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INTRODUCTION

In this dissertation we study sets of transfinite sequences on two symbols, say 0 and 1. In particular, we are concerned with certain order and topological properties which can be introduced in these sets.

Sets of transfinite sequences of 0 and 1, simply ordered by first differences, are referred to as "Universal Ordered Sets" since it can be shown (by virtue of the Axiom of Choice) that every simply ordered set is isomorphic to a subset of such a set.

Motivated by set-theoretical inclusion, we introduce the notion of strong first differences which yields a partial order in sets of transfinite sequences of 0 and 1. This enables us to employ these sets as "Universal Partially Ordered Sets" in connection with the representation of partially ordered sets. Moreover, based on the notion of weakly dense subsets, this representation can be accomplished by sequences which may be of length less than the cardinality of the partially ordered set to be represented.
We find that representation of simply ordered sets by transfinite sequences of 0 and 1 is closely related to the Generalized Continuum Hypothesis. Thus, under this hypothesis we are able to give a necessary and sufficient condition for such a representation. Furthermore, we find that certain statements involving the existence of a representation (by first differences) of simply ordered sets and the cardinality of their Hausdorff dense subsets are equivalent to the Generalized Continuum Hypothesis. Also, we find that the minimal length of sequences of 0 and 1 required for representation of a simply ordered set is independent of the Generalized Continuum Hypothesis if the method of strong first differences is used and is dependent on this hypothesis if the method of first differences is used.

The order by first differences yields immediately an interval topology in a set of transfinite sequences of 0 and 1. Likewise, such a set, viewed as a cartesian product of two-element discrete spaces, can be topologized by the usual product topology. Besides these topologies we introduce here the natural and the aleph topologies for sets of transfinite sequences of 0 and 1. The natural topology is based on the notion of an initial segment of a sequence and...
the aleph topology on that of an aleph limit point. The latter, in turn, is based on the concept of an initial segment and cardinality. Under some conditions on confinality of the order type of the sequences we find necessary and sufficient conditions for the existence of aleph limit points.

We find also that the interval, the product, and the natural topologies coincide if and only if the sequences are of type $\omega$. Moreover, this condition is also necessary and sufficient for the metrizability of the first two topologies, whereas confinality of the ordinal type of the sequences to $\omega$ is a necessary and sufficient condition for the metrizability of the natural topology. Furthermore, an aleph topology is metrizable if and only if it is a metrizable natural topology. Finally, we find that under the Generalized Continuum Hypothesis if the sequences are of regular ordinal type then there are exactly three distinct aleph topologies.

The proofs in this dissertation are based on the Zermelo-Fraenkel set-theoretical axioms, including the Axiom of Choice but excluding the Generalized Continuum Hypothesis unless expressly stated.
PART I

REPRESENTATION OF ORDERED SETS
SECTION I

REPRESENTATION OF PARTIALLY ORDERED SETS

In this section we consider the problem of representing partially ordered sets by sequences of 0 and 1. We partially order these sequences by strong first differences and simply order them by first differences. Our representation is based on the existence of weakly dense and very weakly dense subsets of partially ordered sets.

DEFINITION 1. Let \( D \) be a subset of a partially ordered set \((P, \leq)\) and let \( p \) and \( q \) be any two elements of \( P \) with \( q \not\leq p \). Then we say that \( D \) is weakly dense in \( P \) if there exists an element \( d_1 \) of \( D \) such that

\[
(1) \quad d_1 \leq q \quad \text{and} \quad d_1 \not\leq p
\]

Furthermore, we say that \( D \) is very weakly dense in \( P \) if there exists an element \( d_2 \) of \( D \) such that

\[
(2) \quad d_2 \leq q \quad \text{and} \quad d_2 \not\leq p
\]
As usual, in a partially ordered set \((P, \leq)\) we will call \(q\) an immediate successor of \(p\), and \(p\) an immediate predecessor of \(q\), if \(p < q\) and there exists no element \(s\) of \(P\) such that \(p < s < q\).

**REMARK 1.** We observe that any partially ordered set is weakly dense, and hence very weakly dense, in itself.

**LEMMA 1.** Let \((P, \leq)\) be a partially ordered set with a very weakly dense subset \(D\). Then for every element \(p\) of \(P\) there exists at most one immediate successor \(q\) of \(p\) with the property that \(d \in D\) and \(d < q\) implies \(d < p\).

**PROOF.** Suppose, on the contrary, that there exist in \(P\) two distinct immediate successors, \(q\) and \(r\) of \(p\) with the above mentioned property. Clearly \(r \not< q\), and since \(D\) is very weakly dense in \(P\) there exists a \(d \in D\) such that \(d \leq r\) and \(d \not< q\). But \(p < q\) and hence \(d \leq p\) is impossible. Thus, we have found a \(d \in D\) such that \(d \leq r\) and \(d \not< p\), contrary to the assumption that \(r\) had the property mentioned in the statement of the lemma. This proves the lemma.
THEOREM 1. For every ordinal number $\lambda$, a partially ordered set $(P, \leq)$ has a very weakly dense subset of power $\aleph_\lambda$ if and only if it has a weakly dense subset of power $\aleph_\lambda$.

PROOF. Let $D$ be a very weakly dense subset of $P$ with $|D| = \aleph_\lambda$. For each $d \in D$ adjoin to $D$ the (unique) immediate successor of $d$ which has the property mentioned in Lemma 1 whenever such a successor exists. Call the resulting set $E$. Clearly $|E| = \aleph_\lambda$.

We show that $E$ is weakly dense in $P$.

Let $p$ and $q$ be two elements of $P$ with $q \nleq p$. Then by (2) there exists a $d \in D$ (and hence $d \in E$) such that $d \leq q$ and $d \nleq p$. If $d \nleq p$ then (1) is satisfied. On the other hand, if $d = p$ then we have $p < q$.

Now, for the case where $d = p$, if $q$ is not the immediate successor of $p$ then there exists an $r \in P$ such that $p < r < q$. Hence, in view of (2) and the fact that $q \nleq r$, there exists an $e \in D$ (and hence $e \in E$) with $e \leq q$ and $e \nleq r$. Consequently, we have $e \leq q$ and $e \nleq p$. 
satisfying (1). However, if \( q \) is an immediate successor of \( p \), then in view of the construction of \( E \) either there exists an \( e \in D \) (and hence \( e \in E \)) with \( e \leq q \) and \( e \not\leq p \) or else we must have \( q \in E \), in which case \( q \leq q \) and \( q \not\leq p \). In either case (1) is satisfied. This shows that \( E \) is weakly dense in \( P \).

Since a weakly dense subset is clearly very weakly dense as well the theorem is proved.

**Lemma 2.** Let \( D \) be a weakly dense subset of a partially ordered set \((P, \leq)\) and let \( p \) and \( q \) be elements of \( P \). Then \( p \leq q \) if and only if for every element \( d \) of \( D \) we have \( d \leq p \) implies \( d \leq q \).

**Proof.** Suppose that for every \( d \in D \) we have \( d \leq p \) implies \( d \leq q \). If \( p \not\leq q \) then since \( D \) is weakly dense in \( P \) there exists a \( d_0 \in D \) such that \( d_0 \leq p \) and \( d_0 \not\leq q \), contrary to hypothesis. Hence, \( p \leq q \), as desired.

The converse is obvious.

**Remark 2.** It is easy to verify that any subset of a partially ordered set which characterizes the
order in that set in the manner of Lemma 2 must necessarily be a weakly dense subset.

DEFINITION 2. Let \((s_i)\) and \((t_i)\) be two sequences of 0 and 1 of the same finite or transfinite type. We say that \((s_i)\) is less than or equal to \((t_i)\) according to ordering by strong first differences, and we write

\[(3) \quad (s_i) \rightarrow^* (t_i)\]

if \(s_i = 1\) implies \(t_i = 1\) for every index \(i\).

As usual, if \((s_i) \rightarrow^* (t_i)\) and \((s_i) \not\rightarrow^* (t_i)\) then we write \((s_i) \nrightarrow^* (t_i)\). Clearly \(\rightarrow^*\) partially orders any set of sequences of 0 and 1 of the same type.

THEOREM 2. Let \((P, \leq)\) be a partially ordered set with a very weakly dense subset of power less than or equal to \(\aleph_\lambda\). Then \(P\) is isomorphic to a set of sequences of 0 and 1 of type \(\omega_\lambda\) partially ordered by strong first differences.

PROOF. By Theorem 1, \(P\) has a weakly dense subset \(D\) with \(D \leq \aleph_\lambda\). Consider the mapping \(f\)
from $P$ onto a subset $K$ of the power set of $D$
where for every $p \in P$ we have

$$(4) \quad f(p) = \{ d \mid d \in D \text{ and } d \leq p \}$$

We show that if $K$ is partially ordered by
inclusion then $f$ is an isomorphism.

In view of (4) and Lemma 2 it is clear that
$p \leq q$ if and only if $f(p) \subseteq f(q)$. Furthermore,
if $f(p) = f(q)$ then for every $d \in D$ we have
d $\in f(p)$ if and only if $d \in f(q)$. However, in
view of (4) and Lemma 2, this implies that $p = q$.
This shows that $f$ is an isomorphism.

Now let $(d^j)_{j<\nu}$ be a well-ordering of $D$
with $\nu \leq \omega^\lambda$. Let $g$ be a mapping from $P$ onto
a set $S$ of sequences of 0 and 1 of type $\omega^\lambda$
such that $g(p) = (p_i)_{i<\omega^\lambda}$ for every $p \in P$,
where

$$(5) \quad p_i = \begin{cases} 
1 & \text{if } d^i \in f(p) \\
0 & \text{otherwise}
\end{cases}$$

We show that $g$ is an isomorphism between
$(P, \leq)$ and $(g(P), \mathcal{P}^*)$.

For $p$ and $q$ in $P$ we have seen that $p \leq q$
if and only if $f(p) \subseteq f(q)$. But in view of (5) we
have \( f(p) \subset f(q) \) if and only if \( p_i = 1 \) implies \( q_i = 1 \). However, by Definition 2 this is equivalent to \( (p_i) \preceq (q_i) \).

On the other hand, if \( (p_i) = (q_i) \) then by (5) we have \( f(p) = f(q) \) which, as we have seen, implies \( p = q \). Hence \( g \) is an isomorphism, as desired.

**DEFINITION 3.** Let \((s^i)\) and \((t^i)\) be two sequences of 0 and 1 of the same finite or transfinite type. We say that \((s^i)\) is less than or equal to \((t^i)\) according to ordering by first differences and we write

\[
(s^i) \preceq (t^i)
\]

if \((s^i)\) is equal (identical) to \((t^i)\) or if there exists an index \( j \) such that \( s_j = 0 \) and \( t_j = 1 \) and \( s_i = t_i \) for every index \( i < j \).

Clearly \( \preceq \) simply orders any set of sequences of 0 and 1 of the same type.

**REMARK 3.** In view of Definitions 2 and 3 we see that if \((p_i)\) and \((q_i)\) are two sequences of
the same type then \((p_1) \rightarrow^* (q_1)\) implies \((p_1) \rightarrow (q_1)\).

In view of the above we obtain the following result of Marczewski [1] as an immediate consequence of Theorem 2.

**THEOREM 3.** Let \((P, \preceq^*)\) be a partially ordered set. Then there exists a simple order, \(\preceq\), in \(P\) such that \(p \preceq^* q\) implies \(p \preceq q\) for every two elements \(p\) and \(q\) of \(P\).

**PROOF.** Since, by Remark 1, \(P\) is a weakly dense subset of itself there exists an isomorphism \(f\) from \(P\) onto a set of sequences of 0 and 1 of type \(\omega^\lambda\) ordered by strong first differences, where \(\omega^\lambda = P\). For every \(p\) and \(q\) of \(P\) let \(p \preceq q\) if \(f(p) \preceq f(q)\). Clearly \(\preceq\) is a simple order in \(P\) which, in view of Remark 3 satisfies the conditions of the theorem.

We turn next to the problem of necessary conditions for representation of a partially ordered set by sequences of 0 and 1 partially ordered by strong first differences.
THEOREM 4. Let \((S_{\omega^*}, \leq^*)\) be the set of all sequences of 0 and 1 of type \(\omega^*\), partially ordered by strong first differences. Then \((S_{\omega^*}, \leq^*)\)
has a weakly dense subset of power \(\aleph_{\omega^*}\).

PROOF. For each \(j < \omega^*\) let \(d^j = (\delta^j_i)_{i < \omega^*}\) where \(\delta^j_i\) is the familiar Kronecker \(\delta\). Let \(D\) be the set of all such \(d^j\). Clearly \(D = \aleph_{\omega^*} \) and \(D \subseteq S_{\omega^*}\). We show that \(D\) is a weakly dense subset of \(S_{\omega^*}\).

Let \(s = (s_i)_{i < \omega^*}\) and \(t = (t_i)_{i < \omega^*}\) be two elements of \(S_{\omega^*}\) with \(s \not\supseteq^* t\). Then, in view of (1) there exists an index \(j\) such that \(s_j = 1\) and \(t_j = 0\). Consider \(d^j = (\delta^j_i)_{i < \omega^*}\). Clearly, \(d^j \supseteq^* s\) and \(d^j \not\supseteq^* t\). Hence, in view of Definition 1, \(D\) is weakly dense in \(S_{\omega^*}\) as desired.

COROLLARY 1. Let \(S\) be any set of sequences of 0 and 1 of type \(\omega^*\), partially ordered by strong first differences. Then there exists a set \(S'\) of sequences of 0 and 1 of type \(\omega^*\) with \(S \subseteq S'\) such that \(S = S'\) and \(S'\) has a weakly dense subset of power less than or equal to \(\omega^*\).
PROOF. If \( S \leq \kappa_\lambda \) let \( S' = S \) and since 
\( S \) is weakly dense in itself the conclusion of the 
corollary follows. If \( \bar{S} > \kappa_\lambda \) let \( S' = S \cup D \) 
where \( D \) is the weakly dense subset of \( S_{\omega_\lambda} \) men-
tioned in Theorem 4. Since \( \bar{D} = \kappa_\lambda \) we have 
\( \bar{S}' = \bar{S} \cup \bar{D} = \bar{S} \) and \( D \) is clearly weakly dense 
in \( S' \), as desired.

In connection with the above corollary, the 
following theorem shows that in general the above-
mentioned set \( S \) need not have a weakly dense subset 
\( E \) such that \( \bar{E} \leq \kappa_\lambda \).

THEOREM 5. There exists a set of sequences 
of 0 and 1 of type \( \omega \) partially ordered by 
strong first differences which has no weakly dense 
subset of power less than or equal to \( \kappa_0 \).

PROOF. Let \( P \) be the set of all those se-
quencces of 0 and 1 of type \( \omega \) which have in-
finitely many zeros and infinitely many ones, and 
let \( D \) be any denumerable subset of \( P \). We show 
that \( D \) is not weakly dense in \( P \).
Let $D = (d^j)_{j < \omega}$ and let $d^j = (d^j_i)_{i < \omega}$ for every $j < \omega$. Consider the two sequences $(i_k)$ and $(j_k)$ of natural numbers defined inductively as follows:

- $i_0$ is the first index such that $d^0_{i_0} = 0$
- $j_0$ is the first index such that $d^0_{j_0} = 1$

Moreover, for each natural number $n$

- $i_n$ is the first index such that $d^n_{i_n} = 0$
- $j_n$ is the first index such that $d^n_{j_n} = 1$

Clearly, for every natural number $k$ we have $d^k_{i_k} = 0$ and $d^k_{j_k} = 1$ with $j_{k-1} < i_k < j_k$.

Next we define a sequence $p = (p_m)_{m < \omega}$ as follows:

$$p_m = \begin{cases} 
1 & \text{if } m = i_k \text{ for some } k \\
0 & \text{otherwise (including } m = j_k) 
\end{cases}$$
Since for each k we have $d^k_{i_k} = 0$ and $d^k_{j_k} = 1$ we see that $p$ is not comparable to any element of $D$. Furthermore, since we have $j_{k-1} < i_k < j_k$ for every index $k$ it follows that $p \in P$. Thus, in view of Definition 1, it is clear that $D$ cannot be a weakly dense subset of $P$. This proves the theorem.

**Definition 4.** Let $(P, \leq)$ and $(S, \leq^*)$ be two partially ordered sets. A one-to-one function $f$ from $P$ onto $S$ is called a quasi-isomorphism if $p \leq q$ implies $f(p) \leq^* f(q)$ for every two elements $p$ and $q$ of $P$.

We conclude this section with a short proof of a stronger version of a theorem of J. Popruzenko [2] using a technique suggested by E. Mendelson [3].

**Theorem 6.** Every partially ordered set of power $\aleph^\lambda_\lambda$ is quasi-isomorphic to a set $S$ of sequences of 0 and 1 of type $\omega^\lambda_\lambda$, each with a last non-zero term and ordered by first differences.
Moreover, for every ordinal \( \tau < \omega_\lambda \) there exists an element \((s_i)_{i<\omega_\lambda}\) of \( S \) such that \( s_\tau = 1 \) and \( s_i = 0 \) for every \( i > \tau \).

**PROOF.** Let \((p^j)_{j<\omega_\lambda}\) be a well-ordering of a partially ordered set \((P, \leq)\) of power \( \aleph_\lambda \).

For \( p^j \in P \) let \( f(p^j) = (p^j_i)_{i<\omega_\lambda} \) with

\[
p^j_i = \begin{cases} 1 & \text{if } p^i \leq p^j \text{ and } i \leq j \\ 0 & \text{otherwise} \end{cases}
\]

For \( \tau < \omega_\lambda \), clearly \( f(p^\tau) = (p^\tau_i)_{i<\omega_\lambda} \) is such that \( p^\tau_\tau = 1 \) and \( p^\tau_i = 0 \) for \( i > \tau \). Thus, \( f \) maps \( P \) onto \( S \) (the range of \( f \)) and \( S \) satisfies the conditions set forth in the theorem.

Next, we prove that \( f \) is a one-to-one mapping.

Let \( f(p^j) = (p^j_i)_{i<\omega_\lambda} = (p^k_i)_{i<\omega_\lambda} = f(p^k) \).

Then \( p^j_j = 1 \) implies \( p^k_j = 1 \) which implies \( p^j \leq p^k \). Similarly, by symmetry, \( p^k_i = 1 \) implies \( p^k \leq p^j \). Hence, \( p^j = p^k \) as desired.

Finally, we prove that \( f \) preserves order.
Let $j, k < \omega_\lambda$ and $p^j < p^k$. Then for every $i \leq k$ we see that $p^j_i = 1$ implies $p^i \leq p^j \leq p^k$ and consequently $p^k_i = 1$. However, $p^j_k = 0$ and $p^k_k = 1$. Therefore $f(p^j) \not\rightarrow f(p^k)$.

Thus, $f$ is the desired quasi-isomorphism and the theorem is proved.
SECTION II

REPRESENTATION OF SIMPLY ORDERED SETS AND
THE GENERALIZED CONTINUUM HYPOTHESIS

In this section we consider the special case
of representation of simply ordered sets. We intro­
duce some stronger concepts of denseness and we
investigate the relationship of the Generalized
Continuum Hypothesis to necessary and sufficient
conditions for representation of a simply ordered
set by sequences of 0 and 1 ordered by first
differences

DEFINITION 5. Let $D$ be a subset of a simply
ordered set $(S, \leq)$ and let $p$ and $q$ be any two
elements of $S$ such that $p < q$. Then, as usual,
we say that $D$ is dense in $S$ if there exists an
element $d_1$ of $D$ such that

$$(6) \quad p < d_1 < q$$

Moreover, following Hausdorff [4], we say that
$D$ is Hausdorff-dense in $S$ if there exist two
elements $d_2$ and $d_3$ of $D$ such that

(7) \[ p \leq d_2 < d_3 \leq q \]

**REMARK 4.** In a simply ordered set $(S, \leq)$ Definition 1 assumes the following form: a subset $D$ of $S$ is weakly dense in $S$ if for every two elements $p$ and $q$ of $S$ with $p < q$ there exists an element $d_4$ of $D$ such that

(8) \[ p < d_4 \leq q \]

Moreover, $D$ is very weakly dense in $S$ if for every two elements $p$ and $q$ of $S$ with $p < q$ there exists an element $d_5$ of $D$ such that

(9) \[ p \leq d_5 \leq q \]

Clearly, in view of (6), (7), (8), and (9) denseness implies Hausdorff-denseness which implies weak denseness which in turn implies very weak denseness.

**THEOREM 7.** For every simply ordered set $(S, \leq)$
and for every ordinal number \( \lambda \) the following three statements are equivalent:

(i) \( S \) has a very weakly dense subset of power less than or equal to \( \kappa_\lambda \),

(ii) \( S \) has a weakly dense subset of power less than or equal to \( \kappa_\lambda \),

(iii) \( S \) has a Hausdorff-dense subset of power less than or equal to \( \kappa_\lambda \).

**PROOF.** Assume that \( S \) has a very weakly dense subset \( D \) of power less than or equal to \( \kappa_\lambda \). Let \( E \) be the set formed by adjoining to \( D \) all the immediate successors and immediate predecessors (in \( S \)) of elements of \( D \). Clearly, \( E \leq \kappa_\lambda \).

We show that \( E \) is a Hausdorff-dense subset of \( S \).

Let \( p \) and \( q \) be elements of \( S \) with \( p < q \). Then by (9) there exists a \( d \in D \) (and hence \( d \in E \)) such that \( p \leq d \leq e \). First, we assume that \( p \leq d < q \). Then if \( d \) is not the immediate predecessor of \( q \) there exists an \( r \in S \) such that \( p \leq d < r < q \) and by (9) there exists an \( e \in D \) (and hence \( e \in E \)) such that \( p \leq d < r \leq e \leq q \). Thus, \( p \leq d < e \leq q \) satisfying (7). On the other
hand, if $d$ is the immediate predecessor of $q$ then we have $q \in E$ and $p < d < q < q$ which again satisfies (7). A similar argument suffices if we assume that $p < d < q$. Since $p = d = q$ cannot occur $E$ is Hausdorff-dense in $S$, as desired.

Clearly, (iii) implies (ii) which implies (i). Thus, the theorem is proved.

In the above theorem the various kinds of denseness cannot include denseness in the usual sense. This is easy to see since in view of (6) any set which has at least one element with an immediate successor can have no dense subset.

REMARK 5. In view of Theorem 2 any simply ordered set which has a very weakly dense subset of power less than or equal to $\aleph_\lambda$ may be represented by sequences of 0 and 1 of type $\omega_\lambda$ ordered by strong first differences. In view of Remark 3 this representation is also a representation according to ordering by first differences.

THEOREM 8. Let $(S_{\omega_\lambda}, \rightarrow)$ be the set of all sequences of 0 and 1 of type $\omega_\lambda$, simply ordered
by first differences. Moreover, let $T_{\omega}$ be the subset of $S_{\omega}$ such that if $(t_i)_{i<\omega}$ is an element of $T_{\omega}$ then there exists an index $j$ such that $t_j = 1$ and $t_i = 0$ for every $i > j$.

Then $T_{\omega}$ is weakly dense in $(S_{\omega}, \leq)$ and furthermore, $T_{\omega}$ is a subset of every set which is weakly dense in $(S_{\omega}, \leq)$.

**PROOF.** First, we show that $T_{\omega}$ is weakly dense in $(S_{\omega}, \leq)$. Let $p = (p_i)_{i<\omega}$ and $q = (q_i)_{i<\omega}$ be two elements of $S_{\omega}$ with $p \leq q$. Then, in view of Definition 3 there exists an index $j$ such that

\begin{equation}
(10) \quad p_j = 0 \text{ and } q_j = 1 \text{ and } p_i = q_i \text{ for } i < j
\end{equation}

Let $t = (t_i)_{i<\omega}$ be the element of $T_{\omega}$ given by

\begin{equation}
(11) \quad t_i = \begin{cases} 
  p_i = q_i & \text{for } i < j \\
  1 & \text{for } i = j \\
  0 & \text{for } i > j 
\end{cases}
\end{equation}

Comparing (10) and (11) we see that
which shows that $T_{\omega_\lambda}$ is weakly dense in $(S_{\omega_\lambda}, \subset^3)$, as desired.

It remains to show that $T_{\omega_\lambda}$ is a subset of every weakly dense subset of $(S_{\omega_\lambda}, \subset^3)$. Let $t = (t_i)_{i<\omega_\lambda}$ be an element of $T_{\omega_\lambda}$ and let $j$ be the index such that

\[(12) \quad t_j = 1 \text{ and } t_i = 0 \text{ for } i > j\]

Consider the element $s = (s_i)_{i<\omega_\lambda}$ of $S_{\omega_\lambda}$ given by

\[(13) \quad s_i = \begin{cases} 
  t_i & \text{for } i < j \\
  0 & \text{for } i = j \\
  1 & \text{for } i > j
\end{cases}

In view of (12) and (13) it is clear that $s$ is the immediate predecessor of $t$. It follows by (8) that $t$ must be an element of any weakly dense subset of $(S_{\omega_\lambda}, \subset^3)$. Thus, the theorem is proved.

Theorems 4 and 5 proved the existence of a partially ordered set $S$ with a weakly dense subset of power $\aleph_0$. Moreover, $S$ contains a subset $P$ such that every weakly dense subset of $P$ is of
power greater than $\aleph_0$. The following theorem shows that this is not the case for simply ordered sets.

**THEOREM 9.** Let $(S, \leq)$ be a simply ordered set with a weakly dense subset $D$ of power less than or equal to $\aleph_\lambda$. Then every subset of $S$ contains a weakly dense subset of power less than or equal to $\aleph_\lambda$.

**PROOF.** Let $T$ be a subset of $S$ and let $A$ be the set of all elements of $T$ which have an immediate predecessor (in $T$). Let $a \in A$ and let $b$ be its immediate predecessor (in $T$). Since $D$ is weakly dense in $S$, in view of (8) there exists a $d \in D$ such that $b < d < a$. Thus, there exists a one-to-one mapping from $A$ into $D$ and therefore

$\mathfrak{A} \leq \mathfrak{D} \leq \aleph_\lambda$

Next, for each ordered pair $(d_1, d_2) \in (D \times D)$ let $T[d_1, d_2]$ be the set of all elements $t \in T$ such that

$\mathfrak{d}_1 \leq t \leq \mathfrak{d}_2$
Let $B$ be a set consisting of one element from each non-empty $T[d_1,d_2]$. Then

(16) \[ B \leq D \times D \leq \kappa_\lambda \cdot \kappa_\lambda = \kappa_\lambda \]

Finally, let $C = A \cup B$. Clearly $C \subseteq T$ and by (16) and (14) we have $C \leq \kappa_\lambda$. We show that $C$ is weakly dense in $T$.

Let $t$ and $u$ be elements of $T$ with $t < u$. If $t$ is the immediate predecessor (in $T$) of $u$ then $u \in A$ and we have $t < u \leq u$ with $u \in C$ as desired. On the other hand, if $t$ is not the immediate predecessor (in $T$) of $u$ then there exists a $v \in T$ with $t < v < u$. Now, since $D$ is weakly dense in $S$ we have elements $d_1$ and $d_2$ of $D$ such that

(17) \[ t < d_1 \leq v < d_2 \leq u \]

which, in view of (15) shows that $T[d_1,d_2] \neq \emptyset$. Thus, there exists a $b \in B$ (and hence $b \in C$) such that $b \in T[d_1,d_2]$ which, in view of (15) and (17), shows that $t < b \leq u$, as desired. Thus, the theorem is proved.

In view of Theorem 7, Theorem 9 remains valid.
if weakly dense is replaced by very weakly dense or by Hausdorff-dense.

In what follows we let \( \lim_{i<\lambda} 2^{\kappa_i} = \kappa_\tau \) in case \( \lambda = 0 \).

**COROLLARY 2.** Any simply ordered set \((S, \leq)\) which is isomorphic to a set of sequences of 0 and 1 of type \( \omega_\lambda \) simply ordered by first differences contains a Hausdorff-dense subset of power less than or equal to \( \kappa_\tau = \lim_{i<\lambda} 2^{\kappa_i} \).

**PROOF.** In view of Theorems 7, 8, and 9 we see that \( S \) must contain a Hausdorff-dense subset \( D \) such that

\[
(18) \quad D \leq \bigoplus_{i<\omega_\lambda} 2^i = \lim_{i<\omega_\lambda} 2^i = \lim_{i<\omega_\lambda} \kappa_i = \kappa_\tau
\]

which proves the corollary.
Let us recall [5, p. 387] that

\[(19) \quad \kappa_\lambda \leq \lim_{i<\lambda} 2^{\kappa_i}\]

for every ordinal number \( \lambda \).

**LEMMA 3.** The Generalized Continuum Hypothesis

(which asserts that \( 2^{\kappa_\alpha} = \kappa_{\alpha+1} \) for every ordinal number \( \alpha \)) is equivalent to

\[(20) \quad \kappa_\lambda = \lim_{i<\lambda} 2^{\kappa_i}\]

for every ordinal number \( \lambda \).

**PROOF.** If we assume the Generalized Continuum Hypothesis then

\[\lim_{i<\lambda} 2^{\kappa_i} = \lim_{i<\lambda} \kappa_{i+1} = \kappa_\lambda\]

for every ordinal number \( \lambda \).

On the other hand, if \( \kappa_\lambda = \lim_{i<\lambda} 2^{\kappa_i} \) for every ordinal \( \lambda \), and if \( \alpha \) is an arbitrary ordinal number, then letting \( \lambda = \alpha + 1 \) we obtain

\[\kappa_{\alpha+1} = \kappa_\lambda = \lim_{i<\lambda} 2^{\kappa_i} = \lim_{i<\alpha+1} 2^{\kappa_i} = 2^{\kappa_\alpha}\]
which, since \( \alpha \) is arbitrary, implies the Generalized Continuum Hypothesis, as desired.

**THEOREM 10.** Under the assumption of the Generalized Continuum Hypothesis, for every ordinal number \( \lambda \), a simply ordered set is isomorphic to a set of sequences of 0 and 1 of type \( \omega_\lambda \) and ordered by first differences if and only if it has a very weakly dense subset of power less than or equal to \( \aleph_\lambda \).

**PROOF.** If a simply ordered set \( (S, \leq) \) is isomorphic to a set of sequences of 0 and 1 of type \( \omega_\lambda \) ordered by first differences then by Corollary 2 it has a Hausdorff-dense subset \( D \) of power less than or equal to \( \lim_{i<\lambda} 2^{\aleph_i} \) which, in view of (20), implies that \( \bar{D} \leq \aleph_\lambda \). Consequently, by Theorem 7 \( (S, \leq) \) has a very weakly dense subset of power less than or equal to \( \aleph_\lambda \). On the other hand, if \( (S, \leq) \) has a very weakly dense subset of power less than or equal to \( \aleph_\lambda \) then by Remark 5 it is isomorphic to a set of sequences of 0 and 1 of type \( \omega_\lambda \) ordered by strong first differences and hence ordered by first differences as well.

This proves the theorem.
REMARK 6. It follows from Remark 5 and Theorem 10 that under the assumption of the Generalized Continuum Hypothesis we have: for every ordinal number \( \lambda \), a simply ordered set is isomorphic to a set of sequences of 0 and 1 of type \( \omega_\lambda \) ordered by first differences if and only if it is isomorphic to a set of sequences of the same type and kind ordered by strong first differences.

THEOREM 11. The Generalized Continuum Hypothesis is equivalent to the statement: for every ordinal number \( \lambda \) if a simply ordered set is isomorphic to a set of sequences of 0 and 1 of type \( \omega_\lambda \) ordered by first differences then it has a Hausdorff-dense subset of power less than or equal to \( \aleph_\lambda \).

PROOF. Clearly, in view of Theorems 7 and 10, the Generalized Continuum Hypothesis implies the statement mentioned in the theorem.

Next, assume the statement mentioned in the theorem. Consider the set \( S_{\omega_\lambda} \) of all sequences of 0 and 1 of type \( \omega_\lambda \) simply ordered by first differences. Obviously, \( S_{\omega_\lambda} \) is a simply ordered set of the type mentioned in the theorem. But, by
Theorem 8. \( T^{\omega_{\lambda}} \) is a subset of every weakly dense subset of \( S^{\omega_{\lambda}} \). Hence, in view of Corollary 2, we have

\[
\lim_{i<\lambda} 2^{\aleph_i} = \aleph_{\tau} \leq \aleph_{\lambda}
\]

This, by (19) and (20), implies the Generalized Continuum Hypothesis, as desired.

**THEOREM 12.** The Generalized Continuum Hypothesis is equivalent to the statement: for every ordinal number \( \lambda \) if a simply ordered set has a very weakly dense subset of power less than or equal to \( \aleph_{\tau} = \lim_{i<\lambda} 2^{\aleph_i} \) then it is isomorphic to a set of sequences of 0 and 1 of type \( \omega_{\lambda} \) ordered by first differences.

**PROOF.** Clearly, in view of (20), Theorem 10, and Remark 6, the Generalized Continuum Hypothesis implies the statement mentioned in the theorem.

Next, assume the statement mentioned in the theorem. Consider a well-ordered set \((W, \leq)\) of power \( \aleph_{\tau} \). Obviously, \( W \) is weakly dense in itself. Thus, \((W, \leq)\) is a simply ordered set of
the kind described by the statement mentioned in the theorem. It follows that \((W, \leq)\) is isomorphic to a well-ordered subset \(V\) of the set \(S^{\omega_\lambda}\) of all sequences of 0 and 1 of type \(\omega_\lambda\) ordered by first differences. But by Lemma 2 of \([6]\), we have \(\overline{V} \leq \aleph_\lambda\) and hence it follows that

\[
\lim_{i<\lambda} 2^{\aleph_1} = \aleph_\tau = \overline{W} = \overline{V} \leq \aleph_\lambda
\]

This, in view of (19) and (20), implies the Generalized Continuum Hypothesis, as desired.
SECTION III

THE MINIMAL LENGTH OF SEQUENCES
REPRESENTING SIMPLY ORDERED SETS

In this section we consider the problem of finding the minimal length of sequences of 0 and 1 necessary to represent simply ordered sets. We consider first sequences ordered by strong first differences and then sequences ordered by first differences.

THEOREM 13. Let \((S, \mathbin{\rightarrow}^*)\) be a set of sequences of 0 and 1 of type \(\omega\) simply ordered by strong first differences. Then there exists a set \(S'\) of sequences of 0 and 1 of type \(\omega\) simply ordered by strong first differences with \(S \subseteq S'\) and \(S = S'\) and such that \(S'\) has a very weakly dense subset of power less than or equal to \(\aleph_\lambda\).

PROOF. If \(\overline{S} \leq \aleph_\lambda\) then letting \(S = S'\) we derive the conclusion of the theorem. Thus, we shall assume that \(\overline{S} > \aleph_\lambda\).
For every sequence $s = (s_i)_{i < \omega_\lambda}$ of 0 and 1 of type $\omega_\lambda$ we denote by $[s]$ the set of all ordinal numbers $i$ such that $s_i = 1$. Clearly, in view of Definition 2, we have

$$(s \equiv^* t) \iff ([s] \subset [t])$$

Next, for each ordinal number $j < \omega_\lambda$ let

\begin{equation}
S_j = \bigcup \{ [s] : (s \in S) \land (s_j = 0) \}
\end{equation}

Finally, let $d^j = (d^j_i)_{i < \omega_\lambda}$ be defined as follows:

\begin{equation}
d^j_i = \begin{cases} 1 & \text{if } i \in S_j \\ 0 & \text{if } i \notin S_j \end{cases}
\end{equation}

and let $D = \{ d^j \mid j < \omega_\lambda \}$. Then clearly

\begin{equation}
\overline{D} = \mathcal{K}_\lambda
\end{equation}

Let $S' = S \cup D$. Obviously, $S \subset S'$ and in view of (23) and our assumption that $\overline{S} > \mathcal{K}_\lambda$ we have $\overline{S} = \overline{S'}$.

Next, we show that $D$ is very weakly dense in $S \cup D$. 
Let \( s = (s_i)_{i<\omega_\lambda} \) and \( t = (t_i)_{i<\omega_\lambda} \) be two elements of \( S' \) with \( s \not\preceq^* t \). If \( s \notin D \) or \( t \notin D \) then (9) is satisfied, as desired. Thus, we let \( s \in S \) and \( t \in S \). Since \( s \not\preceq^* t \) there exists an index \( j \) such that \( s_j = 0 \) and \( t_j = 1 \).

Consider \( d^j = (d^j_i)_{i<\omega_\lambda} \) as given by (22). Since \( s_j = 0 \), by (21) we have \([s] \subseteq S_j \). Consequently, in view of (22), we have \( s^j_i = 1 \) implies \( d^j_i = 1 \) for every index \( i \). Hence \( s \not\preceq^* d^j \).

Now, suppose that for some index \( i \) we have \( d^j_i = 1 \). Then by (22) we have \( i \in S_j \). Thus, in view of (21) there exists an element \( u = (u_i)_{i<\omega_\lambda} \) of \( S \) such that \( u_j = 0 \) and \( u_i = 1 \). However, since \( u_j = 0 \) and \( t_j = 1 \) and since \( S \) is simply ordered it follows that \( u \not\preceq^* t \). Thus, since \( u_i = 1 \) we have \( t_i = 1 \). Hence \( d^j_i = 1 \) implies \( t_i = 1 \), showing that \( d^j \not\preceq^* t \). Consequently, we have \( s \not\preceq^* d^j \) implying that \( D \) is very weakly dense in \( S \cup D \), as desired.

It remains to prove that \( S \cup D \) is simply ordered by strong first differences.

Let us observe that, as shown above, \( s \not\preceq^* d^j \) for every element \( s = (s_i)_{i<\omega_\lambda} \) of \( S \) with \( s_j = 0 \).
Moreover, \( d^j \geq^* t \) for every element \( t = (t_i)_{i < \omega^\lambda} \) of \( S \) with \( t_j = 1 \). Thus, in order to show that \( S \cup D \) is simply ordered we have to show that \( D \) is simply ordered.

Let \( d^j \) and \( d^k \) be two elements of \( D \) such that for each \( s \in S \) neither \( d^j \geq^* s \geq^* d^k \) nor \( d^k \geq^* s \geq^* d^j \) is the case. Then by (21) and (22) we have

\[
(s_j = 0) \iff (s \geq^* d^j) \iff (s \geq^* d^k) \iff (s_k = 0)
\]

for every \( s \in S \). Thus, in view of (21) we have \( S_j = S_k \), which by (22) shows that \( d^j = d^k \). Hence \( D \) is simply ordered and the theorem is proved.

**COROLLARY 3.** In view of Theorems 9 and 13 every simply ordered set which is isomorphic to a set of sequences of 0 and 1 of type \( \omega^\lambda \) ordered by strong first differences must contain a weakly dense subset of power less than or equal to \( \kappa^\lambda \).

**DEFINITION 6.** Let \( S \) be a simply ordered set.
We call the smallest cardinal number \( \kappa^\alpha \) such that \( S \) has a weakly dense subset of power \( \kappa^\alpha \) the
separability cardinal of $S$. We denote this by $\text{sep} S$.

**LEMMA 4.** Let $(S, \rightarrow^*)$ be a set of sequences of $0$ and $1$ of type $\nu$ ordered by strong first differences and let $\mu$ be an ordinal number with $\nu \leq \mu$. Then $S$ is isomorphic to a set $T$ of sequences of $0$ and $1$ of type $\mu$ ordered by strong first differences.

**PROOF.** Let $f$ be a one-to-one mapping from $\mu$ onto $\nu$. Define a function $g$ on $S$ as follows: for every element $s = (s_i)_{i<\nu}$ of $S$ let $t = (t_i)_{i<\mu} = g(s)$ where $t_i = s_{f(i)}$. Let $T = g(S)$. In view of Definition 2 it is clear that $g$ is an isomorphism between $S$ and $T$, as desired.

**THEOREM 14.** Let $(S, \leq)$ be a simply ordered set of separability cardinal $\aleph_\lambda$. Then $S$ is isomorphic to a set of sequences of $0$ and $1$ of type $\omega_\lambda$ ordered by strong first differences, and is not isomorphic to any set of such sequences of type smaller than $\omega_\lambda$.

**PROOF.** From Remark 5 it follows immediately that $S$ is isomorphic to a set of sequences of $0$ and $1$ of type $\omega_\lambda$ ordered by strong first
differences. Suppose, however, that $S$ is isomorphic to a set of sequences of 0 and 1 of type $v < \omega_{\lambda}$ ordered by strong first differences. Let $\omega_\tau$ be the initial ordinal such that $v = \omega_\tau$. But then, in view of Lemma 4, we see that $S$ is isomorphic to a set of sequences of 0 and 1 of type $\omega_\tau$ ordered by strong first differences. Consequently, in view of Corollary 3, $S$ must have a weakly dense subset of power less than or equal to $\aleph_\tau$ with $\aleph_\tau < \aleph_{\lambda}$. But this contradicts the assumption that the separability cardinal of $S$ is $\aleph_{\lambda}$. Thus, the theorem is proved.

In view of the above we see that the minimal length of sequences of 0 and 1 ordered by strong first differences which can represent a simply ordered set $S$ is determined solely by the power of the weakly dense subsets of $S$. We shall show, however, that in the case of ordering by first differences the minimal length of sequences of 0 and 1 which can represent $S$ does not depend solely on the power of the weakly dense subsets of $S$. In particular, the minimal length depends on the well-ordered subsets of $S$ and on the Generalized Continuum Hypothesis as well.
DEFINITION 7. We call the smallest cardinal such that a simply ordered set \((S, \leq)\) has a representation by sequences of 0 and 1 of type \(\omega^\beta\) ordered by first differences the dyadic cardinal of \(S\). We denote this by \(\text{dy } S\).

DEFINITION 8. We call the least upper bound of the powers of well-ordered subsets of \(S\) the well-ordering cardinal of \(S\). We denote this by \(\text{wo } S\).

THEOREM 15. For every infinite simply ordered set \((S, \leq)\) we have

\[
\text{wo } S \leq \text{dy } S \leq \text{sep } S
\]

Moreover, there exist simply ordered sets \(W, P, Q,\) and \(R\) such that (with appropriate assumptions concerning the Generalized Continuum Hypothesis) we have

\[
\text{wo } W = \text{dy } W = \text{sep } W
\]

and

\[
\text{wo } P < \text{dy } P = \text{sep } P
\]

and
PROOF. The left hand inequality of (24) follows from Lemma 2 of [6], in view of which no set of sequences of 0 and 1 of type $\omega_\lambda$ ordered by first differences can have a well-ordered subset of power greater than $\aleph_\lambda$. The right hand side of (24) follows from Theorem 7 and Remark 5.

To prove (25) we consider an infinite well-ordered set $(W, \leq)$. Since, in view of Definition 5 $W$ is its only Hausdorff-dense subset it follows that $\text{wo } W = \text{ sep } W$. Consequently, (25) follows from (24).

To prove (26) we assume the Generalized Continuum Hypothesis which, in view of Theorem 10, implies the right hand equality in (26). To complete the proof we consider the set $P$ given by

\[(29) \quad P = \{(x, y) \mid x \text{ real and } y = 0 \text{ or } y = 1\}\]

simply ordered by first differences. Let $K$ be any
well-ordered subset $P$ and let

$$U = \{ (x, y) \mid (x, y) \in K \text{ and } y = 0 \}$$

$$V = \{ (x, y) \mid (x, y) \in K \text{ and } y = 1 \}$$

Clearly, $K = U \cup V$. However, $U$ and $V$ are each isomorphic to a well-ordered subset of the real numbers and hence $K = U + V = \mathbb{N}_0$. Thus, we have

$$(30) \quad \text{wo } P = \mathbb{N}_0$$

Since we have established $dy P = \text{sep } P$ it remains to show that $\text{sep } P > \mathbb{N}_0$. Let $D$ be a Hausdorff-dense subset of $P$. For every real number $x$ we see that $(x, 0)$ is the immediate predecessor of $(x, 1)$ in $P$. However, $D$ must contain every element of $P$ which has an immediate successor or predecessor. Thus, we have $D = P$. Consequently,

$$(31) \quad \text{wo } D = \text{wo } P = 2^{\mathbb{N}_0} > \mathbb{N}_0$$

which, in view of (30), establishes the left hand inequality in (26).

To prove (27) we assume that $2^{\mathbb{N}_0} > \mathbb{N}_1$ and we consider $Q = P$ where $P$ is the set given by (29). We show that $Q$ can be represented by sequences
of 0 and 1 of type $\omega_1$ simply ordered by first differences. To this end it is enough to observe that every element $q \in \mathbb{Q}$ can be represented as a sequence of 0 and 1 where a sequence of 0 and 1 of type $\omega$ representing the first coordinate of $q$ is followed by $\omega_1$ copies of the second coordinate of $q$. Thus, by (30) we have $\overline{\omega_0} = \kappa_0$ and by (31) we have $\overline{\text{sep}} \mathbb{Q} = 2^{\kappa_0} > \kappa_1$. Finally, we have $\overline{\text{dy}} \mathbb{Q} \leq \omega_1$. However, (31) and Corollary 2 show that $\overline{\text{dy}} \mathbb{Q} = \kappa_0$ is impossible. This establishes (27).

To prove (28) we again assume that $2^{\kappa_0} > \kappa_1$ and we take $R$ to be the set $S_{\omega_1}$ of all sequences of 0 and 1 of type $\omega_1$ simply ordered by first differences. Then, in view of Theorem 8, $T_{\omega_1}$ is a minimal weakly dense subset of $R$ and by Corollary 2 we have

$$\overline{\lim}_{i<1} 2^{\kappa_1} = 2^{\kappa_0} > \kappa_1$$

But $\overline{\text{dy}} R \leq \kappa_1$ which establishes the right hand inequality of (28).
Furthermore, the set \( X = (x^j)_{j < \omega_1} \), where for each index \( j < \omega_1 \) we have \( x^j = (x^j_i)_{i < \omega_1} \) given by

\[ x^j_i = \begin{cases} 1 & \text{if } i \leq j \\ 0 & \text{otherwise} \end{cases} \]

is easily seen to be a well-ordered subset of \( S_{\omega_1} \) of power \( \aleph_1 \). Hence \( \omega_0 R = dy R \), which completes the proof.

**THEOREM 16.** The Generalized Continuum Hypothesis is equivalent to the statement: \( dy S = sep S \) for every simply ordered set \( S \).

**PROOF.** Assume the Generalized Continuum Hypothesis. Then, by Theorem 10, \( dy S = sep S \). On the other hand, if \( dy S = sep S \) then, by Theorem 11, the Generalized Continuum Hypothesis holds.
PART II

TOPOLOGIES ON SPACES OF
TRANSFINITE DYADIC SEQUENCES
SECTION IV

ALEPH LIMIT POINTS FOR SETS OF SEQUENCES OF ZERO AND ONE

In this section we consider sets of transfinite sequences of 0 and 1, also called spaces of transfinite dyadic sequences. We introduce the notion of limit points of subspaces of dyadic spaces without introducing at the outset a topology. We depend instead on the concept of cardinality.

In what follows we call the sequence \( (s_i)_{i<j} \) the \( j \)-segment of a sequence \( (s_i)_{i<v} \) for \( j \leq v \) where, as usual, \( i, j, \) and \( v \) are ordinal numbers.

**DEFINITION 9.** Let \( S \) be a set of sequences of 0 and 1 of type \( \omega_\lambda \). A sequence \( (s_i)_{i<\omega_\lambda} \) of 0 and 1 is called an \( \aleph_\alpha \)-limit point of \( S \) if for every ordinal number \( j < \omega_\lambda \) the power of the subset of \( S \) of sequences with \( j \)-segment equal to \( (s_i)_{i<j} \) is greater than or equal to \( \aleph_\alpha \).
LEMMA 5. Let $S$ be a set of sequences of 0 and 1 of type $\omega^\lambda$ and let $\alpha$ and $\beta$ be two ordinal numbers with $\alpha \leq \beta$. Then $s = (s^i_i < \omega^\lambda$ is an $\mathcal{K}_\alpha$-limit point of $S$ if $s$ is an $\mathcal{K}_\beta$-limit point of $S$.

PROOF. Clearly, if the subset $S_j$ of the elements of $S$ with $j$-segment equal to $(s^i_i < j$ is of power greater than or equal to $\mathcal{K}_\beta$ then $\mathcal{K}_\alpha \leq S_j$ for $\alpha \leq \beta$. Thus, indeed, if $s$ is an $\mathcal{K}_\beta$-limit point of $S$ then, in view of Definition 9, $s$ is an $\mathcal{K}_\alpha$-limit point of $S$, as desired.

Following [7] we introduce:

DEFINITION 10. An ordinal number $v$ is called confinal with a limit ordinal number $u$ if $v$ is the limit of an increasing sequence of type $u$ of ordinals.

Clearly, the ordinal $v$ in the above is a limit ordinal.

LEMMA 6. Let $v$ be an ordinal number confinal
with the limit ordinal number \( u \), and let \( u \) be confinal with the limit ordinal number \( w \). Then \( v \) is confinal with \( w \).

**PROOF.** Since \( v \) is confinal with \( u \) let \( v \) be the limit of the increasing sequence \((\lambda_i)_{i < u}\). Likewise, let \( u \) be the limit of the increasing sequence of ordinals \((\tau_i)_{i < w}\). We show that \( v \) is the limit of the increasing sequence \((\lambda_{\tau_i})_{i < w}\).

Let \( k < v \). Then there exists an ordinal \( j < u \) such that \( k < \lambda_j < v \). Furthermore, there exists an ordinal \( i \) such that \( j < \tau_i < u \). Then \( k < \lambda_j < \lambda_{\tau_i} < v \), showing that \( v \) is the limit of the increasing sequence \((\lambda_{\tau_i})_{i < w}\). Hence, \( v \) is confinal with \( w \), as desired.

**THEOREM 17.** A cardinal number \( \aleph_\lambda \) is the least upper bound of a set of power \( \aleph_\alpha \) of cardinal numbers less than \( \aleph_\lambda \) if and only if \( \omega_\lambda \) is confinal with a limit ordinal number \( u \) such that \( u \leq \omega_\alpha < \omega_\lambda \).

**PROOF.** Let

\[
(32) \quad \aleph_\lambda = \text{lub} \left\{ \aleph_{\tau_i} \mid i < \omega_\alpha \right\}
\]
where $\omega_\alpha < \omega_\lambda$ and $\tau_i < \tau_\lambda$ for every $i < \omega_\alpha$.

Let

$$w_j = \omega_{\tau_j} \cup \left( \bigcup_{i < j} w_i + 1 \right) \text{ for } j < \omega_\alpha$$

Clearly, the sequence $(w_i)_{i < \omega_\alpha}$ is increasing and, since $w_j \geq \omega_{\tau_j}$ for every $j < \omega_\alpha$, it follows from (32) that the limit of this sequence is greater than or equal to $\omega_\lambda$. Let $u$ be the smallest ordinal less than $\omega_\alpha$ such that $w_u \geq \omega_\lambda$, if such exists, and let $u = \omega_\alpha$ otherwise. We show that $u$ is a limit ordinal and that $\omega_\lambda$ is the limit of the sequence $(w_i)_{i < u}$.

Assume that $w_u < \omega_\lambda$. Then, in view of (33) and the fact that the sequence is increasing, we have

$$w_{j+1} = \omega_{\tau_{j+1}} \cup (w_j + 1)$$

Since $\omega_{\tau_{j+1}} < \omega_\lambda$, it follows that $w_{j+1} < \omega_\lambda$. Hence, $u < j + 1$ for every $j < \omega_\alpha$, showing that $u$ is a limit ordinal.

Next, assume that $\lim_{i < u} w_i < \omega_\lambda$. If $u < \omega_\alpha$ it would follow from (33) that $w_u < \omega_\lambda$, contrary
to the choice of $u$. If $u = \omega_\alpha$ then, as mentioned above, $\lim_{i<\omega_\alpha} w_i \geq \omega_\lambda$. Hence, in either case, we have $\lim_{i<u} w_i \geq \omega_\lambda$. However, in view of the definition of $u$ we must have $\lim_{i<u} w_i < \omega_\lambda$. Thus, $\lim_{i<u} w_i = \omega_\lambda$, showing that $\omega_\lambda$ is confinal with $u$, where $u \leq \omega_\alpha < \omega_\lambda$, as desired.

On the other hand, let $\omega_\lambda$ be the limit of an increasing sequence $(v_i)_{i<u}$ of type $u < \omega_\lambda$ with $\bar{u} = \omega_\alpha < \omega_\lambda$. We show that

$$\omega_\lambda = \text{lub} \left\{ \bar{v}_i \mid i < u \right\}$$

Suppose, on the contrary, that

$$\text{lub} \left\{ \bar{v}_i \mid i < u \right\} = \omega_\gamma < \omega_\lambda$$

Then, since $\bar{v}_i \leq \omega_\gamma$ for every $i < u$, we have

$$\omega_\delta = \lim_{i<u} v_i = \bigcup_{i<u} \bar{v}_i \leq \omega_\gamma \cdot \bar{u} < \omega_\lambda$$

which is a contradiction. Hence

$$\text{lub} \left\{ \bar{v}_i \mid i < u \right\} = \omega_\lambda$$

Thus, the theorem is proved.
THEOREM 18. Let $S$ be a set of sequences of 0 and 1 of type $\omega_\lambda$ such that $\omega_\lambda$ is conflinal with no ordinal less than or equal to $\nu$. Then $S$ has no $\aleph_\alpha$-limit point for any ordinal number $\alpha$.

PROOF. Let $t = (t_i)_{i<\omega_\lambda}$ be any sequence of 0 and 1 of type $\omega_\lambda$. For each element $s = (s_i)_{i<\omega_\lambda}$ of $S$ with $s \neq t$ let $o(s)$ be the smallest ordinal such that $s_i \neq t_i$.

Let

$$j = \operatorname{lub}_{(s \in S) \land (s \neq t)} o(s).$$

We show that $j < \omega_\lambda$. Assume the contrary, that $j = \omega_\lambda$. Then, by (34) and the fact that $\nu < \aleph_\lambda$ we see that $\aleph_\lambda = \operatorname{lub} \{ o(s) \mid s \in S \}$ and it follows from Theorem 17 that $\omega_\lambda$ is conflinal with an ordinal $u < \nu$. This contradicts the assumption of the theorem, and hence $j < \omega_\lambda$.

Thus, it follows that the $(j+1)$-segment of every element of $S$ (except $t$ itself if $t \in S$) is different from $(t_i)_{i<j+1}$. This shows that $t$ is not an $\aleph_\alpha$-limit point of $S$ for any $\alpha$. Thus, the theorem is proved.
LEMMA 7. Let \( v \) be an ordinal number and let \( u \) be the smallest limit ordinal number with which \( v \) is confinal. Then \( u \) is an initial ordinal.

PROOF. Let \( v \) be the limit of the increasing sequence \( (w_i)_{i<u} \) and assume, by way of contradiction that \( u \) is not an initial number. Let \( \omega_\lambda = \overline{u} \) with \( \omega_\lambda < u \), and let \( f \) be a one-to-one function from \( \omega_\lambda \) onto \( u \). Let

\[
(35) \quad \tau_j = w_f(j) \cup \left( \bigcup_{i<j} \tau_i + 1 \right) \quad \text{for} \quad j < \omega_\lambda
\]

Clearly, the sequence \( (\tau_j)_{j<\omega_\lambda} \) is increasing and
\[
\lim_{j<\omega_\lambda} \tau_j \geq v.
\]
Let \( n \) be the smallest ordinal such that \( \tau_n \geq v \) if such exists, and let \( n = \omega_\lambda \) otherwise. As in the proof of Theorem 17, it follows that \( \lim_{j<n} \tau_j = v \). But \( n \leq \omega_\lambda < u \), contradicting the assumption that \( u \) is the smallest ordinal with which \( v \) is confinal. This proves the theorem.

THEOREM 19. Let \( \alpha \) and \( \lambda \) be ordinal numbers and let \( \omega_\tau \) be the smallest ordinal with which \( \omega_\lambda \) is confinal. If \( \kappa_\tau \leq \kappa_\alpha \leq 2^{\kappa_\lambda} \) then there exists a set \( S \) of sequences of 0 and 1 with \( \overline{S} = \kappa_\alpha \) such that \( S \) has an \( \kappa_\alpha \)-limit point.
PROOF. Let $\omega_\lambda$ be the limit of the increasing sequence of ordinals $(w_i)_{i<\omega_\tau}$ and since $\kappa_\alpha \leq 2^{\kappa_\lambda}$ let $T$ be a set of sequences of 0 and 1 of type $\omega_\lambda$ such that $T = \kappa_\alpha$. For each $j < \omega_\tau$ let $S_j$ be the set formed by preceding each element of $T$ with $w_j$ zeros. Since $w_j < \omega_\lambda$ it follows from [5, p. 331] that $w_j + \omega_\lambda = \omega_\lambda$ for every $j < \omega_\tau$. Hence, $S_j$ is a set of sequences of 0 and 1 of type $\omega_\lambda$ with $S_j = \kappa_\alpha$. Let

$$S = \bigcup_{j<\omega_\tau} S_j$$

Then $S = \kappa_\alpha \cdot \kappa_\tau = \kappa_\alpha$.

We show that the sequence $(z_i)_{i<\omega_\lambda}$ where $z_i = 0$ for every $i < \omega_\lambda$ is an $\kappa_\alpha$-limit point of $S$. Let $k < \omega_\lambda$ be given. Then there exists a $j < \omega_\tau$ such that $k < w_j$. Clearly, $S_j$ is a subset of $S$ of power $\kappa_\alpha$ whose elements have $k$-segment equal to $(z_i)_{i<k}$. Thus, the theorem is proved.

THEOREM 20. Let $S$ be a set of sequences of 0 and 1 of type $\omega_\lambda$ and let $\omega_\tau$ be the smallest ordinal with which $\omega_\lambda$ is confinal. Moreover, let
t = (t_i)_{i<\omega_\lambda} \text{ be an } \kappa_\alpha \text{-limit point of } S \text{ with } 
alpha \leq \tau. \text{ Then } t \text{ is an } \kappa_\tau \text{-limit point of } S.

PROOF. Omitting the trivial case where \alpha = \tau, we assume, by way of contradiction, that 
t = (t_i)_{i<\omega_\lambda} \text{ is an } \kappa_\alpha \text{-limit point of } S \text{ with } 
\alpha < \tau \text{ and } t \text{ is not an } \kappa_\tau \text{-limit point of } S.

Then, in view of Definition 9, there exists an ordinal 
j < \omega_\lambda \text{ such that the subset } S_j \text{ of } S \text{ of the } 
elements with j-segment equal to (t_i)_{i<j} \text{ is of } 
\text{power less than } \kappa_\tau. \text{ Thus, } \kappa_\alpha \leq S_j < \kappa_\tau.

Then, in view of Theorem 18, we see that \( t \) cannot be 
an \kappa_\alpha \text{-limit point of } S_j. \text{ Hence, there exists an } 
\text{ordinal } k \text{ such that the subset } S_k \text{ of } S_j \text{ of the } 
elements with k-segment equal to (t_i)_{i<k} \text{ is of } 
\text{power less than } \kappa_\alpha. \text{ However, since } \kappa_\alpha \leq S_j \text{ we see that } k > j. \text{ But this contradicts the fact that } 
t \text{ is an } \kappa_\alpha \text{-limit point of } S \text{ since the } 
\text{set of all elements of } S \text{ with } k\text{-segments equal to } (t_i)_{i<k} \text{ is equal to } S_k \text{ and } S_k < \kappa_\alpha.

Thus, the theorem is proved.

COROLLARY 4. Let \( S \) be a set of sequences of 
0 \text{ and } 1 \text{ of type } \omega_\lambda \text{ and let } \omega_\tau \text{ be the smallest }
ordinal with which $\omega_\lambda$ is confinal. Then
$$t = (t_i)_{i < \omega_\lambda} \text{ is an } \mathcal{A}_\alpha \text{-limit point of } S \text{ with } \alpha \leq \tau \text{ if and only if } t \text{ is an } \mathcal{A}_{\tau} \text{-limit point of } S.$$  

**Proof.** If $t$ is an $\mathcal{A}_{\tau}$-limit point of $S$ then in view of Lemma 5, for $\alpha \leq \tau$ we see that $t$ is an $\mathcal{A}_\alpha$-limit point of $S$. On the other hand, if $t$ is an $\mathcal{A}_\alpha$-limit point of $S$ with $\alpha \leq \tau$ then in view of Theorem 20 we see that $t$ is an $\mathcal{A}_{\tau}$-limit point of $S$.

**Theorem 21.** Let $\alpha$ and $\lambda$ be ordinal numbers such that:

1. $\omega_\alpha$ is confinal with no ordinal less than or equal to $2^{\mathcal{A}_\beta}$ for any $\beta < \lambda$,

2. $\omega_\alpha$ is confinal with no ordinal less than or equal to $\omega_\lambda$.

Then a set $S$ of sequences of 0 and 1 of type $\omega_\lambda$ contains an $\mathcal{A}_\alpha$-limit point if and only if $\bar{S} \geq \mathcal{A}_\alpha$.

**Proof.** Let $S$ be a set of sequences of 0 and 1 of type $\omega_\lambda$ with $\bar{S} = \mathcal{A}_\alpha$. Assume, by way of
contradiction, that no element of $S$ is an $\aleph_\alpha$-limit point of $S$. For each $j < \omega_\lambda$ let $V_j$ be the subset of $S$ such that if $v = (v_i)_{i < \omega_\lambda}$ is an element of $V_j$ then the subset of $S$ of elements with $j$-segment equal to $(v_i)_{i < j}$ is of power less than $\aleph_\alpha$. (We note that if this is the case then every element of such a subset of $S$ is an element of $V_j$). Then

$$\overline{V_j} \leq \left( \text{lub}_{i < 2^j} (c_i) \right) \cdot 2^j$$

where $c_i$ is a cardinal less than $\aleph_\alpha$ for every $i < 2^j$. Since by (i) $\omega_\alpha$ is not cofinal with any ordinal less than or equal to $2^j$ it follows from Theorem 17 that $\overline{V_j} < \aleph_\alpha$.

We show that

$$(36) \quad S = \bigcup_{j < \omega_\lambda} V_j$$

Let $s = (s_i)_{i < \omega_\lambda}$ be an element of $S$. Then $s$ is not an $\aleph_\alpha$-limit point of $S$. Hence there exists an ordinal $j$ such that the subset of $S$ of elements with $j$-segment equal to $(s_i)_{i < j}$ is of
power less than $\kappa_\alpha$. Thus $s \in V_j$, which establishes (36).

However,

$$\bigcup_{j<\omega_\lambda} V_j \leq \left(\text{lub} \left\{ \bar{V}_j \right\}\right) \cdot \kappa_\lambda$$

and $\bar{V}_j < \kappa_\alpha$ for each $j < \omega_\lambda$. But $\omega_\alpha$ is not confinal with any ordinal less than or equal to $\omega_\lambda$. Consequently, in view of Theorem 17, we have

$$\bigcup_{j<\omega_\lambda} V_j < \kappa_\alpha$$

contrary to the assumption that $S = \kappa_\alpha$. Thus, $S$ must contain an $\kappa_\alpha$-limit point, as desired.

On the other hand, if $S < \kappa_\alpha$ then clearly $S$ can have no $\kappa_\alpha$-limit point. Thus, the theorem is proved.

**THEOREM 22.** Let $\alpha$ and $\lambda$ be ordinal numbers such that $\kappa_\alpha = 2^{\kappa_\lambda}$ and

(i') $\omega_\alpha$ is confinal with an ordinal less than or equal to $2^{\kappa_\beta}$ for some $\beta < \lambda$. 
Then there exists a set $S$ of sequences of 0 and 1 of type $\omega^\lambda$ with $S = \kappa_\alpha$ such that $S$ has no $\kappa_\alpha$-limit point.

PROOF. Let $\omega_\alpha$ be the limit of the increasing sequence of ordinals $(w_i)_{i < u}$ where $u \leq 2^{\aleph_\beta}$. If $u = \omega_\alpha$ then there exists a set $S$ of sequences $(s_i)_{i < \omega^\lambda}$ such that $s_i = 0$ for $\omega_\beta \leq i < \omega^\lambda$ and $S = \kappa_\alpha$. This set clearly has no $\kappa_\alpha$-limit point.

Assume that $u < \omega_\alpha$. Then, as in the proof of Theorem 17, we have $\kappa_\alpha = \operatorname{lub} w_i$. Let $B = \{b^j \mid j < u\}$ be a set of sequences of 0 and 1 of type $\omega_\beta$. Since $\sup w_i \leq \kappa_\alpha \leq 2^{\kappa_\lambda}$ for each $i < u$, we let $T_i$ be a set of sequences of 0 and 1 of type $\omega^\lambda$ such that $T_i = w_i$. Let $U_i$ be the set formed by preceding each element of $T_i$ by $b^i$.

Since by [5, p. 331] we have $\omega_\beta + \omega^\lambda = \omega^\lambda$, it follows that $U_i$ is a set of sequences of 0 and 1 of type $\omega^\lambda$ with $U_i = w_i$. Let

$$S = \bigcup_{i < u} U_i$$

Then, since $u < \kappa_\alpha$ we have $S = \operatorname{lub} w_i = \kappa_\alpha$. 

We show that $S$ has no $\mathcal{K}_\alpha$-limit point. To this end we observe that if $k$ is an ordinal such that $\omega_\beta \leq k < \omega_\lambda$ then the subset of $S$ of elements with $k$-segment equal to any given $k$-segment (say $(s_i)_{i < k}$) is either empty of a subset of the $U_j$ such that $b^j = (s_i)_{i < \omega_\beta}$. Hence, the subset of $S$ of elements with $k$-segment equal to $(s_i)_{i < k}$ is of power less than or equal to $\omega_j < \mathcal{K}_\lambda$. This shows that $S$ has no $\mathcal{K}_\alpha$-limit point, as desired.

**Theorem 23.** Let $\alpha$ and $\lambda$ be ordinal numbers such that $\mathcal{K}_\alpha \leq 2^{\mathcal{K}_\lambda}$ and

(ii') $\omega_\alpha$ is confinal with $\omega_\lambda$.

Then there exists a set $S$ of sequences of 0 and 1 of type $\omega_\lambda$ with $S = \mathcal{K}_\alpha$ such that $S$ contains no $\mathcal{K}_\alpha$-limit point.

**Proof.** If $\omega_\alpha = \omega_\lambda$ then the set

$$D = \{d^j \mid j < \omega_\lambda\}$$

where $d^j = (d^j_i)_{i < \omega_\lambda}$ is a set of power $\mathcal{K}_\alpha$. Moreover, the sequence $z = (z_i)_{i < \omega_\lambda}$ where $z_i = 0$ for every $i < \omega_\lambda$ is clearly the only $\mathcal{K}_\alpha$-limit point of $D$ and $z \notin D$, as desired.
Assume that $\omega_\lambda < \omega_\alpha$, and let $\omega_\alpha$ be the limit of the increasing sequence of ordinals 
$(w_i)_{i<\omega_\lambda}$. Then, as in the proof of Theorem 17, we have $\omega_\alpha = \operatorname{lub} \{w_i\}_{i<\omega_\lambda}$. Since $\bar{w}_j \leq \omega_\alpha \leq 2^{\omega_\lambda}$ for each $j < \omega_\lambda$, we let $T_j$ be a set of sequences of 0 and 1 of type $\omega_\lambda$ with $T_j = w_j$. Let $S_j$ be the set formed by preceding each element of $T_j$ by the sequence $(t^j_i)_{i<\omega_\lambda+1}$ where $t^j_i = 0$ for $i < j$ and $t^j_i = 1$ for $i = j$. Let 

$$S = \bigcup_{j<\omega_\lambda} S_j$$

Then, since $\omega_\lambda < \omega_\alpha$ we have $S = \operatorname{lub} \{w_i\}_{i<\omega_\lambda} = \omega_\alpha$. 

Now, let $s = (s_i)_{i<\omega_\lambda}$ be an $\omega_\alpha$-limit point of $S$. Since the sequence $(\bar{w}_i)_{i<\omega_\lambda}$ is non-decreasing we have 

$$(37) \quad \bigcup_{j<k} S_j \leq \bar{w}_k \leq \omega_\alpha$$

If $V_j$ is the subset of $S$ of elements whose $j$-segment is equal to $(s_i)_{i<j}$ then $V_j = \omega_\alpha$. Hence, in view of (37), given any ordinal $k < \omega_\lambda$, the set $V_j$ must contain elements of $S_i$ for some $i > k$. This shows that the $j$-segment of $s$ is the
zero sequence for each $j$. Thus, $s$ is itself the zero sequence. But then, in view of the construction of $S$, we have $s \notin S$. Hence, $S$ does not contain its $\aleph_\alpha$-limit point, as desired.

Thus, the theorem is proved.

In what follows we denote by $S^\alpha$ the set of all $\aleph_\alpha$-limit points of $S$.

COROLLARY 5. Let $\alpha$ and $\lambda$ be ordinal numbers such that conditions (i) and (ii) of Theorem 21 are satisfied, and let $S$ be a set of sequences of 0 and 1 of type $\omega_\lambda$. Then the set of elements of $S$ which are not $\aleph_\alpha$-limit points of $S$ is of power less than $\aleph_\alpha$.

PROOF. If $S - S^\alpha \geq \aleph_\alpha$ then, by Theorem 21, $S - S^\alpha$ contains an $\aleph_\alpha$-limit point, contrary to the definition of $S^\alpha$.

THEOREM 24. Let $\alpha$, $\beta$, and $\lambda$ be ordinal numbers and let $S$ be a set of sequences of 0 and 1 of type $\omega_\lambda$. Then

$$(38) \quad (S^\alpha)^3 \subset S^\alpha$$
Moreover, let $\beta \leq \alpha$ and let $\alpha$ be an ordinal number such that:

(i) $\omega_{\alpha}$ is cofinal with no ordinal less than or equal to $2^\gamma$ for any $\gamma < \lambda$,

(ii) $\omega_{\alpha}$ is cofinal with no ordinal less than or equal to $\omega_{\lambda}$.

Then

(39) $$(S^\alpha)^\beta = S^\alpha$$

and

(40) $$S^\alpha \subset (S^\beta)^\alpha$$

PROOF. To prove (38) let $s = (s_i)_{i<\omega_{\lambda}}$ be an element of $(S^\alpha)^\beta$ and let $j < \omega_{\lambda}$ be given. Then there exists an element $t = (t_i)_{i<\omega_{\lambda}}$ of $S^\alpha$ such that $(t_i)_{i<j} = (s_i)_{i<j}$. Thus, the set of elements of $S$ with $j$-segment equal to $(t_i)_{i<j}$, and hence equal to $(s_i)_{i<j}$, is of power greater than or equal to $\aleph_\alpha$. Hence, it follows that $s \in S^\alpha$, showing (38). To prove that the equality need not hold in (38) it suffices to consider a set $S$ with a single $\aleph_\alpha$-limit point $t$. Then, clearly, $S^\alpha = \{t\}$ and $(S^\alpha)^\beta = \emptyset$.

Next, if $\bar{S} < \aleph_\alpha$ then $S^\alpha = \emptyset$ and both (39)
and (40) follow immediately. If \( \overline{S} \geq \aleph_\alpha \) we let
\[ s = (s_i)_{i<\omega} \in S^\alpha \quad \text{and} \quad j < \omega \gamma. \]
Then the set \( S_j \) of elements of \( S \) with \( j \)-segment equal to
\( (s_i)_{i<j} \) is of power \( \aleph_\alpha \). Since conditions (i) and (ii) of Corollary 5 are satisfied we have
\[ S - S^\alpha < \aleph_\alpha, \]
and hence it follows that
\[ S_j \cap S^\alpha = \aleph_\alpha > \aleph_\beta. \]
Thus, \( s \) is an \( \aleph_\beta \)-limit point of \( S^\alpha \), as desired. Hence, in view of (38), we obtain (39).

To show (40) we first observe that if \( A \subset B \) then \( A^\alpha \subset B^\alpha \). However, by Lemma 5 we have
\[ S^\alpha \subset S^\beta \quad \text{for} \quad \beta \leq \alpha. \]
Hence, by (39) we have
\[ S^\alpha = (S^\alpha)^\alpha \subset (S^\alpha)^\beta \]
as desired.

Finally, we show that (40) cannot be extended to equality. Let \( \lambda = \beta = 0 \) (i.e., the sequences are of type \( \omega \)), let \( \alpha = 1 \), and let \( S \) be the set of all sequences with finitely many ones. Then
\[ \overline{S} = \aleph_0. \]
Thus, \( S^\alpha = S^1 = \emptyset \). However, \( S^\beta = S^0 \) is the set of all sequences of 0 and 1 of type \( \omega = \omega_0 \), and clearly \( (S^\beta)^\alpha = (S^0)^1 = S^0 \neq \emptyset \).

Thus, the theorem is proved.
THEOREM 25. Let \( S \) be a set of sequences of 0 and 1 of type \( \omega_\lambda \) and let \( \alpha \) be a limit ordinal number. Then

\[
S^\alpha = \bigcap_{\beta < \alpha} S^\beta
\]

PROOF. It follows from Lemma 5 that

\[
S^\alpha \subseteq \bigcap_{\beta < \alpha} S^\beta
\]

Assume, by way of contradiction, that there exists an element \( s = (s_i)_{i<\omega_\lambda} \) of \( \bigcap_{\beta < \alpha} S^\beta \) such that \( s \notin S^\alpha \). Then there exists an ordinal \( j \) such that the subset \( S_j \) of \( S \) of elements with \( j \)-segment equal to \( (s_i)_{i<j} \) is of power less than \( \kappa^\alpha \). Let \( \kappa^\gamma = S_j \). But then \( s \notin S^{\gamma+1} \), with \( \gamma + 1 < \alpha \), contrary to the assumption. Thus, the theorem is proved.

In view of [8, pp. 68-71] we consider the following:

DEFINITION 11. An initial ordinal number \( \omega_\lambda \) is called weakly inaccessible if \( \omega_\lambda \) is confinal with no ordinal less than \( \omega_\lambda \) and \( \kappa^{\alpha+1} < \kappa^\lambda \) for every \( \alpha < \lambda \). Furthermore, \( \omega_\lambda \) is called
strongly inaccessible if \( \omega_\lambda \) is confinal with no ordinal less than \( \omega_\lambda \) and \( 2^{\kappa_\alpha} < \kappa_\lambda \) for every \( \alpha < \lambda \).

In view of Theorems 21, 22, and 23 and the results for sequences of type \( \omega \) given in [9], we would expect to prove the following:

**PROPOSITION.** Let \( \alpha \) and \( \lambda \) be ordinal numbers such that

(i) \( \omega_\alpha \) is confinal with no ordinal less than or equal to \( 2^{\kappa_\beta} \) for any \( \beta < \lambda \),

(ii') \( \omega_\alpha \) is confinal with \( \omega_\lambda \).

Then every set \( S \) of sequences of 0 and 1 of type \( \omega_\lambda \) such that \( S \geq \kappa \) has an \( \kappa_\alpha \)-limit point.

We observe that under the conditions (i) and (ii') \( \omega_\lambda \) must be strongly inaccessible. This follows from the fact that \( \omega_\alpha \), and hence, by Lemma 6, \( \omega_\lambda \) as well, cannot be confinal with any ordinal less than or equal to \( 2^{\kappa_\beta} \) for any \( \beta < \lambda \).
In [10] and [11] it is shown that using the usual axioms of set theory, including the Axiom of Choice and the Generalized Continuum Hypothesis, it is impossible to prove the existence of (weakly or strongly) inaccessible cardinals. Thus, we may assume that all cardinals are accessible, and in this case the above proposition is vacuously true.

On the other hand, it is shown in [12] that if \((\forall \alpha)(2^{\aleph_\alpha} \geq \aleph_{\alpha+1})\) cannot be proved without the Axiom of Choice then the assumption that there exist inaccessible cardinals is consistent with the axioms of set theory, if these are consistent. Nevertheless, so far it is not known that one cannot postulate the existence of inaccessible cardinals.

In view of the above, if it is possible to postulate the existence of inaccessible cardinals, it may still be the case that the above proposition is neither provable nor disprovable.
SECTION V

THE INTERVAL, NATURAL, AND PRODUCT TOPOLOGIES FOR SETS OF SEQUENCES OF ZERO AND ONE

In this section we consider the interval, the natural and the product topologies on sets of transfinite sequences of 0 and 1. We first consider the interval topology generated by simply ordering the sequences by first difference, $\delta$, next the natural topology whose basis is determined by the j-segments of the sequences, and then the product topology as the topology of transfinite dyadic spaces.

We will continue to denote by $S_{\omega_\lambda}$ the set of all sequences of 0 and 1 of type $\omega_\lambda$ and will denote by $T_I(\omega_\lambda)$ the interval topology in $S_{\omega_\lambda}$.

It is shown in [7, p. 460] that $S_{\omega_\lambda}$ is order complete, i.e., each of its non-empty subsets has a least upper bound and a greatest lower bound.
Thus, it follows from [13, p. 162] that the space 
\((S_{\omega_\lambda}, T_\lambda)\) is compact.

In view of [13, p. 57] we see that \((S_{\omega_\lambda}, T_\lambda)\)
is a Hausdorff space, and thus, it follows from
[13, p. 141] that this space is also normal.

**THEOREM 26.** The topological space \((S_{\omega_\lambda}, T_\lambda)\)
is totally disconnected.

**PROOF.** Let \(s = (s_i)_{i<\omega_\lambda}\) and \(t = (t_i)_{i<\omega_\lambda}\)be elements of \(S_{\omega_\lambda}\) such that \(s \not\rightarrow t\). Then there
exists an index \(j < \omega_\lambda\) such that \(s_j = 0\), \(t_j = 1\)
and \(s_i = t_i\) for every \(i < j\). Let \(a = (a_i)_{i<\omega_\lambda}\)and \(b = (b_i)_{i<\omega_\lambda}\) be given as follows:

\[
a_i = \begin{cases}
  s_i = t_i & \text{for } i < j \\
  0 & \text{for } i = j \\
  1 & \text{for } i > j
\end{cases}
\]

\[
b_i = \begin{cases}
  s_i = t_i & \text{for } i < j \\
  1 & \text{for } i = j \\
  0 & \text{for } i > j
\end{cases}
\]

Then, clearly, \(s \not\rightarrow a \not\rightarrow b \not\rightarrow t\) and \(a\) is the immediate
predecessor of \(b\). Thus, the sets \(U = \{x \mid x \not\rightarrow b\}\)
and $V = \{ x \mid a < x \}$ form a separation of $S_{\omega}^\lambda$ such that $s \in U$ and $t \in V$.

Thus, the theorem is proved.

**LEMMA 8.** The topological space $(S_{\omega_0}, T_{I}(\omega_0))$ is homeomorphic to the Cantor Ternary Set with its relative topology as a subset of the real numbers.

**PROOF.** It is a consequence of formula (31) of [14, p. 144] that $(S_{\omega_0}, =3)$ is order-isomorphic to the Cantor Set with the usual ordering among real numbers. Thus, it remains to show that the order topology on the Cantor Set $C$ is identical to the relative topology on $C$.

Let us observe that, on the one hand, an open interval $\{ x \mid x \in C$ and $a < x < b \}$, with $a$ and $b$ elements of $C$, is equal to $C \cap (a,b)$ and hence is open in the relative topology. On the other hand, let $(a,b)$ be an open interval in $[0,1]$.

Let $c = \text{lub } C \cap [0,a]$ and let $d = \text{glb } C \cap [b,1]$. Since $C$ is a closed subset of the real numbers it follows that $c$ and $d$ are elements of $C$. It
is easily verified that

\[ \mathcal{C} \cap (a, b) = \{ x \mid x \in \mathbb{C} \text{ and } c < x < d \} \]

and is thus an open set in the interval topology. Simple modifications of the above argument produce similar results for intervals which contain the end points 0 or 1. Thus, it follows that the two topologies mentioned are identical, as desired.

THEOREM 27. The topological space \((S_{\omega_1}, T_1(\omega_1))\) satisfies the first axiom of countability if and only if \(\lambda = 0\).

PROOF. The fact that \((S_{\omega_0}, T_1(\omega_0))\) is a first axiom space is an immediate consequence of Lemma 8.

Next, let \(\lambda \geq 1\) and let \(s = (s_i)_{i < \omega_1}\) be given as follows:

\[
(41) \quad s_i = \begin{cases} 
1 & \text{if } i < \omega_1 \\
0 & \text{if } \omega_1 \leq i < \omega_\lambda
\end{cases}
\]

\((s_i)\) is identically 1 if \(\lambda = 1\). Assume, by way of contradiction, that a sequence of open intervals \((I_j)_j < \omega) forms a basis for \(T_1(\omega_\lambda)\) at \(s\).
Let \( a_j^j = (a_i^j)_{i<\omega} \) be the left hand end point of \( I_j \) for each \( j < \omega \), and let \( o(j) \) be the smallest ordinal such that \( a_j^j o(j) \neq s o(j) \). Since \( a_j^j \neq s \) it follows from (41) that \( o(j) < \omega_1 \) for every \( j < \omega \). Let

\[
(42) \quad k = \text{lub} \ o(j)_{j<\omega}
\]

In view of the fact that \( \omega_1 \) is not confinal with \( \omega \) we have \( k < \omega_1 \). Let \( t = (t_i^j)_{i<\omega} \) be given as follows:

\[
t_i^j = \begin{cases} 
1 & \text{if } i \leq k \\
0 & \text{if } i > k
\end{cases}
\]

Then, in view of (41) and (42), we have \( a_j^j \rightarrow t \rightarrow s \) for every \( j < \omega \). Thus, the set \( \{ x \mid t \rightarrow x \} \) is an open set which has \( s \) as an element and contains no member of the basis, which is a contradiction.

Hence, the theorem is proved.

As a result of Lemma 8 and Theorem 27 we obtain the following:

**COROLLARY 6.** The topological space \( (S_{\omega_1}, I(\omega_\lambda)) \) is metrizable if and only if \( \lambda = 0 \).
DEFINITION 12. We denote by \( S(\omega_\lambda, (s_i)_{i < j}) \) the set of all sequences of 0 and 1 of type \( \omega_\lambda \) with \( j \)-segment equal to \( (s_i)_{i < j} \). We call such a set a basic set.

LEMMA 9. Every basic set is a closed interval in \( (S_{\omega_\lambda}, \supseteq) \).

PROOF. It is easily verified that
\[
S(\omega_\lambda, (s_i)_{i < j}) = [a, b]
\]
where \( a = (a_i)_{i < \omega_\lambda} \) and \( b = (b_i)_{i < \omega_\lambda} \) are given by:
\[
a_i = \begin{cases} s_i & \text{for } i < j \\ 0 & \text{for } i \geq j \end{cases}
\]
(43)
\[
b_i = \begin{cases} s_i & \text{for } i < j \\ 1 & \text{for } i \geq j \end{cases}
\]

LEMMA 10. Let \( S(\omega_\lambda, (s_i)_{i < j}) \) and \( S(\omega_\lambda, (t_i)_{i < k}) \) be two basic sets with \( j \leq k \).

Then
\[
S(\omega_\lambda, (t_i)_{i < k}) \cap S(\omega_\lambda, (s_i)_{i < j}) = \emptyset
\]
or else
\[
S(\omega_\lambda, (t_i)_{i < k}) \subseteq S(\omega_\lambda, (s_i)_{i < j})
\]
PROOF. If \((t_i)_{i<j} \neq (s_i)_{i<j}\) then it follows from Definition 12 that
\[ S(\omega^i, (t_i)_{i<k}) \cap S(\omega^j, (s_i)_{i<j}) = \emptyset \]
Next, let \((t_i)_{i<j} = (s_i)_{i<j}\). Then, since \(j \leq k\), if a sequence has its \(k\)-segment equal to \((t_i)_{i<k}\) it must have its \(j\)-segment equal to \((s_i)_{i<j}\). Hence
\[ S(\omega^i, (t_i)_{i<k}) \subseteq S(\omega^j, (s_i)_{i<j}) \]
as desired.

In view of Lemma 10, the family of all basic sets in \(S^i\) is a base for a topology in \(S^i\). Thus, we introduce:

**DEFINITION 13.** We call the topology generated by the family of all basic sets in \(S^i\) the natural topology for \(S^i\) and we denote it by \(T_N(\omega^i)\).

**THEOREM 28.** Every open interval in \((S^i, \mathcal{T})\) is a union of basic sets.

**PROOF.** Let \(s \in (a, b)\), where \(a = (a_i)_{i<\omega^i}\), \(b = (b_i)_{i<\omega^i}\), and \(s = (s_i)_{i<\omega^i}\). Then there
exists an index \( j < \omega \lambda \) such that \( a_j = 0 \) and \( s_j = 1 \). Likewise, there exists an index \( k < \omega \lambda \) such that \( s_k = 0 \) and \( b_k = 1 \). Let

\[
    n(s) = \max \{ j, k \} + 1
\]

Then, clearly, \( s \in \mathcal{S}(\omega \lambda, \langle s_i \rangle_{i < n(s)}) \subset (a, b) \) and it follows that

\[
    (a, b) = \bigcup_{s \in (a, b)} \mathcal{S}(\omega \lambda, \langle s_i \rangle_{i < n(s)})
\]

Hence, the theorem is proved.

From Theorem 28 we obtain the following, where \( T_I(\omega \lambda) \) and \( T_N(\omega \lambda) \) are, respectively, the interval and natural topologies for \( \omega \lambda \):

**COROLLARY 7.** For every ordinal number \( \lambda \) we have \( T_I(\omega \lambda) \subset T_N(\omega \lambda) \).

**THEOREM 29.** A necessary and sufficient condition for \( T_I(\omega \lambda) = T_N(\omega \lambda) \) is that \( \lambda = 0 \).

**PROOF.** First assume that \( \lambda = 0 \). Let

\[
    \mathcal{S}(\omega 0, \langle s_i \rangle_{i < j}) = [a, b]
\]

where \( a \) and \( b \) are given by (43) above. Since
j < ω either \( a_i = 0 \) for every \( i < ω \) or else
by (41) there exists a last index \( n \) for which
\( a_n = 1 \). In the latter case the sequence
\( c = (c_i)_{i < ω} \) given by

\[
c_i = \begin{cases} 
a_i & \text{for } i < n \\
0 & \text{for } i = n \\
1 & \text{for } i > n
\end{cases}
\]

is an immediate predecessor of \( a \). Likewise, it
follows from (42) that either \( b \) is the last element
of \( S_ω \omega_0 \) (identically 1), or else \( b \) has an
immediate successor. Thus, we see that
\([a, b] = S(ω_0, (s_i)_{i<j}) \) is an open interval in
\((S_ω \omega_0, \mathcal{T}_I)\). Hence, in view of Corollary 7 we
have \( T_I(ω_0) = T_N(ω_0) \), as desired.

Next, assume \( λ ≥ 1 \) and let \( s = (s_i)_{i<ω_λ} \)
be given as follows:

\[
s_i = \begin{cases} 
1 & \text{for } i < ω \\
0 & \text{for } ω ≤ i < ω_λ
\end{cases}
\]

Then \( S(ω_λ, (s_i)_{i<ω}) = [a, b] \) where \( a = s \).
However, in the interval topology \( s \) is the limit
of the sequence \( (a^j)_{j<ω} \) with \( a^j = (a^j_i)_{i<ω_λ} \)
given by

\[ a^j_i = \begin{cases} 1 & \text{for } i < j \\ 0 & \text{for } j \leq i < \omega \lambda \end{cases} \]

where, clearly, \( a^j \notin [a, b] \) for \( j < \omega \). Thus, \([a, b]\) is not open in \( T_I(\omega \lambda) \), and \( T_I(\omega \lambda) \neq T_N(\omega \lambda) \).

Hence, the theorem is proved.

**THEOREM 30.** The topological space \((S_{\omega \lambda}, T_N(\omega \lambda))\) is totally disconnected, and hence Hausdorff. Moreover, it is compact if and only if \( \lambda = 0 \).

**PROOF.** The fact that \((S_{\omega \lambda}, T_N(\omega \lambda))\) is totally disconnected, and hence Hausdorff, follows from Corollary 7 and the fact that \((S_{\omega \lambda}, T_I(\omega \lambda))\) is totally disconnected by virtue of Theorem 26. Moreover, in view of Theorem VI of [15, p. 27], Corollary 7, and the fact that \((S_{\omega \lambda}, T_I(\omega \lambda))\) is compact and Hausdorff, it follows that \((S_{\omega \lambda}, T_N(\omega \lambda))\) is compact if and only if \( T_I(\omega \lambda) = T_N(\omega \lambda) \), and therefore, by Theorem 29, if and only if \( \lambda = 0 \).

Thus, the theorem is proved.
COROLLARY 8. The topological space \((S_{\omega_\lambda}, T_I(\omega_\lambda))\) is homeomorphic to \((S_{\omega_\lambda}, T_N(\omega_\lambda))\) if and only if \(\lambda = 0\).

PROOF. As mentioned above, \(T_I(\omega_\lambda)\) is compact. Thus, the corollary follows from Theorem 30.

THEOREM 31. The topological space \((S_{\omega_\lambda}, T_N(\omega_\lambda))\) satisfies the first axiom of countability if and only if \(\omega_\lambda\) is confinal with \(\omega\).

PROOF. Assume that \(\omega_\lambda\) is confinal with \(\omega\) and let \(\omega_\lambda\) be the limit of the increasing sequence of ordinals \((w_j)_{j<\omega}\). Let \(s = (s_i)_{i<\omega_\lambda}\) be an element of \(S_{\omega_\lambda}\). Then, in view of the proof of Lemma 10, the family

\[
\{ S(\omega_\lambda, (s_i)_{i<w_j}) | j < \omega \}
\]

is a countable basis for \(T_N(\omega_\lambda)\) at \(s\).

Next, assume that \(\omega_\lambda\) is not confinal with \(\omega\) and let \(s = (s_i)_{i<\omega_\lambda}\in S_{\omega_\lambda}\). Assume, by way of contradiction, and in view of Lemma 9, that \(([a_j, b_j])_{j<\omega}\) is a countable family of basic sets
which form a basis for \( T_N(\omega_\lambda) \) at \( s \). Since, in view of (42), each sequence \( a^j = (a^i_j)_{i<\omega_\lambda} \) terminates in zeros, and each sequence \( b^j = (b^i_j)_{i<\omega_\lambda} \) terminates in ones, it is not the case that \( a^j = s = b^k \) for any \( j, k < \omega_\lambda \). Hence, we assume that \( a^j \neq s \) for each \( j < \omega_\lambda \), and we let \( o(j) \) be the smallest ordinal such that \( a^j_{o(j)} \neq s_{o(j)} \).

Let \( k = \text{lub} o(j) \). Since \( \omega_\lambda \) is not confinal with \( \omega \) it follows that \( k < \omega_\lambda \). Thus, the basic set \( S(\omega_\lambda, (s_i)_{i<k+1}) \) contains no \( a^j \), contradicting the assumption that \( (a^j, b^j)_{j<\omega} \) is a basis for \( T_N(\omega_\lambda) \) at \( s \).

Hence, the theorem is proved.

In view of Corollary 7 and Lemma 9 we see that every basic set is both open and closed in \( T_N(\omega_\lambda) \). Thus, it follows from [13, p. 113] that \( (S\omega_\lambda, T_N(\omega_\lambda)) \) is regular.

**THEOREM 32.** The topological space \( (S\omega_\lambda, T_N(\omega_\lambda)) \) is metrizable if and only if \( \omega_\lambda \) is confinal with \( \omega \).

**PROOF.** It follows from Theorem 30 that for \( (S\omega_\lambda, T_N(\omega_\lambda)) \) to be metrizable it is necessary,
by [13, p. 120], that $\omega_\lambda$ be confinal with $\omega$. We show that this condition is sufficient.

As mentioned above, $(S_{\omega_\lambda}, T_N(\omega_\lambda))$ is Hausdorff and regular. Thus, in view of Theorem 18 of [13, p. 127], it remains to show that $(S_{\omega_\lambda}, T_N(\omega_\lambda))$ has a $\sigma$-discrete base under the assumption that $\omega_\lambda$ is confinal with $\omega$. Let $\omega_\lambda$ be the limit of the increasing sequence of ordinals $(w_j)_{j<\omega}$. For each $j<\omega$ let $B_j$ be the family of all basic sets $S(\omega_\lambda, (t_i)_{i<w_j})$ for every $w_j$-segment $(t_i)_{i<w_j}$. Every element $s = (s_i)_{i<w_\lambda}$ of $S_{\omega_\lambda}$ has a neighborhood $S(\omega_\lambda, (s_i)_{i<w_j})$ which is equal to one element of $B_j$ and, by Lemma 10, is disjoint from all other elements of $B_j$. Thus, the family $B_j$ is discreet for each $j<\omega$. Since $\omega_\lambda = \lim_{j<\omega} w_j$ we see that $\bigcup_{j<\omega} B_j$ is a $\sigma$-discrete base for $T_N(\omega_\lambda)$. Thus, $(S_{\omega_\lambda}, T_N(\omega_\lambda))$ is metrizable, as desired.

DEFINITION 14. Let $\omega_\lambda$ be the limit of the increasing sequence $(w_j)_{j<\omega}$ and let $s = (s_i)_{i<w_\lambda}$ and $t = (t_i)_{i<w_\lambda}$ be two elements of $S_{\omega_\lambda}$. We
define the distance $d(s,t)$ between $s$ and $t$ as follows:

$$d(s,t) = 0 \text{ if } s = t$$

and

$$d(s,t) = \frac{1}{j} \text{ if } s \neq t$$

where $j$ is the smallest number such that

$$(s_i)_{i < w_j} \neq (t_i)_{i < w_j}. \tag{1}$$

**THEOREM 33.** Let $\omega^\lambda$ be the limit of the increasing sequence of ordinals $(\omega^j)_{j < \omega}$. Then the function $d$ given by Definition 14 is a metric for $S_{\omega^\lambda}$. Moreover, the metric topology generated by $d$ is equal to the natural topology $T_N(\omega^\lambda)$.

**PROOF.** To show that $d$ is a metric for $S_{\omega^\lambda}$, we first show that the triangle inequality holds. To this end let $r = (r_i)_{i < \omega^\lambda}$, $s = (s_i)_{i < \omega^\lambda}$, and $t = (t_i)_{i < \omega^\lambda}$ be distinct elements of $S_{\omega^\lambda}$ such that $d(r,s) = \frac{1}{j}$ and $d(s,t) = \frac{1}{k}$. Let $u$ be the smallest ordinal such that $r_u \neq s_u$, let $v$ be the smallest ordinal such that $s_v \neq t_v$, and let $m = \min\{j,k\}$. Since $j$ is the smallest ordinal such that $(r_i)_{i < w_j} \neq (s_i)_{i < w_j}$ and $k$ is
the smallest ordinal such that \((s_i)_{i<w_k} \neq (t_i)_{i<w_k}\) we have

\[ n < j \implies w_n < u \]

and

\[ n < k \implies w_n < v \]

Therefore

\[ n < m \implies w_n < \min\{u,v\} \]

which implies

\[ (r_i)_{i<w_n} = (s_i)_{i<w_n} = (t_i)_{i<w_n} \]

Thus

\[ d(s,t) \leq \frac{1}{n} \leq \frac{1}{j} + \frac{1}{k} = d(r,s) + d(s,t) \]

and therefore the triangle inequality holds. The fact that \(d(s,t) \geq 0\) and \(d(s,t) = 0\) if and only if \(s = t\) and \(d(s,t) = d(t,s)\) is easily verified.

Hence, \(d\) is a metric on \(S^{\omega^\lambda}\), as desired.

On the other hand, for \(s \in S^{\omega^\lambda}\) we have

\[ \{ t \mid d(s,t) < \frac{1}{n} \} \]

\[ = \{ t \mid (t_i)_{i<w_n} = (s_i)_{i<w_n} \} \]

\[ = S(\omega^\lambda,(s_i)_{i<w_n}) \]

Thus, the metric topology for \(S^{\omega^\lambda}\) generated by
d is the natural topology $\mathcal{T}_N(\omega_\lambda)$, and the theorem is proved.

Following [16] we next consider the transfinite dyadic space obtained by regarding $S\omega_\lambda$ as the Cartesian product of $\omega_\lambda$ copies of the set \{0, 1\}, each with the discrete topology. We denote this space by $(S\omega_\lambda, \mathcal{T}_P(\omega_\lambda))$. It is easily verified that a subbasic open set in this topology is the set of all sequences $(s_i)_{i<\omega_\lambda}$ with $s_j = k$ where $k = 0$ or $k = 1$ for some $j < \omega_\lambda$. Likewise, a basic open set in this topology consists of the set of all sequences with fixed values (0 or 1) at finitely many specified coordinates.

As a product of compact spaces $(S\omega_\lambda, \mathcal{T}_P(\omega_\lambda))$ is compact. Since each subbasic set is both open and closed, and since distinct elements of $S\omega_\lambda$ may be separated by subbasic sets, it follows that $(S\omega_\lambda, \mathcal{T}_P(\omega_\lambda))$ is Hausdorff and totally disconnected.

In view of the fact [13, p. 92] that a product of first axiom spaces is first axiom if and only if all but countably many of the coordinate spaces are indiscreet, it follows that $(S\omega_\lambda, \mathcal{T}_P(\omega_\lambda))$ is a
first axiom space if and only if \( \lambda = 0 \). As mentioned in [16], the topological space \((S_\omega, T_P(\omega_0))\) is homeomorphic to the Cantor Ternary Set. Therefore, \((S_\omega^\lambda, T_P(\omega_\lambda))\) is metrizable if and only if \( \lambda = 0 \).

**THEOREM 34.** For every ordinal number \( \lambda \),

\[ T_P(\omega_\lambda) \subseteq T_N(\omega_\lambda) \]

Furthermore,

\[ T_I(\omega_\lambda) = T_N(\omega_\lambda) = T_P(\omega_\lambda) \]

if and only if \( \lambda = 0 \).

**PROOF.** We first show that an open set \( U \) in \( T_P(\omega_\lambda) \) is open in \( T_N(\omega_\lambda) \). Let \( s = (s_i)_{i < \omega_\lambda} \) be an element of \( U \). Then there exists a basic open set \( B \) of \( T_P(\omega_\lambda) \) such that \( s \in B \subseteq U \).

Hence, there exists a finite set of ordinals \( \{v_i \mid i < n\} \) such that \( s_{v_i} = t_{v_i} \) for \( t = (t_i)_{i < \omega_\lambda} \) if and only if \( t \in B \). Let \( j = \max \{v_i \mid i < n\} \). Clearly \( j < \omega_\lambda \) and hence

\[ s \in S(\omega_\lambda, (s_i)_{i < j+1}) \subseteq B \subseteq U \]

Thus, \( U \) is open in \( T_N(\omega_\lambda) \), as desired.
Next, $T_I(\omega_\lambda) = T_N(\omega_\lambda) = T_P(\omega_\lambda)$ for $\lambda = 0$ since, as mentioned above, for $\lambda = 0$ each of these three spaces is homeomorphic to the Cantor Ternary Set. Moreover, if $\lambda > 0$ then $T_N(\omega_\lambda)$ is not compact, showing that $(S_{\omega_\lambda}, T_N(\omega_\lambda))$ is homeomorphic neither to $(S_{\omega_\lambda}, T_P(\omega_\lambda))$ nor to $(S_{\omega_\lambda}, T_I(\omega_\lambda))$ for $\lambda > 0$.

Furthermore, from Theorem VI of [15, p. 27] we see that, since both $(S_{\omega_\lambda}, T_I(\omega_\lambda))$ and $(S_{\omega_\lambda}, T_P(\omega_\lambda))$ are compact and Hausdorff, neither topology is a proper subtopology of the other. Also, if $\lambda > 0$ then the set $\{ x \mid x \not\in \delta_{i < \omega_\lambda} \}$ is an open interval which contains no basic open set of $T_P(\omega_\lambda)$. Hence, $T_I(\omega_\lambda) \neq T_P(\omega_\lambda)$ for $\lambda > 0$. 
In this section we consider topologies based on the notion of aleph limit points introduced in Section IV. We also compare these topologies with the interval, natural, and product topologies introduced in Section V.

**DEFINITION 15.** We call a subset \( S \) of \( S_{\omega_\lambda} \) an \( \mathcal{A}_\alpha \)-closed set if \( S \) contains all its \( \mathcal{A}_\alpha \)-limit points, i.e., if \( S^* \subseteq S \). We call a subset \( G \) of \( S_{\omega_\lambda} \) an \( \mathcal{A}_\alpha \)-open set if \( S_{\omega_\lambda} - G \) is an \( \mathcal{A}_\alpha \)-closed set.

**THEOREM 55.** The set of all \( \mathcal{A}_\alpha \)-open subsets of \( S_{\omega_\lambda} \), denoted by \( T(\omega_\lambda, \mathcal{A}_\alpha) \), forms a topology for \( S_{\omega_\lambda} \).

**PROOF.** In view of Definition 15 it suffices to
verify the following four properties for $\mathcal{K}_\alpha$-closed sets:

(i) $\emptyset$ is clearly $\mathcal{K}_\alpha$-closed.

(ii) $S_{\omega_\lambda}$ is clearly $\mathcal{K}_\alpha$-closed.

(iii) If $S$ and $G$ are $\mathcal{K}_\alpha$-closed then $S \cup G$ is $\mathcal{K}_\alpha$-closed.

To prove (iii) let $s = (s_i)_{i<\omega_\lambda}$ be an $\mathcal{K}_\alpha$-limit point of $S \cup G$ and assume, by way of contradiction, that $s$ is neither a $\mathcal{K}_\alpha$-limit point of $S$ nor $G$. Then, in view of Definition 9, there exists an ordinal $j < \omega_\lambda$ such that the subset $S_j$ of $S$ of sequences with $j$-segment equal to $(s_i)_{i<j}$ is of power less than $\mathcal{K}_\alpha$. Likewise, there exists an ordinal $k < \omega_\lambda$ such that the subset $G_k$ of $G$ of sequences with $k$-segment equal to $(s_i)_{i<k}$ is of power less than $\mathcal{K}_\alpha$. Assume that $j \leq k$ and let $S_k$ be the subset of $S$ of sequences with $k$-segment equal to $(s_i)_{i<k}$. Then $S_k \subseteq S_j$ and hence $S_k < \mathcal{K}_\alpha$. But $S_k \cup G_k$ is the subset of $S \cup G$ of sequences with $k$-segment equal to $(s_i)_{i<k}$ and $S_k \cup G_k < \mathcal{K}_\alpha$, contradicting the assumption that $s$ is an $\mathcal{K}_\alpha$-limit point of
Thus, it follows that if $S$ and $G$ are $\aleph_\alpha$-closed then $S \cup G$ is likewise $\aleph_\alpha$-closed.

(iv) If $(S_j)_{j \in I}$ is a family of $\aleph_\alpha$-closed sets then $\bigcap_{j \in I} S_j$ is $\aleph_\alpha$-closed.

To prove (iv) we observe that if $s = (s_i)_{i < \omega_\lambda}$ is an $\aleph_\alpha$-limit point of $\bigcap_{j \in I} S_j$ then the subset of $\bigcap_{j \in I} S_j$ of sequences with $\kappa$-segment equal to $(s_i)_{i < k}$ is of power greater than or equal to $\aleph_\alpha$ and is also a subset of $S_j$ for every $j \in I$. Thus, $s$ must be an $\aleph_\alpha$-limit point of $S_j$ for every $j \in I$, as desired.

Hence, the theorem is proved.

THEOREM 36. Let $\alpha$, $\beta$, and $\lambda$ be ordinal numbers with $\alpha \leq \beta$. Then

$$T_N(\omega_\lambda) \subset T(\omega_\lambda, \aleph_\alpha) \subset T(\omega_\lambda, \aleph_\beta)$$

PROOF. The fact that $T(\omega_\lambda, \aleph_\alpha) \subset T(\omega_\lambda, \aleph_\beta)$ is an immediate consequence of Lemma 5. To show that $T_N(\omega_\lambda) \subset T(\omega_\lambda, \aleph_\alpha)$ it is sufficient to show that every $\aleph_\alpha$-limit point of a subset $S$ of $S_{\omega_\lambda}$ is also a limit point of $S$ with respect to the natural
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Let $s = (s_i)_{i < \omega}$ be an $X$-limit point of $S \subseteq \mathbb{N}^\omega$. Then it follows from Definition 9 that $s$ is a limit point of $S$ in the natural topology, as desired.

**Theorem 37.** Let $\alpha$, $\beta$, and $\gamma$ be ordinal numbers such that $0 < \gamma < \alpha$ and $\gamma < \alpha$. Then $(\omega^\gamma, T(\omega^\gamma, \gamma))$ is not homeomorphic to $(\omega^\gamma, T(\omega^\gamma, \alpha))$.

**Proof.** In view of Theorem 19, there exists a subset $S$ of $\omega^\gamma$ such that $S = \omega^\gamma$ and $S$ has an $\omega^\gamma$-limit point. Then $(\omega^\gamma, T(\omega^\gamma, \gamma))$ is not homeomorphic to $(\omega^\gamma, T(\omega^\gamma, \alpha))$.

Thus, in view of Definition 13, we see that $s = (s_i)_{i < \omega}$ is a limit point of $S$ in the natural topology, as desired.

**Theorem 38.** The topologies $T(\omega^\gamma, \gamma)$ and $T(\omega^\gamma, \alpha)$ are equal if and only if $\alpha < \gamma$.

where $\omega^\gamma$ is the smallest ordinal with which $\gamma$ is conflnal.

This proves the theorem.
PROOF. Let $\omega_\lambda$ be the limit of the increasing sequence of ordinals $(\omega_i)_{i<\omega_\tau}$, and assume that $\aleph_\alpha \leq \aleph_\tau$. In view of Theorem 35, it remains to show that $T(\omega_\lambda, \aleph_\alpha) \subseteq T_N(\omega_\lambda)$. To this end it is sufficient to show that every limit point $s = (s_i)_{i<\omega_\lambda}$ of a subset $S$ of $\omega_\lambda$ in the natural topology is also an $\aleph_\alpha$-limit point of $S$. Assume, on the contrary, that $s$ is a limit point of $S$ in the natural topology and is not an $\aleph_\alpha$-limit point of $S$ for $\aleph_\alpha \leq \aleph_\tau$. Then there exists an index $j < \omega_\lambda$ such that the set $S_j$ of all elements of $S$ with $j$-segment equal to $(s_i)_{i<j}$ is of power less than $\aleph_\alpha$. For each element $t = (t_i)_{i<\omega_\lambda}$ of $S_j$ with $t \neq s$ let $o(t)$ be the smallest ordinal such that $s o(t) \neq t o(t)$. Let

$$k = \text{lub} \{o(t) \mid t \in S_j \land (t \neq s)\}$$

Clearly $k \geq j$ and, since $S_j < \aleph_\alpha < \aleph_\tau$ and $\omega_\tau$ is the smallest ordinal with which $\omega_\lambda$ is confinal, it follows from Theorem 17 that $k < \omega_\lambda$. But then

$$S(\omega_\lambda, (s_i)_{i<k+1}) \cap S_j \setminus \{s\} = \emptyset$$
and since \( k \geq j \) it follows that

\[
S(\omega_\lambda, (s_i)_{i \leq k+1}) \cap S - \{s\} = \emptyset
\]

discontrary to the assumption that \( s \) is a limit point of \( S \) in the natural topology. Hence, we see that if \( \mathcal{K}_\alpha \leq \mathcal{K}_\tau \) then \( \mathcal{T}_N(\omega_\lambda) = \mathcal{T}(\omega_\lambda, \mathcal{K}_\alpha) \).

On the other hand, it follows from Theorem 37 that if \( \mathcal{K}_\tau < \mathcal{K}_\alpha \) then \( \mathcal{T}(\omega_\lambda, \mathcal{K}_\tau) \neq \mathcal{T}(\omega_\lambda, \mathcal{K}_\alpha) \).

However, from the above we have \( \mathcal{T}(\omega_\lambda, \mathcal{K}_\tau) = \mathcal{T}_N(\omega_\lambda) \).

Thus, if \( \mathcal{K}_\tau < \mathcal{K}_\alpha \) then \( \mathcal{T}(\omega_\lambda, \mathcal{K}_\alpha) \neq \mathcal{T}_N(\omega_\lambda) \).

Hence, the theorem is proved.

**COROLLARY 9.** A necessary and sufficient condition for \( \mathcal{T}_1(\omega_\lambda) = \mathcal{T}(\omega_\lambda, \mathcal{K}_\alpha) \) is that \( \lambda = \alpha = 0 \).

**PROOF.** From Theorem 36 and Corollary 7 we have

\[
\mathcal{T}_1(\omega_\lambda) \subset \mathcal{T}_N(\omega_\lambda) \subset \mathcal{T}(\omega_\lambda, \mathcal{K}_\alpha)
\]

The result then follows from Theorems 29 and 37.

**REMARK 7.** If \( \mathcal{K}_\alpha > 2^{\mathcal{K}_\lambda} \) then no subset of \( S_{\omega_\lambda} \) has an \( \mathcal{K}_\alpha \)-limit point. Thus, \( \mathcal{T}(\omega_\lambda, \mathcal{K}_\alpha) \)

is the discrete topology for \( \mathcal{K}_\alpha > 2^{\mathcal{K}_\lambda} \).
In view of Theorems 29, 34, 36, 37, and 38, Corollaries 7, 8, and 9, and Remark 7, we have the following:

COROLLARY 10. Let \( \lambda, \tau, \alpha, \beta, \gamma \), and \( \delta \) be ordinal numbers such that \( \omega_\tau \) is the smallest ordinal with which \( \omega_\lambda \) is confinal and

\[
\omega_\tau < \omega_\alpha < \omega_\beta \leq 2^{\aleph_\lambda} < \omega_\gamma \leq \omega_\delta
\]

Then

\[
T_N(\omega_\lambda) = T(\omega_\lambda, \mathcal{K}_\tau) \subset T(\omega_\lambda, \mathcal{K}_\alpha)
\]

\[
\subset T(\omega_\lambda, \mathcal{K}_\beta) \subset T(\omega_\lambda, \mathcal{K}_\gamma) = T(\omega_\lambda, \mathcal{K}_\delta)
\]

where each inclusion indicates proper inclusion.

Moreover,

\[
T_T(\omega_\lambda) \subset T_N(\omega_\lambda)
\]

and

\[
T_P(\omega_\lambda) \subset T_N(\omega_\lambda)
\]

where equality holds in each case if and only if \( \lambda = 0 \).

In what follows we call a subset of \( S_{\omega_\lambda} \) which is closed relative to the natural topology \( T_N(\omega_\lambda) \) an \( N \)-closed set.
THEOREM 39. Let $\alpha$ and $\lambda$ be ordinal numbers. Then a subset of $S_{\omega_\lambda}$ is $\aleph_\alpha$-closed if and only if it is the union of an $N$-closed set and a set which has no $\aleph_\alpha$-limit point.

PROOF. By Theorem 36 and Definition 15 an $N$-closed set and a set with no $\aleph_\alpha$-limit point are both $\aleph_\alpha$-closed. Hence, their union is an $\aleph_\alpha$-closed set.

On the other hand, let $S$ be an $\aleph_\alpha$-closed set. Then $S^{\alpha} \subset S$ and hence

$$S = S^{\alpha} \cup (S - S^{\alpha})$$

Let $\omega_\tau$ be the smallest ordinal with which $\omega_\lambda$ is confinal. Then, by Theorem 24 we have

$$(S^{\alpha})^\tau \subset S^{\alpha}$$

showing that $S^{\alpha}$ is $\aleph_\tau$-closed. Thus, by Theorem 38, we see that $S^{\alpha}$ is $N$-closed. Clearly, $S - S^{\alpha}$ has no $\aleph_\alpha$-limit point, and thus the theorem is proved.

COROLLARY 11. Let $\alpha$ and $\lambda$ be ordinal numbers satisfying conditions (i) and (ii) of Theorem 21. Then a subset of $S_{\omega_\lambda}$ is $\aleph_\alpha$-closed if and only
if it is the union of an \( N \)-closed set and a set
of power less than \( \kappa_\alpha \).

PROOF. In view of conditions (i) and (ii) it
follows from Theorem 21 that a subset of \( S_{\omega_\lambda} \) has
no \( \kappa_\alpha \)-limit point if and only if it is of power
less than \( \kappa_\alpha \). Thus, the corollary follows from
Theorem 39.

COROLLARY 12. Let \( \alpha \) and \( \lambda \) be ordinal num-
bbers. Then \( T(\omega_\lambda, \kappa_\alpha) \) is generated by \( T_N(\omega_\lambda) \)
together with those subsets of \( S_{\omega_\lambda} \) whose comple-
ments have no \( \kappa_\alpha \)-limit point.

PROOF. In view of Theorem 39, an \( \kappa_\alpha \)-open
set is the intersection of an \( N \)-open set and a set
whose complement has no \( \kappa_\alpha \)-limit point. Thus,
\( T_N(\omega_\lambda) \) together with those subsets of \( S_{\omega_\lambda} \) whose
complements have no \( \kappa_\alpha \)-limit point form a subbase
for \( T(\omega_\lambda, \kappa_\alpha) \).

COROLLARY 13. Let \( \alpha \) and \( \lambda \) be ordinal num-
bbers satisfying conditions (i) and (ii) of Theorem 21.
Then \( T(\omega_\lambda, \kappa_\alpha) \) is generated by \( T_N(\omega_\lambda) \) together
with those subsets of $S_{\omega^\lambda}$ whose complements are of power less than $\mathcal{K}_\alpha$.

PROOF. As indicated in the proof of Corollary 11 a subset of $S_{\omega^\lambda}$ has no $\mathcal{K}_\alpha$-limit point if and only if it is of power less than $\mathcal{K}_\alpha$. Thus, the corollary follows from Corollary 12.

THEOREM 40. Let $\alpha$ and $\lambda$ be ordinal numbers such that:

(i) $\omega_\alpha$ is confinal with no ordinal less than or equal to $2^{\mathcal{K}_\beta}$ for any $\beta < \lambda$,

(ii) $\omega_\alpha$ is confinal with no ordinal less than or equal to $\omega_\lambda$,

(iii) $\alpha$ is a limit ordinal.

Then a subset of $S_{\omega^\lambda}$ is $\mathcal{K}_\alpha$-closed if and only if it is $\mathcal{K}_\beta$-closed for some $\beta < \alpha$.

PROOF. We first observe that from Theorem 37 it follows that an $\mathcal{K}_\beta$-closed set is $\mathcal{K}_\alpha$-closed for $\beta < \alpha$. 
Now let $S$ be an $\mathcal{H}_\alpha$-closed set. Then from Theorem 41 we have

$$S = S^\alpha \cup (S - S^\alpha)$$

where $S^\alpha$ is $\mathcal{N}$-closed. Since conditions (i) and (ii) of Corollary 5 are satisfied we have

$$S - S^\alpha < \mathcal{H}_\alpha.$$ Let $S - S^\alpha = \mathcal{H}_\gamma$ and let

$$\gamma = \gamma + 1 < \alpha.$$ Then, by Theorem 56 it follows that $S^\alpha$ is $\mathcal{H}_\gamma$-closed, and since $S - S^\alpha < \mathcal{H}_\gamma$ we see that $S - S^\alpha$ is $\mathcal{H}_\gamma$-closed. Hence, $S$ is $\mathcal{H}_\gamma$-closed, as desired.

**COROLLARY 14.** Let $\alpha$ and $\lambda$ be ordinal numbers satisfying conditions (i), (ii), and (iii) of Theorem 40. Then

$$T(\omega_\lambda, \mathcal{H}_\alpha) = \bigcup_{i<\alpha} T(\omega_\lambda, \mathcal{H}_i)$$

**PROOF.** In view of Theorem 39, a subset of $S^\omega_\lambda$ is $\mathcal{H}_\alpha$-open if and only if it is $\mathcal{H}_\gamma$-open for some $\gamma < \alpha$, from which the proof of the corollary follows.

**THEOREM 41.** The topological space $(S^\omega_\lambda, T(\omega_\lambda, \mathcal{H}_\alpha))$ satisfies the first axiom of
countability if and only if either $\alpha = 0$ and $\omega^\lambda$ is confinal with $\omega$ or $\beth_\alpha > 2^{\mathcal{H}_\lambda}$.

**PROOF.** First, assume that $\beth_\alpha \leq 2^{\mathcal{H}_\lambda}$. Then, in view of Lemma 5 and Theorem 19 there exists a subset $S$ of $\omega^\lambda$ such that $S$ has an $\mathcal{H}_\alpha$-limit point $s$. If $\alpha > 0$ then the complement of any countable set is open. Hence, no sequence (of type $\omega$) in $S - \{s\}$ converges to $s$ in the topology $T(\omega^\lambda, \mathcal{H}_\alpha)$. On the other hand, if $\mathcal{H}_\alpha = \mathcal{H}_0$ and $\omega^\lambda$ is not confinal with $\omega$ then it follows from Theorem 18 that no sequence (of type $\omega$) in $S - \{s\}$ converges to $s$ in the topology $T(\omega^\lambda, \mathcal{H}_\alpha)$. Thus, in view of Theorem 8 of [13, p. 72], if $\beth_\alpha \leq 2^{\mathcal{H}_\lambda}$ and either $\beth_\alpha > \mathcal{H}_0$ or $\omega^\lambda$ is not confinal with $\omega$ then $T(\omega^\lambda, \mathcal{H}_\alpha)$ is not first axiom.

Next, if $\alpha = 0$ and $\omega^\lambda$ is confinal with $\omega$ then by Theorem 38 the topology $T(\omega^\lambda, \mathcal{H}_\alpha)$ is the natural topology $T_N(\omega^\lambda)$ which, by Theorem 31, is first axiom when $\omega^\lambda$ is confinal with $\omega$. If $\beth_\alpha > 2^{\mathcal{H}_\lambda}$ then, by Remark 7, the topology $T(\omega^\lambda, \mathcal{H}_\alpha)$ is the discrete topology, and therefore is first axiom.

Thus, the theorem is proved.
COROLLARY 15. The topological space
\((S^j, T(c_j, K^*))\) is metrizable if and only if
either \(\alpha = 0\) and \(\omega_{\lambda}\) is confinal with \(\omega\) or
\(\chi_\alpha > 2^{\chi_{\lambda}}\).

PROOF. The necessity of the above conditions
follows from Theorem 41. On the other hand, if
\(\alpha = 0\) and \(\omega_{\lambda}\) is confinal with \(\omega\) then as in the
proof of Theorem 41, \(T(\omega_{\lambda}, \Gamma_\alpha) = T_N(\omega_{\lambda})\) which
is metrizable by Theorem 52. Moreover, if \(\chi_\alpha > 2^{\chi_{\lambda}}\)
then \(T(\omega_{\lambda}, \Gamma_\alpha)\) is discrete and hence again is
metrizable.

COROLLARY 16. The topological space
\((S^\lambda, T(\omega_{\lambda}, \Gamma_\alpha))\) is Hausdorff and totally dis-
connected. Moreover, it is compact if and only if
\(\lambda = 0\) and \(\alpha = 0\).

PROOF. By Corollary 10 we have
\[ T_I(\omega_{\lambda}) \subseteq T_N(\omega_{\lambda}) \subseteq T(\omega_{\lambda}, \Gamma_\alpha) \]
Since \(T_I(\omega_{\lambda})\) is Hausdorff and totally disconnected
it follows that \(T(\omega_{\lambda}, \Gamma_\alpha)\) is likewise Hausdorff
and totally disconnected. Moreover, by Theorem 29
we have \(T_I(\omega_{\lambda}) = T_N(\omega_{\lambda})\) if and only if \(\lambda = 0\).
and by Theorem 38 we have \( T_N(\omega_0) = T(\omega_0, \kappa_\alpha) \) if and only if \( \alpha = 0 \). Hence, by Theorem VI of [15, p. 27], we see that \( (S_{\omega_\lambda}, T(\omega_\lambda, \kappa_\alpha)) \) is compact if and only if \( \alpha = \lambda = 0 \), as desired.

We close this section by considering the consequences of the Generalized Continuum Hypothesis on the aleph topologies of a set of sequences of 0 and 1.

Following [7] we call an initial ordinal number which is confinal with no smaller ordinal number a regular ordinal. An initial ordinal number which is not regular is called singular.

**REMARK 8.** It follows from [7, p. 404] that \( \omega_\lambda \) is regular if \( \lambda \) is a non-limit ordinal. Moreover, if \( \lambda \) is a limit ordinal then either \( \omega_\lambda \) is singular or else \( \omega_\lambda \) is weakly inaccessible (cf. Def. 11) and \( \omega_\lambda = \lambda \).

**THEOREM 42.** Let \( \omega_\lambda \) be a regular ordinal number. Then under the assumption of the Generalized Continuum Hypothesis any aleph topology on the set
\( S \omega_\lambda \) must be one of the following three distinct topologies:

(a) \( T(\omega_\lambda, \mathcal{N}_\lambda) \)

(b) \( T(\omega_\lambda, \mathcal{N}_{\lambda+1}) \)

(c) \( T(\omega_\lambda, \mathcal{N}_{\lambda+2}) \)

where the topology given in (a) is the natural topology \( T_N(\omega_\lambda) \), the topology given in (b) is generated as mentioned in Corollary 13, and the topology given in (c) is the discrete topology.

PROOF. Since \( \omega_\lambda \) is regular it follows that \( \omega_\lambda \) is the smallest ordinal with which \( \omega_\lambda \) is confinal. Thus, we see by Theorem 38 that \( T(\omega_\lambda, \mathcal{N}_\alpha) = T_N(\omega_\lambda) \) for \( \alpha \leq \lambda \).

Next, in view of Remark 8, it follows that \( \omega_{\lambda+1} \) is regular, and hence, by the Generalized Continuum Hypothesis, \( \omega_{\lambda+1} \) satisfies conditions (i) and (ii) of Theorem 21. Thus, \( T(\omega_\lambda, \mathcal{N}_{\lambda+1}) \) is generated as mentioned in Corollary 13. Furthermore, by Theorem 37, it is not homeomorphic to either \( T(\omega_\lambda, \mathcal{N}_\lambda) \) or \( T(\omega_\lambda, \mathcal{N}_{\lambda+2}) \), which in turn are not homeomorphic.
Finally, by the Generalized Continuum Hypothesis we have $\aleph_{\lambda+2} > 2^{\aleph_\lambda}$. Thus, in view of Remark 7, for $\alpha \geq \lambda + 2$ we have $T(\omega_, \aleph_\alpha) = T(\omega_, \aleph_{\lambda+2})$, which is the discrete topology.

**THEOREM 43.** Let $\omega_\lambda$ be a singular ordinal number and let $\omega_\tau$ be the smallest ordinal number with which $\omega_\lambda$ is confinal. Then, under the assumption of the Generalized Continuum Hypothesis, corresponding to every ordinal number $\alpha$ with $\tau \leq \alpha \leq \lambda + 2$ there exists a distinct topology $T(\omega_, \aleph_\alpha)$ on the set $S_{\omega_\lambda}$.

**PROOF.** As in the proof of Theorem 42 we have

- $T(\omega_, \aleph_\alpha) = T_N(\omega_\lambda)$ for $\alpha \leq \tau$ and
- $T(\omega_, \aleph_\alpha) = T(\omega_, \aleph_{\lambda+2})$ for $\alpha \geq \lambda + 2$.

Thus, it follows from Theorem 37 that for every two ordinals $\alpha$ and $\beta$ with $\tau \leq \alpha \leq \lambda + 2$ and $\tau \leq \beta \leq \lambda + 2$ the topologies $T(\omega_, \aleph_\alpha)$ and $T(\omega_, \aleph_\beta)$ are homeomorphic if and only if $\alpha = \beta$.

In conclusion, we observe that given any cardinal number $\aleph_\gamma$ there exists a set $S_{\omega_\lambda}$ of sequences of 0 and 1 such that (with or without the
Generalized Continuum Hypothesis) the set of all distinct aleph topologies on $S_{\omega_\lambda}$ is of power greater than or equal to $\kappa_\gamma$. Indeed, if $\lambda = \omega_\gamma + \omega$ then $\omega_\lambda$ is cofinal with $\omega$ and therefore $S_{\omega_\lambda}$ has distinct topologies $T(\omega_\lambda, \kappa_\alpha)$ for every $\alpha$ such that $\kappa_\alpha \leq 2^{\kappa_\lambda}$, and in particular for every $\alpha \leq \lambda$. Thus, the set of all aleph topologies on $S_{\omega_\lambda}$ is of power greater than or equal to $\lambda = \omega_\gamma + \omega = \kappa_\gamma$. 
LIST OF REFERENCES


