UNGAR, Edward William, 1936–
THE EFFECT OF FLUID INJECTION AND
SUCTION ON THE LAMINAR BOUNDARY
LAYER/SHOCK WAVE INTERACTION.

The Ohio State University, Ph.D., 1966
Engineering, aeronautical

University Microfilms, Inc., Ann Arbor, Michigan
THE EFFECT OF FLUID INJECTION AND SUCTION ON THE
LAMINAR BOUNDARY LAYER/SHOCK WAVE INTERACTION

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By


* * * * * * *

The Ohio State University
1966

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ACKNOWLEDGMENTS

I would like to express my appreciation to Professor L. S. Han of the Mechanical Engineering Department, The Ohio State University, for our many discussions and his suggestions during the conduct of this research. They served both as sources of continuous encouragement and insight to the author. The able assistance of Dr. R. Reeves and his staff in the Numerical Computation Laboratory was essential to the performance of the numerical integrations reported in this dissertation. The valuable comments of Professors R. S. Brodkey, T. Y. Li, and S. M. Marco on the draft of this dissertation were quite helpful to the author.

I would like to thank J. M. Allen and F. L. Bagby of Battelle Memorial Institute for their encouragement during this work. The fellowship assistance provided by Battelle Memorial Institute during part of this research was an invaluable aid to the author.

My wife Barbara provided considerable assistance and strength to the author during the conduct of this research. Particular thanks are due for her perserverance and typing assistance during the preparation of the rough draft. My children, Michele and Mark, received less than their fair share of attention during this research effort.

The help provided by Mrs. Jane Nolan in the final typing of this dissertation is appreciated.
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NOMENCLATURE

a  Sonic velocity

$a_0 ... a_4$  Coefficients in the outer velocity profile, Equation (34)

B  Integrated blowing function defined by Equation (40)

$b_0 ... b_3$  Coefficients in the inner velocity profile, Equation (37)

c  Wall velocity gradient

c$\_0 ... c\_4$  Coefficients in the attached flow velocity profile, Equation (107)

$c_r$  Shear stress coefficient,

$C_p$  Pressure coefficient,

$D_{ij}$  Coefficients in the differential Equations (55), (56) and (57)

$E_{ij}$  Coefficients in the differential Equations (115), (116) and (117)

$F(M_0)$  Function defined by Equation (77)

$k_{su}$  Sutherland constant

L  Distance from the plate leading edge to shock impingement point

M  Mach number

$M_{es}$  Mach number at separation

$M_0$  Mach number far from the interaction

$M_{ou}$  Mach number upstream of interaction

$M_{od}$  Mach number downstream of interaction

m  Mass flux in the boundary layer
NOMENCLATURE—Continued

p                Pressure
Pr               Prandtl number
R_0              \( a^0 \delta_{js} / \nu^0 \)
R_x              \( a^0 \tau / \nu^0 \)
Re_x             \( \mu_e \tau / \nu_e \)
Re_{x,L}         \( \mu_0 L / \nu_0 \)
Re_{x,0}         \( \mu_e x_0 / \nu_e \)
Re_{\delta^*}   \( \mu_e \delta^* / \nu_e \)
T                Temperature
U                Transformed velocity in X direction, Equation (6)
u                Velocity in the direction parallel to the wall
U                \( U/U_e \)
U_s              Transformed velocity in X direction at the separating streamline
s                \( U_s/U_e \)
V                Transformed velocity in Y direction, Equation (6)
v                Velocity in the direction normal to the wall
w                \( \mu_e / a^0 \)
w_s              Value of \( \mu_e / a^0 \) at separation
X                Transformed coordinate, Equation (5)
x                Distance measured parallel to the wall
X                \( X/ \delta_{js} \)
NOMENCLATURE—Continued

\( x_s \)  
Value of \( x \) at separation

\( Y \)  
Transformed coordinate, Equation (5)

\( y \)  
Distance measured normal to the wall

\( Y_s \)  
Transformed distance to separating streamline

\( y_s \)  
Distance measured normal to the wall to the separating streamline

\( \beta_e \)  
\( \rho_w v_w/\rho_e u_e \)

\( \beta_i \)  
\( v_w/U_e \)

\( \gamma \)  
Ratio of specific heats

\( \Delta \)  
Upper limit of integration of momentum Equation (10)

\( \delta \)  
Boundary layer thickness

\( \delta_i \)  
Transformed boundary layer thickness

\( \delta_{i,s} \)  
Transformed boundary layer thickness at separation or at the start of the interaction for the complete interaction

\( \delta_s \)  
Value of \( \delta \) at separation or at the start of the interaction for the complete interaction

\( \delta^* \)  
Boundary layer displacement thickness

\( \delta_i^* \)  
Transformed displacement thickness

\( \delta_{i,i} \)  
Inner layer displacement thickness, Equation (19)

\( \delta_{i,i}^* \)  
\( \delta_i/\delta_{i,s} \)

\( \gamma \)  
\( y/\delta_i \)

\( \gamma_s \)  
\( Y_s/\delta_i \)

\( \theta \)  
Prandtl-Meyer turning angle
NOMENCLATURE—Continued

\( \theta_i \) Transformed momentum thickness, Equation (12)

\( \theta_{ii} \) Inner layer momentum thickness, Equation (19)

\( \theta^* \) Boundary layer energy thickness

\( \theta_i^* \) Transformed energy thickness

\( \Lambda \) Pohlhausen parameter, Equation (64)

\( \lambda \) Chapman-Rubesin constant

\( \mu \) Absolute gas viscosity

\( \nu \) Kinematic gas viscosity

\( \rho \) Gas density

\( \gamma \) Shear stress

\( \phi \) Flow angle at the edge of the boundary layer

\( \psi \) Stream function

Subscripts

\( w \) Wall

\( e \) Outer edge of the boundary layer

\( o \) Value at beginning of interaction

Superscript

\( o \) Stagnation value
I. INTRODUCTION

The interaction of a shock wave with a boundary layer produces an interesting class of flows in which the pressure distribution and boundary layer growth are coupled. The disturbance produced by the shock wave impingement is transmitted upstream via the subsonic gas in the boundary layer. Boundary layer thickening turns the external supersonic flow and induces a static pressure gradient. The induced, adverse pressure gradient further thickens the boundary layer and, even with relatively weak shock waves, causes separation of laminar boundary layers. It can be expected that fluid injection or suction at the wall should have significant effects on the boundary layer growth and pressure distribution throughout the interaction. In particular, small amounts of injection or suction within the separated region should strongly influence the boundary layer behavior.

The research reported here was directed toward the development of a theoretical model to describe injection and suction effects on the shock wave/laminar boundary layer interaction on an adiabatic, flat plate. This model is confined to sufficiently small injection and suction rates that boundary layer theory is considered valid. The injectant is assumed to have the same composition as the fluid in the main stream.

Boundary layer/shock wave interactions were first observed by Ferri (1). Early experiments which defined the nature of the
interaction were conducted by Liepmann (2), Fage and Sargent (3),
Ackeret, Feldman and Rott (4), and Gadd, Holder, and Regan (5).
Chapman, Kuehn, and Larson (6), and Hakkinen et al. (7) obtained
detailed data on pressure distributions and flow separation. Greber
(8) obtained some experimental data which illustrates the effects of
suction on the interaction.

Figure 1 schematically illustrates the interaction of a shock
wave with a laminar boundary layer without suction or blowing.
Qualitatively, the pressure rises rapidly until the boundary layer
separates. After separation, the pressure gradient diminishes and
can essentially vanish upstream of the shock impingement point.
Appreciable boundary layer thickening occurs in this plateau region.
Downstream of the shock impingement point, reattachment occurs along
with significant compression. The circulation region is bounded by
the separating streamline. Injection and suction can be expected to
open the circulation region since mass cannot cross the separating
streamline. Also, as observed by Greber (8), suction should shorten
the interaction region while injection can be expected to lengthen
the interaction region. This general description is equally applica-
table to a compression corner in which the incident shock wave is
induced by the turning of the supersonic, external inviscid flow.

Though not considered in the present research, the shock wave
interaction tends to promote transition to turbulent flow in the
boundary layer. The effect of transition on the interaction is
illustrated by the data of Chapman, Kuehn, and Larson (6) and
Figure 1. Schematic of laminar boundary layer/shock wave interaction.
Greber (8). Greber further observed that suction tends to retard transition in the interaction region. The occurrence of transition in experiments limits the extent of comparison possible with a laminar boundary layer theory.

The development of a theoretical model suitable for the description of injection and suction effects must rely on a suitable description of the ordinary interaction without injection or suction. This description must be cast into a model which can easily incorporate injection and suction effects. It is the development of this model that is the subject of this research. Consequently considerable attention must be devoted to past research on ordinary interactions.

Considerable past effort has been expended to develop theoretical models of the ordinary interaction. These models must both agree with available data and be suitable for prediction of pressure distributions. Integral methods, and, in particular the theory of Lees and Reeves (9), have been applied with the most over-all success. Bray, Gadd, and Woodger (10) accounted for suction within the context of the mixing theory of Crocco and Lees (11,12).
II. SUMMARY OF INTERACTION THEORIES

Three basic approaches have been followed in past efforts to develop theoretical models. These approaches have relied on methods of small disturbance theory, mixing theory, and on integral methods of boundary layer analysis. Since the present research must lead to a general model of a boundary layer/shock wave interaction, it is appropriate to begin with a review of the various approaches which have been applied to the analysis of ordinary interactions.

Small Disturbance Methods

Howarth (13) analyzed the effects of a pressure disturbance generated in a uniform supersonic stream which is bounded on one side by a uniform subsonic stream. Both streams are semi-infinite and the interface is planar. Assuming small disturbances in a perfect, compressible fluid, Howarth principally showed that the disturbances are propagated upstream through the subsonic stream. To simulate a boundary layer, Tsien and Finsten (14) replaced the semi-infinite subsonic stream with a subsonic stream of finite thickness. Small disturbances of an inviscid, perfect fluid are assumed, as in Howarth's previous analysis. Results obtained by Tsien and Finsten predict considerably less upstream influence of the shock wave than is inferred from measured pressure distributions. However, their results did demonstrate the qualitative behavior of the streamwise pressure
distribution and the magnitude of some transverse pressure differences across the subsonic stream. They explained the differences between laminar and turbulent interactions on the basis of differences in the thickness of the subsonic layer. This explanation, based on differences in the subsonic layer thickness, was later disputed by Liepmann, Roshko, and Dhawan (15).

Some of the real boundary layer effects neglected by Tsien and Finsten (14) are clearly noted by Liepmann et al. (15). They observed experimentally that marked differences exist between laminar and turbulent interactions. They also noted that in the laminar interaction the disturbance extends about 50 boundary layer thicknesses upstream of the shock impingement point and that separation almost always occurs. In contrast, the pressure disturbance was found to extend only about 5 boundary layer thicknesses upstream in a turbulent interaction and no separation was observed.

Noting the discrepancies between the theory of Tsien and Finsten (14) and experimental data, Lighthill (16) extended the small disturbance theory to include a prescribed velocity distribution in the boundary layer. Surprisingly, Lighthill's calculations predict even less upstream influence of the shock wave than Tsien and Finsten. However, Lighthill neglected flow separation and viscous effects near the wall. Lighthill, in a later paper (17), noted the importance of separation and upstream boundary layer thickening on the pressure distribution. In this paper, Lighthill evaluated the
upstream influence of a flow separation in inviscid subsonic and supersonic flows.

For the case where flow separation does not occur, Lighthill (18) incorporated viscous effects into an improved small disturbance model. Viscous effects are confined, in this model, to a thin, low velocity region within the boundary layer. Propagation of the pressure disturbance in the "exterior" region of the boundary layer is analyzed. The resulting theory meets many of the objections to previous small disturbance theories. Differences between laminar and turbulent interactions are attributed to differences in the extent of the "inner" viscous region, rather than the subsonic region.

Philosophically at least, as noted by Lighthill (18), the "inner" viscous region can be extended to include a flow circulation region to simulate separation. However, this approach would neglect the inherent limitation of small disturbance models; i.e., that the nature of the disturbance remains essentially linear. In general, the effects of boundary thickening and separation are large and nonlinear. These difficulties are resolved by the use of mixing layer and boundary layer integral methods to describe the interaction.

**Mixing Layer Method**

The shortcomings of the small disturbance theories clearly demonstrated the need for a model which properly accounts for the nonlinear features of the flow. The Crocco-Lees Theory (11,12) introduces the idea that the mixing process between the external,
essentially isentropic flow and the boundary layer primarily controls the interaction. The mixing process consists of the entrainment of fluid in the viscous flow close to the wall. Mixing is assumed to take place in a uniform, dissipative region next to the wall. This mixing layer replaces the boundary layer by a region in which average boundary layer properties are preserved. The streamwise growth of the dissipative layer, resulting from flattening of the velocity profile, reverse flow, and entrainment of fluid, deflects the external supersonic flow. The deflection of the supersonic flow induces a variation in the static pressure. Mathematically, the growth of the dissipative layer, obtained from conservation of mass and momentum, is coupled to the pressure distribution by the Prandtl-Meyer relation.

Heat transfer and suction effects were considered by Bray (19,20) and Bray, Gadd, and Woodger (10).

Appendix A describes the basic Crocco-Lees formulation and a digital computer program developed for calculations with the theory. The formulation is basically the one presented by Bray et al. (10). This formulation was programmed for computation on the IBM 7094 computer in The Ohio State University Computation Center.

In the mixing layer method the weighting factors which determine average boundary layer properties are contained within the Crocco-Lees parameters. These weighting factors should reflect the nature of the actual velocity profile. The use of exact solutions to the boundary layer equations as the source for these weighting factors would tend to reproduce the boundary layer flow characteristics.
Originally, the Crocco-Lees parameters were derived from the upper-branch of the Falkner-Skan similar solutions (11,12). These are solutions to the boundary layer equations obtained by a Stewartson transformation (21) of the compressible, laminar boundary layer equations. The values of parameters derived from an exact solution to the transformed boundary layer equations depend on the chosen definition of boundary layer thickness (velocity ratio at the edge of the boundary layer). The difference between choices of the boundary layer thickness definition concerned Crocco and Lees (11), Crocco (12), Cheng and Bray (22), and Cheng and Chang (23) in their development and application of the theory based on Falkner-Skan upper-branch solutions. Finally, Quick (24) noted that the results of computations are relatively insensitive to the boundary layer thickness definition as long as the definition is reasonable and consistently followed.

Stewartson (25) noted that the lower-branch Falkner-Skan solutions, which correspond to back-flow, might provide a suitable representation of the flow in a separated region. Following this suggestion, Bray et al. (10) used the lower-branch Falkner-Skan solutions to evaluate Crocco-Lees parameters in the separated flow region of the interaction. By considering heat transfer and suction distributions consistent with the similar profile requirements, Bray et al. obtained modified velocity profiles from the similar boundary layer solutions. Using these modified profiles, they were able to take some account of heat transfer and suction in the
interaction. They found that cooling and suction both reduced the length of the interaction region. The results obtained by this method appear to agree qualitatively with intuition and available data; however, quantitative agreement was not obtained.

Figure 2 shows the pressure distribution in a shock wave/boundary layer interaction calculated with the computer program described in Appendix A. The conditions selected for the calculation match Bray, Gadd, and Woodger's Case 4 (10). The Crocco-Lees parameters tabulated by Bray et al. (10), for upper- and lower-branch Falkner-Skan solutions were used in the calculation. The calculated pressure distribution has the same general appearance as experimentally observed pressure distributions. There is a very pronounced pressure maximum in the plateau region, which is physically unrealistic. As described in Appendix A, the calculation begins with a specified value of \( (m/\mu^*) \) at the separation, where \( (m/\mu^*) \) is essentially the form of a Reynolds number based on the separation boundary layer thickness. The calculation requires the use of a double iteration procedure to match the Mach numbers at the beginning and end of the interaction. This procedure is somewhat cumbersome if direct comparisons with experimental data are desired. Attempts to make direct comparisons between the Crocco-Lees theory and experimental data during the present research were thwarted by lack of prior knowledge of the separation value of \( (m/\mu^*) \) for the conditions corresponding to the experiments. In one calculation a separation value of \( (m/\mu^*) \) was obtained from the velocity profile, boundary layer
Figure 2. Pressure distribution calculated by the mixing layer method and the Cross-Loes parameters derived by Bray, Gall, and Woodgar.
growth, and pressure rise to the separation point predicted by the
theory developed in the present research and described later in this
dissertation. With this input, the computer program described in
Appendix A was used to compare the Crocco-Lees theory with the experi-
mental results of Chapman, Kuehn, and Larson (6) for $M_{ou} = 2.4$,
$P/p^0 = 2.15$, $Re_L = 5.4 \times 10^4$ (see Figure 16). The calculations
indicated that the interaction region would be considerably longer
than was observed experimentally. In fact, the calculated interaction
would have to extend beyond the plate leading edge in order to
accommodate the required pressure rise. This resulted from inadequate
mixing and the resultant growth of the mixing layer between separa-
tion and the shock impingement point. Consequently, the computations
were stopped without ever matching the shock wave pressure rise.
Further comparisons with experimental data were not attempted.

Glick (24) introduced a significant modification to the Crocco-
Lees theory in order to improve the quantitative agreement with
available experimental data. Upstream of separation, Glick replaced
the mixing rate parameter derived from the Falkner-Skan solutions
with a correlation relation. This correlation reflects the changing
character of the boundary layer as separation is approached and is
derived from nonsimilar boundary layer analyses. Downstream of
separation Glick incorporated Chapman's dividing streamline concept
(26) and the results of Chapman, Kuehn, and Larson (6) into a semi-
empirical method to obtain the Crocco-Lees parameters. This approach
produces good correlations with experimental data.
Erdos and Pallone (27) analyzed the separated, supersonic flow over a forward facing step by the Crocco-Lees method. They obtained the parameters from the results of Chapman, Kuehn, and Larson's (6) experiments for both laminar and turbulent boundary layers. In this approach, the mixing layer theory provides a method of extrapolating experimental data. Erdos and Pallone suggest the use of nonsimilar solutions to the boundary layer equations for correlation relations where experimental data are not available.

The lack of universal Crocco-Lees parameters and the difficulties of extrapolation from a semi-empirical theory suggest the need for a new approach. This new approach relies on the more common boundary layer integral methods and does not require the use of coefficients derived from experimental data.

**Integral Methods**

The use of integral methods of obtaining solutions to the boundary layer equations can eliminate the necessity for experimentally derived coefficients. In the simple Karman-Pohlhausen integral method (28) the velocity profile is related to a single parameter which, in turn, is related directly to the local pressure gradient by the wall boundary condition. This simple relation between the single parameter, which determines the velocity profile, and the pressure gradient is obviously inadequate to handle the shock wave/boundary layer interaction. For example in the plateau region, the pressure gradient essentially vanishes, yet the velocity
profile is not the flat plate velocity profile. Thus Martellucci and Libby (29) were not able to predict the occurrence of the pressure plateau by application of the Karman-Pohlhausen method.

Curle (30) extended Thwaites' technique (31) to laminar, supersonic separating and separated boundary layers. Stewartson's lower-branch Falkner-Skan solutions (25) are used to derive the Thwaites correlation functions for the separated region. Curle obtained excellent agreement with experiments in the attached region. However, in the separated region, Curle did not predict the characteristic pressure plateau.

Two-parameter, modified Pohlhausen methods have been applied to achieve a good representation of the pressure distribution in a laminar boundary layer/shock wave interaction. In particular, Makofski (32) obtained good results with a velocity distribution given by a fifth order polynomial. However, these approaches are complex and difficult to use in flow calculations. Nielsen, Lynes, Goodwin, and Holt (33) apply the integral method of Dorodnitsyn (34) to boundary layer/shock wave interactions. The Dorodnitsyn method allows the use of higher approximations to the integral equations. Nielsen et al. used a fourth approximation which is essentially equivalent to the use of a four-parameter velocity profile. They achieved a good match with experimental data in the attached flow region but had to artificially impose the condition of constant pressure to achieve an adequate representation of the pressure plateau after separation.
To eliminate the direct dependence of the pressure gradient on the velocity profile, an additional equation is required beyond the Karman momentum equation in a single parameter method. A number of approaches which supply this equation by introducing at least one additional moment of the momentum equation have been suggested (35). Tani (36) used a second moment equation to replace the Pohlhausen wall boundary condition and a quartic velocity profile related to the wall velocity gradient. He used the Stewartson transformation (21) to extend the results to compressible flows. Abbott, Holt, and Nielsen (37) applied Tani's method with a quartic velocity profile to shock wave/boundary layer interactions but obtained large pressure maxima in the plateau region and consequently did not obtain realistic pressure distributions in the separated region. Honda (38) used a fourth order polynomial distribution to obtain good results in the attached flow region. He patched the attached and separated flows together assuming a linear growth of the reverse flow region beyond separation. His model does not include a laminar boundary layer reattachment.

Lees and Reeves (9) follow Tani with the use of a second moment equation. In addition, following the idea that exact solutions to the boundary layer equations can be used to improve the results of integral methods, they use the velocity profiles described by the Falkner-Skan similar solutions. Upper-branch solutions are used for the attached flow and lower-branch solutions for the
separated flow. Lees and Reeves do indeed obtain good correlation with available experimental data. In many respects, the theory of Lees and Reeves represents a significant level of development in the analysis of shock wave/laminar boundary layer interactions without injection or suction.
III. MODEL FORMULATION

The model developed here for analysis of injection and suction effects on the shock wave-boundary layer interaction is based on integral methods for approximately satisfying the compressible laminar boundary layer equations. The static pressure variation results from deflection of the supersonic stream by the growing boundary layer. It is implicitly assumed in this research that Reynolds numbers are sufficiently large and transverse velocities and pressure gradients sufficiently small that boundary layer theory is applicable. The details of the flow at the shock wave-boundary layer merger are important only in a region which is small compared to boundary layer dimensions and are neglected. Finally, the formulation is restricted to cases where cross-flow is negligible. While conventional methods are considered adequate for the attached flow, a new model is developed here to describe wall injection and suction effects on the separated flow.

For the steady, laminar flow of a compressible fluid, the continuity equation is

\[ \frac{3}{3x} (\rho u) + \frac{3}{3y} (\rho v) = 0 \]  

(1)

and the momentum equation is

\[ \rho u \frac{3u}{3x} + \rho v \frac{3v}{3y} = -\frac{3p}{3x} + \frac{3}{3y} (\mu \frac{3u}{3y}) \]  

(2)
The appropriate boundary conditions are
\[ u(x, o) = 0 \]
\[ v(x, o) = v_w(x) \]
\[ \lim_{y \to \infty} u = u_e(x) \] (3)

The wall velocity, \( v_w \), is a prescribed function of \( x \) and the external velocity, \( u_e \), is obtained from additional equations describing the interaction. The positive \( x \) direction is taken in the streamwise direction with the origin at either the start of interaction or the separation point.

The Cohen and Reshotko modification (39) to the Stewartson transformation (21) is used to transform Equations (1), (2), and (3) into an equivalent incompressible form. In applying the transformation it is assumed that the fluid is a perfect gas, that changes in the external flow which are induced by boundary layer growth are isentropic, and that the fluid viscosity follows the law
\[ \frac{\mu}{\mu^0} = \lambda \frac{T}{T^0}, \] (4)

where \( \lambda \) is the Chapman-Rubesin constant given by
\[ \lambda = \sqrt{\frac{T_0}{T^0}} \left( \frac{T^* + k_{su}}{T_0 + k_{su}} \right), \]

and \( k_{su} \) is the Sutherland constant (\( k_{su} = 198.6^\circ R \) for air). For an adiabatic wall and Prandtl number of unity, \( \lambda = 1 \).
A stream function and transformed coordinates are defined by
\[
\begin{align*}
\frac{\partial \psi}{\partial y} & = \frac{\rho u}{\rho_0} \\
\frac{\partial \psi}{\partial x} & = -\frac{\rho v}{\rho_0} \\
X & = \int \lambda \frac{\rho}{\rho_0} \frac{\partial y}{\partial \psi} \, dx \\
Y & = \frac{\alpha_0}{\alpha} \int \frac{\rho}{\rho_0} \, dy
\end{align*}
\]  
(5)

Also, velocities are defined in the transformed plane by the relations
\[
\begin{align*}
U & = \frac{\partial \psi}{\partial Y} \\
V & = -\frac{\partial \psi}{\partial X}
\end{align*}
\]  
(6)

Then, for adiabatic wall conditions, the transformed continuity equation is
\[
\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0
\]  
(7)

and the momentum equation is
\[
U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = U_t \frac{\partial U}{\partial X} + v \frac{\partial^2 U}{\partial Y^2}
\]  
(8)

with the boundary conditions
\[
\begin{align*}
U(X,0) & = 0 \\
V(X,0) & = V_w(X) \\
\lim_{Y \to \infty} U & = U_e(X)
\end{align*}
\]
Separated Boundary Layer

In the separated flow, the boundary layer can be divided into two regions by the separating streamline, as shown in Figure 3. The position of the separating streamline, \( y_s \), is defined by the mass balance

\[
\int_0^{y_s} \rho u \, dy = \int_0^x \rho_\omega \nu_\omega \, dx,
\]

where \( x = 0 \) at separation. The inner region defined by \( y < y_s \) contains recirculating fluid and the mass injected beyond the separation point. The outer region defined by \( y > y_s \) sustains the separation velocity profile and must account for velocity profile changes resulting from the growth of the recirculation region. The proposed theoretical model emphasizes the importance of these separate regions by requiring conservation of momentum in each and by merging the velocity profile at the streamline dividing the two regions. Wall injection and suction will primarily effect the inner region.

Two first-order, nonlinear differential equations in terms of the independent variable \( X \) are obtained by requiring that the momentum equation (Equation 8) be (i) satisfied on the average over the entire boundary layer (Karman momentum equation), and (ii) satisfied on the average between the wall and the separating streamline. This approach is in the spirit of a method developed by Pallone (40) for the analysis of nonsimilar, compressible boundary layers. In Pallone's method, the boundary layer is divided into \( N \) strips and it is required that the momentum equation be satisfied exactly over each strip. The boundary layer history is then obtained from the
Figure 3. Separating and separated boundary layer with transformed coordinates.
simultaneous solution of the N momentum equations. The use of the separating streamline in the present research constituting a basic departure from previous work (instead of the fixed boundary layer position in Pallone's method), greatly simplifies the application of the method to the porous wall problem and as to be shown later, clarifies some of the physical features of the flow.

The momentum equation can be rewritten as

\[
\frac{2}{\partial X} [U(U_e - U)] + \frac{2}{\partial Y} [V(U_e - U)] + \left[ U_e - U \right] \frac{dU_e}{dX} = - \frac{\partial}{\partial Y} \frac{\partial U}{\partial X} \tag{10}
\]

Assurance that the momentum equation be satisfied across the entire boundary layer is obtained by requiring that the integral of Equation (10) across the boundary layer is satisfied. Multiply Equation (10) by dY and integrate from Y = 0 to Y = A, where A is sufficiently large that U = U_e and the boundary conditions are given by Equation (9). The resulting equation is the Karman momentum equation with an added wall velocity parameter,

\[
\frac{\Theta_i U_e}{\nu^*} \frac{d\Theta_i}{dX} + \frac{\Theta_i}{\nu^*} \frac{dU_e}{dX} (z + \Sigma_i) = \frac{\Theta_i}{\sigma^*} \left( \frac{\partial \sigma^*}{\partial Y} \right) + \beta_i \frac{\Theta_i U_e}{\nu^*}, \tag{11}
\]

where quantities in Equation (11) are defined by

\[
\begin{align*}
\Sigma_i &= \int_{\delta_i}^{\eta} (1 - \sigma) dY \\
\Theta_i &= \int_{\sigma}^{\Sigma_i} \sigma (1 - \sigma) dY \\
\sigma &= U / U_e \\
\eta &= Y / \delta_i \\
\beta_i &= \nu_{\omega} / U_e \\
\end{align*} \tag{12}
\]
It is noted that the blowing parameter in the incompressible plane, \( \beta_1 \), is related to the quantity

\[
\beta_c = \frac{P_\infty v_\infty}{P_e u_e}
\]  

(13)

via the transformation defined by Equation (5). The transformation for \( \beta_1 \) is shown in Appendix B to be

\[
\beta_1 = \left(1 + \frac{r_{-1}}{M_e^2} \right) \beta_c
\]

(14)

Generally, \( \beta_c \) can be considered as a prescribed function of \( x \).

A second equation is now obtained by requiring that Equation (8) be satisfied on the average between the separating streamline and the surface. This is accomplished by replacing the upper integration limit, \( \Delta \), by the distance to the separating streamline, \( Y_s \). The resulting equation is

\[
\int_0^{Y_s} \frac{2}{\delta X} [U(u_e - U)] dY + \int_0^{Y_s} \frac{2}{\delta Y} [V(u_e - U)] dY + \int_0^{Y_s} \frac{2}{\delta X} (u_e - U) dY = \int_0^{Y_s} \frac{\partial \Sigma}{\partial Y} dY.
\]

(15)

Noting that \( Y_s \) depends on \( x \), the first integral in Equation (15) can be written as

\[
\int_0^{Y_s} \frac{2}{\delta X} [U(u_e - U)] dY = \frac{2}{\delta X} \int_0^{Y_s} [U(u_e - U)] dY
\]

\[
- \left[ u_s (u_e - U_s) \right] \frac{dY_s}{dX},
\]

(16)

where \( U_s \) is the velocity at the separating streamline and \( \frac{dY_s}{dX} \) represents the growth of the recirculation zone.
Since \( Y_s \) is measured to a streamline,

\[
\frac{V_s}{U_s} = \frac{dY_s}{dX}.
\]  

(17)

Substitution of Equations (16) and (17) into Equation (15) leads to

\[
\frac{\Theta_{id}}{\nu^2} \frac{d}{dX} \frac{dU_c}{dY} + \frac{\Theta_{id}}{\nu^2} \frac{dU_c}{dX} (\frac{\delta_{zd}}{\nu}) \left( \frac{\partial \sigma}{\partial \eta} \right)_{o} - \left( \frac{\partial \sigma}{\partial \eta} \right)_{l} = \frac{\Theta_{id}}{\nu^2} \left( \frac{\partial \sigma}{\partial \eta} \right)_{o} - \left( \frac{\partial \sigma}{\partial \eta} \right)_{l} + \rho_d \frac{\Theta_{id}}{v^2} U_c,
\]

(18)

where displacement and momentum thickness associated with the re-circulation zone are defined by

\[
\delta_{zd} = \int_{0}^{Y} \left( \frac{\sigma}{1 - \sigma} \right) \, dY,
\]

\[
\Theta_{zd} = \int_{0}^{Y} \sigma \left( \frac{\sigma}{1 - \sigma} \right) \, dY.
\]

(19)

Thus the form of the momentum integral equation for a region bounded by the wall and a streamline is similar in appearance to the Karman equation. Equation (18) shows that the difference in velocity gradient across the inner region will significantly effect the flow.

In an ordinary boundary layer analysis, the external velocity profile is a prescribed function of \(X \) (or \(x \)). In the interaction zone, the external velocity variations are induced by the growth of the boundary layer. Thus additional equations are required. The first added equation is a mass balance equation which relates the growth of the boundary layer to the angle of the streamline at the edge of the boundary layer. The second equation expresses the
Prandtl-Meyer relationship between the angle of the streamline and the external Mach number.

Figure 4 shows a control volume for the boundary layer mass balance. The angle of the external stream, \( \Phi \), is measured from the plane parallel to the wall. A mass balance yields

\[
- \int_0^\delta \left( \frac{\partial u}{\partial x} \right) dy + \rho_v v_v - \rho_e u_e = 0.
\]  

(20)

Since \( \delta = \delta(x) \),

\[
- \frac{d}{dx} \int_0^\delta (\rho u) dy + \rho_e u_e \frac{d\delta}{dx} + \rho_v v_v - \rho_e u_e = 0.
\]  

(21)

Also, as seen from Figure 4

\[
\tan \Phi = \frac{v_v}{u_e}.
\]  

(22)

Then for small \( \Phi \), manipulation of Equation (21) yields

\[
\Phi = \rho_e \frac{d\delta^*}{dx} - \left[ \frac{d}{dx} \left( \ln \rho_e u_e \right) \right] \int_0^\delta \frac{\rho u}{\rho_e u_e} dy.
\]  

(23)

where

\[
\delta^* = \int_0^\delta \left( 1 - \frac{\rho u}{\rho_e u_e} \right) dy.
\]  

(24)

Equation (23) becomes identical to the mass-balance relation presented by Lees and Reeves (9) when the blowing parameter, \( \rho_c \), is set equal to zero. The last term in Equation (23) represents the effect of streamwise variations in free-stream conditions and the velocity
Figure 4. Mass balance control volume.
profile on $\phi$. It is also noted that $\beta_\infty$ has a direct effect on the flow turning angle $\phi$.

It remains to express Equation (23) in terms of the variables defined in the incompressible plane, free-stream Mach number, and the boundary layer-induced flow-turning angle $\phi$. The transformation is presented in Appendix B for the case of an adiabatic wall. The resulting transformed equation is

\[
\frac{\lambda \delta_i'}{M_e^2} \frac{\delta_i''}{\delta_i'} \left[ \frac{3y-1}{x} + \left( \frac{3y-1}{x} \right)^2 \left( \frac{\theta_i' + \frac{3y-1}{x} \delta_i''}{\theta_i'} \right) \right] \left( \frac{\theta_i'}{\theta_i} \right) + \frac{\lambda \delta_i'}{M_e^2} \frac{\delta_i''}{\delta_i'} \left( \frac{\theta_i'}{\theta_i} \right) \frac{d M_e}{d \lambda} = \phi - \frac{\lambda \delta_i'}{M_e^2} \frac{\delta_i''}{\delta_i'}
\]

(25)

The next step is to relate the flow-angle $\phi$ to the Mach number at the outer edge of the boundary layer, $M_e$. It is assumed that the inviscid outer flow is essentially shock-free and isentropic on each side of the impinging shock wave. The flow angle is then given by the isentropic Prandtl-Meyer relation

\[
\phi = \Theta(M_o) - \Theta(M_e),
\]

(26)

where the subscript $o$ denotes a condition in the unperturbed stream; i.e., $M_o = M_{\text{hu}}$ upstream of the shock impingement point and $M_o = M_{\text{od}}$
downstream of the shock impingement point, and the Prandtl-Meyer turning angle is given by (41)

$$
\Theta(M) = \sqrt{\frac{\gamma + 1}{\gamma - 1}} \tan^{-1} \sqrt{\frac{\gamma - 1}{\gamma + 1}} (M^2 - 1)
$$

(27)

$$
-\tan^{-1} \sqrt{M^2 - 1}.
$$

Then for $M^2 - 1 \sim \tan \phi$ and $\phi$ small compared with unity, the value of $\phi$ can be obtained from the linearized relation (9)

$$
\phi = \frac{1}{1 + \gamma - 1} \left[ 1 - \frac{M_0}{M_0^*} \right] (28)
$$

The equations of motion to be solved in the separated zone can be rewritten and summarized as

$$
\frac{d}{dx} \left( \frac{\Theta_i}{\delta_i} \right) + \left( \frac{\Theta_i}{\delta_i} \right) \frac{d}{dx} \delta_i + \left[ 2 + \frac{\delta_i^*}{\Theta_i} \right] \left( \frac{\Theta_i}{\delta_i} \right) \frac{dM_0}{dx} = \frac{1}{R_0 \delta_i M_0} \left( \frac{\delta_i}{\gamma} \right)_0 + \beta_i
$$

(29a)

(momentum equation across the entire boundary layer)

$$
\frac{d}{dx} \left( \frac{\Theta_{ii}}{\delta_i} \right) + \frac{\Theta_{ii}}{\delta_i} \frac{d}{dx} \delta_i + \left[ 2 + \frac{\delta_{ii}^*}{\Theta_{ii}} \right] \left( \frac{\Theta_{ii}}{\delta_i} \right) \frac{dM_0}{dx}
$$

$$
= \frac{1}{R_0 \delta_i M_0} \left[ \left( \frac{\partial \delta_i}{\partial \gamma} \right)_0 - \left( \frac{\partial \delta_i}{\partial \gamma} \right)_i \right] \beta_i
$$

(30)

(momentum equation from the wall to the separating streamline)
where

\[ R_0 = \frac{\alpha \delta_{is}}{y^o} \]

\[ \bar{\delta}_d = \frac{\delta_d}{\delta_{is}} \]

\[ \bar{X} = \frac{X}{\delta_{is}} \]  \hspace{3cm} (32)

and \( \delta_{is} \) is the transform of the boundary layer thickness at the separation point.

**Velocity profile**

The separation zone is characterized by the growth and collapse of the reverse flow region between the wall and the separating streamline. For the present work, the velocity profile is divided into two parts by the separating streamline. The outer profile starts at separation and is modified as it is displaced away from the wall by the underlying recirculation zone. The alterations to the velocity profile
are characterized by the single parameter describing the position of
the separating streamline, \( \gamma_s \). The inner profile includes all of
the reversed flow and is directly modified by blowing, which changes
the mass flux between the separating streamline and the wall.

Many representations of the velocity profile are possible, and
many have indeed been applied in past researches. The complexity of
the flow in the separated zone precludes the use of the simple, fourth-
order polynomial representation. Lees and Reeves (9) have success-
fully applied the lower-branch, Falkner-Skan similar velocity profiles
to the separated zone in a laminar boundary layer-shock wave inter-
action. Each similar profile characterizes a particular value of the
pressure exponent in the Falkner-Skan equations. Lees and Reeves
evaluated the velocity profile dependent boundary layer functions as
functions of the distance from the wall to the point of zero velocity.

The present research utilizes separate polynomial representa-
tions of the velocity profile for the inner flow region below the
separating streamline and for the outer region above the separating
streamline. This choice of profile representations permits an ex-
plicit dependence of the profile on the blowing parameter, \( \beta_1 \).

Finally, the Pohlhausen wall boundary condition (as modified
for blowing)

\[
\rho_i \left( \frac{\partial \gamma}{\partial \gamma} \right) \bigg|_{\gamma=0} = \frac{\bar{\delta}}{U_e} \frac{d U_e}{d x} + \left( \frac{U^0}{U_e \bar{\delta}} \right) \left( \frac{\partial^2 \gamma}{\partial \gamma^2} \right) \bigg|_{\gamma=0}, \quad (33)
\]

is discarded since it implies a unique relation between the velocity
profile parameter and the pressure gradient. Actually, the need for
this boundary condition is abrogated by the use of the inner region
momentum equation. Thus the requirement that the momentum equation
be satisfied on the average between the wall and the separating
streamline serves the same purpose (in eliminating the need for
Equation (33)) as Tani's second moment equation.

For the outer region, $\gamma_s \leq \gamma \leq 1$, the velocity profile is taken
to be a fourth-order polynomial,

$$\psi = a_0 + a_1 \gamma + a_2 \gamma^2 + a_3 \gamma^3 + a_4 \gamma^4,$$

where the coefficients are to be obtained from the boundary condi-
tions,

$$\gamma = 1; \quad \psi = 1, \quad \frac{\partial \psi}{\partial \gamma} = 0 = \frac{\partial^2 \psi}{\partial \gamma^2}$$

$$\gamma = \gamma_s; \quad \psi = \psi_s, \quad \lim_{\gamma \to \gamma_s} \frac{\partial \psi}{\partial \gamma} = 0,$$

where $\psi_s$ is the velocity-ratio on the separating streamline. The
last condition is actually an initial condition, requiring that the
outer velocity profile match the separation profile at the point of
separation, where $\gamma_s = 0$. Here, the condition of zero shear at
separation is imposed. Thus, as it should, the outer velocity profile
"remembers" that it grew from a pressure-gradient-induced separation
rather than flow off a bluff body or backward-facing step. The
extension to the backward-facing step is obviously to let $\gamma_s$ be a
prescribed value (not zero) at $X = 0$ and let \( \lim_{X \to 0} \left( \frac{\partial \psi}{\partial \gamma} \right)_{\gamma = \gamma_s} \neq 0 \), but
instead take a value dependent on the upstream flow. This is analogous
to the specification of the separation velocity profile by Baum, King, and Denison (42) in their numerical calculation of the flow field immediately behind a bluff body.

The condition

$$\lim_{\gamma_s \to c} \left( \frac{2a}{\gamma_s^2} \right) = 0$$

is not definitive. For example, it is satisfied by any one of the relations

$$\frac{2a}{\gamma_s^2} = 0, \gamma_s^2, \gamma_s^4, \ldots$$

or any multiple of the shear stress. Since by Equation (34) $a_1$ would be equal to $\frac{2a}{\gamma_s^2} = 0$ if the outer velocity profile were extrapolated to $\gamma = 0$, $a_1$ is a logical choice for a second parameter (where $\gamma_s$ is the first parameter) to describe the velocity profile. An additional equation such as a second moment equation, would be required to specify the second parameter. However, the simplicity of a single-parameter analysis was considered more important than the added refinements of a two-parameter analysis for the present research. As described below, the velocity profiles used in the present research already depend on a second-parameter which is related to mass injection.

Imposing the conditions of Equation (35), Equation (34) becomes

$$\sigma = 1 - (1 - \sigma_s) \left( \frac{\gamma - \gamma_s}{1 - \gamma_s} \right)^3 + \frac{a_1}{1 + \gamma_s^2} (\gamma - \gamma_s) (1 - \gamma)^3$$

(36)

for $\gamma_s \leq \gamma \leq 1$, where $a_1$ is to be determined by an additional relation within the constraint of Equation (35).
The inner velocity profile is represented by a cubic,

\[
\frac{\nabla}{s} = b_1 + b_2 \left( \frac{\eta}{\eta_s} \right) + b_3 \left( \frac{\eta}{\eta_s} \right)^2 + b_4 \left( \frac{\eta}{\eta_s} \right)^3 ,
\]

where for convenience the variables are normalized to the separating streamline. The values of the coefficients are determined from the conditions

\[
\eta = 0 \quad ; \quad \nabla = 0
\]

\[
\eta = \eta_s \quad ; \quad \nabla = \nabla_s \left( \frac{\partial \sigma}{\partial \eta} \right) = \left( \frac{\partial \sigma}{\partial \eta} \right)_{\gamma^* \gamma^*} = \left( \frac{\partial \sigma}{\partial \eta} \right)_{\gamma^* \gamma^*} ,
\]

where the conditions at \( \eta = \eta_s \) insure good matching to the outer profile.

Recalling that the mass flux between the wall and the separating streamline is equal to the mass injected at the wall,

\[
\int_0^{\gamma_s} \rho u d\gamma = \int_0^X \rho \omega v_w \, dX .
\]

where \( X = 0 \) at separation. Then, in terms of variables of the transformed coordinates,

\[
\int_0^{\gamma_s} \nabla d\gamma = \frac{1}{\frac{\varepsilon_s}{p^0 M_e}} \int_0^X p^0 M_e \rho_w \, dX .
\]

Introducing the function, \( B \),

\[
B \equiv \frac{1}{\frac{\varepsilon_s}{p^0 M_e}} \int_0^X p^0 M_e \rho_w \, dX
\]
the condition for locating the separating streamline is simply

$$\int_0^\gamma \mathcal{V} \, d\eta = B$$  \hspace{1cm} (42)

Thus, an explicit dependence of the velocity profile on the blowing parameter is introduced by the mass balance between the wall and the separating streamline. This dependence requires that either the separating streamline must be displaced further from the wall by blowing or the velocity increased. In either case, it is the total mass injected which affects the velocity profile.

Combining Equations (38) and (42) with Equation (37) leads to the velocity profile,

$$\mathcal{V} = \left[ -\frac{12}{5-3\gamma_s} \mathcal{V}_s - \frac{\gamma_s (1-\gamma_s)^3}{5-3\gamma_s} \mathcal{A}_1 + 12 \left( \frac{B}{\gamma_s} \right) \left( \frac{\gamma_s - \gamma}{\gamma_s} \right) \right] \left( \frac{\gamma}{\gamma_s} \right)$$

$$+ \left[ \frac{3 (\gamma + 3\gamma_s)}{5-3\gamma_s} \mathcal{V}_s + \frac{3 \gamma_s (1-\gamma_s)^3}{5-3\gamma_s} \mathcal{A}_1 - 4 B \left( \frac{B}{\gamma_s} \right) \left( \frac{\gamma_s - \gamma}{\gamma_s} \right) \right] \left( \frac{\gamma}{\gamma_s} \right)^2$$

$$+ \left[ -\frac{4 (1+3\gamma_s)}{5-3\gamma_s} \mathcal{V}_s - \frac{2 \gamma_s (1-\gamma_s)^3}{5-3\gamma_s} \mathcal{A}_1 + 12 \left( \frac{B}{\gamma_s} \right) \left( \frac{\gamma_s - \gamma}{\gamma_s} \right) \right] \left( \frac{\gamma}{\gamma_s} \right)^3$$  \hspace{1cm} (43)

for $0 \leq \gamma \leq \gamma_s$ with the separating streamline velocity ratio given by

$$\mathcal{V}_s = \frac{2 \gamma_s (\gamma - 3\gamma_s) + 4 B \left( \frac{B}{\gamma_s} \right)^2 \left( 1 + 3\gamma_s \right)}{2 \gamma_s (\gamma - 3\gamma_s) + 3(1-\gamma_s)(1+3\gamma_s) - \mathcal{A}_1 \left( \frac{\gamma_s}{\mathcal{V}_s} \right) \left( 1 - \gamma_s \right) \mathcal{A}_1}$$  \hspace{1cm} (44)

It is seen that injection ($B > 0$) increases the velocity ratio on the separating streamline.

Application of Equations (36), (43), and (44) require selection of an appropriate function for $\mathcal{A}_1$. The simplest function which
satisfies the limit relation in Equation (35) is \( a_1 \equiv 0 \). The choice \( a_1 \equiv 0 \) implies that if the outer velocity profile is extrapolated to \( \eta = 0 \), then \( \frac{du}{d\eta} \eta \equiv 0 \) throughout the separated flow. The outer velocity profile therefore remains similar to the separation profile, though it is displaced from the wall by the inner profile. The velocity-profile-dependent functions appearing in the differential equations (29), (30), and (31) are evaluated below for \( a_1 \equiv 0 \).

**Combined equations, \( a_1 \equiv 0 \).** -- The ratio of the displacement thickness associated with the recirculation zone to the boundary layer thickness is obtained directly from Equation (19),

\[
\frac{\delta_{\text{w}}^*}{\delta_{\text{w}}} = \int_0^{\gamma_s} \left(1 - \theta \right) d\eta .
\]

It follows directly from Equation (42) that

\[
\frac{\delta_{\text{w}}^*}{\delta_{\text{w}}} = \gamma_s - B
\]

which indicates physical displacement of the flow by the inner circulation region. The added mass flow indicated by \( B \) reduces the mass flow deficiency in the boundary layer (for \( B > 0 \)) and thereby reduces the displacement thickness.

The momentum thickness ratio associated with the recirculation zone is

\[
\frac{\Theta_{\text{w}}}{\delta_{\text{w}}} = \int_0^{\gamma_s} \left( 1 - \theta \right) d\eta .
\]
It follows from the inner velocity profile, given by Equation (43), that

\[
\frac{\Theta_{ii}}{\delta_i} = -\frac{6}{35} \gamma_i \delta_i^2 + \frac{343}{35} (1-\gamma_i) \frac{\gamma_i^3 \delta_i^3 (1-3\gamma_i)}{(1-\gamma_i)^3 (1+3\gamma_i)}
\]

\[-\frac{4}{105} (1-\gamma_i) \frac{\gamma_i^3 (1-3\gamma_i)}{(1-\gamma_i)^4 (1+3\gamma_i)^3}
\]

\[B \left[ 1 - \frac{12}{105} \frac{B}{\gamma_i} \right] \]

(48)

Finally, the total displacement thickness ratio is given by

\[
\frac{\delta_i^*}{\delta_i} = \frac{\delta_{ii}^*}{\delta_i} + \int_0^1 (1-D) \, d\gamma_i,
\]

(49)

where the integral in Equation (49) must be evaluated from the outer velocity profile given by Equation (36). The displacement thickness ratio is then

\[
\frac{\delta_i^*}{\delta_i} = \gamma_i - B + \frac{1}{5} (1-\gamma_i) \frac{(1-\gamma_i)^2 (1+2\gamma_i)}{1+2\gamma_i - 3\gamma_i^2}.
\]

(50)

The momentum thickness ratio is similarly given by

\[
\frac{\Theta_i}{\delta_i} = \frac{\Theta_{ii}^*}{\delta_i} + \int_0^1 \overline{U} (1-D) \, d\gamma_i,
\]

(51)

where the integral is evaluated with the aid of Equation (36). Then

\[
\frac{\Theta_i}{\delta_i} = \frac{\Theta_{ii}^*}{\delta_i} + \frac{1}{5} (1-\gamma_i) \frac{(1-\gamma_i)^2 (1+2\gamma_i)}{1+2\gamma_i - 3\gamma_i^2}
\]

\[-\frac{1}{7} (1-\gamma_i)^2 \frac{(1-\gamma_i)^3 (2+2\gamma_i - 5\gamma_i^2)}{(1+2\gamma_i - 3\gamma_i^2)}.
\]

(52)
Finally, the shear stress functions in Equations (29) and (30) are evaluated from the inner velocity profile, Equation (43), to yield

\[
\left( \frac{\partial \sigma}{\partial \gamma} \right)_0 = -3 \frac{D^s}{\gamma^s} + 2 \frac{B}{\gamma^2} + 2 \left( 1 - D^s \right) \frac{\gamma^s \left( 1 - 3 \gamma^s \right)}{(1 - \gamma^s)(1 + 2 \gamma^s \cdot 3 \gamma^s^2)},
\]

and

\[
\left[ \left( \frac{\partial \sigma}{\partial \gamma} \right)_0 - \left( \frac{\partial \sigma}{\partial \gamma \gamma^s} \right) \right] = -6 \frac{D^s}{\gamma^s} + 12 \frac{B}{\gamma^2}.
\]

The flow field is completely determined by the differential equations (29), (30), and (31), the definition of B given by Equation (41), the turning angle relation given by Equation (28), and the velocity-profile-dependent functions given by Equations (44), (46), (48), (50), (52), (53), and (54). An additional quadrature is required to transform the \( \xi \)-system to the physical \( x \)-system. The blowing parameter, \( \beta_1 \), must be specified by an auxiliary equation. The value of \( \beta_C \) (related to \( \beta_1 \) by Equation (14)) will frequently be specified as a function of position measured from a fixed location (like the plate leading edge). The direct influence of the total mass injected (as measured by B) can be seen in Equations (45) to (54).

The set of equations to be solved include three, nonlinear, first-order differential equations. In this section the equations are rewritten in terms of the velocity-profile-dependent functions, the nature of the solutions considered, and the singularity associated with separation is discussed.
By utilizing the velocity-profile-dependent functions, the differential equations (29), (30), and (31) can be rewritten in the form

\[ D_{11} \frac{d \eta_1}{d y} + D_{12} \frac{d \bar{\eta}_2}{d y} + D_{13} \frac{d \bar{\eta}_3}{d y} = D_{14} \]  

(55)

\[ D_{21} \frac{d \eta_2}{d y} + D_{22} \frac{d \bar{\eta}_2}{d y} + D_{23} \frac{d M_0}{d y} = D_{24} \]  

(56)

\[ D_{31} \frac{d \eta_3}{d y} + D_{32} \frac{d \bar{\eta}_3}{d y} + D_{33} \frac{d M_0}{d y} = D_{34} \]  

(57)

where the \( D_{ij} \)'s are coefficients derived from the functions of the preceding section. These functions are listed in Appendix C. Equations (55), (56), and (57), along with Equation (26) and a specification of the blowing parameter provide a complete set of equations for the separated flow region.

The quadrature required to transform the streamwise coordinate back to the physical plane is given by the inverse of the transformation defined by Equation (5). Thus at the surface, \( y = 0 = Y \), and

\[ x = \int_0^X \frac{1}{a} \alpha \frac{P_0}{P_0} d \bar{X} \]  

(58)

In terms of dimensionless variables

\[ \beta_x = \frac{R}{\bar{X}} \int_0^Y \left[ 1 + \frac{\eta - 1}{R} M^2 \right]^{\frac{1}{(\gamma - 1)}} d \bar{X} \]  

(59)
where

\[ R_x = \frac{a^* x}{v^*} \tag{60} \]

and sonic velocity and pressure ratios were expanded in terms of Mach number.

**Separation singularity.** -- The possibility of a singularity in the solution for the stream function \( \psi \) at separation is pointed out by Stewartson and others (43, 44, 45, 35). This singularity arises as a result of logarithmic terms which appear in the expansion of \( \psi \) about the separation point. The singularity has been found in some calculations performed for prescribed pressure distributions. Stewartson (43) suggests that if a boundary layer is to continue downstream of separation, then the main stream must adjust so that in fact no singularity occurs. In the interaction problem being considered here, the main stream does adjust to the boundary layer growth and separation.

The singularity of specific concern here occurs if the separating streamline leaves the wall at a normal angle. Lees and Reeves (9) and Nielsen et al. (33) found in their analyses that in a boundary/shock wave interaction, the separating streamline leaves the wall at a nonzero angle, consistent with Oswatitsch's local regular solution to the Navier Stokes equation (46). Lighthill also evaluates the streamline separation angle from physical arguments (35). The angle between the separating
streamline and the wall can also be obtained from the boundary layer equations.¹

¹ The momentum equation is

\[ \rho u \frac{\partial u}{\partial x} + \rho \nu \frac{\partial u}{\partial y} = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right). \]

Along the wall, \( y = 0 \) and the momentum equation is

\[ \sigma = -\frac{\partial P}{\partial x} + \mu \omega \left( \frac{\partial^2 u}{\partial y^2} \right)_o + \left( \frac{\partial u}{\partial y} \right)_o \left( \frac{\partial \mu}{\partial y} \right)_o. \]

At separation

\[ \tau_o = \mu \omega \left( \frac{\partial u}{\partial y} \right)_o = \sigma = \left( \frac{\partial u}{\partial y} \right)_o, \]

and then

\[ \sigma = -\frac{\partial P}{\partial x} + \mu \omega \left( \frac{\partial^2 u}{\partial y^2} \right)_o. \]

The velocity may be expanded as

\[ u = a_1 y + a_2 y^2 + a_3 y^3 + \ldots. \]

Then

\[ \left( \frac{\partial^2 u}{\partial y^2} \right)_o = \frac{1}{\mu \omega} \frac{\partial P}{\partial x} = 2 a_2. \]

Also the stream function is defined by

\[ u = \frac{\partial \psi}{\partial y}. \]
so that
\[
\psi = \frac{a_1}{2} y^2 + \frac{a_2}{3} y^3 + \frac{a_3}{4} y^4 + \ldots + g(x)
\]

At \( y = 0 \), \( \psi = 0 \) so that \( g(x) = 0 \). Thus
\[
\psi = \frac{\partial g}{\partial y}(\frac{a_1}{2} y + \frac{a_2}{3} y^2 + \frac{a_3}{4} y^3 + \ldots )
\]

The path of the separating streamline, \( \psi = 0 \) is
\[
\frac{a_1}{2} y + \frac{a_2}{3} y^2 + \frac{a_3}{4} y^3 + \ldots = 0
\]

Therefore the slope of the separating streamline at \( y = 0 \) is
\[
\left( \frac{d\psi}{dx} \right)_{\text{sep}} = -\frac{3}{2} \left( \frac{d\psi}{dx} \right)_{\text{sep}} = -\frac{2}{2}
\]

Since \( a_1 = \left( \frac{\partial u}{\partial y} \right)_\phi = \frac{1}{\mu \omega} \gamma \omega \), the slope of the separating streamline is
\[
\left( \frac{d\psi}{dx} \right)_{\text{sep}} = -3 \left( \frac{d\psi}{dx} \right)_{\text{sep}}
\]

This same result has been previously obtained by Oswatitsch (46) and by Lighthill (35).
Of course, if $\beta_c \neq 0$ at separation then the normal velocity does not vanish at the wall and $\frac{d\gamma_s}{dX} \rightarrow \infty$ at separation.

A singularity appears in Equations (55) to (57) for $\gamma_s$ at separation for the case where $\beta_c \equiv 0$. The singularity arises because $\gamma_s \sim \frac{1}{x}$ as separation is approached. This singularity is considered a serious shortcoming of the velocity profile obtained from the selection $\alpha_t \equiv 0$. Noting that $\gamma_s = 0$ at separation, the values of the derivatives at separation are evaluated directly. Thus

$$\left(\frac{d(M_c)}{dX}\right)_{x=0} = -\frac{20}{R \delta_i},$$

(61)

$$\left(\frac{d\delta_i}{dX}\right)_{x=0} = \frac{110}{R \delta_i M_\infty} + 4 \frac{d\gamma_s^2}{dX},$$

(62)

and since $\frac{d\gamma_s^2}{dX}$ remains bounded

$$\left(\frac{d\gamma_s^2}{dX}\right)_{x=0} = \frac{3 \phi}{5 \delta_i} + \frac{15 M_\infty}{2 R \delta_i (1 + \frac{4}{5} \frac{M_\infty^2}{M_\infty^2})} \left\{ \frac{3 \gamma_s - 1}{2} + \frac{4 (4 \gamma_s - 1) \left[ \frac{1 + \frac{3 \gamma_s - 1}{5} \frac{M_\infty^2}{M_\infty^2}}{1 - \frac{4 \gamma_s - 1}{5} \frac{M_\infty^2}{M_\infty^2}} \right]}{1 + \frac{4 \gamma_s - 1}{5} \frac{M_\infty^2}{M_\infty^2}} \right\} - \frac{165}{4 R \delta_i M_\infty} \left[ \gamma_s - \frac{2}{3} \right]$$

+ $\frac{3 \gamma_s - 1}{\delta_i} \frac{M_\infty^2}{1 + \frac{4 \gamma_s - 1}{5} \frac{M_\infty^2}{M_\infty^2}}$.

(63)

If a Pohlhausen parameter is defined by

$$\Lambda \equiv \frac{\delta_i}{\gamma_s} \frac{dU_c}{dX},$$

(64)

then Equation (61) implies that the value of the Pohlhausen parameter at separation is $-20$. This is more negative than most estimates of the separation value of the Pohlhausen parameter.
There are two major difficulties associated with the velocity profile derived for \( a_1 = 0 \). The first is related to the singularity in \( \gamma_s \) and the second is related to the value of the pressure gradient at separation. First, the singularity in \( \gamma_s \) arises because all of the coefficients \( D_{11} \), \( D_{21} \), and \( D_{31} \) approach zero at separation.

For \( \partial_0 \equiv 0 \),

\[
D_{11} = \delta_i \frac{d}{d \gamma_s} \left( \frac{\phi_i}{\delta_i} \right),
\]

\[
D_{2i} = \delta_i \frac{d}{d \gamma_s} \left( \frac{\phi_{2i}}{\delta_i} \right),
\]

and

\[
D_{3i} = \delta_i \left[ \frac{d}{d \gamma_s} \left( \frac{\phi_i}{\delta_i} \right) + \frac{\gamma_s - 1}{1 - \gamma_s} \frac{d}{d \gamma_s} \left( \frac{\phi_i}{\delta_i} \right) \right].
\]

Since the inner region vanishes as separation is approached,

\[
\lim_{\gamma_s \to 0} \frac{d}{d \gamma_s} \left( \frac{\phi_i}{\delta_i} \right) = 0 .
\]

This limit is approached independent of the function selected for \( a_1 \).

It is found that for \( a_1 \equiv 0 \),

\[
\lim_{\gamma_s \to 0} \frac{d}{d \gamma_s} \left( \frac{\phi_i}{\delta_i} \right) = 0 ,
\]

\[
\lim_{\gamma_s \to 0} \frac{d}{d \gamma_s} \left( \frac{\phi_i}{\delta_i} \right) = 0 ,
\]

which leads to \( D_{11} = 0 = D_{31} \) and produces the singularity.
Physically, the portion of the velocity profile represented by the external flow can be considered as a continuation of the attached flow velocity profile. If the outer velocity profile was extended to the attached flow, then upstream of separation $a_1$ would be a positive number proportional to the wall shear, while downstream of separation the wall shear is proportional to $b_1$. However, $a_1$ continues to influence the external velocity profile downstream of separation. It decreases to zero at separation and then remains identically zero in the separated zone. This is illustrated in Figure 5a for separation and reattachment. The discontinuity in the slope of $a_1$ produces a discontinuity in the rate of change in shear in the streamwise direction.

The second difficulty with the velocity profile is related to the separation pressure gradient. As separation is approached from positive values of $\gamma_s$, the inner momentum equation should properly approach the Pohlhausen wall condition (Equation 33)). In addition the values of $\left(\frac{\partial \sigma}{\partial \gamma_s}\right)$ along various paths should converge to a single value. Consider Equation (30) with $\phi = 0$. The limit of each term is zero as $\gamma_s \to 0$. However, if the equation is divided through by $\gamma_s$ then the limit, as separation is approached, is

$$\frac{\bar{d}_s}{M_e} \frac{d M_e}{d x} = \frac{1}{R_0 \bar{S}_s M_e} \lim_{\gamma_s \to 0} \left\{ \frac{1}{\gamma_s} \left[ \left( \frac{\partial \sigma}{\partial \gamma_s} \right)_0 - \left( \frac{\partial \sigma}{\partial \gamma_s} \right)_{\gamma_s = 0} \right] \right\}$$

$$= - \frac{1}{R_0 \bar{S}_s M_e} \left( \frac{\partial \sigma}{\partial \gamma_s^2} \right)_{\gamma_s = 0}.$$
a. Variation of $a_1$ and $b_1$ through an interaction.

b. Schematic of $\eta_6$ variation near separation, $a_1=0$.

c. Schematic of $\eta_5$ variation near separation, $a_1=-2 \frac{1+3\eta_5}{(1-\eta_5)^3} \left( \frac{\eta_6}{\eta_5} \right)$.

Figure 5: Variation of velocity profile parameters.
which is equivalent to the Pohlhausen condition of Equation (33).

With $\beta_1 = 0$, the velocity profile based on $a_1 = 0$ leads to

$$
\lim_{\eta_1 \to 0} \left( \frac{\partial^2 \psi}{\partial \eta_1^2} \right)_{\eta_1 = 0} = 28,
$$

$$
\lim_{\eta_1 \to 0} \left( \frac{\partial^2 \psi}{\partial \eta_1^2} \right)_{\eta_1 = \gamma} = 12,
$$

and the limit of the inner momentum equation yields the average value,

$$
\lim_{\eta_1 \to 0} \left\{ \frac{1}{\eta_1} \left[ \frac{\partial \sigma}{\partial \gamma} \right]_{\eta_1 = \gamma} - \left( \frac{\partial \sigma}{\partial \gamma} \right)_{\eta_1 = 0} \right\} = \left( \frac{\partial^2 \psi}{\partial \eta_1^2} \right)_{\eta_1 = 0} = 20.
$$

This is illustrated by the sketch in Figure 5b.

A similar difficulty with $\left( \frac{\partial^2 \psi}{\partial \eta_1^2} \right)$ was encountered by Nielsen et al. (33). In their case, the inner flow was the region between the wall and the $\square = 0$ line. They eliminated the difficulty by using a quadratic representation of the inner velocity profile.

As a result of the arguments above, a functional relationship was sought for $a_1$ which satisfied the conditions (i) that $a_1$ is multiple of the wall shear stress and (ii) the function leads to an inner profile for which $\lim_{\eta_1 \to 0} b_1 = 0$ for $\beta_1 = 0$. Thus, the first requirement provides continuation of the attached velocity profile into the outer velocity profile. By the second condition, the velocity profile approaches a quadratic at points of separation and reattachment for $\beta_1 = 0$. If $\beta_1 \neq 0$, then reattachment does not occur and separation is truly a singular point. Inspection of
Equations (43) and (53) show that the above conditions are satisfied by the relation

$$a_1 = -2 \frac{\gamma_s^{1.5}}{(1-\gamma_s)^{1.5}} \left( \frac{\Omega_s}{\gamma_s} \right).$$

(65)

Use of the condition specified by Equation (65) allows the external velocity profile to adjust to growth of the inner circulation region. In comparison to $a_1 = 0$, Equation (65) increases $\Omega_s$ and consequently tends to flatten the outer velocity profile for a fixed $\gamma_s$. Other choices of the $a_1$ variation are possible. For example, a choice which explicitly incorporates $B$ might be interesting. A two-parameter analysis could be used to investigate the variation of $a_1$.

Combined equations, $a_1 = -2 \frac{\gamma_s^{1.5}}{(1-\gamma_s)^{1.5}} \left( \frac{\Omega_s}{\gamma_s} \right)$. -- With $a_1$ given by Equation (65), the inner velocity profile of Equation (43) becomes

$$\Omega = \left[ -2 \Omega_s + 1.2 \left( \frac{\Omega_s}{\gamma_s} \right) \left( \frac{\gamma_s}{1-\gamma_s} \right) \right] \left( \frac{\gamma_s}{1-\gamma_s} \right)^{1.5}$$

$$+ \left[ 3 \Omega_s - 3.2 \frac{B}{(\epsilon-3\gamma_s)} \right] \left( \frac{\gamma_s}{1-\gamma_s} \right)^{1.5}$$

$$+ \left[ 1.2 \left( \frac{\gamma_s}{1-\gamma_s} \right) \left( \frac{1+\gamma_s}{\epsilon-3\gamma_s} \right) \right] \left( \frac{\gamma_s}{1-\gamma_s} \right)^{1.5}$$

(66)

which reduces to a quadratic function if $B = 0$.

Also, the outer velocity profile of Equation (36) becomes

$$\Omega = 1 - \left( 1 - \Omega_s \right) \left( 1 - \gamma_s \right)^{1.5} \left( \frac{1+3\gamma_s}{1-\gamma_s} \right) - 2 \left( \frac{\gamma_s}{1-\gamma_s} \right)^{1.5} \Omega_s^3$$

(67)

with Equation (44) leading to

$$\Omega_s = \frac{2 \gamma_s^2 (\epsilon-3\gamma_s) + 4 \left( \frac{B}{\gamma_s} \right) (1-\gamma_s) (1+3\gamma_s)(1+3\gamma_s)}{2 \gamma_s^2 (\epsilon-3\gamma_s) + (1-\gamma_s) (1+3\gamma_s)(1+3\gamma_s)}.$$

(68)
With the above velocity profile, the displacement thickness ratio of the inner region remains

$$\frac{\delta_{id}^*}{\delta_i} = \gamma_s - B$$  \hspace{1cm} (46)

while the total boundary layer displacement thickness ratio becomes

$$\frac{\delta_{id}^*}{\delta_i} = \gamma_s - B + \frac{(1 - \alpha_s)(1 - \gamma_s)(2 + 3\gamma_s)}{5(1 + 3\gamma_s)} + \frac{1}{10} (1 - \gamma_s)^2 \left( \frac{\delta_2}{\gamma_s} \right)$$  \hspace{1cm} (69)

The momentum thickness ratio associated with the inner region of the boundary layer becomes

$$\frac{\Theta_{id}^*}{\Theta_i} = -\frac{2}{\gamma_s} \gamma_s \delta_i^2 + B \left[ 1 + \frac{1}{5} (1 - \gamma_s) \left( \frac{\delta_2}{\gamma_s} \right) \right] - \frac{2\gamma_s(1 - \gamma_s) + 4\gamma_s^2}{105(5 - 3\gamma_s)^2} \left( \frac{\delta_2}{\gamma_s} \right).$$  \hspace{1cm} (70)

and the momentum thickness ratio for the entire boundary layer becomes

$$\frac{\Theta_i}{\Theta_i} = \frac{\Theta_{id}^*}{\Theta_i} + \frac{(1 - \alpha_s)(1 - \gamma_s)(2 + 3\gamma_s)}{5(1 + 3\gamma_s)} + \frac{1}{10} (1 - \gamma_s)^2 \left( \frac{\delta_2}{\gamma_s} \right)$$

$$- \frac{1}{4} (1 - \gamma_s)^2 \left( \frac{\delta_2}{\gamma_s} \right) \left( \frac{\delta_2}{\gamma_s} \right) - \frac{1}{6} (1 - \gamma_s)^3 \left( \frac{\delta_2}{\gamma_s} \right)^2$$

$$- \frac{(1 - \alpha_s)(1 - \gamma_s)^3(5 + 7\gamma_s)}{42(5 - 3\gamma_s)} \left( \frac{\delta_2}{\gamma_s} \right).$$  \hspace{1cm} (71)

The shear stress functions become

$$\left( \frac{\partial\sigma}{\partial\gamma} \right)_0 = -2\left( \frac{\delta_2}{\gamma_s} \right) + 12\frac{B}{\gamma_s} \left( \frac{3 - \gamma_s}{5 - 3\gamma_s} \right).$$  \hspace{1cm} (72)

and

$$\left[ \left( \frac{\partial\sigma}{\partial\gamma} \right)_0 - \left( \frac{\partial\sigma}{\partial\gamma} \right)_{\gamma_s} \right] = -6\left( \frac{\delta_2}{\gamma_s} \right) + \frac{96B}{\gamma_s^2(5 - 3\gamma_s)}.$$  \hspace{1cm} (73)
The values of the coefficients $D_{ij}$ in the differential equations (55), (56), and (57) were evaluated from Equations (68) to (73) and are tabulated in Appendix D. It is noted that, as illustrated in Figure 5c, the revised velocity profile yields

$$\lim_{\eta_s \to \infty} \left( \frac{\partial^2 u}{\partial \eta^2} \right) = \lim_{\eta_s \to \infty} \left( \frac{\partial^2 u}{\partial \eta^2} \right)_{\eta_s}$$

$$= \lim \left\{ \frac{1}{\eta_s} \left[ \left( \frac{\partial u}{\partial \eta} \right)_{\eta_s} - \left( \frac{\partial u}{\partial \eta} \right)_{\eta_s} \right] \right\}$$

$$= 12$$,

which is identical to the value obtained from a fourth order polynomial Pohlhausen calculation at separation.

When $\beta_1 = 0$ at separation, the velocity profile, derived with $a_1$ given by Equation (65), yields a regular solution to Equations (55) to (57) at separation ($\eta_s = 0$). Thus, as illustrated in Figure 5b, $(d\eta_s/d\Gamma)$ remains bounded at separation. The values of the derivatives at separation (for $\beta_1 = 0$) are founded by direct substitution of $\eta_s = 0$ to be

$$\left( \frac{dM_e}{d\Gamma} \right)_{\eta_s = 0} = -\frac{12}{R_0 \delta_s^2},$$

$$\left( \frac{d\bar{S}_r}{d\Gamma} \right)_{\eta_s = 0} = \frac{66}{R_0 M_e \delta_s} + \frac{1}{3} \bar{S}_r \frac{d\eta_s}{d\Gamma},$$

and
Equation (74) implies a separation value of the Pohlhausen parameter of -12, as noted previously.

Equation (76) predicts the initial slope of the separating streamline. With $\phi$ positive at the separation point, Equation (76) always predicts positive values of $\left(\frac{d\eta}{dx}\right)_{x=0}$ provided $\left[\frac{1}{3} \frac{d\eta}{dx} - \frac{\phi}{\lambda}\right] \geq 0$. Figure 6 shows a plot of

$$F(M_e) = R_e \frac{\delta_e}{M_e} \left[\frac{1}{3} \frac{d\eta}{dx} - \frac{\phi}{\lambda}\right]$$

(77)

based on Equation (76) for two values of the ratio of specific heats ($\gamma$) and $\delta_e = 1$. From Figure 6, it can be seen that for $\gamma = 1.4$ and 1.3 and $M_e$ less than 2.3 and 2 respectively, it is possible to obtain $\left(\frac{d\eta}{dx}\right)_{x=0} < 0$. Physically this implies that separation cannot be sustained. Thus Figure 6 indicates a region of $M_e$, $R_e$, $\delta_e$, and $\phi$ in which separation cannot occur. Note also that $\left(\frac{d\eta}{dx}\right)_{x=0} > 0$ from Equation (76) is a necessary but not sufficient condition that the
Figure 9: Values of the function $F(M_0)$, calculated at the point of separation.
flow separates. For the selection \( a_1 = 0 \), a similar argument applies to Equation (63) and the value of \( \left( \frac{d^2 T}{dx^2} \right)_{x = 0} \).

Flow calculations in separated region

Equations (55) to (57) and the necessary auxiliary equations (Equations (28), (58), (59) and a specification of \( \beta_c \)) were programmed for computation on the IBM 7094 computer in the Ohio State University Computation Center. A fourth-order Runge-Kutta integration method was used. Details of the program are included in Appendix E.

Zero injection. — Figure 7 shows the calculated pressure distribution for the case:

\[ R_0 = 227, \quad M_{ou} = 2.4, \quad M_{od} = 1.91, \quad M_{es} = 2.27, \quad \beta_c = 0. \]

This case corresponds to an experiment conducted by Chapman, Kuehn, and Larson (6) in which a shock wave was reflected from the laminar boundary layer on a flat plate. The value of \( R_0 \) used in this calculation was chosen to match the slope of the pressure distribution at the separation point. Verification of the velocity profile is sought from comparison of the measured and calculated plateau pressures, pressure rise after shock impingement, and from the calculated position of the reattachment point. The measured and calculated pressure distributions are compared in the figure for \( a_1 = 0 \) and for

\[ a_1 = -2 \left( \frac{1 + \frac{1}{2} \gamma_s}{1 - \frac{1}{2} \gamma_s} \right)^2 \left( \frac{U_s}{U_e} \right). \]

The latter curve is considered an excellent match to the experimental data for \( \frac{x - x_r}{L} < 0.5 \) and within the data scatter for the rest of the separated flow region. The discontinuities in
Figure 7. Comparison of measured and calculated pressure distribution in separated flow.

\[ P_\delta = 0.27 \]
\[ M_{ou} = 2.4 \]
\[ M_{oo} = 1.91 \]
\[ M_{es} = 2.27 \]
\[ \delta = 1.4 \]
\[ \beta = 0 \]

\[ a_\delta = \frac{1+5\beta^2}{(1-\gamma_e)^2} \]

Chapman, Kuehn, and Larson (Reference 6)

Shock Impingement Point

Reattachment
the calculated curves at the shock impingement point result from the step change in \( M_0 \). The continuous pressure changes across the shock wave are neglected in the analysis and replaced by the step change. The calculation was stopped at the reattachment point. The calculated position of reattachment is indicated at \( \left( \frac{x - x_0}{L} \right) = 1.355 \). This value is slightly less than the value of 1.44 calculated by Lees and Reeves (9) for the length of the separated region at the same flow condition. However, the calculated lengths are within 6.25 per cent of each other and are thus considered in excellent agreement.

Figure 7 shows that a pressure plateau is achieved upstream of shock impingement. This plateau region allows the proper amount of fluid entrainment to occur prior to the shock impingement point. The magnitude of mixing achieved is related to the boundary layer and circulation zone growth. Reattachment begins immediately after shock impingement. The extent of the plateau region determines the reattachment pressure rise. The reattachment pressure rise must add to the subsequent attached flow pressure rise to produce the final shock pressure rise. For fixed initial Mach number and Reynolds number, the length of the plateau region will increase with increasing shock strength.

The calculated pressures for \( a_1 = -2 \left( \frac{\gamma - 1}{\gamma - 1} \right) \left( \frac{x - x_0}{L} \right) \) exhibit a maximum and then decrease slightly in the plateau region. This is considered to be a defect in the velocity profile related to insufficient flattening of the inner velocity profile as the external boundary layer grows at constant pressure. The pressure maximum is
very pronounced for the model based on $a_1 = 0$. Calculations by Lees and Reeves (9), based on similar profiles, yield a monotonically increasing pressure in the plateau region for the flow conditions used in Figure 7. Conditions under which this pressure maximum occurs are discussed in detail in connection with the flow calculations for the complete interaction.

The pressure distribution calculated for $a_1 = 0$ is considered in poor agreement with the data. The initial slope is too high, the pressure falls too sharply after the peak and rises too sharply after the shock impingement point. The values of $R_o$, $M_{ou}$, $M_{OD}$, and $M_{es}$ were incompatible for the $a_1 = 0$ calculation. Consequently, the flow did not attach in that calculation, though attachment was achieved in other calculations. To achieve attachment for $a_1 = 0$ the separation-to-shock impingement distance should be reduced. This reduction would introduce a larger deviation from the experimental data in Figure 7. As a result of this poor agreement with data and the difficulties described previously, the model based on $a_1 = 0$ was discarded in favor of the model based on $a_1 = -2 \frac{\gamma_s - 1}{\gamma_s} \left( \frac{U_s}{\gamma_i} \right)$.

Recirculation region and separation point. -- Figure 8 shows the thickness of the inner recirculation region for the same flat plate conditions as considered above. For $a_1 = 0$ the initial slope of the separating streamline is infinite and as noted above reattachment does not occur. The separating streamline leaves the wall and reattaches at finite slope for

$$a_1 = -2 \frac{\gamma_s - 1}{\gamma_s} \left( \frac{U_s}{\gamma_i} \right)$$.
Figure 8. Recirculation region thickness and separating streamline velocity.
For an adiabatic wall, the slope of the separating streamline at separation is

\[ \left( \frac{d\gamma_s}{d\lambda} \right)_{sep} = \left( \frac{d\gamma_s}{dX} \right)_{sep} . \]  

(78)

The momentum equation leads to the slope of the separating streamline at separation being given by:

\[ \left( \frac{d\gamma_s}{d\lambda} \right)_{sep} = -3 \left( \frac{d\tau_w}{d\lambda} \right)_{sep} . \]  

(79)

For the case where the velocity distribution is based on \( a_1 = -2 \left( \frac{i+3\gamma_s}{(i-\gamma_s)^2} \right) \), the shear stress gradient is given by the expression

\[ \left( \frac{d\tau_w}{d\lambda} \right)_{sep} = \frac{4\rho_0 a_1^2}{\rho_o} \left[ \frac{M_e}{(1 + \frac{e}{c} M_o^3)^\frac{\nu_x}{\nu_y}} \right] \frac{d\gamma_s}{d\lambda} . \]  

(80)

The pressure gradient, in the transformed coordinates, is

\[ \frac{dP}{d\lambda} = \frac{-\frac{\nu_x}{c} M_e}{(1 + \frac{e}{c} M_o^3)^\frac{\nu_x}{\nu_y}} \frac{dM_e}{d\lambda} . \]  

(81)

Recalling that \( \left( \frac{dM_e}{d\lambda} \right)_{sep} \) is given by Equation (74), Equations (79) to (81) lead to the result that

\[ \left( \frac{d\gamma_s}{d\lambda} \right)_{sep} = \left( \frac{d\gamma_s}{dX} \right)_{sep} . \]

In other words, the velocity distribution based on \( a_1 = -2 \left( \frac{i+3\gamma_s}{(i-\gamma_s)^2} \right) \) leads to a separating streamline slope which is related to \( \left( \frac{d\gamma_s}{dX} \right) \).
In exactly the same way as the value obtained directly from the momentum equation.

Figure 8 also shows the separating streamline velocity history calculated for \( a_1 = -2 \left( \frac{\Omega_s}{\sqrt{\gamma_s}} \right)^{1/3} \). It increases rapidly and then at a diminishing rate until shock impingement. The compression during reattachment occurs at a relatively uniform rate. The maximum value for \( \Omega_s \) is calculated to be about 0.4. As expected, this is less than the Chapman value of 0.587 (26), which is predicted for the separating streamline velocity behind a rearward facing step with zero initial boundary layer thickness.

Figure 9 shows the calculated pressure distribution for a separated flow with the following conditions:

\[
R_o = 1000, \quad M_{bu} = 2.0, \quad M_{OD} = 1.78, \quad M_{es} = 1.928, \quad \beta < 0.
\]

The experimental data shown in the figure were obtained from Hakkinen et al. (7). The value of \( M_{es} = 1.928 \) matches the measured separation condition. In addition, the value of \( R_o \) was inferred from the measured boundary layer thickness. There was no attempt made, as in the previous calculation, to refine the choice of \( R_o \) to match the separation pressure gradient.

The calculated pressures are slightly higher than the measured values, but show the same trends. Within the accuracy of the data, the comparison is considered quite favorable. In particular, the accuracy of prediction of the reattachment point is considered to be excellent. For the flow conditions, the calculated pressure
Figure 9. Comparison of measured and calculated pressure distribution in separated flow.
monotonically increases in the plateau region until the shock impingement point is reached. The values of $\gamma_s$ and $\sigma_s$ varied in a manner similar to the pressure calculation. The maximum values of $\gamma_s$ and $\sigma_s$ were obtained at the shock impingement point and were 0.334 and 0.144 respectively.

It can be concluded that the model provides a suitable description of the flow in the separated region with no injection. The effect of injection on the separated flow is considered in the next section. After discussion of an attached boundary layer model, the complete interaction will be considered in a later section.

**Effect of injection.** -- The effect of injection through a slot on the separated flow was analyzed for one of the cases of the previous section;

$$R_o = 227, \ M_{ou} = 2.4, \ M_{OD} = 1.91, \ M_{eS} = 2.27.$$  

As noted above, calculations of complete interactions will be discussed in a later section. In this calculation, the flow upstream of the shock impingement point was the same as in the previous calculation. Slot injection was assumed to occur downstream of the shock impingement point. The resulting reduction in shock strength and modifications to the flow downstream of the shock impingement point were evaluated.

The calculation was conducted in two parts. First, a separated flow calculation was conducted for the flow field parameters above and with a slot extending from $(x - x_S)/S_S = 36$ to $(x - x_S)/S_S = 38.$
This slot is downstream of the shock impingement point, \( \frac{x - x_s}{5_s} = 23.9 \). A value of \( \beta_c = 0.001 \) was chosen for the calculation. No additional shock waves were considered to be formed by the slot injection. The effect of the slot was to return the pressure gradient to zero before attachment could occur.

The second part of the calculation involved an iteration. Values of the downstream Mach number, \( M_{OD} \) were selected. For each value of \( M_{OD} \), the separated flow calculation yielded a Mach number at which the pressure gradient vanished. The purpose of the iteration was to match this Mach number to \( M_{OD} \).

Figure 10 shows the Mach number at which the pressure gradient vanishes as a function of the selected downstream Mach number. Values obtained in the first part of the calculation and in successive iterations are shown in the figure. The desired solution is found at the intersection of a curve through the computed Mach numbers and the 45-degree line shown in the figure. It is seen from Figure 10 that increasing values of \( M_{OD} \), corresponding to decreasing shock strengths, produce rapidly decreasing values of the final plateau Mach number. This results from the decrease in the pressure gradient downstream of shock impingement associated with a decrease in shock strength.

Two calculations were conducted for selected plateau Mach numbers in excess of 1.967, but are not shown in Figure 10. In these calculations the value of \( \eta_s \) began to increase in the region where the friction coefficient was still negative. Thus the thickness of
Mach Number at which Pressure Gradient Vanishes

Figure 10. Iterations for downstream Mach Number. Discrepancy in shock location.

Selected Downstream Mach Number, Mo

With fixed upstream conditions, the shock strength decreases.

Shock Strength
the circulation zone decreased and then began to increase. As the
values of $\gamma_s$ increased, the corresponding values of the friction
coefficient became more negative. This leads to the physically un-
realistic situation that the circulation region never ends. The
plateau Mach numbers obtained from these particular calculations were
not considered to be meaningful.

Figure 10 implies that the solution plateau Mach number for
the case above is slightly higher than $M_{OD} = 1.963$. The last cal-
culation at $M_{OD} = 1.963$ was considered to be close enough for the
purposes of this discussion.

Figure 11 shows the calculated wall shear stress coefficient,
\[ c_s = \frac{\gamma}{\frac{1}{2} \rho u_s^2}, \]
as a function of plate position. The end of the flow circulation
region is indicated by the wall velocity gradient (and shear stress
coefficient) passing from negative to positive values. For this case,
the circulation region was calculated to end at \( (x - x_0)/L = 1.913 \).

Figure 12 shows a comparison of the calculated pressures for
the separated flow with and without injection. The calculation for
$\beta_c = 0$ discussed previously, was terminated at the point of reat-
tachment. The plateau pressure corresponding to $M_{OD} = 1.91$ (for
$\beta_c = 0$) is about 0.147. Thus, the effect of injection was to in-
crease the length of the interaction region, reduce the over-all
pressure ratio (shock strength required for upstream flow), and reduce
the pressure gradient downstream of the shock impingement point.
Figure 11. Variation of the shear coefficient for a boundary layer interaction with slot injection.

\[ R_0 = 2.27 \]
\[ M_{on} = 2.4 \]
\[ M_{es} = 2.27 \]
\[ M_{00} = 1.963 \]
Figure 12: Comparison of calculated pressure distributions for slot injection and no injection with upstream conditions held constant.
Figure 13 shows a comparison of the position of the separating streamline with and without injection. Injection increases the length of interaction region and opens the downstream end of the circulation zone. The downstream opening of the circulation zone must be sufficient to permit the outflow to balance the flow into the region at the slot.

The calculation discussed above demonstrates some of the essential effects of injection. Additional calculations require merger of the separated flow model with an attached flow model.

**Effect of suction.** -- In the previous section, it is shown that boundary layer injection reduces the shock strength associated with a prescribed upstream separation length. Conversely, it can be expected that suction would increase the shock strength associated with a prescribed boundary layer separation. From the viewpoint of fixed shock strength, suction would either diminish the length of the separated flow region or eliminate it altogether. If separation occurs, then the flow circulation region is bounded by the separating streamline, as before. Also according to the model developed herein, the local velocity profiles are defined by $\gamma_s$ and $\beta$.

Since $\gamma_s$ defines the position of a streamline, there cannot be any flow into the circulation region from the external portion of the boundary layer. Also, since reattachment of the separating streamline will occur, the inner recirculation region must be considered to be closed by the separating streamline.
Figure 13. Comparison of the path of the separating streamline for slot injection and no slot injection with upstream conditions held constant.
If the circulation region remained closed, then suction would diminish the contained mass as a function of time and steady-state separation from the wall would not be possible. Thus with suction, separation may appear to begin at a point above the wall and/or a flow instability can be induced. The fluid which enters the circulation zone as a result of the displaced streamline, or as a result of the instability, must exactly balance the amount of fluid sucked out at the wall. The beginning of the circulation region is defined as the point where the wall shear stress vanishes. This physical picture is in agreement with the one set forth by Jones and Watson (36) to describe the effect of suction at the boundary layer separation point.

From the viewpoint of the separated flow model, the values of $\gamma_s$ and $B$ are not zero at the beginning of the circulation region. Instead $B$ has the initial positive value required to balance the integrated suction. Beyond the slot, the value of $B$ should be zero and reattachment should occur in a manner similar to the case of zero injection.

Even if possible flow instabilities are neglected, it is not possible to apply directly the model developed in this research to a calculation forward from the apparent separation point. This is because of the initial condition on the separating streamline defined by Equation (35). If a value of $B$ is selected and a corresponding value of $\gamma_s$ determined from the attached flow, then velocity profiles would have to be derived for the initial value of $\left(\frac{\partial \gamma}{\partial r}\right)_0$. Each set of flow conditions would require new velocity profile relations.
An alternate approach would be to integrate backwards from the attachment point. This approach requires prior knowledge of the attachment pressure. Obtaining prior knowledge of the attachment pressure is no more difficult than obtaining prior knowledge of the separation pressure. An iteration is necessary in either case to balance the upstream and downstream flows with the shock strength. Upstream integration would begin with the usual initial conditions \( \gamma_s = 0 = B \) and \( \gamma_i = 1 \) obtained from the attachment pressure. The value of \( R_0 \) corresponding to the attachment (or the downstream attached flow) would have to be used as a parameter and the distance from attachment to the slot and to the shock wave impingement point would have to be obtained by a second iteration. Presumably the distance from the slot to the shock wave would be known from the definition of the problem.

Injection or suction opens one end of the circulation zone. If injection occurs, then the separation zone upstream of the shock wave is lengthened for a given shock strength and reattachment does not occur. Conversely, suction can be expected to shorten the separation length downstream of the shock wave for a given shock strength and the separating streamline does not originate at the wall.

**Uniform injection**

It was shown in the previous discussion that the expected regular solutions to the differential equations (55) to (57) are obtained provided that (i) the proper choice is made for the velocity
profile and (ii) $\beta_1 = 0$ in the vicinity of separation. In this section, uniform injection is considered throughout the interaction region with the shock wave produced by an external source and reflected from the flat plate. Since there is a finite velocity at the surface, any streamline could be considered as a separating streamline. However, the condition of Equation (35) specifies that the recirculation zone begins at the point of zero shear stress. Thus, the particular streamline which defines the parameter $\gamma_s$ is uniquely defined. Reattachment of this streamline is, of course, not possible.

Since fluid is injected with zero velocity in the streamwise direction, the streamlines are normal to the wall and $\left( \frac{\partial \gamma}{\partial \nu} \right)_{\nu=0} \rightarrow \infty$. The order of the singularity must be determined and extrapolation formulas derived to obtain the values of the dependent variables a short distance beyond separation of the bounding streamline.

It is assumed that near $\gamma_s = 0$, the injection integral $B$ can be written as

$$\begin{align*}
B &= K \gamma_s \nu_s, \\
&= \frac{K \gamma_s \nu_s}{\nu}.
\end{align*}$$

(82)

where the constant $K$ will be evaluated by the analysis below.

Near $\nu=0$,

$$B \approx \beta_1 \nu.$$

(83)

so that according to Equation (82),

$$\beta_1 = \frac{K \gamma_s \nu_s}{\nu}.$$
For $\gamma_s$ small, Equation (68) leads to

$$U_s \approx \frac{10 \gamma_s^2}{5 - 4K}.$$  \hspace{1cm} (85)

Substitution of Equation (85) into Equations (83) and (84) leads to

$$\beta_i = \left( \frac{10K}{5 - 4K} \right) \frac{\gamma_s^3}{X}.$$  \hspace{1cm} (86)

and

$$B = \left( \frac{10K}{5 - 4K} \right) \gamma_s^3.$$  \hspace{1cm} (87)

With the aid of Equations (85), (86), and (87), the coefficients in Equations (55), (56) and (57) were evaluated for $\gamma_s$ small. As before, the inner momentum equation was divided through by $\gamma_s$ to eliminate it as a common multiplier. The dimensionless boundary layer thickness $\bar{\Sigma}_i$ was taken to be essentially unity for $\gamma_s$ close to zero.

The resulting differential equations are

$$\frac{d \delta_i}{d \tau} = - \left( \frac{60 - 192K}{5 - 4K} \right) \beta_i,$$  \hspace{1cm} (88)

$$\frac{d \delta_i}{d \tau} = \frac{22}{R_0 M_e} \left( \frac{15 - 48K}{5 - 4K} \right) + 0.048 \frac{\beta_i}{\gamma_s^2},$$  \hspace{1cm} (89)

or equivalently

$$\frac{d \bar{\Sigma}_i}{d \tau} = \frac{22}{R_0 M_e} \left( \frac{15 - 48K}{5 - 4K} \right) + 0.048 \left( \frac{10K}{5 - 4K} \right) \frac{\gamma_s}{X}.$$  \hspace{1cm} (90)
and
\[ \frac{1}{3} \left(1 - \frac{8}{\xi - 4K}\right) \frac{d\eta_5}{d\xi} = -0.08533 \frac{\beta_i}{\eta_5^2}. \quad (91) \]

It is noted that, although singularities appear in Equations (89), (90), and (91), Equation (88) is regular indicating that the pressure gradient remains bounded (for \( K \neq 5/4 \)).

Equation (91) can be rewritten as
\[ 3 \eta_5^2 \frac{d\eta_5}{d\eta_5^3} = \frac{0.768 \beta_i}{\frac{8}{\xi - 4K} - 1} d\eta_5. \quad (92) \]

which can be integrated for the case \( \beta_1 = \text{constant} \) to yield
\[ \eta_5^3 = \frac{0.768 \beta_i}{\frac{8}{\xi - 4K} - 1} \eta_5 + A \quad . \quad (93) \]

Taking \( \eta_5 = 0 \) to be the point of separation leads to \( A = 0 \). Equation (93) can be rewritten as
\[ \eta_5 = \left[ \frac{0.768 \beta_i}{\frac{8}{\xi - 4K} - 1} \right]^{1/3} \eta_5^{1/3}. \quad (94) \]

The constant \( K \) can now be evaluated because of the requirement that both Equations (86) and (94) be satisfied. This requirement implies that
\[ \frac{10K}{\xi - 4K} = \frac{0.768}{\frac{8}{\xi - 4K} - 1} \quad , \quad (95) \]

which leads to
\[ K = 0.0896 \pm 0.516. \quad (96) \]
For injection, $B > 0$ which implies that $K > 0$ by the definition of $K$ in Equation (82). Thus the value of $K$ for uniform injection is obtained using the plus sign in Equation (96) and is 0.606. Substitution of $K = 0.606$ into Equation (94) leads to

$$\gamma_s = 0.753 \beta_{i^3} R^{1/3}.$$  \hspace{1cm} (97)

Substituting Equation (97) and $K = 0.606$ into Equation (90) leads to

$$\frac{d\bar{e}_i}{dX} = -\frac{12\beta}{R_{e}M_x} + 0.1075 \beta_{i^3} R^{1/3},$$  \hspace{1cm} (98)

so that $\frac{d\bar{e}_i}{dX}$ blows up as $X^{2/3}$ as $X \rightarrow 0$. Noting that $\bar{e}_i = 1$ when $X = 0$, Equation (98) can be integrated to yield

$$\bar{e}_i = 1 - \frac{12\beta}{R_{e}M_x} X + 0.0759 \beta_{i^3} R^{1/3}.$$  \hspace{1cm} (99)

The use of Equation (99) is limited to values of $X$ very close to zero corresponding to $\gamma_s$ close to zero. For $X$ very small, the last term in Equation (99) should be larger than the second term leading to $\bar{e}_i > 1$.

Finally, for $K = 0.606$, Equation (88) becomes

$$\frac{dM_x}{dX} = \frac{2.18}{R_{e}},$$  \hspace{1cm} (100)

which implies an expansion rather than a compression at separation.

The result is identical to the result obtained by the alternate procedure of applying the Pohlhausen wall condition directly. Thus, the inner momentum equation approaches the proper limit. The pressure
gradient obtained from Equation (100) corresponds to the point of zero wall shear stress. Equation (100) indicates that a recirculation region can be produced by injection in an expanding flow.

On a flat plate with uniform injection, streamlines separate from the wall at each point. However, the shear stress is not zero anywhere. A shock wave which impinges on the plate and reflects creates a compression region. The adverse pressure gradient in the compression region can be expected to establish a recirculation in the already separated flow. The beginning of the recirculation region is defined, in the usual manner, by a point of zero shear stress. Equation (100) implies that once a recirculation zone is initiated by the impinging shock wave it will extend all the way to the leading edge of the plate. In a more complex flow, in which the plate follows an expansion region, the recirculation will extend into the expansion region until the pressure gradient satisfies Equation (100).

**Attached Boundary Layer**

Several methods have been shown to provide an adequate representation of the attached flow region of a shock wave/boundary layer interaction. In addition to the research cited previously, Gadd (47) obtained good comparison with experimental data by extending Stratford's method (48), based on inner and outer solutions to the boundary layer equations, to compressible flow. In the present research, emphasis was placed on the selection of a method which properly
matched the separated flow model at separation and reattachment. At separation and reattachment the separated flow velocity profile reduces to a quartic profile and the inner momentum equation reduces to the Pohlhausen wall boundary condition. Therefore, the logical extension to the attached flow, which insures continuity of the pressure gradient and boundary layer growth, involves the use of Pohlhausen's method.

An alternate approach involves the use of Tani's method (36) for the attached flow. Tani's method predicts earlier separation than Pohlhausen's method and thus agrees somewhat better with observed flow separations. By the use of a quartic velocity profile in Tani's method, the separation and attachment profiles match the limiting separated flow velocity profile. However, the use of the second moment equation in place of the Pohlhausen wall boundary condition requires discontinuities in the derivatives of flow variables at separation and reattachment.

The equations required to apply both Tani's and Pohlhausen's methods to the attached flow with possible wall blowing or suction are described below. The Prandtl-Meyer relation (Equation (28)), the Karman momentum equation (29), and the mass-balance equation (31), remain valid for the attached boundary layer.

**Tani method**

The Stewartson-Cohen and Reshotko transformation of the momentum equation is given for Pr = 1 and an adiabatic wall by
Equation (8). To obtain the second moment equation, Equation (8) is multiplied by $U$ and integrated from $Y = 0$ to $Y = \Delta$ where $\Delta$ is sufficiently large that $U = U_e$. Note that as a consequence of the continuity equation (7) and integration by parts

$$
\int_0^\Delta U \frac{\partial U}{\partial Y} dY = \frac{1}{\tau} \nu \omega \frac{\partial u^2}{\partial x} - \frac{1}{\tau} \int_0^\Delta (u^2 - \bar{u}^2) \frac{\partial U}{\partial X} dY.
$$

(101)

The resulting second moment equation, including the effect of blowing, is

$$
\frac{u_e}{\nu^2} \frac{d \Theta^*}{dX} + \frac{\Theta^*}{\nu^2} \frac{d U_e}{dX} = \frac{4 \Theta^*}{U_e} \int_0^\Delta (\frac{\partial U}{\partial Y})^2 dY + \frac{2 \Theta^*}{\nu^2} \nu \omega
$$

(102)

where $\Theta^*$ is the energy thickness defined by

$$
\Theta^* = \int_0^{\xi_i} \vartheta (1 - D^2) dY.
$$

(103)

Finally, the second moment equation can be written in the more convenient form

$$
\frac{\xi_i}{\nu^2} \frac{d}{dX} \left( \frac{\Theta^*}{\xi_i} \right) + \left( \frac{\Theta^*}{\xi_i} \right) \frac{d \xi_i}{dX} + 3 \left( \frac{\Theta^*}{\xi_i} \right) \frac{d M_e}{dX}

= \frac{2}{R_e M_e \xi_i} \int_0^{\Delta} (\frac{\partial \Theta}{\partial Y})^2 dY + \beta_i.
$$

(104)

where

$$
\frac{\Theta^*}{\xi_i} = \int_0^\Delta \vartheta (1 - D^2) dY.
$$

(105)
Following Tani, a quartic velocity profile is assumed which satisfies the boundary conditions,

\[
\begin{align*}
\gamma &= 0 \ ; \ \Omega = 0 \\
\gamma &= 1 \ ; \ \Omega = 1, \ \frac{\partial \Omega}{\partial \gamma} = 0 = \frac{\partial^2 \Omega}{\partial \gamma^2}.
\end{align*}
\]

Thus the velocity distribution

\[
\Omega = c_0 + c_1 \gamma + c_2 \gamma^2 + c_3 \gamma^3 + c_4 \gamma^4
\]

becomes

\[
\Omega = \gamma^2 (c - 8 \gamma + 3 \gamma^2) + c_1 \gamma (1 - \gamma)^3,
\]

where

\[
c_1 = \left( \frac{\partial \Omega}{\partial \gamma} \right)_{\gamma=0},
\]

is the parameter which prescribes the velocity profile. For convenience, the subscript one is dropped in the following discussion.

Based on the above quartic profile, the various boundary layer functions can be evaluated as functions of the local wall velocity gradient, \(c\). Thus

\[
\frac{\partial \Omega}{\partial x} = \frac{x}{5} - \frac{c}{20}
\]

\[
\frac{\partial \Omega}{\partial y} = \frac{4}{35} + \frac{c}{105} - \frac{c^2}{252}
\]

\[
\frac{\partial \Omega}{\partial z} = \frac{87c}{5005} + \frac{73}{5005} c - \frac{23}{5460} c^2 - \frac{1}{2860} c^3
\]

\[
(\frac{\partial \Omega}{\partial \gamma})_{\gamma=0} = c
\]

\[
(\frac{\partial^2 \Omega}{\partial \gamma^2})_{\gamma} = \frac{4 \delta}{35} - \frac{4}{35} c + \frac{3}{35} c^2
\]
Utilizing these velocity-profile-dependent functions, Equations (29), (104), and (31) can be written in the form

\[ E_{11} \frac{d\alpha}{dX} + E_{12} \frac{d\delta}{dX} + E_{13} \frac{dM}{dX} = E_{14} \]  \hspace{1cm} (115) 

\[ E_{21} \frac{d\alpha}{dX} + E_{22} \frac{d\delta}{dX} + E_{23} \frac{dM}{dX} = E_{24} \]  \hspace{1cm} (116) 

\[ E_{31} \frac{d\alpha}{dX} + E_{32} \frac{d\delta}{dX} + E_{33} \frac{dM}{dX} = E_{34} \]  \hspace{1cm} (117) 

where

\[ E_{11} = \left( \frac{1}{105} - \frac{c}{124} \right) \overline{\delta}_u \]  \hspace{1cm} (118) 

\[ E_{12} = \frac{\theta_u}{\overline{\delta}_u} \]  \hspace{1cm} (119) 

\[ E_{13} = \frac{\delta_u}{M_e} \left( 2 \frac{\theta_u}{\overline{\delta}_u} + \frac{\delta_u'}{\overline{\delta}_u} \right) \]  \hspace{1cm} (120) 

\[ E_{14} = \frac{c}{R_0 M_e \overline{\delta}_u} + \beta_u \]  \hspace{1cm} (121) 

\[ E_{21} = \left( \frac{73}{5005} - \frac{3}{2750} c - \frac{3}{2860} c^2 \right) \overline{\delta}_u \]  \hspace{1cm} (122) 

\[ E_{22} = \frac{\theta_u}{\overline{\delta}_u} \]  \hspace{1cm} (123) 

\[ E_{23} = 3 \left( \frac{\theta_u}{\overline{\delta}_u} \right) \frac{\overline{\delta}_u}{M_e} \]  \hspace{1cm} (124) 

\[ E_{24} = \frac{2}{R_0 M_e \overline{\delta}_u} \left( \frac{48}{35} - \frac{4}{35} c + \frac{3}{35} c^2 \right) + \beta_u \]  \hspace{1cm} (125) 

\[ E_{31} = \lambda \overline{\delta}_u \left[ \frac{\theta_u}{1 + \frac{1}{M_e} \left( \frac{\theta_u}{\overline{\delta}_u} - \frac{c}{124} \right) - \frac{1}{24} \right] \]  \hspace{1cm} (126) 

\[ E_{32} = \lambda \left[ \left( \frac{\theta_u}{\overline{\delta}_u} \right)^2 + \frac{\theta_u}{M_e} \left( \frac{\theta_u}{\overline{\delta}_u} \right) \right] \]  \hspace{1cm} (127)
\[ E_{33} = \frac{\lambda}{1 - \gamma - 1} M_e \left\{ \frac{3\gamma - 1}{2} + (\gamma - 1) \left( \frac{\partial}{\partial y} \right) \left[ \frac{1 - \frac{3\gamma - 1}{4} M_e^2}{1 + \frac{3\gamma - 1}{4} M_e^2} \right] \right\} \]

\[ - \left( 1 - \frac{\partial}{\partial y} \right) \left[ \frac{1 + (3 + \frac{3\gamma - 1}{4} M_e^2) (\frac{\partial}{\partial y} M_e^2)}{(1 + \frac{3\gamma - 1}{4} M_e^2) M_e^2} \right] \right\} \] (128)

\[ E_{34} = \phi - \frac{\lambda \beta_i}{1 + \frac{\gamma - 1}{4} M_e^2} \] (129)

Equations (115), (116), and (117), along with the turning angle relation of Equation (28) and a specification of the blowing parameter provide a complete set of equations for the attached region. The quadrature given by Equation (59) provides the transformation back to physical coordinates.

At separation, \( c = 0 \), \( \beta = 1 \), and \( M_e \) is a specified value. The derivatives of the variables \( c \), \( \beta \), and \( M_e \) as evaluated by the differential equations (115), (116), and (117), remain bounded. Thus, the calculation could be started directly at separation -- even for \( \beta_i \neq 0 \). Also, the outer separated velocity profile given by Equation (36) and the quartic profile of Equation (108) are identical at separation (\( c = \gamma_i = \psi_s = 0 \)). Thus, all of the boundary layer functions take identical values as separation is approached from either side.

**Pohlhausen method**

The Pohlhausen wall boundary condition with blowing is given by Equation (33). Equation (33) can be rewritten in the form
The equations for Tani's method can be converted to Pohlhausen's method by replacing Equation (104) by Equation (130) and retaining the quartic velocity profile.

For the velocity distribution given by Equation (108), Equation (130) becomes

\[
\frac{dM_e}{d\gamma} = \frac{M_e c \beta_i}{\delta_i} - \frac{12 - c c}{R_0 \delta_i^2} .
\]  
(131)

Thus, for a flat plate with \( \beta_i = 0 \), \( c = 2 \) yields the constant pressure solution. The separation condition of \( c = 0 \) leads to the usual separation value of the Pohlhausen parameter, -12.

Tani's method is replaced by Pohlhausen's method by the substitution of coefficients in Equation (116) which represent Equation (131) instead of Equation (104). Thus for Pohlhausen's method

\[
E_{21} = 0 ,
\]  
(132)

\[
E_{22} = 0 ,
\]  
(133)

\[
E_{23} = 1.0 ,
\]  
(134)

\[
E_{24} = - \frac{12 - c c}{R_0 \delta_i^2} + \frac{\beta_i M_e c}{\delta_i} .
\]  
(135)

replace Equations (122) through (125). These expressions are used in Equations (115), (116), and (117) along with the other coefficients.
for the quartic velocity profile given in the previous section —
Equations (110), (111), (118) to (121), and (126) to (129). As before
Equations (28) and (59) complete the set of required equations. With
the Pohlhausen method, values of flow variables and their derivatives
match the values obtained from the separated flow model at separation
and attachment.

**Flow calculations in attached region**

The attached flow equations were programmed together with the
separated flow relations for computation on the IBM 7094 computer at
The Ohio State University computation center. The complete program,
including the SCATRAN language listing, is described in Appendix E.
Two versions of the program were developed so that both the second
momentum equation and the Pohlhausen wall condition could be applied.

Integration begins at the beginning of the interaction and is
carried downstream towards separation. This permits the use of a
value of \( R_0 \) which is related to Reynolds number via the flat plate
boundary layer growth. The initial value of \( c \) for the Pohlhausen
method is 2 and 1.857 for Tani's method (36). The initial value of
\( \bar{S}_q \) is unity. The starting value of \( M_\infty \) is \( M_{ou} \) minus the minimum
Mach number disturbance which will yield proper signs for the initial
values of the derivatives.

Constrained by initial flat plate values of \( \bar{S}_q \) and \( c \) and by
the use of a simple integral method, there is a minimum Mach number
disturbance which makes \( \phi \) large enough to yield a negative value of
This difficulty was apparently not encountered by Nielsen et al. (33). However, they used a higher order approximation with the integral method of Dorodnitsyn (34). Other investigators have integrated from the separation point so that \( \bar{\delta} \) or \( \delta \) can float at the beginning of interaction and \( M_e \) can be matched more closely to \( M_{ou} \).

For the Pohlhausen method, with \( c = 2 \) and \( \bar{\delta} = 1 \), the condition that \( \frac{dc}{dx} < 0 \) reduces to the condition that the relation

\[
\varphi > \frac{2}{R_0 M_e} \left[ 2.55 + \frac{\frac{V-1}{2} M_e^2}{1 + \frac{V-1}{2} M_e^2} \right]
\]

be satisfied. Since the turning angle \( \varphi \) depends on \( M_e \) and \( M_{ou} \), Equation (136) prescribes a maximum value of \( M_e \) \((M_e < M_{ou})\) for which computation can begin. For large values of \( R_0 \), computation can begin with \( \varphi \) close to zero which corresponds to \( M_e \) close to \( M_{ou} \). The deviation of the starting value of \( M_e \) from \( M_{ou} \) is reflected in a calculational error in the initial pressure.

Figure 14 shows pressure distributions in the attached region calculated by both Tani and Pohlhausen methods. With \( M_{ou} = 2.0 \) and \( R_0 = 748 \) the initial Mach number was taken to be 1.98 to satisfy Equation (136). A choice of \( M_e = 2.349 \) satisfied Equation (136) for \( M_{ou} = 2.4 \) and \( R_0 = 172.5 \). Though slightly higher starting Mach numbers could be used for Tani's method, the difference was small and not sufficient to warrant separate computations. Computations were terminated at the separation point. As expected, Tani's method yielded earlier flow separation at lower pressure levels than Pohlhausen's method.
Figure 14. Comparison of pressure distributions in the attached flow as calculated by the methods of Tani and Pohlhausen.
Figure 15 compares the boundary layer growth, in transformed coordinates, obtained from the calculations with Tani and Pohlhausen methods. Although the rate of boundary layer growth is higher with the Tani calculation, the total thickening is less than is obtained from the Pohlhausen calculation.

The over-all differences in the results of the computations are not too large in comparison to the boundary layer growth and pressure rise in the complete interaction. In addition, the differences are in the direction that they would tend to cancel when combined with the separated flow relations. Thus, the Pohlhausen method predicts later separation but with a larger pressure rise and greater boundary layer thickening prior to separation.

For those flows in which there is no injection or suction at the wall, the boundary layer reattaches. At reattachment, the attached flow calculation is resumed and continued until the pressure gradient vanishes. The details of the computational logic are described in Appendix E.

The relatively small computational differences and the difficulties related to derivative discontinuities at both separation and attachment suggest the use of the Pohlhausen method in computations of the complete interaction. Use of the Pohlhausen method ensures smooth transition of all flow variables between the attached and separated flows.
Figure 15. Comparison of boundary layer thickening in the attached flow as calculated by the methods of Tani and Pohlhausen.
IV. FLOW CALCULATIONS FOR COMPLETE INTERACTION

The theory developed in the present research was used to examine the effects of injection and suction on complete laminar interactions which include attached and separated flows. Since interactions without injection or suction represent a base point in this research, ordinary interactions are examined first. Effects of injection and suction are identified by the results of several illustrative computations. Several integration procedures were developed to permit integration to begin at the start of interaction with only a single iteration. A separate procedure involving an upstream integration is developed for analysis of suction.

The complete interaction, as described by the attached and separated flow equations, was programmed for digital computations. The program description and SCATRAN language listing are given in Appendix 2. Machine runs were made on the IBM 7094 computer at The Ohio State University Computation Center.

All of the complete interaction runs were conducted with the separated flow velocity profile given previously for

\[ a_1 = -2 \frac{1 + 3 \gamma_s^3}{(1 - \gamma_s^3)^3} \left( \frac{U_2}{U_1} \right) \]

In addition, all of the computational results presented for the complete interaction were obtained with the attached flow described
by the Pohlhausen method. Some runs were made with the attached flow described by Tani's method. However, the pressure gradient discontinuity at separation and attachment with Tani's method made it difficult to match the over-all pressure rise.

**Ordinary Interaction**

For interactions without injection or suction, the computations were begun at the upstream end of the interaction. An iteration was required to match the interaction pressure rise to the inviscid-flow pressure rise. In each of the computations performed in this study, the distance from the plate leading edge to the shock wave impingement point was a prescribed, fixed distance. Iterations were done by three different procedures involving location of the start of the interaction and the initial boundary layer thickness.

The advantage of a procedure which begins computation at the upstream end of the interaction is that the boundary layer thickness, and hence $R_0$, is uniquely related to $Re_{X_0}$ via the undisturbed flat plate boundary layer growth. In two of the iteration procedures which were used, the location of the upstream end of the interaction was moved about until the proper pressure rise was obtained. Increasing the distance between the upstream end of the interaction and the shock wave impingement point increased the calculated pressure rise. The difference between the two procedures was that in one $Re_L$ was fixed, while in the other $Re_{X_0}$ was fixed. A third procedure, using a fixed location of the start of the interaction, was used to
obtain comparisons with experimental data. In this last procedure, \( Re_L \) was also fixed and \( R_o \) was varied independently. Decreasing \( R_o \) increased the over-all pressure rise. This procedure is suggested by the sensitivity of the computed pressure distributions to the value of \( R_o \) and the uncertainties in the boundary layer thickness obtained in the experiments. Comparisons of final values of \( R_o \) with values obtained from flat plate boundary layer growth considerations were made to check the consistency of results.

Figure 16 compares a calculated pressure distribution with some measurements by Chapman, Kuehn, and Larson (6). The iteration was done with a fixed position for the start of the interaction. The poor comparison with the initial pressure results from the difficulties mentioned earlier regarding the initial value of \( M_e \). Aside from this point, the over-all comparison with the data appears to be good. A value of \( R_o = 168 \) was obtained from the iteration. This value of \( R_o \) is consistent with flat plate boundary layer growth.

Figures 17 and 18 show the calculated variations of the shear stress coefficient \( c_f \) and the parameters \( c \) and \( \gamma_s \) through the interaction. The reduction in \( c_f \) at the end of the interaction in Figure 17 reflects the effect of boundary layer thickening. In Figure 18, \( c \) returns to the flat plate value at the end of the interaction. This is, of course, required since \( c = 2 \) is the only permissible value at \( (dM_e/d\overline{x}) = 0 \) according to the Pohlhausen wall boundary condition.

The circulation region, bounded by \( \gamma_s \), is closed by separation and attachment. The beginning of reattachment is a discontinuity which reflects the step change in \( \phi \) at the shock impingement point.
Figure 16. Measured and calculated pressure distribution for a complete interaction, $M_{\infty} = 2.4$, $M_0 = 1.91$, $Re_L = 5.4 \times 10^4$, $\gamma = 1.4$, $Pe = 0$. 
Figure 17. Calculated shear stress coefficient for a complete interaction, $M_{ax} = 2.4$, $M_{ud} = 1.91$, $Re_L = 5.4 \times 10^4$, $\gamma = 1.4$, $\beta_0 = 0$. 
Figure 18. Variation of the velocity profile parameters through the complete interaction, $M_{ou} = 2.4$, $M_{od} = 1.91$, $Re_L = 5.4 \times 10^4$, $\gamma = 1.4$, $\beta_c \approx 0$. 
Pressure maximum in plateau region

A pressure maximum, followed by a small pressure drop, appears in the plateau region in Figure 16. The condition that a pressure maximum occurs is that \( \frac{dM_e}{dX} = 0 \). A pressure maximum would generally occur well into the separated flow region. Inspection of a number of computations in which pressure maxima were observed showed that they occurred approximately in the range \( 0.4 < \gamma_s < 0.5 \). Substituting values of \( \gamma_s = 0.4 \) and \( 0.5 \) into the separated flow model and letting \( \frac{dM_e}{dX} = 0 \) leads to the conditions that

\[
R_o M_e \bar{\delta}_l \Phi \approx \begin{cases} \approx 183 & (\gamma_s = 0.4) \\ \approx 130 & (\gamma_s = 0.5) \end{cases}
\]

for \( \beta = 1.4 \) and \( \beta_c = 0 = \bar{B} \). If the values on the right-hand side of Equation (137) are exceeded, then \( \frac{dM_e}{dX} > 0 \) and the pressure gradient is positive. In Figure 16, this condition prevailed in the region \( 0.65 < \frac{X}{L} < 1.0 \) where the pressure drop is observed.

Lees and Reeves (9) also considered the condition controlling the occurrence of a pressure maximum. To compare Equation (137) with the results of Lees and Reeves, the left-hand side of Equation (137) is written in terms of \( Re_{\delta^*} \). With the aid of the boundary layer thickness relations in Appendix B, the modified Stewartson transformation leads to

\[
R_o M_e \bar{\delta}_l = \frac{(\frac{\delta_e}{\delta_{le}})(\frac{\delta_i}{\delta_{le}})(\frac{\delta_i^*}{\delta_e^*}) Re_{\delta^*}}{(1 + \frac{\delta_{le}}{\delta_i^*} M_e)^2 \left[ 1 - \frac{\delta_{le}}{\delta_i^*} M_e \right] \left( 1 - \frac{\delta_i^*}{\delta_i} \right)^2}
\]

(138)
where
\[
\frac{\delta_0}{\delta_{i,0}} = \left(1 + \frac{\gamma - 1}{2} M_{o u}^2 \right)^{\frac{3\gamma - 1}{2(\gamma - 1)}} \left[ 1 - \frac{\gamma - 1}{1 - \frac{\gamma - 1}{2} M_{ou}^2} \left( 1 - \left( \frac{\delta_0}{\delta_{i,0}} \right)^{\frac{\gamma}{\gamma - 1}} \left( \frac{\beta}{\delta_{i,0}} \right) \right) \right]^{\frac{2}{\gamma - 1}},
\]
(139)

and
\[
\frac{\delta''}{\delta_i} = \left(1 + \frac{\gamma - 1}{2} M_e^2 \right)^{\frac{3\gamma - 1}{2(\gamma - 1)}} \left[ 1 + \frac{\gamma - 1}{2} M_e^2 \left( 1 - \frac{\gamma - 1}{2} M_e^2 \right)^{\frac{\gamma}{\gamma - 1}} \left( \frac{\beta}{\delta_i} \right) \right].
\]
(140)

To use Equations (137), (138), and (140) it remains to estimate the Mach number and \( \phi \) at the pressure plateau. Following Lees and Reeves, the semiempirical relation for the pressure coefficient on the plateau, given by Chapman, Kuehn, and Larsen (6) is used:
\[
C_p = \gamma (M_{ou}^2 - 1)^{-\frac{1}{2}} \left( R e_{x,0} \right)^{-\frac{1}{4}}.
\]
(141)

Equation (141) is suitable for those interactions, which are sufficiently strong that pressure maxima might occur. The pressure is related to \( M_e \) via the isentropic relations and \( \phi \) is obtained from Equation (28).

Figure 19 shows the critical values of \( Re_{\delta^*} \) as a function of \( M_e \). The calculations were done for an initial Reynolds number, \( R e_{x,0} = 10^4 \). Curves are plotted for \( \gamma' s = 0.4 \) and 0.5. It can be expected that pressure maxima will occur if \( Re_{\delta^*} \) equals or exceeds the values given by the curves. For comparison, similar curves developed by Lees and Reeves (9) for the two-moment method with Falkner-Skan velocity profiles and a simple Tani quartic method are
Figure 19. Conditions for the occurrence of a pressure maximum in the plateau region.
shown. In general, the present method can be expected to produce pressure maxima at lower values of $Re_{\theta}$ than the Lees and Reeves method. This is unfortunate, but apparently related to the use of an inner momentum equation instead of a second moment of the momentum equation. Fortunately, as can be seen in Figure 16, the rate of pressure fall-off beyond the pressure maximum is sufficiently small that its effect is not too serious.

Figure 20 shows another comparison of calculated and measured pressure distributions. The same difficulty is noted with the initial pressure as before. Although the calculated curve generally falls below the data, the comparison is considered reasonable. For the conditions of this calculation a pressure maximum did not occur in the plateau region.

Figures 21 and 22 show the calculated shear stress coefficient and boundary layer velocity profile parameters. Again, the smooth variation of $c_f$ and the termination at a value of $c_f$ less than the starting value in Figure 21 is noted. The return of the velocity profile to the flat plate profile is again indicated by $c = 2$ at the end of the interaction.

It can be concluded that the model provides a reasonable quantitative description of the ordinary shock wave/laminar boundary layer interaction. The principal shortcomings relate to the matching of the initial pressure and the pressure maximum in the plateau region. The problem with the initial pressures is balanced by the relative simplicity of integrating directly from the start of
Interaction, $M_{0n} = 2.0$, $M_{0p} = 1.78$, $Re = 2.96 	imes 10^5$, $Re = 4.82 	imes 10^5$

Figure 21. Calculated shear stress coefficient for a complex.

Distance from Leading Edge, $x$, inches

Shear Stress Coefficient, $C_x 	imes 10^5$
Figure 22. Variation of the velocity profile parameters through distance from leading edge, in inches.
interaction. The occurrence of pressure maxima is inherent in the model and can limit its range of applicability.

Effect of Injection

The effect of fluid injection into the circulation region was investigated with the model. Available experimental data are generally for large injection rates and are thus not suitable for comparison with a boundary layer theory. In order that boundary layer theory remain valid, it is necessary that transverse velocities \((v/\nu_c)\) remain of the order of the boundary layer thickness. This implies that the wall injection rates remain small; of the order \((28)\)

\[
\beta_c < \frac{1}{\sqrt{Re_x}} .
\] (142)

In addition, injection generally induces shock waves and alters shock wave angles in the inviscid stream. Calculations were restricted to sufficiently small values of \(\beta_c\) that the injection had negligible effect on the shock structure.

Figures 23, 24 and 25 show the calculated effect of injection on a laminar boundary layer/shock wave interaction. The Reynolds number at the shock impingement point is the same for all the cases shown in Figures 23-25. The upstream ends of the interactions were found by the iteration scheme, which was discussed previously, to match the shock wave pressure rise. For the calculations shown in Figures 23-25 injection is confined to a small slot downstream of the shock impingement point. The slot width is of the order of magnitude of a boundary layer thickness and small compared to the over-all length of the interaction.
Figure 23. Comparison of computed pressure distributions for various amounts of fluid injection through a slot, $M_{ou} = 2.0$, $M_{oo} = 1.78$, $Re_e = 2.96 \times 10^5$, $\gamma = 1.4$. 
Figure 25: Boundary layer thickness variation with fluid injection through a slot, $M_{\infty} = 2.0$, $M_{op} = 1.78$, $Re_{L} = 2.96 \times 10^5$, $\gamma = 1.4$. 

Slot Location: $x = 2.2$ inches 
Slot Width = 0.01 inch
Figures 23-25 show that, as expected, injection appreciably increases the interaction length and pushes separation upstream. As the start of the interaction and separation are moved upstream, larger values of the function \( R_0 M_0 \frac{\partial \xi}{\partial z} \) are attained in the pressure plateau region. Consequently, pressure maxima appear for \( \beta_c = 10^{-3} \) and \( 10^{-2} \) although there was no maximum observed for \( \beta_c = 0 \) in Figure 23. The levels of the plateau pressures are essentially unaffected by the fluid injection.

Figure 24 shows the variation of the velocity profile parameters through the interaction. The value of \( c \) was plotted through the complete interaction. It varies smoothly through separation (and attachment for \( \beta_c = 0 \)), but is discontinuous at the beginning and end of the slot. Though \( c \) returns to the flat plate value of 2 for the case \( \beta_c = 0 \), it does not fully recover for \( \beta_c \neq 0 \). For \( \beta_c \neq 0 \), \( c > 0 \) at the end of the interaction indicates an end to the circulation region. However, the velocity profile at the end of the interaction is distorted by the injection and interaction. This distortion is related to the opening of the circulation region by the fluid injection. Figure 24 also shows that, as noted previously, the separating streamline does not reattach if injection occurs. The end of the flow circulation is indicated by \( \left( \frac{2c}{\sqrt{\gamma}} \right)_{\gamma = \infty} = 0 \), which occurs when the right-hand side of Equation (72) vanishes. Thus, the integrated blowing parameter \( B \) is related directly to the termination of flow circulation. At this point the velocity profile depends on \( \gamma \) and \( B \). Consequently, it is not possible to return to
Pohlhausen's method at the end of the circulation region. If Pohlhausen's method were used at this point, then the pressure gradient would be incorrectly given by the gradient corresponding to Equation (74).

Figure 25 shows the variation of the boundary layer thickness through the interaction. Considerable thickening occurs prior to the shock impingement point. After the shock impingement point, the boundary layer thickness is reduced by compression and the vanishing of the flow circulation. The final thickness exceeds the initial thickness. Fluid injection significantly increases the amount of thickening that occurs during the interaction.

Figure 26 compares the pressure distributions obtained with the 0.01-inch slot and $\beta_c = 10^{-2}$ with uniform injection downstream of the shock impingement point and $\beta_c = 10^{-4}$. The relative values were selected because they produce about the same amount of integrated blowing, $B$, at the downstream end of the interaction. The two calculations produced identical results upstream of the shock impingement point and only slightly different results downstream of the shock impingement point. It can be concluded from Figure 26, that $B$ has a more important effect on the pressure distribution than $\beta_c$ or slot location.

Figure 27 shows the variations of $\gamma_s$ and $c$ through the interactions for slot and distributed injection. The primary difference between the two calculations is the previously noted discontinuity in $c$ at the beginning and end of the slot. This results in some
Figure 26. Comparison of computed pressure distributions for slot and distributed injection, $Ma_u = 2.0$, $Ma_D = 1.78$, $Re_c = 2.96 \times 10^5$, $\gamma = 1.4$. 

- $x < 1.96$ inch, $p_c = 0$
- $x \geq 1.96$ inch, $p_c = 10^{-2}$
- Slot Location = 2.2 inches
- Slot Width = 0.01 inch

Separation

Shock Impingement Point
Figure 27. Variation of the velocity profile parameters for slot and distributed injection, $M_{eu} = 2.0$, $M_{eo} = 1.78$, $Re = 2.96 \times 10^5$, $\delta = 1.4$. 
differences between the two calculations in the values of $c$ and the location of the end of flow circulation.

Figure 28 shows the effect of uniform injection downstream of the shock impingement point on a shock wave interaction at $M_{ou} = 6$ and $M_{oD} = 5$. The calculation was done with $Re_{x_o}$ fixed at $4.3 \times 10^5$ corresponding to $R_o = 100$ for laminar boundary layer growth on a flat plate and $\gamma = 1.4$. The shock strength used in this calculation is not enough to produce a pronounced pressure plateau upstream of the shock impingement point for $\rho_c = 0$. In addition, the flow is separated for only about half in the interaction length with $\rho_c = 0$. Uniform injection downstream of the shock impingement point prevents reattachment and increases the length of the interaction. Flow separation is pushed significantly upstream, which leads to the development of a pressure plateau. The calculated pressure gradient was positive along much of the plateau. However, the gradient was sufficiently small that the pressure on the plateau appears constant in Figure 28.

Figure 29 shows the results of a computation similar to the one just described but with $\gamma = 1.3$. The Mach number ratio $M_{ou}/M_{oD}$ was the same (6/5) as before and the value of $R_o = 100$ was maintained. With $R_o = 100$ and $\gamma = 1.3$ the corresponding value of $Re_{x_o}$ is $2.6 \times 10^5$. With maintenance of a constant Mach number ratio, the results in Figure 29 represent a larger shock wave pressure ratio than the results in Figure 28. In spite of this increase in pressure ratio, the length of interaction region is somewhat shorter with $\gamma = 1.3$, $\rho_c = 0$ than with $\gamma = 1.4$, $\rho_c = 0$. As before, the flow is
Figure 20. Effect of distributed injection on the pressure distribution for $M_{ou} = 6$, $M_{op} = 5$, $Re_{x,0} = 4.3 \times 10^5$, $\delta = 1.4$. 

Pressure Ratio, $P_0 / p_0 \times 10^3$
Figure 29. Effect of distributed injection on the pressure distribution for $M_{ou} = 6$, $M_{od} = 5$, $Re_{x_0} = 2.6 \times 10^5$, $\sigma = 1.3$. 
separated for about half of the interaction length for $\beta_0 = 0$. The effect of fluid injection is to again increase the length of the interaction and push separation upstream. A pressure maximum appears in the plateau with a slight drop in the calculated pressure prior to the shock impingement point. Again, the pressure level at the shock impingement point is relatively unaffected by fluid injection.

**Effect of Suction**

Suction can be expected to reduce or eliminate flow separation and reduce the over-all length of the interaction. These effects have been observed in the experiments performed by Greber (8) on a flat plate with suction through slots. Unfortunately direct comparisons between Greber's data and a laminar flow theory are difficult because boundary layer transition occurred in many of his experiments. The reduction in interaction length was also found by application of the Crocco-Lees mixing theory by Bray et al. (10).

Application of the model developed in this research was accomplished by integration in the upstream direction from the downstream end of the interaction. The program for the complete interaction described in Appendix E was used for the calculations. As before, the Pohlhausen method was used for the attached flow. The computation begins with the flow attached at the downstream end of the interaction. A double iteration procedure is required to match the upstream Mach number, $M_{ou}$, and the value of $R_0$ at the upstream end of the interaction. With suction, the upstream end of the interaction is
opened up so that the streamline which is followed attaches but never separates from the wall. Thus, there is enough mass admitted to the circulation region to balance the suction rate.

Figure 30 shows the calculated effect of suction through a slot in the flow circulation region on the pressure distribution. As expected, the pressure gradients in Figure 30 are steeper downstream of the shock impingement point with suction than without suction. However, suction increased the length of the plateau region and the total interaction. This is contrary to expectations and represents a major deviation from previous results. The suction calculation for $\rho_s = -0.01$ was carried out for two slot locations in the flow circulation region. The effect of slot location, as shown by the results in Figure 30, is secondary.

Figure 31 shows the effect of suction on the boundary layer growth. As expected, suction decreases the boundary layer thickness at the end of the interaction. Resumption of normal boundary layer growth at the downstream end of the interaction is noted in Figure 31. Since the initial condition at the downstream end is taken as $c = 2$, the pressure gradient vanishes and the boundary layer growth rate should be the same as for a normal flat plate. This same effect is observed in the $\rho_s \equiv 0$ calculation since $c$ returns to 2 for the attached flow at the downstream end of the interaction. The physically unrealistic lengthening of the interaction region is also shown in Figure 31.
Figure 30. Effect of suction on the pressure distribution for $M_0 = 2.0$, $M_0 = 1.78$, $Re = 2.96 \times 10^5$, $\gamma = 1.4$. 
Figure 31. Boundary layer thickness variation with suction through a slot, $M_{ou} = 2.0$, $M_{od} = 1.78$, $Re_{L} = 2.96 \times 10^5$, $k = 1.4$. 

Slot Location
$2.18 \text{ in} \leq x \leq 2.20 \text{ in.}$
For the cases investigated, suction lengthened rather than shortened the interaction. It is assumed in this investigation, as in prior investigations, that the nature of the flow separation is independent of the inducing agent; i.e. the shock wave impingement. As discussed in relation to fluid injection, a similar assumption at attachment would not be correct because injection altered the velocity profile at the end of the circulation region. Thus, with injection, the Pohlhausen wall boundary condition would not have properly predicted the pressure gradient at the downstream end of the flow circulation region. In the present method of analyzing suction, the attachment velocity profile is assumed to be unaffected by the suction. Because the streamline which is being followed does not originate at the wall, the circulation region is open at the upstream end and the situation is just a reversal of the injection situation. Thus $B$ is greater than zero at the upstream end of the interaction indicating a net flow towards the slots. As a result, the velocity profile at the upstream end of the flow circulation region is affected by the suction and the pressure gradient is not given by the Pohlhausen wall boundary condition. Since a streamline (though not the one to which $\eta_4$ is measured) separates from the wall at this point, this alteration to the velocity profile results in a contradiction to the assumption that the nature of the flow separation is unaffected by the inducing agent.

Although a downstream effect of injection is considered to be reasonable, a similar upstream effect of suction is not. A substantial
upstream effect, of the type observed, would require that the influence of suction be signalled upstream even beyond the beginning of the interaction. Thus, at some point between the plate leading edge and the upstream end of the interaction, velocity profile changes would have to occur to accommodate the suction.

The difficulty at separation suggests the need for experimental data and the use of a new approach to the suction calculation. Integration should begin at the upstream end of the interaction and proceed downstream through separation. Downstream of the suction slot $\gamma_s$ should be measured to the streamline for which $B = 0$. This new streamline can then be followed to attachment indicating the end of the flow circulation. The attached flow calculation would then be carried out to the end of the interaction. Unfortunately applying this procedure in a simple step would limit the approach to narrow slots which could be approximated by a step change in $B$. This step change would produce discontinuities in the derivatives of all of the

For the inner velocity profile in Equation (66), the position indicated by Equation (42) of a new streamline, $\gamma_s'$, for which $B' = 0$ is given by

$$\gamma_s' = -\frac{2}{3} \frac{A_x}{A_3} + \left( \frac{2}{3} \frac{B_x}{A_3} \right) - 2 \frac{A_1}{A_3} \left( \gamma_s - \frac{B}{\gamma_s (5-3 \gamma_s)} \right)$$

where

$$A_1 = -2 \frac{B}{\gamma_s} + 12 \left( \frac{B}{\gamma_s^3} \right) \left( \frac{3-\gamma_s}{5-3 \gamma_s} \right)$$

$$A_2 = 3 \frac{B}{\gamma_s^3} - 48 \frac{B}{\gamma_s^5 (5-3 \gamma_s)}$$

$$A_3 = 12 \left( \frac{B}{\gamma_s^2} \right) \left( \frac{1+\gamma_s}{5-3 \gamma_s} \right)$$

and $B$ is the integrated suction for the slot.
flow variables. Thus it would be more desirable to solve continuously for the new value of \( \gamma \) corresponding to \( B = 0 \). However, if this procedure were followed then the inner region would be bounded by the locus of \( \beta = 0 \) and not by a streamline. If the inner region is not bounded by a streamline, then Equation (17) and the inner momentum Equation (18) are no longer valid. Thus it is necessary to drop the inner momentum equation and replace it by another relation. It is suggested that the second moment of the momentum equation might serve this purpose. In this suggested procedure some of the influence of \( B \) on the velocity profile is lost.
V. CONCLUSIONS

The basic flow model developed here adequately describes the ordinary shock wave/laminar boundary layer interaction. This model utilizes an additional momentum equation for the flow circulation close to the wall and polynomial velocity profiles matched at the separating streamline. It is described by Equations (55) to (58), (26) and the coefficients in Appendix D. Though a pressure maximum in the plateau region is observed at lower values of $Re_{\infty}$ than is observed with the Lees and Reeves theory, the magnitude of the pressure drop beyond the maximum is relatively small. At separation and attachment, the inner momentum equation is equivalent to the Pohlhausen wall boundary condition. The use of Pohlhausen's method for the attached flow eliminated discontinuities in derivatives at separation and attachment. Pohlhausen's method (as described by Equations (115)-(117), (110), (111), (118)-(128), (126)-(129), and (132)-(135)) was used for the attached flow description in the calculations of complete interactions.

The significant boundary layer thickening which occurs between separation and the shock impingement point reduces the validity of the thin layer assumption inherent in the boundary layer equations. Review of the computational results (see for example Figures 25 and 31) indicates that the boundary layer thickens by factors of 3 to 4 prior to the shock impingement point. For the relatively large
Reynolds numbers considered in the illustrative calculations this thickening should not cause large deviations from ordinary boundary layer theory. At the shock impingement point the maximum values of the turning angle $\phi$ are obtained. Maximum values of about 0.1 were obtained in the calculations, which is consistent with the initial assumptions on $\phi$.

The boundary layer equations contain the assumption that $(v/u_\infty)$ is of the order of the boundary layer thickness, $\delta$. Values of $\beta_c$ selected for the calculations were restricted to conform to this assumption. However, large transverse velocities can occur, even for $\beta_c = 0$, at the separation and reattachment points. The favorable comparisons between experiment and theory and the consistency of the predicted slope of the separating streamline at separation suggest that the boundary layer equations do provide a suitable description of the flow.

The model predicts that flow injection significantly lengthens the interaction region and pushes separation upstream. As a result of injection, the separating streamline does not reattach and the velocity profile at the end of the circulation region is modified. Thus the pressure gradient at the downstream end of the circulation region does not correspond to the Pohlhausen wall boundary condition. Injection destroys the relative symmetry of separation and attachment, which is associated with the ordinary shock wave/boundary layer interaction.
Since injection increases the length of the interaction, it increases the values of the parameter \( R_0 M_e \delta \phi \) which are achieved in the plateau region. Consequently injection tends to promote the occurrence of a pressure maximum in the plateau region. The extent of injection which can be analyzed with the model is limited by the occurrence of pronounced pressure maxima.

As expected, the integrated injection, \( B \), has a more significant effect on the over-all interaction than the local injection rate, \( \beta_c \). This effect of \( B \) is seen in the modification to the velocity profile in and downstream of the injection. In addition, the value of \( \eta \) at the end of the interaction must be sufficiently large that the mass flow indicated by \( B \) can pass between the wall and the separating streamline. The displacement of the separating streamline at the end of (and beyond) the interaction represents a diffusion of injected mass into the gas stream. Through the explicit dependence of the velocity profile on \( B \), it is felt that some of the benefits of a two-parameter velocity profile were achieved.

Suction calculations were performed by integrating in the upstream direction. However, when this was done, the effect of suction on the velocity profile modified the nature of the initial flow separation. It is felt that these results reflected the effect of an inversion of the integration procedure rather than a true physical picture.

Experimental investigations in the range of injection and suction rates consistent with boundary layer theory are necessary.
These experiments should be conducted with sufficiently low Reynolds numbers that the interaction can be studied free of transition effects. Re-evaluation of the theoretical results in the light of experimental data is necessary. Particular attention should be devoted to the velocity profiles and pressure gradients at the ends of the flow circulation region with suction and injection.

Several extensions to the present investigation are possible. The most obvious is the suggested modification to perform suction calculations. This modification would replace the inner momentum equation with the second moment of the momentum equation. For suction calculations, the streamline to which $\gamma_s$ is measured would always be one which bounds a region with zero net flow. It is expected that replacement of the inner momentum equation with the second moment equation would also increase the values of Re$_s$ at which pressure maxima occur. This would extend the useful range of the theory.

Another modification which should improve the accuracy of the theory, would be to add the second moment equation in place of the assumed function for \( a_i \) while retaining the inner momentum equation. Thus the ordinary interaction would be described by a two-parameter method. With $B \neq 0$ the method would be, in effect, a three parameter method. Though this type of modification should improve the accuracy of the calculations it would certainly add significantly to its complexity.
Finally, it is suggested that the basic model could be extended to include heat transfer and with somewhat more effort, injection of a foreign gas. The model could also be extended to axially symmetric geometries which have considerable practical interest.
CROCCO-LEES MIXING CALCULATIONS

To aid in the evaluation of the mixing layer method, the governing equations were programmed for numerical computations with the IBM 7094 computer in The Ohio State University Computation Center. The program is written in the SCATRAN language. The formulation of Bray, Gadd, and Woodger (10) is used directly for the program. The equations yield the rate of growth of the mixing layer and the resultant turning of the external supersonic stream. They are derived from consideration of continuity of mass and momentum for a uniform flow layer between the wall and the inviscid stream. Properties of flow variables in the mixing layer are related to the boundary layer velocity profile via the Crocco-Lees parameters defined below.

Following Bray et al. (10) the momentum and continuity equations in the dissipative region are

\[
\rho u \frac{2u}{\partial x} + \rho v \frac{2u}{\partial y} = -\frac{dp}{\partial x} + \frac{2}{\partial y} (\mu \frac{2u}{\partial y})
\]

and

\[
\frac{2}{\partial x} (\rho u) + \frac{2}{\partial y} (\rho v) = 0
\]

In the inviscid outer flow, the Bernoulli equation applies,

\[
\frac{dp}{\partial x} = -\rho_e u_e \frac{du_e}{\partial x}
\]
Integrating the momentum equation across the dissipative layer thickness $\delta$ leads to

$$\frac{dI}{d\gamma} = u_e \frac{dm}{d\gamma} + \rho_e u_e \delta \frac{d\delta}{d\gamma} - \gamma + \rho_e u_e^2 C_Q,$$

where $I$ is the momentum flux in the viscous layer given by

$$I = \int_0^\delta \rho u^2 dy,$$

$m$ is the mass flux,

$$m = \int_0^\delta \rho u dy,$$

and $C_Q$ is the suction parameter,

$$C_Q = -\rho_e.$$

Including blowing or suction, a mass balance for the dissipative layer produces

$$\frac{dm}{d\gamma} = \rho_e u_e \left[ \frac{d\delta}{d\gamma} - \Phi - C_Q \right],$$

where $\Phi$ is the deflection angle of the outer streamline of the dissipative layer. Defining a mixing coefficient, $C_M$, as

$$C_M \equiv \frac{d\delta}{d\gamma} - \Phi - C_Q,$$

the mass balance becomes

$$\frac{dm}{d\gamma} = \rho_e u_e C_M.$$

It remains to relate the thickening of the dissipative layer to the pressure distribution. For isentropic, supersonic external flows the Prandtl-Meyer relation accomplishes the purpose. This is
represented by Bray et al. (10) by a second-order approximation to the simple wave-flow equation,

\[ \phi = (1 - \frac{\omega}{\overline{\omega}})(a + b \frac{\omega}{\overline{\omega}}) \]

where

\[ v = \frac{u_e}{a^0} \]
\[ a = (\overline{\bar{M}}^2 - 1)^{1/2} - b \]
\[ b = \frac{r - 1}{4} (\overline{\bar{M}}^2 - 1)^{3/4} + \frac{r - 1}{2} (\overline{\bar{M}}^2 - 1)^{1/4} + \frac{r + 1}{4} (\overline{\bar{M}}^2 - 1)^{1/4} \]
\[ \bar{M} = \overline{\bar{w}} (1 - \frac{r - 1}{2} \bar{w})^{1/2} \]
\[ \overline{\bar{w}} = \begin{cases} \omega_{\text{up}} & \text{upstream of shock} \\ \omega_{\text{down}} & \text{downstream of shock} \end{cases} \]

Now a set of parameters are defined which describe the properties of the dissipative layer and are independent of Mach number. The compressible boundary layer equations are transformed by the Stewartson transformation and the correlation parameters are obtained from solutions to the incompressible boundary layer equations.

The first of these Crocco-Lees parameters is the momentum deficiency ratio defined by

\[ \chi \equiv \frac{1}{m} \frac{u_e}{u_e} \]

where \( \chi \) is always less than unity for the dissipative layer.

The second parameter, \( C \), is related to the rate of entrainment of fluid into the boundary layer and is defined as

\[ C \equiv \frac{m}{\mu_e} C_m \]
The third parameter, \( \sigma \), is related to the wall shear and is defined as
\[
\sigma = \frac{m c_f}{2 (1 - \lambda) C \mu_c}
\]
The parameter \( \sigma \) clearly vanishes with \( c_f \) at a point of separation.

Finally, a shape parameter, \( \psi \), is introduced which depends on the shapes of the velocity and temperature profiles,
\[
\psi = t \left( \frac{\rho_{en} \gamma \bar{E}}{m} - \lambda \right)
\]
where \( t \) is the temperature ratio at the edge of the boundary layer given by
\[
t = 1 - \frac{\nu}{\varepsilon} \omega^2
\]

The mass and momentum equations for the dissipative layer can now be written in terms of the Crocco-Lees parameters defined above. Beginning with the momentum equation,
\[
\frac{d\delta}{dx} = \frac{B}{m} \frac{d\omega}{dx} + \frac{\psi}{\omega t} \frac{d\omega}{dx}
\]
where
\[
B \equiv (1 - \lambda)(1 - \sigma) + C_A / C_M
\]

From differentiation of the definition of the shape factor and Bernoulli's equation, the rate of growth of the dissipative layer is
\[
\frac{d\delta}{dx} = \frac{5}{\psi + \kappa t} \left\{ \frac{1}{m} \frac{d\omega}{dx} \left[ B \frac{d\psi}{dx} + \psi - \kappa \sigma (1 - \lambda) + t + t \frac{C_A}{C_M} \right] \right.
\]
\[
+ \frac{1}{\omega \varepsilon} \frac{d\omega}{dx} \left[ \psi \frac{d\psi}{dx} + \psi \right] \right\}
\]
where
\[
j = \gamma \omega^2 \psi + \kappa t (\omega^2 - \lambda)
\]
This last expression for \( \frac{d\delta}{dx} \) can be equated to
\[
\frac{d\delta}{dx} = C_M + \Phi + C_Q
\]
to obtain the interaction equation
\[
\frac{1}{\omega t} \frac{d\omega}{dx} \left[ \psi \frac{d\psi}{dx} + J \right] + \frac{d}{dx} \left[ B \frac{d\psi}{dx} - E \right] = 0,
\]
where
\[
E = \frac{t \Phi}{C_M} + t \sigma (1-\kappa) - \psi,
\]
\( \kappa \) is a dimensionless distance referenced to the separation point,
\[
\kappa \equiv \frac{X - X_s}{\delta},
\]
and \( \Phi \) is essentially a Reynolds number based on boundary layer thickness defined by
\[
\Phi \equiv \frac{m}{\mu_\infty}.
\]
Finally the mass and momentum equations can be written respectively as
\[
\frac{d\xi}{dx} = \frac{\kappa \delta_\infty \rho_\infty}{\mu_\infty a_\infty} \frac{\rho}{\rho_\infty} \frac{C\omega}{\Phi},
\]
and
\[
\frac{dX}{dx} = \frac{B}{\Phi} \frac{d\xi}{dx} + \frac{\psi}{\omega t} \frac{d\omega}{dx}.
\]
Following Bray et al. (10) the interaction and momentum equations are used in the form
\[
\frac{d\psi}{d\xi} = \frac{J B + \psi E}{\xi (J \kappa' + \psi)},
\]
and
\[
\frac{d\omega}{d\xi} = \frac{\omega t (-B + \kappa' E)}{\xi (J \kappa' + \psi)},
\]
where
\[
\kappa' = \frac{dX}{d\Phi}.
\]
A second-order Runge-Kutta integration routine is used to solve the latter two simultaneous, nonlinear differential equations.

The parameters $X$, $C$, $\sigma$ and the derivative $X'$ are tabulated functions of $\psi$. The tabulated functions are obtained from solutions to the boundary layer equations or from experiment. For the calculations described in the text the upper-branch Falkner-Skan solution was used for attached flow and the lower-branch for separated flow. The Crocco-Lees parameters derived from these solutions are tabulated by Bray et al. (10).

The results of the integration are related to position by simple quadrature based on the mass balance with the equation

$$\frac{dX}{d\xi} = \frac{\psi P_s \omega_s}{\xi_s \xi \omega (\psi_s + X \tau_s)}$$

which incorporates the Stewartson transformation, and where, for $\gamma = 1.4$

$$P = t^{\gamma/2}$$

and the subscript $s$ refers to the separation point.

In addition, the boundary layer thickness ratio is given by

$$\frac{\delta}{R_s} = \frac{\xi (\psi + X \tau)}{\xi_s (\psi_s + X \tau_s) \xi \omega}$$

Finally position can be obtained from

$$\frac{d\xi}{d\psi} = \frac{\mu \alpha \rho}{\sigma^{\gamma/2} \rho \omega} \xi$$

or for $\gamma = 1.4$

$$\frac{d\xi}{d\psi} = \frac{t_\omega \omega \chi \xi}{R \omega \xi \omega} \xi$$
where

\[ x_L = \text{distance to the shock impingement point} \]

\[ \text{Re}_{x_L} = \text{Reynolds number based on the distance to the shock impingement and free-stream undisturbed velocity, density, and viscosity} \]

\[ w_0 = \text{value of w before interaction} \]

\[ t_0 = \text{value of t before interaction} \]

Integration begins at the separation point, with specified values of \( \psi \), \( w_{ou} \) and the separation values of the Crocco-Lees parameters, and proceeds upstream to the start of the interaction. The program is written on the basis of \( \psi = 1.4 \) though this does not represent any basic limitation. An initial value of \( w \) is assumed at the separation point. The program computes the attached flow and compares \( w \) to \( w_{ou} \) at the end of the interaction. This point is defined by \( \psi' < 0 \) or \( \psi < \psi_{min} \). The program adjusts \( w_s \) until a solution is obtained which matches \( w_{ou} \) at the start of the interaction.

The tolerance on the match to \( w_{ou} \) was selected to produce no more than a 2 per cent error in the initial pressure. After an acceptable value of \( w_s \) is obtained at separation, the equations are integrated in the downstream direction. The object of the downstream integration is to match \( w_{OD} \). The distance between separation and the shock impingement point is adjusted by the program until a suitable match of \( w \) and \( w_{OD} \) is obtained at the end of the interaction. This point is defined by

\[ \frac{dw}{d\psi} \geq \epsilon \text{ or } w \geq w_{OD}, \text{ where } \epsilon \text{ is close to zero.} \]

Detailed analysis of the systematic errors resulting from the computational tolerances

\[ \psi_{min} \text{ is the value of } \psi \text{ corresponding to the boundary layer with zero pressure gradient.} \]
was not carried out because of the obvious gross inadequacies in the method, as noted in the text in connection with Figure 2.

As an example of the computation, the curve shown in Figure 2 was computed for $M_{OU} = 3$, $M_{OD} = 2.65$ and $\alpha = 560$. The computation was begun at the separation point with the separation values of the Crocco-Lees parameters given by Bray et al. (10). A step-size $DELCHI = \Delta \psi = 4$ was used for the attached flow and a value $DELCHI = 8$ was used downstream of separation since the variables changed slower after separation. An initial separation value of $w_s = 1.761$ was assumed. The program repeated the attached flow computation four times to obtain a final value of $w_s = 1.764$. An initial assumption of $X = 30$ at the shock impingement point was used to start the computation downstream of separation. The downstream computation was repeated six times to obtain a revised value of $X = 30.75$ at shock impingement. This particular run was executed in 0.3 minute on the IBM 7094 with a total run time of 0.7 minute. The results accurately matched Bray et al.'s computational Case 4.

The definitions of terms used in the program and a SCATRAN language listing of the program follow:

$A = a$

$AD = \text{value of } "a" \text{ downstream of shock impingement}$

$AU = \text{value of } "a" \text{ upstream of shock impingement}$

$B = b$

$BC = B$

$BD = \text{value of } "b" \text{ downstream of shock impingement}$
\[ BLR = \frac{\xi}{\xi_s} \]

\[ BU = \text{value of } \theta \text{ upstream of shock impingement} \]

\[ C = c \]

\[ CHI = \chi \]

\[ CHIS = \chi \text{ at the separation point} \]

\[ D(N) = \text{estimate of } \left( \frac{d\chi}{d\xi} \right) \]

\[ DEL = \text{increment in iteration of } \theta \text{ at the separation point} \]

\[ DELCHL = \text{integration step size} \]

\[ E = E \]

\[ EP = \text{tolerance on } \frac{d\omega}{d\xi} \rightarrow 0 \text{ in downstream integration, } \xi \]

\[ ETA = \text{increment in iteration on separation to shock impingement point distance} \]

\[ ETAL = \text{tolerance on iteration for separation to shock impingement point distance -- minimum value of ETA} \]

\[ F = \text{tolerance on iteration for } \theta \text{ at separation point -- minimum value of DEL} \]

\[ I = \text{index in Crocco-Lees parameters lookup table} \]

\[ IDV = \text{number of columns in Crocco-Lees parameter table look-up} (5) \]

\[ J = \text{index in Crocco-Lees parameter look-up table} \]

\[ JC = J \]

\[ K(N) = \text{estimate of } \left( \frac{d\chi}{d\xi} \right) \]

\[ KAPPA = \chi \]

\[ KAPPAP = \chi' \]

\[ KAPPAS = \chi \text{ at the separation point} \]

\[ LS = L \]

\[ MB = \bar{\bar{M}} \]

\[ MBD = \bar{\bar{M}} \text{ downstream of the shock impingement point} \]
MBU = \( \bar{M} \) upstream of the shock impingement point

N = index controlling integration

P = \( t^{3.5} \)

PB = \( (t/t_0)^{3.5} \)

PO = \( t_0^{3.5} \)

PHI = \( \Phi \)

PIN = \( t^{3.5} \) corresponding to \( w_{oD} \)

PS = \( t^{3.5} \) at the separation point

PSI = \( \Psi \)

PSIMIN = minimum permissible value of \( \Psi \)
          (corresponds to flat plate value)

PSIP = \( \frac{d\Psi}{d\xi} \)

PSIS = separation value of \( \Psi \)

R = tolerance on \( w \) at the downstream end of the integration

REL = \( Re_{x,L} \)

SIG = \( \sigma \)

T = \( t \)

TO = \( t_0 \)

TS = value of \( t \) at separation

U(I,J) = entries in Crocco-Lees parameter table

UNC = interpolation variable in Crocco-Lees parameter table look-up

W = \( w \)

WB = \( \bar{w} \)

WIN = \( w_{oD} \)

WO = \( w_0 \)
\[ WP = \frac{d w}{d F} \]

\[ WS = w \text{ at the separation point} \]

\[ X = x \]

\[ XL = \frac{x - x_s}{\delta_s} \]

\[ XSH = \left( \frac{x - x_s}{\delta_s} \right) \text{ evaluated at the shock impingement point.} \]
*** RUN

*** SCATRAN

*** DUMP LOWER CORE

FLOATING(MB,MBU,MBD,K,KAPPA,KAPPAP,KAPPAS,LS,JC)-
DIMENSION(K(3),D(3),U(135,IDV))- START
IDV=5-
READ INPUT, INP, ( (U(I,J),J=0,1,J,L,5), I=0,1,I,L,27))- F INP
( 5F10.5)- WRITE OUTPUT, CLP, ( (U(I,J),J=0,1,J,L,5), I=0,1,I,L,27))- F CLP
( 1H, 5F9.5)- READ INPUT, 7, (WO,WIN,PO,PIN,MBU,MBD,
AU,AD,BU,BD,PSIS,KAPPAS,PSMIN,
CHIS,WS,CELCHI,DEL,F,EP,ETA,ETAL,
XSH,LS,REL,TO)- WRITE OUTPUT, 2, (WO,WIN,PO,PIN,MBU,
MBD,AU,AD,BU,BD,PSIS,KAPPAS,
PSIMIN, CHIS, WS, DELCHI, DEL, F,
EP, ETA, ETA, XSH, LS, REL, TO
DELCHI = DELCHI
N = 0
SEP
TS = 1 - 0.2 * WS * WS
PS = TS * P * 3.5
W = WS
PSI = PSIS
CHI = CHIS
X = 0
XL = 0
T = TS
INT
I = 0
J = 0
TAB
I = I + 1
PROVIDED (I * GE. 27) TRANSFER TO (ERROR)
PROVIDED(PSI,U(I,J))\*TRANSFERTO(TAB)-
UNC=(PSI-U(I,J))/(U(I-1,J)-U(I,J))-^J=1-
KAPPA=U(I,J)+(U(I-1,J)-U(I,J))\*UNC-^J=2-
KAPPA_P=U(I,J)+(U(I-1,J)-U(I,J))\*UNC-^J=3-
C=U(I,J)+(U(I-1,J)-U(I,J))\*UNC-^J=4-
SIG=U(I,J)+(U(I-1,J)-U(I,J))\*UNC-
PROVIDED(XL,XSH),TRANSFERTO(SUBS)-
MB=MBD-
A=AD-
B=BD-
WB=WIN-
TRANSFERTO(RE)
SUBS  MB=MBU-
    A=AU-
    B=BU-
    WB=WO-
RES   PHI=(1-(W/WB))*(A+(B*W)/WB)-
    JC=(1.4*PSI*W*W)+(KAPPA*T*(W*W-T))-
    BC=(1-KAPPA)*(1-SIG)-
    E=(CHI*PHI/C)+((1-KAPPA)*SIG*T)-PSI-
    N=N+1-
    PROVIDED(N.G.2),TRANSFERTO(ERROR)-
    K(N)=((JC*BC)+(PSI*E))/(CHI*(JC*KAPPAP+PSI))-
    D(N)=W*T*(-BC+(KAPPAP*E))/(CHI*
               (JC*KAPPAP+PSI))-\n    PROVIDED(N.E.2),TRANSFERTO(DER)-
    CHI=CHI+DELCHI-
    PSI=PSI+(K(N)*DELCHI)-
W=W+(D(N)*DELCHI)-
T=1-0.2*W*W-
TRANSFERTO(INT)-
DER
PSIP=0.5*(K(1)+K(2))-
WP=0.5*(D(1)+D(2))-
PSI=PSI-(K(1)*DELCHI)-
W=W-(D(1)*DELCHI)-
PSI=PSI+(PSIP*DELCHI)-
W=W+(WP*DELCHI)-
N=0-
T=1-0.2*W*W-
P=T*P*3.5-
Pb=P/PO-
XL=XL+(PS*WS*CHI*DELCHI)/(CHIS* 
C*P*W*(PSIS+KAPPAS*TS))- 
X=X+(WO*(TO*P*1.5)*LS*CHI*DELCHI)/
\[(\text{REL}\times C\times P\times W) - \]
\[\text{BLR} = (\text{CHI} \times (\text{PSI} + \text{KAPPA}\times T)\times PS\times WS)/(\text{CHIS}) \]
\[\ast ((\text{PSI} + \text{KAPPA}\times TS)\times P\times W) - \]
\[\text{WRITEOUTPUT}, \text{ANS}, (\text{CHI} \times \text{XL}\times X, \text{DEL}\times \text{ETA}\times \text{PSI}, \]
\[W, \text{PSIP}, W, \text{P}, \text{PHI}, WS, P, PB, XSH, BLR) - \]
\[F \text{ ANS} \]
\[(1H0\times F8.3\times 2F11.4\times 2F10.5\div 6F12.6\times 4F7.3) - \]
\[\text{PROVIDED}(\text{DELCHI}\times G, 0)\times \text{TRANSFERTO}(\text{TEX}) - \]
\[\text{PROVIDED}(\text{PSIP} \times \text{LE}, 0)\times \text{TRANSFERTO}(\text{TED}) - \]
\[\text{PROVIDED}(\text{PSI} \times G, \text{PSIMIN})\times \text{TRANSFERTO}(\text{INT}) - \]
\[\text{PROVIDED}(\text{DEL} \times G, 0)\times \text{TRANSFERTO}(\text{ADJ}) - \]
\[\text{PROVIDED}(\text{ABS} \times \text{DEL} \times G, F)\times \text{TRANSFERTO}(\text{HAL}) - \]
\[\text{TRANSFERTO}(\text{UP}) - \]
\[\text{TED} \]
\[\text{PROVIDED}(\text{DEL} \times L, 0)\times \text{TRANSFERTO}(\text{ADJ}) - \]
\[\text{PROVIDED}(\text{ABS} \times \text{DEL} \times LE, F)\times \text{TRANSFERTO}(\text{UP}) - \]
\[\text{HAL} \]
\[\text{DEL} = -\text{DEL}/2 - \]
\[\text{ADJ} \]
\[WS = WS + \text{DEL} - \]
TRANSFERTO(SEP) -

PROVIDED(XL, LE, (XSH + (PS, WS, CHI, 
DELCHI) / (CHIS * C * PW * (PSIS + KAPPA * TS))),
TRANSFERTO(INT) -

PROVIDED(WP, L, EP), TRANSFERTO(INT) -

PROVIDED(W, LE, (WIN + R)), TRANSFERTO(TEW) -

PROVIDED(ETA, G, 0), TRANSFERTO(SHOCK) -

ETA = -ETA / 2 -

TRANSFERTO(SHOCK) -

TRANSFERTO(SHOCK) -

TEW PROVIDED(W, GE, (WIN - R)), TRANSFERTO(END) -

PROVIDED(ETA, L, 0), TRANSFERTO(SHOCK) -

ETA = -ETA / 2 -

SHOCK XSH = XSH + ETA -

PROVIDED(.ABS, ETA, G, ETAL), TRANSFER
TO(SEP) -

WRITEOUTPUT, 2, (PS, IP, EP, ETA, ETAL) -
TRANSFER TO (END) -

UP DELCHI = - 2 * DELCHI -

TRANSFER TO (SEP) -

ERROR WRITE OUTPUT, 3, (I, N) -

END CALL SUBROUTINE() = ENDJOB() -

END PROGRAM (START) -

*** DATA
The mass balance is expressed by Equation (23) as

\[ \Phi = \rho_c + \frac{d \delta^*}{dx} - \left[ \frac{d}{dx} \left( \ln \rho_e u_e \right) \right] \int_0^\delta \frac{\rho u}{\rho_e u_e} \, dy, \]

where

\[ \delta^* = \int_0^\delta \left( 1 - \frac{\rho u}{\rho_e u_e} \right) \, dy. \]

This equation must be transformed into a form which utilizes the variables in the incompressible plane to be compatible with the transformed momentum equations.

The blowing parameter, \( \beta_c \), is defined by

\[ \beta_c = \frac{\rho \nu}{\rho_e u_e}. \]

Now, the transformation equations (5) lead to

\[ u = \frac{a^*}{a^0} \, U, \]

and

\[ \nu = \lambda \frac{a^*}{a^0} \frac{\rho^o}{\rho_e} \frac{\rho^o}{\rho^o} \, \nu. \]

Therefore

\[ \beta_c = \lambda \left( \frac{\rho^0}{\rho_e} \right) \left( \frac{\rho_e}{\rho^0} \right) \beta_u, \]

and for a perfect gas

\[ \beta_c = \lambda \left( \frac{T^0}{T_e} \right) \beta_u, \]

which can be written in terms of Mach number as

\[ \beta_c = \frac{\lambda}{1 - \frac{\rho_e}{\rho^0} M^2_e} \beta_u. \]
Let
\[ H \equiv \frac{\lambda}{1 - \frac{\rho u}{\rho_c u_c}} \]
so that
\[ \rho_c = H\rho_c \]
and the transformation variable, \( H \), is a function of position \( x \).

The displacement thickness is
\[ \delta^* = \int_{0}^{\delta} (1 - \frac{\rho u}{\rho_c u_c}) d\gamma \]
which can be written as
\[ \delta^* - \delta = -\int_{0}^{\delta} \frac{\rho u}{\rho_c u_c} d\gamma \]
Transforming the right-hand side
\[ \delta^* - \delta = -\frac{\alpha^o \rho^o}{\alpha_e \rho_e} \int_{0}^{\delta_i} \alpha \gamma d\gamma \]
which can also be written as
\[ \delta^* - \delta = \frac{\alpha^o \rho^o}{\alpha_e \rho_e} (\delta_i^* - \delta_i) \]
or as
\[ \delta^* = \delta - \frac{\alpha^o \rho^o}{\alpha_e \rho_e} (\delta_i - \delta_i^*) \]

The boundary layer thickness \( \delta \) can be transformed by
\[ \delta = \int_{0}^{\delta_i} d\gamma = \int_{0}^{\delta_i} \frac{\alpha^o \rho^o}{\alpha_e \rho_e} \gamma d\gamma \]
Then, for an adiabatic wall and Prandtl number of unity
\[ \delta = \frac{\alpha^o \rho^o}{\alpha_e \rho_e} \int_{0}^{\delta_i} \gamma d\gamma = \frac{\alpha^o \rho^o}{\alpha_e \rho_e} \int_{0}^{\delta_i} \left(\frac{T}{T_e}\right) d\gamma \]
or in terms of the stagnation temperature ratio,
\[ \delta = \frac{\alpha^o \rho^o \frac{T^o}{T_e}}{\alpha_e \rho_e \frac{T_e}{T_e}} \int_{0}^{\delta_i} \left(\frac{T}{T_e}\right) d\gamma \]
Now,
\[
\frac{T}{T_0} = 1 - \frac{\gamma - 1}{2} \omega^2,
\]
where
\[
w = \frac{u}{a_e^a}.
\]
The temperature ratio can be further written as
\[
\frac{T}{T_0} = 1 - \frac{\gamma - 1}{2} \omega^2 \left( \frac{u}{u_e} \right)^2,
\]
Then, since
\[
u = \frac{a_e^\alpha}{a_e^2} U,
\]
the temperature ratio can be further rewritten as
\[
\frac{T}{T_0} = 1 - \frac{\gamma - 1}{2} \omega^2 \Omega^2.
\]
The boundary layer thickness then becomes
\[
\delta = \frac{a_e^\alpha e^\alpha T^a}{a_e^2 \rho_e T_e} \left[ \delta_i - \frac{\gamma - 1}{2} \omega^2 \int \delta_i \Omega^2 d\gamma \right].
\]
It is convenient now to introduce the incompressible momentum thickness,
\[
\Theta_i = \int_0^{\delta_i} \Omega (1 - \Omega) d\gamma,
\]
which is
\[
\Theta_i = \delta_i - \delta_i^* - \int_0^{\delta_i} \Omega^2 d\gamma,
\]
and the integral can be written as
\[
\int_0^{\delta_i} \Omega^2 d\gamma = \delta_i - \delta_i^* - \Theta_i.
\]
Therefore
\[
\delta = \frac{a_e^\alpha e^\alpha T^a}{a_e^2 \rho_e T_e} \left[ \delta_i - \frac{\gamma - 1}{2} \omega^2 \left( \delta_i - \delta_i^* - \Theta_i \right) \right].
\]
Reverting back to Mach number as a variable, with the relation
\[
\omega^2 = \frac{T_e}{T_0} M_e^2,
\]
the displacement thickness is given by the relation

\[ \delta^* = \frac{a_e^0 P_0^0}{a_e^0 \rho_e^0} \delta_i - \frac{\gamma - 1}{2} M_e^2 \frac{a_e^0 P_0^0}{a_e^0 \rho_e^0} \left( \delta_i - \delta_i^* - \Theta_i \right) \]

\[ - \frac{a_e^0 P_0^0}{a_e^0 \rho_e^0} \left( \delta_i - \delta_i^* \right) \]

From perfect gas relations

\[ \frac{a_e^0 P_0^0}{a_e^0 \rho_e^0} = \frac{P_0^0}{\rho_e^0} \sqrt{\frac{T_e}{T_0^0}} \]

For the external stream

\[ \frac{P_0^0}{\rho_e^0} = \left( 1 + \frac{\gamma - 1}{2} M_e^2 \right)^{\gamma/\gamma - 1} \]

and since

\[ \frac{T_0^0}{T_e} = 1 + \frac{\gamma - 1}{2} M_e^2 \]

the relation becomes

\[ \frac{a_e^0 P_0^0}{a_e^0 \rho_e^0} = \left( 1 + \frac{\gamma - 1}{2} M_e^2 \right)^{\frac{\gamma + 1}{\gamma - 1}} \]

Therefore the displacement thickness becomes

\[ \delta^* = \left( 1 + \frac{\gamma - 1}{2} M_e^2 \right)^{\frac{\gamma + 1}{\gamma - 1}} \delta_i - \frac{\gamma - 1}{2} M_e^2 \left( 1 + \frac{\gamma - 1}{2} M_e^2 \right)^{\frac{\gamma + 1}{\gamma - 1}} \left( \delta_i - \delta_i^* \right) \]

\[ - \Theta_i \left( 1 + \frac{\gamma - 1}{2} M_e^2 \right)^{\frac{\gamma + 1}{\gamma - 1}} \left( \delta_i - \delta_i^* \right) \]

Also since \( \delta^* \) is independent of \( Y \),

\[ \frac{d \delta^*}{d x} = \lambda \frac{a_e^0 P_0^0}{\rho_e^0} \frac{d \delta^*}{d x} \]

so that

\[ \frac{d \delta^*}{d x} = \lambda \frac{a_e^0 P_0^0}{\rho_e^0} \frac{d}{d x} \left\{ \left( 1 + \frac{\gamma - 1}{2} M_e^2 \right)^{\frac{\gamma + 1}{\gamma - 1}} \left[ \left( 1 + \frac{\gamma - 1}{2} M_e^2 \right)^{\frac{\gamma + 1}{\gamma - 1}} \delta_i^* \right. \right. \]

\[ + \left. \left. \frac{\gamma - 1}{2} M_e^2 \Theta_i \right] \right\} \]
Carrying out the indicated differentiation

\[
\frac{d\delta_1^*}{dx} = \lambda \frac{a_s}{a^*} \frac{P_e}{p^*} \left\{ \frac{\gamma + 1}{2} (1 + \frac{\gamma - 1}{2} M_e^2)^{1 - \frac{\gamma}{2(\gamma - 1)}} M_e \left[ (1 - \frac{\gamma - 1}{2} M_e^2) \delta_1^* 
+ \frac{\gamma - 1}{2} M_e^2 \Theta_i \right] \frac{dM_e}{dX} + (1 - \frac{\gamma - 1}{2} M_e^2)^{\frac{\gamma - 1}{2(\gamma - 1)}} \left[ (1
+ \frac{\gamma - 1}{2} M_e^2) \frac{d\delta_i^*}{dx} + \delta_i^* (\gamma - 1) M_e \frac{dM_e}{dX}
+ (\gamma - 1) \Theta_i M_e \frac{dM_e}{dX} + \frac{\gamma - 1}{2} M_e^2 \frac{d\Theta_i}{dx} \right] \right\}. 
\]

Noting that for a perfect gas

\[
\frac{a_s}{a^*} \frac{P_e}{p^*} = (1 + \frac{\gamma - 1}{2} M_e^2)^{-\frac{3 - \gamma}{2(\gamma - 1)}}
\]

and collecting terms leads to

\[
\frac{d\delta_1^*}{dx} = \frac{\lambda M_e}{1 + \frac{\gamma - 1}{2} M_e^2} \left[ \frac{\gamma + 1}{2} \delta_1^* + \frac{(\gamma - 1)(\gamma + 1) M_e^2}{1 + \frac{\gamma - 1}{2} M_e^2} \Theta_i \right]
\]

\[
+ (\delta_1^* + \Theta_i) (\gamma - 1) \frac{dM_e}{dX} + \lambda \frac{d\delta_i^*}{dx}
\]

\[
+ \lambda \frac{\frac{\gamma - 1}{2} M_e^2}{1 + \frac{\gamma - 1}{2} M_e^2} \frac{d\Theta_i}{dx}.
\]
Note that \( \frac{d \delta_i}{dx} \) does not depend explicitly on \( \frac{d \xi}{dx} \). The expression can be finally written as

\[
\frac{d \delta_i}{dx} = \frac{\lambda \delta_i \, M_e}{1 + \frac{\epsilon-1}{2} \, M_e^2} \left\{ (\gamma - 1) \left( \frac{\varphi_i}{\delta_i} \right) \left( \frac{1 + \frac{\epsilon-1}{2} \, M_e^2}{1 + \frac{\epsilon-1}{2} \, M_e^2} \right) \right. \\
\left. + \frac{3 \gamma - 1}{2} \, \frac{\delta_i}{\xi_i} \right\} \frac{d M_e}{dx} \\
+ \lambda \, \frac{d \delta_i}{dx} + \lambda \, \frac{\epsilon-1}{2} \, M_e^2 \frac{d \varphi_i}{dx}.
\]

Now the last term in the mass balance can be considered in two parts. First note that

\[
-\frac{d}{dx} \left( \ln \rho_e u_e \right) = \rho_e u_e \frac{d}{dx} \left( \frac{1}{\rho_e u_e} \right),
\]

which upon transforming the derivative becomes

\[
-\frac{d}{dx} \left( \ln \rho_e u_e \right) = \rho_e u_e \, \lambda \, \frac{a_x}{a_0} \, \frac{P_e}{P_0} \frac{d}{dx} \left( \frac{1}{\rho_e u_e} \right),
\]

which, for a perfect gas, becomes

\[
-\frac{d}{dx} \left( \ln \rho_e u_e \right) = \lambda \, M_e \left( \frac{P_e}{P_0} \right)^{\gamma} \frac{d}{dx} \left( \frac{1}{\sqrt{T_e}} \right) \frac{1}{M_e}.
\]

In terms of Mach number

\[
-\frac{d}{dx} \left( \ln \rho_e u_e \right) = \lambda \, M_e \left( 1 + \frac{\epsilon-1}{2} \, M_e^2 \right)^{\frac{\gamma-1}{\gamma+1}} \frac{d}{dx} \left[ \frac{(1 + \frac{\epsilon-1}{2} \, M_e^2)^{\frac{\gamma+1}{\gamma-1}}}{M_e} \right].
\]
Carrying out the differentiation leads to
\[- \frac{d}{dx} (\ln \rho_x u_x) = \lambda \frac{\gamma + 1}{\gamma} \left(1 + \frac{\rho_x}{\rho_e} \left(\frac{M_e}{M_x}\right)^{\frac{\gamma - 1}{\gamma}}\right) \frac{dM_x}{dx} + \lambda \frac{\gamma + 1}{\gamma} \frac{M_e}{M_x} \frac{dM_x}{dx} .\]

The remainder of the last term is
\[\int \frac{\delta \rho u}{\rho_x u_x} \, dx = \frac{a_\infty}{\rho_x} \int_0^\infty \delta u \, dy ,\]
which can be rewritten as
\[\int \frac{\delta \rho u}{\rho_x u_x} \, dx = (1 + \frac{\rho_x}{\rho_e} \left(\frac{M_e}{M_x}\right)^{\frac{\gamma + 1}{\gamma}}) \left(\delta u - \delta u^*\right) .\]

The last term in the mass balance can now be written as
\[- \frac{d}{dx} (\ln \rho_x u_x) \int \frac{\delta \rho u}{\rho_x u_x} \, dy = \lambda \left[ \frac{\gamma + 1}{\gamma} \left(\delta u - \delta u^*\right) \left(1 + \frac{\rho_x}{\rho_e} \left(\frac{M_e}{M_x}\right)^{\frac{\gamma - 1}{\gamma}}\right) \right] \frac{dM_x}{dx} .\]

The mass balance may now be expressed as
\[\frac{\lambda \delta u \left(\frac{M_e}{M_x}\right)}{1 + \frac{\rho_x}{\rho_e} \left(\frac{M_e}{M_x}\right)^{\frac{\gamma - 1}{\gamma}}} \frac{dM_x}{dx} = \frac{\delta u}{\delta_i} \frac{dM_x}{dx} \left(1 - \frac{\delta u}{\delta_i}\right) \left(1 + \frac{\rho_x}{\rho_e} \left(\frac{M_e}{M_x}\right)^{\frac{\gamma - 1}{\gamma}}\right) + \lambda \frac{\gamma + 1}{\gamma} \frac{M_e}{M_x} \frac{dM_x}{dx} + \lambda \frac{d\delta u}{dx} .\]

\[+ \lambda \frac{\gamma + 1}{\gamma} \frac{M_e}{M_x} \frac{d\theta_i}{dx} = \Phi - \frac{\lambda \rho_i}{1 + \frac{\gamma - 1}{\gamma} \frac{M_e}{M_x}} .\]
which can be written more conveniently as

\[
\frac{\lambda \delta \frac{M_e}{1 + \frac{x^2 - 1}{2} M_e^2}}{\frac{\lambda \delta \frac{M_e}{1 + \frac{x^2 - 1}{2} M_e^2}}{2}} + \left( \gamma - 1 \right) \left( \frac{\Theta_\delta}{\delta} \right) \left[ \frac{1 + \frac{x^2 - 1}{2} M_e^2}{1 + \frac{x^2 - 1}{2} M_e^2} \right]
\]

\[
= \left[ 1 + \left( 3 + \frac{3x^2 - 1}{2} M_e^2 \right) \left( \frac{x^2 - 1}{2} M_e^2 \right) \right] \left( 1 - \frac{\delta}{\delta} \right) \frac{dM_e}{dX}
\]

\[
+ \lambda \left[ \frac{\delta}{\delta} + \frac{x^2 - 1}{2} M_e^2 \left( \frac{\Theta_\delta}{\delta} \right) \right] \frac{d\delta}{dX} + \lambda \delta \frac{d}{dX} \left( \frac{\delta}{\delta} \right)
\]

\[
+ \lambda \delta \frac{\frac{x^2 - 1}{2} M_e^2}{1 + \frac{x^2 - 1}{2} M_e^2} \frac{d}{dX} \left( \frac{\Theta_\delta}{\delta} \right) = \frac{\lambda \Theta_\delta}{1 + \frac{x^2 - 1}{2} M_e^2}
\]

This last equation represents the mass balance in terms of variables defined in the incompressible plane, the free-stream Mach number, and the turning angle induced by the boundary layer growth.
The coefficients in the differential equations describing the separated flow model are evaluated by means of the velocity profile-dependent functions. The resulting coefficients are:

\[
D_{11} = D_{21} + \frac{1}{4} \left\{ \frac{(1 - D_5)(1 - \gamma_s)}{1 + 2 \gamma_s - 3 \gamma_s^2} \left[ \frac{3}{5} + \frac{1}{7} \frac{(1 - D_5)(1 + \gamma_s - 2 \gamma_s)}{1 + \gamma_s - 3 \gamma_s^2} \right] 
- \frac{1}{7} \frac{(1 - D_5)(1 - \gamma_s)(\gamma + 14 \gamma_s)}{1 + 2 \gamma_s - 3 \gamma_s^2} + \frac{2}{7} \frac{(1 - D_5)(1 - \gamma_s)(1 - 3 \gamma_s)(2 + 7 \gamma_s + 7 \gamma_s^2)}{1 + \gamma_s - 3 \gamma_s^2} \right\} 
+ \frac{(1 - \gamma_s)(1 - D_5)}{1 + 2 \gamma_s - 3 \gamma_s^2} \left[ -2 - \frac{(1 - \gamma_s)(1 - 2 \gamma_s)}{1 + 2 \gamma_s - 3 \gamma_s^2} \right] \left[ \frac{1}{5} (2 + 3 \gamma_s) 
- \frac{1}{7} \frac{(1 - D_5)(1 - \gamma_s)(2 + 7 \gamma_s + 7 \gamma_s^2)}{1 + 2 \gamma_s - 3 \gamma_s^2} \right] \right\} 
- \frac{1}{7} \frac{D_5 (1 - \gamma_s)(1 - 2 \gamma_s)(1 + 2 \gamma_s - 3 \gamma_s^2) + 4 \frac{D_5}{\gamma_s} (1 - \gamma_s)(1 + 2 \gamma_s - 3 \gamma_s^2) + 4 \frac{D_5}{\gamma_s^2} (1 - \gamma_s)(1 + 2 \gamma_s - 3 \gamma_s^2)}{2 \gamma_s^2 (5 - 3 \gamma_s) - 3 (1 - \gamma_s)(1 + 2 \gamma_s - 3 \gamma_s^2)} 
- \frac{D_5 (3 - 10 \gamma_s + 9 \gamma_s^2)}{2 \gamma_s^2 (5 - 3 \gamma_s) + 3 (1 - \gamma_s)(1 + 2 \gamma_s - 3 \gamma_s^2)} \left[ \frac{(1 - \gamma_s)^2}{1 + 2 \gamma_s - 3 \gamma_s^2} \right] \left[ \frac{1}{5} (2 + 3 \gamma_s) 
- \frac{2}{7} \frac{(1 - D_5)(1 - \gamma_s)(2 + 7 \gamma_s + 7 \gamma_s^2)}{1 + 2 \gamma_s - 3 \gamma_s^2} \right] \right\} 
\]
\[ D_{21} = \frac{3}{35} \left\{ -\frac{6}{35} \, \hat{D}_{s}^{2} - \frac{(1 - \hat{D}_{s})}{(1 - \gamma_{s}) (1 + 2 \gamma_{s} - 3 \gamma_{s}^{2})} \left[ 3 \gamma_{s}^{2} - 12 \gamma_{s}^{3} \right. \right. \]

\[ - \frac{\gamma_{s}^{3} (1 - 3 \gamma_{s}) (1 - 10 \gamma_{s} + 9 \gamma_{s}^{2})}{(1 - \gamma_{s}) (1 + 2 \gamma_{s} - 3 \gamma_{s}^{2})} \left[ \frac{3}{35} \, \hat{D}_{s}^{2} - \frac{2}{105} \, \right. \]

\[ - \frac{\gamma_{s}^{2} (1 - 3 \gamma_{s})}{(1 - \gamma_{s}) (1 + 2 \gamma_{s} - 3 \gamma_{s}^{2})} \right] - \frac{2}{105} \frac{\gamma_{s}^{3} (1 - 3 \gamma_{s}) (1 - \gamma_{s})}{(1 - \gamma_{s})^{2} (1 + 2 \gamma_{s} - 3 \gamma_{s}^{2})} \left[ 2 \gamma_{s} \right. \]

\[ - \frac{- \gamma_{s}^{2} - \frac{\gamma_{s}^{2} (1 - 3 \gamma_{s}) (1 - 10 \gamma_{s} + 9 \gamma_{s}^{2})}{(1 - \gamma_{s}) (1 + 2 \gamma_{s} - 3 \gamma_{s}^{2})}}{\left[ \frac{12 \hat{B}_{s}}{105} \right]^{2}} \]

\[ - \frac{4}{105} \frac{\frac{8 (1 - \gamma_{s})}{(1 - \gamma_{s}) (1 + 2 \gamma_{s} - 3 \gamma_{s}^{2})}}{2 \gamma_{s} - \gamma_{s}^{2} - \frac{\gamma_{s}^{2} (1 - 3 \gamma_{s}) (1 - 10 \gamma_{s} + 9 \gamma_{s}^{2})}{(1 - \gamma_{s}) (1 + 2 \gamma_{s} - 3 \gamma_{s}^{2})}} \]

\[ + \left[ \frac{2 \gamma_{s} (10 - 9 \gamma_{s}) - 4 \frac{\hat{B}_{s}}{105} (1 - \gamma_{s}) (1 + 2 \gamma_{s} - 3 \gamma_{s}^{2}) + 4 \frac{\hat{B}_{s}}{105} (1 - 10 \gamma_{s} + 9 \gamma_{s}^{2})}{2 \gamma_{s}^{2} (1 - 3 \gamma_{s})} \right. \]

\[ + 3 \left( 1 - \gamma_{s} \right) (1 + 2 \gamma_{s} - 3 \gamma_{s}^{2}) \]

\[ - \frac{\hat{D}_{s} (3 - 10 \gamma_{s} + 9 \gamma_{s}^{2})}{2 \gamma_{s}^{2} (1 - 3 \gamma_{s}) + 3 (1 - \gamma_{s}) (1 + 2 \gamma_{s} - 3 \gamma_{s}^{2})} \left[ - \frac{12 \, \hat{B}_{s}}{35} \, \hat{D}_{s} \right. \]

\[ - \frac{\gamma_{s}^{3} (1 - 3 \gamma_{s})}{(1 - \gamma_{s}) (1 + 2 \gamma_{s} - 3 \gamma_{s}^{2})} \left[ \frac{3}{35} \, \hat{D}_{s}^{2} - \frac{2}{105} \, \frac{\gamma_{s}^{2} (1 - 3 \gamma_{s})}{(1 - \gamma_{s}) (1 + 2 \gamma_{s} - 3 \gamma_{s}^{2})} \right. \]

\[ + \frac{\gamma_{s}^{3} (1 - 3 \gamma_{s})}{(1 - \gamma_{s}) (1 + 2 \gamma_{s} - 3 \gamma_{s}^{2})} \left[ \frac{3}{35} \, \hat{D}_{s}^{2} + \frac{2}{105} \, \frac{\gamma_{s}^{2} (1 - 3 \gamma_{s})}{(1 - \gamma_{s}) (1 + 2 \gamma_{s} - 3 \gamma_{s}^{2})} \right] \]

\[ - \frac{4 \, \frac{\hat{B}_{s} \gamma_{s}^{2} (1 - 3 \gamma_{s})}{(1 - \gamma_{s}) (1 + 2 \gamma_{s} - 3 \gamma_{s}^{2})} \right\} \]
\[ D_{31} = \lambda \left( \frac{\bar{D}_{11}}{1 - \frac{1}{4} \frac{M_e^2}{M_0^2}} \right) + 3 \eta_s \left( \frac{2 \eta_s (10 - \eta_5) - 4 \frac{B}{15}(1 - \eta_5)(1 + 2 \eta_5 - 3 \eta_5^2) + 4 \frac{B}{\eta_s} (1 - 10 \eta_5 + 9 \eta_5^2)}{2 \eta_s^2 (5 - 3 \eta_5) - \frac{3}{3} (1 - \eta_5)(1 + 2 \eta_5 - 3 \eta_5^2)} \right) \]
\[ D_{22} = \left( \frac{\Theta_{\delta}}{\delta} \right) - B \left\{ 1 - \frac{2\xi_s}{705} \frac{B}{\gamma_s} - \frac{4\xi_s}{35} \frac{D_s}{\gamma_s} - \frac{4\xi_s}{705} \frac{(1-\eta_s) \gamma_s^3 (1-3\gamma_s)}{(1-\eta_s)(1+2\gamma_s-3\gamma_s^2)} \right\} \]
\[ + \frac{4}{705} \left\{ \frac{(1-\eta_s)^3 (1+2\gamma_s-3\gamma_s^2)}{2\gamma_s^2 (5-3\gamma_s) + 3(1-\gamma_s)(1+2\gamma_s-3\gamma_s^2)} \right\} \left[ - \frac{12}{95} \frac{\gamma_s^3 (1-3\gamma_s)}{(1-\eta_s)(1+2\gamma_s-3\gamma_s^2)} \left( \frac{2}{35} \frac{D_s}{\gamma_s} \right) \right] \]
\[ - \frac{2}{705} \left( 1 - \frac{\gamma_s^2 (1-3\gamma_s)}{(1-\gamma_s)(1+2\gamma_s-3\gamma_s^2)} \right) + \left( 1 - \frac{\gamma_s^2}{(1-\gamma_s)(1+2\gamma_s-3\gamma_s^2)} \right) \left( \frac{\frac{3}{85}}{B} \right) \]
\[ + \frac{2}{705} \frac{\gamma_s^2 (1-3\gamma_s)}{(1-\gamma_s)(1+2\gamma_s-3\gamma_s^2)} \right\} + \frac{4\xi_s}{705} \frac{D_s}{\gamma_s} \left( \frac{1-\gamma_s}{(1-\gamma_s)(1+2\gamma_s-3\gamma_s^2)} \right) \left\} \right\} \]
\[ D_{32} = \lambda \left\{ \left( \frac{\delta_i}{\delta_i} \right) + \frac{\chi - 1}{\gamma^3} M_0^2 \right\} D_{12} + B + \left( \frac{4\xi_s}{1-\gamma_s} \frac{D_s}{\gamma_s} \left( \frac{1-\gamma_s}{(1-\gamma_s)(1+2\gamma_s-3\gamma_s^2)} \right) \right\} \]
\[ \left\} \right\} \]
\[ D_{13} = \frac{\delta_i}{M_0} \left[ \left( \frac{\delta_i}{\delta_i} \right) + \left( \frac{\Theta_{\delta_i}}{\delta_i} \right) + D_{12} \right] \]
\[ D_{23} = \frac{\delta_i}{M_0} \left[ \left( \frac{\delta_i}{\delta_i} \right) + \left( \frac{\Theta_{\delta_i}}{\delta_i} \right) + D_{22} \right] \]
\[ D_{33} = \lambda \frac{\delta_{i}}{M_0} \left\{ \frac{\chi - 1}{\gamma^3} M_0^2 \left\{ \frac{3\chi - 1}{2} + (\gamma - 1) \left( \frac{1 - \frac{\chi - 1}{\chi + 1} M_0^2}{1 + \frac{\chi - 1}{\gamma^3} M_0^2} \right) \right\} \right\} \left( \frac{\Theta_{\delta_{i}}}{\delta_{i}} \right) \]
\[ - \frac{1 + (3 + \frac{3\chi - 1}{\chi + 1} M_0^2 \left( \frac{\chi - 1}{\gamma^3} M_0^2 \right)}{(1 + \frac{\chi - 1}{\gamma^3} M_0^2) M_0^2} \right\} \left\} \right\} \left( \frac{\Theta_{\delta_{i}}}{\delta_{i}} \right) \]
\[ - \frac{\delta_{i}}{M_0} \left[ \left( \frac{\delta_{i}}{\delta_{i}} \right) + \frac{\chi - 1}{\gamma^3} M_0^2 \left( \frac{\Theta_{\delta_{i}}}{\delta_{i}} \right) \right] \]
For \( \rho_d \neq 0 \),

\[
D_{14} = \frac{1}{R_0 \xi_d M_e} \left[ -3 \frac{\xi_d}{\gamma_s} + 8 \frac{B}{\gamma_s^2} + 2 \left( 1 - \alpha_s \right) \frac{\gamma_s (1 - 3 \gamma_s)}{(1 - \gamma_s)(1 + 2 \gamma_s - 3 \gamma_s^2)} \right]
\]

\[
+ \rho_{d'} - \frac{\rho_{d'}}{B} \left( \frac{\xi_{d'}}{\xi_d} \right) + \frac{\rho_{d'}}{B} D_{12}
\]

\[
D_{24} = \frac{1}{R_0 \xi_d M_e} \left[ -6 \frac{D_s}{\gamma_s} + 12 \frac{B}{\gamma_s^2} \right] + \rho_{d'} - \frac{\rho_{d'}}{B} \left( \frac{\xi_{d'}}{\xi_d} \right) + \frac{\rho_{d'}}{B} D_{22}
\]

\[
D_{34} = \Phi - \frac{\rho_{d'}}{1 + \frac{D_s}{\gamma_s} M_e^2} - \frac{\rho_{d'}}{B} \left[ \left( \frac{\xi_{d'}}{\xi_d} \right) + \frac{\gamma_s - 1}{1 - \gamma_s - 1} \frac{M_e^2}{\gamma_s} \left( \frac{\xi_{d'}}{\xi_d} \right) \right]
\]

\[
+ \frac{\rho_{d'}}{B} D_{32}
\]

For \( \rho_d = 0 \),

\[
D_{14} = \frac{1}{R_0 \xi_d M_e} \left[ -3 \frac{D_s}{\gamma_s} + 8 \frac{B}{\gamma_s^2} + 2 \left( 1 - \alpha_s \right) \frac{\gamma_s (1 - 3 \gamma_s)}{(1 - \gamma_s)(1 + 2 \gamma_s - 3 \gamma_s^2)} \right]
\]

\[
D_{24} = \frac{1}{R_0 \xi_d M_e} \left[ -6 \frac{D_s}{\gamma_s} + 12 \frac{B}{\gamma_s^2} \right]
\]

\[
D_{34} = \Phi
\]
COEFFICIENTS IN SEPARATED FLOW DIFFERENTIAL EQUATIONS, \( a_1 = -2 \frac{1+3 \gamma_\ell}{(1-\gamma_\ell)} \left( \frac{Q_f}{\gamma_\ell} \right) \)

The coefficients in the differential equations describing the separated flow model are evaluated by means of the velocity profile-dependent functions for \( a_1 = -2 \frac{1+3 \gamma_\ell}{(1-\gamma_\ell)^3} \left( \frac{Q_f}{\gamma_\ell} \right) \). The resulting coefficients are:

\[
\begin{align*}
D_{11} &= C_{12} \frac{\delta_i}{\delta_i} \\
D_{12} &= \frac{\Theta_{ii}}{\delta_i} - B C_{13} \\
D_{13} &= \left[ 2 \left( \frac{\Theta_{ii}}{\delta_i} \right) + \frac{\delta_{ii}^x}{\delta_i} - B C_{13} \right] \frac{\delta_i}{M_i} \\
D_{14} &= \frac{1}{R_0 \delta_i M_i} \left[ -2 \left( \frac{\Theta_{ii}}{\gamma_\ell} \right) + 12 \frac{B}{\gamma_\ell^3} \left( \frac{3-\gamma_\ell}{3-3\gamma_\ell} \right) \right] + \rho_i (1 - C_{13}) \\
D_{21} &= C \gamma \frac{\delta_i}{\delta_i} \\
D_{22} &= \frac{\Theta_{ii}}{\delta_i} - B C_8 \\
D_{23} &= \left[ 2 \left( \frac{\Theta_{ii}}{\delta_i} \right) + \frac{\delta_{ii}^x}{\delta_i} - B C_8 \right] \frac{\delta_i}{M_i} \\
D_{24} &= \frac{1}{R_0 \delta_i M_i} \left[ -6 \left( \frac{\Theta_{ii}}{\gamma_\ell} \right) + \frac{q_6 B}{\gamma_\ell^3 (3-3\gamma_\ell^3)} \right] + \rho_i (1 - C_8) \\
D_{31} &= (C_5 C_{16} + C_{12} C_{17}) \frac{\delta_i}{\delta_i} \\
D_{32} &= \left[ C_{15} + B (C_6 C_{16} - C_{13} C_{17}) \right] \\
D_{33} &= \left[ C_{14} - \frac{\delta_{ii}^x}{M_i} (C_6 C_{16} - C_{13} C_{17}) \right] \\
D_{34} &= C_{18} + \rho_i (C_6 C_{16} - C_{13} C_{17}) 
\end{align*}
\]
where

\[
C_1 = \frac{2\gamma_5 (10 - 9\gamma_5) - \Omega (7 - 2\gamma_5 + 9\gamma_5^2) + 4 \left( \frac{B}{\gamma_5} \right) (1 - 10\gamma_5 + 9\gamma_5^2) - 4 \left( \frac{B}{\gamma_5} \right) (1 - \gamma_5) (1 + 3\gamma_5)}{2 \gamma_5^2 (5 - 3\gamma_5) + (1 - \gamma_5) (1 + 3\gamma_5) (5 - 3\gamma_5)}
\]

\[
C_2 = \frac{4}{\gamma_5} \left[ \frac{(1 - \gamma_5)^2 (1 + 3\gamma_5)}{2 \gamma_5^2 (5 - 3\gamma_5) + (1 - \gamma_5) (1 + 3\gamma_5) (5 - 3\gamma_5)} \right]
\]

\[
C_3 = \frac{2 (5 - 6\gamma_5) - \left( \frac{\Omega}{\gamma_5} \right) (7 - 2\gamma_5 + 9\gamma_5^2) + 4 \left( \frac{B}{\gamma_5} \right) (1 - 10\gamma_5 + 9\gamma_5^2) - 8 \left( \frac{B}{\gamma_5} \right) (1 - \gamma_5) (1 + 3\gamma_5)}{2 \gamma_5^2 (5 - 3\gamma_5) + (1 - \gamma_5) (1 + 3\gamma_5) (5 - 3\gamma_5)}
\]

\[
C_4 = \frac{C_2}{\gamma_5}
\]

\[
C_5 = 1 - \frac{1}{5 (1 + 3\gamma_5)} \left[ (1 - \gamma_5) (1 + 3\gamma_5) C_1 - (1 - \Omega) (1 - 6\gamma_5) \right]
+ \frac{3 (1 - \gamma_5) (2 + 3\gamma_5) (1 - \Omega)}{1 + 3\gamma_5} - \frac{1}{5} (1 - \gamma_5) \left( \frac{\Omega}{\gamma_5} \right) + \frac{1}{10} (1 - \gamma_5)^2 C_3
\]

\[
C_6 = 1 + \frac{(1 - \gamma_5) (2 + 3\gamma_5)}{5 (1 + 3\gamma_5)} C_2 - \frac{1}{10} (1 - \gamma_5)^2 C_4
\]

\[
C_7 = -\frac{2}{15} \Omega^2 - \frac{4}{15} \gamma_5 \Omega C_1 + \frac{2 B}{5 (5 - 3\gamma_5)} \left[ (3 - \gamma_5) C_1 + \frac{\Omega (\gamma - 6\gamma_5 + 3\gamma_5^2)}{5 - 3\gamma_5} \right]
+ \frac{44 (11 - 13\gamma_5 + 4\gamma_5^2)}{21 (5 - 3\gamma_5)} \left( \frac{B}{\gamma_5^2} \right) - \frac{44 (1 + \gamma_5)}{21 (5 - 3\gamma_5)^2} \left( \frac{B}{\gamma_5} \right)
\]

\[
C_8 = 1 - \frac{4}{15} \gamma_5 \Omega C_2 + \frac{2 (3 - \gamma_5) (\Omega + C_2 B)}{5 (5 - 3\gamma_5)}
- \frac{576 (11 - 13\gamma_5 + 4\gamma_5^2)}{105 (5 - 3\gamma_5)^2} \left( \frac{B}{\gamma_5} \right)
\]
$$C_9 = \left( 1 - \frac{\sigma_3}{\gamma} \right) \left[ \frac{1 - 6 \gamma_5}{5} - \frac{3 (1 - \gamma_5)(2 + 3 \gamma_5)}{5 (1 + 3 \gamma_5)} - \frac{(1 - 3 \gamma_5)(5 - 2 \gamma_5)}{7 (1 + 3 \gamma_5)} \right] + \frac{\sigma_3}{7} \frac{(1 - 3 \gamma_5)(1 - \gamma_5)(2 + 7 \gamma_5 + 7 \gamma_5^2)}{(1 + 3 \gamma_5)^2} - (1 - \gamma_5) \left( \frac{\sigma_3}{\gamma} \right) \left[ \frac{1}{5} \right] - \frac{(1 - 3 \gamma_5)(1 + 7 \gamma_5^2)}{14 (1 + 3 \gamma_5)} - \frac{(1 - 3 \gamma_5)(1 - \gamma_5)(5 + 7 \gamma_5)}{14 (1 + 3 \gamma_5)^2} - \frac{(1 - \gamma_5)^3}{14} \left( \frac{\sigma_3}{\gamma} \right) \left[ \frac{1}{5} \right]$$

$$C_{10} = - (1 - \gamma_5) \left[ \frac{2 + 3 \gamma_5}{5} - \frac{2}{7} \frac{(1 - \gamma_5)(2 + 7 \gamma_5 + 7 \gamma_5^2)}{1 + 3 \gamma_5} \right] - \frac{(1 - \gamma_5)(5 + 7 \gamma_5)}{42} \left( \frac{\sigma_3}{\gamma} \right) \left[ \frac{1}{5} \right]$$

$$C_{11} = (1 - \gamma_5) \left[ \frac{1}{10} - \frac{(1 - \gamma_5)(5 + 7 \gamma_5)}{42 (1 + 3 \gamma_5)} - \frac{2}{63} (1 - \gamma_5) \frac{\sigma_3}{\gamma_5} \right]$$

$$C_{12} = C_7 + C_9 + C_1, C_{10} + C_3, C_{11}$$

$$C_{13} = C_8 + C_2, C_{10} + C_4, C_{11}$$

$$C_{14} = \frac{\lambda \bar{\delta}_4 \bar{M}_e}{1 - \bar{v}^2 \bar{M}_e^2} \left\{ \frac{3 \bar{v} - 1}{2} \left( \frac{1 + \frac{3 \bar{v} - 1}{2} \bar{M}_e^2}{1 - \bar{v}^2 \bar{M}_e^2} \right) \left( \frac{\Theta_i}{\delta_i} \right) \right\}$$

$$- \left[ 1 + \frac{3 + \frac{3 \bar{v} - 1}{2} \bar{M}_e^2}{(1 - \bar{v}^2 \bar{M}_e^2) \bar{M}_e^2} \right] \left( 1 - \frac{\delta_4^+}{\delta_i} \right)$$

$$C_{15} = \lambda \left[ \frac{\delta_4^+}{\delta_i} + \frac{\bar{v} - 1}{1 - \bar{v}^2 \bar{M}_e^2} \left( \frac{\Theta_i}{\delta_i} \right) \right]$$
\[ C_{16} = \lambda \]

\[ C_{17} = \lambda \frac{x^{-1} \frac{M_0}{2}^2}{1 + x^{-1} \frac{M_0}{2}^2} \]

\[ C_{18} = \phi - \frac{\lambda \rho_i}{1 + x^{-1} \frac{M_0}{2}^2} \]
APPENDIX E
COMPUTER PROGRAM FOR BOUNDARY LAYER CALCULATIONS

The attached and separated flow relations were programmed for computation on the IBM 7094 computer in The Ohio State University Computation Center. The program is written in the SCATRAN language available at the OSU Computation Center.⁴ All together, four separate programs were written. These include two programs based on the separated flow relations of Equations (55) to (57) with auxiliary relations given by Equations (28), (58) and the $\beta_c$ specification. One separated flow program uses the flow coefficients listed in Appendix C ($a_1 \equiv 0$) and the other uses the flow coefficients listed in Appendix D ($a_1 = -2 \frac{i-3 \gamma_s}{(1-\gamma_s)^3} \left( \frac{\sigma_s}{\gamma_s} \right)$). The third and fourth programs combine the Tani and Pohlhausen attached flow relations with the separated flow relations using the flow coefficients of Appendix D.

All of the integrations are carried out by a fourth-order Runge-Kutta integration method. A fixed step-size in the independent variable $X$ is used. During initial trials with the programs a step-size of 0.1 was chosen; i.e. 0.10 of the initial incompressible boundary layer thickness. Comparison of computation results for larger and smaller step-sizes indicated that adequate accuracy was achieved. However, the step-size calculation must be re-evaluated


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for each new set of flow conditions to insure adequate computation accuracy.

Selection of a minimum step-size is dictated primarily by machine time considerations for each case calculated. The IBM 7094 can accomplish about 750 integration steps per minute. The attached flow portion runs somewhat faster. Since the flow is usually separated for most of the interaction length, the separated flow relations determine the machine time for the combined attached/separated flow program.

The remainder of this section is devoted to a more detailed description, including the SCATRAN program listing, of the combined flow program. The basic elements of the program are integration routines for attached and separated flow. The relation between these elements is schematically illustrated in Figure 32.

Input data includes a Reynolds number based on boundary layer thickness at the beginning of the interaction \( \Re_b \), free stream Mach number, shock strength, shock impingement point, and data required to evaluate \( \beta_c(\text{x}) \). Scale factors and reference distances are used to relate program variables to the physical dimension of the problem. The source listing and definition of variables describes these terms. The initial value of external Mach number is selected as a disturbance to the free stream Mach number to start the interaction. Integration begins at the beginning of the interaction and proceeds downstream if \( \text{\%} \geq 0.0 \). If \( \text{\%} < 0.0 \), then the integration is begun at the downstream end of the interaction and is carried on upstream.
Figure 32

Relation Between Basic Elements of the Combined Flow Program
The calculation begins with an attached flow and proceeds to the separation point (attachment if \( \theta_c < 0 \)) as indicated by \( c \leq 0 \). Equations (74) to (76) are used in the program at the separation point. Beyond separation, the general separated flow relations are used. The calculation proceeds until, either attachment as indicated by \( \eta_s \leq 0 \), or until the pressure gradient vanishes at the end of the interaction region. If attachment occurs, then the attached flow calculation is begun again (with \( c = 0 \) as an initial condition) and proceeds until the end of the interaction, as indicated by the pressure gradient vanishing.

The definitions of terms and the SCATRAN language listing of the combined flow program are given below along with the instruction differences which represent the Tani and Pohlhausen methods for the attached flow.

\[
\begin{align*}
B & = B \\
BETA & = \theta_i \\
BETAC & = \theta_c \\
BP(J) & = \text{estimates of } \frac{dB}{dX} \\
BPP & = \frac{dB}{dX} \\
C & = c \\
CA & = \text{coefficient in } \theta_c(x) \\
CB & = \text{coefficient in } \theta_c(x) \\
CC & = \text{coefficient in } \theta_c(x) \\
CD & = \text{coefficient in } \theta_c(x) \\
CF & = c_f
\end{align*}
\]
\[ C_{FPAS} = c_f \text{ from previous integration step} \]
\[ C_{FP} = \text{rate of change of } c_f, \quad \frac{c_f - C_{FPAS}}{\Delta x} \]
\[ C_{P(J)} = \text{estimates of } \frac{d c_f}{d x} \]
\[ C_{PP} = \frac{d c_f}{d x} \]
\[ C_1 \ldots C_{18} \text{ functions } C_1 \text{ to } C_{18} \text{ defined in Appendix D} \]
\[ D = \text{determinant} \quad \begin{vmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{vmatrix} \]
\[ \text{DELB} = \overline{\delta}_i \]
\[ \text{DELBP} = \frac{d \overline{\delta}_i}{d x} \]
\[ \text{DELBS} = \text{initial value of } \overline{\delta}_i \]
\[ \text{DELIN} = \frac{\overline{\delta}_i^*}{\delta_i} \]
\[ \text{DELO} = \frac{\overline{\delta}_i^*}{\delta_i} \]
\[ \text{DELRS} = \text{initial value of } \frac{\delta_i}{\delta_i} \]
\[ \text{DELT} = \frac{\delta}{\delta_0} \]
\[ \text{DRP} = \frac{d R_p}{d x} \]
\[ \text{DRX(J)} = \text{estimate of } \frac{d R_p}{d x} \]
\[ \text{DP(J)} = \text{estimate of } \frac{d \overline{\delta}_i}{d x} \]
\[ \text{DXB} = \text{step size in } x \]
\[ D_{11} \ldots D_{14} = \text{coefficients in Equation (55)}; D_{11}, D_{12}, D_{13}, D_{14} \]
\[ D_{21} \ldots D_{24} = \text{coefficients in Equation (56)}; D_{21}, D_{22}, D_{23}, D_{24} \]
\[ D_{31} \ldots D_{34} = \text{coefficients in Equation (57)}; D_{31}, D_{32}, D_{33}, D_{34} \]
\[ \text{EP(J)} = \text{estimate of } \frac{d \eta_s}{d x} \]
\[ \text{EPS} = \text{tolerance on } \frac{d \eta_s}{d x} \text{ at the end of the interaction} \]
\[ \text{ETAS} = \eta_s \]
ETASP = \frac{d\eta_s}{dx}

E11...E14 = coefficients in Equation (115), E_{11}, E_{12}, E_{13}, E_{14}

E21...E24 = coefficients in Equation (116), E_{21}, E_{22}, E_{23}, E_{24}

E31...E34 = coefficients in Equation (117), E_{31}, E_{32}, E_{33}, E_{34}

F1...F62 = repeated algebraic functions defined in program

GAM = \gamma

G1...G12 = repeated algebraic functions defined in program

H1...H3 = repeated algebraic functions defined in program

J = index controlling integration

K = number of slots

LAM = \lambda

ME = M_e

MEP = \frac{dM_e}{dx}

MES = initial value of M_e and value at separation and attachment

MO = M_o

MOD = M_{OD}

MOU = M_{OU}

MP(J) = estimate of \frac{dM_e}{dx}

N = index denoting each slot

N1 = determinant, \begin{vmatrix} D_{14} & D_{12} & D_{13} \\ D_{24} & D_{22} & D_{23} \\ D_{34} & D_{32} & D_{33} \end{vmatrix}

N2 = determinant, \begin{vmatrix} D_{11} & D_{14} & D_{13} \\ D_{21} & D_{24} & D_{23} \\ D_{31} & D_{34} & D_{33} \end{vmatrix}
\[ \text{N3} = \text{determinant,} \begin{vmatrix} D_{11} & D_{12} & D_{14} \\ D_{21} & D_{22} & D_{24} \\ D_{31} & D_{32} & D_{34} \end{vmatrix} \]

\[ \text{PHI} = \phi \]

\[ \text{RO} = R_0 \]

\[ \text{RX} = R_x \]

\[ \text{SCALE} = \text{conversion factor which multiplies} \ (x/\delta_o) \ \text{to obtain physical dimensions} \]

\[ \text{THETIN} = \Theta_{i,j} / \delta_i \]

\[ \text{THETO} = \Theta_i / \delta_i \]

\[ \text{TOT} = \theta_0 / \theta \]

\[ \text{US} = U_s \]

\[ \text{XB} = \chi \]

\[ \text{XM} = \text{distance from the start of integration to an intermediate slot position (XMID) measured in} \ (x/\delta_o) \]

\[ \text{XMID} = \text{distance from the leading edge of a slot to an intermediate position in the slot at which the injection rate may change} \]

\[ \text{XR} = x/\delta_o \]

\[ \text{XREF} = \text{distance from plate leading edge to initial position of integration} \]

\[ \text{XS} = \text{distance from start of integration to slot leading edge measured in} \ (x/\delta_o) \]

\[ \text{XSSCALE} = \text{distance from plate leading edge} \]

\[ \text{XSE} = \text{distance from the start of integration to the end of a slot measured in} \ (x/\delta_o) \]

\[ \text{XSH} = \text{distance from plate leading edge to the shock impingement point,} \ L \]

\[ \text{XSL\text{\textunderscore}ND} = \text{width of the slots} \]

\[ \text{XSLT(N)} = \text{distance from plate leading edge to the start of the Nth slot.} \]
*** RUN

*** DUMP LOWER CORE

*** SCATRAN

FLOATING(ME, MEP, MP(1), MP(2), MP(3), MP(4), MO, MOU, MOD, MES, LAM,
N1, N2, N3)


DIMENSION(XSLOT(20))

SEP

READINPUT7, (MOU, MOD, RO, GAM, LAM, DXB, EPS, SCALE, XMID, XSLEND,
CA, CB, CC, CD, MES, XSH, XREF, C)

WRITEOUTPUT2, (MOU, MOD, RO, GAM, LAM, DXB, EPS, SCALE, XMID, XSLEND,
CA, CB, CC, CD, MES, XSH, XREF, C)

READINPUT8, (K)

READINPUT7, (XSLOT(N), N=1, 1, N, LE, K)

WRITEOUTPUT3, (K)

WRITEOUTPUT2, (XSLOT(N), N=1, 1, N, LE, K)

N=1
TRANS

\[ X\text{SLOT}(N) = \frac{(X\text{SLOT}(N) - X\text{REF})}{\text{SCALE}} \]

\[ N = N + 1 \]

**PROVIDED(N \geq K), TRANSFER TO (TRANS)**

\[ N = 1 \]

\[ X\text{MID} = \frac{X\text{MID}}{\text{SCALE}} \]

\[ X\text{SLEND} = \frac{X\text{SLEND}}{\text{SCALE}} \]

\[ X\text{S} = X\text{SLOT}(1) \]

\[ X\text{M} = X\text{MID} + X\text{SLOT}(1) \]

\[ X\text{SE} = X\text{SLEND} + X\text{SLOT}(1) \]

\[ X\text{SH} = \frac{(X\text{SH} - X\text{REF})}{\text{SCALE}} \]

\[ \text{ETAS} = 0.0 \]

\[ \text{DELB} = 1.0 \]

\[ \text{ME} = \text{MES} \]

\[ \text{BETA} = 0.0 \]

\[ B = 0.0 \]

\[ J = 0 \]
ATTACH

SUCK

XM=0

CM=0

RX=0

CF=0

DXB=0

G1=(GAM-1.0)*G1/2.0

G2=(GAM-1.0)*G1

G3=G2/TOT

G4=((3.0*GAM)-1.0)/2.0

TOT=1.0+G2

G5=1.0+(G1*G4/2.0)

MO=MOD

TRANSFERTO(ATTACH)

MO=MO

TRANSFERTO(SUCK)

G1=(GAM-1.0)*G1

G2=(GAM-1.0)*G1

G3=G2/TOT

G4=((3.0*GAM)-1.0)/2.0

TOT=1.0+G2

G5=1.0+(G1*G4/2.0)

HI=MO

MO=MO

TRANSFERTO(ATTACH)

MO=MO

TRANSFERTO(SUCK)
\[ H_2 = (\sqrt{(H_1 - 1.0)}) - \]
\[ H_3 = 1.0 + ((GAM - 1.0) \cdot H_1) / 2.0 \]
\[ G_{11} = G_4 / (GAM - 1.0) \]
\[ G_{12} = TOT \cdot P \cdot G_{11} \]
\[ D E L O = 0.4 - (C / 20.0) \]
\[ \text{THETO} = (4.0 / 35.0) + (C / 105.0) - (C^*C / 252.0) \]
\[ \text{DELR} = G_{12} \cdot (1.0 - (G_3 \cdot (1.0 - D E L O - \text{THETO}))) \]
\[ \text{DELR}_S = \text{DELR} \]
\[ \text{TANI} \quad \text{PROVIDED}(C \cdot L \cdot 0.0), \text{TRANSFERTO}(UPOUT) \]
\[ \text{PROVIDED}(N \cdot GE \cdot K), \text{TRANSFERTO}(MNTAN) \]
\[ \text{PROVIDED}((XR / DXB) \cdot L \cdot (XSLOT(N+1) / DXB)) \cdot \text{TRANSFERTO}(MNTAN) \]
\[ N = N + 1 \]
\[ X_S = XSLOT(N) \]
\[ X_M = (X_{MID} + XSLOT(N)) \]
\[ X_{SE} = (X_{SLEND} + XSLOT(N)) \]
\[ \text{MNTAN} \quad D E L O = 0.4 - (C / 20.0) \]
\begin{align*}
\text{THETO} &= \left( \frac{4 \cdot 0}{35 \cdot 0} \right) + \left( \frac{C}{105 \cdot 0} \right) - \left( \frac{C \cdot C}{252 \cdot 0} \right) - \\
E_{11} &= \text{DELB} \times \left( \frac{1 \cdot 0}{105 \cdot 0} \right) - \left( \frac{C}{126 \cdot 0} \right) - \\
E_{12} &= \text{THETO} - \\
E_{13} &= \text{DELB} \times \left( \frac{2 \cdot 0 \times \text{THETO} + \text{DELO}}{\text{ME}} \right) - \\
E_{14} &= \left( \frac{C}{\text{RO} \times \text{DELB} \times \text{ME}} \right) + \text{BETA} - \\
E_{21} &= 0 \cdot 0 - \\
E_{22} &= 0 \cdot 0 - \\
E_{23} &= 1 \cdot 0 - \\
E_{24} &= \left( \frac{12 \cdot 0 - (6 \cdot 0 \cdot C)}{\text{RO} \times \text{DELB} \times \text{DELB}} \right) + \left( \frac{\text{BETA} \times \text{ME} \times C}{\text{DELB}} \right) - \\
G_{1} &= \text{ME} \times \text{ME} - \\
G_{2} &= \left( \frac{(\text{GAM} - 1 \cdot 0 \cdot G_{1})}{2 \cdot 0} \right) - \\
\text{TOT} &= 1 \cdot 0 + G_{2} - \\
G_{3} &= G_{2} / \text{TOT} - \\
E_{31} &= \text{LAM} \times \text{DELB} \times (-0 \cdot 05 + G_{3} \times \left( \frac{1 \cdot 0}{105 \cdot 0} \right) - \left( \frac{C}{126 \cdot 0} \right) \right) - \\
E_{32} &= \text{LAM} \times \left( \frac{\text{DELO} + (G_{3} \times \text{THETO})}{\text{ME}} \right) - \\
E_{33} &= \left( \frac{\text{LAM} \times \text{DELB} \times \text{ME} / \text{TOT}}{\text{G4} + (\text{GAM} - 1 \cdot 0) \times \text{THETO} \times (1 \cdot 0 + (G_{4} \times G_{1} / 2 \cdot 0))} \right)
\end{align*}
/TOT) - ((1.0 - DELO) * (1.0 + (G2 * (3.0 + (G4 * G1)))) / (G1 * TOT)) -

PHI = (H2 / H3) * (1.0 - (ME / MO)) -

E34 = PHI - (LAM * BETA / TOT) -

D = E11 * ((22 * E33) - (E32 * E23)) + E21 * ((E32 * E13) - (E12 * E33)) + E31 * ((

E12 * E23) - (E13 * E22)) -

N1 = E14 * ((E22 * E33) - (E32 * E23)) + E24 * ((E32 * E13) - (E12 * E33)) + E34 * (

E12 * E23) - (E13 * E22)) -

N2 = E11 * ((E24 * E33) - (E23 * E34)) + E21 * ((E34 * E13) - (E14 * E33)) + E31 * (

E14 * E23) - (E13 * E24)) -

N3 = E11 * ((E22 * E34) - (E32 * E24)) + E21 * ((E32 * E14) - (E12 * E34)) + E31 * (

E12 * E24) - (E22 * E14)) -

J = J + 1 -

PROVIDED (J * G4) * TRANSFER (ERROR) -

PROVIDED (D * E0 * U) * TRANSFER (ERROR) -

CP(J) = N1 / D -

DP(J) = N2 / D -
MP(J)=N3/D-
PROVIDED(J,E,4)*TRANSFERTO(ARULE)-
PROVIDED(J,NE,1)*TRANSFERTO(ARUKU)-
XB=XB+(DXB/2.0)-
C=C+(CP(J)*DXB/2.0)-
DELB=DELB+(DP(J)*DXB/2.0)-
ME=ME+(CP(J)*DXB/2.0)-
G12=T0T.P*G11-
DRX(J)=RO*G12/LAM-
RX=RX+(DRX(J)*DXB/2.0)-
XR=RX/(RO*DELS)-
PROVIDED((XR/DXB)*G0.0*G0.0)*TRANSFERTO(ASLOTA)-
BETA=0.0-
TRANSFERTO(TANI)-

ASLOTA
PROVIDED((XR/DXB)*G0.0*G0.0)*TRANSFERTO(APOSTA)-
PROVIDED((XR/DXB)*G0.0*G0.0)*TRANSFERTO(AMIDA)-
BETAC = CA + CB * (XR - XS)

TRANSFERTO(ABCA)

AMIDA

BETAC = CC + CD * (XSE - XR)

TRANSFERTO(ABCA)

APOSTA

BETAC = 0.0

ABCA

BETA = TOT * BETAC / LAM

TRANSFERTO(TANI)

ARUKU

PROVIDED(JNE.2), TRANSFERTO(ATHIR)

C = C + ((CP(2) - CP(1)) * DXB / 2.0)

DELB = DELB + ((DP(2) - DP(1)) * DXB / 2.0)

ME = ME + ((MP(2) - MP(1)) * DXB / 2.0)

G12 = TOT * P * G11

DRX(2) = RO * G12 / LAM

RX = RX + (DRX(2) - DRX(1)) * DXB / 2.0

XR = RX / (RO * DELRS)

PROVIDED((XR / DXB) . GE. (XS / DXB), TRANSFERTO(ASLOTB)
BETA=0.0 -
TRANSFERTO(TANI) -

ASLOTB PROVIDED((X/R/DXB) • G*(XSE/DXB)) * TRANSFERTO(APOSTB) -
PROVIDED((X/R/DXB) • G*(XM/DXB)) * TRANSFERTO(AMIDB) -
BETAC=CA+(CB*(XR-XS))-
TRANSFERTO(ABCB) -

AMIDB BETAC=CC+(CD*(XSE-XR))-
TRANSFERTO(ABCB) -

APOSTB BETAC=0.0-

ABCB BETA=TOT*BETAC/LAM-
TRANSFERTO(TANI) -

ATHIR XB=XB+(DXB/2.0)-
C=C-(CP(2)*DXB/2.0)+(CP(3)*DXB)-
DELB=DELB-(DP(2)*DXB/2.0)+(DP(3)*DXB)-
ME=ME-(MP(2)*DXB/2.0)+(MP(3)*DXB)-
G12=TOT*P*G11-
\[
\begin{align*}
\text{DRX}(4) &= \text{RO} \times \text{G12} / \text{LAM} = \\
\text{CPP} &= (\text{CP}(1) + (2 \times \text{CP}(2)) + (2 \times \text{CP}(3)) + \text{CP}(4)) / 6.0 - \\
\text{DELP} &= (\text{DP}(1) + (2 \times \text{DP}(2)) + (2 \times \text{DP}(3)) + \text{DP}(4)) / 6.0 - \\
\text{MEP} &= (\text{MP}(1) + (2 \times \text{MP}(2)) + (2 \times \text{MP}(3)) + \text{MP}(4)) / 6.0 - \\
\text{DRP} &= (\text{DRX}(1) + (2 \times \text{DRX}(2)) + (2 \times \text{DRX}(3)) + \text{DRX}(4)) / 6.0 - \\
\text{C} &= \text{C} + (\text{DXB} \times (\text{CPP} - \text{CP}(3))) - \\
\text{DELB} &= \text{DELB} + (\text{DXB} \times (\text{DELP} - \text{DP}(3))) - \\
\text{ME} &= \text{ME} + (\text{DXB} \times (\text{MEP} - \text{MP}(3))) - \\
\text{RX} &= \text{RX} + (\text{DXB} \times (\text{DRP} - \text{DRX}(3))) - \\
\text{J} &= 0 - \\
\text{G1} &= \text{ME} \times \text{ME} - \\
\text{G2} &= ((\text{GAM} - 1.0) \times \text{G1}) / 2.0 - \\
\text{TOT} &= 1.0 + \text{G2} - \\
\text{POP} &= 1.0 / (\text{TOT} \times (\text{GAM} / (\text{GAM} - 1.0))) - \\
\text{XR} &= \text{RX} / (\text{RO} \times \text{DELRS}) - \\
\text{G12} &= \text{TOT} \times \text{POP} \times \text{G1} - 
\end{align*}
\]
\[
\begin{align*}
\text{DLE}R &= G_{12} \cdot (1.0 - G_{3} \cdot (1.0 - DELO - \theta_{1}))) - \\
G_{13} &= TQT \cdot P \cdot 1.5 - \\
C_{F} &= 2.0 \cdot (R_{0} \cdot DEB \cdot M \cdot G_{13}) - \\
X_{SCA} &= \text{XR} \cdot \text{SCA} + \text{XRE} - \\
D&L = DEB \cdot DLEB / DLEB - \\
\text{P}ROV\text{I}D\text{E} \left[ (XR / DXB) \cdot G \cdot (XS / DXB) \right], \text{T}R\text{A}N\text{S}FERT\text{O} (A\text{S}L\text{O}T\text{D}) - \\
\text{B}E\text{T}A &= 0.0 - \\
\text{T}R\text{A}N\text{S}FERT\text{O} \{ \text{UP}O\text{T}U \} - \\
\text{A}S\text{L}O\text{T}\text{D} \text{P}ROV\text{I}D\text{E} \left[ (XR / DXB) \cdot G \cdot (XS / DXB) \right], \text{T}R\text{A}N\text{S}FERT\text{O} (A\text{P}O\text{S}\text{T}\text{D}) - \\
\text{P}ROV\text{I}D\text{E} \left[ (XR / DXB) \cdot G \cdot (XM / DXB) \right], \text{T}R\text{A}N\text{S}FERT\text{O} (A\text{M}\text{I}\text{D}) - \\
\text{B}E\text{T}\text{A}C &= CA + (C_{B} \cdot (\text{XR} - \text{XS})) - \\
\text{T}R\text{A}N\text{S}FERT\text{O} (A\text{B}C\text{D}) - \\
\text{A}M\text{I}\text{D} \text{B}E\text{T}\text{A}C &= CC + (C_{D} \cdot (XSE - XR)) - \\
\text{T}R\text{A}N\text{S}FERT\text{O} (A\text{B}C\text{D}) - \\
\text{A}P\text{O}\text{S}\text{T}\text{D} \text{B}E\text{T}\text{A}C &= 0.0 - \\
\text{A}B\text{C}\text{D} \text{B}E\text{T}\text{A} &= \text{TOT} \cdot \text{BETAC} / LAM - 
\end{align*}
\]
UPGUT  WRITEOUTPUT,AANS,(XB,XR,RX,C,CPP,ME,MEP,DELB,DELBP,CF,XSCALE,BETA,XS,XSH,PHI,TOT,POP,DELT)-

FAANS  (1H0.3F11.3,6F11.5/9F11.5)-
PROVIDED(ME,G,MOU),TRANSFERTO(END)-
PROVIDED(ME,L,MOD),TRANSFERTO(END)-
PROVIDED(C,G,0.0),TRANSFERTO(MTEST)-
MES=ME-
TRANSFERTO(INCON)-

MTEST  PROVIDED(MEP,L,EPS),TRANSFERTO(TANI)-
TRANSFERTO(END)-

INCON  PHI=(H2/H3)*1.0-(ME/MO)-
G5=1.0+(G1*G4/2.0)-
J=0-
EP(J)=((3.0*PHI)/(LAM*DELB))+(36.0*ME*(G4+(4.0*(GAM-1.0))*G5/(35.0*TOT))-(0.6*(1.0+(3.0+(G4*G1))*G2)/(G1*TOT)))/(RO*TOT*DELB*DELB))-(198.0*(0.4+(4.0*G2/(35.0*TOT)))/(RO*ME*DELB))-
\[
\begin{align*}
DP(J) &= (66.0/(RO*ME*DELB)) + (EP(J)*DELB/3.0) - \\
MP(J) &= -12.0/(RO*DELB*DELB) - \\
XB &= XB + (DXB/10.0) - \\
ETAS &= (EP(J)*DXB)/10.0 - \\
ME &= ME + (MP(J)*DXB)/10.0 - \\
DELB &= DELB + (DP(J)*DXB)/10.0 - \\
\text{INT} &= \text{PROVIDED} (\text{ETAS} \leq 0.0) \rightarrow \text{TRANSFER} \rightarrow \text{(OUT)} - \\
\text{INT} &= \text{PROVIDED} (\text{ETAS} \geq 1.0) \rightarrow \text{TRANSFER} \rightarrow \text{(ERROR)} - \\
\text{INT} &= \text{PROVIDED} (\text{N} \geq K) \rightarrow \text{TRANSFER} \rightarrow \text{(MAIN)} - \\
\text{INT} &= \text{PROVIDED} ((XR/DXB) \leq (XSL(N+1)/DXB)) \rightarrow \text{TRANSFER} \rightarrow \text{(MAIN)} - \\
N &= N + 1 - \\
XS &= XSL(N) - \\
XM &= XMD + XSL(N) - \\
XSE &= XSL(N) - \\
\text{MAIN} &= \text{F1} = 1.0 - \text{ETAS} - \\
F2 &= \text{ETAS} \times \text{ETAS} - \\
\end{align*}
\]
F16 = 8 \times (1.0 + F11 - F15) - 
THETIN = F9 + F16 - 
F17 = DELO - DELIN - 
F18 = F7 \times F7 \times F1 \times (2.0 + (7.0 \times ETAS) + (7.0 \times F2)) / (7.0 \times F4 \times F4) - 
F19 = (F1 \times F1 \times F1 \times US \times US) / (63.0 \times F2) - 
F20 = (F7 \times F1 \times F1 \times (5.0 + (7.0 \times ETAS)) \times US) / (42.0 \times F4 \times ETAS) - 
THETO = THETIN + F17 - F18 - F19 - F20 - 
F21 = (10.0 - 9.0 \times ETAS) \times 2.0 \times ETAS - 
F22 = US \times (7.0 - 22.0 \times ETAS + 9.0 \times F2) - 
F23 = 4.0 \times F14 \times (1.0 - 10.0 \times ETAS + 9.0 \times F2) - 
F24 = 4.0 \times F14 \times F1 \times F4 \times ETAS - 
F25 = 2 \times F2 \times F5 + F6 \times F5 - 
C1 = (F21 - F22 + F23 - F24) / F25 - 
C2 = (4 \times F1 \times F6) / (F25 \times ETAS) - 
F26 = 2.0 \times (5.0 - 6.0 \times ETAS) - 
C3 = (F26 - (F22 / ETAS) + (F23 / ETAS) - (2.0 \times F24 / ETAS)) / F25 -
C4 = C2 / ETAS

F27 = F1 * F8 * C1

F28 = F7 * (1 * 0 - 6 * 0.0 * ETAS)

F29 = 3.0 * F1 * F8 * F7 / F4

F30 = (F27 - F28 + F29) / (5 * F4)

C5 = 1.0 * U = F30 - (F1 * US) / (5.0 * ETAS) + (F1 * F1 * C3 / 10.0)

C6 = 1.0 + (F1 * F8 * C2) / (5.0 * F4) - (F1 * F1 * C4 / 10.0)

F31 = 2.0 * US * US / 15.0

F32 = 4.0 * U * ETAS * US * C1 / 15.0

F33 = (3.0 - ETAS) * C1

F34 = US * (9.0 - (8.0 * ETAS) + (3.0 * F2)) / F5

F35 = 144.0 * F12 * F14 / (21.0 * F5 * ETAS)

F36 = 144.0 * (1.0 + ETAS) * F14 / (21.0 * F5 * F5)

F37 = 0.4 * B / F5

F38 = F37 * (F33 + F34 + F35 - F36)

C7 = -F31 - F32 + F38
F39 = 4.0 * ETAS * US * C2 / 15.0
F40 = 0.4 * (3.0 - ETAS) * (US + (B * C2)) / F5
F41 = ((576.0 * F12 * F14) / (105.0 * F5 * F5))
C8 = 1.0 - F39 + F40 - F41
F42 = F7 / F4
F43 = (1.0 - (6 * ETAS)) / 5.0
F44 = 0.6 * F1 * F8 / F4
F45 = F7 * (5.0 - 21.0 * F2) / (7.0 * F4)
F46 = F7 * F1 * (2.0 + 7 * ETAS + 7 * F2) * 6.0 / (7.0 * F4 * F4)
F47 = F42 * (F43 - F44 - F45 + F46)
F48 = F1 * US / ETAS
F49 = F7 * (1.0 + (7.0 * ETAS)) / (14.0 * F4)
F50 = F7 * F1 * (5.0 + (7.0 * ETAS)) / (14.0 * F4 * F4)
F51 = F48 / 21.0
F52 = F48 * (0.2 - F49 - F50 - F51)
C9 = F47 - F52
F53=F1/F4-
F54=F8/5.0-
F55=2.0*F7*(2.0+(7*ETAS)+(7*F2))/(7.0*F4)-
F56=F1*(5.0+(7.0*ETAS))*US/(42.0*ETAS)-
C10=F53*(F55-F54+F56)-
F57=(F7*(5.0+(7.0*ETAS)))/(42.0*F4)-
F58=2.0*F48/63.0-
C11=F1*F1*(0.1-F57-F58)-
C12=C7+C9+(C1*C10)+(C3*C11)-
C13=C8+(C2*C10)+(C4*C11)-
D11=C12*DELB-
D12=THETO-(B*C13)-
D13=DELB*{(2*THETO)+DELO-(B*C13)}/ME-
F59=2.0*US/ETAS-
F60=(12.0*F14*F10)/(F5*ETAS)-
D14=(-F59+F60)/(RO*DELB*ME)+BETA*(1.0-C13)-
D21=DELB*C7-
D22=THETIN-(B*C8)-
D23=DELB*((2*THETIN)+DELIN-(B*C8))/ME-
F61=6.0*US/ETAS-
F62=((96.0*F14)/(ETAS*F5))-D24=(-F61+F62)/(RO*DELB*ME)+BETA*(1.0-C8)-
G1=ME*ME-
G2=((GAM-1.0)*G1)/2.0-
TOT=1.0+G2-
G3=G2/TOT-
G4=((3.0*GAM)-1.0)/2.0-
G5=1.0+(G1*G4/2.0)-
G6=(GAM-1.0)*G5*THETO/TOT-
G7=(1.0+(3.0+G4*G1)*G2)*(1.0-DELO)/(TOT*G1)-
G8=LAM*DELB*ME*(G4+G6-G7)/TOT-
G9=DELO+(G2*THETO/TOT)-
PROVIDED(XR,XSH), TRANSFER(TO(UP)

MO = MOD

TRANSFER(RES)

UP

MO = MOU

RES

H1 = MO * MO

H2 = SQRT(H1 - 1.0)

H3 = 1.0 + (((GAM - 1.0) * H1) / 2.0)

PHI = (H2 / H3) * (1.0 - (ME / MO))

GJL0 = RO * DELB * ME

C14 = G8

C15 = LAM * G9

C16 = LAM

C17 = LAM * G3

D31 = DELB * ((C5 * C16) + (C12 * C17))

D32 = C15 + B * ((C6 * C16) - (C13 * C17))

D33 = C14 + (B * DELB / ME) * ((C6 * C16) - (C13 * C17))
C18 = PHI - (BETA * LAM / TOT) - 
D34 = C18 + BETA * ((C6 * C16) - (C13 * C17)) - 

DER

D = D11 * ((D22 * D33) - (D32 * D23)) + D21 * ((D13 * D32) - (D12 * D33)) + D31 * 
(D12 * D23) - (D13 * D22) - 
N1 = D14 * ((D22 * D33) - (D32 * D23)) + D24 * ((D32 * D13) - (D12 * D33)) + D34 * 
(D12 * D23) - (D22 * D13) - 
N2 = D11 * ((D24 * D33) - (D23 * D34)) + D21 * ((D34 * D13) - (D33 * D14)) + D31 * 
(D14 * D23) - (D13 * D24) - 
N3 = D11 * ((D24 * D34) - (D24 * D32)) + D21 * ((D34 * D14) - (D34 * D12)) + D31 * 
(D12 * D24) - (D14 * D22) - 
J = J + 1 - 
PROVIDED(J . G . 4) * TRANSFER TO ERROR - 
PROVIDED(D . E . G . 0) * TRANSFER TO ERROR - 
EP(J) = N1 / D - 
DP(J) = N2 / D - 
MP(J) = N3 / D -
Provided(j.e.4), transfer to border-
Provided(j.ne.1), transfer to ruku-

FIR

xb=xb+(dxb/2.0)-
etas=etas+(ep(j)*dxb/2.0)-
delb=delb+(dp(j)*dxb/2.0)-
me=me+(mp(j)*dxb/2.0)-
g11=g4/(gam-1.0)-
g12=tot.p.g11-

DRX(J)=RO*G12/LAM-
RX=RX+(DRX(J)*DXB/2.0)-
XR=RX/(RO*DELRS)-

Provided((xr/dxb) .ge. (xs/dxb)), transfer to slotA-

beta=0.0-
b=0.0-
BP(1)=0.0-

Transfer to INT-
SLOTA  PROVIDED((XR/DXB)*G*(XSE/DXB))*TRANSFERTO(POSTA)-
   PROVIDED((XR/DXB)*G*(XM/DXB))*TRANSFERTO(MIDA)-
   BETAC=CA+C*B*(XR-XS)-
   TRANSFERTO(BCA)-
MIDA  BETAC=CC+CD*(XSE-XR)-
   TRANSFERTO(BCA)-
POSTA BETAC=0.0-
BCA  BETA=TOT*BETAC/LAM-
   BP(1)=(BETA/DELB)-B*((DP(1)/DELB)+(MP(1)/ME))-B=B+(BP(1)*DXB/2.0)-
   TRANSFERTO(INT)-
RUKU  PROVIDED(J*NE.2)*TRANSFERTO(THIR)-
   ETAS=ETAS+((EP(2)-EP(1))*DXB/2.0)-
   DELB=DELB+((DP(2)-DP(1))*DXB/2.0)-
   ME=ME+((MP(2)-MP(1))*DXB/2.0)-
   RX=RX-(DRX(1)*DXB/2.0)-
G11 = G4/(GAM-1.0) -
G12 = TOT*G11 -
DRX(2) = RU*G12/LAM -
RX = RX + (DRX(2)*DXB/2.0) -
XR = RX/(RO*DELS) -
PROVIDED((XR/DXB) . GE. (XS/DXB)) , TRANSFERTO(SLOTB) -
BETA = 0.0 -
B = 0.0 -
BP(2) = 0.0 -
TRANSFERTO(INT) -

SLOTB PROVIDE((XR/DXB) . GE. (XSE/DXB)) , TRANSFERTO(POSTB) -
PROVIDED((XR/DXB) . GE. (XM/DXB)) , TRANSFERTO(MIDB) -
BETAC = CA + CB*(XR- XS) -
TRANSFERTO(BCB) -

MIDB BETAC = CC + CD*(XSE-XR) -
TRANSFERTO(BCB) -
POSTB

BETAC = 0.0

BCB

BETA = TOT * BETAC / LAM

BP(2) = (BETA / DELB) - B * ((DP(2) / DELB) + (MP(2) / ME)) -

B = B + ((BP(2) - BP(1)) * DXB / 2.0)

TRANSFERTO(INT)

THIR

XB = XB + (DXB / 2.0)

ETAS = ETAS - (EP(2) * DXB / 2.0) + (EP(3) * DXB)

DELB = DELB - (DP(2) * DXB / 2.0) + (DP(3) * DXB)

ME = ME - (MP(2) * DXB / 2.0) + (MP(3) * DXB)

RX = RX - (DRX(2) * DXB / 2.0)

G11 = G4 / (GAM - 1.0)

G12 = TOT * P * G11

DRX(3) = RO * G12 / LAM

RX = RX + (DRX(3) * DXB)

XR = RX / (RO * DELRS)

PROVIDED((XR / DXB) .GE. (XS / DXB)) , TRANSFERTO(SLOTC)

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BETA=0.0
B=0.0
BP(3)=0.0
TRANSFER TO (INT)

SLOTC
PROVIDED((XR/DB).G.(XSE/DB)).TRANSFER TO (POSTC)
PROVIDED((XR/DB).G.(XM/DB)).TRANSFER TO (MIDC)
BETAC=CA+(CB*(XR-XS))
TRANSFER TO (BCC)

MIDC
BETAC=CC+(CD*(XSE-XR))
TRANSFER TO (BCC)

POSTC
BETAC=0.0

BCC
BETA=TOT*BETAC/LAM
BP(3)=(BETA/DELB)-B*((DP(3)/DELB)+(MP(3)/ME))
B=B-(BP(2)*DB/2.0)+(BP(3)*DB)
TRANSFER TO (INT)

BDER
PROVIDED(B.E.0.0).TRANSFER TO (RULE)
BP(4) = (BETA/DELB) - B*(DP(4)/DELB) + (MP(4)/ME) -
BPP = (BP(1) + (2.0*BP(2)) + (2.0*BP(3)) + BP(4))/6.0 -
B = B + D*BP*(BPP - BP(3)) -

RULE

G11 = G4/(GAM - 1.0) -
G12 = TOT*P*G11 -

DRX(4) = RO*G12/LAM -
ETASP = (EP(1) + (2.0*EP(2)) + (2.0*EP(3)) + EP(4))/6.0 -
DELB = (DP(1) + (2.0*DP(2)) + (2.0*DP(3)) + DP(4))/6.0 -
MEP = (MP(1) + (2.0*MP(2)) + (2.0*MP(3)) + MP(4))/6.0 -
DRP = (DRX(1) + (2.0*DRX(2)) + (2.0*DRX(3)) + DRX(4))/6.0 -
ETAS = ETAS + (D*BP*(ETASP - EP(3))) -
DELB = DELB + (D*BP*(DELB - DP(3))) -
ME = ME + (D*BP*(MEP - MP(3))) -
RX = RX + (D*BP*(DRP - DRX(3))) -

J = 0 -

PROVIDED(ETAS.LE.0.0), TRANSFER TO(ERROR) -
G1 = ME * ME -
G2 = ((GAM - 1.0) * G1) / 2.0 -
TOT = 1.0 + G2 -
POP = 1.0 / (TOT * P * (GAM / (GAM - 1.0))) -
XR = RX / (RO * DELRS) -
G12 = TOT * P * G11 -
DELR = G12 * (1.0 - G3 * (1.0 - DELO - THET0)) -
CFPAS = CF -
C = - F59 + (12.0 * F14 * F10 / (ETAS * F5)) -
G13 = TOT * P * 1.5 -
CF = 2.0 * C / (RO * DELB * ME * G13) -
CFPR = (CF - CFPAS) / DXB -
XSCALE = XR * SCALE + XREF -
DELT = DEL * DELR / DELRS -
PROVIDED: (XR / DXB) \( \geq \) (XS / DXB), TRANSFERTO (SLOTD) -
BETA = 0.0 -
\begin{align*}
&\text{SLOTD} & B = 0.0 - \\
&\text{TRANSFERTO(OUT)} - \\
&\text{provided((XR/DB) \cdot G \cdot (XSE/DB)) \cdot TRANSFERTO(POSTD)} - \\
&\text{provided((XR/DB) \cdot G \cdot (XM/DB)) \cdot TRANSFERTO(MIDD)} - \\
&\text{BETAC} = CA + (CB \cdot (XR - XS)) - \\
&\text{TRANSFERTO(BCD)} - \\
&\text{MIDD} & \text{BETAC} = CC + (CD \cdot (XSE - XR)) - \\
&\text{TRANSFERTO(BCD)} - \\
&\text{POSTD} & \text{BETAC} = 0.0 - \\
&\text{BCD} & \text{BETA} = \text{TOT} \cdot \text{BETAC} / \text{LAM} - \\
&\text{OUT} & \text{WRITEOUTPUT} \cdot \text{ANS} \cdot (XB, XR, RX, ETAS, ETASP, ME, MEP, DELB, DELBP, DELT,} \\
&\text{BETA, B, XS, XSH, PHI, TOT, POP, US, XSCALE, C, CF, CFPR, DELIN, DELO,} \\
&\text{THETIN, THETO)} - \\
&\text{F_ANS} & (1H0 \cdot 3F11.3 \cdot 7F11.5 / 8F11.5 / 1F11.3 \cdot 7F11.6) - \\
&\text{PROVIDED(ETAS \leq 0.0) \cdot TRANSFERTO(ENDSEP)} - \\
&\text{PROVIDED(((XR/DB) \leq \max((XSH+DB)/DB))) \cdot TRANSFERTO(INT)} - 
\end{align*}
PROVIDED([ETASP/DEXB].G.0.0),TRANSFER(END)-
PROVIDED(XR.LE.(XSH+DXB)),TRANSFER(INT)-
PROVIDED(MEP.L.EPS),TRANSFER(INT)-
TRANSFER(END)-

C=0.0-
J=0-
MES=ME-
TRANSFER(INTANW)-

WRITEOUTPUT,2,(ETAS,RX)-

CALLSUBROUTINE()=ENDJOB()-

ENDPROGRAM(SEP)-

*** DATA
REFERENCES

(1) Ferri, A. "Experimental Results with Aerofoils Tested in the High-Speed Tunnel at Guidonia," Atti di Guidonia No. 17, 1939, Translated as NACA TM 946.


