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CONJUGATE PHASING AND BEAM TAGGING.

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A COHERENT TRANSMITTING
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PHASING AND BEAM TAGGING

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

by

Charles Herbert Brenner, BSEE, MSEE

*****

The Ohio State University
1966

Approved by

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ACKNOWLEDGMENTS

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"Differential Doppler," Antenna Laboratory Report 1072-8, 15 April 1964

"Pattern Handbook, Volume IV: Far-Field Patterns of a Linear Antenna Radiating in the Presence of Rectangular Cylinders, " Antenna Laboratory Report 1522-14, 1 August 1965


"Pattern Handbook, Volume VI; Far-Field Patterns of a Linear Antenna Radiating in the Presence of Circular Cylinders, " Antenna Laboratory Report 1522-16, 1 August 1965

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SYMBOLS

\( \alpha_i \) Phase angle of the sinusoidal tagging frequencies, rad

\( \beta \) Wave number \(-2\pi/\lambda, \text{m}^{-1}\)

\( \beta_i \) Phase angle of the sinusoidal tagging frequencies, rad

\( \gamma \) Phase error between the carrier frequency signal of a tagged element and either the resultant phasor formed by the remaining elements or the other phasor tagged with the same frequency, rad

\( \gamma_i \) Phase of carrier signal received at tagging receiver due to signal \( e_i(t) \), rad

\( \bar{\gamma}_i \) Phase of carrier signal radiated from the \( i^{th} \) element, rad

\( \delta \) Elevation angle, deg

\( \xi_i \) Peak phase deviation of the \( i^{th} \) tagged element, rad

\( \eta \) Phase shift per step of the DPM, deg/step

\( \theta_i \) Angle between the plane wavefront passing through the reference element and a line between the \( i^{th} \) element and the reference element, rad

\( \theta_i^R \) \( \theta_i \) measured at the receiving frequency, rad

\( \theta_i^T \) \( \theta_i \) measured at the transmitting frequency, rad

\( \bar{\theta}_i \) Value of \( \theta_i \) when the signal is initially acquired, rad

\( \lambda \) Wavelength \(-c/\lambda, \text{m}\)

\( \nu \) Angular difference between signal and antenna axis, deg

\( \nu_o \) Null-to-null beamwidth of 30 ft parabola, deg

\( \Delta \nu_o \) Null-to-null beamwidth of interferometer grating lobe pattern, deg
SYMBOLS (Cont.)

$\xi_0$  Beam tagging phase modulation variation, rad

$\Delta\xi_1$  Phase error produced by the differential doppler frequency times the effective time delay, rad

$\Delta\xi_2$  Phase error produced by doppler frequency shift, rad

$\Delta\xi_3$  Phase error produced by phase instabilities in the array, rad

$\Delta\xi_4$  Phase error produced by error in measurement of $N_j$, rad

$\Delta\xi_5$  Phase error produced by frequency multiplication error, rad

$\rho$  $(R - Vt)$, m

$\rho_i$  Peak phase deviation of $i$th tagged element, rad

$\sigma_i$  $(\gamma_i + A_i + B_{i-1})$, rad

$\phi_i$  Phase delay, rad

$\phi_R$  Receiving phase delay, rad

$\phi_R$  Receiving phase delay at time of initial signal acquisition, rad

$\phi_i^R_{\text{pv}}$  Principal value of $\phi_i^R$, rad

$\phi_i^T$  Transmitting phase delay, rad

$\phi_i^T$  Transmitting phase delay at the time of initial signal acquisition, rad

$\chi_i$  Squinting error, rad

$\psi$  Phase variation produced by beam tagging, rad

$\psi_A$  Azimuth angle, deg
SYMBOLS (Cont.)

ω  Carrier frequency, rad/sec
ω_c  Carrier frequency, X band range, rad/sec
ω_i  i\textsuperscript{th} tagging frequency, rad/sec
ω_s  Angular velocity of satellite, rad/sec
ω_T  Arbitrary tagging frequency, rad/sec
Ω_{sw}  Switching frequency that periodically varies the carrier phase by ΔΩ, rad/sec
ΔΩ  Magnitude of periodic carrier phase shift, deg
A  Attenuation above free space loss, dB
A_i  μ_i sin(ω_i t + α_i), rad
B  Bandwidth, Hz
B_i  μ_i sin(ω_i t + β_i), rad
(C/N)  Carrier-to-noise ratio, dimensionless
c  Velocity of light, m/sec
d  Distance between two arbitrary elements, m
d_i  Distance between the reference and the i\textsuperscript{th} elements, m
d_{max}  Maximum distance between elements in O.S.U. array, 25.9 m
ΔD_{rel}  Relativistic corrected differential doppler frequency, Hz
ΔD  Differential doppler frequency, Hz
ΔD  Rate of change of differential doppler frequency, Hz/sec
SYMBOLS (Cont.)

\( D_a \) \quad \text{Parabolic antenna diameter, 30 ft}

\( \overline{D^2} \) \quad \text{Mean square surface deviation of the antennas from a true parabola, } m^2

\( e_{DPM}(t) \) \quad \text{Digital phase modulator output signal}

\( e_i(t) \) \quad \text{Signal radiated from the } i^{\text{th}} \text{ element}

\( e_R(t) \) \quad \text{Signal at tagging receiver}

\( f \) \quad \text{Carrier frequency, Hz}

\( f_R \) \quad \text{Receiving center frequency, Hz}

\( f_T \) \quad \text{Transmitting center frequency, Hz}

\( f_o \) \quad \text{Digital phase modulator center frequency, Hz}

\( f_{sw} \) \quad \text{Digital phase modulator commutating frequency, Hz}

\( f_1, f_2, f_3 \) \quad \text{Beam tagging frequencies, Hz}

\( \Delta f \) \quad \text{Doppler frequency shift (1-way), Hz}

\( \dot{\Delta f} \) \quad \text{Rate of change of doppler frequency shift, Hz/sec}

\( \Delta f_o \) \quad \text{Difference between transmit and receiver center frequencies } (f_T-f_R), \text{ Hz.}

\( \bar{\Delta f} \) \quad \text{Doppler frequency shift when signal is initially acquired, Hz}

\( F \) \quad \text{Noise figure of the receiver, dimensionless}

\( G \) \quad \text{Actual antenna gain (dimensionless)}

\( G_o \) \quad \text{Theoretical gain for a perfect antenna (dimensionless)}

\( G_R \) \quad \text{Receiving antenna gain (dB)}
### SYMBOLS (Cont.)

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<td>Transmitting antenna gain (dB)</td>
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<td>$G_g$</td>
<td>Universal gravity constant ($m^3/kg/sec^2$)</td>
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<td>$H$</td>
<td>Height of satellite above the surface of the earth (mi)</td>
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<tr>
<td>$k$</td>
<td>Number of increments to null the linear phase detector when signal is initially acquired</td>
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<tr>
<td>$k_o$</td>
<td>Ratio satellite velocity to height above earth, ($V/R$)</td>
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<td>$K$</td>
<td>Amplitude when all $K_i$'s are equal</td>
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<td>$K_i$</td>
<td>Amplitude at the tagging receiver of the signal from the $i^{th}$ element</td>
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<td>$\overline{K_i}$</td>
<td>Amplitude of the signal radiated from the $i^{th}$ element</td>
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<td>$L$</td>
<td>Loss factor (dimensionless)</td>
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<td>$L_R$</td>
<td>Receiving loss factor (dimensionless)</td>
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<td>$L_T$</td>
<td>Transmitting loss factor (dimensionless)</td>
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<td>$M$</td>
<td>Number of elements in an arbitrary DPM</td>
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<td>$M_R$</td>
<td>Number of elements in the receive DPM</td>
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<td>$M_T$</td>
<td>Number of elements in the transmit DPM</td>
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<td>$M_e$</td>
<td>Frequency multiplication error (dimensionless)</td>
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<td>$M_a$</td>
<td>Mass of the earth, kg</td>
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<td>$m$</td>
<td>Modulation index</td>
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<td>$N$</td>
<td>Number of elements in an array</td>
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<td>$N_i$</td>
<td>$\left( \phi_i^R - \phi_i^R/\pi \right) / 2\pi$</td>
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<td>$\overline{N_i}$</td>
<td>Value of $N_i$ when signal is initially acquired</td>
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<td>$\Delta N_i$</td>
<td>Error is $N_i$, an integer</td>
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<td>$P_0$</td>
<td>Position of the reference element</td>
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<td>$P_i$</td>
<td>Position of the $i$th slave element</td>
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<td>$R$</td>
<td>Resultant phasor of the $N$ elements</td>
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<td>$R_s$</td>
<td>Distance from the center of array to the satellite</td>
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<td>$R_1$</td>
<td>Distance from element one to the satellite</td>
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<td>$R_2$</td>
<td>Distance from element two to the satellite</td>
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<td>$R_{12}$</td>
<td>$(R_1 - R_2)$</td>
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<td>$R_e$</td>
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<td>$</td>
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<td>$\Delta R$</td>
<td>Amplitude variation produced by digital tagging</td>
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<td>$(S/N)$</td>
<td>Signal-to-noise ratio</td>
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<td>$S_T(t)$</td>
<td>Switching function, $\pm 1$</td>
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<td>$t_i$</td>
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<td>$T_{2R}$</td>
<td>Two-way propagation time delay</td>
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<td>$t_a, t_b$</td>
<td>Integration limits</td>
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<td>$T_{RC}$</td>
<td>Phase detector time constant</td>
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<td>$T_S$</td>
<td>Orbital period of the satellite</td>
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<td>$T_{off}$</td>
<td>Time interval during which no elements are being tagged</td>
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SYMBOLS (Cont.)

T<sub>on</sub>  Time interval during which one slave element is being tagged
T  Reciprocal of the carrier frequency
T<sub>A</sub>  Ambient temperature, 290 °K
T<sub>R</sub>  Receiver noise temperature, °K
T<sub>e</sub>  Equivalent system noise temperature, °K
T<sub>M</sub>  Mixer noise temperature, °K
## ILLUSTRATIONS

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CHAPTER I
INTRODUCTION

With the advent of orbiting satellites and deep space probes, interest in arrays has increased and diversified. The purpose of this analysis is to develop a cohering system for a large active array composed of a few widely spaced elements. However, before entering into a discussion of the cohering system, the capabilities and applications of arrays containing many small, closely-spaced elements are briefly described for the sake of completeness.

A. Closely spaced arrays with many small elements

Arrays of this type may contain from as few as five or ten to many thousands of elements with effective apertures of about one-half wavelength, and interelement separation of one-half to one wavelength. Development of this type of array has been stimulated by several different applications;

1) It can be used to produce a radar system in which the beam is electronically scanned at very high rates. Beam scanning can be
achieved either by sweeping the frequency of transmitted signal[1], or by varying the phase of the signal transmitted from each element by either digital[2] or analog[3] methods.

2) It can be designed to have multiple target capability by assigning portions of the array to track particular targets while the rest of the array is searching for new targets.

3) It can operate as a retrodirective array[4,5], (of which the classic example is the Van Atta array[6,7]), by pointing the transmitted signal wavefront in the same direction as the received signal wavefront. Usually this type of array is implemented by using some form of phase conjugation[8,9].

B. Widely spaced arrays with a few large elements

Interest in arrays with large apertures stems from the fact that they may be used to overcome some of the limitations of a single large aperture.

With a single antenna, the system (S/N) ratio can be increased up to a point by increasing the physical size of the aperture. This approach is limited because the RMS surface tolerance of the aperture must be maintained to within 1/16 of the operating wavelength in order to achieve the nominal antenna gain[10]. As the size of the antenna increases, eventually a point is reached where
the surface accuracy requirement can no longer be met because of
the elasticity of the antenna support structure. At the present
time, the largest steerable (one axis only) antenna is a 300 ft
diameter parabola operating at 21 cm[11].

As the antenna aperture size increases, the beamwidth
decreases proportionately which introduces several more problems.
First, on initial target acquisition, the location (or look angles) of
the target must be known very accurately because mechanical sluing
of the antenna to hunt for a moving target becomes more difficult
as the size of the aperture increases. Second, to maintain a moving
target within a very narrow beamwidth requires an excellent mount
and control system. Third, position scintillation of the received
signal becomes a problem at low elevation angles when beamwidths
on the order of .016° or less are used[12].

The system (S/N) ratio can also be increased by raising the
transmitted power. At the present time, the largest commercially
available klystron is rated at 15 MW peak and 75 kW average
power. Further increases above this level can be obtained by
paralleling several transmitters, but arcing within the feed horn
waveguide eventually limits the number of transmitters that can be
connected in parallel[13].
By using an array, some of the limitations of a single large aperture may be overcome. For example, it should be possible to increase the system \((S/N)\) ratio above the current state-of-the-art for a single aperture or alternatively, the same system \((S/N)\) ratio can be achieved with reduced acquisition and tracking requirements and improved reliability through element redundancy.

Using an array of large apertures is not a panacea. First, the additional electronic equipment required to control several elements adds to the system cost. Second, a more fundamental problem occurs when the elements within the array are separated by many wavelengths. In general, signals from a point source in space travel different path lengths to reach each of the elements in the array. When the carrier frequency signals are compared, it is seen that the unequal propagating path lengths produce different phase shifts at each element. Since this varies as a function of the location of the point source, the maximum system \((S/N)\) ratio cannot be achieved by directly combining the signals from all the elements. The reduction in \((S/N)\) ratio may be eliminated by providing a receiving system which coherently combines the received...
signals by equalizing the phase of the carrier frequency component from all the elements.

An example of a cohering type receiving array is The Ohio State University (O. S. U.) antenna system shown in Fig. 1-1 [14] which is composed of four 30 ft parabolic antennas located at the corners of a 60 ft square. While operating at S band (2.27 GHz), it was used to experimentally demonstrate that the signals from four widely spaced elements can actually be cohered by using phase-locked-loops to bring three IF signals into phase alignment with the fourth or reference signal. Theoretical calculations show that by cohering the signals received by the O. S. U. four-element array, the (S/N) ratio can be improved by from 5.26 to 5.76 dB over that achieved by using a single element [15]. This increase is quite close to the theoretical maximum of 6 dB.

Similar types of coherent receiving arrays using phase locked loops have been operated in the UHF and HF regions with four and five elements respectively [16, 17].

C. Coherent transmitting array

This study analyzes a system used to produce a coherent transmitting array. The phrase, "coherent transmitting array" is a slight misnomer, because in reality the desired operation is to
Fig. 1-1--Four-element O.S.U. transmitting array
control the phase of the signal radiated from each element in the array in such a manner that the signals from all the elements are coherent or in phase at a distant target.

In the coherent transmitting array, signals radiated from the various elements usually travel unequal distances in reaching the target. The effect of the different propagation path lengths is similar in both the transmitting and receiving array but the phasing techniques required to produce a coherent signal are different.

In an N element receiving array, the signals from each element are fed to common point where they are adjusted in phase to allow coherent addition. Thus, the phase error or phase difference between any two signals is easily measured at the cohering receiver. In the case of the coherent transmitting array, the problem is more difficult because phase adjustments must be made at each of the transmitting elements in such a way that the signals add coherently at a desired point in space. Since the signal received at the distant target is formed by the phasor combination of all the transmitted signals, it is not possible to adjust the phase of the individual signals at the target. Therefore, the receiving array cohering scheme which uses a phase locked loop for each element, is not directly applicable to this case.
This study analyzes a cohering system for the transmitting array which uses a combination of phase conjugation and beam tagging techniques [18]. Before the operation of the combined system is discussed, the fundamentals of the phase conjugation, and beam tagging are briefly described below.

Phase conjugation is an open-loop system which requires reception of a test or sample signal from the target. Therefore, if a passive satellite is used, it must be illuminated by a remote site or by the transmitting array operating in a pulsed mode. Phasing information is obtained by detecting the signal radiated from the target at each of the elements in the array. For example, in an N element array, the phase of the signals received at the (N-1) slave elements is compared with one element called the master. The (N-1) phase differences are conjugate* and applied to the carrier signals transmitted by the (N-1) slave elements. In Chapter II, both analog and digital methods of performing phase conjugation are discussed.

*Conjugation of the phase difference reverses the sign of the phase angle, i.e., if the received signal is \( V e^{+j\psi} \), then the transmitted signal should be \( V e^{-j\psi} \).
Phase conjugation alone is not sufficient to produce a coherent condition when an orbiting satellite is used. The doppler frequency shift, initial acquisition phase error, frequency multiplication inaccuracy, and equipment and propagation path phase instabilities, all produce small phase errors which cannot be corrected by phase conjugation.

Beam tagging on the other hand is a closed-loop system which is used to remove the small phase errors left uncorrected by the phase conjugation system. The elements of the array are PM or AM modulated in an appropriate manner so that an error signal developed at the satellite is proportional to the phase error. The signal is reradiated or scattered (active or passive satellite) back to the transmitting array where it is detected. The detected error signals are used to adjust the phase of the (N-1) slave elements so that the residual phase errors are eliminated. Chapter III analyzes several types of beam tagging systems.

Throughout this analysis, the active array is discussed in terms of the O. S. U. array, composed of four 30 ft parabolic antennas driven by four 10 kW transmitters. When this array is cohered, the (S/N) ratio increases by 6 dB, which produces the same field strength as 40 kW radiated from a single 60 ft parabola.
The 6 dB improvement obtained by coherently phasing the O. S. U. four-element array represents the theoretical maximum. In actual practice, this limit can never quite be reached, for several reasons. The present system design assumes that the transmitting and receiving propagation paths have equal delays, even though turbulence of the propagating media can cause slight changes in the path phase delay during the time the signal propagates from the satellite to the array and back again. The degree of phase coherence at the target is reduced as the received (S/N) ratio decreases because the accuracy of the phase measurements which control the transmitted phases is degraded. These factors show that in a practical transmitting array, the cohering gain is always slightly below the theoretical maximum.
CHAPTER II
CONJUGATE PHASING

A. Introduction

In Chapter I, the general operation of a coherent transmitting array using a combination of conjugate phasing and beam tagging techniques was introduced. This chapter analyzes the design and implementation of a phase conjugation system for the O.S.U. four-element array.

The far field patterns of the individual elements and the array of isotropic point sources are considered first because they effect the design of the conjugate phasing system.

1. Antenna patterns of the individual element and the array of isotropic point sources

The O.S.U. array is composed of four 30 ft parabolic antennas located at the corners of a 60 ft square.

The normalized far field of a uniformly illuminated parabolic antenna, \( E(\nu) \), is given by [19]

\[
E(\nu) = \frac{2cJ_1\left(\frac{\pi Da c}{fT}\sin \nu\right)}{\pi fT Da \sin \nu}
\]

(2-1)
where

\[ D_a = \text{diameter of aperture} \]
\[ f_T = \text{operating frequency} \]
\[ \nu = \text{angle with respect to the normal to the aperture} \]
\[ J_1 = \text{first order Bessel function} \]

Equation (2-1) was used to calculate the far field pattern of a 30 ft parabolic antenna operating at 8.33 GHz shown in Fig. 2-1. This figure shows that the half-power and null-to-null (\( \nu_0 \)) beamwidths are approximately 0.27° and 0.54° respectively.

Fig. 2-1--Normalized far field pattern of a 30 ft parabolic antenna with uniform illumination
The far field pattern of the array of isotropic point sources, found by replacing each element with an isotropic point source, is solely a function of geometrical location of the elements in the array. By choosing the center of the array (see Fig. 2-2) as the phase reference point, the equation for the array factor is given by

\( E(\delta, \psi) = 2 \{ \cos[0.5 \beta d \cos \delta \cos \psi] + \cos[0.5 \beta d \cos \delta \sin \psi] \} \)

where

\( \psi = \text{azimuth angle} \)
δ = elevation angle

d = 60√2 ft

β = 2π/λ = 2πf/c

and the isotropic sources have unit excitation. If the signal direction is varied throughout all its possible positions, the angles ψ and δ range from ±π and 0 to π/2 respectively. The variations in ψ and δ cause E(δ, ψ) to pass through many peaks and nulls (0 < |E(δ, ψ)| < 2) which produces what are frequently called interferometer grating lobes. When the azimuth angle, ψ, is zero or 90°, E(δ, ψ) is given by

\[(2-3)\ E(δ, ψ = 0° or 90°) = 2 \cos[0.5 \beta d \cos δ] \.

For these special azimuth angles, the angular distance between the interferometer pattern nulls, Δν₀, is given by

\[(2-4)\ Δν₀ = \frac{c}{(fd \sin δ)} \text{ rad.}\]

For a constant frequency and elevation angle, the angular separation between the interferometer lobes, Δν₀, is minimum when the distance between elements is maximum. With d equal to its maximum value, 60√2 ft (25.9 m) and f equal to 8.33 GHz, Δν₀ equals 0.305°, 0.112° and 0.079° for elevation angles, δ, of 15°, 45° and 90° ( zenith) respectively.
The far field pattern of the array is obtained by multiplying the element pattern by the pattern of the array of isotropic point sources. Thus, the null-to-null beamwidth of the main beam of the array pattern contains almost seven (6.9) interferometer lobes at the zenith position compared to slightly under two (1.79) at 15° elevation.

When the array transmits, the signals radiated from the elements cancel at various points in space which produce perfect nulls in the interferometer pattern, whereas in reception, the interferometer lobes have perfect nulls only if the signal is received from a point source [20]. The largest angle subtended by a signal source in space is approximately 0.0024° for the Echo II passive satellite which is 135 ft in diameter. Since this is approximately 33 narrower than the narrowest interferometer lobe, all orbiting satellites may be treated as point sources.

2. Array bandwidth

Two possible methods of cohering the signals from an array are time delay compensation and phase adjustment at a single frequency.

The basic advantage of cohering by using time delay compensation is that the array may be wideband since the phases of the signals are aligned at all frequencies. If the time delay is inserted at the carrier frequency, the O.S.U. array with a maximum separation of $60\sqrt{2}$ ft between elements, requires a continuously variable time
delay circuit with a maximum delay of 0.086 μs. To control the phase of the carrier frequency signal (nominally 8 GHz) to within ±10°, the time delay circuit must be adjustable to within 0.0035 ns. Alternatively, the time delay could be produced at some arbitrary IF frequency. Since the maximum distance between elements is equivalent to 720 wavelengths at 8.33 GHz, the time delay circuit must be capable of producing a maximum time delay equal to 720 times the reciprocal of the IF frequency (Hz) used. For example, consider these requirements in terms of a time delay circuit operating at 100 MHz. In order to maintain a phasing accuracy of greater than 10°, the time delay must be controlled to within 0.28 μs while the maximum available time delay must be 7.2 μs. Also, the time delay circuit must be wide band (greater than 8 MHz), so that it does not appreciably narrow the array bandwidth.

Since the requirements for both the carrier and IF frequency time delay networks appear to be beyond the present state-of-the-art, a more practical approach is to cohere the signals at only a single frequency (usually the carrier frequency) by phase adjusting techniques. A limitation of this method is that the signals are perfectly cohered at only a single frequency. For example, consider the case of a carrier frequency signal with a single modulating tone. If the phasing system aligns the carrier frequency components,
the sidebands produced by the tone modulation have progressively larger phase errors as the modulating frequency increases. Since the phase error reduces the amplitude of the sidebands, a phase adjusting type array has a finite bandwidth which is solely determined by the location of the elements and is independent of the carrier frequency used. Figure 2-3 shows that the half-power bandwidth of the O. S. U. array[21] has a minimum value of approximately 8 GHz at the horizon and increases with increasing elevation angles until it is theoretically infinite at the zenith.

Since the minimum bandwidth of 8 MHz is more than adequate for most systems, the cohering schemes discussed use phase adjusting rather than time delay techniques.

3. **Digital phase modulator**

The digital phase modulator (DPM), which is formed by a tapped delay line (TDL) and a ring counter connected by appropriate gating signals, is introduced at this point because it is an essential element in all the phase conjugation methods analyzed. In Fig. 2-4, an eight-element TDL is driven at a constant frequency, $f_o$, which allows the $45^\circ$ phase shifting increments to be formed by fixed LC element networks. Only seven phase shifting elements are required because the eighth element would produce a total phase shift equal
Fig. 2-3--O.S.U. four-element array bandwidth vs elevation angle

Fig. 2-4--Eight-element tapped delay line and commutator
to 360° which is equivalent to zero degrees. The eight signals from the TDL are fed to a commutating gate which selects one of them as the output signal. By taking the output signal from succeeding taps further to the right (CW rotation of the commutator), the output is negatively phase shifted by -45° times the number of elements moved. Similarly, by taking the output signal from taps further to the left, the signal is positively phase shifted by +45° times the number of elements moved. Therefore, this device is capable of producing unlimited phase shifts in either a positive or negative direction.

Figure 2-5 shows an eight-element DPM in which the commutating gate is electronically controlled by connecting a ring counter to the TDL with appropriate gating circuits. The ring counter contains the same number (eight) of bistable multivibrators as there are taps on the delay line. The bistables, which are connected together as a closed loop shift register, are initially loaded with one "1" and seven "0"'s. These binary levels control gates in which a binary "0" closes the gate and the single binary "1" opens one gate and connects the corresponding signal from the TDL to the output. The position of the binary "1" is advanced or retarded by one element for every input pulse depending on the sense of the direction gate. Therefore, an M element DPM can
electronically advance or retard the phase of a fixed frequency signal in digital increments of \( \frac{\pm 360}{M} \) degrees where the number of elements, \( M \), is arbitrary.

If an \( M \) element DPM is periodically advanced at a rate, \( f_{sw} \) steps/sec, the fundamental component of the output frequency is shifted by an amount equal to \( \pm \frac{f_{sw}}{M} \) where the + and - signs correspond to driving the commutator in a CCW or CW direction respectively. Since the frequency shift is produced by periodically incrementing the phase in discrete steps, additional spurious harmonics are produced as is shown in the spectral analysis of a periodically shifted DPM presented in Appendix III.
4. **Phase conjugation methods**

The classical methods of producing phase conjugation by using single and double frequency conversion are described in the following articles [22, 23, 24]. In this analysis the phase conjugation operation is accomplished by using phase detectors and digital phase modulators.

B. **Phase conjugation with slowly moving targets**

The performance of a phase conjugation system using a slowly moving target is described for two modes, pulsed operation with \( f_T = f_R \) and CW operation with \( f_T \neq f_R \) where \( f_T \) and \( f_R \) are the transmitting and receiving carrier frequencies respectively.

In an \( N \) element array, one element is arbitrarily called the reference element and the remaining \((N-1)\) elements are called slaves. Usually only the reference and \( i^{th} \) slave elements are shown because the functions associated with the other slave elements are similar.

The restriction of a slowly moving target simplifies the analysis by reducing the effects of the doppler frequency shift and the motion of the target during the propagation delay time to negligible amounts.
1. **Pulsed operation, \( f_T = f_R \)**

When the transmitting and receiving frequencies are approximately equal, pulsed operation is necessary because it is impossible to detect a signal when the transmitters are operating since the locally radiated signals saturate the receivers. In this case, phasing information obtained from the received signals must be stored (e.g., by sample and hold networks), since the receiving and transmitting functions occur at different times.

Operation of the phase conjugation system is described using the block diagram given in Fig. 2-6 which shows only the reference and \( i^{\text{th}} \) slave elements of an \( N \) element array.

The signal received at the \( i^{\text{th}} \) element is delayed in time with respect to the reference element by an amount equal to

\[
(2-5) \quad t_i = \left( \frac{d_i \sin \theta_i}{c} \right) \text{ sec}
\]

where

- \( \theta_i \) - elevation angle of the plane wavefront passing through the reference element
- \( d_i \) - distance between the reference and the \( i^{\text{th}} \) slave elements.

For the O.S.U. array, the maximum distance, \( d_i \), equals \( 60\sqrt{2} \) ft which corresponds to a maximum delay time, \( t_i \), of 0.0865\( \mu s \) when
\[ \theta_i \text{ equals } 90^\circ. \text{ At a particular frequency, } f_T, \text{ the time delay, } t_i, \text{ can be represented in terms of an equivalent phase delay, } \phi_i^T, \text{ given by} \]

\[ (2-6) \quad \phi_i^T = \left[ \frac{2\pi d_i f_T \sin \theta_i}{c} \right] \text{ rad.} \]

With \( f_T \text{ equal to } 8.33 \text{ GHz and } \theta_i \text{ equal to } 60\sqrt{2} \text{ ft}, \) the equivalent phase delay has a maximum value of 720 wavelengths when \( \theta_i \text{ equals } 90^\circ. \text{ Since the value of } \phi_i^T \text{ is greater than one wavelength for most values of } \theta_i, \phi_i^T \text{ can be written as} \]
\[(2-7) \quad \phi_i^T = \left[ 2\pi N_i + \phi_i^T \right] \text{ rad} \]

where \(N_i\) is an integer and the magnitude of \(\phi_i^T \) is less than \(\pi\).

In Eq. (2-7), \(N_i\) corresponds to the total integral number of wavelengths and \(\phi_i^T \) corresponds to the principal value of \(\phi_i^T \).

If the phase of the signal at the reference element is arbitrarily assumed to be equal to zero, the output of the linear phase detector shown in Fig. 2-6 equals \(-\phi_i^T \). This value must be stored (e.g., sample and hold network) since when the transmitters are radiating, no signal can be received. The conjugate phase shifter produces the negative or opposite phase shift from that measured by the linear phase detector. For example, if the angle \(-\phi_i^T \) is detected, the conjugate phase shifter inserts a phase shift of \(+\phi_i^T \) in the carrier frequency signal. The modifying word, conjugate, is used to describe the phase shifter because when the phase angles are expressed as complex exponentials, reversing the sign of the phase angle is equivalent to the mathematical operation of conjugation.

When the received signal plane wavefront passing through the reference element makes an angle \(\theta_i\), the detected phase angle is \(-\phi_i^T \) and the transmitted phase angle is \(+\phi_i^T \). In propagating to the plane wavefront defined by \(\theta_i\), the signal from the \(i^{th}\) slave element is delayed by an amount equivalent to \(\phi_i^T \). Combining the
effect of the initial transmitted phase angle and the propagation phase delay gives the following expression

\[(2-8) \quad (\phi_i^T - \phi_i^T) = \phi_i^T - (2\pi N_i + \phi_i^T) = -2\pi N_i\]

which shows that the signals radiated from the reference and \(i^{th}\) slave elements add coherently in the direction \(\theta_i\) since \(2\pi N_i\) is equivalent to zero phase shift.

2. CW operation, \(f_T \neq f_R\)

In a practical array, operating CW, the transmitting and receiving frequencies must be sufficiently separated so that it is possible to filter out the transmitted signal from the received signal.

When the transmitting and receiving frequencies are unequal, the total phase delays from the \(i^{th}\) slave element to the plane wave-front passing the reference element are given by

\[(2-9) \quad \phi_i^T = \frac{2\pi d_i f_T \sin \theta_i^T}{c}\]

and

\[(2-10) \quad \phi_i^R = \frac{2\pi d_i f_R \sin \theta_i^R}{c} \quad \text{rad.}\]

If an ideal phase conjugation system measured the phase angle, \(\phi_i^R\), and conjugated it, the transmitted signal direction would be "squinted" by the angle \(X_i\), shown in Fig. 2-7. The squinting angle, \(X_i\), given by
(2-11) \( \chi_i = (\theta_i^R - \theta_i^T) = \theta_i^R - \sin^{-1} \left[ \frac{f_T}{f_R} \sin \theta_i^R \right] \),

is found by setting \( \phi_i^T \) equal to \( \phi_i^R \) and equating Eqs. (2-9) and (2-10).

This shows that for unequal transmit and receive frequencies, a more sophisticated phasing system is required to reduce the squinting error, \( \chi_i \), to zero.

![Diagram showing pointing error produced by unequal transmit and receive frequencies](image)

**Fig. 2-7--Pointing error produced by unequal transmit and receive frequencies**

The squinting error can be eliminated by setting \( \theta_i^T \) equal to \( \theta_i^R \). Using Eqs. (2-9) and (2-10), subject to the requirement that \( \theta_i^T \) equals \( \theta_i^R \), gives the following equation for \( \phi_i^T \),

(2-12) \( \phi_i^T = \left( \frac{f_T}{f_R} \right) \phi_i^R \).

In general, \( \phi_i^R \) can be represented as
\( (2-13) \quad \phi_i^R = 2\pi N_i + \phi_i^R |_{pv} \)

where \( \phi_i^R |_{pv} \) is the principal value of \( \phi_i^R \), and \( N_i \) is an integral number of wavelengths. Substituting Eq. (2-13) into Eq. (2-12) gives

\( (2-14) \quad \phi_i^T = (2\pi N_i) (f_T/f_R) + \phi_i^R |_{pv} (f_T/f_R) . \)

In Eq. (2-14), the ratio \( (f_T/f_R) \) can be written as

\( (2-15) \quad (f_T/f_R) = (f_R + \Delta f_o)/f_R = 1 + \Delta f_o/f_R \)

where \( \Delta f_o = (f_T - f_R) \). Neglecting the \( 2\pi N_i \) term gives the final form of Eq. (2-14) as

\( (2-16) \quad \phi_i^T = 2\pi N_i(\Delta f_o/f_R) + \phi_i^R |_{pv}(f_T/f_R) . \)

a. **Analog phase conjugation**

The value of \( \phi_i^T \) specified by Eq. (2-16) can be implemented using the analog phase conjugation system shown in Fig. 2-8 in which the static phase shifts in both the reference and \( i^{th} \) slave element signal channels are assumed to be equal.

The first term in Eq. (2-16), \( (2\pi N_i)(\Delta f_o/f_R) \), is produced by a variable phase shifter which is altered in discrete increments according to the value of \( N_i \), where \( N_i \) represents the integral number
Fig. 2-8--Analog phase conjugation system with $f_T \neq f_R$

of wavelengths between the $i^{th}$ slave element and the plane wavefront passing through the reference element. When the received signal is first acquired, the initial value of $N_i$, $\bar{N}_i$, is computed using servo resolvers which convert the antenna azimuth and elevation angles into a quantity proportional to $\bar{N}_i$. After the value of $\bar{N}_i$ is determined,
a small computer is used to multiply the value of $\bar{N}_1$ by $2\pi(\Delta f_0/f_R)$ to determine the appropriate phase shift. If the variable phase shifter is a DPM, a suitable number of pulses are produced to digitally approximate the value of $2\pi \bar{N}_1(\Delta f_0/f_R)$ but if a phase shifter with range $\pm 180^\circ$ is used, then the computer must also calculate the principal value of $2\pi \bar{N}_1(\Delta f_0/f_R)$.

Figure 2-9 shows a graph of the antenna pointing accuracy required to measure the equivalent path delay to within $\pm 1/2$ wavelength [25]. Since the tracking accuracy of the O.S.U. antennas is approximately $0.05^\circ$, the graph shows that the calculation of $\bar{N}_1$ is marginal for elevation angles greater than $60^\circ$. Fortunately, in most cases, the satellite can be acquired at low elevation angles where the required pointing accuracy is reduced. If the value of $\bar{N}_1$ is incorrect by an integral amount, $\Delta \bar{N}_1$, a phase error equal to $(2\pi \Delta \bar{N}_1)(\Delta f_0/f_R)$ rad is produced. For example, if $\Delta \bar{N}_1$ equals one, a phase error of $28^\circ$ is produced with $f_R$ equal to $7.73$ GHz and $\Delta f_0$ equal to $600$ MHz.

Once the initial value of $N_i$, $\bar{N}_i$, is measured, the value of $N_i$ is automatically incremented by the abrupt phase transitions of the linear phase detector which occur once every time the equivalent phase delay changes by one wavelength. This insures that the
transition of the linear phase detector and the incrementation of \( N_i \) are both synchronized.

![Graph showing required tracking accuracy vs elevation angle.](image)

**Fig. 2-9** -- Pointing accuracy required to measure \( \phi_i^R \) to within one-half-wavelength vs elevation angle.

The second term in Eq. (2-16), \( \phi_i^R \left|_{\text{pv}} (f_T/f_R) \right. \), is produced by the combination of the linear phase detector which measures the angle \( \phi_i^R \left|_{\text{pv}} \right. \) and the conjugate phase shifter which produces a phase shift equal to \( -\phi_i^R \left|_{\text{pv}} (f_T/f_R) \right. \). When the constant, \( (f_T/f_R) \),
is greater than one, the range of the phase shifter must be greater than \( \pm 180^\circ \).

Now, let us examine the operation of the analog phase conjugation system as the incident wavefront angle, \( \theta_i \), slowly varies. The variation of \( \phi_i^R, \phi_i^R|_{pv}, \phi_i^T, \phi_i^R|_{pv}(f_T/f_R) \) and \( 2\pi N_i(\Delta f_0/f_R) \) vs \( \phi_i^R \) is shown in Fig. 2-10. As the angle \( \theta_i \) increases, both \( \phi_i^R \) and \( \phi_i^R|_{pv} \) increase equally until they reach the value \( \pi \). At this point, three abrupt phase shifts occur.

1) The value of \( \phi_i^R|_{pv} \) changes from \( +\pi \) to \( -\pi \).

2) The value of \( \phi_i^R|_{pv}(f_T/f_R) \) changes from \( +\pi(f_T/f_R) \) to \( -\pi(f_T/f_R) \).

3) The value of \( N_i \) is incremented by one, which produces a phase shift equal to \( 2\pi(\Delta f_0/f_R) \).

The last two phase shifts leave the value of \( \phi_i^T \) unchanged. As the value of \( \phi_i^R \) increases beyond \( \pi \), \( \phi_i^R|_{pv} \) increases until \( \phi_i^R \) reaches \( 3\pi \) at which point the abrupt phase transition cycle repeats. For larger values of \( \phi_i^R \), the cycle repeats with abrupt phase changes occurring at odd multiples of \( \pi \).

Ideally, the abrupt phase transitions occurring at odd multiples of \( \pi \), leave the value of \( \phi_i^T \) unchanged. In a practical system it is extremely difficult to produce these phase changes without producing switching transients. Since a maximum of 668 transitions are
produced in going from the horizon to the zenith, the phase switching transient problem is not a severe system limitation.

![Graph showing phase variations and their relationships](image-url)

**Fig. 2-10** -- Variation of $\phi_i^R$, $\phi_i^R|_{pv}$, $\phi_i^T$, and $N_i$ vs $\phi_i$ with $f_T/f_R = 1.2$
b. Digital phase conjugation

In this case it is desirable to write $\phi_i^T$ in a slightly different form than in Eq. (2-16). Assume that the signal is initially acquired when the plane wavefront elevation angle equals $\bar{\theta_1}$, where the bar is used to indicate the initial condition. At $\theta_1 = \bar{\theta_1}$, $\phi_i^R$ is given by

$$
(2-17) \quad \phi_i^R(\theta_1) = \left( \frac{2\pi d_i f_R \sin \bar{\theta_1}}{c} \right) = 2\pi \bar{N_1} + \phi_{i_1}^R |_{pv}.
$$

For values of $\theta_1$ greater than $\bar{\theta_1}$, $\phi_i^R$ is given by

$$
(2-18) \quad \phi_i^R(\theta_1 > \bar{\theta_1}) = \left[ 2\pi \bar{N_1} + \phi_{i_1}^R |_{pv} \right] + \left( \frac{2\pi d_i f_R}{c} \right) \left[ \sin \theta_1 - \sin \bar{\theta_1} \right].
$$

From Eqs. (2-12) and (2-18), the desired form for $\phi_i^T$ is given by

$$
(2-19) \quad \phi_i^T(\theta_1 > \bar{\theta_1}) = \left\{ 2\pi \bar{N_1} (\Delta f_o/f_R) + \phi_{i_1}^R |_{pv} (f_R/f_T) \right\} + \left\{ (f_T/f_R) \left[ \frac{2\pi d_i f_R}{c} \left( \sin \theta_1 - \sin \bar{\theta_1} \right) \right] \right\}.
$$

The first term in Eq. (2-19) represents the phase delay, $\phi_i^T$, when the signal is initially acquired and the second term represents the total transmit phase delay from the time of acquisition.
The digital phase conjugation system shown in Fig. 2-11 digitally approximates Eq. (2-19). Its operation is described as 

θ_i slowly increases for two cases, θ_i = 0° and θ_i > 0°.

When θ_i = 0°, the operation is simplified because the first (initial condition) term is equal to zero. The phase variations, of \( \phi_i^R, \phi_i^T \), the digital approximation to \( \phi_i^T \), and the linear phase detector output vs \( \phi_i^R \) are shown in Fig. 2-12.

Assume that; both channels have identical phase shifts in their feed lines, the ring counters in both DPM's are preset with a binary "1" in element number one, and \( \theta_i \) equals zero. As \( \theta_i \) increases, \( \phi_i^R \) and \( \phi_i^R |_{pv} \) both increase and the output of the linear phase detector increases correspondingly until its positive threshold at \((180/M_R)\) degrees is reached. At this point a pulse is produced which retards the receive DPM by \((360/M_R)\) degrees. As \( \theta_i \) increases further, the positive threshold is reached again and a second pulse is produced which retards the receive DPM by an additional \((360/M_R)\) degrees. This cycle repeats itself whenever the output of the linear phase detector reaches the threshold level. In Fig. 2-12 this sequence is shown by a sawtooth waveform along the abscissa.
Fig. 2-11--Digital phase conjugation system with \( f_T \neq f_R \)
Phase conjugation is achieved by using common pulses to drive the transmit DPM in the opposite direction from the receive DPM. The number of elements in the transmit DPM, \( M_T \), is found by the closest integer approximation to the expression, \( (f_T/f_R)M_R \), where \( M_T \) and \( M_R \) are integers. Since each pulse produces a
phase shift of \((360/M_T)\) degrees, as the number of elements, \(M_T\), increases, the digital staircase phase function becomes a better approximation to \(\phi_i^T\).

When \(\bar{\theta}_i > 0^\circ\), the analysis is more complicated because the initial condition (first) term in Eq. (2-19) is non-zero. The initial phase delay, \(-\bar{\phi}^T_i\), is given by

\[
\bar{\phi}^T_i = 2\pi \bar{N}_i \left( \Delta f_o/f_R \right) + \Phi^R_i \left|_{pv} \right. \left( f_T/f_R \right).
\]

In Eq. (2-20), the first term, \(2\pi \bar{N}_i \left( \Delta f_o/f_R \right)\), can be produced if the correct value of \(\bar{N}_i\) is measured. Since the description of the measurement of \(\bar{N}_i\) given for the analog phase conjugation system (p. 28) also applies to the digital system, the discussion is not repeated.

Measurement and implementation of the phase angle specified by the second term in Eq. (2-20), \(\Phi^R_i \left|_{pv} \right. \left( f_T/f_R \right)\), is explained with the aid of the waveforms shown in Fig. 2-13. The variations of \(\Phi^R_i\) and \(\Phi^R_i \left|_{pv} \right. \bar{N}_i\), and \(2\pi \bar{N}_i \left( \Delta f/f_R \right)\) as \(\phi^R_i\) increases are shown in Figs. 2-13a, b, and c respectively. If the \(M_R\) element receive DPM and phase detector loop is broken, the phase difference at the linear detector varies between \(-180^\circ\) as \(\phi^R_i\) increases. However, in closed-loop operation, when the phase difference reaches the threshold established by \((180/M_R)\) degrees, the DPM is shifted
Fig. 2-13--Selected waveforms showing digital generation of initial phase angle with $M_T = 8$, $M_R = 5$ and $f_T/f_R = 1.6$
Fig. 2-13--Selected waveforms showing digital generation of initial phase angle with $M_R = 8$, $M_T = 5$ and $f_T/f_R = 1.6$.
Fig. 2-13--Selected waveforms showing digital generation of initial phase angle with $M_R=8$, $M_T=5$ and $f_T/f_R = 1.6$. 

**Diagram Description:**
- **Figure (f):** $\phi_i^T$ AND $\phi_i$ VS $\phi_i^R$.
- **Figure (g):** LINEAR PHASE DETECTOR OUTPUT CLOSED LOOP VS $\phi_i^R$. 

The diagrams illustrate the phase relationships and output characteristics of a linear phase detector in a closed-loop system, with specific parameters given for the initial phase angle generation.
by an amount \((360/M_R)\) in the proper direction to reduce the phase error.

For any arbitrary initial value of \(\phi^R_{i_p}\), Fig. 2-13d shows the number of discrete phase shifts, \(k\), required to reduce \(\phi^R_{i_p}\) to a value within the threshold limits. The \(k\) discrete increments also advance (or retard) the transmit DPM which produces a phase shift equal to \((2\pi/M_T)\) \(k\) as shown in Fig. 2-13e.

Figure 2-13f shows the digital approximation to the desired initial phase angle, \(\phi^T_i\), where the shaded areas represent the difference in the initial phase angle caused by the digital quantization levels. The actual initial phase angle established by the digital system is given by

\[
(2-21) \quad \phi^T_i = (2\pi \Delta f_o/f_R) N_i + (2\pi/M_T) k
\]

where the difference between this value and the one given in Eq. (2-20) is less than or equal to \((180/M_T)\) degrees.

Operation of the digital phase conjugation system as \(\theta_i\) slowly increases with \(\theta_i > 0^\circ\) is shown in Fig. 2-14 for the particular but arbitrary initial condition, \(\phi^R_i = 19/16\pi\). When the signal is acquired, the initial phase condition given by Eq. (2-21) is established in a negligibly short time. First, the linear phase detector senses the phase error equal to \(-13\pi/16\). Then the
closed-loop, receive DPM-phase detector system advances the receive DPM by three increments of $\pi/4$ each which reduces the value of the phase detector output to $-\pi/16$.

The transmit initial condition is formed by two terms. Since $N_i$ equals one, the first term in Eq. (2-21), $(2\pi/M_T)k$, equals $-1, 2\pi$ since $k$ equals three, $(2\pi/M_T)$ equals $5\pi/2$, and the minus sign arises from the fact that transmit and receive DPM's are shifted in opposite directions. The sum of these two phase terms produces $0^\circ$ for the initial phase angle and a phase difference of $+0.1\pi$ between the actual and digitized values of $\phi_i^T$. This phase difference acts as a biasing level to produce the appropriate initial condition so that the digital approximation to $\phi_i^T$ is symmetrical about the true value of $\phi_i^T$. As $\theta_i$ increases from $\bar{\theta}_i$, the operation is similar to that shown in Fig. 2-12.

The above discussion assumed that $(f_T/f_R)$ was equal to $(M_R/M_T)$. In many communications systems, the transmit and receive frequencies are fixed by various operating requirements. Therefore, in most cases, the ratio $(f_T/f_R)$ can only be approximated by the rational fractional, $(M_T/M_R)$, which produces a multiplication error, $M_e$, defined by

\begin{equation}
M_e = (M_R/M_T) - (f_T/f_R).
\end{equation}
Fig. 2-14—Phase variations in a digital phase conjugation system with $\bar{\theta}_1 \neq 0^\circ, M_R = 8, M_T = 5,$ and $(f_T/f_R) = 1.6$

When $M_e$ is not equal to zero, the difference between $\phi_i^T$ and the digital approximation to $\phi_i^T$ increases as $\theta_1$ varies, which produces an increasing phase error.
c. Summary

Since phase conjugation is an open-loop system, it cannot correct for the phase errors produced by an error in the initial value of $N_1$ and the frequency multiplication error. Therefore, these errors must be corrected with a closed loop phasing system.

C. Satellite orbital parameters

Before the operation of the conjugate phasing systems is extended to the case of a rapidly moving target (e.g., an orbiting satellite), factors which are dependent only on the motion of the satellite and the geometry of the array are analyzed. These factors are applied in the analysis of phase conjugation systems operating with rapidly moving targets given in Section D.

The target is assumed to be an orbiting satellite moving in a circular orbit around the center of the earth. This assumption is valid for many present day satellites. For example, Echo II has an eccentricity of approximately 0.06 and an apogee point equal to 1.18 earth radii which produces apogee and perigee values equal to 720 mi and 578 mi above the surface of the earth respectively.

The path of the orbiting satellite and the active array are assumed to lie in the same plane as shown in Fig. 2-15 because the planar geometry simplifies the calculations. Only a two-element
array is considered because it is sufficient to determine the effects of satellite motion.

Fig. 2-15--Planar two-element array with satellite traveling in a circular orbit
1. **Propagation time delay and PRF**

First, equations for the propagation time delay and the pulse repetition rate are found.

Using the law of cosines, the distance from the center of the array to the satellite, $R_s$, is given by

\[
R_s = \sqrt{A - B \cos \omega_s t}
\]  

(2-23)

where

- $R_e$ = radius of the earth
- $H$ = height of the satellite above the surface of the earth
- $V$ = velocity of the satellite in circular orbit
- $\omega_s = V/(R_e + H)$
- $A = (R_e + H)^2 + R_e^2$
- $B = 2R_e(R_e + H)$.

The two-way propagation time delay, $T_{2R}$, is given by

\[
T_{2R}(H) = 2R_s/c.
\]  

(2-24)

In pulsed radar operation, the maximum pulse repetition frequency (PRF) is given by

\[
PRF(H) = 1/(2T_{2R}).
\]  

(2-25)
Figures 2-16 and 2-17 show curves of the two-way propagation delay time and the maximum PRF respectively versus elevation angle for five orbital heights. Equations (2-23), (2-24) and (2-25) show that both these curves have even symmetry about the zenith position, $\delta = 90^\circ$.

2. Doppler and doppler rate

The doppler frequency shift and the rate of change of doppler frequency are found by multiplying the first and second derivatives of $R_g$, $(\dot{R}_s$ and $\ddot{R}_s)$ by $(f/c)$. The resulting equations for the doppler frequency shift, $\Delta f$, and the rate of change of doppler frequency, $\dot{\Delta} f$, are given by

\begin{equation}
\Delta f = \frac{(f/c)}{\left(\frac{B\omega_s}{2}\right) \left(\frac{\sin \omega_s t}{R_s}\right)}
\end{equation}

and

\begin{equation}
\dot{\Delta} f = \frac{(f/c)}{\left(\frac{B\omega_s}{2}\right) \left(\frac{\cos \omega_s t}{R_s}\right) - \left(\frac{B \sin \omega_s t}{2R_s^3}\right)}
\end{equation}

where $f$ is the frequency of the signal source.

The value of $\omega_s$ used in Eqs. (2-26) and (2-27) is a function of the velocity of the satellite. Since the satellite moves in a conservative central force field, its period is equal to [26]
Fig. 2-16--Two-way propagation time delay vs elevation angle for five orbit heights and pulse width for a 40% duty cycle.
Fig. 2-17--Pulse repetition rate for a 40% duty cycle vs elevation angle for five orbit heights

\[(2-28) \quad T_S = 2\pi \sqrt{\frac{(R_e + H)^3}{G(M)}}\]

where

- **G** - universal gravity constant \(6.67 \times 10^{-11} \text{ m}^3/\text{kg sec}^2\)
- **M** - earth mass \(5.97 \times 10^{24} \text{ kg}\).

From Eq. (2-28), the satellite velocity, \(V\), is given by

\[(2-29) \quad V(H) = \sqrt{\frac{G(M)}{(R_e + H)}}\]
For orbital heights of 100, 300, 600, 1200, and 2400 miles, \( V(H) \) is equal to 4.83, 4.72, 4.56, 4.29 and 3.87 mi/sec respectively.

Figures 2-18 and 2-19 show curves of the doppler frequency shift and the doppler rate respectively vs elevation angle for five orbital heights with \( f \) equal to 8 GHz. Equations (2-26) and (2-27) show that the doppler frequency curve has odd symmetry about the zenith while the doppler rate curve has even symmetry.

3. **Differential doppler**

The differential doppler frequency represents the rate of change of the phase delay between two elements in the array. This is an important parameter in any cohering system since it can be related to the rate at which phase corrections must be performed. Both planar and non-planar geometries are used to evaluate the differential doppler frequency.

a. **Planar geometry**

The planar geometry case is analyzed using the configuration given in Fig. 2-15. Since \( R_s >> d \), the vectors \( R_1 \) and \( R_2 \) are approximately parallel which allows the distance, \( R_{12} \), to be written as

\[
R_{12} = (R_1 - R_2) \approx d \sin \theta.
\]
Fig. 2-18--Doppler frequency shift vs elevation angle for five orbit heights
Fig. 2-19—Doppler rate vs elevation angle for five orbit heights

PEAK 9.8 Hz AT 90°

f = 8.0 GHz

ELEVATION ANGLE (deg)

100 mi

300 mi

600 mi

1200 mi

2400 mi

I-WAY DOPPLER RATE $\Delta f (kHz/sec)$
Using the law of sines and the geometry shown in Fig. 2-15, Eq. (2-30) may be written in the form

\[
R_{12} = \left[ d(R_e + H) \right] \left( \frac{\sin \omega_s t}{R_s} \right).
\]

The differential doppler frequency, \( \Delta D \), is given by

\[
\Delta D = \left( \frac{f}{c} \right) \frac{d}{dt} (R_{12}).
\]

By taking the indicated derivative of \( R_{12} \) as given in Eq. (2-31), the equation for \( \Delta D \) becomes,

\[
\Delta D = \left( \frac{f \omega_s d (R_e + H)}{c} \right) \left[ \left( \frac{\cos \omega_s t}{R_s} \right) - \frac{B}{2} \left( \frac{\sin^2 \omega_s t}{R_s^3} \right) \right].
\]

The rate of change of the differential doppler frequency, \( \Delta D \), is found by taking the time derivative of Eq. (2-33). The resulting expression for \( \dot{\Delta D} \) is given by

\[
\dot{\Delta D} = \left[ -\frac{f \omega_s d (R_e + H)}{c} \right] \left[ \left( \frac{\omega_s \sin \omega_s t}{R_s} \right) + \left( \frac{R_s \cos \omega_s t}{R_s^2} \right) \right] + \left[ \frac{B \omega_s \cos \omega_s t \sin \omega_s t}{R_s^3} \right] + \left( \frac{3BR_s \sin^2 \omega_s t}{2R_s^4} \right).
\]

Equations (2-33) and (2-34) show that both \( \Delta D \) and \( \dot{\Delta D} \) are directly proportional to the separation between elements, \( d \).

Curves of the differential doppler frequency and the differential doppler rate are shown in Figs. 2-20 and 2-21 respectively versus elevation angle for three orbital heights with \( f \) equal to 8.0 GHz and
d equal to 60 ft. In Fig. 2-22 the differential doppler frequency is plotted vs time which shows that the lower the satellite height, the more peaked the differential doppler frequency response.

If the curve for $\Delta D(H)$ is integrated with respect to time from $t = (\theta = 90^\circ)$ to $t = (\theta = 0^\circ)$ with $H$ equal to a constant, the integral equals $\left(\frac{fd_1}{c}\right)$, independent of $H$. Mathematically this can be expressed by
Fig. 2-21--Differential doppler rate vs elevation angle for three orbit heights

\[ f = 8.0 \text{GHz} \]
\[ d = 60 \text{ft} \]
Fig. 2-22--Differential doppler frequency vs time for three orbit heights

\[(2-35) \int_{t=(\theta=0^\circ)}^{t=(\theta=90^\circ)} [\Delta D(H)] \, dt = \frac{f \lambda_i}{c} = d_i/\lambda\]

where \((f \lambda_i/c)\) equals the number of wavelengths between the reference and \(i^{th}\) slave elements measured at frequency, \(f\). This shows that in an active array the total phase correction depends only on the number of wavelengths between the elements. However, the phase correction
rate is determined by the shape of the differential doppler frequency curve which varies as a function of the height and velocity of the satellite.

b. Non-planar geometries

The differential doppler frequency analysis which was previously limited to planar geometries is now extended to include non-planar two-element arrays. In this analysis the satellite is assumed to travel in a straight line with constant velocity rather than a circular path. The assumption of a constant (vector) velocity was made for two reasons; the differential doppler frequency can be calculated exactly using the special theory of relativity and the non-relativistic \((v/c\) approaching zero) calculations are simplified. For a satellite in a near circular orbit, the direction of the velocity vector continually varies but its magnitude is nearly constant. When the actual velocity is replaced by a constant velocity, the shape of the differential doppler curve is slightly changed but the peak value and the integral with respect to time from the zenith to the horizon remain the same. The last relation stems from the fact that the variation of the phase delay from the zenith to the horizon position is a constant independent of the path traveled.
Figure 2-23 shows the system geometry used for the non-planar array which requires two coordinate systems, one for the satellite and the other for the array. The satellite, which radiates a spherical wave at a constant frequency $f$, is located at the origin of the $O'$ coordinate system. This coordinate system moves at velocity $V$ with respect to the $O$ coordinate system. At time, $t = 0$, both coordinate systems are coincident and the velocity vector, $V$, is directed along the positive $x$ axis. The two-element array is defined in the $O$ coordinate system with the reference
element at point \( P_0 \) and the \( i^{th} \) slave element at point \( P_i \). The \( O \) and \( O' \) coordinate systems are oriented so that the Lorentz transformation equations may be directly applied.

Equations (2-36) through (2-38) are obtained from reference [27]. The relativistic differential doppler frequency, \( \Delta D_{\text{rel}} \), is given by

\[
\Delta D_{\text{rel}} = \frac{f_0 \gamma}{c} \left\{ \frac{1}{\left( \frac{1}{2}\sqrt{\gamma} \right)^{3/2}} \left[ \begin{array}{c}
\left| \vec{p} \right|^2 (\vec{V} \cdot \vec{d}_i) - \left[ (\vec{R} \cdot \vec{V}) - \left| \vec{V} \right|^2 t \right] (\vec{p} \cdot \vec{d}_i) \\
+ \left( \frac{\left| \vec{V} \right|^2}{c^2} \right) \left[ (\vec{R} \cdot \vec{V})^2 - \left| \vec{R} \right|^2 \right] (\vec{V} \cdot \vec{d}_i) \\
- \left[ (\vec{R} \cdot \vec{V}) (\vec{V} \cdot \vec{d}_i) - (\vec{R} \cdot \vec{d}_i) \right] \left[ (\vec{R} \cdot \vec{V}) + \left| \vec{V} \right|^2 t \right] \\
\left( \left| \vec{p} \right|^2 + \frac{\left| \vec{V} \right|^2}{c^2} \left( \vec{R} \cdot \vec{V} \right)^2 - \left| \vec{R} \right|^2 \right) \right) \right\}^{3/2}
\]

where \( \vec{p} = (\vec{R} - \vec{V}t) \) and \( \gamma = \frac{1 - \left| \vec{V} \right|^2 / c^2}{\left| \vec{V} \right|^2 / c^2} \). Since \( \left| \vec{V} \right|^2 / c^2 < 7 \times 10^{-10} \), \( \gamma \) is approximately equal to one.

Examination of the terms in Eq. (2-36) shows that the relativistic correction is on the order of \( (\left| \vec{V} \right|^2 / c^2) \Delta D \). Therefore, the relativistic differential doppler, \( \Delta D_{\text{rel}} \), can be approximated by

\[
(2-37) \quad \Delta D \cong \left( \frac{f_0}{c} \right) \left[ \left| \vec{p} \right|^2 (\vec{V} \cdot \vec{d}_i) - \left[ (\vec{R} \cdot \vec{V}) - \left| \vec{V} \right|^2 t \right] (\vec{p} \cdot \vec{d}_i) \right] / \left| \vec{p} \right|^3.
\]

Replacing the vector dot products by their appropriate trigonometric
relations and setting $\kappa = 90^\circ$ and $\delta = 0^\circ$ gives the final form of $\Delta D$,

$$
\Delta D = \frac{(f_0 d_i) k_0 \cos \delta}{c \sqrt{1 + (k_0 t)^2}} \times \\
\left\{ \cos \gamma + k_0 t \left[ \frac{-k_0 t \cos \psi + \sin \delta \sin \mu + \cos \mu \tan \delta}{1 + (k_0 t)^2} \right] \right\}
$$

where $k_0 = (V/H)$.

Equation (2-38) was used to calculate the curves shown in Figs. 2-24 through 2-27 with $k_0 = 0.01$ sec$^{-1}$, $f = 8$ GHz and $d = 50$ ft. The curves show that a symmetrical differential doppler curve with maximum amplitude occurs when the satellite travels a path parallel to a line connecting the two elements. When the satellite travels in a non-parallel direction, the differential doppler curve is reduced in amplitude and skewed to one side and in the limiting case, if the satellite travels at a right angle to the array, the differential doppler curve is antisymmetrical.

D. **Phase conjugation with moving targets**

The phase conjugation systems that were analyzed in terms of operation with a slowly moving target in Section B are now considered in terms of operation with an orbiting satellite for both pulsed and CW operation. When an orbiting satellite is used as a target, several new sources of phase error are introduced, including the
Fig. 2-24--Normalized differential doppler vs time with $\mu = 0^\circ$, $\delta = 0^\circ$ and $\psi = 0^\circ$, $45^\circ$, $67.5^\circ$ and $90^\circ$.

$k_0 = 0.01$ sec$^{-1}$

$f = 8$ GHz

$d_t = 50$ ft.

Fig. 2-25--Normalized differential doppler vs time with $\mu = 30^\circ$, $\delta = 0^\circ$, and $\psi = 0^\circ$, $45^\circ$, $67.5^\circ$ and $90^\circ$.
Fig. 2-26--Normalized differential doppler vs time with $\mu = 60^\circ$, $\delta = 0^\circ$ and $\psi = 0^\circ$, $45^\circ$, $67.5^\circ$ and $90^\circ$

Fig. 2-27--Normalized differential doppler vs time with $\mu = 90^\circ$, $\delta = 0^\circ$ and $\psi = 0^\circ$, $45^\circ$, $67.5^\circ$ and $90^\circ$
propagation time delay, the differential doppler frequency and the overall doppler frequency shift. The magnitude of these phase errors is analyzed below for pulsed and CW operation.

1. **Pulsed operation** \((f_T = f_R)\)

When an active frequency translating satellite is used as a target, there is no requirement for pulsed operation, but when a passive satellite is illuminated by only a single source, pulsed operation is necessary in order to receive the radar echo required for tracking. Therefore, the phase conjugation system is analyzed using typical Echo II values, obtained from the curves shown in Figs. 2-16 through 2-22 with \(H\) equal to 600 miles.

In pulsed operation with an orbiting satellite, the two-way propagation delay time establishes a limit on the maximum pulse width. Figure 2-28 shows the envelope of the transmitted and received waveforms for a 40% duty cycle with elevation angles equal to 15° and 90°. The shortest pulse width, approximately equal to 5 ms, occurs when the satellite is directly overhead. The \((S/N)\) ratio is optimized if the bandwidth of the phase detector equals the reciprocal of the pulse width\([28]\). With the phase detector RC time constant set equal to 5 ms, the measured phase error is delayed by approximately the same amount. Therefore, at the end
Fig. 2-28—Transmitted and received waveforms vs time for a 40% duty cycle with $\delta = 15^\circ$ and $90^\circ$.
of a pulse, the detected phase error is approximately equal to the actual phase error 5 ms before the end of the pulse. At low elevation angles where the pulse width is greater than 5 ms, the phase information in the beginning portion of the pulse is not used.

In pulsed operation, the phase error signal at the end of a received pulse is sampled, conjugated, and applied to the transmitter as a phase correction. This correction remains constant during the entire transmitting pulse interval and changes its value only at the sampling times. Since the transmitters radiate signals with constant phase corrections, the beam is directed back to a fixed point in space which is determined by the location of the satellite approximately 5 ms before the pulse ends at the satellite. During the time delay between when the effective phasing signal is radiated by the satellite and when the end of the next transmitted pulse reaches the satellite, the phase error increases because the transmitting elements are phased on a pulse-to-pulse basis. Since the differential doppler frequency, $\Delta D$, is approximately constant during this short intervals, the phasing error, called $\Delta \xi_1$, can be approximated by,

\[
(2-39) \quad \Delta \xi_1 = \int_{t_a}^{t_b} \Delta D(t) \, dt \approx \left[ \Delta D(t_a) \right] (t_b - t_a).
\]
The curves in Figs. 2-24 through 2-27 show that the differential doppler frequency varies according to the system geometry but that the curve for $d_1$ parallel to $V$ forms the envelope of all the other curves. The differential doppler frequency, $\Delta D$, and the propagation time delay, $(t_b - t_a)$, curves have maximum values at elevation angles equal to $90^\circ$ and $0^\circ$ respectively but since the differential doppler curve dominates the propagation time delay, the maximum value of $\Delta \xi_1$ occurs at the zenith position.

Figure 2-29 shows the variation in the maximum phase error $\Delta \xi_1$, produced by the product of the differential doppler frequency times the effective time delay, versus the elevation angle. The curve of $\Delta \xi_1$ was calculated for $H$ equal to 600 mi using Eq. (2-35) with values of $\Delta D$ obtained from Fig. 2-22 and $(t_b - t_a)$ equal $(2T_{2r} + 5)$ where $T_{2r}$ is given in Fig. 2-16.

In the four-element array, if one slave element has the phase error shown in Fig. 2-29, the other two slaves have similar phase error curves but with their amplitude reduced by 0.50. The dashed curve in Fig. 2-29 shows the maximum phase variation over the pulse width which validates the assumption made in Chapter III that phase is approximately constant over the period of one pulse.
MAX PHASE VARIATION IN ONE PULSE PERIOD

Fig. 2-29--Maximum phase error produced by propagation time delay vs elevation angle
The doppler frequency shift produces the second source of phase error. When a passive satellite is used, the frequency of the signal scattered varies by the one-way doppler frequency shift between the illuminating source and the satellite. The effect of the doppler frequency shift is shown by using the expression for \( \phi_i^T \) given in Eq. (2-12) and letting \( f_R \) equal \( (f_T + \Delta f) \), which gives the expression

\[
(2-30) \quad \phi_i^T = \left[ \frac{f_T}{f_T + \Delta f} \right] \phi_i^R
\]

where \( \Delta f \) is the one-way doppler frequency shift. Equation (2-30) may be written in the form

\[
(2-31) \quad \phi_i^T \approx \phi_i^R + \left( \frac{\Delta f}{f_R} \right) \phi_i^R
\]

which shows that the phase error, \( \Delta \xi_2 \), produced by the doppler frequency shift, is given by

\[
(2-32) \quad \Delta \xi_2 = \left( \frac{\Delta f}{f_T} \right) \phi_i^R
\]

The phase error, \( \Delta \xi_2 \), increases at low elevation angles because both \( \phi_i^R \) and \( \Delta f \) are largest at the horizon. Figure 2-30 shows the maximum value of the phase error, \( \Delta \xi_2 \), and the sum of the maximum phase errors produced by the propagation time delay and the doppler frequency shift versus the elevation angle.
Fig. 2-30--Maximum phase error produced by doppler frequency shift vs elevation angle
The third source of phase error, $\Delta \xi_3$, comes from slow phase drifts within the array itself. Phase stability measurements have not been made at X band but preliminary measurements performed at S band indicate that the RMS phase variations can be maintained to within $10^\circ$ RMS. Since the phase errors, $\Delta \xi_1$, $\Delta \xi_2$ and $\Delta \xi_3$, remain relatively constant for times greater than several pulse periods, a closed-loop phasing system can be used to reduce these phase errors below the maximum value given in Fig. 2-30.

a. **Pulsed, analog phase conjugation**

Figure 2-6 shows a block diagram of the analog phase conjugation system in which the pulse control generator produces the signals needed for gating the transmitter and receiver, and sampling the linear phase detector. The combination of the linear phase detector and the conjugate phase shifter produces the primary phasing while the analog or digital phase shifter is used to initially calibrate the phase of the slave channel and to insert beam tagging phase corrections at the sampling times. This analog phase conjugation system is easily incorporated into the present receiving array because it requires little additional equipment.
b. **Pulsed, digital phase conjugation**

Figure 2-31 shows a block diagram of a digital phase conjugation system operating in a pulsed mode. The sample and hold, and conjugate phase shifting networks used in the analog system are replaced by an analog to 16 level digital converter and a DPM. At the end of a received pulse, the linear phase detector output is converted to a digital level and the ring counter in the DPM is advanced until its position corresponds with the digital level established by the phase detector which conjugates the phase in a digital manner. The analog or digital phase shifter is used to calibrate the initial phase of the slave channel and insert the beam tagging correction signals. In actual practice this phase shifter could be the second DPM which is required for CW operation.

2. **CW operation, \((f_T \neq f_R)\)**

In CW operation with unequal transmitting and receiving frequencies, the frequency separation between \(f_T\) and \(f_R\) must be sufficient to allow adequate filtering so that the transmitted signals do not saturate the receivers centered at frequency \(f_R\).

Generally, CW operation is used with an active frequency translating type satellite but it can also be used with a passive satellite if a second transmitting site illuminates the target.
Fig. 2-31--Digital phase conjugation system operating in pulsed mode with $f_T = f_R$
Normally this second site is formed by the other terminal of a duplex communications link. In analyzing CW operation, Echo II orbital parameters are used so that the results may be readily compared with those obtained for pulsed operation.

When a conjugate phasing system operating CW with unequal transmit and receive frequencies uses an orbiting satellite as a target, several sources of phase error occur. The cause and magnitude of the various phasing errors are described below.

1) When an orbiting satellite is used as a target, the position of the satellite changes a small amount during the time a signal propagates from the satellite to the array and back again which produces a phase error, called $\Delta \xi_1$. During the two-way propagation time, the differential doppler frequency (rate of change of phase between two elements) is approximately constant. This allows the phase error, $\Delta \xi_1$, to be approximated by the product of the differential doppler frequency and the effective time delay which is approximately equal to the sum of the two-way propagation time delay and the time constant of the linear phase detector. Thus, the expression for the phase error, $\Delta \xi_1$, becomes

\[(2-33) \quad \Delta \xi_1 = 2\pi (T_{2R} + T_{RC}) \ (\Delta D) \ \text{rad}\]
where

\[ \Delta D \text{ - differential doppler frequency} \]

\[ T_{2R} \text{ - two-way propagation time delay} \]

\[ T_{RC} \text{ - linear phase detector RC time constant -(5 ms)} \]

Figure 2-32 shows the variation of the maximum value of \( \Delta \xi \) vs the elevation angle where the values of \( T_{2R} \) and \( \Delta D \) were obtained from Figs. 2-16 and 2-20 respectively with \( H \) equal to 600 miles.

**Fig. 2-32--**Maximum phase error produced by propagation time delay vs elevation angle for CW operation
In CW operation, a continuous phasing signal is available which reduces the effective time delay compared to pulsed operation. Comparison of Figs. 2-29 and 2-32 shows that the phasing error caused by propagation time delay is approximately 50% less for CW operation than it is for pulsed operation.

2) When a passive satellite is used, the frequency of the signal scattered varies according to the one-way doppler frequency shift. This frequency shift produces a phase error which in this case is analyzed under the heading of frequency multiplication errors.

3) The phase instability of the signal processing equipment produces similar phase errors, called $\Delta \xi_3$, in both pulsed and CW operation.

4) When the transmitting and receiving frequencies are unequal, the value of $N_1$ must be measured when the signal is initially acquired. If the initial value of $N_1$ is measured incorrectly by an amount $\Delta N_1$, a constant phase error, $\Delta \xi_4$, given by

$$\Delta \xi_4 = 2\pi (\Delta N_1) \left( \frac{f_T - f_R}{f_R} \right) \text{ rad}$$

is produced. In general, the integer, $\Delta N_1$, should never be greater
than plus or minus one which corresponds to a phase error of $\pm 28^\circ$.

5) If the phasing system does not exactly compensate for the ratio of the transmitting to receiving frequencies, a phase error, called $\Delta \xi_5$, is produced. Since the phase error, $\Delta \xi_5$, occurs for different reasons in the analog and digital phase conjugation systems, each is discussed separately.

a. CW, analog phase conjugation

Figure 2-8 shows a block diagram of the analog phase conjugation system. A linear phase detector measures the phase difference between the signals at the reference and $i^{th}$ slave elements. Then the phase difference is conjugated and multiplied by a constant equal to $(f_T/f_R)$. The second phase shifter is used to control the "integral number of wavelengths" correction. When the signal is initially received, the value of $\bar{N}_i$ is determined from the antenna look angles by using analog resolvers. After initial acquisition, the phase is incremented by an amount $\left[ \frac{(f_T-f_R)}{f_T} \right] 360$ degrees every time the linear phase detector changes from $+\pi$ to $-\pi$ or vice versa.

Assume that the $2\pi(\Delta f_0/f_R) N_i$ correction is produced without error. Then if an active satellite is used, the phase error, $\Delta \xi_5$, is equal to zero because the signal frequency radiated from the satellite is constant. When a passive satellite is used, the phase error, $\Delta \xi_5$,
is not equal to zero because the scattered signal frequency varies according to the doppler frequency shift between the illuminating source and the satellite. The phase error produced by this frequency shift is maximum at the lowest elevation angle. Equation (2-35) gives the expression for the maximum phase error,

\[(2-35) \quad \Delta \xi_5 = \left( \frac{f_T}{f_R} \right) \left( \frac{2\pi d_1 \Delta f}{c} \right) \text{ rad} \]

where \( \Delta f \) is the one-way doppler frequency shift at the time the signal is initially acquired. The maximum value of \( \Delta \xi_5 \) is equal to 5.7° with \( \Delta f \) equal to 170 kHz, \( d_1 \) equal to 25.9 m and \( \left( \frac{f_T}{f_R} \right) \) equal to 1.08.

b. CW, digital phase conjugation

The digital phase conjugation system shown in Fig. 2-11 uses two digital phase modulators. If the ratio of \( \left( \frac{f_T}{f_R} \right) \) is exactly equal to \( \left( \frac{M_R}{M_T} \right) \), then the phase errors are similar to those in the analog system. When \( \left( \frac{M_R}{M_e} \right) \) is not equal to \( \left( \frac{f_T}{f_R} \right) \), the multiplication error \( M_e \), given by Eq. (2-22), produces a phase error, \( \Delta \xi_5 \), given by

\[(2-36) \quad \Delta \xi_5 = M_e \left( \phi^R_1 \right) \text{ rad} \]

where \( \phi^R_1 \) is the value of \( \phi^R_1 \) at the time the signal is initially
acquired. The phase error, \( \Delta \xi_5 \), represents the total phase error that accumulates due to frequency multiplication error in going from an initial elevation angle corresponding to \( \Phi^R_i \) to the zenith where \( \Phi^R_i \) equals zero. If the transmit and receive digital phase modulators are restricted to having less than 33 elements (for equipment simplicity), the maximum value of \( M_e \) is less than 0.0185 for any arbitrary ratio of \( f_T/f_R \) less than 1.15. With \( M_e \) and \( \Phi^R_i \) equal to their maximum values, 0.0185 and 668 wavelengths respectively, \( \Delta \xi_5 \) is equal to 24.8\( \pi \) radians. This phasing error is sufficiently large that it must be corrected for by a closed-loop system.

For example, the maximum phase error can be corrected by approximately 400 steps of a 32 element DPM. Since these 400 corrections must be performed in a period of nominally 10 minutes, the reciprocal of the correction rate is much longer than the two-way propagation time delay.

A digital phase conjugation system has a quantizing phase error associated with it. For example, with a 32 element DPM, the quantizing increments are 22.5°, which produces a peak-to-peak phase error of \( \pm 11.25° \) (6.5° RMS) in a perfectly phased array.

Figures II-2 and II-3 show that this magnitude of RMS phase error produces a negligible reduction in cohered power.
E. Summary

In pulsed operation, the analog phase conjugation system is superior to the digital system because of its simplicity.

For CW operation, the digital phase conjugation system is preferred for several reasons. First, it requires less equipment than the analog system since the $2\pi N_i \Delta f/f_R$ phase shifter shown in Fig. 2-8 is a small computer. Second, the digital system is easy to phase adjust since the initial phase calibration, the beam tagging correction signals, and the digital PM tagging modulation are all produced by shifting the DPM. The disadvantage of the digital system is the phase error produced when $M_e$ is not equal to zero must be removed by a closed-loop system. This is not a major factor though, because all the phase conjugation systems require closed-loop operation to correct for initial phasing errors ($\Delta N_i \neq 0$) and equipment phase drifts.
CHAPTER III
BEAM TAGGING ANALYSIS

A. Introduction

1. Review of phase conjugation

In Chapter I the active array proposed required both conjugate phasing and beam tagging. The need for both of these functions can be seen if we briefly review the operation of the conjugate phasing system described in Chapter II.

Conjugate phasing is an open-loop system that measures the phase variations of the received signal at each of the slave elements with respect to a single master element. The phase difference between each slave and the reference element is conjugated and multiplied by the ratio of the transmit to receive center frequencies which makes all the transmitted signals approximately equal in frequency at the satellite.

The "approximately equal" stems from phase errors which are not corrected for by the conjugate phasing system. These phase errors can be caused by; propagation time delay, overall doppler frequency shift, phase instability of the array equipment, multiplication ratio, \((f_T/f_R)\), slightly in error and initial acquisition,
phase error. In pulsed operation with equal transmit and receive frequencies, only the first three phase errors occur, whereas in CW operation, they all may be present. These five sources of phase error are discussed below.

1) Propagation time delay produces a phase error since during the time the signal propagates from the satellite and back again, the satellite has changed its position.

2) In pulsed radar operation with a passive satellite, \( f_T \) differs from the frequency scattered by the satellite by the one-way doppler frequency shift, \( \Delta f \). This produces a phase error since the ratio of the actual frequencies, \( \frac{f_T}{(f_T-\Delta f)} \) is not equal to one. (Since the doppler frequency shift can be measured, this phase error can be corrected.)

3) Since conjugate phasing operates open-loop, phase drifts in either the transmitting or receiving portions of the slave channel produce an equal phase error in the signal transmitted from that slave. Good design should minimize the equipment phase instability but experimental measurements are required to verify its actual RMS value.

4) In CW operation with unequal transmit and receive frequencies, a phase error is produced if the initial value of \( N_1 \) in incorrectly computed at the time of signal acquisition.
5) In CW operation with unequal transmit and receive frequencies, a phase error occurs whenever the ratio of elements in the DPM's, \((M_R/M_T)\), is not exactly equal to the frequency ratio, \((f_T/f_R)\).

In summary, the phase conjugation system operates as an open-loop retrodirective array in which any of the above phase errors reduce the degree of phase coherence at the satellite. Since all these phase errors are relatively constant for intervals greater than several times the two-way propagation time delay, they can be reduced by using a closed-loop (beam tagging) phasing system.

2. Beam tagging

In this analysis, beam tagging is defined as a closed-loop system in which the transmitted signals are modulated or tagged such that a single receiving site can detect error signals which are then used to control the phase of the transmitted carrier signals[29,30,31].

Figures 3-1 and 3-2 show simplified block diagrams of beam tagging systems used for pulsed radar and CW duplex link operation respectively in which the conjugate phasing system is not shown since it would needlessly complicate the diagrams.
Fig. 3-1--Block diagram of beam tagging system in pulsed radar operation
Fig. 3-2--Block diagram of beam tagging system in CW duplex link operation
a. **Pulsed radar and CW block diagrams**

Figure 3-1 shows a three-element array operating in a pulsed mode. The tagging function generator, operating in conjunction with modulators, tags the carrier signals radiated from each of the elements. At the satellite, the signals from all the elements are linearly combined. If a passive satellite is used, the signals are simply scattered whereas when an active satellite is used, they are translated in frequency and reradiated. When the signals from the transmitting array are properly tagged, phase error signals are developed which may be detected by a single tagging receiver. Since the signal received at a ground site is the same as that radiated from the satellite, except for the propagation time delay, the tagging receiver may be located at a ground terminal. A closed-loop system is formed by feeding the detected phase error signals to phase shifters in each of the slave elements. In an N element array there are (N-1) slave elements which are identical (see Fig. 3-1) and one reference element which contains no phase shifter.

Figure 3-2 shows a three-element array operating in conjunction with a remote site. In CW operation ($f_T \neq f_R$) with a passive satellite, the signal from a second illuminating source, which is normally the
other terminal of a communications link, is required in order to obtain monopulse tracking information. In this case the tagging receiver may be physically located either of two sites.

1) A tagging receiver placed at the other end of the duplex link can detect the phase error signals which are then encoded as telemetry data and transmitted back to the active array. This method has the disadvantage that the other ground terminal of the duplex communications link must have a tagging receiver and telemetry handling equipment.

2) The cohering array may contain an additional element, called the remote receiver, located such that the signal received from the passive satellite is stronger than the stray (sidelobe) radiation received from the coherent transmitting array. A tagging receiver at this site can then detect the phase signals radiated from the satellite and relay them back to the active array over a leased land line since the phase error signals require much less than a 2 kHz bandwidth. This method seems most desirable for operation with the O.S.U. active array because Ohio University, located 60 miles South of Columbus has a 30 ft parabolic antenna and an X-band receiver which could act as the remote site.
Both of these methods are rather inefficient which illustrates the fact that CW operation with $f_T \neq f_R$ is not particularly desirable with a passive satellite. When an active frequency translating satellite is used, the tagging receiver may be located at the transmitting array which simplifies the system operation considerably.

b. **Beam tagging parameters**

Let us discuss the various possible beam tagging methods. These methods, may be classified with respect to five characteristics parameters;

1) **Tagging modulation type**
   a) amplitude modulation (AM)
   b) phase modulation (PM)

   (analysis indicates that FM is not feasible)

2) **Modulating waveform**
   a) sinusoidal
   b) digital

3) **Phase error correction cycle**
   a) Simultaneous — In the simultaneous correction cycle, an N element array is tagged with (N-1) distinct frequencies which produces (N-1) phase error signals and allows the entire array to be simultaneously phased.
b) Sequential — In the sequential correction cycle, one tagging frequency is used to produce a single phase error signal that adjusts only the tagged element in the array. The entire array is phased by repeating the above procedure on a sequential basis.

4) Phase correction technique

a) Paired — In the paired technique, two elements are tagged by the same tagging frequency, and the phase error signal developed represents the phase error between these two carrier signals.

b) Master slave — In the master-slave technique, one slave element is tagged with a given frequency and the error signal produced represents the phase error between the tagged carrier and the resultant signal formed by the remaining (N-1) elements.

5) Reference signal — Since the detected phase error signals indicate only the magnitude of the phase error, a reference signal is required to determine the sign or direction of the phase error. The reference signal may be obtained by,

a) Opposite modulation detection detects the opposite type of modulation compared to that used for detecting the error signal. For example, if PM (AM) is detected to obtain the error signal; then AM (PM) may be detected to obtain the reference signal,
b) Time delay synchronization delays the applied tagging modulation by an amount equal to the two-way propagation time delay which synchronizes the detected and delayed tagging modulations.

3. Summary

From the above discussion it is seen that there are many possible types of beam tagging configurations. The remainder of this chapter is devoted to the description and analysis of the error curves for several beam tagging configurations. Arrays of two, three and four elements are analyzed. Two elements represent a degenerate case, while four elements corresponds to the O.S.U. array.

B. Simultaneous correction - sinusoidal modulation

All the tagging systems described in this section are developed for the case of simultaneous phase corrections which requires an N element array to be simultaneously tagged with (N-1) distinct tagging frequencies. With appropriate filtering, the tagging receiver detects (N-1) phase error signals, which are then fed to the corresponding phase shifters in the (N-1) transmitting elements.
The selection of the actual frequencies used for tagging is determined by the satellite used.

A passive satellite, like Echo II, reflects both specular and diffuse components of the incidence radiation. The diffuse component arises from the surface irregularities in the satellite, which causes amplitude fading. Measurements made on Echo II [38] indicate that the amplitude fading spectrum has a cutoff frequency around 15 Hz. Thus, for passive satellite operation, the tagging frequencies used should be at least on order of magnitude higher than the random fading cutoff frequency. When an active satellite is used, it is assumed that the random amplitude fading is negligible.

Active satellites pose a new problem. Most present day satellites have output amplifiers which limit, thus destroying any AM variations on the received signals. Therefore, PM/AM beam tagging, in which the phase error information is obtained by AM detection is not applicable to an active satellite. Since in AM/PM beam tagging, the AM detected reference signal is destroyed the reference signal must be obtained by delaying the applied modulation by an amount equal to the propagation time delay. Since the reference signal must be synchronized to within a tenth of the tagging frequency period in order to keep the synchronization error to a tolerable level, the higher the tagging frequency, the more
accurately the propagation time delay must be measured. Therefore, the time delay accuracy and amplitude fading rates serve to establish the upper and lower bounds respectively on the tagging frequencies used.

In the remainder of this section, two types of phase error correction techniques, paired and master-slave, are analyzed for both sinusoidal PM/AM and sinusoidal AM/PM tagging modulations.

1. **PM/AM beam tagging**

In this part, the phasing technique used consists of phase conjugation and sinusoidal PM/AM beam tagging. The phase conjugation system, described in Chapter II, adjusts the signals from all the transmitting elements so that they are at nearly the same frequency when measured at the satellite. Then, the beam tagging system makes phase corrections to eliminate the small frequency error and finally produce phase coherence. The notation, PM/AM, indicates that the transmitting elements are tagged using phase modulation, while the received phase error information is AM detected.

The requirement for PM at the transmitting elements is easily achieved in Klystron amplifiers which are used in most high power X-band transmitters.
Both paired and master-slave phase correction schemes are analyzed in terms of their error curves.

a. Paired beam tagging technique

In paired beam tagging, an array of \( N \) elements is grouped into \((N-1)\) pairs. Since each tagging frequency is fed to a pair of elements, a simultaneously tagged \( N \) element array, requires \((N-1)\) tagging frequencies. Figure 3-3 shows this relation for a four-element array.

In Fig. 3-3, the tagging function generator contains three oscillators at frequencies \( f_1, f_2 \) and \( f_3 \), which are much less than the carrier frequency \( f_c \). Since these frequencies are unequal, their initial phase angles are of no importance. Each tagging frequency is applied to two modulators, which gives rise to the name paired tagging. Analysis shows that there must be \( 180^\circ \) phase difference between the tagging frequency applied to each modulator. In a practical system, the \( 180^\circ \) phase shift can be produced by either a LC network or a phase splitter. The carrier frequency signal, after passing through a continuously variable phase shifter is combined with the tagging frequencies in a phase modulator which causes the carrier frequency to be sinusoidally phase modulated at the tagging frequencies. One element does not contain a phase
Fig. 3-3--Block diagram of paired PM/AM beam tagging
shifter because it establishes a reference phase. At the satellite, signals from all the elements add linearly and are scattered or re-radiated back to the tagging receiver.

In pulsed operation with a passive satellite or CW operation with an active frequency translating satellite, the tagging receiver obtains its signal from the reference element of the array which corresponds to switch 1 in the "pulse" position. When operating a CW duplex link with a passive satellite, the tagging receiver may be located at a remote site which is connected to the array by land lines which corresponds to switch 1 in the "CW" position. Figure 3-3 does not show the alternative mode of CW operation with a passive satellite where the tagging receiver is located at the other end of the duplex link.

Figure 3-4 shows the tagging receiver in greater detail. Phase error information is obtained by AM detecting the received signal with an envelope detector that is assumed to approximate an ideal square law device. The outputs of narrow-band filters, centered at each of the tagging frequencies and are fed by the AM detector, indicate the magnitude of the phase errors. The sense of the phase error can be determined by using a synchronous detector which multiplies the phase error signal together with a reference signal of the same frequency at zero phase and passes only the DC component.
For paired PM/AM beam tagging, the PM detected signal at the tagging frequency does not produce a usable reference signal. Therefore, time delay synchronization in which the tagging signals from the tagging function generator are effectively delayed by an amount equal to the propagation time delay must be used. The phase error signals from the synchronous detectors feed the transmitting element phase shifters.

In this configuration, the beam tagging system forms a multiple-loop servo system with an inherent time delay equal to the propagation transmit time.
1) **Phasor diagrams**

PM beam tagging produces AM modulation on the resultant signal. For a two-element array using paired PM/AM tagging, the amplitude and sense of the AM modulation can be seen by representing the signals by two equal amplitude phasors with an arbitrary phase angle, $\gamma$, between them.

Figure 3-5a shows the applied phase modulating waveform as a function of time. This modulating signal applies phase shifts of equal magnitude but opposite sense to the two phasors. In Figs. 3-5b and 3-5d it is seen that when one phasor is advanced in phase by $\xi_o$, the other phasor is retarded by $\xi_o$, which demonstrates the paired operation. Phasor diagrams are shown for four instants of time corresponding to $\omega_T t = 0, \pi/2, \pi$ and $3\pi/2$. The amplitude variations which result, shown in Figs. 3-5c and 3-5e, indicate that the AM signal has a fundamental component at the tagging frequency plus many higher harmonics. In the next section, it is shown mathematically that the amplitude of the fundamental component which is used to provide the error signal increases as the phase error, $\gamma$, increases. It is important to note that the sense of the AM detected signal changes as the sign of $\gamma$ changes which makes it possible to detect the sense of the phase error.
Fig. 3-5--Phasor diagrams for paired PM/AM beam tagging using two elements
In an N element array, (N-1) tagging frequencies are used. Since the tagging frequencies are harmonically unrelated, the time required to go through one complete tagging cycle is lengthy. Therefore, only a two-element array is easily analyzed using phasor diagrams.

2) Mathematical analysis

The mathematical model of a simultaneously corrected, sinusoidally modulated PM/AM paired beam tagging system is analyzed using the following assumptions:

1) signals add linearly at the satellite

2) carrier frequencies from all the elements are nearly equal so that they can be treated as identical frequencies with slowly varying phase shifts

3) AM tagging error signals are square law detected

4) bandpass filters are centered at each of the tagging frequencies

5) tagging frequencies are selected so combinations of the first and second order mixing frequencies do not fall within the pass-band of the tagging filters.

In this section, the equations are developed in terms of a general N element array, but specific results are given for two, three and four element arrays.
In paired, PM/AM beam tagging the carrier frequency signal is simultaneously phase modulated by two tagging signals which may be written as,

\[ A_i = \xi_i \sin(\omega_i t + \alpha_i) \]
\[ B_i = \rho_i \sin(\omega_i t + \beta_i) \]

where

\[ i = 1, 2, \ldots, N \]
\[ \xi_i, \rho_i - \text{modulating amplitude (} \rho_1 = \xi_N = 0 \text{ because the first and last elements are tagged by only one signal)} \]
\[ \alpha_i, \beta_i - \text{phase angle} \]
\[ \omega_i, -i^{th} \text{ tagging frequency}. \]

The notations \( (A_i, \xi_i, \omega_i, \alpha_i) \) and \( (B_i, \rho_i, \omega_i, \beta_i) \) are used to indicate the differences between the two tagging signals. The signal radiated from an arbitrary element can be written in a simplified form by using Eq. (3-1) which gives the radiated signal from the \( i^{th} \) element, \( e_i(t) \), as

\[ e_i(t) = \bar{K}_i \cos[\omega_c t + \bar{\gamma}_i + (A_i + B_{i-1})] \]

where

\[ \bar{K}_i - \text{amplitude of } i^{th} \text{ signal} \]
\[ \bar{\gamma}_i - \text{phase angle of } i^{th} \text{ signal} \]
\( \omega_c \) - carrier frequency

\( A_i, B_{i-1} \) - tagging signals.

In propagating from the individual elements to the satellite, each signal has a slightly different propagation time delay which can be expressed as an equivalent phase shift at the carrier frequency and the amplitude of the signal is reduced by the path loss. These effects can be represented by replacing \( \gamma_j \) and \( K_j \) by \( y_j \) and \( K_j \) respectively. At the satellite, the total signal formed by the sum of the signals from the \( N \) elements is given by

\[
e_R(t) = \sum_{i=1}^{N} K_i \cos[\omega_c t + (\gamma_i + A_i + B_{i-1})].
\]

When the signal \( e_R(t) \) propagates to the tagging receiver, its amplitude is reduced and it is delayed by the propagation time. Since the time delay produces the identical phase shift in all the terms of \( e_R(t) \), the expression given in Eq. (3-3) may be considered as the signal received at the tagging receiver.

Since the \( i \)th phase term

\[
(3-4) \quad \sigma_i = (\gamma_i + A_i + B_{i-1})
\]

is a slowly varying function of time, it produces a narrow-band power density spectrum centered about the carrier frequency, \( \omega_c \).
By making the following substitutions,

\begin{equation}
C = - \sum_{i=1}^{N} K_i \sin(\gamma_i + A_j + B_{i-1}) \tag{3-5}
\end{equation}

and

\begin{equation}
D = \sum_{i=1}^{N} K_i \cos(\gamma_i + A_j + B_{i-1}) \tag{3-6}
\end{equation}

e_R(t) becomes

\begin{equation}
e_R(t) = C \sin \omega_0 t + D \cos \omega_c t. \tag{3-7}
\end{equation}

In an actual tagging receiver, the carrier frequency is down converted to an IF frequency, amplified, and fed to an envelope detector having a square law characteristic. Down conversion and amplification change \( \omega_c \) to \( \omega'_c \) and \( K_i \) to \( K'_i \) respectively. Since mathematically this change is merely a substitution of variables, the unprimed notation is retained because of its simplicity. The output of a square law detector produces a spectrum with components centered around base band and twice the carrier frequency as shown by the two terms given in Eq. (3-8).

\begin{equation}
e_R^2(t) = \left[ \frac{1}{2} (C^2 + D^2) \right] + \left[ \frac{1}{2} (C^2 - D^2) \cos 2\omega_c t + CD \sin 2\omega_c t \right]. \tag{3-8}
\end{equation}
A low pass filter removes the second harmonic spectrum from,
\[ e_R^2(t), \] giving

\[ e_R^2(t) \Big|_{\omega < \omega_c} = \frac{1}{2} \left( C^2 + D^2 \right) \]

as the output of the envelope detector. Substituting the expressions given in Eqs. (3-5) and (3-6) into Eq. (3-9), the result becomes

\[ e_R^2(t) \Big|_{\omega < \omega_c} = \frac{1}{2} \left\{ \sum_{i=1}^{N} K_i \sin \sigma_i \right\}^2 + \left\{ \sum_{i=1}^{N} K_i \cos \sigma_i \right\}^2 \]

where \( \sigma_i = (\gamma_i + A_i + B_{i-1}) \). This simplifies to

\[ e_R^2 \Big|_{\omega < \omega_c} = \frac{1}{2} \left\{ \sum_{i=1}^{N} K_i^2 + \sum_{i=1}^{N} \sum_{j=1}^{N} K_i K_j \left( \sin \sigma_i \sin \sigma_j + \cos \sigma_i \cos \sigma_j \right) \right\} \]

By using the generalized trigonometric formulas developed in Appendix I, \( e_R^2(t) \) can be separated into the following frequency groups:

(3-12A) Baseband - DC

\[ \frac{1}{2} \left\{ \sum_{i=1}^{N} K_i^2 \sum_{i=1}^{N} \sum_{j=1}^{N} K_i K_j \cos (\gamma_i - \gamma_j) \right\} \]

(3-12B) Fundamental frequency

\[ \frac{1}{2} \left\{ \sum_{i=1}^{N} \sum_{j=1}^{N} K_i K_j \sin(\gamma_i - \gamma_j) [A_j - A_i + B_{j-1} - B_{i-1}] \right\} \]
(3-12C) Sum and difference frequencies

\[
\frac{1}{2} \left\{ \sum_{i=1}^{N} \sum_{j=1}^{N} K_i K_j \cos(\gamma_i - \gamma_j) \left[ A_i A_j + A_i B_{j-1} + A_j B_i - A_{i-1} B_i - A_j B_{i-1} + B_i B_{j-1} \right] \right\}
\]

(3-12D) 3rd order mixing frequencies

\[
\frac{1}{2} \left\{ \sum_{i=1}^{N} \sum_{j=1}^{N} K_i K_j \sin(\gamma_i - \gamma_j) \left[ A_j B_{j-1} (A_i + B_{i-1}) - A_i B_{i-1} (A_j + B_{j-1}) \right] \right\}
\]

(3-12E) 4th order mixing frequencies

\[
\frac{1}{2} \left\{ \sum_{i=1}^{N} \sum_{j=1}^{N} K_i K_j \sin(\gamma_i - \gamma_j) \left[ A_i A_j B_{i-1} B_{j-1} \right] \right\}
\]

The expressions in Eqs. (3-12) through (3-12E) were simplified by using the substitutions given in Eq. (3-3). In Eqs. (3-12B) through (3-12E) it is seen that if all the tagging modulation indices, \( \xi_i \) and \( \rho_j \), are set equal, the modulation index becomes a common factor. The fundamental tagging frequency components are multiplied by the modulation index to the first power as shown by Eq. (3-12B). Similarly, second, third and fourth order mixing products are multiplied by the modulation index raised to the second, third and fourth powers and respectively as shown in Eqs. (3-12C) through (3-12E).
Since phase modulation or NBFM requires a small modulation index, usually less than 0.2, the modulation index raised to the 3rd and 4th power makes the 3rd and 4th order mixing frequencies negligible compared to the fundamental frequency. The second order mixing frequency terms can be eliminated by selecting the tagging frequencies such that both their sum and difference frequencies fall outside the passbands of the narrow-band filters centered at the tagging frequencies. The narrow-band filters also remove the DC component, given by Eq. (3-12A). In response to the input signal, \( e_R(t) \), the output of the tagging receiver can be approximated by the fundamental tagging frequencies given in Eq. (3-12B). By virtue of the symmetry in \( i \) and \( j \), Eq. (3-12B) can be reduced to

\[
(3-13) \left\{ \sum_{i=1}^{N} \sum_{j=i+1}^{N} \sin(\gamma_i - \gamma_j) \left[ A_j - A_i + B_{j-1} - B_{i-1} \right] \right\}
\]

Equation (3-13) contains phase error signals at each of the tagging frequencies. A narrow-band filter at the \( k \)th tagging frequency rejects all the frequencies except \( \omega_k \). Thus, the phase error signal at \( \omega_k \) becomes

\[
(3-14) \quad e_R^2(t) \left|_{\omega = \omega_k} \right. = K_k A_k \sum_{i=1}^{N} K_i \sin(\gamma_i - \gamma_k) + K_{k+1} B_k \sum_{i=1}^{N} K_i \sin(\gamma_i - \gamma_{k+1})
\]
Equation (3-14) is easier to analyze, if we let

\[ (3-15A) \quad K_i = K \quad i = 1, 2, \ldots, N \]

\[ (3-15B) \quad \xi_i = \rho_i = m \quad i = 1, 2, \ldots, N \quad \text{except} \quad (\rho_i = \xi_N = 0) \]

Equation (3-15A) requires that the radiated energy and attenuation of the propagating path be approximately equal for all elements and Eq. (3-15B) requires that all the tagging signals have the same modulation index. Thus, Eq. (3-14) reduces to

\[
(3-16) \quad e_R^2(t) \bigg|_{\omega = \omega_k} = mK^2 \left[ + \sum_{i=1}^{N} \sin(\gamma_i - \gamma_k) \sin(\omega_k t + \alpha_k) \right] \left[ + \sum_{i=1}^{N} \sin(\gamma_i - \gamma_{k+1}) \sin(\omega_k t + \beta_k) \right]
\]

Expansion of the terms in the summation of Eq. (3-16) shows that the maximum phase error signal occurs when \((\alpha_k - \beta_k) = \pi\). Thus, for maximum error signal, Eq. (3-16) reduces to:

\[
(3-17) \quad e_R^2(t) \bigg|_{\omega = \omega_k} = \left\{ mK^2 \left[ \sum_{i=1}^{N} \sin(\gamma_i - \gamma_k) - \sin(\gamma_i - \gamma_{k+1}) \right] \right\} \sin(\omega_k t + \alpha_k)
\]

where the expression within the brackets represents the magnitude of the error signal. With \(N\) equal to two, three and four elements, the
phase errors signals at tagging frequency, $\omega_1$, are:

$N = 2$

$$e_R^2(t) = mK^2 \left[ 2 \sin(\gamma_2 - \gamma_1) \right] \sin(\omega_1 t + \alpha_1)$$

$N = 3$

$$e_R^2(t) = mK \left[ 2 \sin(\gamma_2 - \gamma_1) + \sin(\gamma_3 - \gamma_1) + \sin(\gamma_2 - \gamma_3) \sin(\omega_1 t + \alpha_1) \right]$$

$N = 4$

$$e_R^2(t) = mK^2 \left[ 2 \sin(\gamma_2 - \gamma_1) + \sin(\gamma_3 - \gamma_1) + \sin(\gamma_4 - \gamma_1) + \sin(\gamma_2 - \gamma_3) + \sin(\gamma_2 - \gamma_4) \right] \sin(\omega_1 t + \alpha_1)$$

When $N$ equals two, there is only one tagging frequency, but when $N$ equals three and four, there are two and three tagging frequencies respectively. Equations (3-19) and (3-20) give only the phase error equations at one tagging frequency since examination of Eq. (3-17) shows that the error signals for other tagging frequencies can be found by cyclically advancing the subscripts. In each case, the cycle goes from one to the number of tagging frequencies used. This symmetry is very important because it means that for a given number of elements, the phase error equations are similar. Thus, only one error equation need be investigated for each value of $N$. 
3) **Phase error curves**

Equations (3-18), (3-19) and (3-20), with the phase angle of the reference signal, \( \gamma_1 \), set equal to zero, are used to calculate the phase error curves shown in this section. The amplitude of the error curves are normalized by \((1/mK^2)\), where \( K \) is proportional to the received signal and \( m \) equals the modulation index of the tagging signals. Since \( mK^2 \) is the same in all cases, the graphs may be readily compared. In each case, the phase of the \( N^{th} \) carrier signal is varied, and the phase error signal that results is calculated. When \( N \) is greater than two, there are \((N-2)\) phase angle parameters.

Below each graph there are phasor diagrams which correspond to each curve on the graph. The phase of the \( N^{th} \) phasor (the independent variable) is shown as a dashed line in a position which produces zero error signal and the shaded semicircle is used to indicate the range of phase variations used in constructing the graphs.

Figure 3-6 shows the normalized phase error curve vs \( \gamma_2 \) for a two-element array. The normalized curve is exactly equal to a sine wave of maximum amplitude 2.0. Its maximum effective control range is \( \pm 90^\circ \) and the sensitivity or slope, \((E_2/\gamma_2)\), at the zero crossing point is approximately \((mK^2/30)\) volts\(^2/\)deg.
Figure 3-6--Phase error signal vs $\gamma_2$ for a two-element array using paired PM/AM beam tagging.

Figure 3-7 shows three phase error curves vs $\gamma_3$ with the parameter $\gamma_2$ equal -45°, 0° and 45° for a three-element array. Error curves A and C have zero crossing points at -45° and 45° respectively, since the paired modulation methods aligns the phasor being varied with the phase angle of the phasor preceding it. This causes the resultant phasor sum to be lower than that which would occur if phasor 3 was aligned parallel to the resultant of the sum of
Fig. 3-7--Phase error signals vs $\gamma_3$ for a three-element array using paired PM/AM beam tagging with $\gamma_2 = -45^\circ, 0^\circ, +45^\circ$
phasors 1 and 2 but it must be remembered that another error signal, aligns phasors 1 and 2. Thus the values selected for the parameter \( \gamma_2 \) represents a static open-loop condition rather than a dynamic closed-loop situation.

The maximum effective control range is \( \pm 60^\circ \) and the slope, \( (E_3 / \gamma_3) \), at the zero crossing (approximately equal for all curves) is approximately \( (mK^2/20) \) volts\(^2\)/deg. Thus compared to the two-element array, the system sensitivity is increased but the effective control range is decreased.

Figures 3-8 through 3-10 show phase error curve graphs for a four-element array. Equation (3-20) was used with \( \gamma_4 \) the independent variable and \( \gamma_2 \) and \( \gamma_3 \) as parameters. In each graph, \( \gamma_2 \) is set equal to a constant and the parameter, \( \gamma_3 \), is set equal to, \(-45^\circ, 0^\circ, 45^\circ\). All the curves have zero crossings at a phase angle which aligns the position of phasor four parallel with phasor three. Comparing Figs. 3-8 and 3-10, it is seen that the curves in Fig. 3-10 are the same as those in Fig. 3-8 except that they are reflected about the origin.

The phasor diagrams at the bottom of each graph show the position of the three fixed phasors with the fourth phasor (shown dashed) located at the position of zero error signal. The shaded semi-circle shows the region in which phasor four varies for each diagram.
Fig. 3-8--Phase error signals vs $\gamma_4$ for a four-element array using paired PM/AM beam tagging with $\gamma_2 = -45^\circ$ and $\gamma_3 = -45^\circ$, 0°, +45°.
Fig. 3-9--Phase error signals vs $\gamma_4$ for a four-element array using paired PM/AM beam tagging with $\gamma_2 = 0^\circ$ and $\gamma_3 = -45^\circ, 0^\circ, 45^\circ$
Fig. 3-10--Phase error signals vs $\gamma_4$ for a four-element array using paired PM/AM beam tagging with $\gamma_2 = +45^\circ$ and $\gamma_3 = -45^\circ, 0^\circ, 45^\circ$
The maximum effective control range is approximately $\pm 50^\circ$ and the slope, $(E_4/\gamma_4)$, at the zero crossing is approximately 

$$(K_m^2/17) \text{volts}^2/\text{deg}.$$ 

As the number of elements increases, the maximum effective control range decreases slightly and the slope at the zero crossing increases slightly. The maximum range of control is important because if the phase error exceeds this limit, the error signal developed no longer seeks the true null but hunts for a false null which corresponds to the closed loop system going unstable.

b. Master-slave beam tagging technique

In the master-slave beam tagging technique, an N element array is divided into one reference (master) element and (N-1) slave elements as shown in Fig. 3-11. The carrier frequency signal of the reference element is neither modulated nor phase shifted since it establishes the reference phase. Each of the slave elements are phase modulated by a single sinusoidal modulating frequency. Since the tagging frequency signals are unequal, their initial phase angles are unimportant.

The master-slave PM/AM tagging system, shown in Fig. 3-11 is similar to the paired PM/AM system shown in Fig. 3-3 except for the tagging modulation. Signals from all the elements
Fig. 3-11—Block diagram of a four-element array using master-slave AM/PM beam tagging.
in the array are linearly combined at the satellite and then scattered or reradiated back to the tagging receiver which detects the \((N-1)\) phase error signals. The tagging receiver may obtain the reference signal by alternate modulation detection (PM in this case) or time delay synchronization as shown in Fig. 3-12. As the phase of the tagged carrier signal varies positively and negatively about its null, \(\gamma\) equal to zero, the sense of the detected AM changes but the sense of the detected PM remains the same. Thus, by multiplying these two signals together, the magnitude and direction of the phase error may be obtained.

1) **Phasor diagrams**

Phasor diagrams, similar to those shown in Fig. 3-5 for paired tagging, are given in Fig. 3-13 for a two-element array using master-slave PM/AM tagging. In this case, the phasor diagrams are simplified because the reference phasor is constant and only the slave phasor is varied. The phasor diagrams given in Figs. 3-13b and 3-13c show the effects of the applied phase modulation \(\xi_0 = 30^\circ\) for \(\omega_T t = 0, \pi/2, \pi\) and \(3\pi/2\) with \(\gamma = 45^\circ\) and \(\gamma = -45^\circ\) respectively where the angle \(\gamma\) corresponds to the phase error between the two phasors. Maximum signal strength or perfect coherence occurs when \(\gamma\) equals zero. Figure 3-13d shows the AM
variation produced by the tagging signal as a function of time for $\gamma > 0$ and $\gamma < 0$. These graphs show that the sense of the AM variation depends on the sign of the phase error but it does not show that the amplitude of the AM detected component at the tagging frequency produces a satisfactory control curve. In the next section it is demonstrated mathematically that the master-slave tagging technique does indeed produce a proper control curve. Figure 3-13e shows that the sense of the PM variations is independent of the sign of the phase error, $\gamma$, where the resultant signal, $R$, with no tagging
Fig. 3-13--Phasor diagrams for master-slave PM/AM beam tagging using two elements
modulation is used to establish the phase reference. Positive phase is shown as counterclockwise rotation of the resultant phasor with respect to the phase reference.

For more than two elements, the phasor diagram representation becomes so complex that it is easier to resort to a mathematical analysis.
2) Mathematical analysis

The mathematical model of a simultaneously corrected, sinusoidally modulated, PM/AM beam tagging system using the master-slave correction technique is analyzed in this section. By comparing the block diagrams of paired and master-slave correction techniques shown in Figs. 3-3 and 3-11 respectively, it can be seen that the master-slave correction technique represents a specialized form of the paired system. If the tagging signal that passes through the 180° phase shifters in Fig. 3-3, is removed, the reference element is unmodulated and the three remaining elements are modulated by a single tagging frequency. This corresponds to the master-slave configuration shown in Fig. 3-11. Phase error equations for the master-slave technique may be obtained from those previously developed for the paired technique by setting the proper terms equal to zero. This requires that \( A_i \) and all the \( B_i \) terms in the previous equations be set equal to zero. Equations (3-12A) through (3-12D) with all the \( B_i \) terms set equal to zero, become —

Baseband-DC

\[
\begin{align*}
(3-21A) & \quad \frac{1}{2} \left\{ \sum_{i=1}^{N} K_i^2 + \sum_{i=1}^{N} \sum_{j=1}^{N} K_i K_j \cos (\gamma_i - \gamma_j) \right\} \\
\end{align*}
\]
Fundamental frequency

\[(3-21B) \quad \frac{1}{2} \left\{ \sum_{i=1}^{N} \sum_{j=1}^{N} \sin(\gamma_i - \gamma_j) \right. \left[ A_i - A_j \right] \]

Sum and difference frequencies

\[(3-21C) \quad \frac{1}{2} \left\{ \sum_{i=1}^{N} \sum_{j=1}^{N} K_i K_j \cos(\gamma_i - \gamma_j) [A_i A_j] \right\} \]

The third and fourth order mixing frequencies given by Eqs. (3-12D) and (3-12E) are eliminated in this case. Narrow band filters, centered at the tagging frequencies remove the DC component and the sum and difference frequencies. Equation (3-21B) represents error signals at all the tagging frequencies. The phase error signal at the \(k\)th tagging frequency corresponds to the output of the bandpass filter, centered at \(\omega_k\). Selecting only those terms at frequency \(\omega_k\) in Eq. (3-21B), gives

\[(3-22) \quad e^2_R(t) \bigg|_{\omega_k} = \left[ \zeta_k K_k \sum_{i=1}^{N} \sin(\gamma_i - \gamma_k) \right] \sin(\omega_k t + \alpha_k) \]

where the expression within the brackets represents the magnitude of the phase error. Phase error curves at tagging frequency \(\omega_k\) are given below for two, three and four elements subject to the
simplification $\zeta_1 = m$ and $K_1 = K$. These equations are:

\( N=2 \)

\[
(3-23) \quad e_{R}^{2}(t) \bigg|_{\omega_2} = \left[ mK^2 \sin \gamma_2 \right] \sin(\omega_2 t + \alpha_2)
\]

\( N=3 \)

\[
(3-24) \quad e_{R}^{2}(t) \bigg|_{\omega_3} = \left\{ mK^2 \left[ \sin \gamma_3 + \sin(\gamma_3 - \gamma_2) \right] \right\} \sin(\omega_3 t + \alpha_3^*)
\]

\( N=4 \)

\[
(3-25) \quad e_{R}^{2}(t) \bigg|_{\omega_4} = \left\{ mK^2 \left[ \sin \gamma_4 + \sin(\gamma_4 - \gamma_3) + \sin(\gamma_4 - \gamma_2) \right] \right\} \sin(\omega_4 t + \alpha_4)
\]

In a three-element array there are two tagging frequencies and in a four-element array there are three tagging frequencies. For any particular value of $N$, only one phase error equation is required, since error signals at the remaining tagging frequencies may be found by a cyclic permutation of indices.

3) Phase error curves

Phase error curves for PM/AM master-slave beam tagging are shown in Figs. 3-14 through 3-18. These curves are obtained by using Eqs. (3-23), (3-24) and (3-25) with $\gamma_N$ set equal to zero. In
all cases, the curves are normalized by the factor $mK^2$ so that they are independent of the signal amplitude and tagging modulation index.

\[ \frac{E_2}{K^2m} \]

\[ -180 -120 -60 60 120 180 \]

\[ \gamma_2 \text{ (deg)} \]

**Fig. 3-14--Phase error signals vs $\gamma_2$ for a two-element array using master-slave PM/AM beam tagging**

Figure 3-14 shows that the phase error curve for a two-element array follows a sine curve. Comparing this curve with the similar curve shown in Fig. 3-6 for paired PM/AM tagging shows that the master-slave method has one half the amplitude (equivalently $S/N$ ratio) and the same control range.

Figure 3-15 shows three phase error curves vs $\gamma_3$ for a three-element array. The maximum control region is approximately $\pm 90^\circ$ and the slope at the zero crossing is approximately $(mK^2/30)$ volts$^2$/deg for all three curves. Curves with $\gamma_2$ equal to $-45^\circ$, $0^\circ$ and $45^\circ$ have
axis crossings at $\gamma_3$ equal to -22.5°, 0° and $\pm$22.5° respectively.

Thus, the null or position of zero error signal occurs when the phase of the tagged signal is aligned with the resultant of the
remaining \((N-1)\) signals. This is an important distinction between paired and master-slave tagging.

Figures 3-16 through 3-18 show error curves for a four-element array, with \(\gamma_4\) the independent variable and \(\gamma_2\) and \(\gamma_3\) as parameters. The maximum effective control range is nearly \(\pm 90^\circ\) and the slope at the zero crossing points is approximately \((mK^2/24)\) volts\(^2\)/deg for all the curves. The null in the error signal occurs for the value of \(\gamma_4\) that aligns the phase of the independent variable with respect to the resultant of the remaining three phasors.

The signal-to-noise ratio improvement of paired tagging compared to master-slave tagging can be seen intuitively. In paired tagging, both carrier signals are modulated whereas in master-slave tagging only one carrier is modulated which allows the paired method to radiate twice as much tagging power as master-slave method assuming constant modulation indices.

2. **AM/PM beam tagging**

In this section, the cohering technique consists of conjugate phasing and sinusoidal AM/PM beam tagging. The phase conjugate system described in Chapter II, adjusts the signals from all the transmitting elements so that at the satellite the signals are at approximately the same frequency. Then, the beam tagging system
(a) $\gamma_2 = \gamma_3 - 45^\circ$  
(b) $\gamma_2 = -45^\circ, \gamma_3 = 0^\circ$  
(c) $\gamma_2 = -45^\circ, \gamma_3 = 45^\circ$

Fig. 3-16--Phase error signals vs $\gamma_4$ for a four-element array using master-slave PM/AM beam tagging

with $\gamma_2 = -45^\circ$ and $\gamma_3 = -45^\circ, 0^\circ, 45^\circ$
Fig. 3-17--Phase error signals vs $\gamma_4$ for a four element array using master-slave PM/AM beam tagging with $\gamma_2$ and $\gamma_3 = -45^\circ, 0^\circ, 45^\circ$. 
Fig. 3-18--Phase error signals vs $\gamma_4$ for a four element array using master-slave PM/AM beam tagging with $\gamma_2 = 45^\circ$ and $\gamma_3 = -45^\circ, 0^\circ, 45^\circ$. 
makes phase corrections on each of the (N-1) carrier signals to bring them into phase alignment at the satellite. The notation, AM/PM indicates that the transmitting elements are tagged using amplitude modulation, while the received phase error information is PM detected.

The analysis of AM/PM beam tagging closely parallels that given in the previous discussion for AM/PM beam tagging. There it was shown that master-slave tagging represents a special case of paired tagging. Therefore to avoid repetition, the discussion of paired and master-slave techniques are combined in this analysis.

a. AM/PM tagging system description

System descriptions and block diagrams are given for both paired and master-slave tagging methods.

When operating CW with an active frequency translating satellite or pulsed with a passive satellite, the tagging receiver obtains its signal from the reference element. However, in CW duplex link operation, the tagging receiver may be located at either a remote site connected to the array with land lines or at the other terminal of the duplex link.
1) **Paired system description**

Figure 3-19 shows a block diagram of a four-element array using paired AM/PM beam tagging which is similar to Fig. 3-3 except that it contains AM modulators. The tagging receiver is similar to the one shown in Fig. 3-4 with the exception that the phase error signals are PM detected.

An important advantage of the AM/PM system is that it can operate in conjunction with an active satellite that hard limits its transmitted signals. The limiter removes any amplitude variations but preserves the zero crossing phase information. When operating with a limiting type active satellite, the reference signal must be obtained by time delay synchronization.

2) **Master-slave system description**

Figure 3-20 shows a block diagram of a four-element array using master-slave AM/PM beam tagging. This diagram is similar to Fig. 3-11 except for replacing the PM modulators with AM modulators. The tagging receiver is equivalent to the one shown in Fig. 3-12 if the words PM and AM are interchanged everywhere.

If a passive satellite or a non-limiting type active satellite is used, the reference signal may be obtained by AM detection.
Fig. 3-19--Block diagram of a four-element array using paired beam tagging
Fig. 3-20—Block diagram of a four-element array using master-slave beam tagging

b. Phasor diagrams

Phasor diagrams are shown for a two-element array using both paired and master-slave methods of AM/PM beam tagging.

1) Paired AM/PM beam tagging

Figure 3-21 shows phasor diagrams of the signals from a two-element array using paired AM/PM modulation. The signals are represented by two phasors with an arbitrary phase error, $\gamma$, between them.
Fig. 3-21—Phasor diagrams for paired AM/PM beam tagging of a two-element array
The sinusoidal AM modulating waveform shown in Fig. 3-21a is applied with opposite polarity to the two carrier signals which causes the amplitude of one phasor to increase while the amplitude of the other phasor decreases. Phasor diagrams of four instants of time, corresponding to tagging frequency angles of \( \omega_1t = 0, \pi/2, \pi \) and \( 3\pi/2 \) are shown in Fig. 3-21b and 3-21d for \( \gamma > 0 \) and \( \gamma < 0 \) respectively. The phasor diagrams show that the AM tagging modulation produces incidental phase modulation where the angular position of the resultant phasor, \( R \), with zero tagging signal establishes the reference phase. Positive phase variations occur, when the tagging modulation causes the resultant phasor, \( R \), to move in a counter clockwise direction. Figures 3-21c and 3-21e show the phase variation of the resultant phasor \( R \) as a function of time. The major difference between these two figures is that the sense of the phase error \( \psi(t) \) changes depending on the sign of \( \gamma \).

In an actual system, the error signal is obtained by passing the phase variation \( \psi(t) \) through a narrow band filter centered at the tagging frequency. Therefore, the fundamental component of \( \psi(t) \) at the tagging frequency determines the phase error signal. Error curves, derived from the mathematical analysis, show that the amplitude of the fundamental component of \( \psi(t) \) increases as the phase error, \( \gamma \), between the two carrier signals increases.
2) Master-slave AM/PM beam tagging

Figure 3-22 shows phasor diagrams of the signals in a two-element array using master-slave AM/PM beam tagging. The phasor diagrams shown in Figs. 3-22b and 3-22c are similar to those in Figs. 3-21b and 3-21d except that one phasor (the reference) is not modulated. Since only one phasor is AM modulated, the phase variation of the resultant phasor, R, is smaller in this case. The reduction in the variation of $|\psi|$ is seen by comparing Fig. 3-22d with Figs. 3-21c and 3-21e.

In the master-slave method, the amplitude variation of the resultant phasor, R, may be used to establish the tagging frequency reference signal. Figure 3-22e shows that the sense of the amplitude variation is independent of the sign of the phase error $\gamma$. This property allows the AM detected signal to be used as a phase reference.

c. Mathematical analysis

General mathematical equations are developed for sinusoidally modulated, simultaneous corrected, AM/PM, paired beam tagging. With proper specialization, these same equations apply to master-slave tagging.
(a) AM MODULATION TAGGING SIGNAL vs $\omega_T t$

(b) PHASOR DIAGRAM FOR VARIOUS TIMES WITH $\gamma = 45^\circ > 0$

(c) PHASOR DIAGRAM FOR VARIOUS TIMES WITH $\gamma = -45^\circ < 0$

Fig. 3-22--Phasor diagrams for master-slave AM/PM beam tagging of a two-element array
Fig. 3-22--Phasor diagrams for master-slave AM/PM beam tagging of a two-element array

(d) PHASE VARIATION vs $\omega_{tt}$

(e) AMPLITUDE VARIATION OF $|R|$ vs $\omega_{tt}$, $\gamma \leq 0$
The following assumptions are made:

1) signals add linearly at the satellite

2) carrier frequencies from all the elements are approximately equal so that they can be treated as identical frequencies with slowly varying phase shifts

3) the signal channel is hard limited before detection using a zero order full wave device

4) \((N-1)\) narrow bandpass filters are located at each of the tagging frequencies

5) tagging frequencies are selected so that combinations of second order mixing products (sum and difference frequencies) do not fall within the pass-bands of the tagging filters.

The general expression for the signal radiated from the \(i^{th}\) element is given by

\[
(3-26) \quad e_i(t) = \bar{K}_i \left[ 1 + \xi_i \cos(\omega_i t + \alpha_i) + \rho_{i-1} \cos(\omega_{i-1} t + \beta_{i-1}) \right] \times \sin(\omega_c t + \gamma_i)
\]

\(i = 1, 2, \cdots, N\)

where

\(\bar{K}_i\) - amplitude of \(i^{th}\) carrier

\(\gamma_i\) - phase of \(i^{th}\) carrier
\begin{align*}
\zeta_i, \rho_{i-1} & \quad \text{AM tagging modulation indices} \\
(\zeta_N = \rho_o = 0) \\
\omega_c & \quad \text{carrier frequency}. \\
\end{align*}

The effect of the signals propagating to the satellite and back again is the same as in the PM/AM tagging analysis. Therefore, the signals at the tagging receiver are immediately obtained by replacing \(\overline{\gamma}_i\) and \(\overline{K}_i\) by \(\gamma_i\) and \(K_i\) respectively.

The total signal at the tagging receiver, \(e_R(t)\), is formed by the sum of the signals from all the elements which gives

\[
(3-27) \quad e_R(t) = \left[ \sin (\omega_c t + \gamma_i) \right] \times \\
\quad \times \left\{ \sum_{i=1}^{N} K_i \left[ 1 + \zeta_i \cos(\omega_i t + \alpha_i) + \rho_i \cos(\omega_i t + \beta_i) \right] \right\}.
\]

Since \(e_R(t)\) is a narrow-band signal, it can be written in terms of a constant frequency, \(\omega_c\), with slowly varying envelope, \(A(t)\), and phase, \(\psi(t)\), functions. In terms of envelope and phase variables, \(e_R(t)\) is

\[
(3-28) \quad e_R(t) = A(t) \sin [\omega_c t + \psi(t)]
\]

where \(\psi(t)\) equals
\[ \psi(t) = \tan^{-1} \left( \frac{\sum_{i=1}^{N} K_i \left[ 1 + \zeta_i \cos(\omega_1 t + \alpha_1) + \rho_{i-1} \cos(\omega_{i-1} t + \beta_{i-1}) \right] \sin \gamma_i}{\sum_{i=1}^{N} K_i \left[ 1 + \zeta_i \cos(\omega_1 t + \alpha_1) + \rho_{i-1} \cos(\omega_{i-1} t + \beta_{i-1}) \right] \cos \gamma_i} \right) \]

and \( A(t) \) equals

\[ A(t) = \left( \frac{\sum_{i=1}^{N} K_i \sin \gamma_i \left[ 1 + \zeta_i \cos(\omega_1 t + \alpha_1) + \rho_{i-1} \cos(\omega_{i-1} t + \beta_{i-1}) \right]}{\sum_{i=1}^{N} K_i \cos \gamma_i \left[ 1 + \zeta_i \cos(\omega_1 t + \alpha_1) + \rho_{i-1} \cos(\omega_{i-1} t + \beta_{i-1}) \right]} \right)^2. \]

The signal \( e_R(t) \) is passed through a hard limiter (zero order full wave device) which removes the amplitude variation, \( A(t) \) and preserves the phase variation, \( \psi(t) \) \[33\]. A phase detector is used to recover the signal, \( \psi(t) \). Error signals at each of the tagging frequencies are obtained by passing the phase detector output through narrow-band filters centered at each of the tagging frequencies. The process of phase detection and filtering at a particular tagging frequency, \( \omega_k \), is equivalent to finding the amplitude of the Fourier series component of, \( \psi(t) \) at frequency, \( \omega_k \).

When \( N \) equals two, only a single tagging is required. In this case, the phase variation at the tagging frequency can be evaluated analytically. For \( N \) greater than two, it is much easier to evaluate
the Fourier series components by using a digital computer. In this section, the results were obtained by using an IBM 7094 digital computer.

Before the Fourier series analysis of \( \psi(t) \) can be performed, the fundamental period of \( \psi(t) \) must be determined. Since the tagging frequencies are unequal, the fundamental period of the Fourier series must be sufficiently long to allow all the signals to be periodic in the interval \( T_f \). The period can be found using Eq. (3-31)

\[
\frac{2\pi}{\omega_0} = T_f = 2\pi\left(\frac{k_1}{\omega_1} = \frac{k_2}{\omega_2} = \cdots \frac{k_{N-1}}{\omega_{N-1}}\right)
\]

where the \( k_i \)'s are smallest arbitrary positive integers which allow Eq. (3-31) to be satisfied. The well-known expressions for the Fourier series are repeated for convenience. They are:

\[
(3-32) \quad \psi(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[ a_k \cos\left(\frac{2\pi kt}{T}\right) + b_k \sin\left(\frac{2\pi kt}{T}\right)\right]
\]

where \( \psi(t) \) is periodic in the interval \( T \) and

\[
(3-33) \quad a_k = \frac{2}{T} \int_{-T/2}^{T/2} \psi(t) \cos\left(\frac{2\pi kt}{T}\right) \, dt \quad k = 1, 2, \cdots
\]
\[(3-34) \quad b_k = \frac{2}{T} \int_{-T/2}^{T/2} \psi(t) \sin \left( \frac{2\pi k t}{T} \right) \, dt \quad k = 1, 2, \cdots \]

Examination of Eq. (3-29) shows that \( \psi(t) \) is an even function. Therefore, all the \( b_k \) coefficients in Eq. (3-34) are zero and Eq. (3-33) may be rewritten as

\[(3-35) \quad a_k = \frac{4}{T} \int_{0}^{T/2} \psi(t) \cos \left( \frac{2\pi k t}{T} \right) \, dt .\]

The phase variation, \( \psi(t) \), given by Eq. (3-29) may be simplified by making the following assumptions.

1) All the receiver signals, \( K_i \), are equal

2) in paired tagging

\[\xi_i = \begin{cases} m & i = 1, 2, \cdots, (N-1) \\ 0 & i = N \end{cases}, \quad \rho_i = \begin{cases} m & i = 2, 3, \cdots, N \\ 0 & i = 1 \end{cases} \]

3) in master-slave tagging

\[\xi_i = \begin{cases} m & i = 2, 3, \cdots, N \\ 0 & i = 1 \end{cases}, \quad \rho_i = 0 \quad i = 1, 2, \cdots, N .\]
This makes the phase variation, \( \psi(t) \), independent of the received signal amplitude. Equation (3-36) gives the final expression that was used to calculate the error curves.

\[
(3-36) \quad a_k = \frac{4}{T} \int_0^{T/2} \tan^{-1} \left[ \sum_{i=1}^{N} \frac{[1 + p_i \cos(\omega_i t + \alpha_i) + p_{i-1} \cos(\omega_{i-1} t + \beta_{i-1})] \sin \gamma_i}{\sum_{i=1}^{N} [1 + p_i \cos(\omega_i t + \alpha_i) + p_{i-1} \cos(\omega_{i-1} t + \beta_{i-1})] \cos \gamma_i} \right] \cos \left( \frac{2\pi kt}{T} \right) \, dt.
\]

Only those values of \( k \) which correspond to actual tagging frequencies were computed. It was assumed that energy at other frequencies is rejected by the narrow-band filters centered at each of the tagging frequencies. The value \( a_k \) was computed by numerical integration using Simpson's 3/8 rule.

In the paired modulation method, maximum error signal occurs when the phase of the tagging satisfies the relation \( (\alpha_i + \beta_j) = 180^\circ \). When evaluating Eq. (3-36), \( \alpha_i \) and \( \beta_j \) were set equal to 0° and 180° respectively.

In the preceding PM/AM analysis, the modulation index was a common factor which allowed the curves to be normalized. For the AM/PM analysis, the modulation factor is not a common factor.
Therefore, the phase error curves are plotted with the modulation index, \( m \), as a parameter.

The modulation index used must be small, so that the carrier frequency signal is not overmodulated. This is particularly important in the paired modulation method, where the carrier signal is modulated by two tagging frequencies.

Phase error curves are calculated by varying the phase of one phasor while the remaining phasors maintain a constant phase relation. The modulation index, \( m \), and the ratio of the tagging frequencies, \( \omega_i/\omega_j \), are parameters in all the curves.

d. **Error curves**

This section shows phase error curves for both paired and master-slave beam tagging methods.

1) **Two element arrays**

Figures 3-23 and 3-24 show phase error curves for a two-element array using paired and master-slave beam tagging methods respectively. Phase error curves are shown for modulation indices equal to 0, 0.125, 0.25, and 0.50. In all the error curves, the magnitude of the error signal increases as the phase error, \( \gamma_2 \), increases from \( 0^\circ \) to \( \pm 180^\circ \). The shape of error curve produces a control signal that has no false nulls because there is only one stable point, at \( \gamma_2 = 0^\circ \).
Fig. 3-23--Phase error curves vs $\gamma_2$ for a two-element array using paired AM/PM beam tagging with $m = 0.125, 0.25$ and 0.50
Fig. 3-24--Phase error curves vs $\gamma_2$ for a two-element array using master-slave AM/PM beam tagging with $m = .125, .25, \text{ and } .50$
The nonlinear error curves emphasize the detected error signal for large values of phase error, $\gamma_2$. In a closed-loop servo system, this nonlinearity is equivalent to a variable gain constant that increases with increasing phase error, $\gamma_2$.

At the point of zero phase error, the slope of the paired curves is approximately twice that of the master-slave curves which indicates that the paired method has a higher S/N ratio.

The error curves show that the higher modulation indices produce larger phase error signals.

2) Three-element arrays

In a three-element array, there are two tagging frequencies. Equation (3-36) shows that different tagging frequencies have different phase error signals. This means phase error curves should be shown at each of the tagging frequencies. Analysis of the phase error curves for different tagging frequencies shows that they are approximately equal. Therefore, phase error curves at only one tagging frequency are presented here.

Figures 3-25 and 3-26 show phase error curves for a three-element array using paired tagging. Phase error curves are plotted vs $\gamma_3$ with $\gamma_2$ as a parameter equal to $-45^\circ$ and $0^\circ$. Error curves for $\gamma_2 = +45^\circ$ are not shown because they may be obtained by a reflection about the origin of the curves with $\gamma_2 = -45^\circ$. The error signals
Fig. 3-25--Phase error curves vs $\gamma_3$ with $\gamma_2 = -45^\circ$ for a three-element array using paired AM/PM beam tagging with $m = 0.125, 0.25$ and $0.50$. $\gamma_2 = 45^\circ$
Fig. 3-26--Phase error curves vs \( \gamma_3 \) with \( \gamma_2 = 0^\circ \) for a three-element array using paired AM/PM beam tagging with \( m = .125, .25 \) and .50.
shown were obtained at the tagging frequency, \( \omega_2 \), where the ratio of tagging frequencies, \( \omega_2/\omega_1 \), equals 1.5. Tagging frequency ratios of 2.0 and 3.0 produced similar error curves to those for \( \omega_2/\omega_1 = 1.5 \). This indicates that the error signals are relatively insensitive to the ratio of the tagging frequency.

The phase error signal equals zero at phase angles which align the second and third phasors. This is expected because the error signals were obtained from tagging frequency, \( \omega_2 \). If error curves were shown for tagging frequency, \( \omega_1 \), the phase error signal would equal zero when the first and second phasors are aligned.

The maximum effective control range is approximately \( \pm 135^\circ \) compared to \( \pm 180^\circ \) for the two-element array.

Figures 3-27 and 3-28 show phase error curves for a three-element array using master-slave AM/PM modulation. These curves have the opposite polarity compared to the paired error curves but this is a minor difference because the sense of the error signal can be reversed by inverting the phase reference fed to the synchronous detectors.

The curves shown correspond to the error signal at tagging frequency, \( \omega_2 \). A tagging frequency ratio, \( \omega_2/\omega_1 \), equal to 1.5 was used in these curves but ratios of \( \omega_2/\omega_1 \), equal to 2.0 and 3.0 also gave similar error curves. As in the paired case, the error signals
are relatively insensitive to the ratio of the tagging frequencies.

![Graph showing phase error curves vs $\gamma_3$ with $\gamma_2 = -45^\circ$ for a three-element array using master-slave AM/PM beam tagging with $m = 1.25, .25, .50$](image)

Fig. 3-27--Phase error curves vs $\gamma_3$ with $\gamma_2 = -45^\circ$ for a three-element array using master-slave AM/PM beam tagging with $m = 1.25, .25, .50$

The null in the error signal occurs at an angle such that the third phasor is aligned with the resultant of the other two phasors.

For similar modulation indices, the gain of the phase error curves for the master-slave case is approximately 40% less than in the paired case. The maximum control range is approximately $\pm 140^\circ$ which is slightly larger than in the paired curves.
Fig. 3-28--Phase error curves vs $\gamma_3$ with $\gamma_2 = 0^\circ$ for a three-element array using master-slave AM/PM beam tagging with $m = 1.25, .25$ and $.50$

The zero or null point tends to shift from its desired value as the modulation index increases. For modulation indices equal to $0.5$, the zero shift is less than $2.0$ degrees which is certainly tolerable.

3) Four-element arrays

Figures 3-29 through 3-31 and 3-32 through 3-34 show phase error curves for a four-element array using paired and master-slave beam tagging respectively. In these graphs, $\gamma_4$ is the independent variable while $\gamma_2$ and $\gamma_3$ are parameters. Equation (3-36) shows
that when the sign of all the $\gamma$'s change, the error curves are symmetrical about the origin compared with the curves for the original $\gamma$'s. Therefore error curves of $\gamma_2 = 45^\circ$ and $\gamma_3 = 0^\circ$ are similar to those for $\gamma_2 = 0^\circ$ and $\gamma_3 = 45^\circ$ which reduce the number of phase error curves that must be plotted.

![Diagram with phase error curves](image)

**Fig. 3-29--Phase error curves for a four-element array using paired AM/PM beam tagging with $\gamma_2 = \gamma_3 = 0^\circ$**
For each particular value of the parameters, $\gamma_2$ and $\gamma_3$, error signals are produced at three tagging frequencies. The error signals at each of the tagging frequencies are similar. Therefore, only the error signal at the highest tagging frequency is shown.

Tagging frequency ratios of $\omega_2/\omega_1 = 2.0$ and $\omega_3/\omega_1 = 3.0$ were used. This is a wider frequency separation than would be used in
Fig. 3-31—Phase error curves for a four-element array using paired AM/PM beam tagging with $\gamma_2 = \gamma_3 = 45^\circ$ in a practical system but experience from the three-element array shows that the frequency ratio does not appreciably affect the error curves. Closer spacing of the tagging frequencies is desirable because it requires less bandwidth for the tagging spectrum.

Figures 3-29 through 3-31 show phase error curves for three particular values of $\gamma_2$ and $\gamma_3$ using paired modulation. The minimum control range is approximately $\pm 95^\circ$. As the modulation index increases, the axis crossing point remains fixed at a point
which tends to align the variable vector with the phase of the preceding vector. With four elements, the phase error signals are approximately one half as large as with three elements.

Figures 3-32 through 3-34 show phase error curves for the same three values of $\gamma_2$ and $\gamma_3$ using master-slave beam tagging. The minimum control range is approximately $\pm 125^\circ$. At the zero crossing, the slope of the curve is approximately 20% less than that for the paired method.

Fig. 3-32--Phase error curves for a four-element array using master-slave AM/PM beam tagging with $\gamma_2 = \gamma_3 = 0^\circ$
C. Sequential correction-
digital modulation

In this section, the phasing technique discussed consists of phase conjugation and sequential digital beam tagging. The phase conjugation system, described in Chapter II, adjusts the signals from all the transmitting elements so that they are at approximately the same frequency at the satellite. Nearly identical frequencies may be represented by equal frequencies with different time varying phase functions. In Chapter II it was shown that these phase variations remain relatively constant over periods equal to several

![Graph showing phase error curves](image)

Fig. 3-33--Phase error curves for a four element array using master-slave AM/PM beam tagging with $\gamma_2 = 0^\circ$ and $\gamma_3 = 45^\circ$
times the propagation delay. This allows use of sequential rather than simultaneous phase corrections.

In a sequential tagging system, only one tagging frequency is used. This allows simplification of the tagging system compared to the simultaneous phase correction technique which uses \((N-1)\) tagging frequencies. The simplicity is gained at the expense of reduced phasing accuracy.

Fig. 3-34--Phase error curves for a four-element array using master-slave AM/PM beam tagging with \(\gamma_2 = \gamma_3 = 45^\circ\)
1. **Sequential tagging system**
   
   **description**

   Figure 3-35 shows a block diagram of a four-element array using sequential, digital beam tagging. The slave element modulators are labelled "AM or PM" so that the block diagram applies
to either AM or PM digital beam tagging. The sequential system is discussed in terms of CW operation.

The commutation or sequencing signals, shown in Fig. 3-36 vs time, are used to control the periodic tagging of each of the slave elements. The basic sequencer gating waveform is an asymmetrical square wave with time, $T_{off}$, approximately 1.5 times, $T_{on}$ where time, $T_{on}$ is less than the two-way propagation time delay. When the gating signal for the $i^{th}$ slave is "on", the carrier frequency signal from this element is modulated by the digital tagging generator. The gating signals are commutated so that only one slave element is tagged at a time for a duration equal to $T_{on}$. During the time $T_{off}$, the elements radiate untagged signals. In Fig. 3-35, switch 2 produces the commutation operation.

Figure 3-37 shows a block diagram of the sequential tagging receiver. When the transmitted signal is AM (PM) modulated, the phase error signal is obtained by PM (AM) detecting the received signal at the tagging frequency, $\omega_T$. An active filter integrates the output of the bandpass filter centered at $\omega_T$ which improves the $S/N$ ratio by the ratio of the predetection bandwidth to the reciprocal of the integration period.
Fig. 3-36--Sequencer gating signals vs time for a four-element array
Fig. 3-37--Block diagram of a sequential, digital beam tagging receiver
The phase reference signal may be obtained by either alternate modulation detection or time delay synchronization, according to the position of switch 4. Alternate modulation detection uses a channel similar to the error signal except that the opposite type of modulation is detected. The sense of the phase error signal is obtained by comparing it with the reference signal in a phase detector. The signals are in general either in-phase or 180° out-of-phase, which sets the polarity indicator to either +1 or -1. Multiplication of the magnitude of the phase error signal by the polarity information (-1) gives the final phase error signal which is sampled and stored at the end of an information period.

The magnitude of the PM phase variation is independent of the received signal strength whereas the AM variation is a function of the received signal strength. Therefore, when the error signal is AM detected, its amplitude must be adjusted according to the received signal strength which is measured by placing a narrow-band filter at the IF frequency and integrating its output with an active filter. The amplitude of the signal detected by the active filter is used to normalize the AM error signal.

Figure 3-38 shows typical waveforms in the tagging receiver. The output of the signal channel active filter shows a series of isosceles triangles of various widths. Their widths or amplitudes
Fig. 3-38--Typical tagging receiver waveforms
are proportional to the phase error. At the end of a tagging signal period, \( T_{on} \) the active filter is sampled and dumped. A wide dump pulse is used to clamp the active filter to zero during times when the received signal is not tagged to prevent integration on noise alone. Since the phase error signal is stored as a DC level by the "box car" circuit, its amplitude can change only at the sampling intervals when new phase error information is obtained.

In the above discussion, \( T_{off} \), the time the signals are not tagged is greater than, \( T_{on} \), the time during which one slave element is tagged. This simplifies the discussion because the phase correction is inserted in the carrier frequency of the \( i^{th} \) slave element at a time when the array is not being tagged.

2. **AM/PM and PM/AM phasor diagrams**

The phasor diagrams for digital beam tagging are much simpler than those for sinusoidal beam tagging because the digital phasor diagram has only two distinct positions.

Sequential tagging requires that only one element in the array be tagged at any given time. The remaining \((N-1)\) phasors combine to form a resultant phasor, \( R' \). Since all the signals are assumed to have unity amplitude, the magnitude of the resultant phasor, \( R' \), can vary in magnitude, depending on the particular orientation of the \((N-1)\) phasors,
from zero to \((N-1)\). For example, in a four-element array, \(R'\) can vary between zero and three, causing \(R'\) to become a parameter in the phasor diagrams.

a. Digital AM phasor diagrams

Digital AM usually means there are two amplitude states, one above and one below the unmodulated carrier signal level. Normally the X-band klystron tubes are adjusted for maximum power so that increased exciter power does not significantly increase the output power. In cases where the transmitter is peak power limited, digital modulation may be produced by periodically reducing the radiated power. This type of AM modulation is indicated by \((0, -m)\) where \(m\) corresponds to the reduction factor in the transmitter signal. Whenever possible, the more common case of \((+m, -m)\) digital modulation, is preferable because it produces slightly higher error signals than \((0, -m)\) modulation.

Figure 3-39 shows a phasor diagram of an \(N\) element array using digital \((0, -m)\) amplitude modulation. The modulating waveform is a symmetrical square wave that has two levels, \(0\) and \(-m\). In Fig. 3-39b, the phasor, \(R'\), corresponds to the resultant of the \((N-1)\) untagged phasors. Phasor diagrams are shown for the phase error, \(\gamma\), greater than and less than zero. Each diagram shows two
Fig. 3-39—Phasor diagrams for a digital AM/PM beam tagging system
resultant phasors, R, where one corresponds to the modulating signal equal to zero and the other to -m. Figures 3-39c and 3-39d show that digital amplitude tagging produces both PM and AM variations at the tagging rate. Positive phase is defined by phasor rotation counterclockwise from the reference phase, which is established with zero tagging signal. The sense of the PM phase variations are dependent on the sign of the phase error, \( \gamma \), whereas the AM variations are not.

b. Digital PM phasor diagrams

Figure 3-40 shows phasor diagrams for digital PM/AM beam tagging where the modulating signal is a symmetrical square wave with amplitude \( \pm \xi_0 \). The dashed lines in Fig. 3-40b indicate the unmodulated phasor positions and the two phasors labelled, R, represent the resultant phasor when the modulating signal equals \( \pm \xi_0 \). Digital phase modulation of a single phasor with unity amplitude produces amplitude and phase modulation of the resultant phasor as shown in Fig. 3-40c and 3-40d for \( \gamma > 0 \) and \( \gamma < 0 \). The phase variation, \( \psi(t) \), has the same magnitude and polarity for equal positive and negative values of \( \gamma \). A slight phase bias between the two \( \psi(t) \) curves is caused by the choice of the phase reference as position of the unmodulated resultant.
(a) DIGITAL TAGGING MODULATION SIGNAL VS $\omega_t$

(b) PHASOR DIAGRAMS - $\gamma = 45^\circ$ AND $\gamma = -45^\circ$

(c) AMPLITUDE VARIATION VS $\omega_t$

(d) PHASE VARIATION VS $\omega_t$ - $\gamma > 0$ AND $\gamma < 0$

Fig. 3-40 - Phasor diagrams for a digital PM/AM beam tagging system
3. Mathematical analysis

Mathematical analysis of both AM/PM and PM/AM sequential digital beam tagging systems is performed using the following assumptions:

1) signals add linearly at the satellite
2) carrier frequencies from all the elements are nearly equal so they may be represented by identical frequencies with slowly varying phase shifts.
3) only one element is tagged at a given time
4) switching time is assumed negligible.

At the tagging receiver, the received signal, \( e_R(t) \), is given by

\[
(3-37) \quad e_R(t) = \sum_{i=2}^{N} K_i \cos(\omega_c t + \gamma_i) + e_T(t)
\]

where

\[
e_T(t) = \begin{cases} 
K_1 \left[ 1 - m \left( \frac{1+S(t)}{2} \right) \right] \cos(\omega_c t + \gamma_1') & \text{Amplitude} \\
K_1 \cos[\omega_c t + \gamma_1' + S(t) \xi_0] & \text{Phase}
\end{cases}
\]

\[
S(t) = \begin{cases} 
+1 & 0 \leq \omega_R t \leq \pi \\
-1 & \pi \leq \omega_R t \leq \pi
\end{cases}
\]

The two forms of the tagged signal, \( e_T(t) \), correspond to digital amplitude and phase modulation respectively where \( S(t) \) is the
digital switching operator. The terms within the summation sign in Eq. (3-37) are independent of the tagging signal which allows them to be replaced by a single equivalent phasor, $R'$, given by

$$ R' \cos(\omega_c t + \Gamma) = \sum_{i=2}^{N} K_i \cos(\omega_c t + \gamma_i). $$

By making the following substitutions,

$$ \gamma_i' = (\gamma_i + \Gamma) \quad K_i = 1.0 $$

and rotating the coordinate system by $\Gamma$, Eq. (3-37) reduces to,

$$ e_R(t) = \left\{ R' + \left[ 1 - m \left( \frac{1 + s(t)}{2} \right) \right] \cos \gamma_1 \right\} + \left\{ \left[ 1 - m \left( \frac{1 + s(t)}{2} \right) \right] \sin \gamma_1 \right\} $$

$$ e_R(t) = \left\{ R' + \cos(\gamma_1 + \zeta_0 s(t)) \right\} + \left\{ \sin(\gamma_1 + \zeta_0 s(t)) \right\} $$

where the $\cos \omega_c t$ variation has been omitted. Equations (3-40A) and (3-40B) apply to digital amplitude and phase tagging respectively. By writing these equations in terms of envelope and phase functions, the peak-to-peak amplitude and phase variations produced by the tagging may be calculated.
a. AM/PM digital beam tagging

Equation (3-40A) is used to determine expressions for the peak-to-peak amplitude and phase variations. The digital amplitude tagging produces a phase modulated error signal given by

\[
\psi = \tan^{-1}\left( \frac{\sin \gamma_1}{R' + \cos \gamma_1} \right) - \tan^{-1}\left( \frac{(1-m) \sin \gamma_1}{R' + (1-m) \cos \gamma_1} \right).
\]

The peak-to-peak amplitude variation, \( \Delta R \), which may be used as a reference signal is given by,

\[
\Delta R = \sqrt{\frac{\sin \gamma_1}{R' + \cos \gamma_1}^2 + \sin \gamma_1^2} - \sqrt{\frac{(1-m) \sin \gamma_1}{R' + (1-m) \cos \gamma_1}^2 + (1-m) \sin \gamma_1^2}.
\]

b. PM/AM digital beam tagging

Equation (3-40B) is used to obtain expressions for the peak-to-peak amplitude and phase variations produced by digital phase tagging. The AM detected error signal, \( \Delta R \), is given by

\[
\Delta R = \sqrt{\frac{\cos(\gamma_1 + \zeta_0)}{R' + \cos(\gamma_1 + \zeta_0)}^2 + \sin(\gamma_1 + \zeta_0)^2} - \sqrt{\frac{\cos(\gamma_1 - \zeta_0)}{R' + \cos(\gamma_1 - \zeta_0)}^2 + \sin(\gamma_1 - \zeta_0)^2}.
\]

The reference signal may be obtained from the peak-to-peak phase variation, \( \psi \), which is given by
4. **Error curves**

In the simultaneous sinusoidal beam tagging analysis of Section B, many graphs were required to show the error curves for various numbers of elements. With sequential beam tagging the number of curves required is reduced because only one element is tagged at a given instant which allows the remaining \((N-1)\) phasors to be combined into a single resultant phasor, \(R'\). The phase angles of the individual untagged elements are unimportant because the phase error is measured between the tagged signal and the resultant phasor \(R'\). For example, in a four-element array, the resultant phasor \(R'\), may vary between 3.0 when the three un-tagged signals are in perfect alignment to zero when three signals exactly cancel.

In this section, error and reference signal curves are shown vs the phase error, \(\gamma\). The curves are based on the peak-to-peak amplitude and phase variations produced by the square-wave tagging modulation.

The S/N ratio of the error signal is detected by using a narrow-band filter which passes only the fundamental component of the square wave. Since the peak amplitude of the fundamental component may
be determined by multiplying the peak-to-peak magnitude by \( \frac{2}{\pi} \), \((0.637)\). If the curves in Figs. 3-41 through 3-48 are multiplied by 0.637, they may be directly compared with those given for simultaneous, sinusoidal AM/PM tagging.

a. Digital AM/PM tagging

Figures 3-41 through 3-44 show error curves for \((0, -m)\) digital amplitude modulation with \(R'\) equal to 1.0, 1.5, 2.0 and 3.0 respectively with \(m\) equal to 0.5 and 1.0. At the top of each graph, a phasor diagram shows how the digital \((0, -m)\) modulation produces amplitude variation, \(\Delta R\), and phase variation, \(\psi\). The error signal curves produced by the phase variation, \(\psi\), are shown as solid lines, whereas the reference signal curves produced by the amplitude variation, \(\Delta R\), are shown as dashed lines.

For a given modulation index, \(m\), Figs. 3-42 through 3-44 show that as \(R'\) increases, the magnitude of the error signal decreases, and the maximum error signal occurs at reduced values of phase error. The phase angle, \(\gamma_M\), which produces the maximum error signal is given by

\[
\gamma_M = 57.3 \left\{ \pi - \cos^{-1} \left[ \frac{(2-m)R'}{(R')^2 + 1-m} \right] \right\} \text{ deg.}
\]
Fig. 3-41--Phase error curves for digital amplitude tagging with $R' = 1.0$

For the values of $R'$ and $m$ used in these graphs, $\gamma_M$ varies between $109^\circ$ and $180^\circ$. The value of $\gamma_M$ establishes the maximum control range because of values of $\gamma > \gamma_M$, a closed-loop servo system attempts to lock to a false null.
Figure 3-41 shows that with $R_1$ equal to one, the error signal increases monotonically with increasing $\gamma$, reaching its peak value when $\gamma$ equals $180^\circ$. In a practical system, it becomes exceedingly difficult to measure the phase error as $\gamma$ approaches $180^\circ$ because the phasors tend to cancel in one tagging position.

The reference signal curves shown in Figs. 3-41 through 3-44 have similar shapes. As the magnitude of $\gamma$ increases, the
curves cross zero at the angle, $\gamma_Z$, given by

$$\gamma_Z = 57.3 \left[ \pi - \cos^{-1} \left( \frac{2-m}{2R'} \right) \right] \text{ deg.}$$

Fig. 3-43—Phase error curves for digital amplitude tagging with $R' = 2.0$
If the sense of the error signal is obtained from detection of the alternate modulation, the maximum control range is determined by $\gamma_Z$ rather than $\gamma_M$ because $\gamma_Z$ is less than $\gamma_M$. This occurs because the polarity of the reference signal changes before the phase error signal reaches its maximum value.

![Diagram showing phase error curves](image)

**Fig. 3-44**--Phase error curves for digital amplitude tagging with $R' = 3.0$
b. Digital PM/AM tagging

One element is tagged by digitally phase shifting its carrier signal by $\pm \xi_0$ while the remaining (N-1) untagged elements form a resultant phasor, $R'$. The modulation index, $\xi_0$, and the resultant phasor, $R'$, are parameters in the error curves. Since the tagging signal is produced by phase modulation, the error signal is AM detected and the reference signal is PM detected.

Figures 3-45 through 3-48 show error curves for $R'$ equal to 1.0, 1.5, 2.0 and 3.0 respectively with $\xi_0$ equal to 30° and 90°.

The amplitudes of the phase error signals increase almost linearly up to $\gamma$ equals 90° and reach maximum values for $\gamma$ between 90° and 120°. For further increases in $\gamma$, the error signal decreases until it equals zero for $\gamma$ equal to 180°.

The phase variation $\psi$, may be used as a reference signal. As the phase error, $\gamma$, increases, the reference signal decreases until it equals zero at $\gamma_Z$, given by

$$\gamma_Z = 57.3 \left\{ \pi - \cos^{-1} \left( \frac{\cos \xi_0}{R'} \right) \right\} \text{ deg.}$$

(3-47)

For values of $\gamma$ greater than $\gamma_Z$, the polarity of the reference signal is incorrect.
Fig. 3-45--Phase error curves for digital phase tagging with $R' = 1.0$
Fig. 3-46--Phase error curves for digital phase tagging with $R' = 1.5$
Fig. 3-47--Phase error curves for digital phase tagging with $R' = 2.0$
Fig. 3-48--Phase error curves for digital phase tagging with $R' = 3.0$
An alternative method of tagging is digital phase modulation by 0° and 180° which produces an error signal proportional to the phase variation, \( \psi \), and a reference signal proportional to the amplitude variation, \( \Delta R \). The curves produced by this modulating method are the same as those in Figs. 3-46 through 3-48 for \( \xi_0 = 90° \) if the following changes are made:

1) interchange the words "error" and "reference"
2) shift the abscissa by +90°.

The advantage of this tagging method is that it forms a PM/PM system since both the tagging modulation and the error signal depend on phase variations.

D. Summary

The most important factors in beam tagging systems are briefly reviewed.

1. **Applied tagging modulation**

   Sinusoidal or digital phase modulation may be applied to the X-band klystron amplifiers used in the O. S. U. transmitting elements. Sinusoidal or digital phase modulation is easily produced by injecting a modulating signal into the phase locked loop (Dymec stabilizer) of the klystron. Sinusoidal or digital amplitude modulation is produced by varying the exciter drive power. This method has two limitations,
the power dissipated in the klystron increases and incidental phase modulation may be produced by the nonlinearity of the input-output power curve.

The tagging frequency range used is established by several factors. Lower bounds are established by the fact that the tagging frequencies must be higher than:

1) the scintillation rates produced by a passive satellite (approximately 15 Hz) and

2) the cutoff frequency of the doppler frequency tracking loop formed by the discriminator, VCO and mixer combination. The upper bound on the tagging frequencies is determined by the timing accuracy required when the reference signal is obtained by delaying the applied tagging signal. To satisfy these conditions, tagging frequencies in the range of 2 to 50 kHz are required.

2. Modulation index

As the tagging modulation index is increased, the amount of power devoted to the tagging spectrum increases. This improves the sensitivity (or gain) of the phase error curves which normally improves the degree of coherence. Unfortunately, the improvement in cohered power is achieved at the expense of having less transmitting power available for useful communications.
3. **Error signal detection**

With a passive satellite or a linear active satellite, the error signals may be detected by amplitude or phase techniques depending on the applied tagging modulation. Since most present-day active satellites contain hard limiters which remove any amplitude variations, the error signals must be phase detected in this case.

4. **Simultaneous or sequential correction**

In a simultaneously tagged N element array, (N-1) distinct tagging frequencies are used to produce (N-1) error signals. This creates a multiple input servo system in which each loop has an inherent time delay. However, if the array is sequentially tagged with only a single frequency, the system is reduced to a single loop servo system with an inherent propagation time delay. In this case, the phase corrections are applied at periodic intervals rather than on a continuous basis so that a sampling system is produced.

A limitation of the sequential phasing method is that the phase errors must be relatively constant over intervals several times the propagation time delay. As the number of elements increases, this requirement becomes more serious because the time between phasing for element increases.
5. **Paired and master-slave tagging**

In paired tagging, an N element array is grouped into (N-1) overlapping pairs. These (N-1) pairs are each tagged with a separate frequency which requires (N-1) tagging frequencies. The phase error signal at each tagging frequency represents the phase difference between the two carrier signals tagged with this frequency.

In a master-slave tagging, an N element array is grouped into one reference element and (N-1) slave elements. Each slave element is tagged with a separate frequency and the phase error signal produced at each tagging frequency represents the phase difference between this carrier signal and the resultant formed by the remaining (N-1) elements.

Since for equal modulation levels, paired tagging puts twice as much power in the tagging spectrum, the phase error signals produced are approximately twice as large as for master-slave tagging.

6. **Reference signal**

The error signal requires a reference to establish the direction of the phase error. This signal may be obtained by measuring the propagation time delay and delaying the applied tagging signal by
this amount or by detecting the same type of modulation as is applied at the transmitting elements.

7. **Error curves**

The error curves may be classified with respect to two quantities, range of control and sensitivity.

The range of control is defined as the region in which increasing phase error, $\gamma$, produces an increasing error signal. Figure 3-49 shows a table of the minimum control range for the various tagging methods vs the number of elements. Sometimes the range of control varies with the modulation index. In these cases, the value given in Fig. 3-49 corresponds to the modulation index that produces the minimum control range. In general as the number of elements increases, the range of control decreases.

<table>
<thead>
<tr>
<th>No. of Elements</th>
<th>Master-Slave Sensitivity</th>
<th>Paired Sensitivity</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>PM/AM</td>
<td>AM/PM</td>
</tr>
<tr>
<td>2</td>
<td>2.00</td>
<td>2.00</td>
</tr>
<tr>
<td>3</td>
<td>1.50</td>
<td>1.40</td>
</tr>
<tr>
<td>4</td>
<td>1.15</td>
<td>1.25</td>
</tr>
</tbody>
</table>

Fig. 3-49--Table of minimum control range for various tagging configurations vs the number of elements
The sensitivity of the error signal corresponds to the magnitude of the error signal produced by a given phase error. Since some of the error curves are non-linear, the slope at the null point is used to define the sensitivity. The table given in Fig. 3-50 shows the ratio of paired to master-slave sensitivities for various values of N.

<table>
<thead>
<tr>
<th>No. of Elements</th>
<th>Sinusoidal Tagging-Simultaneous Correction</th>
<th>Digital Tagging Sequential Correction</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>PM/AM</td>
<td>AM/PM</td>
</tr>
<tr>
<td>Paired</td>
<td>Master-Slave</td>
<td>Paired</td>
</tr>
<tr>
<td>2</td>
<td>±90°</td>
<td>±90°</td>
</tr>
<tr>
<td>3</td>
<td>±75°</td>
<td>±90°</td>
</tr>
<tr>
<td>4</td>
<td>±60°</td>
<td>±90°</td>
</tr>
</tbody>
</table>

Fig. 3-50--Ratio of paired to master-slave sensitivities for PM/AM and AM/PM vs number of elements

E. Conclusions

Digital (0°, 180°) PM/PM sequential beam tagging is best suited for the O.S.U. coherent transmitting array for several reasons. Digital (0°, 180°) tagging modulation is easily applied to the transmitting elements and PM is detection of the error signal is well suited to the existing phase locked demodulator system.
Sequential tagging simplifies the tagging receiver requirements.

By using PM detection, the system works equally well with both active and passive satellites.
CHAPTER IV
CONJUGATE PHASING AND BEAM TAGGING (S/N) RATIO ANALYSIS

The performance of the conjugate phasing and beam tagging systems vs (S/N) ratio is determined in this chapter.

A. Effects of atmospheric propagation

Before the (S/N) ratio can be calculated, the effects of the atmospheric (troposphere and ionosphere) must be considered.

When signals from orbiting satellites or deep space probes propagate through the troposphere, absorption by water vapor and oxygen molecules cause additional attenuation above the free-space loss. Figure 4-1 shows this additional attenuation for several different elevation angles as a function of frequency[34]. Figure 4-1 was used to derive the curve of attenuation at 8 GHz vs elevation angle shown in Fig. 4-2.

Precipitation produces an additional attenuation. Various sources have published curves of attenuation (dB/km) vs precipitation rate[35,36] but these results are difficult to apply to the space communication situation for two reasons, the average height
of rain clouds varies considerably and at low elevation angles, the horizontal extent of the rain effects the attenuation. The magnitude of the attenuation may be estimated by considering a typical rain storm with a height of 4 km and a precipitation rate of 10 mm/hr (moderate to heavy rain). At 15° elevation, this typical storm produces a two-way attenuation of approximately 3 dB. Based on
Fig. 4-2--Signal attenuation at 8 GHz for one-way propagation through the atmosphere vs elevation angle

this calculation, it appears that the "worst case" rain storm will not produce more than 10 dB attenuation at low elevation angles.

The system performance is also degraded by the external noise received along with the desired signal. This noise, called sky noise, comes from both the atmosphere and sources in outer space. The sun is the most prominent noise source from outer space but since in most cases it is not in the beamwidth of the antennas, its effect is neglected. Atmospheric noise is also
caused by radiation from the oxygen and water vapor molecules which have absorbed the microwave energy. Figure 4-3 shows the variation of the sky noise temperature vs elevation angle at approximately 6 Hertz[37]. In general, the noise level increases with increasing precipitation rates, e.g. heavy rains can increase the sky noise by an order of magnitude over that for a clear day.

![Sky noise temperature vs elevation angle](image)

Fig. 4-3--Sky noise temperature vs elevation angle

B. Carrier-to-noise ratio

The carrier-to-noise (C/N) ratio analysis uses the Echo II passive satellite as a target for two reasons, the previous array design calculations used Echo II and the (C/N) ratio it produces
is comparable to some present day active satellites. For example, the IDCSP series of satellites, scheduled for launching into a semi-synchronous orbit (18,000 mi) in the second quarter of 1966, have 2.5 W X-band transmitters and 6 dB gain antennas. These satellites will produce approximately a 10 dB poorer (C/N) than the Echo II passive satellite.

In addition to the received signal strength, the (C/N) ratio is also affected by the sky and receiver thermal noises. Figure 4-3 shows that the sky noise on a clear dry day is approximately constant at approximately 10°K for elevation angles between 15° and 90° (zenith). At elevation angles below 15°, the sky noise increases drastically but this does not affect The Ohio State University transmitting array because the elements must be turned off at elevation angles less than 15° for safety reasons.

Receiver thermal noise arises from several sources. Losses before the first amplifier can be expressed as an equivalent noise temperature, $T_L$, given by

\[(4-1) \quad T_L = T_A (L-1)\]

where $T_A$ - ambient temperature $\sim 290°K$

$L$ - loss, dimensionless.

The feed system and diplexer combination produce approximately 0.5 dB loss which corresponds to $T_L$ equal to 35°K. The signal
is then amplified with an X-band tunnel diode amplifier having a 5 dB noise figure (F) and a nominal gain of 15 dB. Using these values in the expression

\[(4-2) \quad T_R = T_A(F-1)\]

gives a receiver noise temperature, $T_R$, of 627°K. The tunnel diode amplifier feeds a crystal mixer with a 8.5 dB noise figure, which is equivalent to a noise temperature of 1770°K.

The noise temperature of the cascaded, diplexer, amplifier mixer network is given by

\[(4-3) \quad T = T_L + T_R/G_L + T_M/G_L G_R\]

Using the values of noise temperature and gain given above, the receiving system noise temperature is equal to 801°K. Adding the average sky noise to this value, gives an effective system noise temperature, $T_e$, of approximately 811°K.

Solving the radar range equation for the produce of the bandwidth times the output (C/N) ratio, $B(C/N)_o$, gives

\[(4-4) \quad B(C/N)_o = \left( \frac{P_T G_T G_R \lambda^2 \sigma}{(4\pi)^3 R^4(t) A L T L_T K T_e} \right)\]
\[ \text{where} \quad P_T - \text{power transmitted}, \ 10 \text{ KW} = 40 \text{ dBw} \]

\[ G_R = G_T - \text{antenna gain of a 30'} \text{parabola}, \ 52.6 \text{ dB} \]

\[ \lambda - \text{free space wavelength}, \ 3.60 \text{ cm at 8.33 GHz} \]

\[ \sigma - \text{radar cross section, Echo II - 135'} \text{sphere,} \]

\[ 31.2 \text{ dB above } 1 \text{ m}^2 \]

\[ L_T - \text{transmit feed line loss, } 0.5 \text{ dB} \]

\[ L_R - \text{receive feed line loss, } 0.5 \text{ dB} \]

\[ R - \text{array to satellite range - m} \]

\[ A - \text{attenuation above free space loss - } 0.5 \text{ dB} \]

\[ T_e - \text{effective noise temperature - } 811^\circ \text{K} \]

\[ K - \text{Boltzmann constant - } 1.38 \times 10^{-23} \]

Since all the terms in Eq. (4-4) are approximately constant except for \( R \), the equation can be rewritten as

\[ (4-5) \quad [B(C/N)_O]_{dB} = 314.3 - 40 \log_{10} [R(\text{meters})] \]

As the range, \( R \), of Echo II varies from a minimum of 600 mi

(965 km) at the zenith to a maximum of 1700 mi (2730 km) at 15°
elevation, the product, \([B(C/N)_O]\), varies between approximately
75 and 57 dB respectively.

The antenna gain \((G_T = G_R = 52.6 \text{ dB})\) at X-band may be
slightly optimistic because it was obtained by frequency scaling the
S-band gain, 43.0 ± 0.5 dB, [38] of The Ohio State University
parabolic antennas. Surface irregularities, can reduce the X-band
gain from that obtained by frequency scaling as shown by the
expression[39].

\[ (4-6) \quad \left( \frac{G}{G_0} \right) = e^{-[16\pi^2(D^2/\lambda^2)]} \]
where \( G_0 \) - theoretical ideal gain

\( G \) - actual gain

\( \lambda \) - operating wavelength - cm

\( \overline{D^2} \) - RMS surface deviation from an ideal parabola - cm.

Equation (4-6) shows that with a constant RMS surface deviation, the gain is reduced as the frequency increases (\( \lambda \) becomes smaller).

Conflicting values of \( \overline{D^2} \) for The Ohio State University antennas are obtained from two sources. The manufacturer (Andrew Corporation) of the antennas specifies \( \overline{D^2} \) equal to less than 0.102 cm compared to values of \( \overline{D^2} \) equal to 0.8, 1.5 and 1.9 cm at elevation angles of 90°, 45°, and 0° respectively which were obtained by an optical survey method[40]. Preliminary X-band antenna gain measurements indicate that the surface accuracy is closer to that specified by the manufacturer than it is to the values calculated by the optical survey method. Therefore, the (C/N) variation specified by Eq. (4-5) is probably only slightly optimistic.

C. Conjugate phasing (S/N) ratio

In conjugate phasing, the signal received at the \( i^{th} \) slave element is compared with the signal received at the reference element. The phase difference between these signals is conjugated and applied to the carrier signal radiated from the \( i^{th} \) slave element.
In The Ohio State University array, signals received at X-band are down converted to 30 MHz where they are amplified with a preamplifier having an 8 MHz bandwidth. The 30 MHz IF signal feeds a phase-locked loop which further down converts the received signal to 5 MHz by using a crystal voltage controlled oscillator (VCO) at 25 MHz. Conjugate phasing information may be obtained from the carrier frequency component of either the 30 MHz IF signal or the VCO output.

1. If the conjugate phasing information is obtained from the 30 MHz IF signal, the bandwidth must be reduced considerably, since with the 8 MHz bandwidth the (S/N) ratio varies between +6 dB and -12 dB when Echo II is used. The minimum possible bandwidth is determined by the ability of the down-converting frequency loop, (STALO) to maintain the IF frequency at nearly 30 MHz. Design calculations indicate that the STALO can maintain the 30 MHz signal to within less than a 1 kHz bandwidth.

If the IF signal is amplified, hard-limited, and narrowed to a 1 kHz bandwidth, the minimum (S/N) ratio is increased to approximately 30 dB. This value includes the 3 dB improvement provided by the limiter under high (S/N) ratio. Figure II-2 shows that a 30 dB (S/N) ratio corresponds to a 1.4° RMS phase jitter on the IF signals which feed the phase detector.
Various factors influence the time constant of the phase detector. Its minimum value is established by the reciprocal of the bandwidth of the signals feeding the phase detector which is approximately 1 ms. The maximum time constant is effected by the mode of operation. In pulsed operation, the time constant must allow the phase detector to nearly reach its steady state before the end of the shortest pulse, which is approximately 5 ms. When operating CW with the digital phase conjugation method, the phase detector must nearly reach its steady state value in the time interval required for the path phase delay to change by \((360/M_R)\) degrees. For example, with \(M_R\) equal to 32 and the differential doppler frequency at its maximum value of 3.8 Hz, the minimum time interval is approximately 8 ms. Since both CW and pulsed operation require the phase detector to respond in times in the order of 5-8 ms, the RC time constant should be several times smaller than this value, or on the order of one to two ms.

When this value is used, the bandwidth of the two signals feeding the phase detector are approximately the same as the phase detector time constant. Therefore, the phase detector does not improve the detected \((S/N)\) ratio. Since the phase variations produced by noise in the slave and reference channels are uncorrelated, the random phase variation in the output of
the phase detector is equal to the square root of the sum of the squares, which corresponds to 2.8° RMS phase jitter.

2. The phase detector input signals may be obtained from the 25 MHz crystal VCO in the phased locked loop (PLL). Since the loop bandwidth is small, (variable between 20 and 100 Hz) the PLL tracks only the carrier frequency signal and rejects the information modulation spectrum. Noise on the received signal generally produces phase variations which are outside the bandwidth of the PLL. This results in a phase error between the incoming signal and the VCO, the RMS value of which is given by [41]

\[
\sigma = \left( \frac{1}{2 \left( \frac{S}{N} \right)} \right) \left( \frac{B_n}{B_{IF}} \right)^{1/2} \text{ rad}
\]

(4-7)

where \(B_n\) and \(B_{IF}\) are the loop noise bandwidth and IF bandwidth respectively. Particular values of \(\sigma\) are not calculated because \(\sigma\) was introduced only to show that the PLL tracks the slow or long term phase variations rather than the rapid phase variations. Therefore, the PLL is ideal for tracking the differential doppler frequency which has a maximum value of approximately 4 Hz.

One limitation in using the VCO output is that a small phase error can be produced when the PLL tracks the frequency wander (less than 1 kHz) in the 30 MHz IF signal. If the phase
errors produced at all the slave PLL's are not identical, their difference produces a phasing error. In general it is expected that this tracking phase error will be less than a few degrees so that it only slightly degrades the phasing accuracy. The phase detector requirements are similar to those when the 30 MHz IF signals are used.

Using the VCO signals to make the phase measurements has two advantages. First, since the output amplitude of the VCO is large, the requirement for additional amplifiers and limiters is eliminated. Second, during signal fades, the phase of the VCO changes slowly because the PLL integrates the bias level produced by the noise. However, when the 30 MHz IF signal approaches zero, the phase probability distribution of a limiter rapidly becomes uniform. The only advantage of using the 30 MHz signal is that it eliminates the effect of the small phase jitter between the received signal and the VCO. Therefore, in future discussions, it is assumed that the phase conjugation signals are obtained from the VCO's.

D. Sequential beam tagging

(S/N) ratio

The (S/N) ratio of the beam tagging system is analyzed only in terms of sequential digital tagging because this method is preferred for the Ohio State University coherent transmitting
array. Digital (0°, 180°) phase modulation is used for several reasons; it is easily applied to the transmitting elements, it can be used with active satellites, it produces a large error signal, and the error signal peaks at ± 90° in all cases.

Digital (0°, 180°) phase tagging produces the incidental AM modulation, ΔR, shown in Fig. 4-4. As the phase error, γ,

Fig. 4-4--Amplitude error curves for PM/PM 0°, 180° digital beam tagging with R' equal to 1.5 and 3.0
increases, \( \Delta R \) decreases until it reaches zero at \( \pm 90^\circ \) and for 
\( |\gamma| \) greater than 90°, the amplitude variations increase, but with 
the opposite sign or polarity. Since the error curve is symmetrical about \( \gamma \) equal to zero, the AM variations may be used as a 
reference signal if they are not removed by limiters in an active 
satellite. It is interesting to note that the variation, \( \Delta R \), is 
almost completely independent of the magnitude of the subresultant 
phasor, \( R' \), formed by the untagged signals.

Figure 4-5 shows the variation in the phase detected error 
signal, \( \psi \), vs the phase error, \( \gamma \), for digital (0°, 180°) tagging 
with the subresultant phasor, \( R' \), equal to 1.5, 2.0 and 3.0. 
These curves show that the detected error signal is a function of 
both the phase error, \( \gamma \), and the magnitude of \( R' \). Since the 
amplitude of the phase error signal varies according to the magni-
tude of the subresultant, \( R' \), the phase correction signal must be 
based on the error curve which produces the smallest phase cor-
rection, \( R' = 1.5 \). For example, when \( \gamma \) equals +20°, the phase 
error, \( \psi \), is between -10° and -29°. Therefore, a phase cor-
rection of +10° would be applied to the phase of the tagged slave 
element. By always making the smaller phase correction, more 
tagging intervals are required to null the phase error. This 
problem is not as serious as it might seem at first glance because 
the phase correction is slowest when the three untagged elements
Fig. 4-5--Phase error curves for PM/PM
0°, 180° digital beam tagging with
$R' = 1.5, 2.0$ and $3.0$

have their maximum amplitude. In this case, a small phase error in the fourth element is not as important as when the other three phasors are badly misaligned.

Now, consider the phasing accuracy of the digital beam tagging system. Assume that the tagging system is required to measure the phase coherence of the tagged element to within ±10°.

When $R'$ is equal to its maximum value of 3.0, a phase error, $\gamma$,
equal to 10° produces a detected phase error, \( \psi \), equal to 7.4° peak-to-peak (approximately 2.6° RMS). In order to accurately measure the phase variation, \( \psi \), the RMS phase variation produced by the noise must be considerably less than 2.6° RMS.

The \((C/N)\) ratio of the sum signal varies depending on the coherence of the four signals radiated from the array. For the case under consideration, with \( R' = 3.0 \) and \( \gamma = 10° \), the four phasors are nearly perfectly aligned which improves the \((C/N)\) ratio by approximately 9.5 dB over the \((C/N)\) ratio for a single element. The 9.5 dB improvement is based on three perfectly coherent phasors, since the tagged signal produces no energy at the carrier frequency. Assume that in order to reliably measure the phase variations produced by the tagging, the RMS phase jitter produced by noise must be at least 10 dB less than the RMS signal level. This limits the maximum phase noise jitter to 0.85° which corresponds to a \((C/N)\) ratio of 35 dB in Fig. II-2. After adding +9.5 dB to the numbers in Eq. (4-6), it is seen that a 35 dB \((C/N)\) ratio requires approximately a 350 Hz bandwidth. Since the "Electrak" phase locked demodulators have a minimum bandwidth of 250 Hz, the \((S/N)\) analysis shows that it should be possible to maintain the cohering phase error to less than ±10°.
The above discussion considered only phasing errors produced by the inability to accurately measure the tagged error signals, but phase errors can also occur because the closed-loop system may have transient and steady state errors and the propagating media may produce phase fluctuations which are outside the bandwidth of the closed-loop system.

If we neglect, the sampling portion of the operation, the tagging system forms a phase-locked loop with an inherent time delay equal to at least the two-way propagation delay. This problem has been analyzed[42] assuming the servo system transfer function uses a simple variable gain constant. The results showed that if the cohering function were performed using beam tagging alone, when the maximum differential doppler frequency occurs, a phase error of at least 17° would occur. However, when the digital phase conjugation system is added to the cohering system, the maximum effective differential doppler frequency is reduced to less than 0.068 cps which reduces the closed-loop servo system phase error to well under 1°.

The propagation media can produce jitter[43]. Frequency components below the cutoff frequency of the closed loop system will be corrected while those above it will remain uncorrected
and therefore produce phasing errors. For example, if the system bandwidth is 1.0 cps, the uncorrected phase jitter is only $1^\circ[44]$.

All these factors indicate that it should be possible to maintain a cohering error of less than $\pm 10^\circ$. 
CHAPTER V
CONCLUSIONS

When communicating from an array of transmitting elements to a distant moving target, e.g. an orbiting satellite, the propagating paths are continually changing and consequently, so are the phases of the carrier frequency signals. At the transmitting elements, appropriate phase corrections may be applied to compensate for the different path phase delays, so that the signals are cohered at the satellite.

In this study, a cohering system for controlling the phases of the transmitting elements is analyzed in terms of The Ohio State University four-element array (30 ft parabolic antennas, 10 KW transmitters, 25.9m maximum distance between elements, transmitting frequency ≈ 8.33 GHz and receiving frequency ≈ 7.73 GHz) operating with the Echo II passive satellite (nominally 600 mi high and 135 ft diameter).

In order to maintain phase coherence as the target moves from the horizon to the zenith, the transmitting elements must be capable of producing a phase variation equivalent to $2\pi$ times the maximum number of wavelengths between the phase centers of
any two elements in the array (Ohio State University array - 1440 rad.).

The rate of change of the phase delay (or advance) as the satellite moves, called the differential doppler frequency, varies linearly with the distance between two elements and increases as the orbital height of the satellite decreases. This is an important design parameter because its peak value determines the maximum phase correction rate required at the transmitting elements (Ohio State University array, Echo II satellite—approximately 4 Hz).

Another design parameter, the propagation time delay between the array and the satellite, is important for two reasons. First, if the phase cohering system operates open-loop, it can not compensate for the effect of the propagation time delay, which creates a phasing error approximately equal to the product of the time delay and the differential doppler frequency (maximum approximately 20°).

Second, if a closed-loop phasing scheme is used, the propagation time delay limits the maximum possible transfer function gain for a stable system, which in turn establishes a minimum steady state error for a given input (minimum) error, 17° at the maximum differential doppler frequency).

Since either the open-loop or the closed-loop phasing systems operating alone produce unacceptably large phasing errors, we are led to a combined system. This study analyzes the performance of
a combined system in which the open-loop (conjugate phasing) system provides the large or coarse phase corrections and the closed-loop (beam tagging) provides the fine phase adjustments.

In the open-loop phase conjugation system, phasing information is obtained from a signal radiated (or scattered) from the satellite by measuring the phase differences between the reference element and each of the \( (N-1) \) slave elements. The accuracy of the phase measurements depends only on the received carrier-to-noise \((C/N)\) ratio and is independent of the number of elements in the array (minimum calculated \((C/N)\) ratio in a 350 Hz bandwidth should permit a measurement accuracy of better than 3°). When the transmitting and receiving frequencies are unequal, the phase measurements made at the receiving frequency must be compensated for this frequency difference. While these corrections can be performed in either a digital or analog manner, a detailed analysis (see Chapter II) showed that the digital method is preferred because it requires less equipment, it is less susceptible to phase drifts and it is more compatible with the preferred digital beam tagging system. One difficulty in the digital phase conjugation scheme is that the frequency multiplication factor, \((M_R/M_T)\), usually only approximates the ratio \((f_T/f_R)\). For example, with \(M_T\) and \(M_R\) less than, say 33, and \((f_T/f_R)\) between 1.02 and 1.20, the maximum frequency multiplication error,
(M_R/M_T - f_T/f_R), is 0.011. With this multiplication error, the
digital phase conjugation system can only reduce the maximum phase
correction rate from 3.8 Hz to approximately 0.04 Hz. Since the
system operates open-loop, it can not compensate for this residual
phase rate or other phase errors produced by propagation time
delays, doppler frequency shifts, equipment phase instabilities,
initial error in the measurement of N_i. Generally, these phase
errors vary slowly with respect to the propagation time delay, so
that it is practical to remove them by using a closed-loop phasing
scheme.

Beam tagging is a type of closed-loop phasing system, in
which the transmitted signal (or signals) are modulated such that
that when they combine at the satellite position, it is possible to
detect a signal proportional to the phasing error. This type of
system has two basic limitations; first, as the number of elements
in the array increases, the error signals produced by tagging
decrease because the resultant formed by the remaining (N-1)
elements increases; and second, when an active hard-limiting type
satellite is used, only PM information is available because the
limiter removes the AM variations.
Various types of beam tagging schemes are discussed in terms of their phase error curves (see Chapter III). The general types considered are; paired, in which the error signal produced is proportional to the phase difference between two phasors tagged with the same frequency; and master-slave, in which the phase error signal is proportional to the phase difference between the tagged phasor and the resultant phasor formed by the remaining (N-1) phasors. In either type, if AM modulation is used, the error signal PM detected whereas if PM modulation is applied, the error signal is AM detected except for the special case of digital (0°, 180°) phase modulation which allows PM detection. The tagging signals may be applied simultaneously, in which case an N element array requires (N-1) tagging frequencies or sequentially in which only one element is tagged at a time using a single tagging frequency.

The phase error curves produced by the various combinations are similar in many respects. As the number of elements increases, the amplitude of the phase error decreases and the range of stable control (increasing phase error produces increasing error signal) decreases. By increasing the modulation index, more power is devoted to the tagging spectrum which increases the error signal. Similarly, paired tagging produces approximately twice as large
an error signal as master-slave tagging because two transmitting elements are modulated rather than one.

From the multitude of tagging configurations analyzed, sequential digital (0°, 180°) phase tagging was selected for use in the coherent transmitting array for several reasons. It is easily applied to the X-band klystron amplifiers; it allows PM detection of the error signal; it develops the largest possible phase error signal because all the carrier power is transferred to the tagging spectrum; less bandwidth is occupied by the tagging frequency spectrum; and more transmitter power is available for useful communications (with a 40% duty cycle it requires only 13% as much power as in simultaneous tagging). Sequential tagging is feasible only because the open-loop phase conjugation system reduces the phase correction rate sufficiently such that the phasor positions are relatively constant for periods greater than several times the propagation delay.

When a perfectly cohered four-element array is sequentially tagged (40% duty cycle) using (0°, 180°) phase modulation, the cohered power level is reduced by approximately 0.8 dB but it should be possible to reduce this value even further by lowering the duty cycle when the phase correction rate is small.
The theoretical maximum improvement obtained by cohering the signals from the Ohio State University four-element array (+6 dB) cannot be achieved in actual practice because the tagging modulation used for closed-loop operation wastes power and the finite signal-to-noise ratio produces a RMS phasing error (approximately 10°) which reduces the cohered power by approximately 0.1 dB. To maintain this small amount of phase jitter, the array must be phase stable to within a few degrees for periods of several seconds.

These results show that the Ohio State University four-element coherent transmitting array should cohere the signals to within 1 dB of the theoretical maximum.
APPENDIX I
GENERAL TRIGONOMETRIC EXPRESSIONS
FOR PM/AM TAGGING

The trigonometric formulas used in the mathematical analysis of sinusoidal PM/AM beam tagging are developed below. This appendix shows that the expression given by Eq. (3-11) may be separated into frequency groups involving common mixing orders.

The general term in Eq. (3-11) is given by

\[(I-1) \quad K_i K_j \sin \sigma_i \sin \sigma_j + \cos \sigma_i \cos \sigma_j\]

where

\[\sigma_i = \gamma_i + A_i + B_{i-1}\]

and

\[(I-2) \quad A_i = a_i \sin (\omega_i t + \alpha_i)\]
\[B_i = b_i \sin (\omega_i t + \beta_i)\]

The first term in Eq. (I-1) is equal to

\[(I-3) \quad K_i K_j \sin \sigma_i \sin \sigma_j\]

Substituting the relations given in Eq. (I-2) into Eq. (I-3), gives

\[(I-4) \quad K_i K_j \sin (\gamma_i + A_i + B_{i-1}) \sin (\gamma_j + A_j + B_{j-1})\]
Expanding Eq. (1-4) using the well-known trigonometric sum and difference formulas gives

\[
(1-5) \quad K_i K_j \left[ \begin{array}{c}
\sin \gamma_i [ \cos A_i \cos B_{i-1} - \sin A_i \sin B_{i-1} ] \\
+ \cos \gamma_i [ \sin A_i \cos B_{i-1} + \cos A_i \sin B_{i-1} ] \\
\sin \gamma_i [ \cos A_i \cos B_{i-1} - \sin A_i \sin B_{i-1} ] \\
+ \cos \gamma_i [ \sin A_i \cos B_{i-1} + \cos A_i \sin B_{i-1} ] \\
\end{array} \right].
\]

Since the terms, \( A_i \) and \( B_i \), represent phase modulating signals with peak amplitudes of less than 30°, the following approximations are valid.

\[
(1-6) \quad \sin A_i \approx A_i \quad \sin B_i \approx B_i \\
\cos A_i \approx \cos B_i \approx 1.0
\]

To simplify the appearance of the equations, the following shorthand notation is used

\[
(1-7) \quad \sin \gamma_i \rightarrow S_i \quad \text{and} \quad \cos \gamma_i \rightarrow C_i
\]

Substituting Eqs. (1-6 and 1-7) into Eq. (1-5) gives

\[
(1-8) \quad K_i K_j \left[ \begin{array}{c}
(S_i - S_i A_i B_{i-1} + C_i A_i + C_i B_{i-1}) \\
\times (S_j - S_j A_j B_{j-1} + C_j A_j + C_j B_{j-1}) \\
\end{array} \right].
\]
Repeating the same steps using

\[(I-9) \quad K_i K_j (\cos \sigma_i \cos \sigma_j)\]

gives

\[(I-10) \quad K_i K_j \left\{ (C_i - S_i A_i - S_i B_{i-1} - C_i A_{i-1}) \right\} \times \left( C_j - S_j A_j - S_j B_{j-1} - C_j A_{j-1} \right)\]

Multiplying out the terms in Eqs. (I-8 and I-10) and adding them together gives the original Eq. (I-1) separated into frequency groups as shown below,

**DC**

\[(I-11) \quad K_i K_j \cos (\gamma_i - \gamma_j)\]

**Fundamental**

\[(I-12) \quad K_i K_j \sin (\gamma_i - \gamma_j) \left[ A_j + B_{j-1} - A_i - B_{i-1} \right]\]

**2nd order mixing products**

\[(I-13) \quad K_i K_j \cos (\gamma_i - \gamma_j) \left[ A_i A_j + A_i B_{j-1} + A_j B_{i-1} + B_i B_{j-1} - A_j B_{j-1} - A_i B_{i-1} \right]\]

**3rd order mixing products**

\[(I-14) \quad K_i K_j \sin (\gamma_i - \gamma_j) \left[ A_j B_{j-1} (A_i + B_{i-1}) - A_i B_{i-1} (A_j + B_{j-1}) \right]\]

**4th order mixing products**

\[(I-15) \quad K_i K_j \cos (\gamma_i - \gamma_j) \left[ A_i A_j B_{i-1} B_{j-1} \right]\]
APPENDIX II
RMS PHASE VARIATION VS (S/N) RATIO

When a monochromatic signal, i.e., \( e(t) = \sin(\omega_0 t + \phi_0) \) is contaminated with narrow-band Gaussian noise (NBGN), the resulting signal is amplitude and phase modulated. The magnitude of the random phase fluctuations are extremely important in any phase detecting system because they effect the minimum detectable phase variations. Therefore, the RMS phase jitter vs (S/N) ratio is calculated in this appendix.

Normally the receiver bandwidth is much less than the carrier frequency. This allows the signal plus noise to be written as

\[
(\text{II-1}) \quad e(t) = V(t) \sin[\omega_0 t + \phi(t)]
\]

where \( V(t) \) and \( \phi(t) \) are slowly varying Gaussian random variables.

The probability density function of the phase variation, \( p(\phi) \), given by

\[
(\text{II-2}) \quad p(\phi) = \frac{e^{-(S/N)}}{2\pi} \left\{1 + \sqrt{4\pi(S/N)} \cos \phi \right. \\
\left. e^{(S/N) \cos^2 \phi} \phi \left(\frac{2S}{N \cos \phi}\right)\right\}
\]
may be obtained by either transforming the rectangular Gaussian probability density function, \( p(x, y) \) into \( p(V, \phi) \) and integrating \( V \) from zero to infinity or by consulting several textbooks\([45, 46]\).

Figure II-1 shows the probability density function, \( p(\phi) \), for several \((S/N)\) ratios.

The RMS value of the phase of variation is calculated using the expression for \( p(\phi) \) given in Eq. (II-2). Since the mean of \( p(\phi) \) equals zero, the variance and the second moment are identical. The mean square phase variation, \( \phi^2 \), is given by

\[
(\text{II-3}) \quad \phi^2 = (\sigma_\phi)^2 = \int_{-\pi}^{\pi} \phi^2 p(\phi) \, d\phi
\]

This integral can be evaluated by either a sophisticated mathematical technique which replaces the integral by an infinite series or by the more practical approach of numerical integration on a digital computer as was done in this analysis.

Figure II-2 shows the RMS phase variation, \( \sigma_\phi \), vs the \((S/N)\) ratio. A curve for the expectation of \( \cos \phi \), \( E(\cos \phi) \), is also shown on this graph because it is required in Eq. (II-4).

For a coherent transmitting array operating at a given \((S/N)\) ratio, the curve in Fig. II-2 gives the minimum possible RMS phase jitter. In a practical system, equipment and propagation phase variations cause the actual RMS phase error to be higher than the value calculated using Eq. (II-3).
Fig. II-1--Probability density, \( p(\phi) \), vs \( (S/N) \) ratio for a sine wave plus narrow-band Gaussian noise.
Fig. II-2--RMS phase jitter vs (S/N) ratio.
In an N element array, phase jitter reduces the cohered power. If the signals from all the elements are assumed to be statistically independent and have equal amplitudes, then the average power is proportional to the autocorrelation function, which is given by\[ 47\]

\begin{equation}
(\text{II}-4)
E(VV^*) = N + (N^2 - N) \left[ E(\cos \phi) \right]^2
\end{equation}

The normalized power loss, $P_L$, given by

\begin{equation}
(\text{II}-5)
P_L = 10 \log \left[ \frac{E(VV^*)}{N^2} \right] = 10 \log_{10} \left[ \left( \frac{1}{N} \right) + \left( 1 - \frac{1}{N} \right) E^2(\cos \phi) \right]
\end{equation}

is used to calculate the curves of normalized power loss vs (S/N) ratio shown in Fig. II-3 with N equal to 2, 4, 8 and infinity. These curves show that for a 10 dB (S/N) ratio, the maximum power loss is only 0.26 dB.
Fig. II-3--Normalized power loss (due to phase jitter) vs ($S/N$) ratio for $N$ equal to 2, 4, 16, and infinity.
APPENDIX III
SPEC TR A L ANALYSIS OF A DIGITAL
PHASE MODULATOR

The digital phase conjugation system uses a digital phase
modulator to alter the phase of the carrier signal in discrete
increments. This appendix presents a spectral analysis of the
harmonics produced by the abrupt phase transitions.

The analysis is simplified by making the following
assumptions.

1. The shifting rate is constant. In actual practice, the
shifting rate varies so slowly that it is essentially con-
stant during the time required to produce a 360° phase
shift.

2. The DPM contains an odd number of elements.
(When the DPM contains an even number of elements,
the analysis is more laborious because the time
intervals at the ends of the period are one-half as long
as the interior time intervals.)

3. The carrier signal goes through an even number
of complete cycles during the time between two phase
shifts.
4. The carrier signal has zero phase angle when the phase transitions occur.

The spectral analysis is performed by evaluating the Fourier coefficients. Using the above four assumptions, the output signal of the DPM, \( e_{DPM}(t) \), may be written as

\[
( \text{III-1} ) \quad e_{DPM}(t) = \cos \left[ 2\pi f_0 t + k(2\pi/M) \right]
\]

where \( (2k-1)T < t < (2k+1)T \)

\[ k = 0, \pm 1, \ldots, \pm \left( \frac{M-1}{2} \right) \]

\( M = \) number of elements in the DPM

\( 2T = \) time interval during which phase is constant, \( (1/f_{sw}) \)

\( f_0 = \) center frequency, \( (\omega_0/2\pi) \)

Figure III-1 shows the abrupt phase variations as a function of time for \( M \) equal to seven. Since \( e_{DPM}(t) \) is an even function, the odd terms in the Fourier series are equal to zero. The coefficient of the even terms, \( b_n \), is given by

\[
( \text{III-2} ) \quad b_n = \frac{1}{MT} \int_{-MT}^{MT} e_{DPM}(t) \cos \omega_n t \, dt
\]

where \( \omega_n = \left( \frac{n\pi}{MT} \right) \).
Substituting Eq. (III-1) into Eq. (III-2) gives

\[
\begin{align*}
(\text{III-3}) \quad b_n &= \frac{1}{MT} \sum_{k=-(M-1)/2}^{(M-1)/2} \int_{(2k-1)T}^{(2k+1)T} \{ \cos[2\pi f_0 t + k(2\pi/M)] \cos \omega_n t \} \, dt \\
&= \frac{1}{MT} \sum_{k=-(M-1)/2}^{(M-1)/2} \int_{(2k-1)T}^{(2k+1)T} \cos[2\pi f_0 t + k(2\pi/M)] \cos \omega_n t \, dt
\end{align*}
\]

Expanding \( \cos[2\pi f_0 t + k(2\pi/M)] \) and integrating the two terms that result gives
\( (\text{III-4}) \quad b_n = \begin{cases} \cos(2\pi k/M) \left[ \frac{\sin(x_n t)}{2x_n} + \frac{\sin(y_n t)}{2y_n} \right] & t_2 \\ -\sin(2\pi k/M) \left[ \frac{-\cos(x_n t)}{2x_n} - \frac{-\cos(y_n t)}{2y_n} \right] & t_1 \end{cases} \)

where \( t_1 = (2k-1)T \) and \( t_2 = (2k+1)T \),
\( x_n = (\omega_0 - \omega_n) \) and \( y_n = (\omega_0 + \omega_n) \).
\( k = \left( \frac{M-1}{2} \right) \)

In Eq. (III-4), the terms with \( y_n \) in the denominator are negligible in comparison to those with \( x_n \) in the denominator. By using trigonometric identities and substituting the limits, the expression for \( b_n \) becomes

\[ (\text{III-5}) \quad b_n = \frac{1}{2M(x_n T)} \sum_{k=-k}^{k} \left[ \frac{+\sin[x_n(2k+1)T + 2\pi k/M]}{-\sin[x_n(2k-1)T + 2\pi k/M]} \right] \]

where the terms containing \( y_n \) are neglected. Equation (III-5) is simplified further by using trigonometric identities to give the form
\[
(\text{III-6}) \quad b_n = \left( \frac{\sin x_n T}{x_n T} \right) \left\{ \frac{1}{M} \sum_{k=-k}^{k} \cos \left[ 2k(x_n T + \pi/M) \right] \right\}.
\]

Now let,

\[
(\text{III-7}) \quad A_n = \frac{1}{M} \sum_{k=-k}^{k} \cos \left[ 2k(x_n T + \pi/M) \right]
\]

and

\[
(\text{III-8}) \quad \omega_o = 2\pi q/T
\]

where \(2q\) is the integral number of complete cycles between phase transitions. Substituting the expressions in Eqs. (\text{III-2}) and (\text{III-8}) into the expression for \(x_n\) in Eq. (\text{III-4}) gives

\[
(\text{III-9}) \quad x_n T = (2\pi q - n\pi/M).
\]

Using Eq. (\text{III-9}), the expression for \(A_n\) becomes

\[
(\text{III-10}) \quad A_n = \left( \frac{1}{M} \right) \sum_{k=-k}^{k} \cos \left[ 2\pi(1-n) k/M \right]
\]

where the term \(2\pi q\) has been omitted. This summation for \(A_n\) is evaluated for three cases; \(n\) equal to \((pM+1)\), \(n\) unequal to \((pM+1)\) or \(Mq\), and \(n\) equal to \(Mq\).
CASE 1: \( n = (pM+1) \) where \( p \) is any arbitrary integer.

For these values of \( n \), Eq. (III-10) becomes

\[
(III-11) \quad A_n = \left( \frac{1}{M} \right) \sum_{k=-k}^{k} \cos(2\pi pk)
\]

Since the term, \( \cos(2\pi pk) \), equals one for all values of \( k \) and the summation from \( -k \) to \( +k \) contains \( M \) terms, \( A_n \) equals one.

CASE 2: \( n \neq (pM+1) \) or \( Mq \)

Let \( r = \left[ \frac{2\pi(1-n)}{M} \right] \). Then Eq. (III-10) can be written as

\[
(III-12) \quad A_n = \frac{1}{M} \sum_{k=-k}^{k} \cos(kr) = \frac{1}{M} \left\{ 1 + 2 \sum_{k=-k}^{k} \cos(kr) \right\}
\]

Let \( B_n \) equal

\[
(III-13) \quad B_n = \sum_{k=-k}^{k} \cos(kr)
\]

The summation for \( B_n \) may be replaced by the expression \[43\]

\[
(III-14) \quad B_n = \left\{ \frac{\cos[(\overline{k}+1)(r/2)] \sin[\overline{k}(r/2)]}{\sin(r/2)} \right\}
\]

Substituting values of \( k \) into Eq. (III-14) and simplifying gives

\[
(III-15) \quad B_n = \left\{ \frac{\sin(Mr/2) - \sin(r/2)}{2 \sin(r/2)} \right\}
\]
When the expression for $r$ is substituted into Eq. (III-15), $B_n$ equals one-half because the term, $\sin (Mr/2)$, equals zero for all values of $n$. In Eq. (III-12), $A_n$ equals zero when this value of $B_n$ is used.

**CASE 3: $n = Mq$**

The case where $n = Mq$ corresponds to the spectral component at the carrier frequency. For this condition, Eq. (III-3) becomes

\[
(III-16) \quad b_n = \left\{ \begin{array}{l}
\frac{1}{MT} \sum_{k=-k}^{k} \cos \left( \frac{2\pi k}{M} \right) \int_{(2k-1)T}^{(2k+1)T} \sin \left( \frac{2\pi k}{M} \right) \cos \left( 2\omega_0 t \right) dt \\
\cos \left( \frac{2\pi k}{M} \right) \int_{(2k-1)T}^{(2k+1)T} \left( \frac{1}{2} \left( \frac{1}{2} + \cos 2\omega_0 t \right) \right) dt
\end{array} \right.
\]

The terms with $2\omega_0$ integrate to zero, giving the result,

\[
(III-17) \quad b_n = \frac{1}{M} \sum_{k=-k}^{k} \cos \left( \frac{2\pi k}{M} \right)
\]

Using the same procedure as in Eqs. (III-12) thru (III-14), it can be shown that $b_n$ equals zero.
The harmonic component, $b_n$, is zero for all except the first case where $n$ equals $(M_p+1)$. Therefore, the spectral components of an $M$ (odd or even*) element digital phase modulator shifted at a constant rate are given by

$$b_n = \begin{cases} \sin \left( \frac{n\pi}{M} \right) & n = (pM+1) \\ 0 & n \neq (pM+1) \end{cases}$$

In Eq. (III-18), the entire frequency spectrum has been shifted so that it is centered about the DPM driving frequency, $f_0$. Thus, the harmonic component with $n$ equal to zero corresponds to $f_0$.

If the DPM is driven in the opposite direction, the sign of the phase transitions changes. This reverses the sign of $k$ in Eq. (III-1) which causes the value of $n$ in Eq. (III-18) to become $(pM-1)$.

Frequency spectrums of four, eight and sixteen element DPM's are shown in Figs. III-2 thru III-4 respectively. In all the graphs, the basic effect of the periodic phase adjustments is to change the driving frequency, $f_0$, by an amount $\pm \left( \frac{f_{sw}}{M} \right)$ where $f_{sw}$ is the rate at which DPM is commutated and the plus or minus sign is determined by the commutation direction. In all cases,

* If the digital phase modulator has an even number of elements, the results obtained are similar to those found above.
Fig. III-2--Spectral components of a four element digital phase modulator.
Fig. III-3--Spectral components of an eight element digital phase modulator.
Fig. III-4--Spectral components of a sixteen element digital phase modulator.
the separation between the harmonic components is equal to $f_{sw}$ and is independent of $M$. As the number of elements increases, the digital phase transitions become smaller, which produces less power in the spurious harmonics. The amount of spurious harmonic power is related to the difference between the power at $f_0$ and $(f_0 \pm f_{sw}/M)$. As $M$ increases, this difference becomes smaller which leaves less power to be divided among the spurious harmonics. For example, when $M$ equals sixteen, only 0.057 dB of the original carrier frequency power at $f_0$ is spread among the spurious harmonics.

The amplitudes of the Fourier harmonics, $b_n$, are easily found because they follow a $(\sin x/x)$ envelope as shown in Eq. (III-18).
APPENDIX IV
SPECTRAL ANALYSIS OF A TAGGED SIGNAL

This appendix analyzes the spectrum produced by sequential PM/PM beam tagging.

In a sequentially tagged four-element array, one slave element is tagged while the remaining three (reference and two slave elements) are untagged. The combination of the three untagged phasors forms a subresultant phasor which may vary in amplitude between zero and three. During the time interval required to measure the error signal, the phase conjugation system is assumed to maintain a constant phase relation between the four phasors. The remaining slave element is digitally tagged by phase shifting its carrier signal by 0° or 180° at a rate, \( \omega_T \).

With a passive satellite, the tagging system described above is sufficient to derive the required phase error information. When an active hard-limiting satellite is used, the AM tagging reference signal is eliminated. In this case the required phase error information can be obtained by PM detection alone if two tagging measurements are made. Assume that the slave element is tagged for an interval of time equal to \( LT \). At the end of this interval, the error
signal is measured and the phase of the element being tagged is incremented by a small amount $\Delta \Omega$. The tagging continues for another interval $LT$, at the end of which time the error signal is measured. By comparing these two error signals, the magnitude and sense of the phase error can be determined. The time interval, $T$, is selected so that it contains an even number of complete cycles at the tagging frequency, $\omega_T$. Theoretically only two phase error measurements are required, one with $\Delta \Omega$ equal to zero and the other with $\Delta \Omega$ unequal to zero. In actual practice, it is desirable to repeat the measurements several times in order to increase the accuracy of the error measurement. Therefore, in this analysis the discrete phase shifts of zero and $+\Delta \Omega$ are assumed to occur at a rate $\Omega_{SW}$ where

\begin{equation}
(IV-1) \quad \omega_T = J\Omega_{SW} \quad J - \text{positive integer}
\end{equation}

Figure IV-1 shows a phasor diagram of the tagging operation. The phasor $R'$ is formed by the resultant of the three untagged phasors. The phase difference between the phasor $R'$ and the tagged carrier signal is $\gamma$. Periodic digital phase shifts of $0^\circ$ and $180^\circ$ cause this phasor to vary between points $A_0$ and $A_1$ respectively. When the discrete phase shift, $\Delta \Omega$, is added, the tagged phasor varies between $B_0$ and $B_1$. The phasor diagram shows that
Fig. IV-1--Phasor diagram of digital, 0°, 180° tagging with discrete phase shift $\Delta \Omega$.

The digital 0° and 180° tagging, produces both amplitude and phase variations.

Figure IV-2 shows the tagging operation vs time. From time, $-LT$ to zero, the discrete phase shift $\Delta \Omega$ is equal to zero. The total resultant phasor is in position $A_0$ and $A_1$ as shown in Fig. IV-1 during the intervals labelled $A_0$ and $A_1$ in Fig. IV-2. During the time from zero to $+LT$, the discrete phase shift, $\Delta \Omega$,
is not equal to zero. Therefore, the total resultant phasor varies between $B_0$ and $B_1$ as shown in Fig. IV-1.

In sequential operation, one element is tagged, the error signal detected, and the phase of the carrier signal from that element is corrected before the next element is tagged. During the time when the signal from one slave element is not being tagged, all four elements radiate CW signals. This means the tagging cycle is applied in a pulse-like manner. The tagging operation is normally performed for a period of time equal to several multiples of $LT$. The ratio of the tagging times, $t_{on}$ to $t_{off}$, establishes the positions of the spectral lines within the $(\sin X/X)$ envelope.

In this analysis, the tagging operation is assumed to be periodic with period equal to $2LT$, where $L$ is an arbitrary even positive integer. The four phasor positions shown in Fig. IV-1 may be represented by

$$
\begin{align*}
(\text{IV-2})
\text{e}(t) &= \begin{cases} 
A_0 \sin(\omega t + \alpha) & -2kT < t < -(2k-1)T \\
A_1 \sin(\omega t - \alpha) & -(2k-1)T < t < -2(k-1)T \\
B_0 \sin(\omega t + \beta) & 2(k-1)T < t < (2k-1)T \\
B_1 \sin(\omega t - \beta) & (2k-1)T < t < 2kT
\end{cases}
\end{align*}
$$

where $k = 1, 2, \cdots, L/2$

$\omega = 2\pi M/T$ where $M$ is a positive integer.
The Fourier components of $e(t)$ are calculated using the well-known equations

\begin{align*}
a_n &= \frac{1}{LT} \int_{-LT}^{LT} e(t) \cos \omega_n t \, dt \\
b_n &= \frac{1}{LT} \int_{-LT}^{LT} e(t) \sin \omega_n t \, dt
\end{align*}

(IV-3)

where $\omega_n = n\pi/LT$. Before $e(t)$ is substituted into the Eqs. (IV-3), the following nomenclature simplifications are made,

1. $\sin \theta \rightarrow S_\theta$ and $\cos \theta \rightarrow C_\theta$

2. \begin{bmatrix} a_n \\ b_n \end{bmatrix} and \begin{bmatrix} C \\ S \end{bmatrix}

where $a_n$ is evaluated using the upper terms, $b_n$ the lower terms.

Substituting $e(t)$ into Eqs. (IV-3) gives

\begin{align*}
B_0 & B_1 A_0 A_1 \\
-2L & \sim A_1 A_0 A_1 B_0 B_1 B_0 \\
0 & \sim B_0 B_1 A_0 A_1 +2L
\end{align*}

Fig. IV-2--Tagging cycle vs time with period equal to $4L$. 
\[ (IV-4) \]
\[
\left( \begin{array}{c} a_n \\ b_n \end{array} \right) = \frac{1}{LT} \sum_{k=1}^{\frac{L}{2}} \left\{ \begin{array}{c} + A_0 \left[ C_\alpha \int_{-t_a}^{t_b} S_{\omega t} (C_\omega) \omega_n t \, dt + C_\omega (C_\omega) \omega_n t \, dt \right] \\ - A_1 \left[ C_\alpha \int_{-t_a}^{t_c} S_{\omega t} (C_\omega) \omega_n t \, dt - C_\omega (C_\omega) \omega_n t \, dt \right] \\ + B_0 \left[ C_\beta \int_{t_b}^{t_c} S_{\omega t} (C_\omega) \omega_n t \, dt + C_\omega (C_\omega) \omega_n t \, dt \right] \\ + B_1 \left[ C_\beta \int_{t_b}^{t_a} S_{\omega t} (C_\omega) \omega_n t \, dt - C_\omega (C_\omega) \omega_n t \, dt \right] \end{array} \right. 
\]

where \( t_a = 2kT \)
\( t_b = (2k-1)T \)
\( t_c = 2(k-1)T \).

The special case when \( \omega_n = \omega \) is easily solved for \( a_n \) and \( b_n \) and gives

\[
(IV-5) \quad a_n = \frac{1}{4} \left[ (A_0 S_\alpha - A_1 S_{-\alpha}) + (B_0 S_\beta - B_1 S_{-\beta}) \right] \\
b_n = \frac{1}{4} \left[ (A_0 C_\alpha + A_1 C_{-\alpha}) + (B_0 C_\beta + B_1 C_{-\beta}) \right].
\]

When \( \omega_n \neq \omega \), the evaluation of \( a_n \) and \( b_n \) is more difficult because the product of two trigonometric functions at different frequencies produces sum and difference frequencies. The sum term is neglected because its amplitude is much smaller than the difference term.
Now examine some general integrals which are useful in

evaluating \( a_n \) and \( b_n \). These integrals are,

\[
\begin{align*}
SC \quad Q &= \int_a^b (S_{\omega t} C_{\omega_n t}) \, dt = \frac{-1}{2(\omega - \omega_n)} \left\{ C \left[ \frac{n(\ell + 1)\pi}{L} \right] - C \left[ \frac{\ell k\pi}{L} \right] \right\} \\
CS \quad Q &= \int_a^b (C_{\omega t} S_{\omega_n t}) \, dt = -Q \\
(IV-6) \quad SS \quad Q &= \int_a^b (S_{\omega t} S_{\omega_n t}) \, dt = \frac{-1}{2(\omega - \omega_n)} \left\{ S \left[ \frac{\ell (k + 1)\pi}{L} \right] - S \left[ \frac{\ell k\pi}{L} \right] \right\} \\
CC \quad Q &= \int_a^b (C_{\omega t} C_{\omega_n t}) \, dt = Q
\end{align*}
\]

where

1. \( a = \ell T, \quad b = (\ell + 1) T \)

2. \( (\omega - \omega_n) = (2ML - n)\pi / LT \)

3. multiples of \( 2\pi M \) are not written in the sine and cosine terms.

4. Terms with \( [1/(\omega + \omega_n)] \) are neglected.

The solution to the problem may now be obtained by performing

the proper summation of the terms given in Eqs. \( IV-6 \). Let,

\[
\begin{align*}
SC \quad |P_{n, \ell}| &= \cos \left[ \frac{n(\ell + 1)\pi}{L} \right] - \cos \left[ \frac{\ell \pi}{L} \right] \\
(IV-7) \quad SS \quad |P_{n, \ell}| &= \sin \left[ \frac{n(\ell + 1)\pi}{L} \right] - \sin \left[ \frac{\ell \pi}{L} \right]
\end{align*}
\]
For the $n^{th}$ harmonic, the terms $P_n, l$ and $P_n, l$ SS must be summed over the proper values of $l$. The limits of integration given in Eq. (IV-4) require four particular sequences of $l$,

\begin{align*}
  l_A &= -L, (L-2), \ldots, -2 \\
  l_B &= -(L-1), -(L-3), \ldots, -1 \\
  l_C &= 0, 2, \ldots, (L-2) \\
  l_D &= 1, 3, \ldots, (L-1)
\end{align*}

(IV-8)

To evaluate the summation of $P_n, l$ and $P_n, l$ over the four sequences of $l$, $l_A, l_B, l_C$, and $l_D$ requires some lengthy manipulations. Therefore, only the results are given here.

(IV-9)

\begin{align*}
  \sum_{l_A}^{SC} P_n, l &= \sum_{l_C}^{SC} P_n, l = \begin{cases} 
    -1 & n \text{ odd} \\
    0 & n \text{ even and } \neq L \\
    +L & n = L
  \end{cases} \\
  \sum_{l_B}^{SC} P_n, l &= \sum_{l_D}^{SC} P_n, l = \begin{cases} 
    -1 & n \text{ odd} \\
    0 & n \text{ even and } \neq L \\
    -L & n = L
  \end{cases}
\end{align*}

\begin{align*}
  \sum_{l_A}^{CC} P_n, l &= -\sum_{l_B}^{CC} P_n, l = -\sum_{l_C}^{CC} P_n, l = \sum_{l_D}^{CC} P_n, l = \begin{cases} 
    0 & n \text{ even} \\
    -\tan\left(\frac{n\pi}{2L}\right) & n \text{ odd}
  \end{cases}
\end{align*}
By substituting the values given in Eq. (IV-9) into Eqs. (IV-4), the expressions for $a_n$ and $b_n$ become

For $n$ odd
\[
\begin{align*}
\frac{a_n}{-1} &= \frac{-1}{2\pi(2ML-n)} \left\{ \left[ (A_0 C_\alpha + A_1 C_{\bar{\alpha}}) - (B_0 C_\beta + B_1 C_{\bar{\beta}}) \right] \\
&\quad + \left[ - (A_0 S_\alpha + A_1 S_{\bar{\alpha}}) + (B_0 S_\beta + B_1 S_{\bar{\beta}}) \right] \tan \left( \frac{n\pi}{2L} \right) \right\} \\
\end{align*}
\]

\[
\begin{align*}
n = (2M+q)L \quad a_n &= \frac{-L}{2\pi(2ML-n)} \left\{ - (A_0 C_\alpha - A_1 C_{\bar{\alpha}}) - (B_0 C_\beta - B_1 C_{\bar{\beta}}) \right\} \\
\end{align*}
\]

For $n$ even and $n \neq (2M+q)L$

\[
\begin{align*}
a_n &= 0 \\
\end{align*}
\]

(IV-10)

For $n$ odd
\[
\begin{align*}
\frac{b_n}{-1} &= \frac{-1}{2\pi(2ML-n)} \left\{ \left[ - (A_0 C_\alpha - A_1 C_{\bar{\alpha}}) + (B_0 C_\beta - B_1 C_{\bar{\beta}}) \right] \tan \left( \frac{n\pi}{2L} \right) \right\} \\
&\quad - (A_0 S_\alpha - A_1 S_{\bar{\alpha}}) + (B_0 S_\beta - B_1 S_{\bar{\beta}}) \\
\end{align*}
\]

\[
\begin{align*}
n = (2M+q)L \quad b_n &= \frac{-L}{2\pi(2ML-n)} \left\{ (A_0 S_\alpha + A_1 S_{\bar{\alpha}}) + (B_0 S_\beta + B_1 S_{\bar{\beta}}) \right\} \\
\end{align*}
\]

For $n$ even and $n \neq (2M+q)L$

\[
\begin{align*}
b_n &= 0 \\
\end{align*}
\]

where $q = \text{odd}$

These very formidable equations simplify nicely when they are expressed in terms of the magnitude and phase of the phasors shown in Fig. IV-1a. The magnitude of the spectral component at each frequency is given by

\[
\begin{align*}
(IV-11) \quad c_n &= \sqrt{a_n^2 + b_n^2} \\
\end{align*}
\]
Using Eq. (IV-11) and the simplified versions of Eq. (IV-10), the spectral components are given by

\[(IV-12a) \quad c_0 = R \quad \text{carrier freq.}\]

\[(IV-12b) \quad c_n = \left(\frac{2}{\pi n}\right) \cos\left(\frac{\Delta \Omega}{2}\right) \quad n = qL\]

\[(IV-12c) \quad c_n = \left(\frac{2}{\pi n}\right) \tan\left(\frac{n\pi}{2L}\right) \sin\left(\frac{\Delta \Omega}{2}\right) \quad n \text{ odd}\]

\[(IV-12d) \quad c_n = 0 \quad n \text{ even}\]

The result given in Eq. (IV-12a) shows that all the power at the carrier frequency is contributed by the untagged subresultant phasor \(R^1\). This is expected because the phasor \(R^1\) is unmodulated. The digital 0°, 180° tagging is similar to biphase modulator. In fact, when \(\Delta \Omega\) equals zero, the spectrum is exactly equal to that of a biphase modulator. The spectral components given by Eq. (IV-12b) are not multiplied by a gain constant because the digitally tagged phasor was assumed to have unity amplitude. When \(\Delta \Omega\) is unequal to zero, the periodic phase switching produces the spectral components given in Eq. (IV-12c). Since this phase switching is done at a lower frequency than the digital phase tagging, the switching spectral components are spaced closer together than the tagging spectral components. Equation (IV-12d) shows that all the even
harmonics are equal to zero which is expected since all the tagging and switching functions are performed using a square wave.

Figures IV-3 thru IV-5 show the spectral components, $c_n$, for $\Delta \Omega$ equal to 22.5°, 45° and 90° respectively. The ratio of the tagging frequency to the switching frequency ($\omega_T/\Omega_{sw}$), was arbitrarily set at ten. The three figures show that as the phase switching increment $\Delta \Omega$ increases, the switching spectral components increase and the tagging spectral components decrease.

![Graph](image)

**Fig. IV-3--**Spectrum of a digitally phase tagged ($0^\circ$, $180^\circ$) signal with $\Delta \Omega = 22.5^\circ$. 
Fig. IV-4--Spectrum of a digitally phase tagged $(0^\circ, 180^\circ)$ signal with $\Delta \Omega = 45^\circ$.

Fig. IV-5--Spectrum of a digitally phase tagged $(0^\circ, 180^\circ)$ signal with $\Delta \Omega = 90^\circ$. 
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