations $\beta_c(\Omega)$ corresponding to APSWF pulses of parameter $c = 2$ and $c = 4$.

Ambiguity distribution along the doppler axis

In the preceding sections we have been concerned with sidelobe levels and concentrations of ambiguity as measured by the $\beta(\Omega)$ criterion for various pulses of duration $T$. In particular the APSWF pulses of parameter $c$ have been shown to have optimally concentrated doppler axis ambiguity distributions. It is true that the total ambiguity volume under any surface $\int_{-\infty}^{\infty} |a(\tau,\nu)|^2 d\tau d\nu = \mathbb{E}^2$ for all signals of equal energy, however, that portion of the ambiguity volume contained in the $\Delta \tau$ width strip on $-\Omega \leq \nu \leq \Omega$ (Fig. 13) is dependent on pulse shape as well as energy. By increasing the concentration $\beta_c(\Omega) = \lambda_0(c)$ for APSWF pulses, it is not clear whether the ambiguity volume in the $\Delta \tau$ width strip increases, decreases, or is constant. Let this volume be $V_{\Omega\Delta\tau}(c)$:

$$V_{\Omega\Delta\tau}(c) = \int_{-\Delta\tau/2}^{\Delta\tau/2} \int_{-\Omega}^{\Omega} |a_c(\tau,\nu)|^2 d\nu d\tau .$$

For an infinitesimally small $\Delta \tau$ width strip centered on $\tau = 0$, one may as well study the distribution $I_\Omega(c)$. 

$$\lambda_0(c)$$
\[ I_\Omega(c) = \int_{-\Omega}^{\Omega} |a_c(0,v)|^2 dv = \lambda_\Omega(c) \int_{-\infty}^{\infty} |a_c(0,v)|^2 dv \]

\[ = \frac{2c}{\pi} \left[ R_{oo}(c_1,1) \right]^2 \int_{-T/2}^{T/2} \frac{E^2 u^4(c,t)}{M_4^4(c)} dt \]

\[ = \frac{2c}{\pi} \left[ R_{oo}(c_1,1) \right]^2 \frac{2E^2}{M_4^4(c)} \frac{T}{2} \int_{0}^{1} [S_o(c,x)]^2 dx \]

\[ = \frac{2c}{\pi} \left[ R_{oo}(c_1,1) \right]^2 \frac{TE^2}{[T R_{oo}(1,c_1,1)]^2} \int_{0}^{1} [S_o(c,x)]^2 dx \]

\[ = 2\Omega E^2 \int_{0}^{1} [S_o(c,x)]^2 dx \]  

(3.42)

But

\[ \int_{0}^{1} [S_o(c,x)]^2 dx = \frac{1}{2} \int_{-1}^{1} [S_o(c,x)]^2 dx \]

\[ = \frac{1}{2} \sum_{r=0}^{\infty} \sum_{q=0}^{\infty} d_r(c) d_q(c) \int_{-1}^{1} P_r(x) P_q(x) dx \]

\[ = \frac{1}{2} \sum_{r=0}^{\infty} \sum_{q=0}^{\infty} d_r^2(c) \left( \frac{2}{2r+1} \right), \]  

(3.43)

where advantage has been taken of the Legendre expansion of the spheroidal wave functions and the orthogonality of the Legendre polynomials has been used to obtain (3.43). See Appendix B.

\[ \frac{I_\Omega(c)}{2\Omega E^2} = \frac{1}{2} \sum_{r=0}^{\infty} d_r^2(c) \left( \frac{2}{2r+1} \right) \]  

(3.44)

\[ \text{even} \]
As shown in Fig. 22, $I_\Omega(c)/2\Omega E^2$ apparently decreases monotonically with increasing $c$. Since $\lambda_0(c)$ monotonically increases with $c$, the total doppler axis ambiguity $\int_{-\infty}^{\infty} |a_c(0,\nu)|^2 d\nu$ decreases faster than $\lambda_0(c)$ increases. Figure 22 implies that for a fixed $\pm \Omega$, when pulse duration $T$ is increased ($c = \pi \Omega T$ increases), the ambiguity surface is stretched out along the $t$ direction in such a manner that the narrow cross-sectional volume centered on $t = 0$ (Fig. 13) shrinks so that the constraint $\int \int |a(t,\nu)|^2 dt d\nu = E^2$ is satisfied. It is next shown that $I_\Omega(c)/2\Omega E^2$ is asymptotic to $c^{-1/2}$ for large $c$.

Doppler axis ambiguity distributions for large $c$

For any doppler axis region $|\nu| \leq \Omega$, however small, when the duration of the APSWF pulse is made sufficiently large, the averaged ambiguity content of the $\Delta t$ wide strip may be made arbitrarily small. That is, for a positive $\epsilon$ however small,

$$\lim_{c \to \infty} I_\Omega(c)/2\Omega E^2 \leq \epsilon . \quad (3.45)$$

Slepian [36, p. 102] has shown that for the function $\psi_\infty(c,x) = C_\infty(c) S_\infty(c,x)$,

$$\frac{1}{2} = \int_{-1}^{1} [\psi_\infty(c,x)]^2 dx , \quad (3.46)$$

where both $C_\infty(c)$ and $N_\infty(c)$ are $c$ dependent parameters. Thus

$$\frac{1}{C_\infty^2 N_\infty^2} = \int_{-1}^{1} [S_\infty(c,x)]^2 dx . \quad (3.47)$$
Figure 22. Averaged ambiguity content of $|v| \leq \Omega$ doppler region for APSWF ambiguity distributions.
Slepian's work gives

\[ \frac{1}{N_\infty^2 (c)} = \pi \frac{1}{c} \left[ 1 + \frac{1}{4c^2} \left( \frac{12}{7} \right) + \ldots \right] \]  

(3.48)

and

\[ C_\infty^\infty (c) = \sum_{j=0}^{\infty} \left( \frac{1}{2c} \right)^j \sum_{k=-2j}^2 (-1)^k A_k^j \frac{(2k)!}{k! \ 2^k} \]  

(3.49)

The \( A_k^j \) coefficients may be simply evaluated \([36, \text{p. 101}]\) giving the result:

\[ C_\infty^\infty (c) = 1 - 0.18750 \left( \frac{1}{2c} \right) - 0.26367 + \left( \frac{1}{2c} \right)^2 - 0.34680 + \left( \frac{1}{2c} \right)^3 + \ldots \]  

(3.50)

If we combine the results (3.47) - (3.50):

\[ \lim_{c \to \infty} \int_0^1 [S_o(c,x)]^2 dx = \frac{1}{2} \lim_{c \to \infty} \int_0^1 [S_o(c,x)]^2 dx \]

\[ \lim_{c \to \infty} \frac{I_\Omega (c)}{2\Omega E^2} = \frac{1}{2} \sqrt{\frac{\pi}{c}} = \frac{1}{2} \sqrt{\frac{1}{\Omega T}} \]  

(3.51)

The result for \( c = 8 \) in (3.51) gives \( I_\Omega /2\Omega E^2 = 0.3113 \) while direct integration of \( \int_0^1 [S_o(c,x)]^2 dx \) yields the value 0.3218. At \( c = 10, (3.51) \) gives \( I_\Omega /2\Omega E^2 = 0.2803 \) whereas use of all the available terms of (3.48) and (3.50) results in the value 0.2860. Thus the convergence to (3.51) is quite fast.

By increasing the parameter \( c \) and maintaining the pulse energy \( E = a_c (0,0) \) independently of \( c \), the APSWF amplitude modulated pulses of duration \( T \) have a doppler axis ambiguity distribution that takes on the appearance of a delta function with first zeroes just outside \( \pm v=\Omega = c/\pi T \). The doppler ambiguity outside these limits is obtained in the next few
equations. For large $c$ it is asymptotic to the value given in (3.52) and demonstrates the rapidity with which $\int |a_c(0,v)|^2 d\nu, |\nu| > \Omega$, tends to zero.

$$\int_{|\nu|>\Omega} |a_c(0,v)|^2 d\nu = \int_{-\infty}^{\infty} |a_c(0,v)|^2 d\nu - \int_{-\Omega}^{\Omega} |a_c(0,v)|^2 d\nu$$

$$= (1 - \lambda_0(c)) \int_{-\infty}^{\infty} |a_c(0,v)|^2 d\nu$$

$$\int_{|\nu|>\Omega} |a_c(0,v)|^2 d\nu = 1 - \lambda_0(c)$$

For large $c$,

$$\frac{\int_{|\nu|>\Omega} |a_c(0,v)|^2 d\nu}{\int_{-\infty}^{\infty} |a_c(0,v)|^2 d\nu} \approx \frac{4\sqrt{\pi c}}{\Delta \tau} e^{-2c(1 - \frac{3}{32c})}, \quad (3.52)$$

where a result quoted by Slepian [37, p. 1746] and others has been applied to $1 - \lambda_0(c)$.

The averaged ambiguity volume over the $\Delta\tau \times 2\Omega$ area of interest was found to vary as $1/ \sqrt{\Delta\tau}$. It is interesting to apply the definition (3.41) to measure the ambiguity volume within the same $\Delta\tau \times 2\Omega$ area under the surface $|a_R(\tau,v)|$ corresponding to a transmitted rectangular pulse.

$$V_{\Omega\Delta\tau}(R) = \frac{\Delta \tau}{2} \int_{-\Delta\tau}^{\Delta\tau} \left[ \int_{-\Omega}^{\Omega} |a_R(\tau,v)|^2 d\nu \right] d\tau \quad (3.53)$$
\[ I_\Omega(R) = \int_{-\Omega}^{\Omega} |a_R(0,\nu)|^2 \, d\nu \]

\[ = E^2 \int_{-\Omega}^{\Omega} \left| \frac{\sin \pi \nu T}{\pi \nu T} \right|^2 \, d\nu \]

\[ \frac{I_\Omega(R)}{2\Omega E^2} = \frac{1}{\pi \Omega T} \left[ \text{Si}(2\pi \Omega T) - \frac{\sin^2 \pi \Omega T}{\pi \Omega T} \right] \quad (3.54) \]

\[ \text{Si}(x) = \text{the sine integral of argument } x. \]

For large \( \Omega T \), or fixed \( \Omega \) and large \( T \), (3.54) is asymptotic to

\[ \frac{I_\Omega(R)}{2\Omega E^2} = \frac{1}{\pi \Omega T} \left( \frac{\pi}{2} \right) = \frac{1}{2 \Omega T}, \quad (3.55) \]

and thus varies as \( 1/T \). If the individual pulse energies \( E \) are kept constant, we find that the averaged ambiguity content of the \( \Delta \tau \times 2\Omega \) area varies as \( 1/T \) for rectangular pulses and as \( 1/\sqrt{T} \) for APSWF amplitude modulated pulses. Increasing pulse duration causes the doppler axis slice of the central peak of \( a_R(\tau, \nu) \) to collapse more rapidly than does that of the ambiguity surface associated with APSWF pulses.

Central peak broadening of \( |a_c(0, \nu)| \)

As previously noted in Fig. 14, for small \( c \) the spheroidal wave functions are near rectangular and have associated doppler axis ambiguity distributions tending toward \( E \frac{\sin \pi \nu T}{\pi \nu T} \) as \( c \to 0 \). The -3 db width \( (\Delta \nu)_{HP} \) of the central peak of these distributions is \( 2.78/\pi T \) while the sidelobe level is \(-13.2\) decibels. One finds in Fig. 23 these values as the coordinates of the left end of the curve describing the sidelobe.
levels of $|a_c(0,v)|$ versus $\pi(\Delta v)_{HF}$. In the same figure is plotted the sidelobe level vs. central peak -3 dB width for the doppler axis ambiguity distributions associated with pulse shapes analogous to the Van Der Maas extension of the Dolph-Chebyshev, and Taylor ($\overline{n} = 4, 6$) aperture current distributions. Recall that the Dolph-Chebyshev extension offers an interesting theoretical standard for minimum $|a(0,v)|$ response peak width versus sidelobe level, ($A - 12$); but that there is no realizable, continuous, finite duration associated Dolph-Chebyshev pulse shape. The curves for the Taylor cases were obtained by applying the known broadening factors $\sigma$ [30] to the known Dolph-Chebyshev distribution width. The Taylor case $\overline{n} = 4$ may not be used to realize levels below -40 dB. The APSWF pulses realize any sidelobe level below -13.2 dB.

In Fig. 24, the broadening of the APSWF central doppler peak, $\sigma_c$, measured with respect to the Dolph-Chebyshev central peak width is compared to like broadening factors $\sigma_T$ for Taylor ambiguity distributions of the same sidelobe level. At the outset one notes that through requiring the concentration property by maximizing $\beta(\Omega) = \lambda_0(c)$ for the APSWF pulses, with a single exception, a penalty is paid in terms of a significantly broader -3 dB width of $|a_c(0,v)|$ as compared to that of the Taylor realizations $|a_T(0,v|\overline{n})|$ of the same sidelobe level. The penalty is greater for larger $\overline{n}$.

For $c < 2.2$ the APSWF distribution realizes the same sidelobe levels as realizable with an $\overline{n} = 2$ Taylor distribution, and does so with less broadening of the -3 dB central peak width. This result does not contradict Taylor's optimum approximation to the Dolph-Chebyshev distribu-
Figure 23. Central peak half-power widths of several doppler axis ambiguity distributions.
Figure 24. Central peak broadening factors of $|a_{Ty}(0,v\mid \overline{n})|$ and $|a_c(0,v)|$. 
tion; however, for that result is premised on forcing the field (in this case \( a_{Ty}(0,v|-n=2)| \) to realize zeroes \( z_n \) at integer multiples of \( 1/T \) starting at \( z_2 = 2/T \). See Fig. 11 for an example for \( n = 4 \). As no special positioning of the APSWF distribution zeroes has been required, for \( c < 2.2 \) the zeroes are apparently positioned along the \( v \) axis in such a way as to realize a better combination of low sidelobe levels and central peak width than can the \( n = 2 \) Taylor solution.

F. Summary

This chapter has applied the Fourier transform relation
\[
u(t) \Leftrightarrow a(0,v)
\]
to investigate the manner in which the \( \tau = 0 \) sampled response of a matched filter may vary when the filter input is a doppler shifted version of the expected signal.

It has been shown that by proper amplitude modulation of the \( T \) duration transmitted signal (assuming white noise and a matched receiver) the associated ambiguity function shape may be controlled so as to minimize the matched filter response to signals of doppler shifts greater than \( \pm \Omega \). The ratio \( \beta(\Omega) \) defined in (3.14) is used to measure this minimum and it is defined as the "concentration" of \( |a(0,v)| \). The larger the value of \( \beta \), the smaller the integral \( \int_{|v|>\Omega} |a(0,v)|^2 dv \). An important result is that \( \beta \) takes on monotonically increasing values as the \( \Omega T \) product increases. For a fixed \( \Omega \), increased concentration of \( |a(0,v)| \) is obtainable only by increasing the transmitted pulse duration and changing the shape so that for a particular \( c = \pi \Omega T, u^2(t) \propto S_o(c, \frac{2\pi}{T}) \). The sidelobe levels of \( |a_c(0,v)| \) for the APSWF pulses are direct functions of the parameter \( c \) as is the broadening of the central region of \( |a_c(0,v)| \).
measured with respect to the reference ambiguity surface constructed from Dolph-Chebyshev pulses. The broadening of $|a_c(0,v)|$ is slightly greater than that for the Taylor associated distribution $|a_T(0,v)|$; however, the APSWF sidelobe behavior (monotonically decreasing) and concentration are desirable features obtained in trade.

The various properties of the optimally concentrated $|a_c(0,v)|$ are all independent of any frequency or phase modulation imposed on the transmitted pulse. In the next chapter, the transform $U^2(\omega) \mathcal{F} a(\tau,0)$, and a time delay domain analog of the $|a_c(0,v)|$ concentration problem is studied and in a practical realization is shown to lead straight-forwardly to a variation of well known radar pulse compression signalling schemes.
CHAPTER IV

CONTROL OF RADAR TARGET RANGE AMBIGUITIES

In Chapters I and II the properties of the ambiguity function characterization of the output of a matched filter were examined for the case where the input is a time-delayed, doppler-shifted replica of the expected waveform. From (1.13) it was shown that when the matched filter output was sampled at the time \( t_0 \), the set of all possible sample values formed a continuous surface over a two-dimensional \( \tau-v \) plane.

When a delayed, doppler shifted signal arrives, the real time recording of the filter response (1.11) has the same form as a cut through the ambiguity surface parallel to the delay axis but offset by a fixed doppler frequency. It is assumed that the doppler frequency shift is essentially constant during the length of one pulse. Figure 2 presents this idea. For a fixed \( v \), when \( t \) takes values \(-\infty < t < \infty\) in (4.1) the continuous signal at the matched filter output \( f(t, \tau, v) \) is

\[
g(t, \tau, v) = e^{-j[(\omega_0 + \Omega)v](t_0 - t + \tau)} \int_{-\infty}^{\infty} f(y) f^{*}(y + \tau + t_0 - t)e^{-j2\pi vy} dy , \quad (4.1)
\]

where

\[
a(\tau, v) = g(t_0, \tau, v) . \quad (4.2)
\]

The frequency domain representation of \( g(t) \) is easily obtained from (4.1) by substitution of the Fourier transform \( F(\omega) \) for \( f(y) \),

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\[ g(t,\tau,0) = \int_{-\infty}^{\infty} F(\omega) F^*(\omega) e^{i\omega(t+t_0-\tau)} \frac{d\omega}{2\pi} \]  

\[ g(t,\tau,0) = \int_{-\infty}^{\infty} U^2(\tau) e^{-j2\pi f(t+t_0-\tau)} df \]  

which is an obvious extension of (1.14) with \( v = 0 \).

It has been shown that the shape of a cut through the ambiguity surface (constant \( v \)) is equivalent to the real time response of a matched filter to its expected signal. Thus the study of the problem of controlling or optimally shaping the filter response is equivalent to the signal design problem for proper ambiguity function shaping.

In this discussion it is proposed to apply some ideas developed in Chapter III to the suppression of range (delay) sidelobes of ambiguity functions. The resulting signals have the property of suppressing the real-time sidelobes of the equivalent matched filter output.

A. Concentration of the delay axis ambiguity distribution

The ambiguity surface associated with time-limited pulses \( f(t) \), \( |t| < T/2 \), is also of limited extent, \( |\tau| < T \), in the delay axis direction but of infinite extent along the doppler axis. Following the general method of the work of Chapter III one may attempt to suppress delay axis sidelobes of \( a(\tau,0) \) by concentrating as much as possible of the ambiguity volume into a central \( \tau \) region. As before, one seeks to maximize the ratio \( \Gamma \)

\[ \Gamma(L,T) = \frac{\int_{-L}^{L} |a(\tau,0)|^2 d\tau}{\int_{-T}^{T} |a(\tau,0)|^2 d\tau} \]  

for \( L \ll T \) .  

(4.5)
By substitution from (1.32) of the Fourier transform of \( a(\tau,0) \) into (4.5) we find:

\[
\Gamma(L,T) = \frac{\int_{-\infty}^{\infty} U^2(f) U^2(g) \int_{-L}^{L} e^{-j2\pi(f-g)} \, d\tau \, df \, dg}{\int_{-\infty}^{\infty} U^2(f) U^2(g) \int_{-T}^{T} e^{-j2\pi(f-g)} \, d\tau \, df \, dg}, \quad (4.6)
\]

\[
\Gamma(L,T) = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U^2(f) U^2(g) \frac{\sin 2\pi L(f-g)}{\pi(f-g)} \, df \, dg}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U^2(f) U^2(g) \frac{\sin 2\pi T(f-g)}{\pi(f-g)} \, df \, dg}, \quad (4.7)
\]

It does not appear feasible to find a class of \( f(t) \) whose Fourier transform square magnitudes \( |F(f)|^2 = U^2(f) \) maximizes \( \Gamma(L,T) \). Note that as posed, this problem differs from the problem of maximization of \( \beta(\Omega), (3.15) \), by the presence of the denominator kernel \( \frac{\sin 2\pi T(f-g)}{\pi(f-g)} \) and by the infinite limits on the integrals involved.

It is quite easy to see that (4.7) becomes a symmetric case with respect to (3.15) if we consider bandlimited \( U^2(f) \) and \( T \gg L \). In effect, one is no longer considering strictly time-limited transmitted pulses but only those \( f(t) \) that are square integrable. In general, the auto-correlation \( a(\tau,0) \) is not now time-limited but one may as before attempt to concentrate a significant portion of the matched filter response in the region \((-L,L)\).

Rewrite (4.5) and (4.7) accordingly,

\[
\Gamma(L) = \frac{\int_{-L}^{L} |a(\tau,0)|^2 \, d\tau}{\int_{-\infty}^{\infty} |a(\tau,0)|^2 \, d\tau}
\]
\[
\int_{-W/2}^{W/2} \int_{-W/2}^{W/2} U^2(f) U^2(g) \frac{\sin 2\pi L(f-g)}{\pi(f-g)} \, df \, dg
\]

(4.8)

\[
\int_{-W/2}^{W/2} \int_{-W/2}^{W/2} U^2(f) U^2(g) \delta(f-g) \, df \, dg
\]

(4.9)

\[\Gamma(L) = \int_{-W/2}^{W/2} S^2(f) \, df\]

From the results of Chapter III one can write down the correct form of \( S(f) \) to maximize \( \Gamma(L) \) as the solution to the integral equation

\[\lambda S(f) = \int_{-W/2}^{W/2} S(g) \frac{\sin 2\pi L(f-g)}{\pi(f-g)} \, dg\]

(4.10)

\[U^2(f) = k^2 S_0 (D = \pi LW, \frac{2f}{W}, |f| < \frac{W}{2}\]

(4.11)

which is \[S(f) = \]

0 elsewhere.

As before \[\lambda = \lambda_0 (D = \pi LW),\]

(4.12)

and \[k^2 = \text{constant, adjusted for a proper signal energy} \, E.\]

By analogy with (3.27), for a chosen \( L \) and \( W \)

\[\Gamma(L)_{\text{max}} = \lambda_0 (D = \pi LW),\]

(4.13)
which for convenience is plotted in Fig. 25. The concentration of $a_D(t,0)$ into its central region $(-L, L)$ is clearly a monotonically increasing function of the $LW$ product.

The general ambiguity function of the bandlimited APSWF spectrum can be written in terms of a yet unspecified spectral phase $\theta(f)$.

$$a_D(t,v) = \int_{-W/2}^{W/2} U(f) e^{-j\theta(f)} U(f+v) e^{j\theta(f+v)} e^{-j2\pi ft} df$$

$$a_D(t,v) = k^2 \int_{-W/2}^{W/2} \sqrt{S_0(D, \frac{2f}{W})} \sqrt{S_0(D, \frac{2(f+v)}{W})} e^{j[\theta(f+v)-\theta(f)]}$$

$$e^{-j2\pi ft} df$$ (4.14)

Since $$a_D(0,0) = E = k^2 \int_{-W/2}^{W/2} S_0(D, \frac{2f}{W}) df$$ (4.15)

$$= k^2 \frac{E}{W}$$

$$k = \sqrt{E/M_0(D)}.$$ (4.16)

Values of $M_0^2(D)/W$ are obtainable from Table 1 by using (4.17).

$$M_0^2(D)/W = \frac{M_T^2(c)/T}{T = W, c = D}$$ (4.17)

Finally,

$$a_D(t,v) = E \frac{W/2}{M_0^2(D)} \int_{-W/2}^{W/2} \sqrt{S_0(D, \frac{2f}{W})} \sqrt{S_0(D, \frac{2(f+v)}{W})}$$

$$e^{j[\theta(f+v)-\theta(f)-2\pi ft]} df.$$ (4.18)
When \( v = 0 \) we find \( a_D(\tau, 0) \) is independent of \( \theta(f) \). In the manner of (3.30) to (3.33), by the change of variable \( 2f/W = x \),

\[
a_D(\tau, 0) = \frac{EW}{2M_W^2(D)} \frac{1}{-1} S_O(D, x) e^{-jD(\frac{\tau}{L})x} \text{dx} \quad (4.19)
\]

\[
= E S_O(D, \frac{\tau}{L}). \quad (4.20)
\]

Figure 26 illustrates the form of \( a_D(\tau, 0) \) for several values of \( D \) and sidelobe levels \( R(D) \) are displayed in Fig. 25 as a function of the \( LW \) product. By choosing appropriately large values of the \( LW \) product, the sidelobes of \( a_D(\tau, 0) \) in theory may be suppressed to any desired level while simultaneously nearly 100% of the energy of \( a_D(\tau, 0) \) will lie within the region \( |\tau| \leq L \).

A simple signal which realizes \( U(f) = \sqrt{S_O(D, 2f/W)} \), \( |f| \leq \frac{W}{2} \), is one for which \( \theta(f) \) is taken as a linear phase term \( \theta_2(f) = 2\pi t_0 f \).

Then

\[
f_2(t) = k \int_{-W/2}^{W/2} \sqrt{S_O(D, 2f/W)} e^{j2\pi f(t-t_0)} \text{df} \]

\[
f_2(t) = \frac{kW}{2} \int_{-1}^{1} \sqrt{S_O(D, x)} e^{jD(t/L)x} \text{dx}. \quad (4.21)
\]

Figure 27 illustrates the shape of the normalized non-time-limited pulses defined in (4.21) for various \( D = \pi LW \). The parameter \( t_0 \) has been set equal to zero with no loss of generality. The ambiguity surface generated by such signals is of the same shape as that of Fig. 15 modified by interchanging the doppler and delay axis symbols and taking \( D = c \).
Figure 25. The delay axis ambiguity characteristics for pulses with ideal spectra $\sqrt{S_0(D,2f/W)}$. 

Sidelobe level - $R(D)$ (db) 

$D = \pi LN$
Figure 26. The normalized delay axis ambiguity distribution $|a_D(\tau,0)|$ for APSWF frequency modulated pulses of parameter $D$, $\tau \geq 0$. 
Figure 27. The normalized non-time-limited AM pulse $f_0(t)$, (4.21), whose Fourier spectrum is the bandlimited APSWF distribution $\sqrt{S_0}(D,2f/W)$. 
In Chapter III, development of optimally concentrated distributions, $a_c(0,v)$, gave time-limited pulses as the natural result. The foregoing treatment of the $a_d(\tau,0)$ concentration problem requires a bandlimited pulse which is incompatible with any requirements of exactly time-limited radar pulses. The following section discusses an interesting method due to Campbell et al. [31] which is used to find an appropriately shaped amplitude modulated time-limited pulse whose spectrum approximates the desired spectral envelope $\sqrt{S_o(D,2f/W)}$, $|f| < W/2$.

B. Spectrum approximation by time-limited pulses

An interesting method of constructing time-limited pulses with specified spectrums has been developed by Hofstetter [38] and elaborated on by Campbell et al. [31]. After a brief explanation of the fundamentals, the method is applied to obtain pulses whose spectra approximate the bandlimited APSWF shape for $D = 4$ and $D = 6$. The ambiguity surface for each case is obtained and the delay axis distribution concentration and sidelobe levels are compared to the exact values given in Fig. 25.

Throughout this discussion we will find it convenient to use radian frequency $\omega = 2\pi f$.

Hofstetter-Campbell method for pulse construction

If $F(z)$ is the complex Fourier transform of $f(t)$,

$$F(z) = \int_{-\infty}^{\infty} f(t) e^{-j(\omega+j\sigma)t} \, dt, \quad z = \omega+j\sigma .$$

(4.22)
Paley and Wiener [39, pp. 12-13] have shown that \( f(t) \) is of limited duration \(|t| < T/2\) if and only if

a) \( F(z) \) is an entire (analytic) function of the exponential type, such that \( |F(z)| < e^{\frac{T}{2}} |z| \), and

b) \( F(z = \omega) \) is absolutely square integrable on the whole \( \omega \) axis.

Hofstetter [38, p. 120] has shown that if \( f(t) \) is time-limited and if all its energy spectrum zeroes occur on the real \( \omega \) axis, it is unique in the sense that no other time-limited pulse possesses the same energy spectrum and autocorrelation function. By certain operations on the signal, one may remove or shift the position of a finite or denumerably infinite number of these zeroes in order to approximate a desired energy spectrum. In every case, the resulting spectrum \( G(\omega) \) corresponds to a duration limited signal \( g(t) \) if the density \( \frac{n(r)}{r} \) of its energy spectrum zeroes \( n(r) \) in a \( z \)-plane circle of radius \( r \) satisfies condition (c);

\[
\lim_{r \to \infty} \frac{n(r)}{r} = \frac{T}{\pi}
\]  

(4.23)

as shown by Levin [40, p. 251].

For real \( f(t) \), the real \( \omega \) axis zeroes of the energy spectrum \( E(\omega) \) are of order two since \( E(\omega) = F(\omega) F^*(\omega) \). Shifting or deleting \( F(\omega) \) zeroes is equivalent to energy spectrum zero manipulation and preserves the duration limited character of the modified \( f(t) \) when conditions (a), (b), and (c) are satisfied.
As an example, we will apply this method to the problem of constructing a time-limited pulse \( g(t) \), \( |t| \leq T/2 \), whose amplitude spectrum \( G(\omega) \) approximates \( U(\omega) = \sqrt{S_0(\omega)} \), \( |\omega| \leq \pi \). The purpose in this choice is to find time-limited pulses possessing optimally concentrated autocorrelation functions as previously explained. The approximation will be constructed by shifting or removing real \( \omega \)-axis zeroes of the amplitude spectrum of some known time-limited signal. A convenient signal is

\[
f(t) = \begin{cases} 
1 & |t| \leq T/2 \\
0 & \text{elsewhere}, 
\end{cases}
\]

with the Fourier transform

\[
F(\omega) = \frac{T \sin \omega T/2}{\omega T/2}.
\]

Let

\[
G(\omega) = \frac{F(\omega) N(\omega)}{D(\omega)} = U(\omega),
\]

with \( N(\omega)/D(\omega) \) the quotient of zero shifting and deleting frequency terms to be determined. In order to continue the example, some arbitrary but convenient values must be chosen; say \( T = 1 \) and \( \pi \omega = 16 \).

For \( D = 4 \), \( U_4 = \sqrt{S_0(4,\omega/16)} = \frac{N_4(\omega)}{D_4(\omega)} \frac{\sin \omega/2}{\omega/2} \) \( |\omega| \leq 16 \).

\[
(4.27)
\]

For \( D = 6 \), \( U_6 = \sqrt{S_0(6,\omega/16)} = \frac{N_6(\omega)}{D_6(\omega)} \frac{\sin \omega/2}{\omega/2} \) \( |\omega| \leq 16 \).

\[
(4.28)
\]

From Fig. 28, one can see that it will be necessary to shift the \( \pm 2\pi \) and \( \pm 4\pi \) zeroes of \( \frac{\sin \omega/2}{\omega/2} \) if a \( G(\omega) \) is to be fit to either \( U_4(\omega) \) or
\( U_6(\omega) \). In both cases, take

\[
D_4(\omega) = D_6(\omega) = (\omega-2\pi)(\omega+2\pi)(\omega-4\pi)(\omega+4\pi)
= (\omega^2 - (2\pi)^2)(\omega^2 - (4\pi)^2).
\]

(4.29)

To complete the approximation, \( N(\omega) \) may be chosen as an even powered polynomial in \( \omega \) of degree equal to or less than that of \( D(\omega) \).

\[
N_4(\omega) = \frac{D_4(\omega)}{\sin \omega/2} \sqrt{S_0(4,\omega/16)}
\quad (4.30)
\]

\[
N_6(\omega) = \frac{D_6(\omega)}{\sin \omega/2} \sqrt{S_0(6,\omega/16)}
\quad (4.31)
\]

Pick \( N_4(\omega) = a\omega^4 + b\omega^2 + c \) and by successive substitutions of \( \omega = 0, 8, 16 \) one obtains

\[
c = (2\pi)^2 (4\pi)^2
\]

\[
64a + b = 56.96786
\]

\[
256a + b = 207.39558
\]

so \( a = 0.78348, \quad b = 6.81349, \quad c = 6234.18138. \quad (4.32)\)

Likewise

\[
N_6(\omega) = 0.17951 \omega^4 + 25.74209 \omega^2 + 6234.18138. \quad (4.33)
\]

The fit of \( G_4(\omega) \) and \( G_6(\omega) \) to \( U_4(\omega) \) and \( U_6(\omega) \), respectively, is demonstrated in Figs. 28 and 32. The results are not particularly good in the former case because the rather large discontinuity of \( U_4(\omega) \) occurs
near the $\omega = \pm 6\pi$ zero of $G_4(\omega)$. A better fit could be obtained by removing, in addition, the $\omega = \pm 6\pi$ zeros of $(\sin \omega/2)/(\omega/2)$ and using a polynomial in $\omega^6$ for $N_4(\omega)$. Figure 29 compares the original rectangular pulse $|t| < 1/2$ which when passed through the filter $N_4(\omega)/D_4(\omega)$ results in the pulselike output $g_4(t)$. The slight "tails" of $g_4(t)$, $|t| > 1/2$, may be explained in terms of the computer approximation errors involved. In Fig. 30 we show that truncation of the tails of $g_4(t)$ does not significantly alter $G_4(\omega)$. Also plotted in Fig. 29 is $u_4(t)$, the non-time-limited inverse Fourier transform of the desired spectrum $\sqrt{S_0(4,\omega/16)}$, $|\omega| \leq 16$. Note that $g_4(t)$ and $u_4(t)$ are quite alike on $|t| < 1/2$ and this implies that a good first approximation of a possible transmitted pulse would be to choose a truncated version of $u_4(t)$, $|t| < 1/2$.

In the case of the second example, Fig. 32 shows that $G_6(\omega)$ is a quite good approximation to the desired spectrum; while Fig. 33 shows that the associated waveform $g_6(t)$ is very nearly time-limited.

Ambiguity surface contours of constant doppler frequency associated with $g_4(t)$ and $g_6(t)$, $|t| < 1/2$, are shown in Figs. 31 and 34 and their delay axis distributions $a_4(\tau,0)$ and $a_6(\tau,0)$ are repeated in Fig. 35. For $D = 6$ the ideally concentrated $a_D(\tau,0)$ corresponding to a bandlimited spectrum $\sqrt{S_0(6,\omega/16)}$, $|\omega| \leq 16$, has a concentration $\lambda_D(D=6) = 0.9999$ contained in $|\tau| \leq L = 0.375$ and a sidelobe level of $-44$ db. The $a_6(\tau,0)$ corresponding to $g_6(t)$ derived in this example realizes a concentration $\Gamma(L = 0.38) > 0.9997$ and a sidelobe level $<-37$ db in addition to the property of being identically zero for $|\tau| > 1$. (Recall that $a_D(\tau,0)$ for a bandlimited spectrum is not identically zero over $-\infty < \tau < \infty$ except at a denumerably infinite number of points.) The
D = 6 case gives a remarkably good approximation to that desired $a_D(\tau,0)$. The example with $D = 4$ yields an ambiguity surface with delay axis concentration $\Gamma(L = 0.24) = 0.978$ and a sidelobe level of -20 db as compared to the ideals of $\Gamma(0.25) = \lambda_0(D = 4) = 0.996$ and sidelobe level -28 db. This approximation is reasonably good despite the poor fit of $G_4(\omega)$ to $U_4(\omega)$ noted earlier. There are, however, significant off-axis sidelobes of $a_4(\tau,\nu)$ which may be attributed to the irregular realization $G_4(\omega)$. It is interesting to note that the -3 db widths of $a_4(\tau,0)$ and $a_6(\tau,0)$ and the -3 db widths of their associated pulses $g_4(t)$ and $g_6(t)$, $|t| < 1/2$, are very nearly equal.

Although the parameters chosen for this example were rather arbitrary, the method should work equally well for scaled values, as perhaps $T = 10^{-3}, \sqrt{S_0(4,\omega/(16\times10^3))}$, which would be of interest in a physical problem. As interesting as this method is, however, the next section points out certain real-life considerations which normally limits the use of amplitude modulated pulses like those just derived.
Figure 28. The APSWF spectrum \( H_4(\omega) = \sqrt{S_0(4, \omega/16)} \) and the approximation \( G_4(\omega) \).
Figure 29. The pulse shape $g_4(t)$ associated with the spectrum approximation $G_4(\omega)$ for $\sqrt{S_0(4,\omega/16)}$. 
Figure 30. A comparison of $G_4(\omega)$ and the $t_w$ portion of the Fourier transform of the exact time-limited pulse $g_4(t)$, $|t| \leq T/2$. 
Figure 31. Constant doppler frequency cross-sections of the ambiguity surface of the time-limited AM pulse $g_4(t)$. 
Figure 32. The APSWF spectrum \( H_6(\omega) = \sqrt{S_h(6,\omega/16)} \) and the approximation \( G_6(\omega) \).
Figure 33. The pulse \( g_6(t) \) associated with the spectrum approximation \( G_6(\omega) \) for \( \sqrt{S_0(6, \omega/16)} \).
Figure 34. Constant doppler frequency cross-sections of the ambiguity surface of the time-limited AM pulse $g_6(t)$. 
Figure 35. Delay axis ambiguity distributions for time-limited AM pulses whose spectra approximate $\sqrt{S_o(D, \omega/16)}$. 
C. Amplitude modulation losses

Waveguide breakdown and amplifier heat dissipation problems are among those factors resulting in high power radars being subjected to an instantaneous peak power limitation which constrains the total energy transmitted per pulse and thus limits the radar target detection capability. The peak instantaneous power limit puts an upper bound on transmitted signal amplitude. Obviously the greatest energy transmission occurs when the transmitted signal envelope is constant with the maximum amplitude allowed for its duration $|t| \leq T/2$. Amplitude modulated, transmitted pulses give rise to a target detection loss (receiver $S/N$ ratio loss) since less energy is contained in the AM pulse than in a rectangular envelope pulse of an equal peak amplitude and equal duration.

With matched filtering and white noise, the $S/N$ ratio at the receiver output is always $2E/N_0$ when the energy of the received pulse is $E$ and the noise power is $N_0/2$ (watts/cycle/second) over the receiver bandwidth.

Thus for the rectangular pulse $f(t) = \sqrt{E/T} e^{j2\pi f_0 t}$, $|t| \leq T/2$,

$$\frac{S}{N_R} = \frac{2E}{N_0} . \quad (4.34)$$

For an amplitude limited pulse $g(t) = \sqrt{E/T} u(t) e^{j2\pi f_0 t}$, $|t| \leq T/2$, with $u(t)$ of unity peak value,

$$\frac{S}{N_{AM}} = \frac{2}{N_0} \int_{-T/2}^{T/2} |g(t)|^2 dt$$

$$= \frac{2E}{N_0 T} \int_{-T/2}^{T/2} u^2(t) dt. \quad (4.35)$$
Define the loss $L_{AM}$ as $10 \log \left[ \frac{(S/N)_{AM}}{(S/N)_R} \right]$ which may be taken as a measure of the loss in a radar target detection probability when AM pulses are transmitted. Clearly,

$$L_{AM} = 10 \log \left[ \frac{1}{T} \int_{-T/2}^{T/2} u^2(t) \, dt \right] \quad \text{(db.)} \quad (4.36)$$

The loss is quite easy to calculate and for $g_4(t)$ and $g_6(t)$ of the preceding discussion is found to be 6.9 db and 6.5 db respectively. The loss is independent of pulse duration $T$. These losses are rather large and are indicative of the general situation in which amplitude modulation and matched filtering of the transmitted pulse is an inefficient method of obtaining a shaped delay axis ambiguity distribution $a(t,0)$.

Table 2 lists typical values of AM losses for several types of amplitude modulated pulses possessing equal peak amplitudes. The data contained in the table implies that generally it would not be feasible to choose AM pulses for the purpose of $a(t,0)$ shaping. However, AM pulse shaping may be feasible to control the shape of $a(0,v)$ since in this case $L_{AM}$ is usually less than three decibels. Temes [41] has given consideration to the engineering trade-offs involved in picking suitably amplitude modulated pulses for $a(0,v)$ shaping.

An alternate method of $a(t,0)$ control involves the transmission of frequency modulated, rectangular envelope pulses of the maximum allowable amplitude consistent with the radar peak power limitation. Matched filtering at the receiver preserves the maximum $S/N$ ratio $= 2E/N_o$ for the transmitted signal energy $E$. The advantages of this type of radar signal have caused the technique to be widely used, principally, in the well known CHIRP radars [24] which employ linearly frequency modulated pulses.
Table 2. The Loss $L_{AM}$ for Amplitude Modulated Radar Pulses

| Pulse Type                | $f(t)$ Form on $|t| \leq T/2$                             | Characteristics                                      | $L_{AM}$(-db) |
|---------------------------|-------------------------------------------------------------|------------------------------------------------------|--------------|
| Rectangular               | $\sqrt{E/T} e^{j2\pi f_0 t}$                                | NO AM losses                                         | 0            |
| Truncated-Gaussian        | $\frac{k(2t)^2}{2T} e^{j2\pi f_0 t}$                       | $k = 1$                                              | -21 db       |
|                           |                                                            | $k = 3$                                              | -38 db       |
|                           |                                                            | $k = 5$                                              | (no distinct sidelobes) 4.02 |
| Taylor                    | $(a_0 + 2 \sum_{n=1}^{\infty} a_n \cos \frac{2\pi n t}{T}) e^{j2\pi f_0 t}$ | $n = 4$                                              | -20 db       |
|                           |                                                            | $n = 6$                                              | -30 db       |
| APSWF                     | $\sqrt{E/H_T S_0(c, \frac{2t}{T})} e^{j2\pi f_0 t}$         | $c = 2$                                              | -18 db       |
|                           |                                                            | $c = 4$                                              | -28 db       |
|                           |                                                            | $c = 6$                                              | -44 db       |
|                           |                                                            | $c = 8$                                              | -62 db       |
| Campbell et al., derived  | $g_4(t)$, Figure 29                                          | Pronounced AM losses, -19 db first sidelobe of $a(\tau,0)$ | 6.93         |
|                           | $g_5(t)$, Figure 33                                          |                                                      | 6.47         |
D. Control of $a(\tau,0)$ by pulse frequency modulation

Unacceptable receiver S/N ratio degradation was seen as the result of applying amplitude modulation to radar pulses for the purpose of attempting to concentrate their matched filter response $a(\tau,0)$ in a region $|\tau| \leq L$. Obviously, since $|F(\omega)|^2 = U^2(f) \leftrightarrow a(\tau,0)$, any choice of $f(t) \leftrightarrow F(\omega)$ which realizes the desired APSWF bandlimited spectral distribution will be acceptable from the viewpoint of $a(\tau,0)$ concentration. In order to minimize receiver S/N ratio losses one would like to transmit rectangular pulses and to retain matched filtering if possible.

Let us choose in (4.38),

$$f(t) = u(t) e^{j\phi(t)} \mathcal{F} U(f) e^{j\theta(f)} = E(f), \quad (4.38)$$

$u(t) = \sqrt{E/T}$, $|t| \leq T/2$; and simultaneously choose $U(f) = \sqrt{S_0(D,2f/W)}$, $|f| \leq W/2$. The conditions under which $u(t)$ and $U(f)$ may be specified independently and $\phi(t)$ and $\theta(f)$ found so that the Fourier transform relationship holds are not presently known. E. N. Fowle has discussed conditions under which independently specified moduli may be approximately obtained and has reported on a method of generating the phase functions $\phi(t)$ and $\theta(f)$ so that the Fourier pair (4.38) is approximately realized. The method, based on the stationary phase principle of integral evaluation, is not derived here as interesting discussions of it are generally available [42, 43]. Fowle's results may be simply stated in the form of the differential equations:

$$\phi'(t) = 2\pi \lambda(t) \quad (4.39)$$
$$\theta'(\lambda) = -2\pi t(\lambda),$$
where \( \lambda(t) \) or \( t(\lambda) \) are functions to be obtained by completion of either integral (4.40) or (4.41) on the desired forms of the signal moduli.

\[
\int_{-\infty}^{t} u^2(\xi) d\xi = \int_{-\infty}^{\lambda} U^2(\eta) d\eta \quad (4.40)
\]

\[
\int_{-\infty}^{t} u^2(\xi) d\xi = \int_{-\infty}^{\lambda} U^2(\eta) d\eta \quad (4.41)
\]

It is assumed that \( u^2(\xi) \) and \( U^2(\eta) \) are absolutely integrable. As a rule, the quality of the transform pair approximation improves with increasing values of the TW product when \( T \) and \( W \) are chosen to be consistent measures of the duration of \( u(t) \) and bandwidth of \( U(f) \) respectively. In this instance \( T \) will be chosen as the time-limited, transmitted pulse duration while \( W \) will be taken as the extent of the desired APSWF spectral modulus. The largest TW products are required for a good approximation to the transform (4.38) when the desired moduli \( u(t) \) and \( U(f) \) are, as in this case, discontinuous at their time and frequency domain end points.

Invariance of the pulse-compression ambiguity concentration product \( PD \)

It is proposed to realize approximately a concentrated, sidelobe suppressed \( a_D(t,0) \), matched filter response to a frequency modulated pulse of finite duration. One requires that

\[
u(t) = \sqrt{E/T} \quad |t| \leq T/2, \quad (4.42)
\]

\[
U(f) = \sqrt{E S_0(D,2f/W)/K} \quad |f| < W/2,
\]

with \( D = \pi LW \) and \( K \) a factor to be selected so that Parseval's relation, (4.43), holds.
\[
\int_{-T/2}^{T/2} u^2(\xi) \, d\xi = \int_{-W/2}^{W/2} U^2(\eta) \, d\eta \quad (4.43)
\]

If the APSWF spectral modulus could be realized exactly for a fixed \(D\), the concentration \(\lambda_0(D)\) and the sidelobe level \(R(D)\) of \(a_D(\tau,0)\) would be known. The region \(|\tau| \leq L\) into which \(a_D(\tau,0)\) is to be concentrated may be made as small as desired if sufficient bandwidth \(W\) is available as it is only necessary to keep \(D = \pi LW\) large in order to obtain \(\Gamma(L) = \lambda_0(D) \rightarrow 1\). As we have seen in Fig. 25 this condition results in a suppressed sidelobe structure of \(a_D(\tau,0)\). In fact, for \(D\) greater than about \(D = 2\), the energy of the APSWF form \(a_D(\tau,0)\) of (4.13) is almost completely contained in the \(|\tau| \leq L\) region and it is thus quite natural to define the duration of the matched filter response as \(2L\). The pulse compression \(P\) is defined as the ratio of the transmitted pulse length to the duration of the autocorrelation of \(f(\tau)\), that is, the \(a_D(\tau,0)\) duration.

\[
P = \frac{T}{2L} = \frac{\pi TW}{2D} \quad \text{for } D > 2 \quad (4.44)
\]

Klauder et al. [24, p. 14] have stated qualitatively that for the near rectangular pulse and spectral envelopes of the CHIRP radar, the energy of the signal autocorrelation function \(a_0(\tau,0)\) on \(|\tau| \leq 1/2W\) and the \(a_0(\tau,0)\) -3 db width compression ratio \(T/(1/W) = TW\) are related such that larger \(TW\) products correspond to greater concentrations of the energy of \(a_0(\tau,0)\) on \(|\tau| \leq L\). Equation (4.44) makes a similar but more quantitative statement of the relationship between pulse compression \(P\), the concentration parameter \(D\), and the pulse duration-bandwidth product \(TW\) for the case of APSWF spectral distributions.
One may write (4.44) in the form

\[ PD = \frac{\pi T W}{2} \quad , \]  

(4.45)

a result which justifies the intuitive notion that the compression-concentration product PD is essentially invariant for a fixed signal TW product. If we choose a value for the TW product and take U(f) to be

\[ \sqrt{S_0(D, 2f/W)} , \quad |f| \leq W/2 ; \]

one has available a well defined choice between signal design for large pulse compression P or for suppressed sidelobes of \( a_D(\tau, 0) \).

In reality, a time-limited signal \( f(t) \) cannot possess a strictly bandlimited spectrum \( F(f) \). It is a propitious situation then that the spectrum approximation given by Fowle’s method improves with increasing TW products since it is also true that the larger the TW value the more interesting are the possibilities of achieving simultaneous compression of the central peak of \( a_D(\tau, 0) \) and control of its surrounding sidelobe levels. We next take up a sample calculation to demonstrate the class of frequency modulations \( \phi'(t) \) required to obtain the APSWF spectral modulus specified in (4.42). We then display some typical TW dependent approximate realizations of these moduli and associated ambiguity surfaces.

Optimum radar pulse frequency modulation characteristics

With reference to the discussion accompanying (4.39), the relation

\[ \phi'(t) = 2\pi \lambda(t) \]  

(4.46)

may be interpreted as defining the frequency deviation \( \lambda(t) \) of the radar pulse instantaneous frequency \( \psi_i(t) \) about a carrier \( \psi_0 \).
From (4.40)
\[ \int_{-T/2}^{T/2} u^2(\xi)d\xi = \int_{-W/2}^{W/2} u^2(\eta)d\eta , \]  
(4.48)

\[ \frac{E}{T} \int_{-T/2}^{T/2} d\xi = \frac{E}{K^2} \int_{-W/2}^{W/2} \lambda(t) S_0(D,2\eta/W)dn . \]  
(4.49)

The constant \( K^2 \) evaluated by Parseval's relation given in (4.43) is
\[ K^2 = \int_{-W/2}^{W/2} S_0(D,2\eta/W)d\eta = \frac{W}{2} \int_{-1}^{1} S_0(D,x)dx . \]  
(4.50)

Next substitute (4.50) into (4.49) and let \( x = 2\eta/W \).
\[ \frac{E}{T} (t + \frac{T}{2}) = E \frac{\int_{-1}^{1} S_0(D,x)dx}{\int_{-1}^{1} S_0(D,x)dx} \]  
(4.51)

\[ \frac{t}{T} + \frac{1}{2} = \frac{1}{2} + \frac{\int_{-0}^{+2\lambda/W} S_0(D,x)dx}{\int_{-1}^{0} S_0(D,x)dx} \]  
(4.51)

The (+) integral value is assigned when \( \lambda > 0 \) and the (-) integral value is assigned for \( \lambda < 0 \). Thus since \( t \) ranges over \((-T/2, T/2)\), \( \lambda(t) \) must be an odd symmetric nonlinear modulation of the transmitter frequency about the carrier \( \psi_0 \) with the exact form of the nonlinearity depending on the concentration parameter \( D \). Note that \( \sqrt{S_0(0,x)} \) is a rectangular spectral modulus and the result of (4.51) for this case cor-
rectly reduces to a linear FM of the transmitted pulse about its carrier
frequency \( \psi_0 \).

\[
\frac{t}{T} = \pm \frac{1}{2} \int_{0}^{\frac{[2\lambda(t)]/W}{2}} dx = \frac{\lambda(t)}{W}
\]

\( D = 0 \quad (4.52) \)

\[
\psi_1(t) - \psi_0 = (W/T)t \quad \text{for } |t| \leq T/2
\]

Figure 36 displays the positive \( t \) portion of the odd-symmetric
frequency modulation characteristic for several values of the APSWF
parameter \( D \). Note that the coordinates of Fig. 36 are actually nor-
malized variables \( x = \frac{2t}{T} \) and \( y' = 2\lambda/W \).

The phase modulation functions associated with (4.51) may be
obtained by numerical integration of the relations between \( \lambda \) and \( t \)
shown in Fig. 36.

Let \( \phi'(s) = 2\pi\lambda(s) \)

\[
\int_{\frac{T}{2}}^{t} \phi'(s) ds = 2\pi(T/2)(W/2) \int_{-1}^{2} \frac{2\lambda(xT)}{(2x^2/W)} dx.
\]

Define \( y'_D(x) = \frac{2\lambda(xT)}{2W} \)

\[
\phi(t) = \frac{\pi TW}{2} \int_{-1}^{2} y'_D(x) dx + \phi_0. \tag{4.54}
\]

The constant \( \phi_0 \) may be chosen arbitrarily without modifying the
transmitted pulse frequency modulation characteristics. For computer
calculations it is convenient to choose \( \phi_0 = -\left(\frac{\pi TW}{2}\right) \int_{-1}^{0} y'_D(x) dx \) so
Figure 36. The positive time portion of the APSWF, odd-symmetric frequency modulation characteristic.
that $\phi(0) = 0$. From the inset on Fig. 37 one can see that this choice of $\phi_0$ causes $\phi(t)$ to be a non-negative even function of $t$ (or $x = 2t/T$). The accompanying curves show the positive $t$ (or $x$) portion of $\phi(t)$.

The transmitted pulse is then

$$f(t) = \sqrt{E/T} e^{j[2\pi \psi t + \frac{\pi TW}{2} y_D (\frac{2t}{T}) + \phi_0]}, \quad |t| \leq T/2,$$

while the spectral distribution, centered about the carrier frequency $\psi_0$ which is suppressed here, is

$$F(\psi) = \sqrt{E/T} e^{j\psi_0} \int_{-T/2}^{T/2} e^{j[\frac{\pi TW}{2} y_D (\frac{2t}{T}) - 2\pi \psi t]} dt.$$

For $x = 2t/T$ and $f = \psi$

$$F(\frac{2f}{W}) = \frac{1}{2} \sqrt{E} \ e^{j\psi_0} \int_{-1}^{1} e^{j \frac{\pi TW}{2} (y_D(x) - \frac{2fx}{W})} dx.$$ 

(4.56)

The normalized spectral distributions $|F(2f/W)|$ are compared in Figs. 38-41 to the desired distributions $\sqrt{S_0(D,2f/W)}$ for several cases of pulse compression $P$ and concentration parameter $D$. The results were obtained by digital computer evaluation of the integral (4.56) in which $y_D(x)$ was generated by inserting in the computer program a sufficiently accurate approximation to the appropriate modulation characteristic of Fig. 37.

In attempting to generate rectangular spectral distributions by linear frequency modulation of a rectangular envelope pulse, Klauder et al. report [24, pp. 10-14] that the realized spectrums contain
Figure 37. The positive time portion of the even function $y_D(x) - y_D(0)$. 

\[ \phi(x) = \frac{T}{2} \int \left[ y_D(x) - y_D(0) \right] \]
significant distortion components which decrease in amplitude with
increasing TW product. This conclusion has been verified as shown in
Fig. 38 when \( y_0(x) = x^2/2 \) (linear FM or \( D = 0 \) case) was used to check
the computer program used in deriving the \( D = 4, 6, \) and \( 8 \) spectral
distributions of Figs. 39, 40, and 41 respectively. Compression ratios
of \( P = 10, 30, \) and \( 50 \) were paired with each \( D \) value allowing the inves­
tigation of the spectral distortion or error from the desired \( \sqrt{S_0(D, 2f/W)} \)
for TW products as high as \( TW = [2(50)(8)]/\pi = 254 \) in the \( D = 8 \) case.
Note that the error of the spectral approximation does decrease with
increasing TW product but is still significant for the largest TW products
investigated.

E. Delay-axis concentrated
ambiguity surfaces

The pulse of (4.55) may be "actively" generated by amplifying and
transmitting a \( T \) duration segment of the output of a voltage controlled
oscillator. The oscillator is "gated on" as its output sweeps nonlinear­
earily from \( \psi_o - W/2 \) to \( \psi_o + W/2 \) cycles per second. The time domain
definition of the ambiguity surface associated with this pulse is simply

\[
\alpha_D(\tau, v) = \frac{E}{T} \int_{-T/2}^{T/2} e^{j[\phi(t) - \phi(t+\tau) - 2\pi vt]} dt
\]

or

\[
\alpha_D(\tau, v) = \frac{E}{2} \int_{-l}^{l} e^{j(\pi TW/2)[y_D(x) - y_D(x+2\tau/T) - (2v/W)x]} dx.
\]

One should bear in mind that the use of this definition implies either a
matched filter receiver (whose passband response for a specific \( P \) and \( D \)
Figure 38. Upper sideband of the carrier frequency centered spectrum of the APSWF frequency modulated pulse, \( D=0 \). (The linear FM case)
Figure 39. The normalized spectrum of an APSWF frequency modulated rectangular pulse, compression ratio $P$, $D=4$. 
Figure 40. The normalized spectrum of an APSWF frequency modulated rectangular pulse, compression ratio $P$, $D=6$. 
Figure 41. The normalized spectrum of an APSWF frequency modulated pulse, compression ratio, $P$, $D=8$. 
exactly matches the amplitude distorted spectral distributions of Figs. 39-41 or else a receiver is assumed capable of a time domain correlation calculation like that of (4.57). In some cases the latter assumption is not unreasonable. Special purpose radars employing optical signal processing techniques allow reasonably easy time domain correlation calculations to be performed even when unusual nonlinearly frequency modulated signals are involved [47, part IV, chpt. 3]. Such processes presently employ TW products in the order of 1000 and thus should allow advantageous combinations of high pulse compression ratios P and low sidelobe levels of \(a_D(\tau,0)\). Thus it is of interest to examine the form of the ambiguity surfaces and, in particular, of the delay axis cut defined there by setting \(v = 0\).

Simpson's Rule for numerical integration has been used to calculate \(a_D(\tau,0)\) for each combination of compression ratio \(P = 10, 30,\) and 50 and parameter values \(D = 4, 6,\) and 8 as shown in Figs. 42, 43, and 44 respectively. Evaluation of (4.56) and (4.57) is completed by dividing the \((-1, 1)\) integration region into a large number of sub-intervals. The number of sub-intervals used was 2000 for TW = 254 and proportionally fewer for smaller TW products.

The first sidelobe level of each \(a_D(\tau,0)\) is compared in each of Figs. 42-44 with the level predicted by theory. The theoretical values would be obtained only if the spectrum of the actively generated pulse became exactly \(\sqrt{S_o(D,2f/W)}\), \(|f| < W/2\). It has been seen in Figs. 39-41 that the realized spectra are not bandlimited and only approximate the desired shape with an error decreasing as the TW product is increased. Spectral distortion is also dependent on the quality of the approximation
Figure 42. The normalized delay axis distribution $a_\theta(\tau,\nu)$ of compression ratio $P$, $D=4$. 

$\frac{2\tau}{T} \left( \frac{P}{10} \right)$
Figure 43. The normalized delay axis distribution $a_D(\tau, \nu)$ of compression ratio $P$, $D=6$. 
Figure 44. The normalized delay axis distribution $a_D(t,v)$ of compression ratio $P$, $D=8$. 
Figure 4.5. Ambiguity surfaces \( a_D(t,v) \) for the APSWF nonlinearily frequency modulated pulse, \( P = 10 \). Constant doppler frequency profiles spaced at \( \Delta v \) cps.
to the \(y_D(x)\) curves used in the calculation of (4.56) and (4.57). As the \(y_D(x)\) curves become more nonlinear with increasing \(D\) one finds that the realized sidelobe levels \(a_D(\tau,0)\) can vary substantially from the corresponding theoretical levels. As an example, the theoretical sidelobe level for the \(a_8(\tau,0)\) case is \(-62\) db but the calculated level is \(-39\) db for the least distorted \(D = 8\) case corresponding to a compression \(P = 50\) or TW product = 254. At the other extreme, the \(-28\) db theoretical sidelobe level of \(a_4(\tau,0)\) is achieved for the moderate \(P = 30\) compression ratio corresponding to a TW product of 76.

The concentration \(\Gamma\) of \(a_D(\tau,0)\) defined by (4.8) is for the \(D = 4\), \(P = 10\) case on the order of \(0.960 \leq \Gamma \leq 0.975\); and for the same \(P = 10\) but \(D = 6\) and 8, \(0.975 \leq \Gamma \leq 0.990\). Theoretical values of \(\Gamma\) for these examples are \(\Gamma(4) = 0.9959\) and \(\Gamma(D \geq 6) \geq 0.9999\). The uncertainty in the calculated results arise from the fact that no sidelobe calculations were carried out in the region \(T/4 \leq |\tau| \leq T\) since the computer time required is excessive. The lower bound on \(\Gamma\) has been obtained by assuming that all sidelobes of \(a_D(\tau,0)\) on \(T/4 \leq |\tau| \leq T\) are of the same height as that of the last lobe in the adjoining \(0 \leq |\tau| \leq T/4\) region. The upper bound on \(\Gamma\) follows by assuming that the sidelobes on \(T/4 \leq |\tau| \leq T\) are of negligible contribution to the energy of \(a_D(\tau,0)\) outside \(|\tau| \leq L\). No sidelobe structure would exist on \(|\tau| \geq T\) since perfectly time-limited pulses are assumed. The data of Figs. 42, 43, and 44 indicated that the concentration \(\Gamma\) of each \(a_D(\tau,0)\) increases with increasing TW products since the realized sidelobe levels tend toward the low theoretical levels with increasing TW. When ambiguity energy is forced out of the sidelobe
structure it must be shifted into the central response peak, somewhat broadening it, and slightly increasing $\Gamma$.

Complete ambiguity surfaces for the APSWF frequency modulated pulses for $D = 4$ and $D = 6$ have been calculated using (4.57) and are shown for a pulse compression ratio of $P = 10$ in Fig. 45. For doppler shifts as small as $2v/W = \pm 0.25$ or $v = \pm W/8$, the surface develops a large (-12 db) asymmetrical sidelobe structure positioned about a significantly broadened (50%) central peak. Thus doppler frequency shifts of the received radar signal when processed by an autocorrelation receiver or by the perfectly matched filter receiver will generally result in decreased target resolution, poorer range accuracy, and an increased danger of multiple target ambiguity due to the poor sidelobe structure control. In practical systems it is necessary to provide a sufficient number of receiver matched filters with their center frequencies chosen so that the range of doppler frequencies over which each must operate is limited to perhaps five or ten percent of the signal bandwidth $W$.

F. Summary

In this chapter, time-limited pulses were sought whose matched filter response or, equivalently, whose autocorrelation function could be optimally concentrated in a specified time slot. This problem was not found to possess an easily determinable exact solution, however, the modified assumption of a bandlimited rather than time-limited radar signal allowed the specification of a spectral envelope whose associated ambiguity surface possesses an optimally concentrated $\tau$-axis cut $a_D(\tau,0)$. The square magnitude of the desired spectral distribution has the shape
of a truncated prolate spheroidal wave function of zero order and parameter \( D = \pi LW \) determined by the length of the time slot \( 2L \) allotted to the autocorrelation peak and by the allowed signal bandwidth \( W \) about some carrier frequency \( \nu_0 \).

Possible approximations of the desired spectra generated by truly time-limited pulses were investigated. Amplitude modulated pulses of duration \( T \) were found which synthesized ambiguity functions possessing highly concentrated \( a(\tau,0) \) cuts. In general it appears that amplitude modulated pulses capable of significant \( a(\tau,0) \) shaping also lead to intolerable receiver S/N ratio losses under transmitter peak power limitations. Alternately, nonlinearly frequency modulated pulses of rectangular envelope are specified and allow one to achieve a reasonable approximation of the desired spectral envelopes without the S/N ratio degradation which accompanies amplitude modulation. The associated ambiguity surfaces possess \( a_D(\tau,0) \) cuts which for large \( D \) have sidelobe levels appreciably above those predicted. This error and the spectral distortions of Figs. 39-41 appear to be related to the combined effect of two factors. First, errors enter the calculation of (4.56) and (4.57) because, for increasing \( D \), the required angle modulation characteristic \( y_D(x) \) becomes increasingly nonlinear and more difficult to approximate by a simple curve. Secondly, for a fixed \( P \), increased \( D \) values mean that evaluation errors will grow since a rapidly varying phase factor will be present in the integrands of (4.56) and (4.57). This Fourier transform error may be controlled by more precise computer calculations at the expense of increasing computation time.
Perhaps the most interesting results of this section deals with the connection of the well known idea of pulse compression with that of the concentration of the matched filter response and its "sidelobe" levels. It is shown that for a $T$ duration frequency modulated pulse whose spectrum approximates the optimal $\sqrt{S_0(D,2f/W)}$, $|f| \leq W/2$, that the available pulse compression ratio may be described approximately by $P = \pi TW/2D$ and is therefore proportional to the $TW$ product and inversely proportional to the concentration parameter $D$. When low sidelobes of $a_D(\tau,0)$ are required, $D$ is large and the available pulse compression $P$ is reduced for a fixed signal $TW$ product.
CHAPTER V

APPROXIMATIONS TO OPTIMAL RANGE

AMBIGUITY DISTRIBUTIONS

The calculation of $a_D(\tau, \nu)$ by the time-frequency autocorrelation definition of (4.57) assumes a receiver with a passband frequency response perfectly matched to the spectrum $F(f) = U(f)e^{j\theta(f)}$ of the actively generated rectangular pulse (4.55). In general, it would be difficult to synthesize a receiver whose lumped element filter frequency response exactly matched any of the spectral amplitudes $|U(f)|$ of Figs. 39-41 since this would require exact duplication of the distortion components shown there.

A. Receiver with smoothed passband filter characteristics

It is practical to assume a receiver filter as a tandem combination of (1) an all-pass, bandpass compression filter centered at some IF and possessing the nonlinear phase characteristic $e^{-j\theta(f)}$ and, (2) an essentially bandlimited filter of linear phase characteristic and an amplitude response $\sqrt{S_0(D,2f/W)}$, $|f| < W/2$, centered on the same IF. Such a filter might appear as in Fig. 46.

For any of the distorted spectral distributions of Figs. 39-41, the associated delay axis ambiguity distribution $a_D(\tau,0)$ may be obtained from the transform

\[134\]
Figure 46. The general form of a smoothed passband receiver for actively generated pulses of linear frequency modulation, Eq. 5.16, or nonlinear APSWF frequency modulation, Eq. 4.55.
\[ a_D(\tau,0) = \int_{-\infty}^{\infty} |U(f)|^2 e^{-j2\pi ft} \, df. \]  

(5.1)

Recall that \( a_D(\tau,0) \) represents the response of a hypothetical matched filter receiver to the pulse (4.55). When this receiver is replaced by the one of Fig. 46, the corresponding smoothed passband receiver response to the same signal is given by

\[ \overline{a}_D(\tau,v) = \int_{-W/2}^{W/2} U(f+v)e^{-j\theta(f)} e^{-j2\pi ft} \sqrt{S_o(D,2f/W)} e^{j\theta(f)} \, df. \]  

(5.2)

Let \( x = 2f/W \). Assume that the doppler shift is zero and that \( PD = \frac{\pi TW}{2} \), for \( P \) and \( D \) as previously defined.

\[ \overline{a}_D(\tau,0) = \int_{-1}^{1} U(x) \sqrt{S_o(D,x)} e^{-j \frac{\pi TW}{2} (2\tau) x} \, dx. \]  

(5.3)

A comparison of calculated first sidelobe levels of \( a_D \) and \( \overline{a}_D \) is made in Table 3. In general, the smoothed receiver filter allows a modest 3 to 5 db improvement in sidelobe suppression over that available through the use of the hypothetical matched filter receiver. The sidelobe levels of \( \overline{a}_D \) show little or no improvement over those of \( a_D \) in the pulse compression case \( P = 50 \). Since the evaluation of (5.3) has been based on a limited number (41) of \( U(x = 2f/W) \) data points on \( 0 \leq x \leq 1 \), the poor results for the \( P = 50 \) case have been interpreted to mean that calculation errors cause \( \overline{a}_D \) to have roughly the same sidelobe levels as those of \( a_D \) which are attributable to the spectral distortions illustrated in Figs 39-41. It is believed that \( \overline{a}_D \) sidelobe levels near the theoretical values could be obtained if the calculation
errors were suppressed by using a larger number of $U(x)$ data points in the Simpson's rule integration scheme used. Unfortunately, the digital computer calculation time required to accurately obtain the larger number of $U(x)$ points is excessive and has not been attempted.

In like manner, the excessive machine time required has prevented the calculation of the large number of $(\tau, v)$ points necessary to allow an adequate description of the cross-ambiguity surface $a_D(\tau, v)$. Intuitively one would expect these surfaces to resemble those of Fig. 45 with a somewhat broadened central response peak and a slightly suppressed sidelobe structure. Note that for a particular $P$ and $D$, the evaluation (5.2) of $a_D(\tau, v)$ could not proceed without a preceding calculation of a very large number of values of the transmitted pulse spectrum $|U(f)|$.

Receiver signal to noise ratio loss

The receiver of Fig. 46 will be slightly mismatched to a received actively generated pulse of the same parameters $P$ and $D$. A slight receiver S/N ratio degradation results from this mismatched condition. Define the loss $L_\perp$ as

$$L_\perp = 10 \log \frac{\bar{\rho}}{\rho}, \quad (5.4)$$

where $\bar{\rho}$ = the peak instantaneous S/N ratio at the mismatched receiver output,

$\rho$ = the peak instantaneous S/N ratio at a matched receiver output.
Assume white noise of $N_0/2$ (watts/cy./sec.) over the receiver passband,

$$\rho = \frac{|a_D(0,0)|^2}{n^2}$$

$$\rho = \frac{|\int_{-W/2}^{W/2} U(f) \sqrt{S_o(D,2f/W)} \, df|^2}{N_o \int_{-W/2}^{W/2} S_o(D,2f/W) \, df}$$

$$\rho = \frac{|\int_{-1}^{1} U(x) \sqrt{S_o(D,x)} \, dx|^2}{\frac{N_o W}{4} \int_{-1}^{1} S_o(D,x) \, dx}$$

In like manner,

$$\rho = \frac{|a_D(0,0)|^2}{n^2}$$

$$\rho = \frac{|\int_{-\infty}^{\infty} |U(x)|^2 \, dx|^2}{N_o W \int_{-\infty}^{\infty} |U(x)|^2 \, dx}$$

Then,

$$L_1 = 10 \log \frac{|\int_{-1}^{1} U(x) \sqrt{S_o(D,x)} \, dx|^2}{\int_{-1}^{1} S_o(D,x) \, dx \int_{-\infty}^{\infty} |U(x)|^2 \, dx}$$

(5.5)
Each of the integrands is an even function of x and thus the integrals may be evaluated over the positive x axis alone. Limited space has not allowed Figs. 39-41 to be extended sufficiently to show that the following expressions are reasonable approximations to the normalized spectral distributions for x > 1.

\[ D = 4 \quad P = 10, 30, 50 \quad U(x) = 0.25 e^{-2(x-1)} \]  
\[ D = 6 \quad P = 10, 30, 50 \quad U(x) = 0.15 e^{-3.15(x-1)} \]  
\[ D = 8 \quad P = 10, 30, 50 \quad U(x) = 0.10 e^{-2.55(x-1)} \]

With (5.8), (5.9), and (5.10) and the data of Figs. 39, 40, and 41 respectively, one may evaluate the integral \( \int_{0}^{\infty} |U(x)|^2 dx \). Table 4 presents calculated loss values \( L_1 \) for all pairs of D = 4, 6, and 8 and P = 10, 30, and 50. It is obvious that the losses are negligible so that the optimal receiver S/N ratio, \( 2E/N_0 \), is not significantly degraded when the APSWF frequency modulated rectangular pulse \( f(t) \) of (4.55) is processed by the smoothed passband receiver of Fig. 46.

B. Simultaneous amplitude and angle modulated radar pulses

It has been noted that the ambiguity surfaces for APSWF non-linearly frequency modulated pulses have poor characteristics with respect to doppler shifts of the radar echo signal. Although the side-lobes of the compressed range axis ambiguity distribution \( a_D(\tau,0) \) can be suppressed to 30 to 40 dB below the level of the central response peak, doppler frequency components of the received pulse may cause the receiver matched filter response to be highly distorted and difficult to interpret.
### Table 3

First Sidelobe Level of Normalized $a_D(\tau,0)$ and $\bar{a}_D(\tau,0)$, Functions of $P$ and $D$ (decibels).

<table>
<thead>
<tr>
<th>$D$</th>
<th>$a_D$</th>
<th>$\bar{a}_D$</th>
<th>$a_D$</th>
<th>$\bar{a}_D$</th>
<th>$a_D$</th>
<th>$\bar{a}_D$</th>
<th>Theoretical</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>-23.4</td>
<td>-25.2</td>
<td>-27</td>
<td>-27</td>
<td>-29.4</td>
<td>-28.4</td>
<td>-28</td>
</tr>
<tr>
<td>6</td>
<td>-26.7</td>
<td>-32.6</td>
<td>-31.8</td>
<td>-34.7</td>
<td>-35.6</td>
<td>-37.5</td>
<td>-44</td>
</tr>
<tr>
<td>8</td>
<td>-28.4</td>
<td>-33.3</td>
<td>-37.3</td>
<td>-42.8</td>
<td>-39.0</td>
<td>-37.5</td>
<td>-62</td>
</tr>
</tbody>
</table>

### Table 4

$S/N$ Ratio Loss $L_1$ for the Receiver with Filter $\sqrt{S_D(\tau,2f/W)}$, $|f| \leq W/2$, and the Radar Pulse (4.55) (decibels).

<table>
<thead>
<tr>
<th>$D^p$</th>
<th>10</th>
<th>30</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.18</td>
<td>0.12</td>
<td>0.11</td>
</tr>
<tr>
<td>6</td>
<td>0.06</td>
<td>0.04</td>
<td>0.04</td>
</tr>
<tr>
<td>8</td>
<td>0.05</td>
<td>0.03</td>
<td>0.03</td>
</tr>
</tbody>
</table>
An important consideration in the choice of signals for a pulse compression radar is that the construction of compression filters with linear time delay characteristics is well understood. As far as the author is aware, the construction of compression filters possessing nonlinear time delay characteristics has been given little attention in the literature. It would seem that under special conditions such filters could be constructed, however, there is a tendency among radar signal designers to consider only linearly frequency modulated pulses which are thus partially compatible with existing pulse compression receivers and with the state-of-the-art in filter synthesis. In view of the preceding discussion it is of interest to investigate the application of the results of Chapter III and IV to the design of linear frequency modulated radar signals.

Suppose that a transmitted signal \( f(t) \) is to process a spectrum \( F(\lambda) = U(\lambda) e^{j \phi(\lambda)} \) so that \( \theta'(\lambda) \) is proportional to \(-\lambda\). From (4.39), with \( \alpha \) a proportionality constant,

\[
\theta'(\lambda) = -\alpha \lambda = -2\pi t(\lambda).
\]

(5.11)

Thus

\[
t(\lambda) = \frac{\alpha \lambda}{2\pi}.
\]

(5.12)

Fowle's technique has shown that the moduli of \( f(t) \) and its spectrum approximately satisfies the equation

\[
\int_{-T/2}^{T/2} u^2(s) \, ds = \int_{-W/2}^{W/2} U^2(\eta) \, d\eta.
\]

(5.13)

Since (5.12) has shown \( t(\lambda) \) to be proportional to \( \lambda \), it follows that \( |f(t)|^2 \) and \( |F(f)|^2 \) are also proportional. It is for this reason that
a linear frequency modulated, rectangular envelope pulse of carrier $f_o$ possesses an approximately rectangular spectral distribution of width $W$ centered on the carrier frequency $f_o$. This fact has been demonstrated by the Chapter IV example of Fig. 38.

Let us assume that a non-rectangular spectral modulus centered about a carrier $f_o$ is required in order to shape the associated delay axis ambiguity distribution $a(\tau,0)$ in some convenient manner. Under this requirement (5.13) implies that amplitude modulation of the transmitted signal cannot be avoided. If the radar is of the peak power limited type, an engineering choice must be made between an allowable receiver S/N ratio degradation and some degree of control of the $a(\tau,0)$ shape. Applying previously derived results to this situation, if the simultaneously amplitude and angle modulated signal is denoted

$$f_{\text{AAM}}(t) \triangleq F_{\text{AAM}}(f) = U_{\text{AAM}}(f) e^{j\theta(f)},$$

the choice

$$U_{\text{AAM}}^2(f) = K_1 S_o(D,2f/W), \quad |f| \leq W/2, \quad (5.14)$$

theoretically suppresses the sidelobes of $a_{\text{AAM}}(\tau,0)$ to a value dependent on $D$ alone. Since the modulus of the pulse envelope is to be proportional to that of the spectral distribution,

$$u_{\text{AAM}}^2(t) = K_2 S_o(D,2t/T), \quad |t| \leq T/2. \quad (5.15)$$

The constants $K_1$ and $K_2$ are adjusted for a given signal energy. Of course both moduli cannot be of limited extent so the analysis continues to assume that a linearly frequency modulated pulse of duration $T$ and nominal bandwidth $W$ is transmitted by the radar.
\[ f_{AAM}(t) = \frac{\sqrt{E S_0(D, 2t/T)}}{M_T(D)} \left( j2\pi(f_0 t + \frac{W}{2T} t^2) e^{-\frac{|t|}{T/2}} \right) \]

The energy of \( f_{AAM}(t) \) is \( E \) and its instantaneous frequency is linearly swept in time from \( f_0 - W/2 \) to \( f_0 + W/2 \) cps during one pulse period. The parameter \( D \) is as defined in Chapter IV, \( D = \pi LW \), with \( 2L \) the length of the central peak of the autocorrelation function of the signal \( f_{AAM}(t) \). Equation (3.24) defines the constant \( M_T(D) \). Note that since \( u^2_{AAM}(t) \) is proportional to \( U^2_{AAM}(f) \), the parameter \( c \) of Chapter III, and \( D \) of the present discussion are identical in value.

\[ c = \pi NT = D = \pi LW \]

The spectrum of the pulse is obtained by the standard Fourier transform

\[ F_{AAM}(f) = \int_{-T/2}^{T/2} \frac{\sqrt{E S_0(D, 2t/T)}}{M_T(D)} e^{j[\pi TW \left( \frac{2t^2}{4T} - \frac{2\pi ft}{T} \right)]} dt, \]

when \( y = 2t/T \),

\[ F_{AAM}(2f/W) = \frac{T}{2} \int_{-1}^{1} \frac{\sqrt{E S_0(D, y)}}{M_T(D)} e^{j\frac{\pi TW}{2} \left( \frac{y^2}{2} - \frac{(2\pi f)y}{W} \right)} dy \]

It is convenient to compare \( |F_{AAM}(2f/W)/F_{AAM}(0)| \) with the desired APSWF bandlimited spectrum. As is well known, the quality of the approximation is related to the signal \( TW \) product, with the larger \( TW \) values giving the better fit as is demonstrated in Figs. 47-49 for various \( PD = (\pi/2) TW \) products.

The assumed receiver for the signal of (5.16) contains a matched filter consisting of a tandem combination of a compression filter with a
linear time delay-frequency characteristic and a bandlimiting weighting filter $\sqrt{S_o(D,2f/W)}$, $|f| \leq W/2$, as in Fig. 46. Both filters are centered on a receiver IF which is ignored here. With ideal phase compensation of the received signal by the receiver compression filter, the response of the tandem combination is simply

$$a_{AAM}(\tau,0) = \int_{-W/2}^{W/2} \sqrt{S_o(D,2f/W)} |F_{AAM}(2f/W)| e^{-j2\pi f\tau} df.$$  

(5.19)

For a given $D$, $|F_{AAM}(2f/W)|$ is proportional to one of the TW dependent approximations of the true pulse spectrum to an "ideal" spectrum as shown in Figs. 47, 48, or 49 for $D = 4, 6, \text{ or } 8$ respectively. With the pulse compression ratio $P = T/2L$, we find

$$a_{AAM}(\tau,0) = \frac{W}{2} \int_{-1}^{1} \sqrt{S_o(D,x)} |F_{AAM}(x)| e^{-jPD(2\tau/T)x} dx.$$  

(5.20)

Of course under the ideal condition $|F_{AAM}(x)| \rightarrow \sqrt{K_1S_o(D,x)}$, so (5.20) would give

$$a_{AAM}(\tau,0) \propto K_1S_o(D,\tau/L).$$  

(5.21)

The filter response to a signal possessing the ideal spectrum would be characterized by the $D$ dependent suppressed sidelobe distribution predicted in Chapter IV. For each of the non-ideal spectrum approximations of Figs. 47-49, there is an associated delay axis distribution approximating the right side of (5.21). Each of these axial distributions has been obtained by selecting forty-one equally spaced ($\Delta x = 0.025$) samples of $|F_{AAM}(x)/F_{AAM}(0)|$ on $0 \leq x \leq 1$, and by applying a Simpson's Rule integration method to (5.20). In Figs. 50-52 a normalized $\tau$-axis ambiguity distribution has been shown for each pair of values of
Figure 47. The normalized spectrum of a linearly frequency modulated, APSWF amplitude modulated radar pulse, compression ratio P, D=4.
Figure 48. The normalized spectrum of a linear frequency modulated, APSWF amplitude modulated radar pulse, compression ratio $P$, $D=6$. 

\[ \sqrt{s_c(D, \frac{2f}{W})} \]

$P = 10$

$30$

$50$
Figure 49. The normalized spectrum of a linearly frequency modulated, APSWF amplitude modulated radar pulse, compression ratio $P$, $D=8$. 
P = 10, 30, and 50 and D = 4, 6, and 8. As in a preceding discussion, the calculation of a sufficient number of points to define the complete ambiguity surfaces $a_{AAM}(\tau, \nu)$ requires the pre-calculation of a larger number of $|F_{AAM}(x)/F_{AAM}(0)|$ points than the forty-one values used to find $a_{AAM}(\tau, 0)$. Unfortunately the machine time required for these computations is excessive and thus the complete surfaces have not been constructed so that no information is available concerning the form of the receiver response to grossly doppler shifted signals.

From Figs. 50-52 it can be seen that the realized (calculated) sidelobe levels for a specific D are very close to those predicted by (5.21). This result has not been achieved without a penalty elsewhere, for one should recall that amplitude modulation in a peak power limited radar leads to a loss in the receiver S/N ratio and thus in target detection and an increased false alarm probability. The rectangular envelope, APSWF frequency modulated radar signal does not suffer AM generated S/N ratio losses; however, it does not allow as good a control of the delay axis ambiguity distribution sidelobe levels as do the simultaneously amplitude and angle modulated pulses just discussed.

Both the nonlinear and the linear frequency modulated pulses possess non-bandlimited spectra; thus the S/N ratio at the output of their respective receivers is slightly degraded by the receiver bandlimiting filters $\sqrt{S_0(D, 2f/W)}$. Using (5.7) this loss $L_1$ has been calculated for the APSWF frequency modulated pulses; and in the same manner $L_1$ may be shown to be $< 0.1$ db for any of the APSWF amplitude modulated, linearly frequency modulated pulses.
Figure 50. The normalized delay axis distribution $a_{AAM}(\tau,\nu)$ of compression ratio $P$, $D=4$. 

\[
\frac{2\tau (P)}{T 10}
\]
Figure 5.1. The normalized delay axis distribution $a_{AAM}(\tau, \nu)$ of compression ratio $P$, $D=6$. 
Figure 52. The normalized delay axis distribution $a_{AAM}(\tau, \nu)$ of compression ratio $P$, $D=8$. 
C. Summary

The transmitted radar signals (4.55) and (5.16) and their receivers represent models of practical radar signal processing systems. For convenience, some characteristics of the two systems have been compared in Table 5 as a function of the TW product. In reviewing the results for the large D case, the sidelobe levels of \(|a_D(\tau,0)|\) for the APSWF-FM pulse are in considerable error from predicted values. These errors are due to distortion present in the pulse spectra shown in Figs. 39-41. The errors propagate through the Fourier calculation (5.3) and are augmented there by numerical evaluation errors caused by rapidly varying phase factor present in that integrand. Both sources of errors may be suppressed at the expense of an increased machine computation time. As there is presently little hope for hardware realization of such a system with errors less than those encountered in the preceding model calculations, it is probably that the APSWF-FM pulse should be used only when moderately low sidelobe levels of the delay axis distribution are required.

When compatibility of the transmitted signal with present linear FM pulse compression equipment is desired, the desired concentration and low sidelobe levels of the delay axis ambiguity distribution may be achieved by transmitting the APSWF-AM, linear FM pulse (5.16). This choice is especially advantageous when sidelobe levels below \(-40\) db are required, however, receiver S/N ratio losses in the order of 3 db must be tolerated if the radar is peak-power limited.
TABLE 5

Comparison of Some Characteristics of Radar Systems Employing Frequency Modulated Pulses

| D  | P  | TW  | $\overline{a}_D$ | $a_{AAM}$ | $\overline{a}_D$ | $a_{AAM}$ | System Loss (db) | First Sidelobe Level $|a(\tau,0)/a(0,0)|$ (db) |
|----|----|-----|-----------------|----------|-----------------|----------|-----------------|-------------------------------|
| 4  | 10 | 25.5| 0               | 2.03     | 0.18            | 0.07     | 0.18            | 2.10                          | -25.2 -26.7 -28.2            |
| 4  | 30 | 76.5| 0               | 2.03     | 0.12            | 0.03     | 0.12            | 2.06                          | -27  -28.4 -28.2             |
| 4  | 50 | 127.5| 0             | 2.03     | 0.11            | 0.03     | 0.11            | 2.06                          | -28.4 -27.4 -28.2           |
| 6  | 10 | 38.2| 0              | 2.91     | 0.06            | 0.02     | 0.06            | 2.93                          | -32.6 -40.4 42.7           |
| 6  | 30 | 114.6| 0            | 2.91     | 0.04            | 0.01     | 0.04            | 2.92                          | -34.7 -41.7 42.7           |
| 6  | 50 | 191.0| 0           | 2.91     | 0.04            | 0.01     | 0.04            | 2.92                          | -37.5 -38.8 42.7           |
| 8  | 10 | 50.9| 0             | 3.53     | 0.05            | 0.005    | 0.05            | 3.53                          | -33.3 -52.2 62.0           |
| 8  | 30 | 152.7| 0            | 3.53     | 0.03            | 0.005    | 0.03            | 3.53                          | -42.8 -64.5 62.0           |
| 8  | 50 | 254.5| 0           | 3.53     | 0.03            | 0.005    | 0.03            | 3.53                          | -37.5 -62.0 62.0           |

*Nominal values of first sidelobe levels for very irregular $a(\tau,0)$ structure resulting from calculation errors in these high TW product cases.
CHAPTER VI

REVIEW AND CONCLUSIONS

In this paper we have defined the ambiguity function as the time-frequency correlation function of a radar signal and a delayed, doppler shifted target echo. We have shown how the ambiguity function is representable as a surface over a two-dimensional \((\tau, \nu)\) plane and how this surface is of use in studying the response of a matched filter receiver to any possible combinations of target echo delay and doppler frequency shift. The ambiguity surface possesses unusual mathematical properties some of which have been stated and some examples of which have been shown in order to introduce the casual reader to the problem of determination of a signal which will possess a specified ambiguity surface.

The general ambiguity function synthesis problem is yet unsolved, however, a very interesting aspect of the problem has been the subject of this paper. Since for the transmitted signal \(f(t)\) and its spectrum \(F(\omega)\) the Fourier transforms

\[
|f(t)|^2 \overset{F}{\longrightarrow} a(\omega, \nu)
\]

and

\[
|F(\omega)|^2 \overset{F}{\longrightarrow} a(\tau, \omega)
\]

hold, one is able to control the axial ambiguity distributions \(a(\omega, \nu)\) and \(a(\tau, \omega)\) by properly choosing \(|f|^2\) and \(|F|^2\). When as we have implicitly
done, one assumes that either a specific target's range or its velocity is known and the other quantity is to be obtained, then it is of interest to make \( a(o,v) \) or \( a(\tau,o) \) sharply peaked and with low surrounding sidelobes. By maximizing the energy of \( a(o,v) \) within some closed doppler frequency region of the \( v \) axis, it was found that very low (theoretical) sidelobes of \( a(o,v) \) could be obtained elsewhere along the axis. The degree of sidelobe suppression is determined by the length of the radar signal \( T \) and the width \( 2f_t \) of the doppler frequency region of interest. Amplitude modulation of the radar pulse is necessary to achieve the desired control of \( a(o,v) \) and thus a peak-power limited radar operating in this mode suffers a slight target detection handicap compared to the same radar operating with no amplitude modulation. From Table 2 it appears that of the several amplitude modulations which might be used to achieve the same sidelobe level of \( a(o,v) \), the pulse shape defined in terms of the prolate spheroidal wave functions (APSWF) suffers slightly less target detection loss but leads to a somewhat (5 to 15%) broader central response peak of \( a(o,v) \) than other types of amplitude modulations investigated.

In Chapter IV, we find that a rectangular envelope transmitted pulse when APSWF nonlinearly frequency modulated and then matched filtered leads to the concentration of \( a(\tau,o) \) and the suppression of its sidelobes. The required frequency modulation characteristic is related to the prolate spheroidal wave functions. A very interesting result of this paper is concerned with the generalization of the idea of pulse compression radar signals by the concept of energy concentration of the signal autocorrelation function. We have shown that for the
APSWF-FM pulses, a definition of the autocorrelation duration based on its energy concentration leads to the result \( PD = \frac{\pi TW}{2} \). That is, when received and matched filtered, the radar pulse is compressed by a factor \( P \) which for a controlled \( a(\tau,0) \) sidelobe level (set by \( D \)) is proportional to the product of the transmitted signal duration \( T \) and nominal bandwidth \( W \).

The construction of a RLC lumped element filter matched to the APSWF-FM pulse is not exactly possible, however, optical signal processing techniques hold promise. In Chapter V we have investigated the S/N ratio losses incurred by slightly nonoptimum receiver processing of the APSWF-FM pulse.

Alternatively, it is easy to show that an APSWF-AM, linearly frequency modulated pulse allows a good approximation to the desired spectral distribution \( \sqrt{S_o(D,2f/W)} \), \( |f| < \frac{W}{2} \). This linearly frequency modulated pulse is compatible with existing CHIRP radar signal processors and allows the realization of controllably low sidelobe levels of the compressed correlation function \( a(\tau,0) \). When transmitted by a peak-power limited radar, this amplitude modulated degrades the receiver S/N ratio by 1 - 3.5 dB compared to an equal duration rectangular envelope pulse. The use of simultaneously amplitude and angle modulated signals should be considered when the advantages of simple receiver realization outweigh the accompanying receiver S/N ratio losses.
APPENDIX A

AN ANTENNA FIELD AND AMBIGUITY FUNCTION ANALOG

The far electric field of an antenna aperture current distribution may be obtained from the Fourier integral, (A-1).

\[ F(k \cos \theta) = \int_{-L/2}^{L/2} |A(z')| e^{-j k z' \cos \theta} e^{j k z \cos \theta} \, dz' \quad (A-1) \]

The inverse transform

\[ j \psi(z) \left| A(z) \right| e^{-j k z \cos \theta} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k \cos \theta) e^{j k z \cos \theta} \, dk \quad (A-2) \]

holds where

- \( k = \frac{2\pi}{\lambda} \), \( \lambda \) = wavelength of radiation,
- \( A(z') \) = aperture current distribution,
- \( L \) = aperture length,
- \( \theta \) = the angle that a ray to the point of field measurement makes with the normal to the aperture at its midpoint.

As has been noted, one may write the Fourier transform pairs:

\[ a(0, \nu) = \int_{-T/2}^{T/2} |f(t)|^2 e^{-j 2\pi \nu t} \, dt \quad (A-3) \]

\[ |f(t)|^2 = \int_{-\infty}^{\infty} a(0, \nu) e^{j 2\pi \nu t} \, d\nu \]
and \[ a(\tau,0) = \int_{-w/2}^{w/2} |F(\tau)|^2 e^{-j2\pi\tau f} df. \] (A-4)

\[ |F(\tau)|^2 = \int_{-\infty}^{\infty} a(\tau,0) e^{j2\pi\tau f} df. \]

Except for an unimportant sign difference in the exponential of (A-1), the following term may be thought of as analogous quantities in (A-1) and (A-3):

\[ L \sim T, \]
\[ A(z) \sim |\tau(\tau)|^2, \]
\[ k \cos \theta \sim 2\pi v, \]
\[ F(k \cos \theta) \sim a(0,v). \] (A-5)

Likewise in (A-1) and (A-4) analogous quantities are:

\[ L \sim W, \]
\[ A(z) \sim |F(\tau)|^2, \]
\[ k \cos \theta \sim 2\pi v, \]
\[ F(k \cos \theta) \sim a(\tau,0). \] (A-6)

It follows that a large body of results pertaining to analysis and synthesis of optimum antenna patterns may be applied to the problem of ambiguity function synthesis.

As an example, it is known that the Dolph-Chebyshev distribution of currents in a linear array of discrete sources results in a far electric field with a minimized mainbeam half-power beamwidth for a given sidelobe ratio. Van Der Maas [49] has shown that the extension
of Dolph's results, for the case of an infinite number of sources that
in the limit is a continuous source of length \( L \), yields the far electric
field

\[
F(k \cos \theta, A) = \cos \left[ \frac{\left( \frac{kL \cos \theta}{2} - (\pi A)^2 \right)^{1/2}}{2} \right], \quad (A-7)
\]

with \( R = \) the desired sidelobe ratio,

\[
R = \cosh \pi A, \quad (A-8)
\]

and

\[
k \cos \theta_{HP} = \frac{2}{L} \left[ (\cosh^{-1} R)^2 - (\cosh^{-1} \frac{R}{\sqrt{2}})^2 \right]^{1/2}. \quad (A-9)
\]

Employing the analog relations of (A-5) one can rewrite (A-7)
through (A-9) to obtain expressions for the doppler axis ambiguity
distributions \( a(0, v|A) \) which are optimum in the Dolph-Chebyshev sense.

\[
a(0, v|A) = \cos \left[ \left( \frac{vT}{2} - (\pi A)^2 \right)^{1/2} \right], \quad (A-10)
\]

\[
R = \cosh \pi A, \quad (A-11)
\]

\[
\Delta v_{HP} = \Delta \text{ the width of the } a(0, v|A) \text{ central peak between } -3 \text{ db points},
\]

\[
= 2 \pi v_{HP},
\]

and

\[
\Delta v_{HP} = \frac{2}{T} \left[ (\cosh^{-1} R)^2 - (\cosh^{-1} \frac{R}{\sqrt{2}})^2 \right]^{1/2}. \quad (A-12)
\]

The field of (A-7) is not absolutely integrable and is not physically realizable by a continuous aperture current distribution.

Walter [32, p. 146] points out that a continuous current distribution
of length \( L \) with an additional (infinite amplitude) point source at
each end, in theory, produces the desired filed. The analogous radar
signal would be a pulse of duration $T$ with infinite leading and trailing edge impulses. The continuous portion of the pulse has a shape related to a modified Bessel function of order one.

Extensive use has been made in Chapter III of the similarities of the antenna far field synthesis problem to the synthesis of radar ambiguity functions.
THE PROLATE SPHEROIDAL WAVE FUNCTIONS

The angular prolate spheroidal wave functions satisfy the differential equation

\[ \frac{d}{dx} \left[ (1-x^2) \frac{d}{dx} S_{mn}(c,x) \right] + \left[ \lambda_{mn} - c^2 x^2 - \frac{m^2}{1-x^2} \right] S_{mn}(c,x) = 0 . \]

(B-1)

Over the range \(|x| < 1\), the functions may be expanded into a series of associated Legendre polynomials.

\[ S_{mn}(c,x) = \sum_{r=0,1}^{\infty} d_r^{mn}(c) P^m_n(x) \quad |x| < 1 \quad (B-2) \]

The prime on the summation sign means that when \(m - n\) is even, sum over even \(r\), and when \(m - n\) is odd, sum over odd \(r\).

The functions \( S_{mn}(c,x) \), \( n = 0,1,2,\ldots \), possess rather unusual properties in that they are, for a given \((m,c)\), orthogonal over both the ranges \((- \infty < x < \infty)\) and \((-1 < x < 1)\) [33, p. 45]. Of more immediate interest is the fact that the energy spectrum \( S(\omega) S^*(\omega) \) of the member \( S_{oo}(c,x) \), \( |x| < 1 \), is the most concentrated of all energy spectrums of finite length functions. By the term "concentration" we mean that for a given frequency \( \pm W/2 \) and function duration \((-1,1)\), \( S_{oo}(c,x) \) is that function which maximizes the ratio \( \gamma \).
Fortunately $S_{00}(c,x)$ has an especially simple expression in terms of the simple Legendre polynomials.

$$S_{00}(c,x) = \sum_{r=0}^{\infty} d_r^{\infty}(c) P_r(x) \quad |x| \leq 1 \quad (B-4)$$

Following the normalization scheme of Flammer, the coefficients $d_r(c)$ may be adjusted so that

$$S_{00}(c,0) = 1 \quad (B-5)$$

The expansion coefficients $d_r(c)$ are obtained from a recursive difference equation which is rather difficult to solve. A limited number of tabulated values of $d_r(c)$ are available in Flammer [34, p. 100] for $c = 0, 0.5, 1.5, 2.5$ and $r = 0(2)14$. Additionally, Stratton et al. [50] have tabulated the expansion coefficients $d_r(h|m|l)$, $h = 0(2)8.0$, with the changes of notation of $h$ replacing Flammer's $c$ and $l$ replacing $n$.

The normalization procedure of Stratton sets $S_{m\lambda}(h,l) = 1$ thus necessitating a conversion to coefficients in Flammer's expansion (B-2) and (B-4) in order to retain $S_{mn}(c,0) = 1$. The conversion relation is given simply by (B-5) for the $m=n=0$ case.

$$d_r^{\infty}(c) = \frac{d_r(h|oo)}{\sum_{j=0}^{\infty} \left(\frac{(-1)^{j/2}}{(j)!}\right) \frac{d_j(h|oo)}{2j \left[\frac{(j)!}{2}\right]}} \quad (B-6)$$
Table (B-1) presents an extension in accuracy and length of Flammer's tabulations of $d_{\nu}(c)$ as obtained by desk calculator computations carried to 8 to 10 decimal places. The results have been rounded off to 6 decimal places. This accuracy has been found sufficient to produce the normalization condition, (B-5), within an absolute error $|\varepsilon| \leq 1 \times 10^{-5}$ for all $0 \leq c \leq 8$ when checked by an IBM 7094 digital computer.

In region $|x| \geq 1$, the angular spheroidal wave function $S_{\infty}(c,x)$ is joined to a second solution of the differential equation (B-1) by a constant,

$$S_{\infty}(c,x) = k_0(c) R_{\infty}(c,x),$$

$$R^{(1)}_{\infty}(c,x) = \text{radial prolate spheroidal wave function of the first kind, } |x| \geq 1.$$

The radial functions may be expanded on $|x| \geq 1$ in a series of spherical Bessel functions $j_{r}(x) = \frac{\pi}{2x} J_{r+\frac{1}{2}}(x)$, where $J_r$ is the familiar Bessel function.

Thus

$$R^{(1)}_{\infty}(c,x) = \sum_{r=0}^{\infty} a_{r}^{\infty}(c) j_{r}(x).$$

The coefficients $a_{r}(c)$ are available from Flammer [34] or as $a_{r}(h|\infty)$ from Stratton, [50]; however, recall that in the latter source the normalization is taken to be $S_{\infty}(c,1) = 1$ rather than Flammer's $S_{\infty}(c,0) = 1$. Accordingly, in order to obtain results consistent with the normalization $a(0,0) = ES_{\infty}(c,0) = 1$, one merely applies (B-6) as in (B-9).
A useful relation of Stratton [p. 56] shows that

\[ d_r(h|\infty) = (-1)^{3r/2} a_r(h|\infty), \quad (B-10) \]

\[ r = 0, 2, \ldots, \]

which in view of (B-6) and (B-9) allows one to conclude immediately

\[ d_r(c) = (-1)^{3r/2} a_r(c), \quad (B-11) \]

\[ r = 0, 2, \ldots. \]

The Joining Factor

For the general solutions of the differential equation (B-1), one has for \(|x| \geq 1, \)

\[ S_{mn}(c, x) = k_{mn}^{(1)}(c) R_{mn}^{(1)}(c, x), \quad (B-12) \]

which was specialized to obtain (B-7) by allowing \(m=n=0\). When \((n-m)\) is even,

\[ k_{mn}^{(1)}(c) = \frac{\sum_{r=0}^{\infty} d_r^{mn}(c) \frac{2m+r}{(r)!}}{2^{n+m} d_0^{mn}(c) c^m \frac{n-m}{2} \frac{m+n}{2}!}. \quad (B-13) \]
A comparable result exists for \((n-m)\) odd [34, p. 30]. If one sets 
\(m = n = 0\),

\[
\kappa_\infty (1)(c) = \frac{\sum_{r=0}^{\infty} d_r(c)}{\lim_{\infty}} \tag{B-14}
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Table 6. Values of $d_r(c) = D_r(c) \times 10^{-Pr(c)}$
REFERENCES


