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DISSERTATION

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By

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*****

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1966

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This paper is dedicated to my wife and family, whose support and encouragement were greatly needed and appreciated. I wish also to express my gratitude to Professor Earl J. Mickle who gave generously of his time and energy to engage in many helpful discussions with me throughout the course of work on this dissertation; I am especially grateful to Professor Mickle for the criticisms and valuable suggestions which he offered during these discussions.
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INTRODUCTION

Lebesgue measure has the nice property of being invariant under translations; one might ask: "Are there other spaces which have a non-trivial measure which is invariant in some sense?" It is this question which is the basis for the theory of Haar measure. As an immediate generalization of the invariance of Lebesgue measure we have the following well-known result: any locally compact Hausdorff topological group has a Haar measure which is unique up to a factor of proportionality.

If $G$ is a topological group and $H$ is a closed normal subgroup, then the coset space $G/H$ with quotient topology is a locally compact Hausdorff topological group and has a Haar measure invariant with respect to $G$. A natural question is the following: if $H$ is any closed subgroup of a locally compact Hausdorff topological group $G$, does there exist a Haar measure on the coset space $G/H$ (with quotient topology)? Andre Weil [17] (numbers in [ ] refer to the bibliography) has found a necessary and sufficient condition for an affirmative answer to this question. Thus the above question reduces to the following: given a closed subgroup $H$ of a locally compact Hausdorff topological group $G$, does
the Weil condition hold so that there is a Haar measure on G/H? However, in a given case, it may not be easy to check whether the Weil condition does or does not hold.

In Chapter 3 we investigate the more general question: if G is a group of autohomeomorphisms of a locally compact Hausdorff space X, does there exist a Haar measure on X which is invariant with respect to G? We state a condition (Condition A) which gives an affirmative answer to this question. It is shown that equicontinuity implies Condition A and in Chapter 4 we look at some other ideas which are related to Condition A.

There are basically two approaches to the theory of Haar measure on groups. One method (used by André Weil) is to use the Axiom of Choice in the form of the Tychonoff product theorem to establish the existence of a Haar measure and then show independently that if a Haar measure exists it is unique. Some mathematicians object to this idea of using the Axiom of Choice to find an element in a set (of Haar measures) consisting of one element. With this objection in mind H. Cartan [4] established the existence and uniqueness of Haar measure (on groups) simultaneously and without appealing to the Axiom of Choice. However, it is the first method which lends itself to the generalization established in Chapter 3.

In Chapter 5 we apply the work of Chapter 3 to groups and coset spaces of groups. An interesting sideline deals
with equicontinuity and uniform equicontinuity in coset spaces.

Mickle and Rado [12,13] have established the existence of a unique Haar measure on the space $\mathcal{H}$ of oriented lines in $E_3$ with respect to the group of rigid motions in $E_3$. Professor Mickle asked if this result is a special case of a more general theory. In this paper his question is answered affirmatively by identifying the space $\mathcal{H}$ with a coset space of its specified group of autohomeomorphisms and then showing (by Weil's result) that this coset space has a unique Haar measure. However, as Professor Mickle pointed out, just a knowledge of the existence and uniqueness of a Haar measure on $\mathcal{H}$ was not sufficient for their work; they needed the explicit formula for the measure and this formula is not given by an abstract existence proof.

In Chapter 6 we ask when this identification with a coset space can be achieved in general; i.e., if $G$ is a group of autohomeomorphisms of a (locally compact Hausdorff) topological space $X$ onto itself, under what conditions can $G$ be topologized in such a way that $G/H$ is homeomorphic to $X$ for a suitable subgroup $H$?
Chapter 1
PRELIMINARY DEFINITIONS AND NOTATION

Definition 1.1 If \( X \) is an abstract space and \( A \subseteq X \) we shall denote \( A' = \{x : x \in X \text{ and } x \notin A\} \). If \( A \subseteq X \) and \( B \subseteq X \) we set \( B - A = B \cap A' \).

If \( X \) is a topological space we shall write \( A^0 \) for the interior of \( A \) and \( cA \) for the closure of \( A \).

Definition 1.2 A topological space \( X \) is said to be locally compact if for each \( x \in X \) there is a compact set \( A \) such that \( x \in A^0 \).

If \( O \) is an open set and \( cO \) is compact we shall say that \( O \) is a compact open set. Thus a Hausdorff \( (T_2) \) space is locally compact if and only if each point has a compact open neighborhood.

Definition 1.3 Let \( X \) be an abstract space and \( u \) a set function defined on the class of all subsets of \( X \). We say \( u \) is an outer measure if

1) \( 0 \leq u(A) \leq \infty \) for all \( A \subseteq X \),
2) \( u(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} u(A_n) \) for \( A_n \subseteq X, n = 1,2,3,\ldots \),
3) \( u(A) \leq u(B) \) when \( A \subseteq B \subseteq X \),
4) \( u(\emptyset) = 0 \).
If \( u \) is an outer measure we say that \( A \subseteq X \) is measurable if \( u(E) = u(E \cap A) + u(E - A) \) for all \( E \subseteq X \).

**Definition 1.4** A non-empty class \( \mathcal{S} \) of subsets of a space \( X \) is called a \( \sigma \)-ring if

1) \( A, B \in \mathcal{S} \implies A - B \in \mathcal{S} \),
2) \( A_n \in \mathcal{S}, n = 1, 2, \ldots \implies \bigcup_{n=1}^{\infty} A_n \in \mathcal{S} \).

If in addition

1*) \( A \in \mathcal{S} \implies A' \in \mathcal{S} \)

we say \( \mathcal{S} \) is a \( \sigma \)-algebra.

**Definition 1.5** A measure is a set function \( u \) defined on a \( \sigma \)-ring \( \mathcal{S} \) such that

1) \( 0 \leq u(A) \leq \infty \) for all \( A \in \mathcal{S} \),
2) \( u \left( \bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} u(A_n) \) when the \( A_n \) are mutually disjoint sets in \( \mathcal{S} \),
3) \( u(\emptyset) = 0 \).

**Definition 1.6** A measure \( u \) is said to be complete if \( A \subseteq E \in \mathcal{S} \) and \( u(E) = 0 \) together imply \( A \in \mathcal{S} \).

By the completion of a measure space \((X, \mathcal{S}, u)\) we mean the measure space \((X, \mathcal{S}^*, u^*)\) where

\[ \mathcal{S}^* = \{ A : \text{there exist } B, C \in \mathcal{S} \text{ and } Z \subseteq C \text{ such that } u(C) = 0 \text{ and } A = B \cup Z \} \]

and \( u^*(B \cup Z) = u(B) \) for all \( A = B \cup Z \in \mathcal{S}^* \).
Definition 1.7  By a regular measure on a locally compact Hausdorff space $X$ we mean a measure space $(X, \mathcal{S}, \mu)$ where $\mathcal{S}$ contains all compact subsets of $X$ and

1) $\mu(O) = \text{LUB}\{\mu(C) : O \supseteq C \text{ compact}\}$ for all open sets $O \in \mathcal{S}$,
2) $E \in \mathcal{S} \implies$ there is an open set $O \in \mathcal{S}$ such that $O \supseteq E$,
3) $\mu(E) = \text{GLB}\{\mu(O) : E \subseteq O \in \mathcal{S}, O \text{ open}\}$ for all $E \in \mathcal{S}$.

Definition 1.8  Let $X$ be a locally compact Hausdorff space. We shall denote by $\mathcal{B}^*$ or $\mathcal{B}^*(X)$ the $\sigma$-ring generated by the class of all compact sets in $X$. $\mathcal{B}$ or $\mathcal{B}(X)$ will denote the $\sigma$-algebra generated by the class of all open sets in $X$. $\mathcal{B}^*$ and $\mathcal{B}$ will be called the classes of Borel* and Borel sets respectively.

By a Borel* measure on $X$ we mean a measure space $(X, \mathcal{B}^*, \mu)$ where $\mu(C) < \infty$ for each compact set $C$.

Similarly a Borel measure on $X$ is a measure space $(X, \mathcal{B}, \mu)$ where $\mu(C) < \infty$ for each compact set $C$.

Definition 1.9  Let $h_1$ be a function with domain $\mathcal{D}_1$ and $h_2$ a function with domain $\mathcal{D}_2$. We shall say $h_2$ is an extension of $h_1$ if $\mathcal{D}_2 \supseteq \mathcal{D}_1$ and $h_1(D) = h_2(D)$ for all $D \in \mathcal{D}_1$.
Definition 1.10 \( L \) or \( L(X) \) will denote the class of continuous real valued functions with compact support on a locally compact Hausdorff space \( X \). A function has compact support if it vanishes outside of some compact set. Whenever we write the symbol \( L \) it will always be assumed that the underlying space \( X \) is locally compact and Hausdorff.

A real valued function \( A \) on \( L \) is called a linear functional if \( A(af + bg) = aA(f) + bA(g) \) for all real \( a, b \) and all \( f, g \in L \). \( A \) is said to be non-negative if \( g \geq 0 \) implies \( A(g) \geq 0 \).

Definition 1.11 If \( X \) is a topological space and \( h \) is a homeomorphism of \( X \) onto itself we shall say simply "\( h \) is an autohomeomorphism of \( X \)."

A class \( G \) of functions is said to be a group of autohomeomorphisms of \( X \) if

1) each \( g \in G \) is an autohomeomorphism of \( X \),
2) \( g \in G \rightarrow g^{-1} \in G \),
3) \( g_1, g_2 \in G \rightarrow g_1 \circ g_2 \in G \).

Definition 1.12 A group \( G \) of autohomeomorphisms of \( X \) is said to be transitive if, given \( x, y \in X \), there is a \( g \in G \) such that \( g(x) = y \). We say \( G \) is weakly transitive if, for every non-empty open set \( O \), \( X \subseteq \bigcup_{g \in G} g(O) \).
Chapter 2
BACKGROUND IN HAAR MEASURE

Definition 2.1 Let \( X \) be a locally compact Hausdorff space and "\( \sim \)" a relation defined between certain subsets of \( X \). By a Haar measure on \( X \) with respect to "\( \sim \)" we mean a regular Borel measure \( u \) which is not identically 0 and is invariant in the sense that \( u(A) = u(B) \) whenever \( A, B \in \mathcal{G} \) and \( A \sim B \).

If \( u \) is a Haar measure, then in virtue of the regularity there exists a compact set \( C \) such that \( 0 < u(C) < \infty \). But each compact set \( C \) has a compact open neighborhood. Hence there is a compact open set \( O \) such that \( 0 < u(O) < \infty \).

A trivial example of a Haar measure is the following.

Example 2.1 Let \( X \) be an arbitrary non-empty set with the discrete topology. Let \( A \sim B \) if and only if \( A = B \). Fix \( x_0 \in X \). Then

\[
u(E) = \begin{cases} 1, & x_0 \in E \\ 0, & x_0 \notin E \end{cases}
\]

is a Haar measure on \( X \) with respect to "\( \sim \)."

Our primary interest will be in congruence relations
defined by a group $G$ of autohomeomorphisms of $X$; i.e., $A \equiv B$ if and only if there is a $g \in G$ such that $g(A) = B$.

**Definition 2.2** A non-negative real valued function $\lambda$ defined on the class of compact sets will be called a content if for compact sets $C$ and $D$

1) $\lambda(C \cup D) \leq \lambda(C) + \lambda(D)$,
2) $C \cap D = \emptyset \rightarrow \lambda(C \cup D) = \lambda(C) + \lambda(D)$,
3) $C \subseteq D \rightarrow \lambda(C) \leq \lambda(D)$.

Clearly the restriction of a regular Borel or Borel* measure to the class of compact sets is a content. One of the powerful results in the theory of regular Borel or Borel* measures is that the converse also holds.

**Theorem 2.1** Let $\lambda$ be a content defined on the compact sets of a locally compact Hausdorff space $X$. For each open set $O$ define

$$u_O(O) = \text{LUB} \{ \lambda(C) : O \supseteq C \text{ compact} \}$$

and for any $A \subseteq X$ define

$$u^*(A) = \text{GLB} \{ u_O(O) : A \subseteq O \text{ open} \}.$$ 

Then

1) $u^*$ is an outer measure on $X$,
2) open sets are measurable,
3) $u^*(O) = u_O(O)$ for each open set $O$,
4) $u^*(C) < \infty$ for each compact set $C$,
5) the restriction of $u^*$ to the class $\mathcal{B}$ or its completion is a regular measure.
Proof For (1), (2), and (3), see Segal [16]. If C is compact, let O be a compact open neighborhood of C. Then
\[ u^*(C) \leq u^*(O) = u_0(O) = \text{LUB} \{ \lambda(C^*): O \supseteq C^* \text{ compact} \} \leq \lambda(O) < \infty. \]
This establishes (4).

Segal [16] shows that for each open set O
6) \[ u^*(O) = \text{LUB} \{ u^*(C): O \supseteq C \text{ compact} \}. \]
(3) and (6) together with the definition of \( u^* \) imply (5).

Theorem 2.2 Let \( G \) be a group of autohomeomorphisms of a locally compact Hausdorff space \( X \) and let \( \lambda \) be a content on \( X \) such that \( \lambda(g(C)) = \lambda(C) \) for all compact sets \( C \) and all \( g \in G \). If \( u^* \) is the outer measure constructed from \( \lambda \) by the method of Theorem 2.1, then \( u^*(g(A)) = u^*(A) \) for all \( A \subseteq X \) and all \( g \in G \). Hence the restriction of \( u^* \) to \( \mathcal{G} \) is a Haar measure on \( X \) with respect to \( G \) provided \( u^* \neq 0 \).

Some authors, notably Halmos and Segal, use a slightly different definition of Haar measure. We are now in a position to show that the theories derived from these definitions must be identical.

Definition 2.3 A measure \( u \) on \( \mathcal{S} \supseteq \mathcal{B}^* \) is said to be H-regular if for all \( E \in \mathcal{S} \)
1) \[ u(E) = \text{LUB} \{ u(C): E \supseteq C \text{ compact} \}, \]
2) there is an open set \( O \in \mathcal{S} \) such that \( O \supseteq E \),
3) \[ u(E) = \text{GLB} \{ u(O): E \subseteq O \text{ open}, O \in \mathcal{S} \}. \]
Definition 2.4 A (Halmos) Haar measure is an invariant $H$-regular Borel* measure $u$ which is not identically 0.

Definition 2.5 A (Segal) Haar measure is an invariant $H$-regular measure $u$ on the completion $M^*$ of $B^*$ such that $u$ is not identically 0 and $u(C) < \infty$ for all compact sets $C$.

Note that compact open sets $O$ are in $B^*$ since $(cO - O)$ is compact and $O = cO - (cO - O)$. Thus, in Definitions 2.4 and 2.5, just as in Definition 2.1, there is a compact open set $O$ such that $0 < u(O) < \infty$.

Theorem 2.3 If $u$ is a regular Borel measure, the completion of a regular Borel measure, or an $H$-regular measure on $M^*$ which is finite on compact sets, then the restriction of $u$ to $B^*$ is an $H$-regular Borel* measure.

Proof This is an immediate consequence of Berberian [2] 61.5.

Theorem 2.4 The completion $u$ of an $H$-regular Borel* measure $u^*$ is an $H$-regular measure on $M^*$ which is finite on compact sets.

Proof Fix $A \in M^*$ and write $A = B \cup Z$ where $B \in B^*$ and there is a $Z_1 \in B^*$ such that $Z \subseteq Z_1$ and $u^*(Z_1) = 0$. Then $u^*(B) = \text{LUB}\{u^*(C) : B \supseteq C \text{ compact}\}$

$\leq \text{LUB}\{u^*(C) : A \supseteq C \text{ compact}\} \leq u(A) = u^*(B)$. Thus $u(A) = \text{LUB}\{u(C) : A \supseteq C \text{ compact}\}$. 
There exist open sets $O_1, O_2 \in \mathcal{B}^* \subseteq \mathcal{M}^*$ such that $O_1 \supseteq B$ and $O_2 \supseteq Z_1$. Then $O_1 \cup O_2 \in \mathcal{M}^*$ and $A \subseteq O_1 \cup O_2$.

If $u(A) = \infty$, then clearly $u(A) = \text{GLB}\{u(O) : A \subseteq O \text{ open}, O \in \mathcal{M}^*\}$. If $u(A) < \infty$, fix $\varepsilon > 0$; there exist open sets $U_1, U_2 \in \mathcal{B}^*$ such that $U_1 \supseteq B$, $U_2 \supseteq Z_1 \supseteq Z$, $u^*(U_1 - B) < \varepsilon/2$, and $u^*(U_2) < \varepsilon/2$. Then $A \subseteq U_1 \cup U_2$ and $u(U_1 \cup U_2 - A) \leq u^*(U_1 - B) + u^*(U_2) < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

**Theorem 2.5** If $u^*$ is an $H$-regular Borel measure on $\mathcal{B}^*$, then $u^*$ can be uniquely extended to a regular Borel measure $u$. The completion of $u$ is also regular.

**Proof** The restriction $\lambda$ of $u^*$ to the compact subsets of $X$ is a content. Let $u$ be the regular Borel measure constructed from $\lambda$ by the method of Theorem 2.1. We must show that $u^*(B) = u(B)$ for all $B \in \mathcal{B}^*$. Let $O$ be an open set in $\mathcal{B}^*$. Then

$$u(O) = \text{LUB}\{u(C) : O \supseteq C \text{ compact}\} = u^*(O).$$

Now for a compact set $C$,

$$u(C) = \text{GLB}\{u(O) : C \subseteq O \text{ open}\} = \text{GLB}\{u(O) : C \subseteq O \text{ open}, O \in \mathcal{B}^*\} = \text{GLB}\{u^*(O) : C \subseteq O \text{ open}, O \in \mathcal{B}^*\} = u^*(C).$$

Then by the $H$-regularity of both $u^*$ and the restriction of $u$ to $\mathcal{B}^*$, it follows that $u(B) = u^*(B)$ for all $B \in \mathcal{B}^*$.

The uniqueness of the extension $u$ now follows from the fact that the values of the measure on compact sets
uniquely determine the values of the measure on all sets in $\mathcal{B}$. The fact that the completion of $\mu$ is regular follows from the construction of $\mu$ as given in Theorem 2.1.

Thus the completion of a regular Borel measure is regular and any regular Borel measure can be extended to an outer measure for which open sets are measurable. Also, if we have a Haar measure $\mu$ on $X$ with respect to a group $G$ of autohomeomorphisms of $X$ (any of the given definitions of Haar measure), then $\mu$ can be restricted or uniquely extended to a Haar measure fitting the other definitions; moreover, $\mu$ can be extended to a $G$-invariant outer measure on $X$.

It should be observed that in a metric space every Borel* measure is $H$-regular (Berberian [2] Problem 57.1 and Theorem 60.1). Hence $H$-regularity is usually not specifically requested by authors working exclusively in metric spaces. If $X$ is a separable metric space then $\mathcal{B} = \mathcal{B}^*$ and regularity also follows immediately for Borel measures.

It is clear that if $f$ is a function which is integrable with respect to a measure space $(X, \mathcal{S}, \mu)$, then $f$ is also integrable with respect to any extension $\nu$ of $\mu$ and $\int f \, d\mu = \int f \, d\nu$. Thus, using the fact that the restriction of a Borel measure to $\mathcal{B}^*$ is a Borel* measure we obtain the following results from Berberian [2].
If \( u \) is a Borel (or Borel*) measure on \( X \), then each \( f \in L \) is integrable with respect to \( u \) (Berberian [2] 66.2). Thus the real valued function \( A \) on \( L \) defined by

\[
A(f) = \int f \, du
\]

is a non-negative linear functional. If \( T \) is an auto-homeomorphism of \( X \), then clearly \( f \in L \rightarrow f \circ T \in L \). If, in addition, \( u \) is regular (respectively \( H \)-regular) then

\[
\int f \, du = \int f \circ T \, du
\]

for all \( f \in L \) by Berberian [2] 66.5. Conversely, if

\[
\int f \circ T \, du = \int f \, du
\]

for all \( f \in L \) then

\[
u(T(D)) = u(D)
\]

for all \( D \in \mathcal{B}^* \) (respectively \( \mathcal{B}^* \)) by Berberian [2] 66.6.

**Theorem 2.6** (Riesz-Markov) If \( A \) is a non-negative linear functional on \( L \), there is one and only one \( H \)-regular Borel* measure \( u \) on \( X \) such that \( A(f) = \int f \, du \) for all \( f \in L \) and, in virtue of Theorems 2.3 and 2.5, there is one and only one regular Borel measure \( \nu \) (which extends \( u \)) on \( X \) such that \( A(f) = \int f \, d\nu \) for all \( f \in L \).
**Definition 2.6** Let $G$ be a group of autohomeomorphisms of a locally compact Hausdorff space $X$. By a Haar integral on $X$ with respect to $G$ we mean a non-negative linear functional $A$ on $\mathcal{L}$ such that $A(f \circ g) = A(f)$ for all $f \in \mathcal{L}$ and $g \in G$, and such that $A$ is not identically 0.

**Definition 2.7** We say a Haar measure $\mu$ on $X$ with respect to "$\cong$" is unique if, given any other Haar measure $\nu$ on $X$ with respect to "$\cong$", there is a real number $c > 0$ such that $\mu(E) = c \nu(E)$ for all $E \in \mathcal{B}$.

Similarly, we say a Haar integral $A$ on $X$ with respect to a group $G$ of autohomeomorphisms of $X$ is unique if, given any other Haar integral $I$ on $X$ with respect to $G$, there is a real number $c > 0$ such that $A(f) = cI(f)$ for all $f \in \mathcal{L}$.

In virtue of the above discussion and the Riesz-Markov Theorem we have the following theorem.

**Theorem 2.7** Let $G$ be a group of autohomeomorphisms of a locally compact Hausdorff space $X$. Then there is a Haar measure on $X$ with respect to $G$ if and only if there is a Haar integral on $X$ with respect to $G$. Moreover, the Haar measure is unique if and only if the Haar integral is unique.

**Proof** (Existence) If $\mu$ is a Haar measure on $X$ with respect to $G$, then $A(f) = \int f \, d\mu$ is a non-negative linear functional on $\mathcal{L}$. It follows from the discussion
preceding Theorem 2.6 that \( A(f) = A(f \circ g) \) for all \( f \in \mathcal{L} \) and all \( g \in G \). If \( A \equiv 0 \) then \( A(f) = \int f d\nu \) where \( \nu \equiv 0 \) but this contradicts the Riesz-Markov Theorem. Thus \( A \) is a Haar integral. Conversely, if \( A \) is a Haar integral on \( X \) with respect to \( G \) and \( \mu \) is the corresponding regular Borel measure such that \( A(f) = \int f d\mu \) for all \( f \in \mathcal{L} \), then clearly \( \mu \not\equiv 0 \) since \( A \not\equiv 0 \). It then follows from the discussion preceding Theorem 2.6 that \( \mu \) is a Haar measure on \( X \) with respect to \( G \).

(Uniqueness) If \( A_1 \) and \( A_2 \) are two Haar integrals on \( X \) with respect to \( G \), let \( \mu_1 \) and \( \mu_2 \) be their corresponding Haar measures. If there is a real number \( c > 0 \) such that \( \mu_1 = c \mu_2 \), then \( A_1(f) = \int f d\mu_1 = c\int f d\mu_2 = cA_2(f) \) for all \( f \in \mathcal{L} \). Conversely, if \( \mu_1 \) and \( \mu_2 \) are Haar measures on \( X \) with respect to \( G \), then their integrals are Haar integrals. If there is a real number \( c > 0 \) such that \( \int f d\mu_1 = c\int f d\mu_2 \) (and certainly \( c\int f d\mu_2 = \int f d(c\mu_2) \)) for all \( f \in \mathcal{L} \) then the same equality holds for the restrictions of these measures to \( C^* \). Then, by Berberian [2] 66.4, \( \mu_1(C) = c\mu_2(C) \) for all compact sets \( C \). Hence it follows from regularity that \( \mu_1 = c\mu_2 \).

Thus we may establish the existence of a Haar measure and then prove it is unique by showing that a Haar integral on \( X \) is unique.
In the sequel, if there is no danger of confusion, we shall not mention the relation "$\approx$" or the group $G$ of autohomeomorphisms of $X$; we shall speak simply of a Haar measure or a Haar integral.

As a final note we observe that if $G$ is weakly transitive and $\mu$ is a Haar measure then $\mu(O) > 0$ for every non-empty open set $O$. 
Chapter 3

EXISTENCE OF HAAR MEASURE

In this chapter we establish the existence of a Haar measure in more general situations than those appearing in the literature. The method of attack was to establish the existence of a $G$-invariant content (which in turn generates a Haar measure) on a locally compact Hausdorff space $X$ where $G$ is a group of autohomeomorphisms of $X$. Then this result is expressed in terms of an abstract relation "$\cong$". However, in this paper, the result is first presented in terms of the abstract relation since the result in terms of a group of autohomeomorphisms will then follow readily.

**Theorem 3.1** Let $\{Y, \mathcal{O}\}$ be a topological space and "$\cong$" a relation defined between certain subsets of $Y$ such that $\emptyset \cong E \rightarrow \emptyset = E$. Let $\mathcal{A}$ be a class of subsets of $Y$ and $\mathcal{Q} = \{B_A : A \in \mathcal{A}\}$ a class of open subsets of $Y$ such that

a) $B_A = \emptyset$ if and only if $A^O = \emptyset$,
b) $B_A \subseteq A$ for all $A \in \mathcal{A}$.

Let $\mathcal{A}^A = \{A : A \in \mathcal{A} \text{ and } A^O \neq \emptyset\}$. A subset $E$ of $Y$ will be said to be totally bounded if for each $A \in \mathcal{A}^A$ there is an integer $n \geq 1$ and there exist sets $A_i \cong A$, $i = 1, \ldots, n$, 

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such that $E \subseteq \bigcup_{i=1}^{\infty} B_{A_i}$. $\mathcal{A}^*$ will denote the class of totally bounded subsets of $Y$. Assume that the following conditions are satisfied.

1) $A_1 \not\subseteq A_2 \in \mathcal{A} \quad \rightarrow \quad A_2 \not\subseteq A_1$ and $A_1 \in \mathcal{A}$.

2) If $A^* \not\subseteq A \in \mathcal{A} \cap \mathcal{A}^*$ then $[(A_i \not\subseteq A_0 \in \mathcal{A}^+, i = 1, \ldots, n, \text{ and } A \subseteq \bigcup_{i=1}^{\infty} B_{A_i}) \rightarrow \text{(there are sets } A_i^* \not\subseteq A_0, i = 1, \ldots, n, \text{ such that } A^* \subseteq \bigcup_{i=1}^{\infty} B_{A_i^*})].$ Hence $A^* \in \mathcal{A} \cap \mathcal{A}^*$.

3) There is a point $y_0 \in Y$ such that the set $\mathcal{A}^+ = \{A: y_0 \in A^+ \text{ and } A \in \mathcal{A}^+\}$ is directed downward by set inclusion; i.e., for each $A_1, A_2 \in \mathcal{A}^+$, there is an $A_3 \in \mathcal{A}^+$ such that $A_3 \subseteq A_1 \cap A_2$.

4) If $E_1$ and $E_2$ are totally bounded sets whose closures are disjoint and compact, then there is an $A_0 \in \mathcal{A}^+$ such that $(A_0 \not\supseteq A \in \mathcal{A}^+ \text{ and } A_1 \not\supseteq A) \rightarrow (B_{A_1} \cap E_1 = \emptyset \text{ or } B_{A_1} \cap E_2 = \emptyset)].$

5) There is a set $A_0 \in \mathcal{A}^+ \cap \mathcal{A}^*$.

Then there is a non-negative real valued function $\lambda$ on $\mathcal{A}^*$ such that for all totally bounded sets $E_1, E_2$ the following conditions are satisfied.

1) $\lambda(E_1 \cup E_2) \leq \lambda(E_1) + \lambda(E_2)$.

2) If the closures of $E_1$ and $E_2$ are compact and disjoint then $\lambda(E_1 \cup E_2) = \lambda(E_1) + \lambda(E_2)$.

3) If $E_1 \subseteq E_2$ then $\lambda(E_1) \leq \lambda(E_2)$. 
4) If $E_1 \equiv E_2$ and $E_2 \in \mathcal{A} \cap \mathcal{A}^*$, then $\lambda(E_1) = \lambda(E_2)$.
5) If $E_1 \in \mathcal{A} \cap \mathcal{A}^*$, then $\lambda(E_1) > 0$.

**Proof** For each non-empty $E \in \mathcal{A}^*$ and $A \in \mathcal{A}^4$, define
t$(E, A) = \text{GLB}\{n : E \subseteq \bigcup_{i=1}^{n} B_{A_i}, A_i \equiv A\}$ and set $t(\emptyset, A) = 0$.

Then for $E_1, E_2 \in \mathcal{A}^*$ and $A \in \mathcal{A}^4$, we have

1*) $0 \leq t(E_1, A) < \infty$ and $0 < t(E_1, A)$ when $E_1 \neq \emptyset$,
2*) $E_1 \subseteq E_2 \implies t(E_1, A) \leq t(E_2, A)$,
3*) $t(E_1 \cup E_2, A) \leq t(E_1, A) + t(E_2, A)$,
4*) $E_1 \equiv E_2 \in \mathcal{A} \cap \mathcal{A}^* \implies t(E_1, A) = t(E_2, A)$,
5*) $A_0 \in \mathcal{A}^4 \cap \mathcal{A}^* \implies t(E_1, A) \leq t(E_1, A_0) + t(A_0, A)$,
6*) if the closures of $E_1$ and $E_2$ are compact and disjoint, then there is an $A_0 \in \mathcal{A}^\circ$ such that

$$t(E_1 \cup E_2, A) = t(E_1, A) + t(E_2, A).$$

All of the above are obvious if either $E_1$ or $E_2$ is the empty set. Otherwise (1*), (2*), and (3*) are obvious from the definition of $t(E, A)$, and (4*) follows from assumptions (1) and (2).

Let $A_0 \in \mathcal{A}^4 \cap \mathcal{A}^*$. If $A_0 \subseteq \bigcup_{i=1}^{n} B_{A_i}, A_i \equiv A$ and $E_1 \subseteq \bigcup_{j=1}^{r} B_{A_j}^*, A_j^* \equiv A_0$, then by assumption (2) there are sets $A_j\equiv A$ such that $B_{A_j}^* \subseteq A_j^* \subseteq \bigcup_{i=1}^{n} B_{A_i}$. Then

$$E_1 \subseteq \bigcup_{i=1}^{n} B_{A_i}. \quad \text{This establishes (5*).}$$

If $E_1$ and $E_2$ have compact disjoint closures, then there is an $A_0 \in \mathcal{A}^\circ$ with the properties of assumption (4). Let $A_0 \equiv A \in \mathcal{A}^\circ$. If $n = t(E_1 \cup E_2, A)$, we have sets $A_i \equiv A, i = 1, \ldots, n$ such that
\[
E_1 \cup E_2 \subseteq \bigcup_{i=1}^{n} B_{\alpha_i} = \bigcup_{i=1}^{n} B_{\alpha_i} \bigcup \bigcup_{i=1}^{n} B_{\alpha_i}.
\]
Since \(B_{\alpha_i} \cap E_1 = \emptyset\) or \(B_{\alpha_i} \cap E_2 = \emptyset\) for \(i = 1, \ldots, n\) we have \(t(E_1 \cup E_2, A) = n \geq t(E_1, A) + t(E_2, A)\). Together with \((3*)\) this establishes \((6*)\).

Fix \(A_0 \in \alpha^1 \cap \alpha^*\). Now, for each \(A \in \alpha^\circ\), define \(t_A\) on \(\alpha^*\) by

\[
t_A(E) = t(E, A)/t(A_0, A).
\]
Then for \(E_1, E_2 \in \alpha^*\) and \(A \in \alpha^\circ\),

1#) \(E_1 \subseteq E_2 \implies t_A(E_1) \leq t_A(E_2)\),

2#) \(t_A(E_1 \cup E_2) \leq t_A(E_1) + t_A(E_2)\),

3#) \(E_1 \subseteq E_2 \in \alpha \cap \alpha^* \implies t_A(E_1) = t_A(E_2)\),

4#) \(0 \leq t_A(E_1) \leq t(E_1, A_0) < \infty\) and if \(E \in \alpha^1 \cap \alpha^*\) then \(t_A(E) \geq 1/t(A_0, E)\),

5#) if the closures of \(E_1\) and \(E_2\) are compact and disjoint, then there is an \(A^* \in \alpha^\circ\) such that \([A^* \ni A \in \alpha^\circ \implies t_A(E_1 \cup E_2) = t_A(E_1) + t_A(E_2)\)].

These follow immediately from \((1*) - (6*)\) listed above.

Let \(\Phi = \prod_{E \in \alpha^*} [\delta_E, t(E, A_0)]\)

where

\[
\delta_E = \begin{cases} 
1/t(A_0, E) & \text{if } E \in \alpha^1 \\
0 & \text{if } E \notin \alpha^1.
\end{cases}
\]

By the Tychonoff Theorem, \(\Phi\) with the product topology is a compact Hausdorff space. Clearly \(t_A \in \Phi\) for each \(A \in \alpha^\circ\). For each \(A^* \in \alpha^\circ\) define

\[
\Lambda(A^*) = \{t_A : A^* \ni A \in \alpha^\circ\}.
\]
If \( A_1, \ldots, A_n \in \mathcal{A}^o \) then, by assumption (3), there is a set \( A^* \in \mathcal{A}^o \) such that \( A^* \subseteq \bigcap_{i=1}^n A_i \). Thus

\[
\bigcap_{i=1}^n c \bigcap_{i=1}^n (A_i) \supseteq c \bigcap_{i=1}^n (A_i) \supseteq c \bigcap (A^*) \neq \emptyset;
\]
i.e., the class \( \{c \bigcap (A) : A \in \mathcal{A}^o\} \) has the finite intersection property. Then, since \( \prod \) is compact, there exists a \( \lambda \in \bigcap_{A \in \mathcal{A}^o} c \bigcap (A) \). Clearly \( 0 \leq \lambda(B) \leq t(B, A_0) < \infty \) for all \( B \in \mathcal{A}^o \) and if \( B \in \mathcal{A}^o \cap \mathcal{A}^* \), then \( 0 < 1/t(A_0, E) \leq \lambda(B) \). This establishes (5).

For each \( E \in \mathcal{A}^* \), the real valued function \( f_E \) defined on \( \prod \) by

\[
f_E(\emptyset) = \emptyset(E)
\]
for all \( \emptyset \in \prod \) is continuous since it is a projection.

For \( E_1, E_2 \in \mathcal{A}^* \), the set

\[
\Delta(E_1, E_2) = \{\emptyset : \emptyset \in \prod, \emptyset(E_1) = f_{E_1}(\emptyset) \leq f_{E_2}(\emptyset) = \emptyset(E_2)\}
\]
is closed since \( f_{E_1} \) and \( f_{E_2} \) are continuous. If \( E_1 \subseteq E_2 \), then

\[
\Delta(E_1, E_2) \supseteq \{t_A : A \in \mathcal{A}^o\}
\]
and so

\[
\Delta(E_1, E_2) \supseteq c \{t_A : A \in \mathcal{A}^o\}.
\]
Thus \( \lambda \in \Delta(E_1, E_2) \) and it follows that \( \lambda(E_1) \leq \lambda(E_2) \). This establishes (3). In a similar fashion we verify that \( \lambda \) satisfies (1) and (4).

If the closures of \( E_1 \) and \( E_2 \) are compact and disjoint then there is an \( A^* \in \mathcal{A}^o \) such that \( \{A^* \supseteq A \in \mathcal{A}^o\} \rightarrow \)}
\[ t_A(E_1 \cup E_2) = t_A(E_1) + t_A(E_2) \]. Reasoning as above the set

\[ K(E_1, E_2) = \{ \emptyset: \emptyset \in \mathcal{F} \text{ and } \emptyset(E_1 \cup E_2) = \emptyset(E_1) + \emptyset(E_2) \} \]

is a closed set. It then follows that

\[ K(E_1, E_2) \supseteq c \bigwedge (A^*) \].

Thus \( \lambda \in K(E_1, E_2) \); i.e., \( \lambda(E_1 \cup E_2) = \lambda(E_1) + \lambda(E_2) \).

This completes the proof of the theorem.

In the following corollary we see that a Haar measure can be constructed from a "content" on the class of compact open sets.

**Corollary 3.1.1** (Banach [1]) Let \( X \) be a non-empty locally compact metric space and let \( \mathcal{A} \) be the class of non-empty compact open sets. Let " \( \preceq \) " be a relation defined between certain subsets of \( X \) such that \( \emptyset \preceq E \to \emptyset = E \) and such that the following conditions are satisfied.

1) " \( \preceq \) " is symmetric and transitive.
2) \( B \preceq A \in \mathcal{A} \to B \in \mathcal{A} \).
3) \( A \preceq B \) and \( A \subseteq \bigcup \mathcal{A}_n, \mathcal{A}_n \in \mathcal{A} \) (finite or infinite sequence) \( \to \) for each \( n \) there exists a \( B_n \preceq \mathcal{A}_n \) such that \( B \subseteq \bigcup B_n \).
4) \( A \in \mathcal{A} \to X \subseteq \bigcup_{B \in \mathcal{A}} B \).
5) If \( \{ S_n \} \) is a sequence of concentric compact open spheres with radii tending to 0 and \( G_n \preceq S_n \), then \( [a_n, b_n \in G_n, a = \lim_{n \to \infty} a_n, \text{ and } b = \lim_{n \to \infty} b_n] \to a = b \).
Then there is an outer measure \( u \) on \( X \) such that

1) \( A \in \mathcal{A} \implies 0 < u(A) < \infty \). Hence \( u(C) < \infty \) for each compact set \( C \) and \( u(0) > 0 \) for each non-empty open set \( 0 \).

2) \( E_1 \supseteq E_2 \implies u(E_1) = u(E_2) \).

3) \( u(A) = \text{GLB} \{ u(O) : A \subseteq O \text{ open} \} \) for each \( A \subseteq X \).

4) \( d(A, B) > 0 \implies u(A \cup B) = u(A) + u(B) \).

Hence open sets are measurable and the restriction of \( u \) to \( \mathcal{B}^* \) is a (Halmos) Haar measure.

**Proof** In the notation of Theorem 3.1 let \( B_A = A \) for each \( A \in \mathcal{A} \). Condition (1) is assumed, (3) holds for any \( x_0 \in X \), and (5) follows from assumption (4).

Note that \( \mathcal{A} = \mathcal{A}^1 \subseteq \mathcal{A}^* \). If \( A^* \nsubseteq A \in \mathcal{A} \), \( A \subseteq \bigcup_{i} A_i \), and \( A_0 \nsubseteq A_1 \in \mathcal{A} \), then by assumption (3) there exist \( A_i \nsubseteq A_i \nsubseteq A_0 \) for \( i = 1, \ldots, n \) such that \( A^* \subseteq \bigcup_{i} A_i \). This establishes condition (2) of Theorem 3.1.

If condition (4) does not hold, fix \( y_0 \in X \) and let \( S_n = S(y_0, 1/n) \) for each \( n \) large enough so that \( S_n \in \mathcal{A} \). There exist sets \( E_1, E_2 \) with compact disjoint closures such that for each \( S_n \) there is an \( A_n \in \mathcal{A} \) and an \( A^*_n \nsubseteq A_n \) with \( y_0 \in A_n \subseteq S_n \) and \( A^*_n \bigcap E_1 \neq \emptyset \neq A^*_n \bigcap E_2 \). By assumption (3) there is a \( G_n \nsubseteq S_n \) such that \( G_n \supseteq A^*_n \). There exist points \( a_n \in G_n \bigcap E_1 \), \( b_n \in G_n \bigcap E_2 \). Then since the \( E_i \) have compact closures, there is a subsequence \( \{ n_i \} \) of natural numbers, an \( a \in \mathcal{C} \), and a \( b \in \mathcal{C} E_2 \) such that \( a = \lim_{i \to \infty} a_{n_i} \) and
$$b = \lim_{n \to \infty} b_n.$$ Then by assumption (5) \(a = b\) which yields the contradiction \(CE_1 \cap CE_2 \neq \emptyset\). Thus condition (4) holds.

Hence we have the function \(\lambda\) of Theorem 3.1. Clearly \(\lambda(\emptyset) = 0\). For \(E \subseteq X\) we define

$$u(E) = \begin{cases} \sum_{n} \lambda(A_n); E \subseteq \bigcup_{n} A_n, A_n \in \mathcal{O} \cup \{\emptyset\} & \text{if there is a sequence } A_n \in \mathcal{O} \text{ such that } E \subseteq \bigcup_{n} A_n \\ \infty & \text{otherwise.} \end{cases}$$

It is easily verified that \(u\) is the desired outer measure. See Banach's paper [1] for a proof.

**Definition 3.1** Let \(G\) be a group of autohomeomorphisms of a topological space \(Y\). We shall say that Condition A is satisfied if for each pair \(B, C\) of disjoint compact sets in \(Y\) there is a non-empty open set \(O\) such that \([g(O) \cap C = \emptyset\) or \(g(O) \cap B = \emptyset]\) for all \(g \in G\).

**Theorem 3.2** Let \(G\) be a weakly transitive group of autohomeomorphisms of a locally compact Hausdorff space \(X\). If Condition A is satisfied then there exists a content \(\lambda\) on \(X\) such that

1) \(\lambda(\pi(C)) = \lambda(C)\) for all \(g \in G\) and all compact \(C\),

2) \(C \not\in \emptyset \Rightarrow \lambda(C) > 0\).

**Proof** In the notation of Theorem 3.1 let \(A \subseteq B\) if and only if there is a \(g \in G\) such that \(g(A) = B\), let \(\mathcal{C}\) be the class of compact sets, and let \(B_A = A^0\). Clearly
\( \mathcal{A} \subseteq \mathcal{A}^* \) since \( G \) is weakly transitive. Conditions (1) and (5) of Theorem 3.1 are obviously satisfied.

If \( g(A) = A^* \in \mathcal{A}, A_0 \in \mathcal{A}^1, A \subseteq \bigcup_{i \in I} A_i^0, \) and \( A_i = g_i(A_0) \) for some \( g_i \in G \), then \( A^* = g(A) \subseteq \bigcup_{i \in I} g_i(A_i^0) = \bigcup_{i \in I} (g_i(A_i))^0 \) and \( g(A_i) = g_i(A_0) \). This establishes condition (2).

Fix an arbitrary point \( y_0 \in X \). If \( y_0 \in A_1^0 \cap A_2^0 \) where \( A_1, A_2 \in \mathcal{A} \), then \( y_0 \in (A_1 \cap A_2)^0 \) and \( A_1 \cap A_2 \in \mathcal{A} \). Thus condition (3) is satisfied.

If \( E_1, E_2 \) are sets with compact disjoint closures, then by Condition A, there is a non-empty open set \( O \) such that

\[ * \quad [g(O) \cap c E_1 = \emptyset \text{ or } g(O) \cap c E_2 = \emptyset] \] for all \( g \in G \).

Since \( X \) is locally compact and Hausdorff there is a non-empty open set \( O^* \) such that \( c O^* \) is compact and \( c O^* \subseteq O \); thus we may rewrite (*) with \( c O^* \) in place of \( O \). By weak transitivity there is a \( g_o \in G \) such that \( y_0 \in g_o(c O^*) \). Since \( c(g_o(O^*)) = g_o(c O^*) \in \mathcal{A} \) we see that condition (4) is likewise satisfied.

The function \( \lambda \) of Theorem 3.1 restricted to \( \mathcal{A} \) is clearly the desired content.

**Corollary 3.2.1** If \( G \) is a weakly transitive group of autohomeomorphisms of a non-empty locally compact Hausdorff space \( X \) and Condition A is satisfied, then there is a Haar measure on \( X \).
Proof Let \( \lambda \) be the content of Theorem 3.2 and \( u \) the outer measure constructed from \( \lambda \) by the method of Theorem 2.1. The restriction of \( u \) to \( \mathcal{B} \) is a regular Borel measure which is invariant with respect to \( G \). (See Theorems 2.1 and 2.2).

If \( C \) is compact and \( O \) is an open set containing \( C \) then \( u_O(O) = \lambda(C) \). Thus \( u(C) \geq \lambda(C) > 0 \) when \( C^0 \neq \emptyset \). Hence \( u \) is a Haar measure.

Theorem 3.3 Let \( G \) be a group of autohomeomorphisms of a locally compact Hausdorff space \( X \neq \emptyset \). If \( X_0 \) is a non-empty closed subspace of \( X \) such that \( g(X_0) = X_0 \) for each \( g \in G \) and if \( u^* \) is a Haar measure on \( X_0 \), then \( u^* \) can be extended to a Haar measure on \( X \) by setting

\[
    u(E) = u^*(E \cap X_0)
\]

for all \( E \in \mathcal{B}(X) \).

Proof \( \mathcal{B}(X) \supseteq \mathcal{B}(X_0) \) since each set which is closed in \( X_0 \) is also closed in \( X \). We must show that \( E \in \mathcal{B}(X) \rightarrow E \cap X_0 \in \mathcal{B}(X_0) \). \( C \) closed in \( X \) \( \rightarrow \) \( C \cap X_0 \) is closed in \( X_0 \); clearly the class of sets \( M \) such that \( M \cap X_0 \in \mathcal{B}(X_0) \) is a \( \sigma \)-algebra. Thus \( u \) is indeed an extension of \( u^* \).

For every open set \( O \),

\[
    u(O) = u^*(O \cap X_0) = \text{LUB}\{u^*(C):O \cap X_0 \supseteq C \text{ compact}\} = \text{LUB}\{u(C):O \supseteq C \text{ compact}\} = \text{LUB}\{u(C):O \supseteq C \text{ compact}\}.
\]
For an arbitrary $A \in \mathcal{B}(X)$,

$$u(A) = u^*(A \cap X_0)$$

$$= \text{GLB}\{u^*(O \cap X_0) : O \cap X_0 \supseteq A \cap X_0, \ O \ \text{open}\}$$

$$= \text{GLB}\{u(O) : O \cap X_0 \supseteq A \cap X_0, \ O \ \text{open}\}$$

$$= \text{GLB}\{u(O \cup X_0) : O \supseteq A \cap X_0, \ O \ \text{open}\}$$

$$= \text{GLB}\{u(O^*): O^* \supseteq A, \ O^* \ \text{open}\}.$$ 

Hence $u$ is a regular Borel measure.

$u$ is non-trivial since $u^*$ is non-trivial; also for any $E \in \mathcal{B}(X)$, and any $g \in G$, 

$$u(g(E)) = u^*(g(E) \cap X_0)$$

$$= u^*(g(E) \cap g(X_0))$$

$$= u^*(g(E \cap X_0))$$

$$= u^*(E \cap X_0)$$

$$= u(E).$$

Hence $u$ is indeed Haar measure on $X$.

Definition 3.2 Let $G$ be a group of autohomeomorphisms of a topological space $Y$. For each $y \in Y$, $G(y)$ will denote the set $\{g(y) : g \in G\}$. This set will be called the orbit of $y$ under $G$.

If $G$ is a group of autohomeomorphisms of a topological space $Y$ and $X_0$ is a non-empty closed subspace of $Y$ such that $g(X_0) = X_0$ for all $g \in G$, then $G$ is a group of autohomeomorphisms of $X_0$ also. If $G$ is weakly transitive on $Y$ then $Y = X_0$; in particular, if $G$ is weakly transitive
on \( Y \), then \( cG(y) = Y \) for all \( y \in Y \) since \( gcG(y) = cgG(y) \) (= \( cG(y) \)) for all \( y \in Y \), \( g \in G \). Thus in the following theorem we may replace "\( cG(x) \)" with "an invariant closed subspace \( X_0 \neq \emptyset \)."

The following theorem is the most general theorem which will be proved in this paper regarding the existence of Haar measure. It will be seen later that Condition A is not necessary for the existence of Haar measure.

**Theorem 3.4** Let \( G \) be a group of autohomeomorphisms of a locally compact Hausdorff space \( X \). Assume there is an \( x \in X \) such that \( G \) is weakly transitive on \( cG(x) \). If Condition A holds on \( cG(x) \) then there exists a Haar measure \( u \) on \( X \) and the restriction of \( u \) to \( \bigotimes (cG(x)) \) is a Haar measure on \( cG(x) \).

**Proof** Clearly \( cG(x) \) is a locally compact Hausdorff space and \( gcG(x) = cG(x) \) for all \( g \in G \). The conclusion then follows from Corollary 3.2.1 and Theorem 3.3.

**Definition 3.3** Let \( X \) be an abstract space. We say that a non-empty family \( \mathcal{U} \) of subsets of \( X \times X \) is a uniformity for \( X \) if

1) \( U \in \mathcal{U} \rightarrow \{(x,x) : x \in X\} \subseteq U \),
2) \( U \in \mathcal{U} \rightarrow U^{-1} \in \mathcal{U} \) where \( U^{-1} = \{(y,x) : (x,y) \in U\} \),
3) \( U \in \mathcal{U} \rightarrow \) there is a \( V \in \mathcal{U} \) such that \( V \circ V \subseteq U \) where \( V \circ V = \{(x,y) : \) there is a \( z \in X \) such that \((x,z) \in V \) and \((z,y) \in V\} \),
4) \( u, v \in \mathcal{U} \rightarrow u \cap v \in \mathcal{U} \),
5) \( u \in \mathcal{U}, \ u \subseteq v \subseteq X \times X \rightarrow v \in \mathcal{U} \).

If \( \mathcal{U} \) is a uniformity for \( X \), then \((X, \mathcal{U})\) will be called a uniform space.

**Definition 3.4** If \((X, \mathcal{U})\) is a uniform space, then for each \( U \in \mathcal{U} \) and \( x \in X \) we define

\[
U[x] = \{ y : (x, y) \in U \}.
\]

It is easily verified that the family \( \mathcal{T} \) of subsets \( T \) of \( X \) such that \( [x \in T \rightarrow \text{there is a } U \in \mathcal{U} \text{ for which } U[x] \subseteq T] \) is a topology for \( X \). This topology will be called the uniform topology. If \( X \) is given the uniform topology then \( x \in U[x]^0 \) for each \( x \in X \) and each \( U \in \mathcal{U} \). See Kelley [9] p. 178-179 for a proof of this fact.

**Definition 3.5** Let \( F \) be a family of functions from a topological space \( X \) into a uniform space \((Y, \mathcal{U})\). We say that \( F \) is equicontinuous at \( x \in X \) provided \( U \in \mathcal{U} \rightarrow \text{there is an open neighborhood } O \text{ of } x \text{ such that } f(O) \text{ is contained in } U[f(x)] \) for every \( f \in F \). If \( F \) is equicontinuous at each \( x \in X \), we shall simply say that \( F \) is equicontinuous.

We are now in a position to generalize a result of I. E. Segal. First we establish the following lemmas.
Lemma 3.5.1 Let \((X, \mathcal{U})\) be a uniform space. If \(A\) is compact, \(B\) is closed (in the uniform topology), and \(A \cap B = \emptyset\), then there exists a \(V \in \mathcal{U}\) such that \(V = V^{-1}\) and \([V[x] \cap A = \emptyset \text{ or } V[x] \cap B = \emptyset]\) for all \(x \in X\).

Proof By Gaal [6] p. 140 there is a \(U \in \mathcal{U}\) such that \((a,b) \in A \times B \implies (a,b) \notin U\). There is a \(V \in \mathcal{U}\) such that \(V = V^{-1}\) and \(V \cdot V \subseteq U\). If \(V[x] \cap A \neq \emptyset \neq V[x] \cap B\) for some \(x \in X\), then there is a pair \((a,b) \in A \times B\) such that \((x,a) \in V\) and \((x,b) \in V\). But \(V = V^{-1}\) and hence \((a,b) \in V \cdot V \subseteq U\). But this is a contradiction.

Lemma 3.5.2 Let \(G\) be a group of autohomeomorphisms of a topological space \(X\). Then \(G\) is weakly transitive if and only if the orbit of every point is dense in \(X\).

Proof Assume that there is an \(x_0 \in X\) such that \(cG(x_0) \neq X\). Let \(O = X - cG(x_0)\). By weak transitivity, there is a \(g \in G\) such that \(x_0 \in g(O)\). But then \(g^{-1}(x_0) \in O\). This is a contradiction.

Conversely, if \(G\) is not weakly transitive, then there is a non-empty open set \(O\) such that \(X \notin \bigcup_{g \in G} g(O) = O^*\). Clearly \(g(O^*) = O^*\) for all \(g \in G\); hence \(g(X - O^*) = g(X) - g(O^*) = X - O^*\) for all \(g \in G\). Fix \(x \in X - O^*\). Then \(cG(x) \subseteq X - O^*\); i.e., the orbit of \(x\) is not dense in \(X\).
Lemma 3.5.3 Let $G$ be a group of autohomeomorphisms of a topological space $X$ whose topology is introduced by a uniformity $\mathcal{U}$. If $G$ is equicontinuous and there is an $x_0 \in X$ whose orbit is dense, then $G$ is weakly transitive.

Proof Let the orbit of $x_0$ be dense in $X$ and assume that $G$ is not weakly transitive. Then there is an $x \in X$ and a non-empty open set $O$ such that $x \notin \bigcup_{g \in G} g(O)$. Since $cG(x_0) = X$, there is a $g_0 \in G$ such that $g_0(x_0) \in O$. Then $x_0 \in g_0^{-1}(O)$ and $\bigcup_{g \in G} g(O) = \bigcup_{g \in G} g g_0^{-1}(O)$. Hence we may assume $x_0 \in O$.

There is a $U \in \mathcal{U}$ such that $U[x_0] \subseteq O$. By Kelley [9] 6.19 there is a pseudo-metric $d$ on $X$ such that for each $r^* > 0$, $V_{d,r^*} = \{(x,y) : d(x,y) < r^*\}$ is a member of $\mathcal{U}$ and for some $r > 0$, $V_{d,r} \subseteq U$. Hence $V_{d,r}[x_0] \subseteq O$. By equicontinuity there is an open neighborhood $O^*$ of $x$ such that

1) $g(O^*) \subseteq V_{d,r}[g(x)]$ for all $g \in G$.

Clearly $G(x) \bigcap O = \emptyset$ and hence

2) $G(x) \bigcap V_{d,r}[x_0] = \emptyset$.

In virtue of (1) and (2),

$(\bigcup_{g \in G} g(O^*))) \bigcap V_{d,r/2}[x_0] = \emptyset$.

Thus $x_0 \notin \bigcup_{g \in G} g(O^*)$; then $G(x_0) \bigcap (\bigcup_{g \in G} g(O^*)) = \emptyset$ and it follows that

$cG(x_0) \bigcap (\bigcup_{g \in G} g(O^*)) = \emptyset$.

This contradicts the density of the orbit of $x_0$. Thus $G$ is weakly transitive.
Theorem 3.5  Let $G$ be a group of autohomeomorphisms of a non-empty locally compact Hausdorff space $X$ whose topology is introduced by a uniformity $\mathcal{U}$. If $G$ is equicontinuous, then there is a Haar measure $\mu$ on $X$.

Proof  Fix $x_0 \in X$ and let $B = cG(x_0)$ be given the relative topology. Clearly $G$ is a group of autohomeomorphisms of the locally compact Hausdorff space $B$. By Kelley [9] p. 182, $\mathcal{U}^* = \{u \cap B \times B : u \in \mathcal{U}\}$ is a uniformity for $B$ which generates the relative topology. Clearly $G$ is equicontinuous on $B$. If $c^*$ is the closure operator in $B$, then $c^*G(x_0) = cG(x_0) \cap B = B$. Thus the orbit of $x_0$ is dense in $B$ and by Lemma 3.5.3, $G$ is weakly transitive on $B$.

Let $C,D$ be disjoint compact sets in $B$. By Lemma 3.5.1 there is a $V \in \mathcal{U}^*$ such that $[V[x] \cap C = \emptyset$ or $V[x] \cap D = \emptyset]$ for all $x \in B$. By equicontinuity on $B$ there is an open neighborhood $O$ of $x_0$ in $B$ such that $g(O) \subseteq V[g(x_0)]$ for all $g \in G$. Thus $[g(O) \cap C = \emptyset$ or $g(O) \cap D = \emptyset]$ for all $g \in G$. Then the conclusion follows from Theorem 3.4.

Lemma 3.5.4  Let $G$ be a group of autohomeomorphisms of a topological space $X$ whose topology is generated by a uniformity $\mathcal{U}$. If $G$ is weakly transitive on $X$ and equicontinuous at some $x_0 \in X$, then $G$ is equicontinuous at each $x \in X$.
Proof. Fix \( U \in \mathcal{U} \); there is a \( V \in \mathcal{U} \) such that \( V = V^{-1} \) and \( V \cdot V \subseteq U \). There exists an open neighborhood 0 of \( x_0 \) such that \( g(0) \subseteq V[g(x_0)] \) for all \( g \in G \).

Fix \( x \in X \); there is a \( g_0 \in G \) such that \( x \in g_0(0) = 0^* \). Then \( g(0^*) \subseteq V[g_0(x_0)] \) for all \( g \in G \) so that \( (g(x), g_0(x_0)) \in V^{-1} = V \) for all \( g \in G \). Hence \( y \in 0^* \rightarrow (g(x), g(y)) \in V \cdot V \subseteq U \); i.e., \( g(0^*) \subseteq U[g(x)] \) for all \( g \in G \).

The following corollaries may be too complicated to be of any importance; however, we shall list them anyhow.

The first is an immediate consequence of the proof of Theorem 3.5 and the other two follow from Lemma 3.5.4.

**Corollary 3.5.1** Let \( G \) be a group of autohomeomorphisms of a locally compact Hausdorff space \( X \). Assume there is an \( x_0 \in X \) such that \( cG(x_0) \) with relative topology has a uniformity \( \mathcal{U} \) with respect to which \( G \) is equicontinuous (on \( cG(x_0) \)). Then there is a Haar measure on \( X \).

**Corollary 3.5.2** Let \( G \) be a weakly transitive group of autohomeomorphisms of a locally compact Hausdorff space \( X \) whose topology is introduced by a uniformity \( \mathcal{U} \). If there is an \( x_0 \in X \) such that \( G \) is equicontinuous at \( x_0 \), then there is a Haar measure on \( X \).

**Corollary 3.5.3** Let \( G \) be a group of autohomeomorphisms of a locally compact Hausdorff space \( X \). Assume there is an \( x_0 \in X \) such that \( G \) is weakly transitive on
Also assume that there is an \( x \in cG(x_0) \) and a uniformity \( U^* \) on \( cG(x_0) \) which generates the relative topology such that \( G \) is equicontinuous at \( x \). Then there is a Haar measure on \( X \).

Next we want to show that the Haar measure in Theorem 3.5 is unique if \( G \) is weakly transitive. The following concept is useful in proving this result.

**Definition 3.6** Let \( f \) be a function from a uniform space \((X, \mathcal{U})\) to a uniform space \((Y, \mathcal{V})\). \( f \) is said to be uniformly continuous if for each \( V \in \mathcal{V} \) there is a \( U \in \mathcal{U} \) such that \((x, y) \in U \implies (f(x), f(y)) \in V\).

**Lemma 3.6.1** Let \((X, \mathcal{U})\) be a uniform space whose uniform topology is locally compact and Hausdorff. Then each \( f \in L \) is uniformly continuous.


**Lemma 3.6.2** Let \( G \) be an equicontinuous group of autohomeomorphisms of a topological space \( X \) whose topology is introduced by a uniformity \( \mathcal{U} \). For each \( V \in \mathcal{U} \) and \( x_0 \in X \), there is a continuous, non-negative real valued function \( f \) such that

1) \( f(x_0) \neq 0 \),
2) \( f(X - V[x_0]) = 0 \),
3) \( f \circ g(x_0) = f \circ g^{-1}(x_0) \) for all \( g \in G \).
Proof  Fix $V \in \mathcal{U}$ and $x_0 \in X$. There is a pseudometric $d$ on $X$ such that $V_{d,r^*} = \{(x,y): d(x,y) < r^*\} \in \mathcal{U}$ for all $r^* > 0$ and there is an $r > 0$ such that $V \supseteq V_{d,r}$ (Kelley [9] 6.19).

For each $x,y \in X$ define
\[ h^*(x,y) = \text{LUB} \{d(g(x),g(y)): g \in G\} \]
and
\[ h(x,y) = \frac{r, h^*(x,y)}{r} . \]
Clearly $h$ is a pseudometric on $X$ and
\[ h(x,y) = h(g(x),g(y)) \]
for all $x,y \in X$ and all $g \in G$. Moreover $h(x,x_0)$ is a continuous function of $x$. For if we fix $x \in X$ and $\varepsilon$ such that $0 < \varepsilon < r$, then there is an open neighborhood $\mathcal{O}$ of $x$ such that $g(\mathcal{O}) \subseteq V_{d,\varepsilon}[g(x)]$ for all $g \in G$. Thus
\[ y \in \mathcal{O} \mapsto |h(y,x_0) - h(x,x_0)| \leq h(x,y) \leq \varepsilon ; \]
this establishes the continuity of $h( ,x_0)$.

Define $f(x) = r - h(x,x_0)$ for all $x \in X$. Then $f$ is a continuous non-negative real valued function. Moreover,
\[ f(x_0) = r > 0 \]
and
\[ x \notin V[x_0] \longrightarrow x \notin V_{d,r}[x_0] \longrightarrow d(x,x_0) \geq r \]
\[ \longrightarrow h(x,x_0) = r \longrightarrow f(x) = 0 . \]

Also
\[ f(g(x_0)) = r - h(g(x_0),x_0) = r - h(x_0,g^{-1}(x_0)) \]
\[ = f(g^{-1}(x_0)) \]
for each $g \in G$. 
**Lemma 3.6.3** If $G$ is a weakly transitive group of autohomeomorphisms of a locally compact Hausdorff space $X$ and $f_1, f_2$ are non-negative functions in $\mathcal{L}$ such that $f_2$ is not the function identically 0, then there exist $g_1, \ldots, g_n \in G$ and positive real numbers $c_1, \ldots, c_n$ such that
\[
f_1(x) \leq \sum_{i=1}^{n} c_i f_2 \circ g_i(x)
\]
for all $x \in X$. Hence if $I$ is a Haar integral on $X$ we have $I(f_2) > 0$.

**Proof** Let $C$ be a compact support for $f_1$. There is an $\eta > 0$ and a non-empty open set $U$ such that $f_2(U) \geq \eta$ and there is a finite collection $g_1, \ldots, g_n \in G$ such that $C \subseteq \bigcup_{i=1}^{n} g_i(U)$. Let $M = \text{LUB} \{f_1(x) : x \in X\}$ and $c_i = M/\eta$ for $i = 1, \ldots, n$. If $x \in C$ there is an $i$ such that $x \in g_i(U)$. Then $g_i^{-1}(x) \in U$ and hence $f_2 \circ g_i^{-1}(x) \geq \eta$, so that $c_i f_2 \circ g_i^{-1}(x) \geq M \geq f_1(x)$.

**Theorem 3.6** Let $G$ be an equicontinuous group of autohomeomorphisms of a non-empty locally compact Hausdorff space $X$ whose topology is generated by a uniformity $U$. Then Haar measure on $X$ is unique if and only if $G$ is weakly transitive.

**Proof** Assume $G$ is not weakly transitive on $X$. By Lemma 3.5.3, the orbit of no point is dense in $X$. Fix $x_0 \in X$ and let $X_0 = cG(x_0) \neq X$; fix $x_1 \in X - X_0$ and let $X_1 = cG(x_1) \neq X$. Each $X_i$ is a locally compact Hausdorff
space whose topology is introduced by the uniformity
\[ \mathcal{U}_i^* = \{ U \cap X_i \times X_i : U \in \mathcal{U} \}. \]
It is easily seen that \( G \) is an equicontinuous group of autohomeomorphisms of each \( X_i \); hence there exist Haar measures \( u_i^* \) on \( X_i \) which can be extended to Haar measures
\[ u_i(E) = u_i^*(E \cap X_i) \]
on \( X \). There is a compact open neighborhood \( O \) of \( x^1 \) such that \( O \subseteq X - X_0 \) since \( X - X_0 \) is open. Then
\[ u_1(O) = u_1^*(O \cap X_1) > 0 \]
since \( O \cap X_1 \) is a non-empty open set in \( X_1 \) and \( G \) is weakly transitive on \( X_1 \) (the closure of \( x_1 \)'s orbit is \( X_1 \)). But
\[ u_0(O) = u_0^*(O \cap X_0) = u_0^*(\emptyset) = 0. \]
Thus \( u_0 \) and \( u_1 \) are distinct Haar measures on \( X \).

To prove the converse it suffices to show that a Haar integral on \( X \) is unique if \( G \) is weakly transitive (Theorem 2.7).

Fix a point \( x_0 \in X \) and let \( U \in \mathcal{U} \). By equicontinuity at \( x_0 \) there is a \( V \in \mathcal{U} \) such that \( gV[x_0] \subseteq U[g(x_0)] \) for all \( g \in G \). We may assume that \( cV[x_0] \) is compact. By Lemma 3.6.2 there is a continuous non-negative real valued function \( f^* \) such that
1) \( f^*(x_0) \neq 0 \),
2) \( f^*(X - V[x_0]) = 0 \),
3) \( f^*(g(x_0)) = f^*(g^{-1}(x_0)) \) for all \( g \in G \).
Since \( cV[x_0] \) is compact, (2) says that \( f^* \in L \).
Let $I$ be any Haar integral on $X$ and consider an $f \in L$ such that $f \geq 0$ and $f \neq 0$. For each $g \in G$ define

$$h(g) = I(f \cdot f^{*}g^{-1}).$$

Let $K = \cup_{\mathcal{B}_{1}} \{ |f(x) - f(y)| : (x, y) \in U \} < \infty$. Now

$$f^{*}g^{-1}(x) \neq 0 \implies g^{-1}(x) \in \mathcal{V}[x_{0}] \implies x \in U[g(x_{0})]$$

$$\implies |f(x) - f \circ g(x_{0})| \leq K$$

$$\implies f(x) \geq f \circ g(x_{0}) - K.$$

Thus, for all $x \in X$,

$$f(x) \cdot f^{*}g^{-1}(x) \geq [f \circ g(x_{0}) - K] \cdot f^{*}g^{-1}(x).$$

Hence, for all $g \in G$,

4) $h(g) \geq \left[ f \circ g(x_{0}) - K \right] \cdot I(f^{*}g^{-1})$

$$= \left[ f \circ g(x_{0}) - K \right] \cdot I(f^{*}).$$

Let $C$ be a compact support for $f$ and $m \in L$ such that $0 \leq m \leq 1$ and $m(C) = 1$ (Berberian [2] 54.3). Fix an arbitrary $\delta > 0$. By Lemma 3.6.1 there exists a $U* \in \mathcal{U}$ such that $|f^{*}(x) - f^{*}(y)| < \delta$ whenever $(x, y) \in U*.$

There is a $V* \in \mathcal{U}$ such that $gV*[x_{0}] \subseteq U*[g(x_{0})]$ for all $g \in G$. By weak transitivity there exist $g_{1}, \ldots, g_{n} \in G$ such that $C \subseteq \bigcup_{i=1}^{n} g_{i}V*[x_{0}].$ There exist $f_{i} \in L$ such that $0 \leq f_{i} \leq 1$ and $f_{i}(X - g_{i}V*[x_{0}]) = 0$ and $\sum_{i=1}^{n} f_{i}(x) = 1$ for all $x \in C$ (Berberian [2] 75.12). Then for all $g \in G$,

$$h(g) = I\left[ (\sum_{i=1}^{n} f_{i}) \cdot f \cdot f^{*}g^{-1} \right] = \sum_{i=1}^{n} I(f_{i} \cdot f \cdot f^{*}g^{-1}).$$

Now

$x \in g_{i}V*[x_{0}] \implies g^{-1}(x) \in g_{i}^{-1}g_{1}V*[x_{0}]$ for all $g \in G$

$$\implies g^{-1}(x) \in U*[g_{i}^{-1}g_{1}(x_{0})]$ for all $g \in G$

$$\implies |f^{*}g^{-1}(x) - f^{*}g_{i}^{-1}(x_{0})| < \delta$$ for all $g \in G.$
Thus for all $x \in X$ and all $g \in G$ we have the following,

$$f_i(x) \cdot f(x) \cdot f \circ g^{-1}(x) \leq f_i(x) \cdot f(x) \cdot [f \circ g^{-1}g_i^*(x_0) + \delta]$$

and hence

5) $h(g) \leq \sum_{i=1}^{n} [f \circ g_i^{-1}g_i^*(x_0) + \delta] \cdot I(f_i \cdot f)$ for all $g \in G$.

Combining (4) and (5) with the fact that

$f \circ g(x_0) = f \circ g^{-1}(x_0)$ for all $g \in G$ and noting that

$I(f^*) > 0$, we obtain

6) $f \circ g(x_0) - K \leq h(g) / I(f^*)$

$$\leq \sum_{i=1}^{n} [I(f_i \cdot f) / I(f^*)] \cdot [f \circ g_i^{-1}g_i^*(x_0) + \delta]$$

for all $g \in G$. But by Lemma 3.5.2 the orbit of $x_0$ is dense in $X$. Thus

7) $f(x) - K \leq \sum_{i=1}^{n} [I(f_i \cdot f) / I(f^*)] \cdot [f \circ g_i^{-1}(x) + \delta]$

for all $x \in X$.

For each $f^*, f_0 \in \mathcal{L}$ such that $f^* \not= 0$, $f_0 \not= 0$, and $f_0 \not= 0$ we define

$$N(f^*, f_0) = \{ c_i \in \mathbb{C} : \text{There is an integer } n_0 \geq 1 \text{ and there are real numbers } c_i \geq 0 \text{ and there are } g_i \in G \text{ such that } f^*(x) \leq \sum_{i=1}^{n_0} c_i f_0 \circ g_i^*(x) \text{ for all } x \in X \}.$$ 

We know that $N(\ , \ )$ exists in virtue of Lemma 3.6.3.

Setting

$$c_0 = I(f) / I(f^*) = \sum_{i=1}^{n} [I(f_i \cdot f) / I(f^*)]$$

we obtain the following from (7).

8) $f(x) \leq (K + \delta c_0)m(x) + \sum_{i=1}^{n} [I(f_i \cdot f) / I(f^*)] \cdot f \circ g_i^{-1}(x)$

for all $x \in X$. 
Then clearly

\[ 9) \quad N(f, f^*) \leq (K + \delta c_o)N(m, f^*) + c_o. \]

But $\delta > 0$ was arbitrary, so

\[ 10) \quad N(f, f^*) \leq K \cdot N(m, f^*) + c_o. \]

If $f(x) \leq \sum \tilde{c}_i f^* \circ \tilde{g}_i(x)$, then $I(f) \leq \sum \tilde{c}_i I(f^* \circ \tilde{g}_i) = (\sum \tilde{c}_i)I(f^*)$. Thus $N(f, f^*) \geq c_o$. Combining this fact with (10) we obtain

\[ 11) \quad I(f^*)N(f, f^*) - K \cdot N(m, f^*)I(f^*) \leq I(f) \leq I(f^*)N(f, f^*). \]

Now we have established the following result. For each $U \in \mathcal{U}$, there is a non-negative $f^* \neq 0$ in $\mathcal{L}$ with the following properties. If $I$ is a Haar integral, $f \in \mathcal{L}$, $f \geq 0$, $f \neq 0$, $C$ is a compact support for $f$, and $m \in \mathcal{L}$ such that $m(C) = 1$ and $0 \leq m \leq 1$, then (11) holds where $K = \text{LUB} \left\{ |f(x) - f(y)| : (x, y) \in U \right\}$.

Now fix $f_o \in \mathcal{L}$ such that $f_o \geq 0$ and $f_o \neq 0$. Consider any $f \in \mathcal{L}$ such that $f \geq 0$ and $f \neq 0$. Then for each positive integer $n$ there is a $U_n \in \mathcal{U}$ such that

\[ |f(x) - f(y)| < 1/n \quad \text{and} \quad |f_o(x) - f_o(y)| < 1/n \]

for all $(x, y) \in U_n$. Let $f^*_n$ be the corresponding function whose existence is asserted in the preceding paragraph. If $C_o, C$ are compact supports for $f_o, f$ respectively and $m_o, m \in \mathcal{L}$ such that $0 \leq m_o \leq 1 = m_o(C_o)$, $0 \leq m \leq 1 = m(C)$, and $I$ is any Haar integral, then for each positive integer $n$ we have
12) \( I(f_n^*)N(f_n^*, f_n^*) - \frac{1}{n}N(m,f_n^*)I(f_n^*) \leq I(f) \)
\[ \leq I(f^*)N(f_n^*, f_n^*) \]

and

\[ I(f_n^*)N(f_0^*, f_n^*) - \frac{1}{n}N(m_0^*, f_n^*)I(f_n^*) \leq I(f_0) \]
\[ \leq I(f_n^*)N(f_0^*, f_n^*) \]

Note that if \( f_1, f_2, f_3 \in \mathcal{L}, f_i \geq 0, \) and \( f_2 \neq 0 \neq f_3, \)
then \( N(f_1, f_3) \leq N(f_1, f_2)N(f_2, f_3). \) Thus

\[ N(m_0^*, f_n^*) \leq N(m_0^*, f_0^*)N(f_0^*, f_n^*) \]
for each positive integer \( n. \) The second of equations (12) tells us that \( N(f_0^*, f_n^*) \geq 0 \) for each positive integer \( n \)
since \( I(f_0) > 0. \) If \( N(m_0^*, f_n^*) = 0, \) then

\[ N(f_0^*, f_n^*) - \frac{1}{n}N(m_0^*, f_n^*) > 0; \]

if \( N(m_0^*, f_n^*) > 0 \) and \( n > N(m_0^*, f_0^*), \) then

\[ N(f_0^*, f_n^*) - \frac{1}{n}N(m_0^*, f_n^*) > N(f_0^*, f_n^*) - \frac{N(m_0^*, f_n^*)}{N(m_0^*, f_0^*)} \geq 0. \]

Hence for \( n > N(m_0^*, f_0^*), \) we obtain the following from (12).

13) \[ \frac{[N(f_n^*, f_n^*) - \frac{1}{n}N(m,f_n^*)]}{N(f_0^*, f_n^*)} \leq \frac{I(f)}{I(f_0)} \]
\[ \leq N(f,f_n^*)/[N(f,f_n^*) - \frac{1}{n}N(m,f_n^*)]. \]

But observe that for all positive integers \( n \)
\[ N(m_0^*, f_n^*)/N(f_0^*, f_n^*) \leq N(m_0^*, f_0^*) < \infty \]
and

\[ N(m,f_n^*)/N(f_0^*, f_n^*) \leq N(m,f_0^*) < \infty. \]

Thus for \( n > N(m_0^*, f_0^*), (13) \) yields

14) \[ \frac{[N(f_n^*, f_n^*)/N(f_0^*, f_n^*)] - \frac{1}{n}N(m,f_0^*)}{N(f_0^*, f_n^*)} \leq \frac{I(f)}{I(f_0)} \]
\[ \leq \frac{N(f,f_n^*)}{N(f_0^*, f_n^*)} \times \frac{1}{1 - N(m_0^*, f_0^*)/n}. \]
Hence \( \lim_{n \to \infty} \frac{N(f, f_n^*)}{N(f_0, f_n^*)} \leq \frac{I(f)}{I(f_0)} \leq \lim_{n \to \infty} \frac{N(f, f_n^*)}{N(f_0, f_n^*)} \); i.e.,

15) \( \frac{I(f)}{I(f_0)} = \lim_{n \to \infty} \frac{N(f, f_n^*)}{N(f_0, f_n^*)} \).

Recall that the \( f_n^* \) were chosen and then it was shown that (15) must hold for any Haar integral \( I \). Thus if \( J \) is any other Haar integral, we have

16) \( \frac{J(f)}{J(f_0)} = \frac{I(f)}{I(f_0)} \).

Hence for fixed \( f_0 \in \mathcal{L} \) such that \( f_0 \geq 0 \) and \( f_0 \neq 0 \), and for arbitrary Haar integrals \( I, J \), (16) holds for any \( f \in \mathcal{L} \) such that \( f \geq 0 \). The extension of (16) to arbitrary \( f \in \mathcal{L} \) is trivial. This completes the proof of Theorem 3.6.

In order to state Segal's result we must introduce the following concepts.

**Definition 3.7** A space \( X \) whose topology is introduced by a uniformity \( \mathcal{U} \) is said to be uniformly locally compact if there is a \( U \in \mathcal{U} \) such that \( cU[x] \) is compact for each \( x \in X \).

**Definition 3.8** A family \( F \) of functions from a uniform space \( (X, \mathcal{U}) \) to a uniform space \( (Y, \mathcal{V}) \) is said to be uniformly equicontinuous if for each \( V \in \mathcal{V} \) there is a \( U \in \mathcal{U} \) such that \( fU[x] \subseteq V[f(x)] \) for all \( x \in X \) and all \( f \in F \).
Corollary 3.6.1 (Segal [16]) Let $G$ be a uniformly equicontinuous group of (uniformly continuous) automorphisms of a uniformly locally compact Hausdorff space $X$ whose topology is introduced by a uniformity $\mathcal{U}$. Then there is a Haar measure on $X$; moreover, this Haar measure is unique if and only if there is a point in $X$ whose orbit is dense in $X$.

Note that if $F$ is a uniformly equicontinuous family, then each $f \in F$ is uniformly continuous. Hence Segal could have omitted "uniformly continuous" from his hypothesis. Furthermore, we can show that if $G$ is a weakly transitive, uniformly equicontinuous group of automorphisms of a locally compact Hausdorff uniform space $X$, then $X$ is uniformly locally compact.

Corollary 3.6.2 (Mibu [11]) Let $X$ be a locally compact Hausdorff space whose topology is introduced by a uniformity $\mathcal{U}$. Let $G$ be a transitive group of automorphisms of $X$ such that $gU[x] = U[g(x)]$ for all $g \in G$, $x \in X$, and for all $U \in \mathcal{U}$ such that $U = U^{-1}$. Then there is a Haar measure on $X$.

In the above corollary, in virtue of Theorem 3.6, we can omit transitivity; then the Haar measure is unique if and only if $G$ is weakly transitive.
Chapter 4

SOME IDEAS RELATED TO CONDITION A

In the following note the similarity of (*) to condition (5) of Corollary 3.1.1.

Theorem 4.1 Let $G$ be a weakly transitive group of autohomeomorphisms of a 1st countable Hausdorff space $X$. Then Condition A is equivalent to

(*) $\lim_{n \to \infty} g_n(x_n) = \lim_{n \to \infty} g_n(y_n)$.

Proof Assume that Condition A holds but (*) does not hold. Then there exist $a, b, c, x_n, y_n \in X$ and $g_n \in G$ such that $x_n \to a$, $y_n \to a$, $g_n(x_n) \to b$, $g_n(y_n) \to c$, and $b \neq c$.

There exist disjoint open neighborhoods $O_1, O_2$ of $b, c$ respectively. We may assume that $g_n(x_n) \in O_1$ and $g_n(y_n) \in O_2$ for all $n$. Now $B = \{b\} \cup \{g_n(x_n) : n\}$, and $C = \{c\} \cup \{g_n(y_n) : n\}$ are disjoint compact sets; thus there is a non-empty open set $O$ such that $[g(O) \cap B = \emptyset$ or $g(O) \cap C = \emptyset]$ for all $g \in G$. There is a $g_0 \in G$ such that $a \in g_0(O)$. We may assume that $x_n, y_n \in g_0(O)$ for all $n$.

But then $g_n(x_n), g_n(y_n) \in g_ng_0(O)$ for all $n$ and hence $g_ng_0(O) \cap B \neq \emptyset \neq g_ng_0(O) \cap C$, a contradiction. Thus Condition A implies (*).
Conversely, assume that (*) holds but that Condition A does not hold. Then there are disjoint compact sets $B$ and $C$ such that for any non-empty open set $O$ there is a $g \in G$ for which $g(O) \cap B \neq \emptyset \neq g(O) \cap C$. Let $\{O_n\}$ be a descending countable base at some $x_0 \in X$. Then there exist $a_n, b_n \in O_n$, $g_n \in G$ such that $g_n(a_n) \in C$ and $g_n(b_n) \in B$. We may assume that there exist points $a \in C$, $b \in B$ such that $g_n(a_n) \to a$ and $g_n(b_n) \to b$. But then (*) implies that $a = b$ or $B \cap C \neq \emptyset$, a contradiction. Thus (*) implies Condition A.

**Corollary 4.1.1** Let $G$ be a weakly transitive group of autohomeomorphisms of a Hausdorff space $X$. If (*) above holds for nets rather than sequences, then Condition A holds.

**Proof** The proof is essentially the same as the proof for sequences.

**Definition 4.1** A group $G$ of autohomeomorphisms of a topological space $X$ is said to be symmetrically continuous if for any net $\{g_\nu\}$ in $G$ and any $x \in X$ we have $g_\nu(x) \to x$ if and only if $g^{-1}_\nu(x) \to x$.

**Definition 4.2** A family $F$ of functions from a topological space $X$ to a topological space $Y$ is said to be evenly continuous if for each $x \in X$, $y \in Y$, and neighborhood $O$ of $y$, there exist neighborhoods $O_1, O_2$ of $x, y$ respectively such that $[g(x) \in O_2 \to g(O_1) \subseteq O]$ for all $g \in F$. 
Definition 4.3 A family $F$ of functions from a topological space $X$ to a topological space $Y$ is said to be strongly evenly continuous if for each $x \in X$, $y \in Y$, and $O$ neighborhood of $y$, there exist neighborhoods $O_1, O_2$ of $x, y$ respectively such that $\left[ f(O_1) \cap O_2 \neq \emptyset \rightarrow f(O_1) \subseteq O \right]$ for each $f \in F$.

Theorem 4.2 Let $G$ be a weakly transitive group of autohomeomorphisms of a locally compact Hausdorff space $X$ which is either compact or locally connected. If Condition A holds then $G$ is strongly evenly continuous.

Proof Assume $X$ is locally connected. Fix $x, y \in X$ and a neighborhood $O$ of $y$. We may assume $O$ has compact closure. There is a compact open neighborhood $O_2$ of $y$ such that $cO_2 \subseteq O$. Then $cO_2$ and $bO = cO - O$ are disjoint compact sets. Thus there is a neighborhood $O_1$ of $x$ such that $g(O_1) \cap cO_2 = \emptyset$ or $g(O_1) \cap bO = \emptyset$ for all $g \in G$. We may assume $O_1$ is connected. Thus, if $g(O_1) \cap O_2 \neq \emptyset$ we have

$$g(O_1) = (g(O_1) \cap O) \cup (g(O_1) - cO).$$

Then, since $O_1$ is connected, we must have $g(O_1) = g(O_1) \cap O$.

If $X$ is compact the proof is just as easily established.
Theorem 4.3 Let $G$ be a transitive group of autohomeomorphisms of a topological space $X$. The following are then equivalent.

a) There is a uniformity $\mathcal{U}$ on $X$ (which generates the topology on $X$) with respect to which $G$ is equicontinuous.

b) $G$ is strongly evenly continuous.

c) $G$ is evenly continuous and symmetrically continuous.

Proof a) $\rightarrow$ b) Fix $x, y \in X$ and a neighborhood $\mathcal{O}$ of $y$. There is a $U \in \mathcal{U}$ such that $U = U^{-1}$ and $U \cdot U \cdot U[y] \subseteq \mathcal{O}$. By equicontinuity there is a neighborhood $\mathcal{O}^*$ of $x$ such that $g(\mathcal{O}^*) \subseteq U[g(x)]$ for all $g \in G$. If $g(\mathcal{O}^*) \cap U[y] \neq \emptyset$, fix a point $z \in g(\mathcal{O}^*) \cap U[y]$. Then $p \in \mathcal{O}^*$ $\rightarrow$ $z, g(p) \in g(\mathcal{O}^*) \subseteq U[g(x)]$ $\rightarrow$ $(z, g(x)), (g(p), g(x)) \in U$ $\rightarrow$ $(z, g(p)) \in U \cdot U$. But $(y, z) \in U$ and hence $(y, g(p)) \in U \cdot U \cdot U$. Hence $g(\mathcal{O}^*) \subseteq U \cdot U \cdot U[y] \subseteq \mathcal{O}$.

b) $\rightarrow$ c) Clearly strong even continuity implies even continuity. If $g_\nu(x) \rightarrow x$ in $X$ let $\mathcal{O}$ be an arbitrary neighborhood of $x$. There exist neighborhoods $\mathcal{O}_1, \mathcal{O}_2$ of $x$ such that $[g(\mathcal{O}_2) \cap \mathcal{O}_1 \neq \emptyset$ $\rightarrow$ $g(\mathcal{O}_2) \subseteq \mathcal{O}]$ for all $g \in G$. There is a $\nu_0$ such that $g_\nu(x) \in \mathcal{O}_2$ for all $\nu > \nu_0$. Then for all $\nu > \nu_0$, we have $x \in g_\nu^{-1}(\mathcal{O}_2) \cap \mathcal{O}_1$ and so $g_\nu^{-1}(\mathcal{O}_2) \subseteq \mathcal{O}$; i.e., $g_\nu^{-1}(x) \in \mathcal{O}$ for all $\nu > \nu_0$. Thus $g_\nu^{-1}(x) \rightarrow x$. 
c) \[\text{Fix } x_o \in X. \text{ For each open neighborhood } O \text{ of } x_o, \text{ let } U_O = \bigcup_{g \in G} [g(x_o) \times g(O)] \]
\[= \{(g(x_o), g(y)) : g \in G, y \in O\} \]

By transitivity each $U_O$ contains the diagonal.

For each such $O$ there is an open set $O^*$ such that $x_o \in O^* \subseteq O$ and $[g(x_o) \in O^* \implies g(O^*) \subseteq O]$ for all $g \in G$. Assert $U_{O^*} \cup U_{O^*} \subseteq U_O$. $(x, y), (y, z) \in U_{O^*} \implies$ there exist $g_1, g_2 \in G$ and $y^*, z^* \in O^*$ such that
\[x = g_1(x_o) \text{ and } y = g_1(y^*) \]
\[y = g_2(x_o) \text{ and } z = g_2(z^*) \]

Now $y^* = g_1^{-1}(y) = g_1^{-1}g_2(x_o) \in O^* \implies g_1^{-1}g_2(O^*) \subseteq O \implies g_2(O^*) \subseteq g_1(O) \implies g_2(z^*) = g_1(w)$ for some $w \in O \implies (x, z) = (g_1(x_o), g_1(w)) \in U_O$. Thus $U_{O^*} \cup U_{O^*} \subseteq U_O$.

Clearly $U_{O_1} \cap U_{O_2} \supseteq U_{O_1 \cap O_2}$ for each pair of open neighborhoods $O_1, O_2$ of $x_o$.

Given $U_O$, by symmetric continuity there exists an open neighborhood $O^*$ of $x_o$ such that $[g(x_o) \in O^* \implies g^{-1}(x_o) \in O]$ for all $g \in G$. Assert $U_O^{-1} \supseteq U_{O^*}$. $(x, z) \in U_{O^*} \implies$ there exist $g \in G$ and $y \in O^*$ such that $(x, z) = (g(x_o), g(y))$. By transitivity there is a $g^* \in G$ such that $g^*(x_o) = g(y)$; then $g^*(x_o) \in g(O^*)$ or $g^{-1}g^*(x_o) \in O^*$. Hence $g^*^{-1}g(x_o) \in O$ or $g(x_o) \in g^*(O)$ so that $(z, x) \in [g^*(x_o) \times g^*(O)] \subseteq U_O$. Hence $(x, z) \in U_O^{-1}$.

Thus the class of all such $U_O$ is a base for a uniformity $\mathcal{U}$ on $X$ (Kelley [9] 6.2). Furthermore, $G$ is equicontinuous at $x_o$ since $g(O) \subseteq U_O[g(x_o)]$ for each $g \in G$.
and each $U_0$. Then by Lemma 3.5.4 G is equicontinuous at each $x \in X$.

It remains to be shown that $\mathcal{U}$ is compatible with the original topology on $X$. For $y \in X$ there is a $g \in G$ such that $g(x_0) = y$; then for each $U_0$ we have $U_0[y] = U_0[g(x_0)] \supseteq g(O)$. Thus each set open in the uniform topology is also open in the original topology. Conversely, if $O$ is an open neighborhood of any $x \in X$, then by even continuity there exist open sets $O_1$ and $O_2$ such that $x_0 \in O_1$, $x \in O_2 \subseteq O$, and $[g(x_0) \in O_2 \implies g(O_1) \subseteq O]$ for all $g \in G$. Then $U_{O_1}[x] = \bigcup_{g \in G} g(O_1) \subseteq \bigcup_{g \in G} g(O_1) \subseteq O$. Hence each set open in the original topology is also open in the uniform topology and the topologies coincide.

Definition 4.4 A group $G$ of autohomeomorphisms of a topological space $X$ is said to satisfy Strong Condition A if for disjoint sets $B$ and $C$, one of which is compact and one of which is closed, there exists a non-empty open set $O$ such that $[g(O) \cap B = \emptyset$ or $g(O) \cap C = \emptyset]$ for all $g \in G$.

Theorem 4.4 Let $G$ be a weakly transitive group of autohomeomorphisms of a locally compact Hausdorff space $X$. Then Strong Condition A and strong even continuity are equivalent.

Proof Assume that Strong Condition A holds. Fix $x, y \in X$ and let $O$ be an open neighborhood of $y$. Let $O^*$ be a compact open neighborhood of $y$ such that $cO^* \subseteq O$. Then
since $O'$ is closed, Strong Condition A says that there is a neighborhood $O_1$ of $x$ such that $[g(O_1) \cap O^* \neq \emptyset \rightarrow g(O_1) \subseteq O]$ for all $g \in G$.

Assume that $G$ is strongly evenly continuous and let $B \cap C = \emptyset$ where $B$ is compact and $C$ is closed. Fix $x \in X$. For each $y \in B$ there exist neighborhoods $O^*_x$ and $O_y$ of $x$ and $y$ respectively such that $[g(O^*_x) \cap O_y \neq \emptyset \rightarrow g(O^*_x) \cap C = \emptyset]$ for all $g \in G$. There exist $y_1, \ldots, y_n \in B$ such that $B \subseteq \bigcup_{i=1}^{n} O_{y_i}$. Let $O_1 = \bigcap_{i=1}^{n} O^*_x$. Then $[g(O_1) \cap B \neq \emptyset \rightarrow g(O_1) \cap C = \emptyset]$ for all $g \in G$.

**Corollary 4.4.1** Let $G$ be a transitive group of autohomeomorphisms of a locally compact Hausdorff space $X$ which is either compact or locally connected. Then the following are equivalent.

a) Condition A is satisfied.

b) Strong Condition A is satisfied.

c) There is a uniformity $\mathcal{U}$ on $X$ (which generates the topology on $X$) with respect to which $G$ is equicontinuous.

d) $G$ is strongly evenly continuous.

e) $G$ is symmetrically continuous and evenly continuous.

**Proof** See Theorems 4.2, 4.3, and 4.4.
Corollary 4.4.2 Let \( G \) be a transitive group of automorphisms of a locally compact Hausdorff space \( X \). If any of the conditions (a) - (e) of Corollary 4.4.1 are satisfied, then there is a Haar measure on \( X \) with respect to \( G \). This Haar measure is unique if one of the conditions (b) - (e) are satisfied; if \( X \) is either compact or locally connected, then the Haar measure is unique when Condition A is satisfied.

Proof See Theorems 4.2, 4.3, 4.4, 3.6, and Corollary 3.5.2, and Corollary 3.2.1.
In this chapter we show that the well-known theory of Haar measure on topological groups and on coset spaces of topological groups indeed follows from the general theory established in Chapter 3. We then apply this theory to the space of oriented lines in Euclidean 3-space.

**Definition 5.1** A topological group is a group $G$ together with a topology $\mathcal{T}$ under which the group operations "inverse" and "product" ($f(x) = x^{-1}$ and $g(x,y) = x \cdot y$ respectively) are continuous functions from $G$ and $G \times G$ (with product topology) respectively, into $G$.

If $G$ is a topological group, then it is clear that, for fixed $a \in G$, the functions

1) $f(x) = x^{-1}$
2) $f(x) = a \cdot x$
3) $f(x) = x \cdot a$

are autohomeomorphisms of $G$. It should also be noted that if $A$, $B$ are compact subsets of $G$, then $A \times B$ is compact in $G \times G$ by the Tychonoff Theorem. Hence $A \cdot B = \{a \cdot b : a \in A, b \in B\}$
is compact in G since the "product" takes $A \times B$ continuously onto $A \times B$.

**Definition 5.2** If $H$ is a subgroup of a group $G$, then $G/H$ will denote the space of left or right cosets of $H$ in $G$ and $\alpha$ will be a standard notation for the projection $\alpha(x) = xH$ or $\alpha(x) = Hx$ depending on whether we are considering left or right cosets.

If $G$ is a topological group, then the collection of all sets $A \subseteq G/H$ such that $\alpha^{-1}(A)$ is open in $G$ is a topology for $G/H$. This topology will be called the quotient topology; we shall always assume that $G/H$ is given the quotient topology.

**Theorem 5.1** Let $H$ be a subgroup of a topological group $G$. Then

1) $\alpha$ is both continuous and an open map;
2) if $G$ is locally compact, then $G/H$ is locally compact;
3) if $G$ is locally compact and $B$ is a compact closed subset of $G/H$, then there is a compact subset $A$ of $G$ such that $\alpha(A) = B$;
4) if $H$ is closed, then $G/H$ is Hausdorff.

If \( H \) is a subgroup of a topological group \( G \), it is clear from the definition of the quotient topology that for fixed \( a \in G \), the function \( f(xH) = axH \) (or \( f(Hx) = Hxa \)) is an autohomeomorphism of \( G/H \). Hence, if \( H \) is a closed subgroup of a locally compact Hausdorff topological group \( G \), we can consider the existence of

1) a (left or right) Haar measure on \( G \) with respect to \( G \) considered as a group of autohomeomorphisms of itself when acting as (left or right) multiplication,

2) a Haar measure on \( G/H \) with respect to \( G \) considered as a group of autohomeomorphisms of \( G/H \) when acting as left or right multiplication depending on whether we are considering the left or right coset space; if fact, we need not assume that \( G \) is Hausdorff in this case.

If \( u \) is a left Haar measure on a topological group \( G \), then clearly \( \nu(E) = u(E^{-1}) \) is a right Haar measure and vice versa. Similarly, if we have a Haar measure on a left coset space \( G/H \) we get a Haar measure on the right coset space \( G/H \) and vice versa.

It is well known that any locally compact Hausdorff topological group \( G \) has a unique Haar measure. If \( H \) is a closed subgroup, then \( H \) is a locally compact Hausdorff topological group with the relative topology; hence \( H \) has
a unique Haar measure. These Haar measures will tell us whether or not \( G/H \) has a Haar measure.

The following result was proved by Poncet [14]. However a different proof is given here.

**Theorem 5.2** If \( H \) is a closed subgroup of a locally compact topological group \( G \) and if for each open neighborhood \( V \) of the identity \( e \) in \( G \) there is an open neighborhood \( U \) of \( e \) such that \( H \cdot U \subseteq V \cdot H \), then the left coset space \( G/H \) has a unique Haar measure.

**Proof** For each open neighborhood \( V \) of \( e \), let
\[
U_V = \{(xH,yH) : x^{-1}y \in H \cdot V \cdot H\}.
\]
Then clearly each \( U_V \) contains the diagonal \( \{(xH,xH) : x \in G\} \). Furthermore, if \( V \) is an open neighborhood of \( e \) then \( V^* = V \cap V^{-1} \) is also an open neighborhood of \( e \) and \( U_V \supseteq U_{V^*} = U_{V^{-1}} \) or \( U_{V^{-1}} \supseteq U_{V^*} \).

Consider \( U_V \). There is an open neighborhood \( V_1 \) of \( e \) such that \( V_1 \cdot V_1 \subseteq V \); there is an open neighborhood \( O^* \) of \( e \) such that \( H \cdot O^* \subseteq V_1 \cdot H \). Let \( O = O^* \cap V_1 \). Then
\[
U_O \cdot U_V = \{(xH,zH) : \text{there is a } y \in G \text{ such that } x^{-1}y \in H \cdot O \cdot H \text{ and } y^{-1}z \in H \cdot O \cdot H\}
\]
\[
\subseteq \{(xH,zH) : x^{-1}z \in H \cdot O \cdot H \cdot O \cdot H\}
\]
\[
\subseteq \{(xH,zH) : x^{-1}z \in H \cdot O \cdot V_1 \cdot H\}
\]
\[
\subseteq \{(xH,zH) : x^{-1}z \in H \cdot V \cdot H\}
\]
\[
= U_V.
\]
Given $U_Y$ and $U_W$ we have

$$U_Y \cap U_W = \{ (xH, yH) : x^{-1}y \in H \cdot V \cdot H \cap W \cdot H \} \supseteq \{ (xH, yH) : x^{-1}y \in H \cdot (V \cap W) \cdot H \} = U_Y \cap W.$$

Thus the class of all such $U_Y$ is a base for a uniformity for $G/H$ (Kelley [9] 6.2). If $O$ is open in the quotient topology and $xH \in O$, then $x \in \mathcal{X}^{-1}(O)$ and $e \in V = x^{-1} \mathcal{X}^{-1}(O)$. There is an open neighborhood $V^*$ of $e$ such that $H \cdot V^* \subseteq V \cdot H$. Then

$$U_{V^*}[xH] = \{ yH : x^{-1}y \in H \cdot V^* \cdot H \} \supseteq \{ yH : y \in x \cdot V \cdot H \} \supseteq \mathcal{X}(xV) = O.$$

Conversely, if $O$ is open in the uniform topology and $xH \in O$, then there is a $U_Y$ such that $U_Y[xH] \subseteq O$. Now

$$\mathcal{X}^{-1}(O) \supseteq \mathcal{X}^{-1}(U_Y[xH]) = \bigcup_{yH \in H \cdot V \cdot H} yH = x \cdot H \cdot V \cdot H$$

and this last set is open in $G$. Hence $x \in (\mathcal{X}^{-1}(O))^O$ and $\mathcal{X}^{-1}(O)$ is open in $G$ so that $O$ is open in the quotient topology. Thus the quotient topology coincides with the uniform topology.

Consider $U_Y$ and let $O = H \cdot V \cdot H$. $\mathcal{X}(O)$ is an open neighborhood of $eH$ in $G/H$. Now $yH \in \mathcal{X}(O) \rightarrow y \in H \cdot V \cdot H \rightarrow (xH, xyH) \in U_Y$ for all $x \in G \rightarrow xyH \in U_Y[xH]$ for all $x \in G$. Thus $x \mathcal{X}(O) \subseteq U_Y[xH]$ for all $x \in G$; i.e., $G$ is an equicontinuous group of autohomeomorphisms of $G/H$. Then by Corollary 3.5.2 and Theorem 3.6 there is a unique Haar measure on $G/H$. 

Corollary 5.2.1  If \( H \) is a closed normal subgroup of a locally compact topological group \( G \), then there exists a Haar measure on \( G/H \) with respect to \( G \).

Corollary 5.2.2  If \( G \) is a locally compact Hausdorff topological group, then there exists a unique Haar measure on \( G \).

Corollary 5.2.3  If \( H \) is a compact closed subgroup of a locally compact topological group \( G \), then there exists a unique Haar measure on \( G/H \).

Proof  Let \( V \) be an open neighborhood of \( e \) in \( G \). Then for any \( h \in H \), \( h \cdot e = h \in Vh \), there exist open neighborhoods \( U_h \) and \( U^*_h \) of \( h \) and \( e \) respectively such that \( U_h \cdot U^*_h \subseteq Vh \).

Since \( H \) is compact, there exist \( h_1, \ldots, h_n \in H \) such that \( H \subseteq \bigcup_{i=1}^{n} U_{h_i} \). Let \( U = \bigcap_{i=1}^{n} U^*_{h_i} \). Then \( H \cdot U \subseteq \bigcup_{i=1}^{n} U_{h_i} \cdot U^*_{h_i} \subseteq \bigcup_{i=1}^{n} V_{h_i} \subseteq V \cdot H \).

Theorem 5.3  If \( H \) is a subgroup of a topological group \( G \) and the quotient topology is introduced by a uniformity \( U \) such that \( H \) is equicontinuous at \( eH \) in \( G/H \), then for each open neighborhood \( V \) of \( e \) there is an open neighborhood \( U \) of \( e \) such that \( H \cdot U \subseteq V \cdot H \). Hence if \( G \) is also locally compact and \( H \) is closed, there exists a unique Haar measure on \( G/H \) with respect to \( G \).
Proof For each open neighborhood $V$ of $e$ there is a $U^* \in \mathcal{U}$ such that $U^*[eH] \subseteq \alpha(V)$. By equicontinuity of $H$ at $eH$ there exists an open neighborhood $O$ of $eH$ such that $hO \subseteq U^*[eH] = U^*[eH]$ for all $h \in H$. Let $V_1 = \alpha^{-1}(O)$. Then $z \in H \cdot V_1 \rightarrow$ there exists an $h \in H$ such that $hz \in V_1 \rightarrow zH = h^{-1}hzH \in h^{-1}O \subseteq U^*[eH] \rightarrow zH \in \alpha(V) \rightarrow z \in V \cdot H$; i.e., $H \cdot V_1 \subseteq V \cdot H$. The existence of a unique Haar measure then follows from Theorem 5.2.

Corollary 5.3.1 Let $H$ be a subgroup of a topological group $G$. Then there exists a uniformity $\mathcal{U}$ for $G/H$ which generates the quotient topology such that $H$ is equicontinuous at $eH$ if and only if there is a uniformity $\mathcal{U}^*$ for $G/H$ which generates the quotient topology such that $G$ is uniformly equicontinuous on $G/H$.

Proof By Theorem 5.3, for each neighborhood $V$ of $e$ there is a neighborhood $U$ of $e$ such that $H \cdot U \subseteq V \cdot H$. Let $\mathcal{U}^*$ be the uniformity constructed in Theorem 5.2. Then for each $U_V$ we have

$$xU_V[yH] = x\{zH : y^{-1}z \in H \cdot V \cdot H\}$$

$$= \{xzH : (xy)^{-1}xz \in H \cdot V \cdot H\}$$

$$= \{zH : (xy)^{-1}z \in H \cdot V \cdot H\}$$

$$= U_V[xyH]$$

for all $x, y \in G$. Thus $G$ is uniformly equicontinuous with respect to $\mathcal{U}^*$. 
In virtue of the above corollary, Theorem 3.5 reduces to Corollary 3.6.1 whenever \( X \) is a coset space of \( G \); it can be shown that if \( G \) is a weakly transitive uniformly equicontinuous group of autohomeomorphisms of a locally compact Hausdorff uniform space \( X \), then \( X \) is uniformly locally compact.

**Definition 5.3** Let \( G \) be a locally compact Hausdorff topological group. For each \( f \in L \) and \( t \in G \) we define \( f^t(x) = f(xt) \) for all \( x \in G \). Clearly \( f^t \in L \).

**Theorem 5.4** If \( I \) is a left Haar integral on a locally compact Hausdorff topological group \( G \), then for each \( t \in G \) there exists a unique real number \( \Delta(t) > 0 \) such that \( \Delta(t)I(f) = I(f^t) \) for all \( f \in L \). Furthermore \( \Delta \) is a continuous homomorphism of \( G \). (See Berberian [2] 77.1 and 77.2.) If \( u \) is a Haar measure on \( G \) then \( u(Eg) = \Delta(g^{-1})u(E) \) for all \( E \in \mathcal{B} \) and \( g \in G \). (Berberian [2] 78.3)

**Definition 5.4** \( \Delta \) as defined in Theorem 5.4 is called the modular function for \( G \). If \( \Delta \equiv 1 \) then \( G \) is said to be unimodular.

We are now in a position to state the fundamental result of Andre Weil. (See Weil [17] p. 45).
Theorem 5.5 (Weil) Let \( H \) be a closed subgroup of a locally compact Hausdorff topological group \( G \); let \( \Delta, \delta \) be the modular functions for \( G \) and \( H \) respectively. Then there exists a Haar measure on \( G/H \) if and only if \( \Delta(h) = \delta(h) \) for all \( h \in H \). Moreover, if a Haar measure exists it is unique.

The following is a well known example.

Example 4.1 Let \( G \) be the group of \( 3 \times 3 \) real matrices with determinant 1. Topologize \( G \) with the metric
\[
d \left[ (x_{ij}), (y_{ij}) \right] = \sqrt{\sum_{i,j=1}^{3} (x_{ij} - y_{ij})^2}.
\]
Then \( G \) becomes a unimodular topological group. Let \( H \) be the subgroup of matrices of the form
\[
\begin{pmatrix}
1/a & 0 & 0 \\
0 & a & b \\
0 & 0 & 1
\end{pmatrix}, \quad a > 0, \ b \text{ real}.
\]
Then \( H \) is not unimodular; hence by the Weil Theorem there is no Haar measure on \( G/H \).

For the sake of completeness we include the following well-known result.

Corollary 5.5.1 Let \( G \) be a locally compact Hausdorff topological group. If \( G \) is abelian, compact, or discrete, then any closed subgroup \( H \) has the corresponding property. Hence, in each of these three cases, \( G/H \) has a unique Haar measure since both \( G \) and \( H \) are unimodular.
Corollary 5.5.2 Let \( H \) be a closed subgroup of a locally compact Hausdorff topological group \( G \). If the restriction to \( \mathcal{B}(H) \) of Haar measure on \( G \) is a Haar measure on \( H \), then \( G/H \) has a unique Haar measure.

Proof The modular functions for \( G \) and \( H \) are clearly identical on \( H \).

Definition 5.5 A topological space is said to be \( \sigma \)-compact if it is a countable union of compact sets.

Corollary 5.5.3 Let \( H \) be a closed subgroup of a locally compact Hausdorff topological group \( G \). If \( u \) is Haar measure on \( G \), \( H \) is Lindelöf (or \( \sigma \)-compact or 2nd countable), and \( u(H) > 0 \), then \( G/H \) has a unique Haar measure.

Proof For any \( E \in \mathcal{B}(H) \subseteq \mathcal{B}(G) \) and \( h \in H \), we have \( u(hE) = u(E) \). Moreover \( u(C) < \infty \) when \( C \) is a compact subset of \( H \).

For any set \( A \subseteq H \) we have \( u(A) = \text{GLB} \{ u(O) : O \supseteq A \} = \text{GLB} \{ u(O \cap H : O \cap H \supseteq A) \} \). Let \( O^* \) be an open set in \( H \) and let \( O \) be an arbitrary non-empty compact open set in \( G \). Then there exists an open set \( O^\# \) in \( G \) such that \( O^* = O^\# \cap H \).

Since \( H \) is Lindelöf there exist \( g_n \in G \) such that \( H \subseteq \bigcup_{n=1}^{\infty} g_n(O) \). Let \( O_n = \bigcup g_i(O) \) and \( U_n = O_n \cap O^\# \). Then \( U_n \cap H \uparrow O^* \) and \( u(U_n \cap H) \uparrow u(O^*) \). For each integer \( n \) there exists a compact set \( C_n \subseteq U_n \) such that \( u(U_n) - u(C_n) = u(U_n - C_n) < 1/n \). Then \( u(U_n \cap H) = u(C_n \cap H) + u([U_n-C_n] \cap H) \).
But \( u([U_n - C_n] \cap H) < 1/n \) and hence \( u(C_n \cap H) \to u(0^*) \).

Note that for each integer \( n \) the set \( C_n \cap H \) is compact in \( H \).

Thus the restriction of \( u \) to \( \mathcal{B}(H) \) is an invariant, non-trivial, regular Borel measure. The conclusion then follows from the preceding corollary.

In order to identify the space \( \mathcal{H} \) of oriented lines in \( E_3 \) with a coset space we must have the following results on semi-direct products of groups.

The proof of the following theorem is straightforward and is omitted.

**Theorem 5.6** Let \( G \) be a group of functions from a group \( H \) into \( H \) such that the identity \( I \) of \( G \) is the identity map of \( H \). If \( G, H \) are (topological) groups, then \( G \times H \) becomes a (topological) group under the multiplication

\[
(*) \quad (g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 g_1(h_2))
\]

if and only if each \( g \in G \) is an automorphism of \( H \) and \( g_1(g_2(h)) = g_1(g_2(h)) \) for all \( g_1, g_2 \in G \) and all \( h \in H \) (and \( F(g, h) = g(h) \) is a continuous function from \( G \times H \) into \( H \)).

We obtain a similar result with the operation \((**)\) defined by inserting "\( g_1(h_2)h_1 \)" in place of "\( h_1 g_1(h_2) \)" in \((*)\).

Just as in Loomis [10] pp. 119-120, we can prove the following result.
Theorem 5.7  Let G and H be locally compact Hausdorff topological groups with modular functions \( \Delta_1, \Delta_2 \) respectively. Assume G is a group of functions from \( H \) into \( H \) such that the identity of G is the identity map of \( H \) and that \( G \times H \) becomes a topological group under the operation \( (*) \) of Theorem 5.6. Then each \( g \in G \) is an automorphism of \( H \) and there exists a non-negative real valued function \( \Delta \) on G such that \( \nu(g(E)) = \Delta(g) \nu(E) \) for all \( E \in \mathcal{B}(H) \) and all \( g \in G \) where \( \nu \) is right Haar measure on \( H \).

Note that \( G \times H \) is locally compact and Hausdorff. The modular function for \( G \times H \) can be computed and is found to be \( \Delta(g, h) = \Delta_1(g) \Delta_2(h) \Delta(g) \).

Theorem 5.8  Let G be a group of functions from a group \( H \) into \( H \) such that the identity I of G is the identity map of \( H \). Assume that G and H are locally compact Hausdorff topological groups and that G is compact. If \( G \times H \) is a topological group under the operation \( (*) \) of Theorem 5.6 and \( \nu \) is left or right Haar measure on \( H \), then

\[
\nu(g(E)) = \nu(E)
\]

for all \( E \in \mathcal{B}(H) \) and all \( g \in G \).

Proof  \( G \times H \) is a locally compact Hausdorff topological group and as such has a unique left (Halmos) Haar measure \( \nu \). Define \( \nu^* \) on \( \mathcal{B}^*(H) \) by the equation
\[ u^*(E) = u(G \times E). \quad u^* \text{ is defined since } \mathcal{B}^*(G) \times \mathcal{B}^*(H) \subseteq \mathcal{B}^*(G \times H) \text{ (Berberian [2], p. 128, Problem 15).} \]

If \( E \) is compact in \( H \) then \( G \times E \) is compact in \( G \times H \) and hence \( u^*(E) < \infty \). \( u^* \) is clearly a measure on \( \mathcal{B}^*(H) \).

Again, if \( E \in \mathcal{B}^*(H) \) then \( G \times E \in \mathcal{B}(G \times H) \); hence given \( \lambda < u^*(E) \), there is a compact set \( C \) in \( G \times H \) such that \( C \subseteq G \times E \) and \( u(C) > \lambda \). Let \( C_1 \) be the projection of \( C \) into \( H \). Then \( C_1 \) is a compact subset of \( E \) and \( u^*(C_1) = u(G \times C_1) \geq u(C) > \lambda \). Thus \( u^* \) is \( H \)-regular by Berberian [2], 61.5.

Let \( e \) be the identity of \( H \). Then for \( (g,h) \in G \times H \), \( E \in \mathcal{B}^*(H) \), we have

\[
(**) \quad u^*(g(E)) = u(G \times g(E)) = u((g,e)(G \times E)) = u(G \times E) = u^*(E)
\]

and

\[
u^*(hE) = u(G \times hE) = u((I,h)(G \times E)) = u(G \times E) = u^*(E).
\]

Thus \( u^* \) is left (Halmos) Haar measure on \( H \) which in turn can be extended to left Haar measure on \( H \); then in virtue of (**) and regularity we see that (#) holds for left Haar measure on \( H \).

From Theorem 5.6 we see that \( G \times H \) is also a tomo-
logical group under the operation (**) of Theorem 5.6. Using a similar proof we establish (#) for right Haar measure on H.

**Corollary 5.8.1** If, in addition to the hypothesis of Theorem 5.8, H is unimodular (left and right Haar measures coincide), then G x H is unimodular when we define multiplication as the operation (*) of Theorem 5.6.

**Proof** Both G and H are unimodular. We then appeal to Theorems 5.7 and 5.8.

**Definition 5.6** $M_3$ shall be standard notation for the multiplicative group of orthogonal $3 \times 3$ real matrices. A matrix $A$ is said to be orthogonal if its rows (and columns) are mutually perpendicular unit vectors and the determinant of $A$ is $\pm 1$. It then follows that $A^{-1} = A^T$ where $A^T$ denotes the transpose of $A$.

**Lemma 5.9.1** If $M_3$ is topologized with the distance function $d_1[(x_{ij}),(y_{ij})] = \sqrt{\sum_{i=1}^{3} \sum_{j=1}^{3} (x_{ij} - y_{ij})^2}$, then $M_3$ is a topological group. Furthermore, $M_3$ is homeomorphic to a closed subspace of Euclidean 9-space and hence $M_3$ is locally compact.

Note that $E_3$ is a locally compact Hausdorff topological group under the operation of vector addition. Also
each $M \in \mathcal{M}_3$ may be considered as a function from $E_3$ into $E_3$ in the following manner.

$$
\begin{pmatrix}
m_{11} & m_{12} & m_{13} \\
m_{21} & m_{22} & m_{23} \\
m_{31} & m_{32} & m_{33}
\end{pmatrix}
\begin{pmatrix}
a_1 \\
a_2 \\
a_3
\end{pmatrix}
= 
\begin{pmatrix}
\sum_{i=1}^{3} m_{1i} a_j \\
\sum_{i=1}^{3} m_{2i} a_j \\
\sum_{i=1}^{3} m_{3i} a_j
\end{pmatrix}
$$

The identity matrix is then clearly the identity map of $E_3$.

**Lemma 5.9.2** \( \mathcal{M}_3 \times E_3 \) is a topological group under the operation \( (A,\bar{a})(B,\bar{b}) = (AB,\bar{a} + A\bar{b}) \).

**Proof** Clearly each $M \in \mathcal{M}_3$ is an automorphism of $E_3$ and the group operation in $\mathcal{M}_3$ is functional composition. Let $M_n \to M_o$ in $\mathcal{M}_3$ and $\bar{x}_n \to \bar{x}_o$ in $E_3$. In virtue of Theorem 5.6 it suffices to show that $M_n(\bar{x}_n) \to M_o(\bar{x}_o)$ in $E_3$. If $M_n = (m^n_{ij})$ and $x_n = (x^n_1, x^n_2, x^n_3)$, then $M_n(\bar{x}_n) = (\sum_{i,j}^n m^n_{ij} x^n_j, \sum_{i,j}^n m^n_{ij} x^n_j, \sum_{i,j}^n m^n_{ij} x^n_j)$. But $m^n_{ij} \to m^o_{ij}$ and $x^n_j \to x^o_j$ so that $m^n_{ij} x^n_j \to m^o_{ij} x^o_j$. Hence $M_n(\bar{x}_n) \to M_o(\bar{x}_o)$.

**Lemma 5.9.3** \( \mathcal{M}_3 \times E_3 \) is unimodular.

**Proof** $M = (m_{ij}) \in \mathcal{M}_3 \to |m_{ij}| \leq 1$ for all $i,j$.

Thus $\mathcal{M}_3$ may be considered as a bounded closed subset of $E_9$. Hence $\mathcal{M}_3$ is compact. $E_3$ is abelian and hence unimodular. The conclusion then follows from Corollary 5.8.1.

**Definition 5.7** For each $(A,\bar{a}) \in \mathcal{M}_3 \times E_3$, we define \((A,\bar{a})\bar{x} = A\bar{x} + \bar{a} \) for all $\bar{x} \in E_3$.  

It is well known that with the above definition, each $(A, \bar{a}) \in M_3 \times E_3$ is a distance preserving transformation of $E_3$. In fact, $M_3 \times E_3$ is the class of all distance preserving transformations of $E_3$ onto $E_3$. Moreover, each $(A, \bar{a}) \in M_3 \times E_3$ takes lines into lines. (See Birkhoff-MacLane [3] p. 259 and p. 287).

Definition 5.8 An oriented line in $E_3$ is a line with one of the two unit vectors parallel to the line designated as the positive direction. $\mathcal{H}$ will denote the class of oriented lines in $E_3$.

If $k \in \mathcal{H}$, $\bar{u}$ is a unit vector specifying the orientation of $k$, and $\bar{a}$ is a point on $k$, then clearly $\bar{p}$ is a point on $k$ if and only if $\bar{p} = t\bar{u} + \bar{a}$ for some real number $t$. Thus an oriented line is determined by specifying

1) a point $\bar{a} \in E_3$,
2) a unit vector $\bar{u}$;

i.e., each oriented line has a representation $(\bar{u}, \bar{a})$ where $\bar{u}$ assigns its direction and $\bar{a}$ is a point through which the line passes. Clearly this representation is not unique since, for any real number $t$, $(\bar{u}, t\bar{u} + \bar{a})$ will be a representation of the same oriented line.

Definition 5.9 For each $k \in \mathcal{H}$ there exists a point $\psi(k)$ on $k$ closest to the origin and a unit vector $\bar{u}(k)$
which assigns the direction or orientation of \( k \). Clearly
\[
d^*(k_1,k_2) = |\overline{u}(k_1) - \overline{u}(k_2)| + |\overline{\phi}(k_1) - \overline{\phi}(k_2)|
\]
is a distance function in \( \mathcal{K} \). Mickle and Rado [12] introduced this metric; hence we shall call the topology defined by this metric the Mickle-Rado topology.

**Definition 5.10** Let \((\overline{u},\overline{s})\) be a representation of an oriented line \( k \). Then for each \((A,a) \in \mathcal{M}_3 \times E_3\) we define
\[
(A,a)(\overline{u},\overline{s}) = (A\overline{u},A\overline{s} + \overline{a});
\]
i.e., the line determined by \((A\overline{u},A\overline{s} + \overline{a})\). Clearly this definition does not depend on the representation of the line \( k \).

Mickle and Rado [12,13] have shown that with the above definition each \((A,a) \in \mathcal{M}_3 \times E_3\) is a homeomorphism of \( \mathcal{K} \) onto itself and that there is a unique non-trivial invariant measure in \( \mathcal{K} \). The following work will show that this result actually follows from the theory of Haar measures on coset spaces.

**Lemma 5.9.4** Let \( \overline{e}_3 = (0,0,1) \) and \( \overline{0} = (0,0,0) \). The class \( H \) of those \((A,\overline{a}) \in \mathcal{M}_3 \times E_3\) which take the line represented by \((\overline{e}_3,\overline{0})\) into the line represented by \((\overline{e}_3,\overline{0})\) is a closed unimodular subgroup of \( \mathcal{M}_3 \times E_3\).

**Proof** Let \( \mathcal{L}_3 \) be the set of those \( A \in \mathcal{M}_3 \) of the form
\[
A = \begin{pmatrix}
a_1 & a_2 & 0 \\
-a_2 & a_1 & 0 \\
0 & 0 & 1
\end{pmatrix}
or
\[
A = \begin{pmatrix}
a_1 & a_2 & 0 \\
a_2 & -a_1 & 0 \\
0 & 0 & 1
\end{pmatrix},
a_1^2 + a_2^2 = 1.
\]
Let \( E^*_1 = \{(0,0,a) : a \text{ real}\} \). It can be verified that \( H \) is just \( \mathcal{A}_3 \times E^*_1 \). Moreover, \( H, \mathcal{A}_3, E^*_1 \) are topological groups with the topology and group operation inherited from \( M_3 \times E_3, M_3, \) and \( E_3 \) respectively. Both \( H \) and \( \mathcal{A}_3 \) are closed subspaces of their parent spaces. Then \( \mathcal{A}_3 \) is compact since \( M_3 \) is compact and Hausdorff.

\( E^*_1 \) is abelian and hence unimodular. Then since the subspace topology for \( H \) coincides with the product of the subspace topologies for \( \mathcal{A}_3 \) and \( E^*_1 \) it follows from Corollary 5.8.1 that \( H \) is unimodular.

**Theorem 5.9** There exists a unique Haar measure on \((M_3 \times E_3)/H\).

**Proof** This follows immediately from the Weil Theorem since both \( M_3 \times E_3 \) and \( H \) are unimodular.

For the remainder of this chapter, \( H \) will be as defined in Lemma 5.9.4.

**Lemma 5.10.1** Define \( \bar{e}_3 \) and \( \bar{0} \) as in Lemma 5.9.4. Let \( k_0 = (\bar{e}_3, \bar{0}) \). Then the function \( f:(M_3 \times E_3)_H \rightarrow K \) defined as \( f(H \cdot (A, \bar{a})) = (A, \bar{a})^{-1}k_0 \) is well-defined, one-to-one, and onto.

**Proof** If \( H(A, \bar{a}) = H(B, \bar{b}) \), then \( (A, \bar{a})(B, \bar{b})^{-1} \in H \) so that \( (A, \bar{a})(B, \bar{b})^{-1}k_0 = k_0 \). Thus \( (B, \bar{b})^{-1}k_0 = (A, \bar{a})^{-1}k_0 \); i.e., \( f(H \cdot (A, \bar{a})) = f(H \cdot (B, \bar{b})) \) and \( f \) is well-defined.

Since any line (in particular \( k_0 \)) can be transformed
into any other line by a rigid motion, it follows that \( f \) is onto.

If \((A, \overline{a})^{-1}k_0 = (B, \overline{b})^{-1}k_0\), then \((B, \overline{b})(A, \overline{a})^{-1}k_0 = k_0\) so that \((B, \overline{b})(A, \overline{a})^{-1}H\). Then \(H \cdot (B, \overline{b}) = H \cdot (A, \overline{a})\) and it follows that \( f \) is one-to-one.

We could give \( \mathcal{K} \) the topology it receives from \((M_3 \times \mathbb{R}^3)/H\) via the mapping \( f \) and establish a unique Haar measure on \( \mathcal{K} \) in this way. But the topology for \( \mathcal{K} \) has already been specified. What we want to show now is that \( f \) is a homeomorphism between \((M_3 \times \mathbb{R}^3)/H\) and \( \mathcal{K} \) when \( \mathcal{K} \) is given the Wickle-Rado topology.

Lemma 5.10.2 Let \( d_2 \) be the usual metric in \( \mathbb{R}^3 \) and let \( d_1 \) be the metric defined in Lemma 5.9.1. Then the (product) topology in \( M_3 \times \mathbb{R}^3 \) coincides with the topology introduced by the metric \( d_3 = \sqrt{d_1^2 + d_2^2} \). Moreover for any \((A, \overline{a}), (B, \overline{b}), (D, \overline{d}) \in M_3 \times \mathbb{R}^3\) we have

\[
d_3 [(D, \overline{d})(A, \overline{a}), (D, \overline{d})(B, \overline{b})] = d_3 [(A, \overline{a}), (B, \overline{b})].
\]

The right coset space \((M_3 \times \mathbb{R}^3)/H\) is metrizable with the metric

\[
\delta [H(A, \overline{a}), H(B, \overline{b})] = \text{GLB} \left\{ d_3 [(D_1, \overline{d_1}), (D_2, \overline{d_2})] : (D_1, \overline{d_1}) \in H(A, \overline{a}) \right. \left. \text{ and } (D_2, \overline{d_2}) \in H(B, \overline{b}) \right\}.
\]

Proof The left invariance of the distance function \( d_3 \) is a straightforward calculation and is omitted. The metrization of the right coset space then follows from Hewitt-Ross [8] 8.14.
Lemma 5.10.3  \( f \) as defined in Lemma 5.10.1 is continuous.

Proof  Let \( H(A_n, a_n)^{-1} \to H(A_0, a_0)^{-1} \) in \( (M_3 \times E_3)/H \). In order to prove continuity we need only show that 
\( (A_n, a_n) \to (A_0, a_0) \) in \( H \). In virtue of Lemma 5.10.2 we may assume 
\( (A_n, a_n)^{-1} \to (A_0, a_0)^{-1} \) in \( M_3 \times E_3 \); then, since \( M_3 \times E_3 \) is a topological group, it follows that 
\( (A_n, a_n) \to (A_0, a_0) \). Hence \( A_n \to A_0 \) and \( a_n \to a_0 \).

\( 0 \) and \( e_3 \) are two points on \( k_0 \). Also note that, in virtue of Lemma 5.0.2 and Theorem 5.6, \( A_n e_3 \to A_0 e_3 \). Now 
\( (A_n, a_n)0 = \bar{a}_n \to \bar{a}_0 = (A_0, a_0)0 \)
and 
\( (A_n, a_n)e_3 = A_n e_3 + \bar{a}_n \to A_0 e_3 + \bar{a}_0 = (A_0, a_0)e_3. \)
Mickle and Rado [12, 13] have shown that then 
\( (A_n, a_n)(e_3, 0) \to (A_0, a_0)(e_3, 0). \)

Lemma 5.10.4  If \( B \in M_3 \), \( I \) is the identity matrix, and \( |3^{rd} \text{ row } B - 3^{rd} \text{ row } I| < r \) in \( E_3 \), then there exists a \( B^* \in M_3 \) such that \( 3^{rd} \text{ row } B = 3^{rd} \text{ row } B^* \) and \( d_1(I, B^*) < r1/3 \).

Proof

Figure 1
If 3rd row $B = 3rd$ row $I$, let $B^* = I$. Otherwise find the great circle on the unit sphere determined by 3rd row $B$ and 3rd row $I$. Find the line $k^*$ perpendicular to the plane of this great circle. (See Figure 1.)

Rotate 3rd row $I$ to 3rd row $B$ about the axis $k^*$. Each point on the unit sphere will traverse an arc on a circle which is centered on $k^*$, the greatest distance being traveled on the great circle. If $B^*$ is the orthogonal matrix which is the image of $I$ under the above rotation then clearly 3rd row $B = 3rd$ row $B^*$ and

$$d_1(I,B^*) \leq \sqrt{\sum_{i=1}^{3} |i\text{th row } I - i\text{th row } B^*|^2}$$

$$\leq \sqrt{3 \cdot |3rd \text{ row } I - 3rd \text{ row } B^*|^2}$$

$$< r\sqrt{3}.$$

**Theorem 5.10** $\mathcal{K}$ is homeomorphic to $(M_3 \times \mathbb{R}^3)/\mathcal{H}$ via the mapping $f$ defined in Lemma 5.10.1.

**Proof** Fix $r > 0$. Then for any $k \in \mathcal{K}$ such that $d^*(k_o,k) < r$ we have the following. There exists a $(B,\bar{b}) \in M_3 \times \mathbb{R}^3$ such that

1. $(B,\bar{b})k = k_o$
2. $(B,\bar{b})\phi(k) = \bar{b}.$

Since $k = (B,\bar{b})^{-1}k_o = (B^T,-B^T\bar{b})(\bar{e}_3,\bar{b}) = (B^T\bar{e}_3,-B^T\bar{b})$ we have $\bar{u}(k) = B^T\bar{e}_3 = 3rd$ row $B$. Clearly $\bar{u}(k_o) = 3rd$ row $I$. Hence $r > |u(k_o) - u(k)| = |3rd$ row $B - 3rd$ row $I|$. By Lemma 5.10.4 there exists a $B^* \in M_3$ such that 3rd row $B = 3rd$ row $B^*$ and $d_1(I,B^*) < r\sqrt{3}$. Let $\bar{b}^* = B^*B^T\bar{b}$. 
Note that $B^T\bar{e}_3 = B^T\bar{e}_3$ since $B^*$ and $B$ have the same last row. Thus $(B^T, -B^T)k_0 = (B^T, -B^T)k_0$ or $(B^*, B^TB)^{-1}k_0 = (B, \bar{B})^{-1}k_0 = k$ so that

(1*) $(B^*, \bar{B}^*)k = k_0$.

Also note that

$(B^T, -B^T)\bar{o} = -B^T = (B^T, -B^T)\bar{o} = \phi(k)$. Thus

(2*) $(B^*, \bar{B}^*)\phi(k) = \bar{o}$.

It follows from (2*) that $\phi(k) = -B^T\bar{b}^*$. Now

$r > |\phi(k) - \phi(k_0)| = |\bar{o} - B^T\bar{b}^*| = |B\bar{o} - \bar{b}^*| = |\bar{o} - \bar{b}^*|.$

Hence $d_3[(1, \bar{o}), (b^*, \bar{b}^*)] = \sqrt{d_1(1, b^*)^2 + d_2(0, b^*)^2} < \sqrt{3r^2 + r^2} = 2r$;

then $\int_{H(I, \bar{o}) \cap H(b^*, \bar{b}^*)} < 2r$. Clearly $f[H(I, \bar{o})] = k_0$ and $f[H(b^*, \bar{b}^*)] = k$. Thus

$f [S_d (H(I, \bar{o}), r)] \supset S_{d^*} [f(H(I, \bar{o}), r/2)]$ for all $r > 0$; i.e., the image of an open neighborhood of $H$ in $M_3 \times E_3/H$ is a (not necessarily open) neighborhood of $k_0$. Let $O$ be an open neighborhood of $H(A, \bar{a})$. Then $O^* = O \cdot (A, \bar{a})^{-1}$ is an open neighborhood of $H$. Hence

$f(O^*) = (A, \bar{a})f(O)$ is a neighborhood of $k_0$. Then $f(O)$ is a neighborhood of $(A, \bar{a})^{-1}k_0 = f[H(A, \bar{a})]$. Thus $f$ is indeed an open map; in virtue of Lemma 5.10.3 this completes the proof of the theorem.

**Corollary 5.10.1** There is a unique Haar measure on $\mathcal{K}$ with respect to $M_3 \times E_3$. 
Proof By Theorem 5.9 there is a unique Haar measure on $\left( M_3 \times E_3 \right)/H$. In virtue of the homeomorphism $f$ we obtain a non-trivial regular Borel measure on $\mathcal{H}$. For any set $Q \subseteq \left( M_3 \times E_3 \right)/H$ we have $f(Q \cdot (A, \bar{a})) = (A, \bar{a})^{-1} f(Q)$. The measure in $\mathcal{H}$ is then invariant with respect to $M_3 \times E_3$ so that it is Haar measure; the uniqueness likewise follows from the homeomorphism $f$.

Both $M_3$ and $E_3$ are $\sigma$-compact; hence $M_3 \times E_3$ is also. Then, since the projection $\alpha$ is continuous, $\left( M_3 \times E_3 \right)/H$ is $\sigma$-compact. Hence $\mathcal{B} = \mathcal{B}^*$. Since we are working with a metric space, any Borel* measure on $\mathcal{H}$ is regular. Thus the above result indeed coincides with that of Mickle and Rado [12,13].

Theorem 5.11 Condition A is not necessary for the existence of a Haar measure.

Proof

![Figure 2]
We need only show that Condition A is not satisfied by $\mathcal{H}$ and $\mathcal{M}_3 \times \mathbb{E}_3$.

Let $s_0 = (0,0,1)$ and $k_0^*$ the line as indicated in the $X_2 - X_3$ plane. Let $k_n \neq k_0^*$ be any sequence of lines in the $X_2 - X_3$ plane converging to $k_0^*$.

Let $g_n$ be the translation parallel to $k_0^*$ which takes $k_n$ into a line through the origin. Let $k_0^\#$ be the line through the origin which is parallel to $k_0^*$. Then $g_n(k_n) \rightarrow k_0^\#$; but $g_n(k_0^*) = k_0^*$. Hence we have found sequences $\{k_n\}$, $\{m_n = k_0^*\}$, and $\{g_n\}$ such that $k_n \rightarrow k_0^*$, $m_n \rightarrow k_0^*$, and $g_n(m_n) \rightarrow k_0^\#$, $g_n(k_n) \rightarrow k_0^\# \neq k_0^*$. Then by Theorem 4.1 Condition A does not hold.
Chapter 6
IDENTIFICATION OF X WITH A COSET SPACE

In this chapter X will denote a (locally compact Hausdorff) topological space and G will denote a transitive group of autohomeomorphisms of X. Under certain conditions there is a Haar measure on X with respect to G. In the space of oriented lines in $\mathbb{E}^3$ we obtained a Haar measure by identifying $\mathcal{H}$ with the coset space $(M_3 \times \mathbb{E}^3)/H$ where $H$ was the subgroup leaving the positively oriented z-axis fixed. In the following definition we see the reason for restricting ourselves to transitive groups.

Definition 6.1 For each $x \in X$, let

$$H_x = \{g : g \in G \text{ and } g(x) = x\}.$$  

Clearly the correspondence

$$(*) \quad gH_x \leftrightarrow g(x) \quad [H_x g \leftrightarrow g^{-1}(x)]$$

is a one-to-one mapping from the left [right] coset space $G/H_x$ onto X. It will be called the natural mapping.

This chapter deals with the following question which is suggested by the introductory paragraph. Can we topologize G so that it becomes a (locally compact Hausdorff) topological group such that, for some $x \in X$, $G/H_x$ with
quotient topology is homeomorphic to \( X \) under the natural map (*)? Let us begin by demanding only that left [right] translations be continuous maps and hence automorphisms.

**Lemma 6.1.1** If a group \( G \) is given a topology such that right [left] translations are continuous, then for any subgroup \( H \) the projection \( \varphi \) of \( G \) into the left [right] coset space \( G/H \) is an open continuous map when \( G/H \) is given the quotient topology.

**Proof** Continuity follows from the definition of the quotient topology. If \( O \) is open in \( G \), then

\[
\varphi^{-1}[\varphi(O)] = O \cdot H = \bigcup_{h \in H} Oh
\]

is open in \( G \) since right translations are open maps.
Thus \( \varphi(O) \) is open in \( G/H \).

**Lemma 6.1.2** If right [left] translations are continuous in a group \( G \), then for any subgroup \( H \) of \( G \) and any \( g_0 \in G \), the left [right] coset spaces \( G/H, G/g_0Hg_0^{-1} \) are homeomorphic under the correspondence

\[
gh \leftrightarrow gHg_0^{-1} \quad \left[ Hg \leftrightarrow g_0Hg \right].
\]

If in addition "inverse" is continuous, then both right and left translations are continuous and the right coset space is homeomorphic to the left coset space under the correspondence \( gh \leftrightarrow Hg^{-1} \).
Proof The correspondence \( gH \longleftrightarrow gHg^{-1} \) is clearly one-to-one and onto since
\[
\begin{align*}
g_1g_0^{-1}(g_0 Hg_0^{-1}) &= g_2g_0^{-1}(g_0 Hg_0^{-1}) \\
\iff g_0 g_2^{-1} g_1 g_0^{-1} &\in g_0 Hg_0^{-1} \\
\iff g_2^{-1} g_1 &\in H \\
\iff g_1^H &= g_2^H.
\end{align*}
\]

Let \( \alpha_1, \alpha_2 \) be the projections into \( G/H, G/g_0 H g_0^{-1} \) respectively. Let \( T(g) = g g_0^{-1} \) for all \( g \in G \). Then the correspondence \( gH \longleftrightarrow gHg^{-1} \) is just the mapping \( \alpha_2 \cdot T \circ \alpha_1^{-1} \). Its inverse is \( \alpha_1 \cdot T^{-1} \alpha_2^{-1} \). Both of these are continuous by Lemma 6.1.1.

If "inverse" and right [left] translations are continuous, then left [right] translations are also continuous since each of the following correspondences is a homeomorphism for fixed \( h \in G \):
\[
\begin{align*}
g &\longleftrightarrow g^{-1} \longleftrightarrow g^{-1} h^{-1} \longleftrightarrow h g.
\end{align*}
\]

Let \( \alpha_L, \alpha_R \) be the projections into the left and right coset spaces respectively: let \( S \) be the "inverse". The correspondence \( gH \longleftrightarrow Hg^{-1} \) is clearly one-to-one and onto; furthermore, it is just the map \( \alpha_L \circ S \circ \alpha_L^{-1} \). Its inverse is \( \alpha_R \circ S \circ \alpha_L^{-1} \). Each of these is continuous by Lemma 6.1.1.

Theorem 6.1 If \( G \) is given a topology such that right [left] translations are continuous and for some \( x \in X \) the left [right] coset space \( G/H_x \) with quotient topology is homeomorphic to \( X \) under the natural map, then for every
If \( x \in X \) the left [right] coset space \( G/H_x \) is homeomorphic to \( X \) under the natural map.

If "inverse" is also continuous, then for every \( x \in X \) both the right and left coset spaces \( G/H_x \) are homeomorphic to \( X \) under the natural map.

**Proof** First observe that for any \( x,y \in X \) and \( g_0 \in G \) such that \( g_0(x) = y \), we have \( H_y = g_0 H_x g_0^{-1} \). By Lemma 6.1.2 each of the following correspondences is a homeomorphism and the conclusion then follows:

\[
gg_0^{-1}(y) = g_0^{-1}g_0(x) = g(x) \leftrightarrow gH_x \leftrightarrow gg_0^{-1}g_0 H_x g_0^{-1} = gg_0^{-1}H_y.
\]

If "inverse is continuous, the conclusion follows from the fact that each of the following is a homeomorphism: \( g^{-1}(x) \leftrightarrow g^{-1}H_x \leftrightarrow H_x g \).

Note that Theorem 6.1 does not hold if we do not demand continuity of translations. For example, fix \( x \in X \) and let \( \mathcal{O} \) be open in \( G \) if and only if \( \mathcal{O} = \{g: g(x) \in \mathcal{O}^*\} \) for some open set \( \mathcal{O}^* \) in \( X \).

**Definition 6.2** \( \mathcal{S}^* \) will denote the following set.

\[
\{S: \text{there exists an } x \in X \text{ and an open in } X \text{ such that } S = \{g: g(x) \in \mathcal{O}\}\}
\]

\[
\mathcal{S} = \mathcal{S}^* \cup \{S^{-1}: S \in \mathcal{S}^*\}.
\]

\( \Omega^* \) and \( \Omega \) will denote the topologies on \( G \) induced by \( \mathcal{S}^* \) and \( \mathcal{S} \) respectively. Thus \( \Omega^* \) consists of all sets of the form \( \bigcup_i S^*_{i,\nu} \), where \( S^*_{i,\nu} \in \mathcal{S}^* \); \( \Omega \) consists of all sets of the form \( \bigcup_i S_{i,\nu} \), where \( S_{i,\nu} \in \mathcal{S} \).
Theorem 6.2 If $G$ is given a topology $\mathcal{T}$ such that right translations are continuous and for some $x \in X$ the left coset space $G/H_x$ is homeomorphic to $X$ under the natural map, then $\mathcal{T} \supseteq \mathcal{O}^*$. If in addition "inverse" is continuous then $\mathcal{T} \supseteq \mathcal{O}^*$.

Proof By Theorem 6.1 $G/H_x$ is homeomorphic to $X$ under the natural map for each $x \in X$. Thus for any $x \in X$ and $O$ open in $X$, the set $\{gH_x : g(x) \in O\} = V$ is open in $G/H_x$. But then $\mathcal{L}^{-1}(V) = \{g : g(x) \in O\}$ is open in $G$. Hence $\mathcal{T} \supseteq \mathcal{O}^*$. If "inverse" is continuous then $(\mathcal{L}^{-1}(V))^{-1} = \{g^{-1} : g(x) \in O\}$ is open in $G$. Hence $\mathcal{T} \supseteq \mathcal{O}^*$.

Theorem 6.3 If $G$ is given a topology $\mathcal{T} \supseteq \mathcal{O}^*$ such that right translations are continuous, then the left coset spaces $G/H_x$ are homeomorphic to $X$ under the natural map if and only if for each $O \in \mathcal{T}$ and each $x \in X$ the set $\{g(x) : g \in O\}$ is open in $X$.

Proof If $O \in \mathcal{T}$ then for each $x \in X$ the set $\{gH_x : g \in O\}$ is open in $G/H_x$. If $G/H_x$ is homeomorphic to $X$ then $\{g(x) : g \in O\}$ is open in $X$. If $G/H_x$ is not homeomorphic to $X$, then since $\mathcal{T} \supseteq \mathcal{O}^*$ there is some set $A \subseteq G/H_x$ which is open in $G/H_x$ but $\{g(x) : gH_x \in A\}$ is not open in $X$. Let $O = \{g : gH_x \in A\} \in \mathcal{T}$. Then $\{g(x) : g \in O\}$ is not open in $X$. 

Lemma 6.4.1 If $G$ is given either of the topologies $\Omega^*$ or $\Omega$, then both right and left translations are continuous. "Inverse" is continuous in $\{G, \Omega\}$.

Proof If $S^* = \{g: g(x) \in O\}$ and $h \in G$, then

$$S^* h = \{gh: gh(x) \in O\}$$

$$= \{gh: gh h^{-1}(x) \in O\}$$

$$= \{g^*: g^*(h^{-1}(x)) \in O\} \in \mathcal{S}^*$$

and

$$h S^* = \{hg: g(x) \in O\}$$

$$= \{hg: h(x) \in h(O)\}$$

$$= \{g^*: g^*(x) \in h(O)\} \in \mathcal{S}^*.$$

If $S = \{g: g^{-1}(x) \in O\}$ and $h \in G$, then

$$h S = \{hg: g^{-1}h^{-1}(x) \in O\}$$

$$= \{g^*: g^*^{-1}(h(x)) \in O\} \in \mathcal{S}$$

and

$$S h = \{gh: h^{-1}g^{-1}(x) \in h^{-1}(O)\}$$

$$= \{g^*: g^*^{-1}(x) \in h^{-1}(O)\} \in \mathcal{S}.$$  

Thus all translations are continuous in both $\Omega^*$ and $\Omega$.

Clearly $S \in \mathcal{S} \iff S^{-1} \in \mathcal{S}$; thus "inverse" is continuous in $\{G, \Omega\}$.

Theorem 6.4 There is a topology for $G$ such that right translations are continuous [and "inverse" is continuous] and such that the left and right coset spaces $G/H_\chi$ are homeomorphic to $X$ under the natural map if and only if
for every finite collection \((x_i, O_i)\), where \(x_i \in X\) and \(O_i\) is open in \(X\), and every \(x \in X\) the set
\[
\{g(x) : g^i(x_i) \in O_i \text{ and } \varepsilon_i = +1 \text{ [or -1]} \text{ for each } i\}
\]
is open in \(X\) [and the set
\[
\{g^{-1}(x) : g^i(x_i) \in O_i \text{ and } \varepsilon_i = +1 \text{ for each } i\}
\]
is open in \(X\)].

In this case \(\bigcap^* \left[ -\Omega \right]\) is the smallest such topology.

**Proof** This follows directly from Theorems 6.2, 6.3, and Lemma 6.4.1.

We now impose the more stringent condition that \(G\) be a topological group. Since we are primarily interested in locally compact Hausdorff spaces and since every such space is uniformizable, the following result is not surprising.

**Theorem 6.5** If \(G\) is given a topology \(\mathcal{I}\) such that some (each) coset space \(G/\Pi x\) is homeomorphic to \(X\) under the natural map and \(\{G, \mathcal{I}\}\) is a topological group, then \(X\) is a uniform space with the collection \(\mathcal{U}\) of all sets of the form

\[
U_0 = \begin{cases} 
\{(x, g(x)) : x \in X, g \in O\}, & e \in O \in \mathcal{I} \\
\text{or} \\
\{(g(x), x) : x \in X, g \in O\}, & e \in O \in \mathcal{I}
\end{cases}
\]
as a subbase for its uniformity \(\mathcal{U}^*\) (where \(e\) is the identity in \(G\)). Moreover, each \(g \in G\) is uniformly continuous.
Proof Clearly each \( U \in \mathcal{U} \) contains the diagonal and \( U^{-1} \in \mathcal{U} \). For any open neighborhood \( O \) of the identity \( e \) in \( G \) there is an open neighborhood \( O^* \) of \( e \) such that \( O^* \cdot O^* \subseteq O \). If \( U_0 = \{(x,g(x)): x \in X, g \in O\} \) let \( U_0^* = \{(x,g(x)): x \in X, g \in O^*\} \). Then

\[
U_0^* \cdot U_0^* = \{(x,g_2g_1(x)): g_1, g_2 \in O^*\} \subseteq U_0.
\]

If \( U_0 = \{(g(x),x): x \in X, g \in O\} \) let \( U_0^* = \{(g(x),x): x \in X, g \in O^*\} \). Then \( U_0^* \cdot U_0^* = \{(g_1g_2(x),x): g_1, g_2 \in O^*\} \subseteq U_0 \). Thus \( \mathcal{U} \) is a subbase for a uniformity \( \mathcal{U}^* \) (Kelley [9] 6.3).

Let \( T \) be an open neighborhood of \( x \) in \( X \). Then \( O = \{g: g(x) \in T\} \) is an open neighborhood of \( e \) in \( G \) and choosing \( U_0 \) according to the first form given above, we have \( U_0[x] = \{g(x): g \in O\} = T \). Thus \( T \) is an open set in the uniform topology. Conversely, if \( T \) is a neighborhood of \( x \) in the uniform topology, then by Kelley [9] 6.5 there exist \( U_1, \ldots, U_n \in \mathcal{U} \) such that \( x \in U_1[x] \cap \ldots \cap U_n[x] \subseteq T \). If \( U_i = \{(y,g(y)): y \in X, g \in O_i\} \), then \( U_i[x] = \{g(x): g \in O_i\} \) and this set is open in \( X \) by Theorem 6.4. If \( U_i = \{(g(y),y): y \in X, g \in O_i\} \) then \( U_i[x] = \{y: g(y) = x \text{ for some } g \in O_i\} = \{g^{-1}(x): g \in O_i\} = \{g^*(x): g^* \in O_i^{-1}\} \) and this set is also open in \( X \) by Theorem 6.4. Thus \( T \) is a neighborhood of \( x \) in the original topology; i.e., the original topology coincides with the uniform topology.

We must verify uniform continuity. Fix \( U_0 \in \mathcal{U} \) and \( f \in G \). If \( U_0 = \{(x,g(x)): x \in X, g \in O\} \) then
\{(x,y):(f(x),f(y)) \in U\} = \{(x,y):f(y) = gf(x) \text{ for some } g \in O\}
= \{(x,y):y = f^{-1}gf(x) \text{ for some } g \in O\}
= \{(x,g^*(x)):x \in X, g^* \in f^{-1}\sigma\}
= U_{f^{-1}\sigma}.

A similar argument holds if \( U = \{(g(x),x):x \in X, g \in O\} \); the extension to arbitrary \( U \in U^* \) is then immediate.

**Lemma 6.6.1** Let \( F \) be a family of functions from a topological space \( X \) to a topological space \( Y \). Then \( F \) is evenly continuous if and only if for each \( x \in X, y \in Y, \) and each neighborhood \( O \) of \( y \) there exists a neighborhood \( O^* \) of \( y \) such that \( O^* \subseteq O \) and \( x \in \bigcap_{f \in \sigma} f^{-1}(O) \).

**Proof** Assume \( F \) is evenly continuous. Then \( x \in X \) and \( y \in O \) open in \( Y \) there exists an \( O^* \) open in \( Y \) and a \( V \) open in \( X \) such that \( x \in V, y \in O^* \subseteq O \), and \( f(V) \subseteq O \) when \( f(x) \in O^* \). Then \( x \in V \subseteq \bigcap_{f \in \sigma} f^{-1}(O) \).

Conversely, fix \( x \in X \) and \( y \in O \) open in \( Y \). There exists an \( O^* \) open in \( Y \) such that \( y \in O^* \subseteq O \) and \( x \in \bigcap_{f \in \sigma} f^{-1}(O) = V \). Then \( f(x) \in O^* \rightarrow f(V) \subseteq f^{-1}(O) \subseteq O \) and it follows that \( F \) is evenly continuous.

**Theorem 6.6** Let \( G \) be an evenly continuous, transitive group of autohomeomorphisms of a topological space \( X \). Then \( \{G, \sigma\} \) is a topological group.
Proof Since translations and "inverse" are continuous we need only verify that "product" is continuous. To do this it is sufficient to show that for each \( g_1, g_2 \in G \) and each \( S \in \mathcal{S} \) such that \( g_1 g_2 \in S \), there exist \( S_1, S_2 \in \mathcal{S} \) such that \( g_1 \in S_2 \), \( g_2 \in S_2 \), and \( S_1 S_2 \subseteq S \).

Let \( O^* \) be an open set in \( X \) and \( S = \{ g : g(x_0) \in O^* \} \in \mathcal{S}^* \). If \( g_1 g_2 \in S \), there exists an \( O^* \) open in \( X \) such that \( g_1 g_2 x_0 \in O^* \subseteq O^* \) and \( g_2 x_0 \in \left[ \bigcap \phi_{x_0}(\text{h}^{-1}(O^*)) \right] = V \). Let \( S_1 = \{ g : g(x_0) \in O^* \} \in \mathcal{S}^* \) and \( S_2 = \{ g : g(x_0) \in V \} \in \mathcal{S}^* \). Then \( g_1 \in S_1 \), \( g_2 \in S_2 \), and \( S_1 S_2 \subseteq S \); for \( (h, g) \in S_1 \times S_2 \)

\[
g(x_0) \in \left[ \bigcap \phi_{x_0}(\text{h}^{-1}(O^*)) \right] \quad \text{and} \quad hg_2 x_0 \in O^* \quad \text{and} \quad hg_2 (x_0) \in O^* \Rightarrow hg(x_0) \in \text{h}^{-1}(O^*) = O^* \Rightarrow hg \in S.
\]

Let \( g_1 g_2 \in S_0 = \{ g : g^{-1}(x_0) \in O^* \} \in \mathcal{S} \). Then by the above there exist \( S_1, S_2 \in \mathcal{S} \) such that \( g_1^{-1} \in S_1^{-1} \) and \( S_2^{-1} S_1^{-1} \subseteq S_0^{-1} \); i.e., \( g_1 \in S_1 \) and \( S_1 S_2 \subseteq S_0 \).

It will be seen later that even continuity is not necessary for \( \{ G, \Omega \} \) to be a topological group.

Corollary 6.6.1 If \( X \) is locally compact, Hausdorff, either compact or locally connected, and Condition A is satisfied then \( \{ G, \Omega \} \) is a topological group.

Proof This is an immediate consequence of the above theorem and Corollary 4.4.1.
Theorem 6.7 If $X$ is $T_0$, $T_1$, $T_2$, or regular, then 
\{G, \Omega\} and \{G, \Omega^*\} are likewise $T_0$, $T_1$, $T_2$, or regular, respectively.

Proof If $X$ is $T_2$, let $g_1, g_2$ be distinct points in $G$. Then for some $x \in X$, $g_1(x) \neq g_2(x)$ and there exist disjoint open neighborhoods $O_1, O_2$ of $g_1(x), g_2(x)$ respectively. Hence \{g : g(x) \in O_1\} and \{g : g(x) \in O_2\} are disjoint open neighborhoods of $g_1$ and $g_2$ in $G$.

A similar proof holds for $T_0$ and $T_1$.

To prove regularity it clearly suffices to show that for $g \in S \in S$ there exists an $S^* \in S$ such that $g \in S^* \subseteq cS^* \subseteq S$. Let $g_1 \in S = \{g : g(x) \in O\}$. There exists an $O^*$ open in $X$ such that $g_1(x) \in O^* \subseteq cO^* \subseteq O$. Then
\[ S^* = \{g : g(x) \in cO^*\} \]
is closed in $G$ since $G - S^* = \{g : g(x) \in X - cO^*\}$ is open. Let $S^* = \{g : g(x) \in O^*\} \subseteq S^* \subseteq S$. Then
\[ g_1 \in S^* \subseteq cS^* \subseteq S. \]
A similar argument holds if $g_1 \in S_0 = \{g : g^{-1}(x) \in O\}$.

We now seek conditions under which \{G, \Omega\} and \{G, \Omega^*\} are locally compact. The following result of Ellis is interesting.

Theorem 6.8 (Ellis [5]) If $Y$ is a locally compact Hausdorff space with a group structure such that right and left translations are continuous, then $Y$ is a topological group.
Corollary 6.8.1 If $X$ is $T_2$ and $\{G, \Omega\}$ is locally compact, then $\{G, \Omega^*\}$ is a topological group. Hence if $\{G, \Omega^*\}$ is locally compact, then $\Omega = \Omega^*$.

Corollary 6.8.2 If $X$ is $T_2$ and $G$ is given a topology such that translations are continuous and $G$ is locally compact, then for any subgroup $H$, $G/H$ is locally compact. Thus, if for some $x \in X$, $G/H_x$ is homeomorphic to $X$, then $X$ is locally compact.

Proof $G$ is a topological group and the conclusion then follows from Theorem 5.1.

Definition 6.3 A set $A$ in a topological space is said to be relatively compact if each net in $A$ has a converging subnet (whose limit need not be in $A$).

Lemma 6.9.1 If $G$ is a topological group and $A$ is a relatively compact subset of $G$, then $cA$ is compact.

Proof Let $\{h_\beta\}$ be a net in $cA$ and let $\{V_\gamma\}$ be the neighborhood system of the identity $e$ in $G$. For each $(\gamma, \beta)$ choose a point $g_{\gamma, \beta} \in h_\beta^{-1}(A) \cap V_\gamma$. This choice is possible since $e \in h_\beta^{-1}(cA) = ch_\beta^{-1}(A) \to V_\gamma \cap h_\beta^{-1}(A) \neq \emptyset$.

The set of all $(\gamma, \beta)$ is a directed set if we define $(\gamma, \beta) > (\gamma', \beta')$ to mean $\beta > \beta'$ and $\gamma > \gamma'$ (i.e., $V_\gamma \subseteq V_{\gamma'}$). Setting $h_{\gamma, \beta} = h_\beta$ for each $(\gamma, \beta)$ we obtain a
subnet \( \{h_{\eta, \rho}\} \) of \( \{h_{\eta}\} \). Now \( \{h_{\eta, \rho}g_{\eta, \rho}\} \) is a net in \( A \). Hence there is a subnet \( \{h_{N(\eta)}g_{N(\eta)}\} \) and a \( g_o \in G \) such that
\[ h_{N(\eta)}g_{N(\eta)} \rightarrow g_o. \]
Clearly \( g_{\eta, \rho} \rightarrow e \) in \( G \) so \( g_{N(\eta)} \rightarrow e. \)
Then, since \( G \) is a topological group, we have \( h_{N(\eta)} \rightarrow g_o. \)
But \( \{h_{N(\eta)}\} \) is a subnet of \( \{h_{\eta, \rho}\} \) and hence is a subnet of
\( \{h_{\eta}\} \). Moreover \( g_o \in cA. \) Thus \( cA \) is compact.

**Theorem 6.9** If \( X \) is \( T_2 \) then \( \{G, -\Omega^*\} \) \( \left[\left[ G, \Omega \right]\right] \) is
locally compact if and only if \( G \) is a topological group
and there is a non-empty relatively compact open set in \( G \).

**Proof** This is an immediate consequence of Theorem 6.8
and Lemma 6.9.1.

**Corollary 6.8.2** says that, in order for \( X \) to be
identified with a coset space of a locally compact Haus­
dorff topological group, it is necessary that \( X \) be locally
compact (and hence regular). Accordingly, the following
theorem is probably more to the point.

**Theorem 6.10** If \( X \) is a regular Hausdorff space and if
there is a non-empty relatively compact open set \( O \) in
\( \{G, -\Omega^*\} \) \( \left[\left[ G, \Omega \right]\right] \) then \( \{G, -\Omega^*\} \) \( \left[\left[ G, \Omega \right]\right] \) is a locally compact \( T_2 \)
topological group.

**Proof** By Theorem 6.7, \( G \) is regular. Thus there exists
a non-empty open set \( O^* \) such that \( cO^* \subseteq O \). Then \( O^* \) is a
compact open set and \( G \) is locally compact since translations are homeomorphisms.
Lemma 6.11.1 (Kelley [9] p. 217) Let \{g_\nu\} be a net in \( G \). Then \( \lim \nu g_\nu = g_0 \) in \( \bigcap^* \) if and only if 
\( \lim_\nu g_\nu(x) = g_0(x) \) for all \( x \in X \).

Lemma 6.11.2 If \( \{g_\nu\} \) is a net in \( G \), then \( \lim \nu g_\nu = g_0 \) in \( \bigcap \) if and only if 
\( \lim_\nu g_\nu(x) = g_0(x) \) and 
\( \lim_\nu g_\nu^{-1}(x) = g_0^{-1}(x) \) for all \( x \in X \).

Proof If \( \lim_\nu g_\nu = g_0 \) in \( \bigcap \) then \( \lim_\nu g_\nu = g_0 \) in 
\( \bigcap^* \) since \( \bigcap \supseteq \bigcap^* \); also \( \lim_\nu g_\nu^{-1} = g_0^{-1} \) in \( \bigcap \) so 
\( \lim_\nu g_\nu^{-1} = g_0^{-1} \) in \( \bigcap^* \). Thus \( \lim_\nu g_\nu(x) = g_0(x) \) and 
\( \lim_\nu g_\nu^{-1}(x) = g_0^{-1}(x) \) for all \( x \in X \).

Conversely, if \( \lim_\nu g_\nu(x) = g_0(x) \) and \( \lim_\nu g_\nu^{-1}(x) = g_0^{-1}(x) \) 
for all \( x \in X \), let \( O \) be an open neighborhood of \( g_0 \) in \( \bigcap \). 
we may assume \( O = \bigcap S_i \) where \( S_i \in \mathcal{S} \). If 
\( S_i = \{g: g(x_i) \in O_i\} \), there exists a \( \nu_i \) such that \( g_\nu(x_i) \in O_i \) 
for all \( \nu > \nu_i \). If \( S_i = \{g:g^{-1}(x_i) \in O_i\} \), there exists a 
\( \nu_i \) such that \( g_\nu^{-1}(x_i) \in O_i \) for all \( \nu > \nu_i \). Now there is 
a \( \nu_0 \) such that \( \nu_0 > \nu_i \) for \( i = 1, \ldots, n \). Thus 
\( \nu > \nu_0 \rightarrow g_\nu \in O_i \); i.e., \( \lim_\nu g_\nu = g_0 \) in \( \bigcap \).

Theorem 6.11 \( \{G, \bigcap^*\} \left[\{G, \bigcap\}\right] \) has a non-empty 
relatively compact open set if and only if there exists a 
finite collection of points \( x_i \in X \) and sets \( O_i \) open in \( X \).
such that for any net \( \{ g_\nu \} \) in \( G \) with \( g^\varepsilon_i(x_i) \in \mathcal{O}_i \) for each \( i \) where \( \varepsilon_i = 1 \) [or \(-1\)] there exists a subnet \( \{g_{N(n)}\} \) and a \( g_0 \in G \) such that \( \lim_{n \to \infty} g_{N(n)}(x) = g_0(x) \) [and \( \lim_{n \to \infty} g_{N(n)}^{-1}(x) = g_0^{-1}(x) \)] for all \( x \in X \), and there exists a \( g^* \in G \) such that \( g^* \in \mathcal{O}_i \) for each \( i \).

**Proof** This is an immediate consequence of the preceding lemmas and the definition of \( \bigcap \) and \( \bigcap^* \).

Relating the above theory to the theory of Haar measure we have the following theorem.

**Theorem 6.12** Let \( G \) be a transitive group of automorphisms of a locally compact Hausdorff space \( X \). Assume that the following conditions hold.

1) For every finite collection \( (x_i, \mathcal{O}_i) \), \( x_i \in X \), \( \mathcal{O}_i \) open in \( X \), and every \( x \in X \), the set \( \{g(x): g(x_i) \in \mathcal{O}_i \text{ for each } i\} \) is open in \( X \).

2) There exists a finite collection \( (x_i, \mathcal{O}_i) \), \( x_i \in X \), \( \mathcal{O}_i \) open in \( X \), such that \( \{g: g(x_i) \in \mathcal{O}_i \text{ for each } i\} \) is not empty and for any net \( \{g_\nu\} \) with \( g_\nu(x_i) \in \mathcal{O}_i \) for each \( i \) there exists a subnet \( \{g_{N(n)}\} \) and \( g_0 \in G \) such that \( \lim_{n \to \infty} g_{N(n)}(x) = g_0(x) \) for all \( x \in X \).

Then \( G \) is a locally compact Hausdorff topological group and \( X \) may be identified with any of the left coset spaces \( G/H_x \). Clearly \( H_x \) is a closed subgroup so that we may apply the *Weil Theorem*: i.e., if \( \Delta \), \( \delta_x \) are the
modular functions for $G$, $H_x$ respectively, then there exists a (unique) Haar measure on $X$ if and only if $\Delta(h) = \delta_h(h)$ for all $h \in H_x$.

In the above theorem we are clearly imposing the topology $\Omega^*$ on $G$. A similar result holds for the topology $\Omega$.

We conclude this chapter with some pertinent examples.

**Example 6.1** Let $X = \mathbb{R} =$ the set of real numbers and let $\mathbb{R}^*$ be the set of non-zero rational numbers. Set $G = \mathbb{R}^* \times \mathbb{R}$; define $(r,a)x = rx + a$ for each $(r,a) \in G$ and each $x \in X$. Then $G$ is a transitive group of autohomeomorphisms of $X$ when we define $(r_1,a_1)(r_2,a_2) = (r_1r_2,r_1a_2 + a_1)$.

If $G$ is given the topology $\Omega^*$, then a net $\{(r_\nu,a_\nu)\}$ in $G$ converges to $(r_0,a_0)$ in $G$ if and only if $r_\nu x + a_\nu$ converges to $r_0 x + a_0$ for all $x \in X$. Letting $x = 0$ and $x = 1$ we see that this is the case if and only if $\{r_\nu\}$ converges to $r_0$ and $\{a_\nu\}$ converges to $a_0$.

Clearly $(r,a)^{-1} = (1/r,-a/r)$ for each $(r,a) \in G$. Now $(r_\nu,a_\nu) \rightarrow (r_0,a_0)$ in $G$ if and only if $r_\nu \rightarrow r_0$ and $a_\nu \rightarrow a_0$. Then $1/r_\nu \rightarrow 1/r_0$ and $-a_\nu/r_\nu \rightarrow -a_0/r_0$ so that "inverse" is continuous. Thus $\Omega^* = \Omega$. If also $(r_\nu^*,a_\nu^*) \rightarrow (r_0^*,a_0^*)$ then $r_\nu r_\nu^* \rightarrow r_0 r_0^*$ and $r_\nu a_\nu^* \rightarrow r_0 a_0^*$ so that $r_\nu a_\nu^* + a_\nu \rightarrow r_0 a_0^* + a_0$: i.e., $(r_\nu,a_\nu)(r_\nu^*,a_\nu^*) \rightarrow (r_0,a_0)(r_0^*,a_0^*)$. Thus $G$ is a topological group.
Remark 6.1.1 \( \bigcap^* = \bigcap \) and \( G \) is a Hausdorff topological group. However \( G \) is not evenly continuous since any interval, no matter how small, can be multiplied by a rational number to an arbitrarily large interval and then translated anywhere on the real line.

Let \( x_i \in X \) and \( O_i \) open in \( X \), \( i = 1, \ldots, n \) be a finite collection such that for some \( (r, a) \in G \), \( (rx_i + a) \in O_i \) for \( i = 1, \ldots, n \). Let \( r_i \) = distance from \( (rx_i + a) \) to \( (x - O_i) \) and let \( r_0 = r_1, \ldots, r_n > 0 \). Fix \( x \in X \); then \( |rx + a - y| < r_0 \) \( \Rightarrow \) there exists a \( b \in R \) such that \( y = (rx + a + b) \) and \( |b| < r_0 \). Then \( (rx_i + a + b) \in O_i \) for \( i = 1, \ldots, n \). Hence \( G/H_x \) is homeomorphic to \( X \) for every \( x \in X \) by Theorem 6.3.

Remark 6.1.2 \( X \) may be identified with every right or left coset space \( G/H_x \).

Continuing, there is clearly an irrational number \( r_0 \) such that \( (r_0x_i + a) \in O_i \), \( i = 1, \ldots, n \). Let \( \{r_j\} \) be a sequence of non-zero rational numbers such that \( r_j \rightarrow r_0 \). Then we may assume that \( (r_jx_i + a) \in O_i \) for \( i = 1, \ldots, n \) and for all \( j \). Clearly no subnet of \( \{(r_j, a)\} \) can converge to an element of \( G \). Thus \( G \) is not locally compact.

Remark 6.1.3 \( G \) is not locally compact even though \( X \) is a locally compact Hausdorff space.
Example 6.2 Consider $G = \mathcal{M}_3 \times E_3$ as a group of autohomeomorphisms of $E_3$ and the topology $\bigtriangleup^*$ on $G$ induced by $E_3$. Let $\{G\nu\} = \{(A\nu, a\nu)\}$ be a net in $G$ such that $g\nu(0, 0, \frac{1}{2}) \in O_1$ = open unit sphere. Then $\{a\nu\}$ is a net in the sphere of radius 2 about the origin. Thus we may assume there exists an $a_0 \in E_3$ such that $a_\nu \longrightarrow a_0$. But $\{A\nu(0, 0, \frac{1}{2})\}$, $\{A\nu(0, \frac{1}{2}, 0)\}$, and $\{A\nu(\frac{1}{2}, 0, 0)\}$ are nets on the surface of the sphere of radius $\frac{1}{2}$. Hence we may assume there exist $\bar{a}, \bar{b}, \bar{c} \in E_3$ such that $|\bar{a}| = |\bar{b}| = |\bar{c}| = \frac{1}{2}$ and $A\nu(0, 0, \frac{1}{2}) \longrightarrow \bar{a}$, $A\nu(0, \frac{1}{2}, 0) \longrightarrow \bar{b}$, $A\nu(\frac{1}{2}, 0, 0) \longrightarrow \bar{c}$. Clearly there is an $A_0 \in \mathcal{M}_3$ such that $A_0(0, 0, \frac{1}{2}) = \bar{a}$, $A_0(0, \frac{1}{2}, 0) = \bar{b}$, and $A_0(\frac{1}{2}, 0, 0) = \bar{c}$. But then $A\nu(\bar{x}) + a_\nu \longrightarrow A_0(\bar{x}) + a_0$ for all $\bar{x} \in E_3$.

Remark 6.2.1 Hence $\{G, \bigtriangleup^*\}$ is a locally compact Hausdorff topological group and $\bigtriangleup^* = \bigtriangleup$ by Theorem 6.10.

Let $\bar{x}_i \in E_3$, $O_i$ open in $E_3$, $i = 1, \ldots, n$ be a finite collection such that for some $(A, \bar{a}) \in G$, $(A\bar{x}_i + \bar{a}) \in O_i$ for $i = 1, \ldots, n$. Let $d(X - O_i, A\bar{x}_i + \bar{a}) = r_i > 0$ and let $r_0 = r_1, \ldots, r_n$. Fix an $\bar{x} \in E_3$. If $d(\bar{y}, A\bar{x} + \bar{a}) < r_0$ then there exists a $\bar{b} \in E_3$ such that $|\bar{b}| < r_0$ and $\bar{y} = A\bar{x} + \bar{a} + \bar{b}$. Then $A\bar{x}_i + \bar{a} + \bar{b} \in O_i$ for $i = 1, \ldots, n$.

Remark 6.2.2 Hence $G/H_{\bar{x}}$ is homeomorphic to $E_3$ for each $\bar{x} \in E_3$ by Theorem 6.3.
Example 6.3 Consider $G = \mathcal{M}_3 \times E_3$ as a group of autohomeomorphisms of the space $\mathcal{H}$ of oriented lines in $E_3$ with the topology $\bigvee^*$ induced by $\mathcal{H}$.

Let $\overline{e}_1 = (1,0,0)$, $\overline{e}_2 = (0,1,0)$, $\overline{e}_3 = (0,0,1)$, and $\overline{0} = (0,0,0)$. Let $k_1$ be the oriented line determined by $(\overline{e}_1, \overline{0})$; i.e., $k_1, k_2, k_3$ are the positively oriented $x, y, z$ axes respectively. Let $O_i$ be the sphere (in $\mathcal{H}$) of radius $1/10$ about $k_i$. Consider a net $\{g_\nu\}$ in $G$ such that $g_\nu(k_i) \in O_i$ for $i = 1, 2, 3$. If $g_\nu = (A_\nu, \overline{a}_\nu)$ then

$$d^*\left((A_\nu, \overline{a}_\nu)k_i, k_i\right) < 1/10 \rightarrow |A_\nu \overline{e}_1 - \overline{e}_1| + |\overline{\phi}\left((A_\nu, a_\nu)k_i\right)| < 1/10$$

for $i = 1, 2, 3$.

Let $\overline{a}_\nu^i$ denote the $i$th column of $A_\nu$ and let $B_\nu$ be the reference system defined by the $\overline{a}_\nu^i$; see Figure 3.

![Figure 3](image-url)
Write \( \bar{a}_\nu = (x_\nu, y_\nu, z_\nu) \) in the reference system \( B_\nu \).

Then \[ |\phi((A_\nu, \bar{a}_\nu)k_1)| = \sqrt{y_\nu^2 + z_\nu^2} < 1/10 \] and \[ |\phi((A_\nu, \bar{a}_\nu)k_2)| = \sqrt{x_\nu^2 + z_\nu^2} < 1/10. \]

Thus \( x_\nu^2 + y_\nu^2 < 1/100 \) and \( y_\nu^2 + z_\nu^2 < 1/100 \) so that \( x_\nu^2 + y_\nu^2 + z_\nu^2 < 1/50 \) or \( |\bar{a}_\nu| < 1/5. \)

Hence \( \{g_\nu\} \) is bounded in the (12-space) metric topology for \( G \); thus there exists a subnet \( \{g_{N(\nu)}\} \) and a \( g_0 \in G \) such that \( g_{N(\nu)} \rightarrow g_0 \) in the \( E_{12} \) metric topology for \( G \).

But the coset spaces of \( G \) with the \( E_{12} \) metric topology are homeomorphic to \( \mathcal{K} \) (Theorem 5.10) and \( G \) is a topological group with this metric topology. Then by Theorem 6.2 \( \mathcal{Q}^* \) is contained in the metric topology. Hence \( g_{N(\nu)} \rightarrow g_0 \) in \( \mathcal{Q}^* \). This shows that the set \( \{g : g(k_i) \in O_i, i = 1, 2, 3\} \) is a relatively compact non-empty open set. We know from Chapter 5 that \( \mathcal{K} \) is Hausdorff and locally compact (hence regular). Then by Theorem 6.10, \( \{G, \mathcal{Q}^*\} \) is a locally compact Hausdorff topological group.

Also, by Theorem 6.3, \( \mathcal{K} \) is homeomorphic to all coset spaces \( G/H_\chi \) since the \( E_{12} \) metric topology contains \( \mathcal{Q}^* \) and \( G/H_\chi \) is homeomorphic to \( \mathcal{K} \) if \( G \) is given the \( E_{12} \) metric topology.

**Example 6.4** If \( G \) is a locally compact Hausdorff topological group, it does not necessarily follow that \( G/H_\chi \) is homeomorphic to \( X \).
Let $X$ be the plane with the following lines deleted:

1) the coordinate axes,
2) the lines through $(1,1)$ parallel to the coordinate axes,
3) the lines through $(-1,-1)$ parallel to the coordinate axes.

Then $X = \{(x,y): x \neq 0 \neq y, |x| \neq 1 \neq |y|\}$.

Figure 4
Let \( \mathcal{B}_1 \) be the class of sets each of which is in \( X \) and is in the 1st or 3rd quadrant and is an open interval lying on some line parallel to the x-axis; let \( \mathcal{B}_2 \) be the class of sets each of which is in \( X \) and is in the 2nd or 4th quadrant and is an open interval lying on some line parallel to the y-axis. See Figure 4. Then \( \mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \) is a base for a topology \( \mathcal{I} \) on \( X \) and \( \mathcal{I} \) is not the discrete topology; i.e., points are not open.

In \( (X, \mathcal{I}) \), properly oriented bounded closed intervals on a line lying entirely in one quadrant are compact. Thus \( (X, \mathcal{I}) \) is locally compact and Hausdorff.

For each pair of non-zero real numbers \( (r_1, r_2) \),

\[
\frac{r_2}{r_1}(x, y) = ((\text{sgn } x)|x|^{r_1}, (\text{sgn } y)|y|^{r_2})
\]

is a one-to-one mapping from \( X \) onto \( X \). In the first quadrant we have

\[
\frac{r_2}{r_1}(a < x < b, y) = (a^{r_1} x^{r_1} < b^{r_1} y^{r_2});
\]

we obtain similar results for the base sets in the other quadrants. Thus each \( \frac{r_2}{r_1} \) is an open map.

Clearly \( \frac{r_2}{r_1} \circ \frac{r_4}{r_3} = \frac{r_2 r_4}{r_1 r_3} \) so that \( (\frac{r_2}{r_1})^{-1} = \frac{1}{r_1} \frac{1}{r_2} \).

Thus the class \( G_1 \) of all such functions is a group of autohomeomorphisms of \( X \). The class

\[
G_2 = \{ \text{rotation } k\pi/2 : k \text{ an integer} \}
\]

is also a group of autohomeomorphisms of \( X \).

Let \( G = \{ g_1 \circ g_2 : g_1 \in G_1, g_2 \in G_2 \} \). It is easily verified that \( (\text{rotation } \pi/2) \circ \frac{r_2}{r_1} = \frac{r_2}{r_1} \circ (\text{rotation } \pi/2) \) and from
this relation it can easily be verified that \( G \) is a group of autohomeomorphisms of \( X \). Clearly \( G \) is transitive on \( X \).

Consider \( z_0 = (2,2) \), \( z_1 = (2,-2) \) and respective neighborhoods \((1 < x < 3, 2) = O_0 \), \((2, -3 < y < -1) = O_1 \). Then \((fr_2 \circ \text{rotation } k\pi/2)(z_i) \in O_i \) for \( i = 0,1 \rightarrow k \equiv 0[4], 2^{r_2} = 2 \), and \( 2^{r_1} = 2 \rightarrow k \equiv 0[4], r_2 = 1, \) and \( r_1 = 1 \). Thus the identity is an open set in \( \{G, \Omega^*\} \). Then \( \Omega^* = \Omega = \text{discrete topology} \); hence \( \{G, \Omega^*\} \) is a locally compact Hausdorff topological group. But no coset space \( G/H_X \) is homeomorphic to \( X \) since \( X \) does not have the discrete topology.
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