DIOPHANTINE INEQUALITIES FOR
QUADRATIC AND OTHER FORMS

DISSERTATION

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By

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§ 1. Non-homogeneous Indefinite Quadratic Forms:

A well known theorem of Minkowski on the product of two linear forms states that if

\[ L_1 = \alpha x + \beta y, \quad L_2 = \gamma x + \delta y \]

are two linear forms with real coefficients and determinant \( \Delta = |\alpha \delta - \beta \gamma| \neq 0 \), then given any real numbers \( c_1, c_2 \) we can find integers \( x, y \) such that

\[ |(L_1 + c_1)(L_2 + c_2)| \leq \frac{\Delta}{4}. \quad (1.1) \]

In terms of quadratic forms Minkowski's theorem asserts that if \( Q(x,y) \) is a real indefinite binary quadratic form with determinant \( \Delta \neq 0 \), then given any real numbers \( x_0, y_0 \) we can find integers \( x, y \) such that

\[ |Q(x + x_0, y + y_0)| \leq \left( \frac{\Delta}{4} \right)^{1/2}. \quad (1.2) \]

Davenport and Heilbronn [15] have proved that for any real \( x_0, y_0 \) we can find integers \( x, y \) to satisfy

\[ 0 < Q(x + x_0, y + y_0) \leq \left( \frac{\Delta |D|}{4} \right)^{1/2}. \quad (1.3) \]

Davenport [13] has extended (1.2) to indefinite ternary quadratic forms and proved that if \( Q(x,y,z) \) is an indefinite ternary
quadratic form with real coefficients and determinant $D \neq 0$, then for any real numbers $x_0, y_0, z_0$ we can find integers $x, y, z$ satisfying

$$|Q(x + x_0, y + y_0, z + z_0)| \leq \left(\frac{27}{100}|D|\right)^{1/3} \quad (1.4)$$

E. S. Barnes [6] has extended (1.3) to indefinite ternary quadratic forms with negative determinants. He has proved

**Theorem:** Let $Q(x, y, z)$ be an indefinite ternary quadratic form with real coefficients and determinant $D < 0$. Then given any real numbers $x_0, y_0, z_0$ we can find integers $x, y, z$ such that

$$0 < Q(x + x_0, y + y_0, z + z_0) \leq (4|D|)^{1/3} \quad (1.5)$$

All the above results are the best possible.

The problem for indefinite quadratic forms in $n$-variables can be stated as follows:

Let $Q(x_1, x_2, \ldots, x_n)$ be an indefinite quadratic form in $n$-variables with real coefficients of determinant $D \neq 0$ and signature $s$. Then to find the best possible constant $c_{n, s}$ depending only on $n$ and $s$ such that given any real numbers $\xi_1, \ldots, \xi_n$ we can find integers $x_1, \ldots, x_n$ satisfying

$$|Q(x_1 + \xi_1, \ldots, x_n + \xi_n)| \leq (c_{n, s}|D|)^{1/n} \quad (1.6)$$

From (1.2) and (1.4) we have

$$c_{2, 0} = \frac{1}{4} \quad (1.7)$$
Birch [7] has proved that
\[ c_{2n,0} = \frac{1}{4} \]  
\[ c_{3,1} = c_{3,-1} = \frac{27}{100} \]  

(1.8)

Watson [21] has found \( c_{n,s} \) for \( n \geq 21 \) and all \( s \). He proved:

**Theorem:** Let \( Q(x_1, x_2, \ldots, x_n) \) be an indefinite non-singular quadratic form with real coefficients in \( n \geq 21 \) variables, with discriminant \( d \neq 0 \) and signature \( s \). Then given any real numbers \( \xi_1, \xi_2, \ldots, \xi_n \) we can find integers \( x_1, x_2, \ldots, x_n \) such that

\[
Q(x_1 + \xi_1, x_2 + \xi_2, \ldots, x_n + \xi_n) \leq \begin{cases} 
\frac{1}{2}(|d|)^{1/n} & \text{for } S \equiv \begin{cases} +3,4 \end{cases} \pmod{8} \\
\frac{1}{2}(|d|/2)^{1/n} & \text{for } S \equiv \begin{cases} +1 \end{cases} \pmod{8} \\
\frac{1}{2}(|d|/3)^{1/n} & \text{for } S \equiv \begin{cases} +2 \end{cases} \pmod{8} \\
\frac{1}{2}(|d|/4)^{1/n} & \text{for } S \equiv \begin{cases} 0 \end{cases} \pmod{8}
\end{cases}
\]  

(1.10)

where the discriminant \( d \) is given by

\[
d = \begin{cases} 
(-1)^{n/2}2^{n\det Q} & \text{for even } n, \\
(-1)^{(n-1)/2}2^{n-1} \det Q & \text{for odd } n.
\end{cases}
\]  

(1.11)

All the above results are the best possible.

The problem corresponding to (1.3) is to find the best possible constants \( \gamma_{n,s} \) such that given any real numbers \( \xi_1, \ldots, \xi_n \)
we can find integers \( x_1, \ldots, x_n \) satisfying

\[ 0 < a(x_1 + \xi_1, \ldots, x_n + \xi_n) \leq (\gamma_{n+8}[D])^{1/n}. \]  

(1.12)

From (1.3) and (1.5) we have

\[ \gamma_{2,0} = 4 \]  

(1.13)

\[ \gamma_{3,1} = 4 \]  

(1.14)

In Chapter I we prove that

\[ \gamma_{3,-1} = 8. \]  

(1.15)

In Chapter II we use (1.15) to prove that

\[ c_{4,2} = c_{4,-2} = \frac{1}{3}. \]  

(1.16)

From Birch's result (1.9) we have

\[ c_{4,0} = \frac{1}{4}. \]  

(1.17)

This settles the case of quaternary quadratic forms.

In Chapter III, we use (1.15) to prove that

\[ \gamma_{4,2} = \frac{16}{3}. \]  

(1.18)

In Chapter IV, we use (1.5) and Blaney's results [8] on asymmetric inequalities for non-homogeneous binary quadratic forms to prove that
\[ \gamma_{4,0} = 16. \] (1.19)

The critical forms in each case are also described.

\section{Other forms:

\subsection{On the density of double lattice covering by spheres in three dimensional space.}

**Definition:** Let \( K \) be a closed bounded set in the \( n \)-dimensional euclidean space \( \mathbb{R}^n \) with a non empty interior. A lattice \( \Lambda \) in \( \mathbb{R}^n \) is said to provide an \( r \)-fold covering of the space by \( K \) if each point of \( \mathbb{R}^n \) is contained in at least \( r \) sets \( K + A, A \in \Lambda \). \( (K,\Lambda) \) is said to be an \( r \)-covering.

\[ \varrho_r(K,\Lambda) = \frac{\nu(K)}{d(\Lambda)}, \] (2.1)

where \( \nu(K) \) is the volume of \( K \) and \( d(\Lambda) \) is the determinant of the lattice \( \Lambda \), is defined to be the density of the \( r \)-covering \( (K,\Lambda) \).

\[ \varrho_r(K) = \inf \varrho_r(K,\Lambda), \] (2.2)

where the infimum is taken over all \( r \)-lattice coverings \( (K,\Lambda) \), is defined to be the density of the thinnest \( r \)-covering of the space by \( K \).

If \( K \) is a circle then W. J. Blundon [9] has found \( \varrho_2(K) \) for \( r = 2,3,4,5 \).

In Chapter \( V \) we obtain bounds for \( \varrho_2(K) \) where \( K \) is a sphere in \( \mathbb{R}^3 \). We prove

\[ \frac{\sqrt{3}}{2} \pi = (\frac{108}{125})^{1/2} \cdot 2 \varrho_1(K) \leq \varrho_2(K) \leq 2 \varrho_1(K) = \frac{5\sqrt{5}}{12} \pi \] (2.3)
§ 2.2. On positive values of homogeneous indefinite binary quadratic forms.

Let \( f(x,y) = ax^2 + bxy + cy^2 \) be an indefinite binary quadratic form with discriminant \( d = b^2 - 4ac > 0 \). Let

\[
\mu_(f) = \inf_{\sqrt{d}} \frac{f(u,v)}{\sqrt{d}},
\]

where the infimum is taken over all integers \( u,v \) such that \( f(u,v) > 0 \).

Mahler (see e.g. Cassels [10], Chapter II) proved that

\[
0 < \mu_+(f) < 1
\]

and that there is no isolation at \( 1 \); that is given \( \epsilon > 0 \), there exists \( f(x,y) \) such that \( \mu_+(f) > 1 - \epsilon \). It will be interesting to determine the nature of distribution of values of \( \mu_+(f) \) for various \( f \). In Chapter VI we show that \( \mu_+(f) \) has no value in the interval \( (\frac{1}{2}, \frac{5}{6}) \); that \( \frac{1}{2} \) and \( \frac{5}{6} \) are assumed and that \( \frac{1}{2} \) is not isolated.

2.3. An inhomogeneous minimum of a class of functions.

Several authors (Barnes [5], Bambah [1], [4], Chalk [12], Mordell [17] and K. Rogers [4], [18]) have proved theorems of the following type:

Theorem A: Let \( f(x,y) \) be a function of a given form. Then for given real \( x_0, y_0 \), there exist integers \( x,y \) such that

\[
|f(x + x_0, y + y_0)| \leq \max\{|f(x_0,0)|, |f(0,y_0)|, |f(x_0, y_0)|, |f(0, y)|, |f(x, y_0)|, |f(1, 0)|\}
\]

(2.6)
In particular, Bambah [1], proved the result for all binary cubic forms with real coefficients and positive discriminants, and Bambah and K. Rogers [4] proved it for functions corresponding to the "regions with hexagonal symmetry." They remarked that their theorem does not have an analogue for "general" regions with four or more asymptotes. Their example (pp. 345) shows that this is true if the right hand side of (2.6) is unchanged.

However, we can interpret the above theorem as stating that

$$\sup_{(x,y)} \inf_{(x_0,y_0)} |f(x + x_0, y + y_0)|$$

is bounded by the maximum of values of $|f(x,y)|$ at a finite number of points, which can be written down as soon as $f(x,y)$ is given. With this interpretation, together with Miss Rajinderjeet Hans, I prove that the results of Theorem A can be generalized to regions with four asymptotes with "octagonal symmetry." Chapter VII consists of this paper, published in Monatshefte für Mathematik, 69 (1965), 193-207.
CHAPTER I
ONE SIDED INEQUALITY FOR INHOMOGENEOUS INDEFINITE
TERNARY QUADRATIC FORMS

§ 1. In this chapter we prove the following theorem:

Theorem: Let \( Q(x,y,z) \) be an indefinite ternary quadratic form with
real coefficients and determinant \( D < 0 \). Then given any real numbers
\( x_0, y_0, z_0 \), we can find integers \( x, y, z \), such that

\[
-(8|D|)^{1/3} < Q(x + x_0, y + y_0, z + z_0) < 0. \quad (1.1)
\]

Equality is needed if and only if

\[
Q(x,y,z) - \rho Q_1 = \beta(x^2 + xy + y^2 - 2yz), \quad (1.2)
\]

where \( \beta > 0 \). For \( Q_1 \) equality occurs if and only if \( (x_0, y_0, z_0) = (0,0,1/2) \ (mod \ 1) \).

§ 2. Some Lemmas:

In the course of the proof we shall use the following lemmas:

Lemma 1: If \( Q(x,y,z) \) is an indefinite ternary quadratic form with
determinant \( D < 0 \), then there exist integers \( u,v,w \) satisfying

\[
0 < Q(u,v,w) \leq (4|D|)^{1/3} \quad (2.1)
\]
Equality is necessary if and only if

\[ Q(x,y,z) \sim f(x^2 + yz), \quad g > 0. \tag{2.2} \]

This is Theorem 2 of Davenport [14].

Lemma 2: Let \( \alpha, \beta, d \) be real numbers with \( \beta^2 > \frac{1}{4}, \ d > 1 \). Then for any real number \( x_0 \) there exists \( x = x_0 \pmod{1} \) satisfying

\[ 0 < -(x + \omega)^2 + \beta^2 \leq d. \tag{2.3} \]

provided that

\[ \beta^2 \leq d + (\lceil \frac{d}{2} \rceil)^2. \tag{2.4} \]

If \( d \) is not an integer strict inequality in (2.4) implies strict inequality in (2.3). If \( d \) is an integer, a sufficient condition for (2.3) to be true with strict inequality is that

\[ \beta^2 < \left( \frac{d + 1}{2} \right)^2. \tag{2.5} \]

(It may be noticed that \( d + (\lceil \frac{d}{2} \rceil)^2 > d + (\frac{d - 1}{2})^2 = (\frac{d + 1}{2})^2 \).)

Proof: If \( \frac{1}{4} < \beta^2 < 1 \), choose \( x = x_0 \pmod{1} \) with \( |x + \alpha| \leq \frac{1}{2} \), so that

\[ 0 < -(x + \omega)^2 + \beta^2 < 1 \leq d. \]
If \( 1 \leq \beta^2 < \left(\frac{d+1}{2}\right)^2 \); choose \( x \equiv x_0 \pmod{l} \) with

\[
\beta - 1 \leq x + \alpha < \beta,
\]

so that

\[
0 < -(x + \alpha)^2 + \beta^2 \leq 2\beta - 1 < d.
\]

If \( d \) is an integer and \( \beta = \frac{d+1}{2} \) we proceed as in (2.6) and get the result perhaps with equality.

Let now

\[
\beta^2 \begin{cases} 
> \left(\frac{d+1}{2}\right)^2 & \text{if } d \text{ is an integer,} \\
\geq \left(\frac{d+1}{2}\right)^2 & \text{if } d \text{ is not an integer,}
\end{cases}
\]

so that in either case

\[
\beta^2 > \left[\frac{d}{2}\right] + 1.
\]  

Choose \( x \equiv x_0 \pmod{l} \) with

\[
\frac{d}{2} \leq |x + \alpha| \leq \frac{[d] + 1}{2},
\]

so that from (2.4), (2.7), (2.8) we have
We have strict inequality in the above if (2.4) is true with strict inequality. This completes the proof of the Lemma.

§ 3. Proof of the Theorem:

Let

\[ m = \inf_{u,v,w \text{ integ.}} Q(u,v,w) \quad (3.1) \]

§ 3.1. Case \( m = 0 \)

Lemma 3: If \( m = 0 \), then the result is true.

Proof: Since \( m = 0 \), given \( \xi_0 \) \((0 < \xi_0 < 1)\) we can find integers \( u,v,w \) such that

\[ 0 < Q(u,v,w) = \xi < \xi_0, \quad (u,v,w) = 1. \quad (3.2) \]

By replacing \( Q \) by an equivalent form we can suppose \( Q(1,0,0) = \xi \). Then \( Q \) can be written as

\[ Q(x,y,z) = \xi(x + hy + gz)^2 - \phi(y,z) \quad (3.3) \]

where \( \phi(y,z) \) is an indefinite binary quadratic form with discriminant
By choosing $\varepsilon_0$ sufficiently small we can ensure that $\Delta \geq 1$. Further by a result of Mahler (see eg. Cassel [10]) we can find relatively coprime integers $v_1, w_1$ such that

$$\varphi(v_1, w_1) = a, \quad 0 < a \leq \Delta. \quad (3.5)$$

By a suitable unimodular substitution we can suppose

$$\varphi(y, z) = a(y + fz)^2 - \frac{\Delta^2}{4a} z^2. \quad (3.6)$$

Therefore,

$$Q(x, y, z) = \varepsilon(x + hy + gz)^2 - a(y + fz)^2 + \frac{\Delta^2}{4a} z^2. \quad (3.7)$$

Choose $z = z_0 \mod 1$ with $|z| \leq \frac{1}{2}$. Then choose $y = y_0 \mod 1$ to satisfy

$$\left(\frac{\varepsilon}{4a} + \frac{\Delta^2}{4a} z^2\right)^{1/2} \cdot 2 < y + fz < \left(\frac{\varepsilon}{4a} + \frac{\Delta^2}{4a} z^2\right)^{1/2} + 1,$$

so that

$$\Delta^2 = \frac{4|\mu|}{\varepsilon} \quad (3.4)$$
\[ \frac{\varepsilon}{4} < \beta^2 = \varphi(y, z) \leq a + \frac{\varepsilon}{4} + (\varepsilon a + \Delta^2 z^2)^{1/2} \]
\[ < \Delta + \frac{1}{4} + (\Delta + \frac{\Delta^2}{4})^{1/2} \]
\[ \leq c \Delta, \]  
\[ (3.8) \]

where \( c \) is an absolute constant independent of \( \Delta, \varepsilon_0 \).

Let \( \alpha = hy + g z \), so that

\[ -Q(x, y, z) = -\varepsilon(x + \alpha)^2 + \beta^2. \]  
\[ (3.9) \]

If \( \frac{\varepsilon}{4} < \beta^2 < \varepsilon \), choose \( x = x_0 \) (mod \( n \)) with \( |x + \alpha| \leq \frac{1}{2} \), so that

\[ 0 < -\varepsilon(x + \alpha)^2 + \beta^2 = -Q(x, y, z) < \varepsilon < \varepsilon_0 \]  
\[ (3.10) \]

If \( \varepsilon \leq \beta^2 < c \Delta \), choose \( x = x_0 \) (mod \( n \)) to satisfy

\[ \frac{\beta}{\varepsilon^{1/2}} - 1 \leq x + \alpha < \frac{\beta}{\varepsilon^{1/2}}, \]

so that,

\[ 0 < -Q(x, y, z) = -\varepsilon(x + \alpha)^2 + \beta^2 \leq 2 \frac{\varepsilon \beta}{\varepsilon^{1/2}} - \varepsilon. \]
where $A$ is an absolute constant. Since $\varepsilon_0$ can be chosen arbitrarily small, the right hand side of (3.10) and (3.11) can be made as small as we please, and the lemma follows.

§ 3.2: Proof Cont'd:

We can now suppose $m > 0$  

Then given $\varepsilon_0$ ($0 < \varepsilon_0 < \frac{1}{16}$), we can find integers $u,v,w$ to satisfy

$$Q(u,v,w) = \frac{m}{1 - \varepsilon},$$

where $0 \leq \varepsilon < \varepsilon_0$. By Lemma 1, we can further assume that

$$Q(u,v,w) = \frac{m}{1 - \varepsilon} < (4|D|)^{1/3}$$  

Since $0 \leq \varepsilon < \varepsilon_0 < \frac{1}{16}$, by definition of $m$ we must have $(u,v,w) = 1$. By replacing $Q$ by an equivalent form we can suppose

$$Q(1,0,0) = \frac{m}{1 - \varepsilon}.$$ Then we can write $Q(x,y,z)$ as

$$Q(x,y,z) = \frac{m}{1 - \varepsilon} \left[ (x + by + gz)^2 - \varphi(y,z)^2 \right],$$
where \( \varphi(y,z) \) is an indefinite binary quadratic form with discriminant

\[
\Delta^2 = \frac{4|D|}{(m(1-\varepsilon))^3} \geq 1. \tag{3.15}
\]

Also, by definition of \( m \) we have for any integers \( x,y,z \) either \( Q(x,y,z) \leq 0 \) or \( Q(x,y,z) \geq m \), i.e. either

\[
(x + hy + gz)^2 - \varphi(y,z) \leq 0 \quad \text{or} \quad (x + hy + gz)^2 - \varphi(y,z) \geq 1 - \varepsilon.
\]

Because of homogeneity it suffices to prove

Theorem A: Let

\[
Q(x,y,z) = (x + hy + gz)^2 - \varphi(y,z). \tag{3.16}
\]

where \( \varphi(y,z) \) is an indefinite binary quadratic form with discriminant

\[
\Delta^2 = 4|D| \geq 1. \tag{3.17}
\]

Let \( 0 < \varepsilon_0 < \frac{1}{16} \) be given arbitrarily small. Suppose for integers \( x,y,z \) we have either
\[
Q(x,y,z) \leq 0, \quad \text{or} \\
Q(x,y,z) \geq 1 - \varepsilon, 
\]  
(3.18)

where \( 0 \leq \varepsilon < \varepsilon_0 < \frac{1}{10} \). Let

\[
da = (8|\mathfrak{d}|)^{1/3},
\]  
(3.19)

so that from (3.17) we have

\[
da^3 = 8|\mathfrak{d}| = 2 \Delta^2 \geq 2. 
\]  
(3.20)

Then given any real numbers \( x_0, y_0, z_0 \) we can find \((x,y,z) \equiv (x_0,y_0,z_0) \pmod{1}\) such that

\[
0 < -Q(x,y,z) = -(x + hy + gz)^2 + \varphi(y,z) \leq \Delta. 
\]  
(3.21)

The equality in (3.21) is needed if and only if \(Q(x,y,z)\) is the form given by (1.2) with \( \Psi = 1 \).

\[\text{§3.3. Proof of Theorem A:}\]

\[\text{Lemma 4: Let } Q(x,y,z) \text{ be given as in Theorem A. Then for integers } y,z \text{ we have either}\]

\[
\varphi(y,z) = 0 \quad \text{or} \quad \varphi(y,z) \geq \frac{1}{4} \quad \text{or} \quad \varphi(y,z) \leq \frac{3}{4} + \varepsilon. 
\]  
(3.22)
Proof: If $\frac{3}{4} \varepsilon + \varphi(y,z) < 0$, then by choosing integer $x$ with $|x + hy + gz| \leq \frac{1}{2}$, we have

$$0 < Q(x,y,z) = (x + hy + gz)^2 - \varphi(y,z) < \frac{1}{4} + \frac{3}{4} - \varepsilon = 1 - \varepsilon,$$

contrary to (3.18).

If $\frac{1}{4} \varepsilon - \varphi(y,z) < 1$, then by choosing integer $x$ with

$$\frac{1}{2} \leq |x + hy + gz| \leq 1,$$

we have

$$0 < Q(x,y,z) < 1 - \varepsilon,$$

contrary to (3.18).

If $0 < \varphi(y,z) \leq \frac{1}{16}$, then for a suitable integer $n$ we have $\varepsilon < \varphi(ny,nz) < \frac{1}{16}$, which is not possible and the lemma follows.

By Markoff Chain Theorem (see e.g. [11], Chapter 2), we can find integers $v_1, w_1$ such that

$$|a = \varphi(v_1, w_1)| \leq \frac{\Delta}{d^2}, \quad (v_1, w_1) = 1. \quad (3.23)$$

Further, if $|a| > \frac{\Delta}{3}$, then $\varphi(y,z)$ is a Markoff form and so represents both $a$ and $-a$. In that case we can assume $a > 0$.

Also if $a = -b < 0$, then from (3.22) we have $b \geq \frac{3}{4} - \varepsilon$.

Thus we are to distinguish the following three cases:
(i) \( 0 < a \leq \frac{\Delta}{\sqrt{5}}. \)

(ii) \( a = -b, \quad \frac{3}{4} - \varepsilon \leq b \leq \frac{\Delta}{3}. \) \hfill (8)

(iii) \( a = 0. \)

Lemma 5: Let \( \nu_1 > 0, \nu_2 > 0 \) be defined by

\[
\nu_1 \Delta = \begin{cases} 
\left( \frac{d + 1}{2} \right)^2 & \text{if } d \text{ is an integer} \\
\frac{d}{2} + \left( \frac{[d]}{2} \right)^2 & \text{if } d \text{ is not an integer.}
\end{cases}
\] \hfill (3.24)

\[
\nu_2 \Delta = \frac{1}{4}. \] \hfill (3.25)

Suppose that there exist \((y, z) = (y_0, z_0) (\mod 1)\) such that

\[
\nu_2 \Delta < \varphi(y, z) \leq \nu_1 \Delta. \] \hfill (3.26)

Then for any real number \( x_0 \) there exists \( x \equiv x_0 (\mod 1) \) satisfying

\[
0 < -\varphi(x, y, z) \leq d \] \hfill (3.27)

Further, strict inequality in (3.26) implies strict inequality in (3.27).

Proof: This follows from Lemma 2 with

\[
\beta^2 = \varphi(y, z), \quad \alpha = hy + gz.
\]
Lemma 6: If $0 < a \leq \frac{\Delta}{\sqrt{5}}$, then (3.21) is true with strict inequality.

Proof: By replacing $\varphi(y,z)$ by an equivalent form we can suppose

$$\varphi(y,z) = a(y + fz)^2 - \frac{\Delta^2}{4a} z^2. \quad (3.28)$$

Choose $z \equiv z_o \pmod{1}$ such that $|z| \leq \frac{1}{2}$.

Lemma 6 then follows from:

Lemma 7: (3.21) is true with strict inequality if

(i) $0 < a < \left(\frac{1 + d}{2}\right)^2 + \frac{1}{4} - \left(\frac{1 + d}{2}\right) - \frac{d^3}{8}\right)^{1/2} = \gamma_1, \quad (3.29)$

or (ii) $\gamma_1 \leq a \leq \frac{\Delta}{\sqrt{5}}$ and $d \geq 2 \quad (3.30)$

or (iii) $0 < a < d + \frac{1}{2} - (d + \frac{1}{4} + \frac{d^3}{8})^{1/2} = \gamma_2 \quad (3.31)$

and $\frac{3\sqrt{2}}{4} \leq \frac{3\sqrt{2}}{4} \leq d < 2.$

or (iv) $\gamma_2 \leq a \leq \frac{\Delta}{\sqrt{5}}$ and $\frac{3\sqrt{2}}{4} \leq \frac{3\sqrt{2}}{4} \leq d < 2.$ \quad (3.32)

Proof of 7(i): If (3.29) holds, choose $y \equiv y_o \pmod{1}$ to satisfy

$$\left(\frac{1}{4a} + \frac{\Delta^2}{4a} z^2\right)^{1/2} < y + fz \leq \left(\frac{1}{4a} + \frac{\Delta^2}{4a} z^2\right)^{1/2} + 1,$$

so that

$$\frac{1}{4} \leq \varphi(y,z) = a(y + fz)^2 - \frac{\Delta^2}{4a} z^2 \leq \frac{1}{4} + a + (a + \frac{\Delta^2}{4})^{1/2} \leq \frac{1}{4} + a + (a + \frac{\Delta^2}{4})^{1/2}$$
i.e.

\[ \frac{1}{4} < \varphi(y, z) \leq \frac{1}{4} + a + (a + \frac{d^3}{8})^{1/2} \]  

(3.33)

Then \( \varphi(y, z) \) satisfies (3.26) with strict inequality, if we have

\[ \frac{1}{4} + a + (a + \frac{d^3}{8})^{1/2} < (\frac{d + 1}{2})^2 \]  

(3.34)

A slight calculation shows that (3.34) is true if (3.29) holds and the proof of 7(i) follows from Lemma 5.

Proof of 7(ii): In case (3.30) holds, we have

\[ \frac{1}{4a} + \frac{\Delta^2}{4a} z^2 \leq \frac{1}{4a} + \frac{\Delta^2}{16a^2} \]

\[ \leq \frac{1}{(1 + d)^2 + 1 - 2[(1 + d)^2 + \frac{d^3}{2}]^{1/2}} \]

\[ + \frac{\frac{d^3}{2}}{[(1 + d)^2 + 1 - 2[(1 + d)^2 + \frac{d^3}{2}]^{1/2}]^2} \]

\[ = \left( \frac{[(1 + d)^2 + \frac{d^3}{2}]^{1/2} - 1}{(1 + d)^2 + 1 - 2[(1 + d)^2 + \frac{d^3}{2}]^{1/2}} \right)^2 \]

\[ < 1, \]
if

$$3\left((1 + a)^2 + \frac{d^3}{2}\right)^{1/2} < d^2 + 2d + 3 \quad \text{or} \quad f(d) = 2d^3 - d^2 + 2d - 12 > 0 \quad (3.35)$$

Now \( f(a) \) is an increasing function of \( a \) and \( f(2) = 4 > 0 \).

Therefore if (3.30) holds we have

$$\frac{1}{4a} + \frac{\Delta^2}{4a^2} z^2 < 1. \quad (3.36)$$

We will now distinguish between the following two subcases:

7(ii)(a): \( \frac{1}{4a} + \frac{\Delta^2}{4a^2} z^2 < \frac{1}{4} \). \quad (3.37)

7(ii)(b): \( \frac{1}{4} \leq \frac{1}{4a} + \frac{\Delta^2}{4a^2} z^2 < 1. \quad (3.38)\)

Subcase 7(ii)(a). In this case choose \( y \equiv y_0 \pmod{1} \) to satisfy

\( \frac{1}{4} \leq |y + fz| \leq 1 \), so that

$$\frac{1}{4} < \Phi(y, z) = a(y + fz)^2 - \frac{\Delta^2}{4a} z^2 \leq a < 2a + \frac{1}{4} \leq \frac{2\Delta}{\sqrt{5}} + \frac{1}{4}. \quad (3.39)$$

\( \Phi(y, z) \) satisfies (3.26), if we have

$$\frac{2\Delta}{\sqrt{5}} + \frac{1}{4} < \left(\frac{d + 1}{2}\right)^2,$$
or if

$$5d^2 - 12d + 20 = 5(d - \frac{6}{5})^2 + \frac{64}{5} > 0;$$

which is true for all $d$.

**Subcase 7(ii)(b):** In this case choose $y \equiv y_0 \pmod{l}$ to satisfy

$$1 \leq |y + fz| \leq \frac{3}{2},$$

so that

$$\frac{1}{4} < Q(y, z) \leq (\frac{9}{4} - \frac{1}{4})a + \frac{1}{4} = 2a + \frac{1}{4} \leq \frac{24a}{\sqrt{5}} + \frac{1}{4} < (\frac{1 + d}{2})^2; \quad (3.40)$$

as in the subcase 7(ii)(a).

Hence if (3.30) is true we can find $y \equiv y_0 \pmod{l}$ such that $Q(y, z)$ satisfies (3.26) and the result follows from Lemma 5.

**Proof of 7(iii):** In this case we choose $y \equiv y_0 \pmod{l}$ as in 7(i), so that (3.33) holds. Now (3.33) implies (3.26) if we have

$$\frac{1}{4} + a + (a + \frac{a^3}{8})^{1/2} < d + \frac{1}{4} \quad (3.41)$$

A slight calculation shows that (3.41) is true if (3.31) holds.

Hence the proof of 7(iii) follows from Lemma 5.

**Proof of 7(iv):** In this case, we have

$$\frac{1}{4a} + \frac{\Delta^2}{4a^2} z^2 \leq \frac{1}{4a} + \frac{\Delta^2}{16a^2}.$$
This is so if

\[
9(4d + 1 + \frac{d^3}{2}) < (4d + 3)^2 \quad \text{or} \\
9d^2 - 32d + 24 < 0 \quad \text{or} \\
(3d - \frac{16 + \sqrt{40}}{3})(3d - \frac{16 - \sqrt{40}}{3}) < 0;
\]

This is so if

\[
\frac{16 - \sqrt{40}}{9} < d < \frac{16 + \sqrt{40}}{9};
\]  

This is true in this case since \( \frac{3\sqrt{2}}{2} \leq d < 2 \) is contained in this interval. Hence (3.36) is satisfied and the result follows as in 7(ii).

This completes the proof of Lemma 7 and hence of Lemma 6.
Lemma 8: If \( a = -b, \frac{3}{4} - \varepsilon \leq b \leq \frac{A}{3} \), then (3.21) is true with strict inequality.

Proof: Since \( a = -b \), we can write \( \varphi(y,z) \) as

\[
\varphi(y,z) = \frac{A^2}{4b^2} z^2 - b(y + fz)^2. \tag{3.43}
\]

Since \( \frac{3}{4} - \varepsilon \leq b \leq \frac{A}{3} \), from (3.20) we have

\[
d^3 = 2A^2 \geq 2\left(\frac{9}{4} - 3\varepsilon\right)^2 > 10; \tag{3.44}
\]

if \( \varepsilon_0 \) is taken sufficiently small.

Lemma 8 then follows from:

Lemma 9: (3.21) is true with strict inequality, if

(i) \( \frac{3}{4} - \varepsilon \leq b \leq \frac{A}{3} \) and \( d^3 > \frac{288}{25} \) \tag{3.45}

(ii) \( \frac{3}{4} - \varepsilon \leq b \leq \frac{A}{3} \) and \( 10 < d^3 \leq \frac{288}{25} \). \tag{3.46}

Proof of 9(i): In case (3.45) holds we have

\[
b \leq \frac{A}{3} < \left(\frac{A^2 + 1}{2}\right)^{1/2} - 1; \tag{3.47}
\]

for (3.47) is true if \( (2A + 3)^2 > 9(A^2 + 1) \) or \( A > \frac{12}{5} \) or

\[
d^3 = 2A^2 > \frac{288}{25}.
\]
Choose \( z \equiv z_0 \, (\text{mod} \, 1) \) with \( \frac{1}{2} \leq |z| \leq 1 \), so that

\[
\frac{\Delta^2}{4b^2} z^2 - \frac{1}{4b} \geq \frac{\Delta^2}{16b^2} - \frac{1}{4b} > \frac{1}{4};
\]  

(3.48)

by using (3.47)

**Now we shall distinguish the following two subcases:**

\( 9(i)(a): \frac{1}{4} < \frac{\Delta^2}{4b^2} z^2 - \frac{1}{4b} \leq 1 \)  

(3.49)

\( 9(i)(b) \frac{\Delta^2}{4b^2} z^2 - \frac{1}{4b} > 1 \)  

(3.50)

**Proof of 9(i)(a):** Choose \( y \equiv y_0 \, (\text{mod} \, 1) \) with \( |y + fz| \leq \frac{1}{2} \), so that

from (3.43) we have

\[
\frac{1}{4} < \varphi(y,z) \leq b + \frac{1}{4} \leq \frac{\Delta}{3} + \frac{1}{4};
\]  

(3.51)

and \( \varphi(y,z) \) satisfies (3.26), if we have

\[
\frac{4\Delta}{3} < d(d+2),
\]

i.e.

\[
d^2 + \frac{28}{9}d + 4 > 0,
\]

which is true for all \( d \). Hence by Lemma 5 the result follows.
Proof of 9(i)(b): In this case choose \( y \equiv y_0 \mod l \) with

\[
\left( \frac{\Delta^2}{4b^2} z^2 - \frac{1}{4b^2} \right)^{1/2} - 1 \leq y + rz < \left( \frac{\Delta^2}{4b^2} z^2 - \frac{1}{4b^2} \right)^{1/2}; \tag{3.52}
\]

so that

\[
\frac{1}{4} < \psi(y,z) \leq \frac{1}{4} - b + \left( \frac{\Delta^2}{4} - b \right)^{1/2}
\leq \frac{1}{4} - \frac{3}{4} + \epsilon + \left( \frac{\Delta^2}{4} - \frac{3}{4} + \epsilon \right)^{1/2}
\leq \Delta - \frac{1}{2}, \tag{3.53}
\]

if \( \epsilon \) is taken sufficiently small.

Thus (3.26) is satisfied with strict inequality if we have

\[
\Delta - \frac{1}{2} < \left( \frac{d + 1}{2} \right)^2
\]

i.e. if \( f(d) = d^2(d - 2)^2 + 6d^2 + 12d + 9 > 0 \);
which is true for all \( d \). Hence the result follows from Lemma 5.

Proof of 9(ii): In this case choose \( z \equiv z_0 \mod l \) with

\( 1 \leq |z| \leq \frac{3}{2} \), so that

\[
\frac{\Delta^2}{4b^2} z^2 - \frac{1}{4b^2} \geq \frac{\Delta^2}{4b^2} - \frac{1}{4b} > \frac{1}{4}; \tag{3.54}
\]

since \( b^2 + b \leq \frac{\Delta^2}{9} + \frac{\Delta}{3} < \Delta^2 \) (since \( \Delta > 1 \)).
Now we will distinguish the following two subcases:

(ii)(a) \[ \frac{1}{4} < \frac{\Delta^2}{4b^2} z^2 - \frac{1}{4b} \leq 1 \] (3.55)

(ii)(b) \[ \frac{\Delta^2}{4b^2} z^2 - \frac{1}{4b} > 1 \] (3.56)

Proof of 9(ii)(a): In this case the result follows as in 9(i)(a).

Proof of 9(ii)(b): In this case we choose \( y = y_0 \pmod{1} \) to satisfy (3.52); so that

\[
\frac{1}{4} < \Phi(y, z) \leq \frac{1}{4} - b + (\Delta^2 y^2 - b)^{1/2} \\
\leq \frac{1}{4} - \frac{3}{4} + \varepsilon + \left( \frac{3}{4} \Delta^2 - \frac{3}{4} + \varepsilon \right)^{1/2} \\
\leq \frac{3\Delta}{2} - \frac{1}{2},
\] (3.57)

if \( \varepsilon_0 \) is taken sufficiently small. \( \Phi(y, z) \) satisfies (3.26) with strict inequality if we have

\[
\frac{3}{2} \Delta - \frac{1}{2} < a + 1,
\]
i.e.

\[
g\Delta^2 < (2a + 3)^2
\]
i.e.
\[ f(d) = 9d^3 - 8d^2 - 24d - 18 < 0 \quad (3.58) \]

Now \( f(d) \) is an increasing function of \( d \) for \( d > 2 \), \( f(d) < 0 \) for \( d = 2 \cdot 3 \) and since

\[
10 < d^3 \leq \frac{288}{25} < (2 \cdot 3)^3;
\]

(3.58) is true. Hence \( \Phi(y,z) \) satisfies (3.26) and the result follows from Lemma 5. This completes the proof of Lemma 9 and hence of Lemma 8.

3.4. Proof of Theorem A cont'd:

From now on we can suppose \( a = 0 \). (3.59)

By a unimodular transformation we can suppose \( \Phi(0,1) = 0 \), so that we can write \( \Phi(y,z) \) as

\[
\Phi(y,z) = \lambda y(z - \Theta y), \quad \lambda \neq 0. \quad (3.60)
\]

By changing \( y \) to \(-y\) if necessary we can suppose \( \lambda > 0 \), and by changing \( z \) to \( z + my \) where \( m \) is a suitable integer we can suppose

\[
0 \leq \Theta < 1. \quad (3.61)
\]

Also by changing \( x \) to \( tx + ry + sz \) where \( r, s \) are suitable integers we can suppose
Lemma 10: If \( \Phi(y,z) \) is given by (3.60), then again (3.21) is true.

Proof: We shall distinguish the following two subcases.

(i) \( y_0 \not\equiv 0 \pmod{1} \) .  
(ii) \( y_0 \equiv 0 \pmod{1} \).

Proof of subcase (i): In this case choose \( z \equiv z_0 \pmod{1} \) to satisfy

\[
\frac{1}{4\lambda y} < z - \theta y \leq \frac{1}{4\lambda y} + 1, \quad \text{if } y > 0; \\
\frac{1}{4\lambda y} - 1 \leq z - \theta y < \frac{1}{4\lambda y}, \quad \text{if } y < 0;
\]

so that in either case

\[
\frac{1}{4} < \Phi(y,z) = \lambda y(z - \theta y) \leq \frac{1}{4} + \lambda |y| \leq \frac{1}{4} + \frac{\lambda}{2}.
\]

Hence (3.26) is satisfied if we have

\[
\frac{1}{4} + \frac{\lambda}{2} < \left( \frac{d + 1}{2} \right)^2 \quad \text{or} \\
2\lambda < d(d + 2) \quad \text{or} \\
2d^3 < d^2(d + 2)^2 \quad \text{since } \Delta^2 = \lambda^2 = \frac{d^3}{2} \quad \text{or} \\
d^2 + 2d + 4 > 0
\]

which is true for all \( d \). The result follows from Lemma 5 with strict inequality.
Proof of subcase (ii): \( y_0 \equiv 0 \pmod{1} \), take \( y = 1 \). Choose \( z \equiv z_0 \pmod{1} \) with

\[
\frac{1}{4\lambda} < z - \theta \leq \frac{1}{4\lambda} + 1;
\]

so that

\[
\frac{1}{4} < \varphi(y, z) = \lambda y (z - \theta y) \leq \lambda + \frac{1}{4}; \tag{3.65}
\]

and (3.26) holds if we have

\[
\lambda + \frac{1}{4} \leq (\frac{d + 1}{2})^2 \quad \text{or} \quad 4\lambda \leq d(d + 2) \quad \text{or} \quad 8d^3 \leq d^2(d + 2)^2 \quad \text{or} \quad (d - 2)^2 \geq 0, \tag{3.66}
\]

which is true for all \( d \) and the result follows from Lemma 5. This completes the proof of the lemma.

3.5. Proof of Theorem A cont'd: The Case of Equality

Lemma 11: Equality is needed in (3.21) if and only if \( \mathbf{Q}(x, y, z) = 0 \) and \( (x_0, y_0, z_0) \equiv (0, \frac{1}{2}) \pmod{1} \).

Proof: The proof shows that equality can occur only in the subcase (ii) of Lemma 10. From (3.66) we must have \( d = 2, \lambda^2 = d^{3/2} = 4 \) or \( \lambda = 2; y_0 = 0 \pmod{1} \). Also from (3.65) \( \theta \) and \( z_0 \) should be such that

\[
\frac{1}{4} < 2y(z - \theta y + z_0) < \lambda + \frac{1}{4} = \frac{9}{4} \quad \text{i.e.} \quad \frac{1}{8} < f(y, z) = y(z - \theta y + z_0) < \frac{9}{8} \tag{3.67}
\]
has no solution in integers $y, z$.

By (3.22) and (3.61) we have

$$
\Phi(1, 1) = 2(1 - \Theta) \geq \frac{1}{4}, \quad \text{i.e.}
$$

$$
0 < \Theta \leq \frac{7}{8} \quad (3.68)
$$

Also, there is no loss of generality in supposing $0 < z_0 \leq 1$.

If $z_0 < \Theta + \frac{1}{8}$, then

$$
\frac{1}{8} < f(1, 1) = 1 - \Theta + z_0 < \frac{9}{8},
$$

and if $z_0 > \Theta + \frac{1}{8}$, then

$$
\frac{1}{8} < f(1, 0) = -\Theta + z_0 < 1 < \frac{9}{8}.
$$

Therefore, if equality is to occur we must have

$$
z_0 = \Theta + \frac{1}{8} \quad (3.69)
$$

If $\Theta = \frac{7}{8}$, so that $z_0 = 1$, then

$$
\frac{1}{8} < f(2, 1) = 2(1 - 2\frac{7}{8} + 1) = \frac{1}{2} < \frac{9}{8}
$$

Therefore, we must have

$$
\Theta < \frac{7}{8}, \ z_0 + \Theta < \frac{15}{8}.
$$

If $\Theta + z_0 > \frac{7}{8}$, then

$$
\frac{1}{8} < f(-1, -2) = 2 - \Theta - z_0 < \frac{9}{8}.
$$
If \( \theta + z_0 < \frac{7}{8} \), then
\[
\frac{1}{8} < f(-1,-1) = 1 - \theta - z_0 < 1 < \frac{9}{8}.
\]

Hence for equality we must have
\[\theta + z_0 = \frac{7}{8} \quad (3.70)\]

From (3.69) and (3.70) we have
\[z_0 = \frac{1}{2}, \quad \theta = \frac{3}{8}\]

Thus, if equality is to occur we must have
\[\Phi(y,z) = 2y(z - \frac{3}{8}y), \quad (y_0, z_0) \equiv (0, \frac{1}{2}) \mod 1 \quad (3.71)\]

Therefore
\[Q(x,y,z) = (x + hy + gz)^2 - 2y(z - \frac{3}{8}y).\]

If \( g \neq 0 \), then
\[0 < Q(0,0,1) = g^2 \leq \frac{1}{4} < 1 - \varepsilon,\]

from (3.62); contrary to (3.16).

Therefore,
\[Q(x,y,z) = (x + hy)^2 - 2yz + \frac{3}{4} y^2.\]

Again, if equality is to occur in (3.21), the inequalities
\[0 < \chi(x,y,z) = -(x + hy + x_0)^2 + 2y(z + \frac{1}{2}) + \frac{3}{4} y^2 < 2 \quad (3.72)\]
Should have no solution in integers \( x, y, z \).

Now
\[
0 < \mathcal{K}(x, 1, 1) = -(x + h + x_0)^2 + \frac{9}{4} < 2,
\]
is solvable for integer \( x \) unless
\[
h + x_0 \equiv \frac{1}{2} \pmod{1} \tag{3.73}
\]

Again,
\[
0 < \mathcal{K}(x, -1, -2) = -(x - h + x_0)^2 + \frac{9}{4} < 2
\]
is solvable for integer \( x \) unless
\[
-h + x_0 \equiv \frac{1}{2} \pmod{1} \tag{3.74}
\]

From (3.73) and (3.74) we have
\[
2h \equiv 0 \pmod{1},
\]
so that by (3.62) we have
\[
h = 0 \text{ or } h = \frac{1}{2}. \tag{3.75}
\]

If \( h = 0 \), we have
\[
\mathcal{K}(2, 1, 2) = 4 - 2(2 - \frac{3}{6}) = \frac{3}{4} < 1 - \varepsilon,
\]
contrary to (3.16).

Hence we must have \( h = \frac{1}{2} \) and so by (3.73), \( x_0 \equiv 0 \pmod{1} \).

Thus, equality can occur only if
\[ Q(x,y,z) = \left(x + \frac{1}{2}y\right)^2 - 2yz + \frac{3}{4}y^2 \]
\[ = x^2 + xy + y^2 - 2yz = Q \]  
(3.76)

and \((x_0, y_0, z_0) \equiv (0,0,\frac{1}{2}) \pmod{1}\).

We next show that equality is necessary for the form (3.76).

Since

\[ Q(x,y,z + \frac{1}{2}) = x^2 + xy + y^2 - y(2z + 1), \]  
(3.77)

is an integer for integral values of \(x,y,z\), it suffices to prove that for integers \(x,y,z\) we cannot have

\[ x^2 + xy + y^2 - y(2z + 1) = -1 \]  
(3.78)

Let, if possible, (3.76) be true for integers \(x,y,z\) so that

\[ x(x + y) + y(y - 1) - 2yz = -1 \]

i.e.

\[ x(x + y) \equiv 1 \pmod{2}. \]

Therefore, \(x\) is odd, \(y\) is even.

Let \(y = 2y_1\), so that (3.76) becomes

\[ x^2 + 4y_1^2 - 4y_1z + 2y_1(x - 1) = -1 \]

or

\[ x^2 \equiv -1 \pmod{4} \]  
(since \(x\) is odd)

which is impossible.

This completes the proof of Lemma 11.

Theorem A follows from Lemmas 6, 8, 10, and 11.

This completes the proof of the theorem.
CHAPTER II
INHOMOGENEOUS MINIMUM OF INDEFINITE QUATERNARY
QUADRATIC FORMS WITH SIGNATURE +2

§ 1. In this chapter we shall prove the following theorem:

Theorem: Let $Q(x,y,z,t)$ be an indefinite quaternary quadratic form with real coefficients, determinant $D \neq 0$ and signature $+2$. Then given any reals $x_0, y_0, z_0, t_0$ we can find integers $x, y, z, t$ such that

$$|Q(x + x_0, y + y_0, z + z_0, t + t_0)| \leq \left(\frac{1}{3}|D|\right)^{1/4} \quad (1.1)$$

Equality is needed if and only if

$$Q(x,y,z,t) \sim \mathcal{Q}_1 = \mathfrak{g}(x^2 + y^2 + 3z^2 - t^2); \quad (1.2)$$

where $\mathfrak{g} \neq 0$. For $Q_1$ equality occurs if and only if $(x_0, y_0, z_0, t_0) \equiv (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \pmod{1}$.

§ 2. Some Lemmas:

In the course of the proof we shall use the following Lemmas:

Lemma 1: If $Q(x,y,z,t)$ is an indefinite quaternary quadratic form of signature $2$ and determinant $D < 0$, then there exist integers $x_1, y_1, z_1, t_1$ such that

$$0 < Q(x_1, y_1, z_1, t_1) \leq \left(\frac{16}{3}|D|\right)^{1/4} \quad (2.1)$$

Equality is necessary if and only if

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\[ q(x,y,z,t) \sim \xi q_2 = \xi(x^2 + xy + y^2 + zt), \quad \xi > 0. \quad (2.2) \]

This is Theorem 2 of Oppenheim [20].

**Lemma 2**: If \( \varphi(y,z,t) \) is an indefinite ternary quadratic form with determinant \( D < 0 \), then we can find integers \( y_2^2, z_2^2, t_2 \) such that

\[ 0 < \varphi(y_2^2, z_2^2, t_2) \leq \left( \frac{2}{9} |D| \right)^{1/3} \quad (2.3) \]

except when

\[ \varphi(y,z,t) \sim \xi(y^2 + zt), \quad \xi > 0 \quad (2.4) \]

This is a Theorem due to Oppenheim [19].

**Lemma 3**: Let \( \varphi(y,z,t) \) be an indefinite ternary quadratic form with determinant \( D < 0 \). Then given any real numbers \( y_0, z_0, t_0 \), we can find \( (y,z,t) \equiv (y_0, z_0, t_0) \) (mod 1) such that

\[ -(8D)^{1/3} \leq \varphi(y,z,t) < 0 \quad (2.5) \]

This is the Theorem of Chapter I.

**Lemma 4**: Let \( \beta, d \) be real numbers with \( d > \frac{1}{4} \) and

\[ \beta^2 \leq d + \frac{[2d]^2}{4}. \quad (2.6) \]

Then for any real number \( x_0 \), there exists \( x \equiv x_0 \) (mod 1) satisfying

\[ |(x + \alpha)^2 - \beta^2| \leq d \quad (2.7) \]

If \( 2d \) is not an integer, strict inequality in (2.6) implies strict inequality in (2.7). If \( 2d \) is an integer, a sufficient condition for (2.7) to be true with strict inequality is that

\[ \beta^2 < d + \frac{(2d - 1)^2}{4} = d^2 + \frac{1}{4} \quad (2.8) \]
This is Lemma 5 of Davenport [13].

Lemma 5: Let \( \chi(z, t) \) be an indefinite binary quadratic form with discriminant \( \Delta^2 > 0 \) and \( \mu > 3 \) be a real number. Then given any real numbers \( z_0, t_0 \) we can find \( (z, t) = (z_0, t_0) \) (mod 1) such that

\[
-\frac{\mu \Delta}{\delta(1 + \mu)(9 + \mu)^{1/2}} \leq \chi(z, t) \leq \frac{\Delta}{\delta(1 + \mu)(9 + \mu)^{1/2}} \tag{2.9}
\]

Equality is needed if and only if \( \mu = 4n - 1, n = 1, 2, 3, ... \), and

\[
\chi(z, t) \sim c\gamma_n = c\left( (n + 2)z^2 - nt^2 \right) \tag{2.10}
\]

where \( c > 0 \). For \( \gamma_n \) equality occurs if and only if \( (z_0, t_0) \equiv (1, 1) \) (mod 1); or \( \mu = \infty \) and \( \chi(z, t) \sim c\gamma' \) or \( c\gamma'' \), where \( c > 0 \), \( \gamma' = zt, \gamma'' = z^2 - t^2 \). For \( \gamma' \) equality occurs if and only if \( (z_0, t_0) \equiv (0, 0) \) (mod 1) and for \( \gamma'' \) if and only if \( (z_0, t_0) \equiv (\frac{1}{2}, \frac{1}{2}) \) (mod 1).

This is Theorem 2 of Blaney [8].

\( \S3. \) Proof of the Theorem:

By replacing \( Q \) by \( -Q \) if necessary we can suppose signature of \( Q \) is 2. Let

\[
m = \inf_{x, y, z, t \text{ integ.}} Q(x, y, z, t) \tag{3.1}
\]

\( Q(x, y, z, t) > 0 \)

\( \S3.1. \) Case \( m = 0. \)

Lemma 6: If \( m = 0 \), then the result is true.

Proof: Since \( m = 0 \), given \( \varepsilon_0 > 0 \), we can find integers \( x_1, y_1, z_1, t_1 \) such that
0 < Q(x_1, y_1, z_1, t_1) = \xi < \xi_0, \quad (x_1, y_1, z_1, t_1) = 1 \quad (3.2)

By replacing \( Q \) by an equivalent form we can suppose

\[ Q(1, 0, 0, 0) = \xi, \text{ so that} \]

\[ Q(x, y, z, t) = \xi (x + hy + gz + ut)^2 + \varphi(y, z, t); \quad (3.3) \]

where \( \varphi(y, z, t) \) is an indefinite ternary quadratic form with determinant \( \frac{D}{\xi} < 0 \).

By Lemma 3 we can find \( (y, z, t) \equiv (y_0, z_0, t_0) \) (mod 1) such that

\[ 0 < \beta^2 = -\varphi(y, z, t) \leq (\frac{8 |D|}{\xi})^{1/3} \quad (3.4) \]

Let \( \alpha = hy + gz + ut \), and choose \( x \equiv x_0 \) (mod 1) with

\[ \frac{\beta}{\sqrt{\xi}} \leq x + \alpha < \frac{\beta}{\sqrt{\xi}} + 1, \]

so that

\[ 0 \leq Q(x, y, z, t) = \xi (x + \alpha)^2 - \beta^2 < \xi + 2\beta\sqrt{\xi} \]

\[ \leq \xi + 2(\frac{8 |D|}{\xi})^{1/6}\xi^{1/2} \]

\[ = \xi + A(|D|)^{1/6}\xi^{1/3} \]

\[ < \xi_0 + A(|D|)^{1/6}\xi_0^{1/3} \quad (3.5) \]

where \( A \) is an absolute constant. Since \( \xi_0 \) can be chosen arbitrarily small, the right hand side of (3.5) can be made as small as we please and the Lemma follows.
§ 3.2. Proof Cont'd:

We can now suppose \( m > 0 \).

Then given \( 0 < \varepsilon_0 < \frac{1}{16} \), we can find integers \( x_1, y_1, z_1, t_1 \) to satisfy

\[
Q(x_1, y_1, z_1, t_1) = \frac{m}{1 - \varepsilon},
\]

where \( 0 < \varepsilon < \varepsilon_0 \). Because of Lemma 1 we can further assume that

\[
Q(x_1, y_1, z_1, t_1) = \frac{m}{1 - \varepsilon} \leq \left( \frac{16}{3} |D| \right)^{1/4}
\]

(3.6)

Since \( 0 < \varepsilon < \varepsilon_0 < \frac{1}{16} \), by the definition of \( m \), we must have

\( (x_1, y_1, z_1, t_1) = 1 \). By applying a suitable transformation to \( Q \) we can suppose that \( Q(1, 0, 0, 0) = \frac{m}{1 - \varepsilon} \). \( Q(x, y, z, t) \) can then be written as

\[
Q(x, y, z, t) = \frac{m}{1 - \varepsilon} \left\{ (x + hy + gz + ut)^2 + \varphi(y, z, t) \right\},
\]

where \( \varphi(y, z, t) \) is an indefinite ternary quadratic form with determinant

\[
\frac{D}{\left( \frac{m}{1 - \varepsilon} \right)^4} \leq \frac{3}{16}
\]

(3.7)

Equality occurs in (3.7) if and only if \( \varepsilon = 0 \) and \( Q \sim mQ_2 \) (by Lemma 1).

Also, by definition of \( m \) we have for any integers \( x, y, z, t \)

either \( Q(x, y, z, t) \leq 0 \) or \( Q(x, y, z, t) \geq m \); i.e. either

\[
(x + hy + gz + ut)^2 + \varphi(y, z, t) \leq 0 \quad \text{or}
\]

\[
(x + hy + gz + ut)^2 + \varphi(y, z, t) \geq m \quad \text{or}
\]
Because of homogeneity it suffices to prove

**Theorem A:** Let

\[ Q(x, y, z, t) = (x + hy + gz + ut)^2 + \Phi(y, z, t) \]  \hspace{1cm} (3.8)

where \( \Phi(y, z, t) \) is an indefinite ternary quadratic form of determinant \( D < 0 \) such that

\[ |D| \geq \frac{3}{16}. \]  \hspace{1cm} (3.9)

Suppose that for integers \( x, y, z, t \) we have either

\[ Q(x, y, z, t) \leq 0 \quad \text{or} \quad Q(x, y, z, t) \geq 1 - \varepsilon, \]  \hspace{1cm} (3.10)

where \( 0 < \varepsilon < \frac{1}{16} \) is given sufficiently small. Let

\[ d = \left( \frac{1}{3} |D| \right)^{1/4}; \]  \hspace{1cm} (3.11)

so that from (3.9) we have

\[ d \geq \frac{1}{2}. \]  \hspace{1cm} (3.12)

Also from Lemma 1, \( d = \frac{1}{2} \) if and only if \( Q(x, y, z, t) = Q_2 \). Then given any real numbers \( x_0, y_0, z_0, t_0 \) we can find \( (x, y, z, t) \equiv (x_0, y_0, z_0, t_0) \pmod{1} \) such that

\[ |Q(x, y, z, t)| \leq d. \]  \hspace{1cm} (3.13)

The equality in (3.13) is needed if and only if \( Q(x, y, z, t) \) is the
§3.3. Proof of Theorem A:

Lemma 7: If \( Q(x, y, z, t) \) is given as in Theorem A, then for integers \( y, z, t \) we have either

\[
\begin{align*}
\varphi(y, z, t) &= 0 \quad \text{or} \\
\varphi(y, z, t) &\leq \frac{1}{4} \quad \text{or} \\
\varphi(y, z, t) &\geq \frac{3}{4} - \varepsilon
\end{align*}
\]  
(3.14)

Proof: The proof is similar to Lemma 4 of Chapter I.

Lemma 8: If \( d = \frac{1}{2} \), then \( Q = Q_2 = x^2 + xy + y^2 + zt \). In this case (3.13) is true with strict inequality.

Proof: If \( (x, y, z, t) \not\equiv (0, 0) \pmod{1} \); without loss of generality we can suppose that \( z \equiv 0 \pmod{1} \). Choose \( z \equiv z_0 \pmod{1} \) with \( 0 < |z| < \frac{1}{2} \).

Choose \( (x, y) \equiv (x_0, y_0) \pmod{1} \) arbitrarily, so that

\[
Q(x, y, z, t) = A + zt \quad (3.15)
\]

Now choose \( t \equiv t_0 \pmod{1} \) such that

\[
|Q(x, y, z, t)| = |A + zt| \leq \frac{1}{4} < \frac{1}{4} < d.
\]

Now suppose \( (z_0, t_0) \equiv (0, 0) \pmod{1} \). Take \( z = t = 0 \). Choose \( y \equiv y_0 \pmod{1} \) with \( |y| \leq \frac{1}{2} \) and then choose \( x \equiv x_0 \pmod{1} \) with \( |x + \frac{y}{2}| \leq \frac{1}{4} \), so that

\[
Q(x, y, z, t) = x^2 + xy + y^2 + zt = (x + \frac{y}{2})^2 + \frac{3}{4} y^2 
\]

\[
\leq \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} = \frac{7}{16} < \frac{1}{2} = d.
\]

This completes the proof of the lemma.
Lemma 9: Let \( \nu_1 > 0, \nu_2 > 0 \) be defined by

\[
\nu_1 = d - \frac{1}{4}
\]  \hspace{1cm} (3.16)

\[
\nu_2 = \begin{cases} 
    d^2 + \frac{1}{4} & \text{if } d \text{ is an integer} \\
    d + \left[ \frac{2d}{4} \right]^2 & \text{if } d \text{ is not an integer}
\end{cases}
\]  \hspace{1cm} (3.17)

Suppose that there exist \((y, z, t) = (y_0, z_0, t_0) \pmod 1\) satisfying

\[-\nu_2 \leq \phi(y, z, t) < \nu_1.\]  \hspace{1cm} (3.18)

Then for any \(x_0\) there exists \(x \equiv x_0 \pmod 1\) such that

\[|\phi(x, y, z, t)| \leq d\]  \hspace{1cm} (3.19)

Further strict inequality in (3.18) implies strict inequality in (3.19).

Proof: If

\[0 \leq \phi(y, z, t) < \nu_1,\]

choose \(x \equiv x_0 \pmod 1\) with \(|x + hy + gz + ut| \leq \frac{1}{2}\), so that

\[0 \leq \phi(x, y, z, t) < \nu_1 + \frac{1}{4} = d.\]

If \(-\nu_2 \leq \phi(y, z, t) < 0\), then the result follows from Lemma 5 with

\[\alpha = hy + gz + ut, \quad \beta^2 = -\phi(y, z, t).\]
Lemma 10: If \( \Phi(y,z,t) \) is equivalent to (2.4), then (3.19) is true with strict inequality.

Proof: Without loss of generality we can suppose

\[ \Phi(y,z,t) = \zeta(y^2 + zt), \quad \zeta > 0; \]

so that

\[ Q(x,y,z,t) = (x + hy + gz + ut)^2 + \zeta(y^2 + zt) \quad (3.20) \]

By replacing \( x \) by \( x + \alpha y + \beta z + \gamma t \), where \( \alpha, \beta, \gamma \) are suitable integers, we can suppose that

\[ |h| \leq \frac{1}{2}, \quad |g| \leq \frac{1}{2}, \quad |u| \leq \frac{1}{2} \quad (3.21) \]

We first assert that \( h = g = u = 0, \quad \zeta \geq 1 \). If \( g \neq 0 \), then

\[ 0 < Q(0,0,1,0) = g^2 \leq \frac{1}{4} < 1 - \xi, \]

from (3.21), contrary to (3.10). Similarly \( u = 0 \).

If \( h \neq 0 \), then

\[ 0 < Q(0,1,1,-1) = h^2 \leq \frac{1}{4} < 1 - \xi, \]

from (3.21), contrary to (3.10). Therefore,

\[ Q(x,y,z,t) = x^2 + \zeta(y^2 + zt). \]

If \( \zeta < 1 - \xi \), then

\[ 0 < Q(0,1,0,0) = \zeta < 1 - \xi, \]

contrary to (3.10). If \( 1 - \xi \leq \zeta < 1 \), then
0 < Q(1,0,1,-1) = 1 - \xi \leq \epsilon < 1 - \xi,
contrary to (3.10). Therefore, we have

\[ Q(x,y,z,t) = x^2 + \xi(y^2 + zt), \quad \xi \geq 1. \tag{3.22} \]

We will now distinguish between the following two subcases:

1. \( \xi \geq 7. \) \tag{3.23}

2. \( 1 \leq \xi < 7. \) \tag{3.24}

Subcase 1: \( \xi \geq 7. \)

If \( (z_0,t_0) \neq (0,0) \) (mod l), without loss of generality suppose \( z_0 \neq 0 \) (mod l). Choose \( z \equiv z_0 \) (mod l) with \( 0 < |z| \leq \frac{1}{2}. \) Choose \( y \equiv y_0 \) (mod l) with \( |y| \leq \frac{1}{2} \) and then choose \( t \equiv t_0 \) (mod l) to satisfy

\[ 0 \leq -(y^2 + zt) \leq |z| \leq \frac{1}{2}. \]

If \( (z_0,t_0) \equiv (0,0) \) (mod l); take \( z = 1, t = -1. \) Choose \( y \equiv y_0 \) (mod l) with \( \frac{1}{2} \leq |y| \leq 1, \) so that

\[ 0 \leq -(y^2 + zt) \leq \frac{3}{4}. \]

In either case we have

\[ \frac{3}{4} \xi \leq \Phi(y,z,t) = \xi(y^2 + zt) \leq 0 \tag{3.25} \]

Thus, \( \Phi(y,z,t) \) satisfies (3.18) if we have

\[ \frac{3}{4} \xi < d^2 + \frac{1}{4} = \left(\frac{\epsilon^3}{12}\right)^{1/2} + \frac{1}{4}. \]
i.e.

\[ f(\xi) = \frac{(3\xi - 1)^2}{3} < \frac{4}{3}. \]  

(3.26)

Since \( f(\xi) \) is a monotonically decreasing function of \( \xi \) and

\[ f(7) = \frac{400}{343} < \frac{4}{3}, \]  

(3.26) is true for \( \xi \geq 7 \). Thus (3.18) is satisfied

in this case and the result follows from Lemma 9.

Subcase (ii): Suppose \( 1 \leq \xi < 7 \).

If \((z_o, t_o) \neq (0,0) \) (mod 1), without loss of generality we
can suppose \( z_o \neq 0 \) (mod 1). Choose \( z \equiv z_o \) (mod 1) with \( 0 < |z| \leq \frac{1}{2} \).

Choose \( y \equiv y_o \) (mod 1) with \( |y| \leq \frac{1}{2} \) and then \( t \equiv t_o \) (mod 1) to

satisfy

\[ |y^2 + zt| \leq \frac{|z|}{2} \leq \frac{1}{4}. \]

If \((z_o, t_o) \equiv (0,0) \) (mod 1), take \( z = t = 0 \). Choose \( y \equiv y_o \) (mod 1)
with \( |y| \leq \frac{1}{2} \). In either case we can find \((y, z, t) \equiv (y_o, z_o, t_o) \)

(mod 1) such that

\[ |\varphi(y, z, t)| \leq \frac{\xi}{4}. \]

(3.28)

Hence (3.18) is true with strict inequality if we have

\[ \frac{\xi}{4} < \min(u_1, u_2) = \min(d - \frac{1}{4}, d^2 + \frac{1}{4}) = d - \frac{1}{4} \]

i.e. if

\[ \frac{\xi + 1}{4} < d = \left(\frac{6^3}{12}\right)^{1/4} \]

i.e. if
\[ f(\xi) = \xi^4 - \frac{52}{3} \xi^3 + 6 \xi^2 + 4 \xi + 1 < 0 \quad (3.29) \]

\( f(\xi) \) has two changes of sign, so it has at most two positive roots. Since \( f(0) > 0, f(1) = \frac{16}{3} < 0; \ f(7) = 64(64 - \frac{243}{3}) < 0, f(\infty) > 0, \)
so that (3.29) is true for \( 1 \leq \xi < 7 \) and the result follows from Lemma 9.

This completes the proof of Lemma 10.

**Lemma 11:** If \( d > \frac{9}{2} \), then (3.19) is true with strict inequality.

**Proof:** By Lemma 3 we can find \( (y, z, t) \equiv (y_0, z_0, t_0) \) (mod 1) such that

\[-(2^4d^4)^{1/3} = -(8|d|)^{1/3} \leq \varphi(y, z, t) < 0. \quad (3.30)\]

\( \varphi(y, z, t) \) satisfies (3.18) if we have

\[ (2^4d^4)^{1/3} < d^2 + \frac{1}{4}, \]

i.e. if

\[ f(d) = (d^2 + \frac{1}{4})^3 - 2^4d^4 > 0. \quad (3.31) \]

\( f(d) \) is an increasing function of \( d \) for \( d \geq \frac{4}{2}, f(5) > 0. \) Thus (3.18) is true if \( d \geq 5 \) and the result follows from Lemma 9.

Let now \( \frac{9}{2} < d < 5 \), so that \( [2d] = 9. \) In this case (3.30) implies (3.18) if we have

\[ (2^4d^4)^{1/3} < d + \frac{81}{4}, \]

This is so if
Since \( f(d) \) is an increasing function of \( d \) and \( f(5) = (25-25^2)^{1/2} - \frac{101}{4} < 0 \), (3.18) is satisfied for \( \frac{2}{5} < d < 5 \). Thus the lemma follows from Lemma 9.

§3.4. Proof of Theorem A cont’d:

From now on we can suppose \( \frac{1}{2} < d \leq \frac{9}{2} \).

\[
\chi(y,z,t) = \zeta(y^2 + zt), \quad \zeta > 0.
\]

Since \( \phi(y,z,t) \not\equiv \zeta(y^2 + zt) \), by Lemma 2 we can find integers \( y_2, z_2, t_2 \) such that

\[
0 < a = \phi(y_2, z_2, t_2) \leq \left( \frac{3}{4} |D| \right)^{1/3} = \left( \frac{27}{4} d^4 \right)^{1/3}
\]

where \( (y_2, z_2, t_2) = 1 \). Also by (3.14) we have

\[
a \geq \frac{3}{4} \varepsilon.
\]

By a unimodular transformation we can suppose \( \phi(1,0,0) = a \), so that

\[
\phi(y,z,t) = a \left\{ (y + f z + t)^2 + \gamma(z,t) \right\}
\]

where \( \gamma(z,t) \) is an indefinite binary quadratic form with discriminant

\[
\Delta = \frac{k|D|}{a^3} \geq \frac{16}{9},
\]

from (3.33). Let
\[ \lambda = \frac{(d - \frac{1}{k})^3}{3d^4} \quad (3.37) \]

Then \( \lambda \leq \frac{9}{64} \), maximum occurring at \( d = 1 \) \( (3.38) \)

For \( \frac{1}{2} < d \leq \frac{9}{2} \),

\[
\lambda \geq \min \left\{ \frac{(\frac{1}{2} - \frac{1}{k})^3}{3(\frac{1}{2})^4} , \frac{\left(\frac{9}{2} - \frac{1}{k}\right)^3}{3(\frac{9}{2})^4} \right\}
\]

\[
= \min \left(\frac{1}{12}, \frac{17^3}{12 \cdot 9^4}\right) = \frac{17^3}{12 \cdot 9^4}
\]

Therefore,

\[
k = \left(\frac{\lambda |D|}{a^3}\right)^{1/3} \geq \left(\frac{17^3}{12 \cdot 9^4}\right)^{1/3} \cdot \frac{4}{9}^{1/3} = \frac{17}{3^{11/3}} > \frac{1}{4} \quad (3.39)
\]

Also,

\[
k = \left(\frac{\lambda |D|}{a^3}\right)^{1/3} = \frac{d - \frac{1}{k}}{a} \quad (3.40)
\]

\[
\Delta^2 = \frac{k |D|}{a^3} = \frac{4k^3}{\lambda} \quad (3.41)
\]

Let \( \mu \) be the positive root of

\[
(\mu + 1)(\mu + 9) = \frac{\Delta^2}{(k - \frac{1}{4})^2} \quad (3.42)
\]

i.e.
\[ \mu = -5 + 4 \sqrt{1 + \frac{\Delta^2}{(4k - 1)^2}} \]
\[ = -5 + 4 \sqrt{1 + \frac{4k^3}{\lambda(4k - 1)^2}} \]
\[ \geq -5 + 4 \sqrt{1 + \frac{4}{\lambda(3 - 1)^2}} \]
\[ = -5 + 4 \sqrt{1 + \frac{27}{64\lambda}} \]
\[ \geq -5 + 4 \sqrt{1 + 3} = 3, \]

since \( \frac{k^3}{(4k - 1)^2} \) has its min at \( k = \frac{3}{4} \) and \( \lambda \leq \frac{9}{64} \) from (3.38) Thus \( \mu \geq 3 \).

By Lemma 5, we can find \((z,t) \equiv (z_o,t_o) \text{ (mod 1)}\) such that
\[ \frac{\mu \Delta}{(1 + \mu)(9 + \mu)^{1/2}} \leq \gamma(z,t) < \frac{\Delta}{(1 + \mu)(9 + \mu)^{1/2}} \]
\[ \text{i.e. } -\mu(k - \frac{1}{4}) \leq (z,t) < k - \frac{1}{4}, \text{ or} \]
\[ -\left\{ \sqrt{(4k - 1)^2 + \frac{4k^3}{\lambda}} - 5k + \frac{5}{4}\right\} \leq \gamma(z,t) < k - \frac{1}{4} \]
from (3.42) and (3.43).

Thus we have chosen \((z,t) \equiv (z_o,t_o) \text{ (mod 1)}\) to satisfy (3.45).

Therefore
\[ Q(x,y,z,t) = (x + hy + gz + ut)^2 + a\left( (y + fz + ut)^2 + \gamma(z,t) \right) \]
\[ \text{where } \gamma(z,t) \text{ satisfies (3.45).} \]
Lemma 12: If \( 0 \leq \gamma(z,t) < k - \frac{1}{4} \), \((3.19)\) is true with strict inequality.

Proof: Choose \( y \equiv y_0 \pmod{1} \) with \(|y + fz + ut| \leq \frac{1}{2} \), so that

\[
0 \leq \Phi(y,z,t) = \frac{1}{4} \left( y + fz + ut \right)^2 + \gamma(z,t)
\]

\[
< a \left( \frac{1}{4} + k - \frac{1}{4} \right) = ak = d - \frac{1}{4}.
\]  
from \((3.40)\).

Now choose \( x \equiv x_0 \pmod{1} \) with \(|x + hy + gz + ut| \leq \frac{1}{2} \), so that

\[
0 \leq \Psi(x,y,z,t) < \frac{1}{4} + d - \frac{1}{4} = d,
\]
from \((3.46)\) and \((3.47)\). This completes the proof of this Lemma.

Therefore from now on we can suppose

\[
0 < \beta = -\gamma(z,t) \leq \left( (4k - 1)^2 + \frac{4k^3}{\lambda} \right)^{1/2} - 5k + \frac{2}{4}  \]  
\( (3.48) \)

Lemma 13: If \( \beta + k \leq 1 \), then \((3.19)\) is true with strict inequality.

Proof: Choose \( y \equiv y_0 \pmod{1} \) with \(|y + fz + ut| \leq \frac{1}{2} \), so that

\[
-(a + \frac{1}{4} - d) = -a(1 - k) \leq \beta \leq \Phi(y,z,t) = a \left( y + fz + ut \right)^2 - \beta
\]

\[
\leq a \frac{1}{4} < ak = d - \frac{1}{4},
\]  
from \((3.39)\) and \((3.40)\).

Therefore \((3.18)\) is satisfied if we have

\[
a + \frac{1}{4} - d < d^2 + \frac{1}{4}.
\]

This is so if
\[ a < d + d^2 \] or

\[ a \leq \left( \frac{27}{4} d^4 \right)^{1/3} < a(d + 1) \] or

\[ f(d) = \frac{(1 + d)^3}{d} > \frac{27}{4} \quad (3.50) \]

\( f(d) \) is a monotonically increasing function of \( d \) for \( d > \frac{1}{2} \). Since

\( f\left(\frac{1}{2}\right) = \frac{27}{4} \), (3.50) and (3.18) are satisfied and the result follows from Lemma 9.

**Lemma 14:** If \( \beta + k > 1 \) and \( \frac{3}{2} \leq d \leq \frac{9}{2} \), then (3.19) is true with strict inequality.

**Proof:** Choose \( y = y_0 \pmod{1} \) such that

\[ \sqrt{\beta + k - 1} \leq y + f_z + u < \sqrt{\beta + k} \quad (3.51) \]

so that,

\[ a + d - \frac{1}{4} - 2a \sqrt{\beta + k} = a(1 + k - 2\sqrt{\beta + k}) \]

\[ \leq \Phi(y, z, t) < ak = \hat{a} - \frac{1}{4} \quad (3.52) \]

Then the result will follow from Lemma 9 if we have

\[ a + d - \frac{1}{4} - 2a \sqrt{\beta + k} > -(d^2 + \frac{1}{4}) \]

i.e.

\[ 2a \sqrt{\beta + k} < d + d^2 + a \quad (3.53) \]

This is so if
\[
4a^2 \left\{ \frac{(4d - 1 - a)^2}{a^2} + \frac{12d^4}{a^3} \right\}^{1/2} < d^2(1 + a)^2 + 2ad(9 + d) - 4a(1 + a)
\]
i.e. if

\[
f(a, d) = 16a^3(1 + d) - 4a^2(1 + a)(15d - d^2 - 4) - 4ad(d^4 + 11d^3 - 31d^2 + 5a - 2) - d^3(1 + d)^4 < 0
\]

(3.54)

where \( \frac{3}{4} - \varepsilon \leq a \leq \left( \frac{27}{4} d^4 \right)^{1/3} \).

\[
\frac{\partial f}{\partial a} = 48a^2(1 + a) - 8a(1 + d)(15d - d^2 - 4) - 4d(d^4 + 11d^3 - 31d^2 + 5a - 2)
\]

\[
= 48(1 + d)(a - \alpha_1)(a - \alpha_2),
\]

(3.55)

where

\[
\alpha_1, \alpha_2 = \frac{(1 + d)(15d - d^2 - 4)}{12(1 + d)}
\]

\[
+ \frac{\left\{ (1 + a)(15d - d^2 - 4)^2 + 12a(1 + d)(d^4 + 11d^3 - 31d^2 + 5a - 2) \right\}^{1/2}}{12(1 + d)}
\]

\[
\alpha_1 \geq \alpha_2.
\]

Now \( \alpha_2 < \frac{3}{4} \), if

\[
(1 + a)\left\{ (15d - d^2 - 4) - 9 \right\} < \left\{ (1 + a)^2(15d - d^2 - 4)^2 + 12a(1 + d) \right\}^{1/2} \cdot (d^4 + 11d^3 - 31d^2 + 5a - 2)
\]

The above is true if
\[ f(d) = 4d^5 + 44d^4 - 130d^3 + 104d^2 + 31d - 51 > 0. \]

Since \( f(d) \) is increasing for \( d > 1 \), \( f(1) > 0 \), we have \( \alpha_2 < \frac{3}{4} \) if \( d > 1 \).

Therefore,

\[ \frac{\partial f}{\partial d} < 0 \quad \text{if} \quad \frac{3}{4} - \varepsilon < d < \alpha_1, \]

\[ \frac{\partial f}{\partial d} > 0 \quad \text{if} \quad \alpha_1 < d < (\frac{27}{4} d^4)^{1/3}. \]

Therefore,

\[
\max_{\frac{3}{4} - \varepsilon < d < (\frac{27}{4} d^4)^{1/3}} f(a,d) = \max \left\{ f\left(\frac{3}{4} d, a\right), f\left((\frac{27}{4} d^4)^{1/3}, a\right) \right\} \tag{3.56}
\]

\[ f\left(\frac{3}{4} d, a\right) = \frac{1}{4} \left[ 27(1 + d) - 9(1 + d)(15d - d^2 - 4) \right. \]
\[ - 12d(d^4 + 11d^3 - 31d^2 + 5d - 2) - 4d^3(1 + d)^4 \left. \right]\]
\[ = \frac{1}{4} \left[ 4d^3(1 + d)^4 + 12d^5 + 132d^4 - 381d^3 + 186d^2 + 48d - 63 \right]\]
\[ < \frac{1}{4} \left[ 2d^2(4d(1 + d)^4 - 113) + 12d^5 + 33d^2(2d - 3)^2 \right. \]
\[ + 15d^3 + 24(2d - 3) + 9 \left. \right]\]
\[ < \frac{1}{4} \left[ d^2\left(\frac{1875}{8} - 113\right) + 12d^5 + 33d^2(2d - 3)^2 \right. \]
\[ + 15d^3 + 24(2d - 3) + 9 \left. \right], \quad \text{if} \quad d \geq \frac{3}{2} \]
\[ < 0 \quad \text{if} \quad d \geq \frac{3}{2}. \tag{3.57} \]

\[ f\left((\frac{27}{4} d^4)^{1/3}, a\right) = 108d^4 (1 + d) - 4(1 + d)(\frac{27}{4} d^4)^2/3(15d - d^2 - 4) \]
\[ - 4(\frac{27}{4} d^4)^{1/3}d(d^4 + 11d^3 - 31d^2 + 5d - 2) \]
\[ - d^3(1 + d)^4. \]
The right hand side is a decreasing function for \( d \geq \frac{3}{2} \) and < 0 for \( d = \frac{3}{2} \). So that

\[
f((\frac{27}{4} d^4)^{1/3}, d) < 0 \text{ if } d \geq \frac{3}{2}.
\] (3.58)

Therefore (3.54) is true from (3.56), (3.57), and (3.58). This completes the proof of the lemma.

**Lemma 15:** If \( \beta + k > 1 \) and \( 1 < d < \frac{3}{2} \), then (3.19) is true with strict inequality.

This lemma follows from

**Lemma 16:** (3.19) is true with strict inequality if

i) \( \beta + k > 1, 1 < d < \frac{3}{2}, \frac{3}{4} - E \leq a \leq 2 \). (3.59)

ii) \( \beta + k > 1, 1 < d < \frac{3}{2}, 2 < a \leq (\frac{27}{4} d^4)^{1/3} \) (3.60)

**Proof of (i):** Choose \( y = y_0 \pmod{1} \) as in (3.51), so that (3.52) is true. Since \([2d] = 2\), the result follows from Lemma 9 if we have

\[
a + d - \frac{1}{4} - 2a\sqrt{\beta + k} > -(d + [\frac{2d}{4}]) = -d - 1.
\]

If

\[
2a\sqrt{\beta + k} < 2d + a + \frac{3}{4}.
\]

If

\[
4a^2\sqrt{(4k - 1)^2 + \frac{hk}{\lambda}} + 5a^2 - 16a^2k < (2d + a + \frac{3}{4})^2.
\]

Using (3.37), (3.40), a slight calculation shows that this is so if
\[ f(a, d) = 4a^3(8d + 3) - 4a^2(28d^2 + d - \frac{57}{16}) \\
+ 4a(48d^4 - 40d^3 - 25d^2 - \frac{15}{8}d + \frac{45}{64}) - (2d + \frac{3}{4})^4 < 0. \]  
(3.61)

\[ \frac{\partial f}{\partial a} = 12a^2(8d + 3) - 8a(28d^2 + d - \frac{57}{16}) + 4(48d^4 - 40d^3 - 25d^2 \\
- \frac{15}{8}d + \frac{45}{64}) \\
= 12(8d + 3)(a - \alpha_1)(a - \alpha_2), \]  
(3.62)

where

\[ \alpha_1, \alpha_2 = \frac{28d^2 + d - \frac{57}{16}}{3(8d + 3)} \]

\[ \left\{ \frac{(28d^2 + d - \frac{57}{16})^2 - 3(8d + 3)(48d^4 - 40d^3 - 25d^2 - \frac{15}{8}d + \frac{45}{64})^{1/2}}{3(8d + 3)} \right\}^{1/2} \]

\[ \alpha_1 \geq \alpha_2 \]

\[ \alpha_2 < \frac{3}{4}; \text{ if} \]

\[ \left\{ 28d^2 + d - \frac{57}{16} - \frac{9}{4}(8d + 3) \right\} \left\{ (28d^2 + d - \frac{57}{16})^2 - 3(8d + 3) \cdot (48d^4 - 40d^3 - 25d^2 - \frac{15}{8}d + \frac{45}{64}) \right\}^{1/2} \]

i.e. if

\[ f(d) = 48d^4 - 40d^3 - 67d^2 + \frac{81}{8}d + \frac{711}{64} < 0 \]  
(3.63)

By the rule of signs \( f(d) \) has at most two positive real roots.

Since \( f(0) > 0, f(1) < 0, f\left(\frac{3}{2}\right) < 0, f(\infty) > 0 \), \( f(d) \) has no root.
between 1 and \( \frac{3}{2} \). Therefore \( f(d) \) does not change sign in this interval and (3.63) is true. Hence,

\[
\max_{\frac{3}{4} \leq a < \infty} f(a,d) = \max \left\{ f\left( \frac{3}{4},d \right), f(2,d) \right\}
\]  

(3.64)

\[
f\left( \frac{3}{4},d \right) = \frac{27}{16}(8d + 3) - \frac{9}{4}(28d^2 + d - \frac{27}{16}) + 3(48d^4 - 40d^3 - 25d^2
\]

\[
- \frac{15}{8}d + \frac{45}{64} - (2d + \frac{3}{4})^4
\]

\[
= 128d^4 - 144d^3 - \frac{303}{2}d^2 + \frac{9}{4}d + \frac{81}{16}(3 - \frac{1}{16})
\]  

(3.65)

Again by the rule of signs it has at most two true real roots and \( f\left( \frac{3}{2},0 \right) > 0, f\left( \frac{3}{4},1 \right) < 0, f\left( \frac{3}{4}, \frac{3}{2} \right) < 0, f(\infty) > 0 \). Thus,

\[
f\left( \frac{3}{4},d \right) < 0 \text{ for } 1 < d < \frac{3}{2}
\]  

(3.66)

\[
f(2,d) = 32(8d + 3) - 16(28d^2 + d - \frac{27}{16}) + 6(48d^4 - 40d^3 - 25d^2
\]

\[
- \frac{15}{8}d + \frac{45}{64} - 16d^4 - 24d^3 - \frac{27}{2}d^2 - \frac{27}{8}d - \frac{81}{256}
\]

\[
< 368d^4 - 344d^3 - \frac{1323}{2}d^2 + 222d + 159
\]  

(3.67)

By the same argument as above the right hand side of (3.67) < 0 for \( 1 < d < \frac{3}{2} \).

Therefore from (3.63), (3.64), (3.66), and (3.67), (3.61) is true and the result follows in this case.

**Proof of 16 (ii):** If \( \beta + k > \frac{9}{4} \), choose \( y = y_0 \) (mod 1) such that

\[1 \leq |y + fz + ut| \leq \frac{3}{2} \]
Therefore,

\[-a(\beta - 1) \leq \varphi(y, z, t) = a \left\{ (y + rz + ut)^2 - \beta \right\} \]

\[\leq a \left( \frac{9}{4} - \beta \right) < ak = a - \frac{1}{4} \]

(3.68)

The result will follow from Lemma 9 if we have

\[a(\beta - 1) < d + 1. \]

(3.69)

Using (3.48), (3.37), (3.40), (3.69) is true if we have,

\[(4d - 1 - a)^2 + \frac{12d^4}{a} < (6d - \frac{1}{4} - a)^2\]

i.e. if

\[r(a, d) = \frac{15}{16} a^3 - 5a^2(d - \frac{3}{8}) - 5a(2d + \frac{3}{4})(2d - \frac{1}{4}) + 12d^4 < 0 \]

(3.70)

\[\frac{\partial r}{\partial a} = 5 \left( \frac{9}{16} a^2 - a(2d - \frac{3}{4}) - (2d + \frac{3}{4})(2d - \frac{1}{4}) \right)\]

\[= \frac{15}{16}(a + a_1)(a - a_2), \]

(3.71)

where \(a_1 > 0,\)

\[a_2 = \frac{2d - \frac{3}{4} + \left\{ (2d - \frac{3}{4})^2 + \frac{9}{4}(2d + \frac{3}{4})(2d - \frac{1}{4}) \right\}^{1/2}}{9} \]

For \(1 < d < \frac{3}{2};\)
\[ \alpha_2 > \frac{2 - \frac{3}{4} + \left(\frac{25}{16} + \frac{2}{4} \cdot \frac{11}{4} \cdot \frac{7}{4^2}\right)}{\frac{9}{8}} \]

\[ = \frac{\frac{5}{4} + \sqrt{\frac{793}{6^4}}}{\frac{9}{8}} > \frac{1}{9}(10 + 28) = \frac{38}{9} > \left(\frac{27}{4} d^4\right)^{1/3} \]

Therefore, in this case we have

\[ 2 < a < \left(\frac{27}{4} d^4\right)^{1/3} < \alpha_2. \]

Consequently \( \frac{\partial f}{\partial a} < 0 \) for \( 2 < a < \left(\frac{27}{4} d^4\right)^{1/3} \). Therefore

\[ \max_{2 < a < \left(\frac{27}{4} d^4\right)^{1/3}} f(a, d) = f(2, d) \]

\[ = \frac{15}{2} - 20(d - \frac{3}{4}) - 10(2d + \frac{3}{4})(2d - \frac{1}{4}) + 12d^4 \]

\[ = 12d^4 - 40d^2 - 30d + \frac{135}{8} \]

\[ = 4d^2(3d^2 - 10) + \frac{135}{8} - 30d \]

\[ < 0 \quad \text{for} \quad 1 < d < \frac{3}{2} \]

Hence (3.70) and so (3.69) is true and the result follows.

Suppose now \( 1 < \beta + k \leq \frac{9}{4} \).

Choose \( y \equiv y_0 \pmod{1} \) with \( \frac{1}{2} \leq |y + fz + ut| \leq 1 \), so that

\[ -a(\beta - \frac{1}{4}) \leq \phi(y, z, t) \leq a(1 - \beta) < ak = d - \frac{1}{4} \quad (3.73) \]

The result will follow from Lemma 9 if we have

\[ a(\beta - \frac{1}{4}) < d + 1 \quad (3.74) \]
Using (3.48), (3.37), and (3.40) (3.74) is true if

\[(4d - 1 - a)^2 + \frac{12d}{a} < (6d - a - \frac{1}{4})^2\]

i.e.

\[f(a,d) = 2a^2(2d + \frac{3}{4}) - 5a(2d - \frac{1}{4})(2d + \frac{3}{4}) + 12d^4 < 0 \quad (3.75)\]

\[\frac{\partial f}{\partial a} = (2d + \frac{3}{4}) \left\{ 4a - 5(2d - \frac{1}{4}) \right\} \leq (2d + \frac{3}{4}) \left\{ 4d + \frac{27}{4}d^4 \right\}^{1/3} - 5(2d - \frac{1}{4}) < 0, \]

if

\[\frac{125}{64} (2d - \frac{1}{4})^3 > \frac{27}{4} d^4\]

if

\[g(d) = \frac{(2d - \frac{1}{4})^3}{d^4} > \frac{27 \cdot 16}{125}, \quad \text{for } 1 < d < \frac{3}{2} \quad (3.76)\]

Now \(g(d)\) is a monotonically decreasing function of \(d\), so that

\[g(d) > g\left(\frac{3}{2}\right) = \frac{11^3 \cdot 16}{4^3 \cdot 3^4} > \frac{27 \cdot 16}{125}, \]

Therefore \(f(a,d)\) is monotonically decreasing and so

\[\max_{2 < a < (\frac{27d^4}{4})^{1/3}} f(a,d) = f(2,d)\]

\[= 8(2d + \frac{3}{4}) - 10(2d - \frac{1}{4})(2d + \frac{3}{4}) + 12d^4\]

\[= 12d^4 - 40d^2 + 6d + \frac{63}{8}\]

\[< 0, \quad \text{for } 1 < d < \frac{3}{2}. \]
Hence (3.75) is true and the result follows.

This completes the proof of Lemma 16 and hence of Lemma 15.

**Lemma 17:** If $\beta + k > 1$ and $\frac{1}{2} < d \leq 1$, then again (3.19) is true.

**Proof:** If $\beta + k > \frac{9}{4}$, choose $y \equiv y_0 \pmod{1}$ with $1 \leq |y + fz + ut| \leq \frac{3}{2}$, so that

$$-a(\beta - 1) \leq d_{y,z,t} = a \left\{(y + fz + ut)^2 - \beta \right\} \leq a\left(\frac{9}{4} - \beta\right) < ak = d - \frac{1}{4} \quad (3.77)$$

The result will follow from Lemma 9 if we have

$$a(\beta - 1) < d + \left\lfloor \frac{2d}{4} \right\rfloor = d + \frac{1}{4},$$

i.e. if

$$a\beta < d + a + \frac{1}{4} \quad (3.78)$$

Using (3.48), (3.37), and (3.40) (3.78) is true if

$$(4d - 1 - a)^2 + \frac{12d^4}{a} < (6d - 1 - \frac{a}{4})^2$$

This is so if

$$f(a,d) = \frac{15}{16} a^3 - a^2(5d - \frac{3}{2}) - 4ad(5d - 1) + 12d^4 < 0 \quad (3.79)$$

$$\frac{df}{da} = \frac{15}{16} a^2 - 2a(5d - \frac{3}{2}) - 4d(5d - 1)$$

$$= \frac{15}{16} (a + \alpha_1)(a - \alpha_2), \quad \alpha_1 > 0,$$

and
\[ \alpha_2 = \frac{5d - \frac{3}{2} + \left(\frac{5d - \frac{3}{2}}{2}\right)^2 + \frac{9}{4}(5d - 1)d^{3/2}}{\frac{45}{16}} > \left(\frac{27}{4} \cdot d^4\right)^{1/3} \]  
for \( \frac{1}{2} < d < 1 \).

Therefore \( \frac{\partial f}{\partial a} < 0 \) if \( \frac{3}{4} - \varepsilon \leq a \leq \left(\frac{27}{4} \cdot d^4\right)^{1/3} \).

Hence

\[ \max_{\frac{3}{4} - \varepsilon \leq a \leq \left(\frac{27}{4} \cdot d^4\right)^{1/3}} f(a, d) = f\left(\frac{3}{4}, d\right) + o(\varepsilon) \quad (3.80) \]

\[ f\left(\frac{3}{4}, d\right) = 12d^4 - 3d(5d - 1) - \frac{9}{16}(5d - \frac{3}{2}) + \frac{15}{16} \cdot \frac{27}{64} \]

\[ = 12d^4 - 15d^2 + \frac{3}{16} d + \frac{27}{32} + \frac{15 \cdot 27}{16 \cdot 64} \]

\[ < 0 \quad \text{for} \quad \frac{1}{2} < d < 1, \quad (3.81) \]

by using the argument about the rule of signs for positive roots.

Hence we are through with strict inequality if \( \beta + k > \frac{9}{4} \).

Now suppose \( 1 < \beta + k \leq \frac{9}{4} \). \quad (3.82)

Choose \( y = y_0 \pmod{1} \) with \( \frac{1}{2} \leq |y + fz + ut| \leq 1 \), so that

\[-a(\beta - \frac{1}{4}) \leq \varphi(y, z, t) = a\left[(y + fz + ut)^2 - \beta\right] \leq a(1 - \beta) \]

\[ < ak = d - \frac{1}{4} \quad (3.83) \]

By Lemma 9, the result will follow if we have

\[ a(\beta - \frac{1}{4}) \leq d + \frac{1}{4} \quad (3.84) \]

Now we will distinguish the following two subcases.
1. \( a < d \)  
   (3.85)

2. \( a \geq d \).  
   (3.86)

Subcase 1: \( a < d \).

In this case from (3.82) and (3.40) we have

\[
a(\beta - \frac{1}{4}) \leq a\left(\frac{9}{4} - k - \frac{1}{4}\right) = 2a - d + \frac{1}{4} < 2d - d + \frac{1}{4} = d + \frac{1}{4},
\]

and the result is true.

Subcase 2: \( a \geq d \).

In this case (3.84) is true if

\[
a^3 \leq d + \frac{1}{4} + \frac{a}{4}
\]

i.e. if

\[
a\left(4k - 1\right)^2 + \frac{4k^3}{\lambda} \leq d + \frac{1}{4} + \frac{a}{4} - \frac{5a}{4} + 5d - \frac{1}{4} = 6d - a - 1
\]

i.e. if

\[
(4d - 1 - a)^2 + \frac{12d^4}{a} \leq (6d - a - 1)^2
\]

i.e. if

\[
\frac{12d^4}{a} \leq (10d - 2a - 2) - 2d
\]

i.e. if

\[
a(a + 1 - 5d) + 3d^3 \leq 0
\]

i.e. if
\[ f(a, d) = a^2 - (5d - 1)a + 3d^3 \leq 0 \]  
(3.87)

\[ f(a, d) = (a - \alpha_1)(a - \alpha_2), \]  
(3.88)

where

\[ \alpha_1, \alpha_2 = \frac{5d - 1 \pm \sqrt{(5d - 1)^2 - 12d^3}}{2}, \]

\[ \alpha_1 > \alpha_2 \]

Now \( \alpha_2 \leq d \) if we have

\[ (5d - 1) - \sqrt{(5d - 1)^2 - 12d^3} \leq 2d \]

if

\[ (3d - 1)^2 \leq (5d - 1)^2 - 12d^3 \quad \text{or} \]

\[ 12d^3 \leq 4d(4d - 1) \quad \text{or} \]

\[ 3d^2 - 4d + 1 = (3d - 1)(d - 1) \leq 0 \]  
(3.89)

which is true since \( \frac{1}{2} < d \leq 1. \)

Therefore,

\[ \alpha_2 \leq d \]  
(3.90)

Also since \( \alpha_1 \alpha_2 = 3d^3 \), we have

\[ \alpha_1 \geq 3d^2 > \left(\frac{27}{4} d^4\right)^{1/3} \quad \text{(since } d > \frac{1}{2}) \]

Therefore,
\[ \alpha_2 \leq d \leq a \leq \left( \frac{27}{16} d^4 \right)^{1/3} < \alpha_1. \] (3.91)

Hence from (3.88) we have \[ f(a,d) \leq 0 \] and the lemma follows from Lemma 9.

4. The case of equality:

Lemma 18: Equality is needed in (3.19) if and only if \[ Q(x,y,z,t) = Q_1 \] and \[ (x_0, y_0, z_0, t_0) = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \text{(mod 1)}. \]

Proof: Equality can occur only if we have equality in (3.88). This from (3.91) will be so only if \[ a = d = \alpha_2. \] Since \[ \alpha_2 = d, \] (3.89) must be true with equality so that

\[ a = d = 1 \] (4.1)

From (3.36), (3.37), (3.40), (3.43), we have

\[ \lambda = \frac{9}{64}, \quad k = \frac{3}{4}, \quad \mu = 3, \quad \Delta^2 = 12. \] (4.2)

Also if equality is to occur we must have equality in (3.48).

Therefore, from Lemma 5 we must have

\[ \mathcal{K}(z,t) \sim c\mathcal{K}_1 = c(3z^2 - t^2), \] (4.3)

where \( c > 0 \) and \( (z_0, t_0) \equiv \left( \frac{1}{2}, \frac{1}{2} \right) \text{(mod 1)}. \)

Since \( \Delta^2 = 12 \) from (4.2), we have \( c = 1. \) Therefore,

\[ Q(x,y,z,t) = (x + hy + gz + ut)^2 + (y + fz + ut)^2 + 3z^2 - t^2 \] (4.4)

By a suitable unimodular transformation we can suppose that

\[ 0 \leq h \leq \frac{1}{2}, \quad |g| \leq \frac{1}{2}, \quad |u| \leq \frac{1}{2}, \quad 0 \leq f \leq \frac{1}{2}, \quad |v| \leq \frac{1}{2} \] (4.5)
Again, if equality is to occur in (3.19), (3.18) should not be true with strict inequality when \( d = 1 \). That is

\[
\frac{5}{4} < F(y, z, t) = (y + fz + yt + \frac{f}{2} + \frac{v}{2} + y_0)^2 + 3(z + \frac{1}{2})^2 - (t + \frac{1}{2})^2 < \frac{3}{4}
\] (4.6)

should have no solutions in integers \( y, z, t \). Now

\[
\frac{5}{4} < F(y, 0, 0) = (y + \frac{f}{2} + \frac{v}{2} + y_0)^2 + \frac{1}{2} < \frac{3}{4}
\]

is solvable for integer \( y \), unless,

\[
\frac{f}{2} + \frac{v}{2} + y_0 = \frac{1}{2} \pmod{1} \] (4.7)

Similarly by considering \( F(y, -1, 0), F(y, 0, -1) \), if equality is to occur we must have

\[
\frac{f}{2} + \frac{v}{2} + y_0 = \frac{1}{2} \pmod{1} \] (4.8)

\[
\frac{f}{2} - \frac{v}{2} + y_0 = \frac{1}{2} \pmod{1} \] (4.9)

From (4.7), (4.8) and (4.9) we have

\[
f \equiv v \equiv 0 \pmod{1}, \quad y_0 \equiv \frac{1}{2} \pmod{1} \] (4.10)

From (4.5) we have

\[
f = v = 0, \quad y_0 \equiv \frac{1}{2} \pmod{1} \] (4.11)

Hence
\( \varphi(y, z, t) = y^2 + 3z^2 - t^2 \quad (4.12) \)

and \((y_0, z_0, t_0) \equiv (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \pmod{1} \).

Again if equality is to occur in (3.19), the inequalities

\[-1 < G(x, y, z, t) = (x + hy + gz + ut + \frac{h}{2} + \frac{g}{2} + \frac{u}{2} + x_o)^2 + (y + \frac{1}{2})^2 + 3(z + \frac{1}{2})^2 - (t + \frac{1}{2})^2 < 1 \quad (4.13)\]

should have no solutions in integers \(x, y, z, t\). Now

\[-1 < G(x, 0, 0, 0) = (x + \frac{h}{2} + \frac{g}{2} + \frac{u}{2} + x_o)^2 + \frac{3}{4} < 1 \]

is solvable for integer \(x\) unless

\[\frac{h}{2} + \frac{g}{2} + \frac{u}{2} + x_o \equiv 0 \pmod{1} \quad (4.14)\]

Similarly, by considering \(G(x, -1, 0, 0), G(x, 0, -1, 0)\) and \(G(x, 0, 0, -1)\), in order that equality should occur, we must have

\[-\frac{h}{2} + \frac{g}{2} + \frac{u}{2} + x_o \equiv \frac{1}{2} \pmod{1} \quad (4.15)\]

\[\frac{h}{2} - \frac{g}{2} + \frac{u}{2} + x_o \equiv \frac{1}{2} \pmod{1} \quad (4.16)\]

\[\frac{h}{2} + \frac{g}{2} - \frac{u}{2} + x_o \equiv \frac{1}{2} \pmod{1} \quad (4.17)\]

From (4.14), (4.15), (4.16) and (4.17) we have,

\[h = g = u \equiv 0 \pmod{1}, \quad x_o = \frac{1}{2} \pmod{1} \quad (4.18)\]

From (4.5) and (4.18) we get,
Therefore, equality can occur only if

\[ Q(x, y, z, t) = x^2 + y^2 + 3z^2 - t^2 = q_1 \]

and

\[ (x_0, y_0, z_0, t_0) \equiv \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \pmod{1}. \]

We next show that equality is needed for the form \( q_1 \) when

\[ (x_0, y_0, z_0, t_0) \equiv \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \pmod{1}. \]

For this it suffices to show that for all integers \( x, y, z, t \) we have

\[ |(x + \frac{1}{2})^2 + (y + \frac{1}{2})^2 + 3(z + \frac{1}{2})^2 - (t + \frac{1}{2})^2| \geq 4 \quad (4.20) \]

i.e. \( |x^2 + y^2 + 3z^2 - t^2| \geq 4 \), for all odd integers \( X, Y, Z, T \).

Since \( X^2 + Y^2 + 3Z^2 - T^2 = 1 + 1 + 3 - 1 \equiv 4 \pmod{8} \) for odd integers \( X, Y, Z, T \); \( (4.20) \) is true and the assertion is proved. This completes the proof of Lemma 18.

The proof of Theorem A follows from Lemmas 8 to 18 and the theorem is proved.
CHAPTER III
POSITIVE VALUES FOR INHOMOGENEOUS QUATERNARY
QUADRATIC FORMS OF SIGNATURE 2

§ 1. In this chapter we prove the following theorem:

Theorem: Let \( Q(x,y,z,t) \) be an indefinite quaternary quadratic form with determinant \( D < 0 \) and signature 2. Then given any real numbers \( x_0, y_0, z_0, t_0 \) we can find integers \( x, y, z, t \) such that

\[
0 < Q(x + x_0, y + y_0, z + z_0, t + t_0) \leq \left( \frac{16}{3} |D| \right)^{1/4} \tag{1.1}
\]

Equality is needed if and only if either

\[
Q(x, y, z, t) \sim Q_1 = \zeta (x^2 + xy + y^2 + zt) \quad \text{or} \tag{1.2}
\]

\[
Q(x, y, z, t) \sim Q_2 = \zeta (x^2 + y^2 + z^2 + t^2). \tag{1.3}
\]

where \( \zeta > 0 \). For \( Q_1 \) equality occurs if and only if

\[
(x_0, y_0, z_0, t_0) \equiv (0, 0, 0, 0) \pmod{1}
\]

and for \( Q_2 \) if and only if

\[
(x_0, y_0, z_0, t_0) \equiv (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \pmod{1}.
\]

§ 2. Some Lemmas:

In the course of the proof we shall use the following Lemmas:

Lemma 1: Let \( \varphi(y, z, t) \) be an indefinite ternary quadratic form with determinant \( D < 0 \). Then given any real numbers \( y_0, z_0, t_0 \) we can find

\[
(y, z, t) \equiv (y_0, z_0, t_0) \pmod{1}
\]

such that

\[
|\varphi(y, z, t)| \leq \left( \frac{27}{100} |D| \right)^{1/3} \tag{2.1}
\]
This is a theorem due to Davenport [13].

**Lemma 2:** Let \( \gamma(z,t) \) be an indefinite binary quadratic form with discriminant \( \Delta^2 > 0 \) and let \( \lambda > 0 \) be a given real. Then for any real numbers \( z_0, t_0 \) we can find \( (z,t) \equiv (z_0, t_0) \) (mod 1) satisfying

\[
\frac{-\Delta}{4\lambda} \leq \gamma(z,t) < \frac{\lambda}{4} \Delta \tag{2.2}
\]

Equality is necessary if and only if \( \lambda^2 = \frac{m + 2}{m} \); \( m = 1, 2, \ldots \) and

\[
\gamma(z,t) \sim c\gamma_m(z,t) = c(z^2 - m(m + 2)t^2) \tag{2.3}
\]

For \( \gamma_m(z,t) \) equality occurs if and only if \( (z_0, t_0) \equiv (\frac{m^2}{2}, \frac{1}{2}) \) (mod 1).

This is Theorem 1 of Blaney [8].

**Lemma 3:** Let \( \alpha, \beta, d \) be real numbers with \( d \geq 1 \). Then for any real number \( x_0 \), there exists \( x \equiv x_0 \) (mod 1) satisfying

\[
0 < (x + \alpha)^2 - \beta^2 \leq d, \tag{2.4}
\]

provided

\[
\beta^2 < (\lceil \frac{d}{2} \rceil)^2. \tag{2.5}
\]

If \( d \) is not an integer, \( (2.4) \) is true with strict inequality. If \( d \) is an integer a sufficient condition for \( (2.4) \) to be true with strict inequality is that

\[
\beta^2 < \left( \frac{d - 1}{2} \right)^2. \tag{2.6}
\]

**Proof:** If \( \beta^2 < (\frac{d - 1}{2})^2 \), choose \( x \equiv x_0 \) (mod 1) with
\[ |\beta| < x + \alpha \leq |\beta| + 1, \quad (2.7) \]

so that

\[ 0 < (x + \alpha)^2 - \beta^2 \leq 2|\beta| + 1 < d, \]

using (2.6).

If \( d \) is an integer and \( \beta^2 = (\frac{d - 1}{2})^2 \), then we choose \( x \equiv x_0 \pmod{1} \) as in (2.7) and get the result perhaps with equality.

Now suppose

\[ \beta^2 \begin{cases} \geq \left(\frac{d - 1}{2}\right)^2 & \text{if } d \text{ is not an integer}, \\ > \left(\frac{d - 1}{2}\right)^2 & \text{if } d \text{ is an integer}, \end{cases} \]

so that in either case

\[ \beta^2 > \left(\frac{[d]}{2} - \frac{1}{2}\right)^2. \quad (2.8) \]

Choose \( x \equiv x_0 \pmod{1} \) to satisfy

\[ \frac{[d]}{2} \leq |x + \alpha| \leq \frac{[d]}{2} + 1, \quad (2.9) \]

so that

\[ 0 < \left(\frac{[d]}{2}\right)^2 - \beta^2 \leq (x + \alpha)^2 - \beta^2 < \left(\frac{[d]}{2} + 1\right)^2 - \left(\frac{[d]}{2} - \frac{1}{2}\right)^2 = [d] \leq d, \quad (2.10) \]

from (2.5) and (2.8).

This completes the proof of the lemma.
§3. Proof of the Theorem:

Let

\[ m = \inf_{x,y,z,t \text{ integ.}} Q(x,y,z,t) \quad (3.1) \]

\[ Q(x,y,z,t) > 0 \]

§3.1. Case \( m = 0 \).

Lemma 4: If \( m = 0 \), then the result is true.

Proof: Since \( m = 0 \); given \( \varepsilon_0 \) \((0 < \varepsilon_0 < 1)\) we can find integers \( x_1, y_1, z_1, t_1 \) such that

\[ 0 < Q(x_1, y_1, z_1, t_1) = \varepsilon < \varepsilon_0, \quad (x_1, y_1, z_1, t_1) = 1 \quad (3.2) \]

By replacing \( Q \) by an equivalent form we can suppose \( Q(1,0,0,0) = \varepsilon. \) Then \( Q(x,y,z,t) \) can be written as

\[ Q(x,y,z,t) = \varepsilon(x + hy +gz + ut)^2 + \Phi(y,z,t); \quad (3.3) \]

where \( \Phi(y,z,t) \) is an indefinite ternary quadratic form with determinant \( \frac{D}{\varepsilon} < 0. \)

By Lemma 3 of Chapter II, we can find \( (y,z,t) \equiv (y_0,z_0,t_0) \) (mod 1) such that

\[ 0 < \beta^2 = -\Phi(y,z,t) \leq \left( \frac{8|D|}{\varepsilon} \right)^{1/3} \quad (3.4) \]

Let \( \alpha = hy + gz + ut \) and choose \( x \equiv x_0 \) (mod 1) with

\[ \frac{\beta}{\sqrt{\varepsilon}} < x + \alpha \leq \frac{\beta}{\sqrt{\varepsilon}} + 1, \]

so that
where \( A \) is an absolute constant. Since \( \varepsilon_0 \) can be chosen arbitrarily small, the right hand side of (3.5) can be made as small as we please and the lemma follows.

§ 3.2. Proof Cont'd:

We can now suppose \( m > 0 \).

Then given \( 0 < \varepsilon_0 < \frac{1}{16} \), we can find integers \( x_1, y_1, z_1, t_1 \) to satisfy

\[
Q(x_1, y_1, z_1, t_1) = \frac{m}{1 - \varepsilon}
\]

where \( 0 \leq \varepsilon < \varepsilon_0 \). Because of Lemma 1 of Chapter II we can further assume that

\[
Q(x_1, y_1, z_1, t_1) = \frac{m}{1 - \varepsilon} \leq (\frac{16}{3} |D|)^{1/4}
\]

(3.6)

Since \( 0 \leq \varepsilon < \varepsilon_0 < \frac{1}{16} \) by definition of \( m \) we must have

\( (x_1, y_1, z_1, t_1) = 1 \). By applying a suitable transformation to \( Q \) we can suppose that

\( Q(1, 0, 0, 0) = \frac{m}{1 - \varepsilon} \). \( Q(x, y, z, t) \) can then be written as

\[
Q(x, y, z, t) = \frac{m}{1 - \varepsilon} \left\{ (x + hy + gz + ut)^2 + \varphi(y, z, t) \right\}
\]

where \( \varphi(y, z, t) \) is an indefinite ternary quadratic form with determinant

\[
\frac{D}{\left( \frac{m}{1 - \varepsilon} \right)^4} \geq \frac{3}{16}
\]

(3.7)
Equality occurs in (3.7) if and only if \( \varepsilon = 0 \), \( Q \sim mQ_1 \) (by Lemma 1 of Chapter II).

Also by definition of \( m \) we have for any integers \( x, y, z, t \) either \( Q(x, y, z, t) \leq 0 \) or \( Q(x, y, z, t) \geq m \), i.e. either

\[
(x + hy + gz + ut)^2 + \varphi(y, z, t) \leq 0 \quad \text{or} \quad (x + hy + gz + ut)^2 + \varphi(y, z, t) \geq 1 - \varepsilon.
\]

Because of homogeneity it suffices to prove:

**Theorem A:** Let

\[
Q(x, y, z, t) = (x + hy + gz + ut)^2 + \varphi(y, z, t), \quad (3.8)
\]

where \( \varphi(y, z, t) \) is an indefinite ternary quadratic form of determinant \( D < 0 \), such that

\[
|D| \geq \frac{3}{16} \quad (3.9)
\]

\( D = \frac{3}{16} \) if and only if \( Q = Q_1 \).

Suppose that for integers \( x, y, z, t \) we have either

\[
Q(x, y, z, t) \leq 0 \quad \text{or} \quad (3.10)
\]

\[
Q(x, y, z, t) \geq 1 - \varepsilon,
\]

where \( 0 \leq \varepsilon < \frac{1}{16} \) is given sufficiently small. Let

\[
da = (\frac{16}{3}|D|)^{1/4}, \quad (3.11)
\]

so that from (3.9) we have
Also \( d = 1 \) if and only if \( Q(x, y, z, t) = Q_1 \). Then given any real numbers \( x_0, y_0, z_0, t_0 \) we can find \( (x, y, z, t) \equiv (x_0, y_0, z_0, t_0) \pmod{1} \) such that

\[
0 < Q(x, y, z, t) \leq d
\]

Equality holds in (3.13) if and only if either \( Q = Q_1 \) or \( Q_2 \).

3.3. Proof of Theorem A:

Lemma 5: If \( Q(x, y, z, t) \) is given as in Theorem A, then for integers \( y, z, t \) we have either

\[
\varphi(y, z, t) = 0 \quad \text{or} \quad \varphi(y, z, t) \leq -\frac{1}{4} \quad \text{or} \quad \varphi(y, z, t) \geq \frac{3}{4}.
\]

Proof: The proof is similar to Lemma 4 of Chapter I.

Lemma 6: If \( d = 1 \), then \( Q = Q_1 = x^2 + xy + y^2 + zt \). In this case (3.13) holds with strict inequality unless \( (x_0, y_0, z_0, t_0) \equiv (0, 0, 0, 0) \pmod{1} \).

Proof: If \( (z_0, t_0) \neq (0, 0) \pmod{1} \); without loss of generality we can suppose that \( z_0 \neq 0 \pmod{1} \). Choose \( z \equiv z_0 \pmod{1} \) with \( 0 < |z| \leq \frac{1}{2} \).

Choose \( (x, y) \equiv (x_0, y_0) \pmod{1} \) arbitrarily, so that

\[
Q(x, y, z, t) = A + zt
\]

Now choose \( t \equiv t_0 \pmod{1} \) with \( 0 < A + zt \leq |z| \leq \frac{1}{2} < 1 = d \). Let now \( (z_0, t_0) \equiv (0, 0) \pmod{1} \). Take \( z = t = 0 \), so that

\[
Q(x, y, z, t) = x^2 + xy + y^2 = (x + \frac{y}{2})^2 + \frac{3}{4} y^2.
\]
Choose \( y \equiv y_0 \pmod{1} \) with \( |y| \leq \frac{1}{2} \). If \( y \neq 0 \), choose \( x \equiv x_0 \pmod{1} \) with \( |x + \frac{y}{2}| \leq \frac{1}{2} \), so that
\[
0 < (x + \frac{y}{2})^2 + \frac{3}{4} y^2 \leq \frac{1}{4} + \frac{3}{16} < 1 = d,
\]

If \( y = 0 \) so that \( y_0 \equiv 0 \pmod{1} \), and

\[
Q(x,y,z,t) = x^2,
\]  

Choose \( x \equiv x_0 \pmod{1} \) with \( 0 < x \leq 1 \), so that
\[
0 < Q(x,y,z,t) = x^2 \leq 1 = d.
\]

Equality can occur only if \( x_0 \equiv 0 \pmod{1} \) i.e. only if 
\( (x_0,y_0,z_0,t_0) \equiv (0,0,0,0) \pmod{1} \). Clearly equality is needed for 
\( Q_1 \) when \( (x_0,y_0,z_0,t_0) \equiv (0,0,0,0) \pmod{1} \).

This proves the lemma.

We can now suppose \( d > 1 \).

**Lemma 7:** Let \( \eta_1 > 0, \eta_2 > 0 \) be defined by
\[
\eta_1 = d - \frac{1}{4}
\]  

\[
\eta_2 = \begin{cases} 
\left(\frac{d - \frac{1}{2}}{2}\right)^2 & \text{if } d \text{ is an integer} \\
\left(\lfloor d \rfloor + \frac{1}{2}\right)^2 & \text{if } d \text{ is not an integer}
\end{cases}
\]

Suppose we can find \( (y,z,t) \equiv (y_0,z_0,t_0) \pmod{1} \) satisfying
\[
-\eta_2 \leq Q(y,z,t) < \eta_1
\]

Then for any \( x_0 \) there exists \( x \equiv x_0 \pmod{1} \) such that
Further strict inequality in (3.21) implies strict inequality in (3.22).

Proof: If \( 0 < \Phi(y,z,t) < \gamma \), choose \( x = x_0 \) (mod 1) with

\[
|x + hy + gz + ut| \leq \frac{1}{2},
\]

so that

\[
0 < Q(x,y,z,t) = (x + hy + gz + ut)^2 + \Phi(y,z,t) < \gamma_1 + \frac{1}{4} = d
\]

If \(-D \leq \Phi(y,z,t) \leq 0\), then the result follows from Lemma 3 with \( \alpha = hy + gz + ut \), \( \beta^2 = -\Phi(y,z,t) \).

This completes the proof of the lemma.

Lemma 8: If \( d > 12 \), then (3.22) is true with strict inequality.

Proof: By Lemma 3 of Chapter II, we can find \((y,z,t) \equiv (y_0,z_0,t_0) \) (mod 1) satisfying

\[
-(\frac{3}{2}d^4)^{1/3} = -(|y|d^3)^{1/3} \leq \Phi(y,z,t) < 0
\]

Therefore (3.21) is satisfied if we have

\[
(\frac{3}{2}d^4)^{1/3} < (\frac{d - 1}{2})^2
\]

i.e. if

\[
f(d) = \frac{(d - 1)^6}{d^4} > 96.
\]

\( f(d) \) is an increasing function of \( d \) for \( d > 1 \) and \( f(13) > 96 \).

Thus (3.21) is true if \( d \geq 13 \) and the result follows from Lemma 7.
Let now $12 < d < 13$, so that $\lceil d \rceil = 12$, $d$ not an integer. In this case (3.23) implies (3.21) if we have

$$\left(\frac{3}{2} d^4\right)^{1/3} < \left(\frac{12}{2}\right)^2 = 36.$$  \hspace{1cm} (3.25)

Since $\left(\frac{3}{2} \cdot 13^4\right)^{1/3} = \frac{13}{2} (156)^{1/3} < \frac{13}{2} (5.4) = 35.1 < 36$, (3.25) is true for $d < 13$ and the lemma follows from Lemma 7.

**Lemma 9**: If $4 < d < 12$, then (3.22) is true with strict inequality.

**Proof**: By Lemma 1 we can find $(y, z, t) = (y_o, z_o, t_o) \mod 1$ such that

$$|\varphi(y, z, t)| \leq \frac{27}{100} |D|^{1/4} = \left(\frac{81}{1600} d^4\right)^{1/3}. \hspace{1cm} (3.26)$$

Hence $\varphi(y, z, t)$ satisfies (3.21) if we have

$$\left(\frac{81}{1600} d^4\right)^{1/3} < \min(\nu_1, \nu_2)$$  \hspace{1cm} (3.27)

Now

$$\left(\frac{81}{1600} d^4\right)^{1/3} < \nu_1 = d - \frac{1}{4} \hspace{1cm} (3.28)$$

if

$$f(d) = \frac{(4d - 1)^3}{d^4} > \frac{81}{25} \hspace{1cm} (3.29)$$

$f(d)$ is a decreasing function of $d$ for $d > 1$. Since $f(12) = \frac{47^3}{12^4} > \frac{81}{25}$, (3.29) is satisfied for $d < 12$. Also

$$\left(\frac{81}{1600} d^4\right)^{1/3} < \nu_2, \hspace{1cm} (3.30)$$

if we have
\[(\frac{81}{1600} d^4)^{1/3} < \begin{cases} 
\frac{(d - 1)^2}{2} & \text{if } 5 \leq d \leq 12 \\
4 = (\frac{d}{2})^2 & \text{if } 4 < d < 5
\end{cases} \quad (3.31)\]

(3.31) holds if

\[f(d) = \frac{(d - 1)^3}{d^2} > \frac{9}{5}\]

\(f(d)\) is an increasing function of \(d\) and \(f(5) = \frac{64}{25} > \frac{9}{5}\). Thus (3.31) is satisfied. Also for \(d < 5\),

\[(\frac{81}{1600} d^4)^{1/3} < (\frac{81}{1600})^{1/3} = \frac{1}{4} (2025)^{1/3} < 4,\]

so that (3.32) is true. Hence (3.30) is satisfied. From (3.28) and (3.30) it follows that (3.27) is true. The lemma then follows from Lemma 7.

Lemma 10: If \(\Phi(y,z,t) \sim \Phi(y^2 +zt), \Phi > 0, 1 < d < 4,\) then again (3.22) holds with strict inequality.

Proof: As in the proof of Lemma 10 of Chapter II we find that in this case we have

\[\Phi(x,y,z,t) = x^2 + \Phi(y^2 +zt), \quad \Phi \geq 1 \quad (3.33)\]

If \((z_o,t_o) \neq (0,0) \pmod{1}\); without loss of generality suppose that \(z_o \neq 0 \pmod{1}\). Choose \(z \equiv z_o \pmod{1}\) with \(0 < |z| < \frac{1}{2}\). Choose any \((x,y) \equiv (x_o,y_o) \pmod{1}\). Take \(t \equiv t_o \pmod{1}\) to satisfy

\[0 < x^2 + \Phi(y^2 +zt) \leq \Phi |z| \leq \frac{\Phi}{2} = \left(\frac{3}{32} d^4\right)^{1/3} < d. \quad \text{(since } d \leq 4).\]

Let now \((z_o,t_o) \equiv (0,0) \pmod{1}\). We now distinguish the following two subcases.
(i) \( y_0 \equiv 0 \pmod{1} \)

(ii) \( y_0 \not\equiv 0 \pmod{1} \).

**Subcase (i).** In this case take \( y = 1, z = 1, t = -1 \). Choose \( x \equiv x_0 \pmod{1} \) with \( 0 < x < 1 \), so that

\[
0 < Q(x,y,z,t) = x^2 < 1 < d.
\]

**Subcase (ii)** In this case take \( z = t = 0 \). Choose \( (x,y) \equiv (x_0, y_0) \pmod{1} \) with \( |x| \leq \frac{1}{2}, 0 < |y| \leq \frac{1}{2} \), so that

\[
0 < Q(x,y,z,t) = x^2 + \frac{y^2}{4} < \frac{1 + \frac{d}{4}}{4} < d, \tag{3.34}
\]

if \( \xi < 4d - 1 \), i.e. if

\[
\frac{3}{4} d^4 < (4d - 1)^3
\]

i.e. if

\[
f(d) = \frac{d^4}{(4d - 1)^3} < \frac{4}{3} \tag{3.35}
\]

\( f(d) \) is an increasing function of \( d \) for \( d > 1 \) and \( f(4) = \frac{256}{15^3} < \frac{4}{3}. \)

Thus (3.34) is true and the lemma follows.

\( \xi \) 3.4. Proof of Theorem A cont'd:

From now on we can suppose that

\[1 < d < 4, \varphi(y,z,t) \not\mid \varphi(y^2 + zt), \xi > 0.\]

By Lemma 2 of Chapter II we can find integers \( y_2, z_2, t_2 \) such that
\[ 0 < a = \Phi(y_2, z_2, t_2) \leq \left( \frac{9|D|}{3} \right)^{1/3} = \left( \frac{27}{64} d^4 \right)^{1/3} = \frac{3}{4} d^{4/3} \quad (3.36) \]

\((y_2, z_2, t_2) = 1\). Also by (3.14) we have

\[ a \geq \frac{3}{4} - \varepsilon. \quad (3.37) \]

By a suitable unimodular transformation we can suppose

\[ \Phi(1, 0, 0) = a, \] so that

\[ \Phi(y, z, t) = a \left\{ (y + fz + vt)^2 + \gamma(z, t) \right\} \quad (3.38) \]

where

\[ \frac{3}{4} - \varepsilon \leq a \leq \frac{3}{4} d^{4/3}, \quad (3.39) \]

and \( \gamma(z, t) \) is an indefinite binary quadratic form with discriminant

\[ \Delta^2 = \frac{4|D|}{a^3} \geq \frac{16}{9} \quad (3.40) \]

from (3.36). Let

\[ k = \frac{d - \frac{1}{4}}{a} \quad (3.41) \]

Then

\[ k \geq \frac{4}{3} \cdot \frac{d - \frac{1}{4}}{d^{4/3}} = \frac{4d - 1}{3d^{4/3}} \geq \frac{\frac{4}{3} \cdot \frac{4}{3} - 1}{3 \cdot \frac{4}{3}^{4/3}} = \frac{5}{4^{4/3}} > \frac{1}{4}, \quad (3.42) \]

for \( 1 < d \leq 4 \). Let

\[ \lambda = \frac{4k - 1}{\Delta} \quad (3.43) \]
so that \( \alpha > 0 \).

By Lemma 2 we can find \((z,t) \equiv (z_0, t_0) (\mod 1)\) such that

\[
-\frac{1}{a^3(4k - 1)} = -\frac{\Delta^2}{4(4k - 1)} = \frac{-\Delta}{4x} \leq \psi(z,t) < \frac{\Delta}{4} = k - \frac{1}{4} \tag{3.44}
\]

i.e.

\[
-\frac{3d^4}{16a^2(4d - 1 - a)} \leq \psi(z,t) < k - \frac{1}{4} \tag{3.45}
\]

Therefore,

\[
Q(x,y,z,t) = (x + hy + gz + ut)^2 + a\left[(y + fz + vt)^2 + \psi(z,t)^2\right] \tag{3.46}
\]

where \(\psi(z,t)\) satisfies (3.45).

Lemma 11: If \(0 < \psi(z,t) < k - \frac{1}{4}\), then (3.22) is true with strict inequality.

Proof: Choose \(y \equiv y_0 (\mod 1)\) with \(|y + fz + vt| \leq \frac{1}{2}\), so that

\[-\frac{\psi}{2} < 0 \leq \psi(y,z,t) < ak = d - \frac{1}{4} = \psi_1,
\]

and the result follows from Lemma 7.

This completes the proof of the Lemma.

Therefore from now on we can suppose

\[
0 \leq \beta = -\psi(z,t) \leq \frac{3d^4}{16a^2(4d - 1 - a)} \tag{3.47}
\]

Lemma 12: If \(0 \leq \beta + k \leq 1\), then (3.22) is true with strict inequality.
Proof: Choose \( y \equiv y_0 \pmod{l} \) with \( |y + fz + vt| \leq \frac{1}{2} \), so that

\[
-a(1-k) \leq -\beta \leq \Phi(y,z,t) = a\left((y + fz + vt)^2 - \beta\right) \leq a\left(\frac{1}{4} - \beta\right)
\]
i.e.

\[
-(a - d + \frac{1}{4}) \leq \Phi(y,z,t) \leq \frac{a}{4} < ak = d - \frac{1}{4}
\]

(3.48)

from (3.42) and (3.41). Therefore (3.22) is satisfied if we have

\[
a - d + \frac{1}{4} < \left(\frac{d}{2} - \frac{1}{2}\right)^2.
\]

(3.49)

This is so if

\[
4a < d^2 + 2d.
\]

Now \( 4a \leq 3d^{4/3} < d(d + 2) \) if

\[
f(d) = \frac{(d + 2)^3}{d} > 27.
\]

(3.50)

(3.51)

f(d) is a monotonically increasing function of \( d \) for \( d > 1 \). Thus

for \( d > 1 \), f(d) > f(1) = 27. Thus (3.49) is true and the lemma follows from Lemma 7.

Lemma 13: If \( \beta + k > 1 \) and \( 2 < d \leq 4 \), then (3.22) is true with strict inequality.

Proof: Choose \( y \equiv y_0 \pmod{l} \) to satisfy

\[
\sqrt{\beta + k - 1} \leq y + fz + vt < \sqrt{\beta + k}
\]

(3.50)

so that

\[
-(2a\sqrt{\beta + k} + \frac{1}{4} - a - d) \leq \Phi(y,z,t) < ak = d - \frac{1}{4}
\]

(3.51)
The result will follow from Lemma 7 if we have

\[ 2a\sqrt{\beta + \frac{a}{4} + \frac{1}{4}} - a - d < \begin{cases} \left(\frac{3}{2}\right)^2 = \frac{9}{4} & \text{if } 3 \leq d \leq 4 \quad (3.52) \\ (\frac{2}{2})^2 = 1 & \text{if } 2 < d < 3. \quad (3.53) \end{cases} \]

We take the two subcases separately.

(i) \( 3 \leq d \leq 4 \). \quad (3.54)
(ii) \( 2 < d < 3 \). \quad (3.55)

Subcase (i) \( 3 \leq d \leq 4 \). In this case (3.52) is satisfied if we have

\[ 4a^2(\beta + k) < (a + d + 2)^2; \quad (3.56) \]

This will be satisfied if

\[ 4a^2 \cdot \frac{3d^4}{16a^2(4d - 1 - a)} + a(4d - 1) < (a + d + 2)^2 \]

i.e. if

\[ 3d^4 < 4(4d - 1 - a)\left\{ a^2 + (5 - 2d)a + (d + 2)^2 \right\} \]

i.e. if

\[ f(a,d) = 4a^3 - 24(d - 1)a^2 + 36(d + 1)^2a + 3d^4 - 4(4d - 1)(d + 2)^2 < 0. \]

i.e. if

\[ f(a,d) = 4a\left\{ a - 3(d - 1) \right\}^2 + 3d^4 - 4(4d - 1)(d + 2)^2 < 0 \quad (3.57) \]
\[ \frac{\partial f}{\partial a} = 12 \left( a - 3(a - 1) \right) \left( a - (a - 1) \right) \]

Since \( a \leq \frac{3}{4} d^{4/3} < 3(d - 1) \); for \( 3 \leq d \leq 4 \) (since \( \frac{d}{(d - 1)^3} \) is a decreasing function of \( d \) for \( d \leq 4 \) and \( f(3) < 64 \)).

Therefore \( \max_{\frac{3}{4} d^{4/3}} f(a,d) \) occurs at \( a = d - 1 \). Therefore

\[
f(a,d) \leq f(d - 1,d) = 16(d - 1)^3 + 3d^4 - 4(4d - 1)(d + 2)^2 = 3d^4 - 108d^2 = 3d^2(d^2 - 36) < 0 \quad \text{(since } d \leq 4) (3.58)\]

Hence (3.52) is satisfied and the result follows from Lemma 7.

Subcase (iii) \( 2 < d < 3 \).

In this case (3.53) is satisfied if we have

\[
2a \sqrt{\beta + k} < d + a + \frac{3}{4} \]

i.e.

\[ 4a^2(\beta + k) < (d + a + \frac{3}{4})^2 \]

This will be satisfied if

\[
4a^2 \frac{3d^4}{16a^2(4d - 1 - a)} + a(4d - 1) < \frac{(4d + 4a + 3)^2}{16} \]

i.e. if
\[ 12d^4 + 16a(4d - 1)(4d - 1 - a) - (4d - 1 - a)(4d + 4a + 3)^2 < 0 \]

i.e. if

\[ f(a,d) = 16a \left[ a - \frac{12d - 7}{4} \right]^2 + 12d^4 - (4d - 1)(4d + 3)^2 < 0 \]

\[ \frac{\partial f}{\partial a} = 16(a - \frac{12a - 7}{4})(3a - \frac{12a - 7}{4}) \]

\[ = 48(a - \frac{12d - 7}{4})(a - \frac{12d - 7}{12}). \]

Now since \( a \leq \frac{3}{4} d^{4/3} < \frac{12d - 7}{4} \) for \( 2 < d < 3; \)

\[ \max_{\frac{3}{4} d^{4/3} \leq a \leq \frac{3}{4} d^{4/3}} f(a,d) = f\left(\frac{12d - 7}{12},d\right) \]

\[ = \left(\frac{12d - 7}{27}\right)^3 + 12d^4 - (4d - 1)(4d + 3)^2 \]

\[ = 12d^4 - 192d^2 + \frac{160}{3} d - \frac{100}{27} \]

\[ = 12a^2(d^2 - 9) + \frac{80}{3} a(2 - d) - \frac{172}{3} a^2 - \frac{100}{27} \]

\[ < 0 \quad \text{for} \ 2 < d < 3. \]

Hence (3.53) is satisfied and the lemma follows from Lemma 7.

Lemma 14: If \( \beta + k > 1 \) and \( 1 < d \leq 2, \) then the result (3.22) is again true.

Proof: If \( \beta + k > \frac{9}{k}, \) choose \( y \equiv y_0 \pmod{1} \) with

\[ 1 \leq |y + fz + vt| \leq \frac{3}{2}, \]

so that
\[ -a(\beta - 1) \leq \phi(y, z, t) = a^{(y + fz + ut)^2 - \beta} \]
\[ \leq a^{\left(\frac{\alpha}{4} - \beta\right)} < ak = d - \frac{1}{4} \quad (3.60) \]

The result will follow from Lemma 7 with strict inequality if we have

\[ a(\beta - 1) < \frac{1}{4}, \]

This is so if

\[ \frac{3d^4}{16a(4d - 1 - a)} < a + \frac{1}{4} \]

if

\[ f(a, d) = 4(4a^2 + a)(4d - 1 - a) - 3d^4 > 0 \quad (3.61) \]

\[ \frac{\partial f}{\partial a} = 4\left(8a + 1)(4d - 1 - a) - (4a^2 + a)\right) \]
\[ = 4\left[-12a^2 + (32d - 10)a + 4d - 1\right] \]
\[ = 4(a + \alpha_1)(\alpha_2 - a), \quad \alpha_1 > 0 \quad \text{and} \]
\[ \alpha_2 = \frac{16d - 5 + \sqrt{(16d - 5)^2 + 12(4d - 1)}}{12}^{1/2} \]
\[ > \frac{11 + \sqrt{121 + 36}}{12} \quad \text{for } d > 1 \]
\[ > \frac{23}{12} > \frac{3}{4} d^{4/3} \geq a \quad \text{for } d \leq 2. \]

Hence \[ \frac{\partial f}{\partial a} > 0 \quad \text{for } \frac{3}{4} - \varepsilon \leq a \leq \frac{3}{4} d^{4/3}. \]
\[ f(a,d) \geq f\left(\frac{3}{4}a\right) + o(\varepsilon) \quad \text{for} \quad \frac{3}{4} - \varepsilon \leq a \leq \frac{3}{4} \quad d^{4/3} \]

\[ = 12(\frac{4}{3}a - \frac{7}{5}) - 3d^4. \]

\[ = 3\left(\frac{16}{3}a - 7 - d^4\right) \]

\[ = 3\left(\frac{7}{3}(a - 1) + d(9 - d^3)\right)^9 \]

\[ > 0 \quad \text{for} \quad 1 \leq d < 2. \]

Thus (3.61) is satisfied and the result follows in this case from Lemma 7.

If \( 1 < \beta + k \leq \frac{9}{4} \), choose \( y \equiv y_0 \pmod{1} \) with

\[ \frac{1}{2} \leq |y + f z + v t| \leq 1, \]

so that

\[ -a(\beta - \frac{1}{4}) \leq \Phi(y, z, t) = \frac{1}{a}(y + f z + v t)^2 - \beta \]

\[ \leq a(1 - \beta) < ak = a - \frac{1}{4} \quad (3.62) \]

Therefore \( \Phi \) satisfies (3.21) if we have

\[ a(\beta - \frac{1}{4}) \leq \frac{1}{4} \]

i.e.

\[ ab \leq \frac{a + 1}{4} \quad (3.63) \]

Now we shall distinguish the following two cases

(i) \( a < \frac{a}{2} \)

(ii) \( a \geq \frac{a}{2} \)
Subcase (i) If \( a < \frac{d}{2} \), then since \( \beta + k \leq \frac{9}{4} \) we have

\[
an\beta \leq a(\frac{9}{4} - k) = \frac{9a}{4} - d + \frac{1}{4}
\]

\[
< \frac{a}{4} + \frac{1}{4},
\]

and the result follows.

Subcase (ii) If \( a \geq \frac{d}{2} \), then \( a\beta \leq \frac{a + 1}{4} \) is satisfied if we have

\[
\frac{3d^4}{16(4d - 1 - a)} \leq \frac{a + 1}{4}, \quad \text{or}
\]

\[
f(a, d) = 4(a^2 + a)(4d - 1 - a) - 3d^4 \geq 0 \tag{3.66}
\]

\[
\frac{\partial f}{\partial a} = 4\left(2a + 1\right)(4d - 1 - a) - (a^2 + a)
\]

\[
= 4\left(3a^2 + 4a(2d - 1) + 4d - 1\right)
\]

\[
= 12(a + \alpha_1)(\alpha_2 - a), \quad \alpha_1 > 0
\]

\[
\alpha_2 = \frac{4d - 2 + \sqrt{(4d - 2)^2 + 3(4d - 1)^2}}{3}
\]

\[
> \frac{3}{4}d^{3/3} \geq a \quad \text{if} \quad 1 < d \leq 2.
\]

so that \( \frac{\partial f}{\partial a} > 0 \). Therefore

\[
f(a, d) \geq f\left(\frac{d}{2}, d\right)
\]

\[
= 4\cdot \frac{d}{2}\left(\frac{d}{2} + 1\right)(4d - 1 - \frac{d}{2}) - 3d^4
\]

\[
= \frac{1}{2}d(7d^2 + 12d - 4 - 6d^3)
\]

\[
= \frac{1}{2}d(2 - d)(6d^2 + 5d - 2)
\]

\[
\geq 0 \quad \text{for} \quad 1 < d \leq 2. \tag{3.67}
\]
Thus (3.63) is satisfied and the result follows from Lemma 7.

This completes the proof of the lemma.


Lemma 15: Equality is needed in (3.22) if and only if $Q(x, y, z, t) = Q_2$ and $(x_o, y_o, z_o, t_o) \equiv (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \pmod{1}$.

Proof: Equality can occur only if we have equality in (3.63). This from (3.67) will be so only if

$$d = 2, \quad a = \frac{d}{2} = 1$$

(3.68)

From (3.41), (3.11), (3.40), (3.43) we have

$$k = \frac{7}{11}, \quad |D| = 3, \quad \Delta^2 = 12, \quad \Lambda = \frac{4k - 1}{\Delta} = \sqrt{3}$$

(3.69)

or $\lambda^2 = 3$.

Also we must have equality in Lemma 2. Since $\lambda^2 = 3 = \frac{1 + 2}{1}$ (m = 1), we must have

$$\Phi(z, t) \sim c \Phi_1 = c(z^2 - 3t^2), \quad c > 0$$

(3.70)

and $(z_o, t_o) \equiv (\frac{1}{2}, \frac{1}{2}) \pmod{1}$.

Since $\Delta^2 = 12$, we get $c = 1$. Therefore

$$\Phi(y, z, t) = (y + fz + vt)^2 + z^2 - 3t^2$$

(3.71)

By a suitable unimodular substitution we can suppose

$$|h| \leq \frac{1}{2}, \quad |g| \leq \frac{1}{2}, \quad |w| \leq \frac{1}{2}, \quad |f| \leq \frac{1}{2}, \quad |v| \leq \frac{1}{2}$$

(3.72)

Again for equality to occur, the inequalities
\[-\frac{1}{4} < F(y, z, t) = (y + fz + vt + \frac{f}{2} + \frac{v}{2} + y_o)^2 + (z + \frac{1}{2})^2 - 3(t + \frac{1}{2})^2 < \frac{7}{4}\]  

(3.73)

should not hold for any integers \(y, z, t\).

\[-\frac{1}{4} < F(y, l, 0) = (y + \frac{3f}{2} + \frac{v}{2} + y_o)^2 + \frac{3}{2} < \frac{7}{4}\]  

is solvable for integer \(y\) unless

\[\frac{3f}{2} + \frac{v}{2} + y_o \equiv \frac{1}{2} \pmod{1}\]  

(3.74)

Similarly

\[-\frac{1}{4} < F(y, l, -1) = (y + \frac{3f}{2} - \frac{v}{2} + y_o)^2 + \frac{3}{2} < \frac{7}{4}\]  

is solvable for integer \(y\) unless

\[\frac{3f}{2} - \frac{v}{2} + y_o \equiv \frac{1}{2} \pmod{1}\]  

(3.75)

Again

\[-\frac{1}{4} < F(y, 0, 0) = (y + \frac{f}{2} + \frac{v}{2} + y_o)^2 - \frac{1}{2} < \frac{7}{4}\]  

is solvable for integer \(y\) (by taking \(y\) such that \(\frac{1}{2} < y + \frac{f}{2} + \frac{v}{2} + y_o < \frac{3}{2}\)) unless

\[\frac{f}{2} + \frac{v}{2} + y_o \equiv \frac{1}{2} \pmod{1}\]  

(3.76)

From (3.74), (3.75) and (3.76) we get
\[ f = v = 0 \pmod{1}, \quad y_o = \frac{1}{2} \pmod{1} \quad (3.77) \]

From (3.72) we get \( f = v = 0, \quad y_o = \frac{1}{2} \pmod{1} \). Therefore
\[ \varphi(y,z,t) = y^2 + z^2 - 3t^2, \quad (3.78) \]
and \((y_o,z_o,t_o) \equiv (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \pmod{1}\).

Again if equality is to occur, the inequalities
\[ 0 < f(x,y,z,t) = (x + hy + gz + ut + \frac{h}{2} + \frac{g}{2} + \frac{u}{2} + x_o)^2 \]
\[ + (y + \frac{1}{2})^2 + (z + \frac{1}{2})^2 - 3(t + \frac{1}{2})^2 < 2 \quad (3.79) \]
should have no solutions in integers \( x,y,z,t \).

\[ 0 < f(x,0,0,0) = (x + \frac{h}{2} + \frac{g}{2} + \frac{u}{2} + x_o)^2 - \frac{1}{4} < 2 \]
is solvable for integer \( x \) unless
\[ \frac{h}{2} + \frac{g}{2} + \frac{u}{2} + x_o \equiv \frac{1}{2} \pmod{1} \quad (3.80) \]

Similarly by considering \( f(x,-1,0,0), f(x,0,-1,0) \) and \( f(x,0,0,-1) \) we see that if equality is to occur we must have
\[ -\frac{h}{2} + \frac{g}{2} + \frac{u}{2} + x_o \equiv \frac{1}{2} \pmod{1} \quad (3.81) \]
\[ \frac{h}{2} - \frac{g}{2} + \frac{u}{2} + x_o \equiv \frac{1}{2} \pmod{1} \quad (3.82) \]
\[ \frac{h}{2} + \frac{g}{2} - \frac{u}{2} + x_o \equiv \frac{1}{2} \pmod{1} \quad (3.83) \]

From (3.80), (3.81), (3.82) and (3.83) we have
\[ h \equiv g \equiv u \equiv 0 \pmod{1}, \quad x_o \equiv \frac{1}{2} \pmod{1}. \]
Hence from (3.72) we have \( h = g = u = 0, x_0 \equiv \frac{1}{2} \) (mod 1).

Thus equality can occur only if

\[
Q(x,y,z,t) = x^2 + y^2 + z^2 - 3t^2 = q_2
\]

and \((x_0,y_0,z_0,t_0) \equiv (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \) (mod 1).

We now show that equality is necessary for this form. For this it suffices to show that for integers \( x,y,z,t \) we have either

\[
(x + \frac{1}{2})^2 + (y + \frac{1}{2})^2 + (z + \frac{1}{2})^2 - 3(t + \frac{1}{2})^2 < 0 \quad \text{or} \quad (3.84)
\]

\[
(x + \frac{1}{2})^2 + (y + \frac{1}{2})^2 + (z + \frac{1}{2})^2 - 3(t + \frac{1}{2})^2 > 2.
\]

This is equivalent to showing that for odd integers \( X, Y, Z, T \) we have either

\[
X^2 + Y^2 + Z^2 - 3T^2 \leq 0 \quad \text{or} \quad (3.84)
\]

\[
X^2 + Y^2 + Z^2 - 3T^2 > 8.
\]

Since \( X^2 + Y^2 + Z^2 - 3T^2 \equiv 1 + 1 + 1 - 3 \equiv 0 \) (mod 8) for odd integers \( X, Y, Z, T \), (3.84) follows. This completes the proof of the lemma.

Theorem A follows from the Lemmas 6 to 15. This completes the proof of the theorem.
CHAPTER IV
POSITIVE VALUES OF NON-HOMOGENEOUS QUATERNARY
QUADRATIC FORM OF SIGNATURE 0

§1. In this chapter we prove the following theorem:

Theorem: Let \( Q(x, y, z, t) \) be an indefinite quaternary quadratic form
with determinant \( D > 0 \) and signature 0. Then given any real numbers
\( x_0, y_0, z_0, t_0 \) we can find integers \( x, y, z, t \) such that

\[
0 < Q(x + x_0, y + y_0, t + t_0) \leq (16D)^{1/4} \quad (1.1)
\]

Equality is needed if and only if either

\[
Q(x, y, z, t) \sim q_1 Q_1 = q(xy + zt) \quad \text{or} \quad (1.2)
\]
\[
Q(x, y, z, t) \sim q_2 Q_2 = q(x^2 - y^2 + z^2 - t^2) \quad \text{or} \quad (1.3)
\]
\[
Q(x, y, z, t) \sim q_3 Q_3 = q(x^2 - y^2 + 2zt) \quad (1.4)
\]

where \( q \neq 0 \). For \( Q_1 \) equality occurs if and only if

\( (x_0, y_0, z_0, t_0) \equiv (0, 0, 0, 0) \pmod{1} \), for \( Q_2 \) if and only if

\( (x_0, y_0, z_0, t_0) \equiv (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \pmod{1} \) and for \( Q_3 \) if and only if

\( (x_0, y_0, z_0, t_0) \equiv (\frac{1}{2}, \frac{1}{2}, 0, 0) \pmod{1} \).

§2. Some Lemmas:

In the course of the proof we shall use the following lemmas.

Lemma 1: Let \( Q(x, y, z, t) \) be an indefinite quaternary quadratic form
of signature 0 with determinant \( D > 0 \). Then there exist integers
\( x_1, y_1, z_1, t_1 \) such that
0 < Q(x_1, y_1, z_1, t_1) \leq \left( \frac{31}{16} d \right)^{1/4} \quad (2.1)

except when \( Q(x, y, z, t) \sim \xi Q_1, \xi \neq 0 \).

This is Theorem 1 of Oppenheim [20].

**Lemma 2:** Let \( \Phi(y, z, t) \) be an indefinite ternary quadratic form with determinant \(-D < 0\). Then given any real numbers \( y_0, z_0, t_0 \) we can find \((y, z, t) = (y_0, z_0, t_0) \mod 1\) such that

\[ 0 < \Phi(y, z, t) < (4D)^{1/3} \quad (2.2) \]

This is Theorem 1 of Barnes [6].

**Lemma 3:** Let \( \Psi(z, t) \) be an indefinite binary quadratic form with discriminant \( \Delta^2 > 0 \). Then given \( \nu > 1 \) and any real numbers \( z_0, t_0 \) there exist \((z, t) = (z_0, t_0) \mod 1\) such that

\[ -\frac{\nu^2 \Delta}{(\nu - 1)^3(\nu + 3)^{1/2}} \leq \Psi(z, t) \leq \frac{\Delta}{(\nu - 1)^3(\nu + 3)^{1/2}} \quad (2.3) \]

This is Theorem 3 of Blaney [8].

**§3. Proof of the Theorem:**

Let

\[ m = \inf_{x, y, z, t \text{ integers}, Q(x, y, z, t) > 0} Q(x, y, z, t) \quad (3.1) \]

**§3.1. Case** \( m = 0 \).

**Lemma 4:** If \( m = 0 \), then the theorem is true.

**Proof:** Since \( m = 0 \); given \( \xi_0 (0 < \xi_0 < 1) \) we can find integers
such that

\[ 0 < Q(x_1, y_1, z_1, t_1) = \varepsilon < \varepsilon_0, \quad (x_1, y_1, z_1, t_1) = 1 \tag{3.2} \]

By replacing \( Q \) by an equivalent form we can suppose \( Q(1, 0, 0, 0) = \varepsilon \). Then \( Q(x, y, z, t) \) can be written as

\[ Q(x, y, z, t) = \varepsilon (x + hy + gz + ut)^2 - \varphi(y, z, t), \tag{3.3} \]

where \( \varphi(y, z, t) \) is an indefinite ternary quadratic form with determinant \( -\frac{D}{\varepsilon} < 0 \).

By Lemma 2, we can find \( (y, z, t) = (y_0, z_0, t_0) \pmod{1} \) such that

\[ 0 < \varphi(y, z, t) = \beta^2 \leq \left( \frac{4D}{\varepsilon} \right)^{1/3} \tag{3.4} \]

Let \( \alpha = hy + gz + ut \) and choose \( x \equiv x_0 \pmod{1} \) with

\[ \frac{\beta}{\sqrt{\varepsilon}} < x + \alpha \leq \frac{\beta}{\sqrt{\varepsilon}} + 1, \]

so that

\[ 0 < Q(x, y, z, t) = \varepsilon (x + \alpha)^2 - \beta^2 \leq \varepsilon + 2\beta\sqrt{\varepsilon} \leq \varepsilon + 2(\frac{4D}{\varepsilon})^{1/6}\varepsilon^{1/2} < \varepsilon_0 + 2(\frac{4D}{\varepsilon})^{1/6}\varepsilon_0^{1/3} \tag{3.5} \]

Since \( \varepsilon_0 \) can be chosen arbitrarily small, the right hand side of (3.5) can be made as small as we please and the lemma follows.

\section*{3.2. Proof cont'd:}

\textbf{We can now suppose} \( m > 0 \) \hfill (3.6)
Lemma 5: If \( Q(x,y,z,t) = mQ_1 = m(xy + zt) \), then the theorem is true.

Equality is needed for \( Q_1 \) if and only if \( (x_0, y_0, z_0, t_0) \equiv (0,0,0,0) \) (mod 1).

Proof: Without loss of generality we can suppose that

\[ Q = Q_1 = xy + zt \quad (3.7) \]

Choose any \( (z,t) \equiv (z_0, t_0) \) (mod 1). Choose \( y \equiv y_0 \) (mod 1) with \( 0 < y < 1 \). Then choose \( x \equiv x_0 \) (mod 1) to satisfy

\[ 0 < Q(x,y,z,t) = xy + zt \leq y \leq 1 = (16D)^{1/4} \]

Equality can occur only if \( y_0 \equiv 0 \) (mod 1). By symmetry for equality we must have

\[ x_0 \equiv y_0 \equiv z_0 \equiv t_0 \equiv 0 \text{ (mod 1) } \quad (3.8) \]

Clearly equality is necessary when \( (x_0,y_0,z_0,t_0) \equiv (0,0,0,0) \) (mod 1).

This completes the proof of the lemma.

From now on we can suppose \( m > 0 \) and

\[ Q \sqrt{mQ_1} = m(xy + zt) \quad (3.9) \]

Then given \( 0 < \varepsilon_0 < \frac{1}{16} \), we can find integers \( x_1, y_1, z_1, t_1 \) to satisfy

\[ Q(x_1,y_1,z_1,t_1) = \frac{m}{1 - \varepsilon} \]

where \( 0 \leq \varepsilon < \varepsilon_0 \). Also by Lemma 1, we can suppose that
\[ Q(x_1, y_1, z_1, t_1) = \frac{m}{1 - \varepsilon} \leq \left( \frac{81}{16} D \right)^{1/4} \]  

(3.10)

Also, by definition of \( m \) we must have \((x_1, y_1, z_1, t_1) = 1\), since \(1 - \varepsilon > \frac{1}{4}\). By a unimodular transformation we can suppose that \(Q(1,0,0,0) = \frac{m}{1 - \varepsilon}\). \(Q(x,y,z,t)\) can then be written as

\[ Q(x,y,z,t) = \frac{m}{1 - \varepsilon} \{ (x + hy + gz + ut)^2 - \Phi(y,z,t) \}, \]  

(3.11)

where \(\Phi(y,z,t)\) is an indefinite ternary quadratic form of determinant

\[ D_1 = -\frac{D}{(1 - \varepsilon)^4} \leq \frac{16}{81} \]  

(3.12)

Also by definition of \( m \) for integers \(x,y,z,t\) we have either \(Q(x,y,z,t) \leq 0\) or \(Q(x,y,z,t) \geq m\); i.e. either

\[ (x + hy + gz + ut)^2 - \Phi(y,z,t) \leq 0 \]  \(\text{or}\)

\[ (x + hy + gz + ut)^2 - \Phi(y,z,t) \geq 1 - \varepsilon. \]

Because of homogeneity it suffices to prove

**Theorem A:** Let

\[ Q(x,y,z,t) = (x + hy + gz + ut)^2 - \Phi(y,z,t) \]  

(3.13)

where \(\Phi(y,z,t)\) is an indefinite ternary quadratic form of determinant

\[ D_1 = -D \leq \frac{16}{81} \]  

(3.14)

Let \( \frac{1}{16} > \varepsilon_0 > 0 \) be given arbitrarily small. Suppose that for integers \(x,y,z,t\) we have either
\[ Q(x,y,z,t) \leq 0 \quad \text{or} \quad Q(x,y,z,t) \geq 1 - \varepsilon \]  \hspace{1cm} (3.15)

where \( 0 \leq \varepsilon < \varepsilon_0 < \frac{1}{16} \). Let

\[ d = (16D)^{1/4}, \]  \hspace{1cm} (3.16)

so that from (3.14) we have

\[ d \geq \frac{4}{3} \]  \hspace{1cm} (3.17)

Then given any real numbers \( x_0, y_0, z_0, t_0 \) we can find \((x,y,z,t) = (x_0, y_0, z_0, t_0) \pmod{1}\) such that

\[ 0 < Q(x,y,z,t) \leq d \]  \hspace{1cm} (3.18)

Equality holds in (3.18) if and only if \( Q = Q_2 \) or \( Q_3 \).

3.3. Proof of Theorem A:

Lemma 6: If \( Q(x,y,z,t) \) is given as in Theorem A, then for integers \( y,z,t \) we have either

\[ \varphi(y,z,t) = 0, \quad \text{or} \quad \varphi(y,z,t) \geq \frac{1}{4} \quad \text{or} \quad \varphi(y,z,t) \leq -\frac{3}{4} + \varepsilon \]  \hspace{1cm} (3.19)

Proof: The proof is similar to Lemma 4, Chapter I.

Lemma 7: Let \( \nu_1 > 0, \nu_2 > 0 \) be defined by

\[ \nu_1 = d - \frac{1}{4} \]  \hspace{1cm} (3.20)

\[ \nu_2 = \begin{cases} 
\frac{(d-1)^2}{2} & \text{if } d \text{ is an integer}, \\
\left(\frac{[d]}{2}\right)^2 & \text{if } d \text{ is not an integer}.
\end{cases} \]  \hspace{1cm} (3.21)
Suppose we can find \((y,z,t) = (y_0,z_0,t_0) \pmod{l}\) satisfying
\[-\mathcal{V}_1 < (y,z,t) \leq \mathcal{V}_2. \tag{3.22}\]

Then for any \(x_0\), there exists \(x = x_0 \pmod{l}\) such that
\[0 < Q(x,y,z,t) \leq d \tag{3.23}\]

Further strict inequality in (3.22) implies strict inequality in (3.23).

Proof: See Lemma 7, Chapter III with \(\Phi(y,z,t)\) replaced by \(-\Phi(y,z,t)\).

Lemma 8: If \(d > 6\), then the theorem is true with strict inequality.

Proof: By Lemma 2, we can find \((y,z,t) = (y_0,z_0,t_0) \pmod{l}\) such that
\[0 < \Phi(y,z,t) \leq (4!d^4)^{1/3} = (4d)^{1/3} = \left(\frac{1}{4}d^4\right)^{1/3} \tag{3.24}\]

Therefore (3.22) is satisfied if we have
\[\left(\frac{1}{4}d^4\right)^{1/3} < \left(\frac{d-1}{2}\right)^2.\]
i.e. if
\[\frac{(d-1)^3}{d^2} > 4\]
or
\[f(d) = d^3 - 7d^2 + 3d - 1 > 0, \tag{3.25}\]

which is true for \(d \geq 7\). Thus (3.22) is true for \(d \geq 7\), and the result follows from Lemma 7.
Let now \( 6 < d < 7 \), so that \( [d] = 6 \), \( d \) not an integer. In this case (3.24) implies (3.22) if we have

\[
\left( \frac{1}{4} d^4 \right)^{1/3} < 9 \quad \text{or} \quad d^2 < 54
\]

(3.26) is clearly true for \( d < 7 \) and the result follows from Lemma 7. Thus the proof of Lemma 8 is complete.

Lemma 9: If \( 3 < d < 6 \), then again the theorem is true with strict inequality.

Proof: By Lemma 1, Chapter III, we can find \((y, z, t) = (y_0, z_0, t_0) \pmod{1}\) such that

\[
|\varphi(y, z, t)| \leq \left( \frac{27}{100} |D_1| \right)^{1/3} = \left( \frac{27}{1600} d^4 \right)^{1/3}
\]

Therefore \(\varphi(y, z, t)\) satisfies (3.22); if we have

\[
\left( \frac{27}{1600} d^4 \right)^{1/3} < \min(\nu_1, \nu_2)
\]

(3.28)

Now

\[
\left( \frac{27}{1600} d^4 \right)^{1/3} < \nu_1 = d - \frac{1}{4}
\]

(3.29)

if

\[
f(d) = \frac{(4d - 1)^3}{d^4} > \frac{27}{25}
\]

(3.30)

Since \(f'(d) = \frac{4(4d - 1)(1 - d)}{d^5} < 0\) for \( d > 1 \), \(f(d)\) is an increasing function of \(d\). Therefore for \(3 < d \leq 6\), \(f(d) \geq f(6) = \frac{23^3}{6^4} > \frac{27}{25}\) and
so (3.29) is true. Also

\[
\left( \frac{27}{1600} \frac{d^4}{a^2} \right)^{1/3} < y_2, \tag{3.31}
\]

if we have

\[
\left( \frac{27}{1600} \frac{d^4}{a^2} \right)^{1/3} < \begin{dcases}
\frac{(d - 1)^2}{2} & \text{if } 4 \leq d \leq 6 \\
\left( \frac{d}{2} \right)^2 & \text{if } 3 < d < 4
\end{dcases} \tag{3.32}
\]

(3.32) holds if

\[
f(d) = \frac{(d - 1)^6}{d^2} > \frac{27}{25}, \tag{3.34}
\]

\(f(d)\) is an increasing function of \(d\) for \(d \geq 1\) and \(f(4) = \frac{3^6}{4^4} > \frac{27}{25}\).

Thus (3.34) is satisfied.

Also for \(d < 4\),

\[
\left( \frac{27}{1600} \frac{d^4}{a^2} \right)^{1/3} < \left( \frac{27}{25} \right)^{1/3} < \frac{9}{4},
\]

so that (3.33) is true. Hence (3.31) is satisfied. From (3.29) and (3.31) we get (3.28). The lemma then follows from Lemma 7.

Lemma 10: If \(Q(y, z, t) \sim \varphi(y^2 + zt), \varphi > 0, d \leq 3\), then again (3.23) holds with strict inequality.

Proof: Without loss of generality we can suppose

\[
(y, z, t) = \varphi(y^2 + zt), \varphi > 0,
\]

so that

\[
Q(x, y, z, t) = (x + hy + gz + ut)^2 - \varphi(y^2 + zt) \tag{3.35}
\]

By replacing \(x\) by \(x + ay + bz + yt\), where \(a, b, y\) are suitable integers we can suppose that
\[ |h| \leq \frac{1}{2}, \quad |s| \leq \frac{1}{2}, \quad |u| \leq \frac{1}{2} \quad (3.36) \]

We first assert that \( h = g = u = 0 \). If \( u \neq 0 \), then

\[ 0 < Q(0,0,0,1) = u^2 \leq \frac{1}{4} < 1 - \varepsilon, \]

contrary to (3.15). Similarly \( g = 0 \). If \( h \neq 0 \), then,

\[ 0 < Q(0,1,-1) = h^2 \leq \frac{1}{4} < 1 - \varepsilon, \]

contrary to (3.15).

Therefore,

\[ Q(x,y,z,t) = x^2 - \zeta (y^2 + zt). \quad (3.37) \]

Choose any \((x,y) \equiv (x_0,y_0) \pmod{1}\). Choose \( z \equiv z_0 \pmod{1} \) with \( 0 < z < 1 \). Now choose \( t \equiv t_0 \pmod{1} \) to satisfy

\[ 0 < x^2 - \zeta y^2 - \zeta zt \leq \zeta z \leq \zeta = (4d)^{1/3} = (\frac{1}{4} d^{4})^{1/3} < \delta, \]

since \( \delta < 3 < 4 \). This proves the lemma.

§ 3.4. Proof of Theorem A cont'd:

From now on we can suppose that

\[ \frac{4}{3} \leq \delta \leq 3; \bar{Q}(y,z,t) = \zeta (y^2 + zt), \quad \zeta > 0 \quad (3.38) \]

By Lemma 2, Chapter II, we can find integers \( y_2, z_2, t_2 \) such that

\[ (y_2, z_2, t_2) = 1 \quad \text{and} \quad 0 < a = \bar{Q}(y_2, z_2, t_2) \leq (\frac{9}{4} d)^{1/3} = (\frac{3}{4} \delta^4)^{1/3} \quad (3.39) \]

By Lemma 6 we have

\[ a \geq \frac{1}{4} \quad (3.40) \]
By a unimodular transformation we can suppose that

$$\varphi(y,z,t) = \Phi \left( (y + fz + vt)^2 + \psi(z,t) \right)$$  \hspace{1cm} (3.41)

where $\psi(z,t)$ is an indefinite binary quadratic form with discriminant

$$\Delta^2 = \frac{4D}{a^3} = \frac{d^4}{4a^3}$$  \hspace{1cm} (3.42)

Without loss of generality we can suppose

$$0 \leq m \leq \frac{1}{2}, \quad |s| \leq \frac{1}{2}, \quad |u| \leq \frac{1}{2}, \quad 0 \leq f \leq \frac{1}{2}, \quad |v| \leq \frac{1}{2}$$  \hspace{1cm} (3.43)

In view of Lemma 7, if we can show that there exist $(y,z,t) \equiv (y_o,z_o,t_o) \pmod{1}$ satisfying

$$-(d - \frac{1}{4}) \varphi(y,z,t) = \Phi \left( (y + fz + vt)^2 + \psi(z,t) \right) \begin{cases} < 1 \text{ if } 2 < d \leq 3 \\ \leq \frac{1}{4} \text{ if } \frac{4}{3} \leq d \leq 2 \end{cases}$$  \hspace{1cm} (3.44)

then the proof of Theorem A will be completed except for the equality part.

Lemma 11: If $2 < d \leq 3$, then again the theorem is true with strict inequality.

Proof: Since $2 < d \leq 3$, we have

$$\frac{1}{4} \leq a \leq \left( \frac{9}{64} d^4 \right)^{1/3} \leq \frac{9}{4}$$  \hspace{1cm} (3.45)

so that

$$\frac{1}{a} \geq \frac{4}{9} > \frac{1}{4}$$  \hspace{1cm} (3.46)

Let
\[ \lambda = \frac{4 - a}{a \Delta}, \quad (3.47) \]

so that \( \lambda > 0 \).

By Lemma 2 of Chapter III, we can find \((z, t) \equiv (z_0, t_0) \pmod{1}\) such that

\[ -\frac{d}{16a^2(4 - a)} = \frac{\Delta^2}{4(4 - a)} = \frac{\Delta}{4\lambda} \leq \gamma(z, t) < \frac{\Delta}{4} = \frac{1}{a} - \frac{1}{4} \quad (3.48) \]

using \((3.42)\) and \((3.47)\).

If \(-\frac{4d - 1}{4a} < \gamma(z, t) < \frac{1}{a} - \frac{1}{4}\), choose \(y \equiv y_0 \pmod{1}\) with

\[ |y + fz + vt| \leq \frac{1}{2}, \quad \text{so that} \]

\[ -(d - \frac{1}{4}) = -a\left(\frac{4d - 1}{4a}\right) < \gamma(y, z, t) = a\left\{ (y + fz + vt)^2 + \gamma(z, t) \right\} < a\left\{ \frac{1}{4} + \frac{1}{a} - \frac{1}{4} \right\} = 1. \]

Thus (A) is satisfied and the result follows in this case. Let now

\[ \frac{4d - 1}{4a} \leq \beta = \gamma(z, t) \leq \frac{d}{16a^2(4 - a)}. \quad (3.49) \]

(A) will be satisfied if we can find \(y \equiv y_0 \pmod{1}\) to satisfy

\[ 0 < (y + fz + vt)^2 - (\beta - \frac{4d - 1}{4a}) < \frac{1}{a} + \frac{4d - 1}{4a} = \frac{4d + 3}{4a} \quad (3.50) \]

In view of Lemma 3, Chapter III, this is possible if we have

\[ 0 \leq \beta - \frac{4d - 1}{4a} < \left( \frac{4d + 3}{2a} - \frac{1}{2} \right)^2 \quad (3.51) \]

This is possible by \((3.49)\) if
A slight calculation shows that (3.52) is true if

\[ f(a, d) = a(13 - 4d - 4a)^2 \leq (4d + 3)^2 \]

and since \( f(a, d) \) is an increasing function of \( a \) for \( a > \frac{13}{4} \).

Therefore, for \( 2 < d < 3 \),

\[ f(\frac{13 - 4d}{12}, d) = (13 - 4d)^3 - 4(d^2 + 4d + 3)(4d + 3 - d^2) \leq \frac{(13 - 8)^3}{144} - 4(4 + 8 + 3)(4^3 + 3 - 9) = \frac{125}{144} - 4 \cdot 15 \cdot 6 < 0. \]

Also for \( 2 < d < 3 \),

\[ f(\frac{9}{4}, d) = 4\left\{9(1 - d)^2 + d^4 - (4d + 3)^2\right\} = 4d^4 - 7d - 42^2 < 0. \]

Hence (3.53) is satisfied, so that (A) holds for \( 2 < d < 3 \) and the lemma follows from Lemma 7.
Lemma 12: If \( \frac{4}{3} \leq a \leq 2 \), then again the theorem is true.

Proof: We have

\[
\frac{1}{4} \leq a \leq (\frac{2}{3} a^4)^{\frac{1}{3}} \leq (\frac{2}{3})^{\frac{1}{3}} \tag{3.55}
\]

We shall distinguish the following three subcases:

(i) \( 1 < a \leq (\frac{2}{3} a^4)^{\frac{1}{3}} \)

(ii) \( \frac{1}{4} \leq a < 1 \) \( \tag{3.56} \)

(iii) \( a = 1 \)

Lemma 13: If \( 1 < a \leq (\frac{2}{3} a^4)^{\frac{1}{3}} \), \( \frac{4}{3} \leq d \leq 2 \), then the theorem is true with strict inequality.

Proof: Let \( \nu > 1 \) be a solution of

\[
f(\nu) = \nu^2 - 6\nu + 8\nu - 3 - \frac{4d^4}{a(a - 1)^2} = 0 \tag{3.59}
\]

Such a \( \nu \) exists, because \( f(1) < 0, f(\infty) > 0 \). Then

\[
\Delta \left( \frac{\nu - 1}{\nu^2} \right)^{\frac{1}{2}} \nu^{2/3} = \frac{a - 1}{4a} \tag{3.60}
\]

By Lemma 3 we can find \( (z, t) = (z_0, t_0) \pmod{1} \) to satisfy

\[
-\nu^2 (a - 1) = \nu^2 \Delta \left( \frac{\nu - 1}{\nu^2} \right)^{1/2} \leq \psi(z, t) = -\beta
\]

\[
-\frac{\Delta}{\nu^2} \left( \frac{\nu - 1}{\nu^2} \right)^{1/2} = -\frac{a - 1}{4a} \tag{3.61}
\]

Now we want to choose \( \nu = \nu_0 \pmod{1} \) so as to satisfy (A). If
\[ \frac{a - \frac{1}{4a}}{4a} < \beta < \frac{4d - \frac{1}{4a}}{4a}, \quad (3.62) \]

choose \( y = y_o \pmod{l} \) with \(|y + fz + vt| \leq \frac{1}{2l} \), so that
\[ -\left(\frac{4d - 1}{4a}\right) \leq -\beta \leq (y + fz + vt)^2 + \psi(z, t) = (y + fz + vt)^2 - \beta \]
\[ \leq \frac{1}{4} - \frac{a - \frac{1}{4a}}{4a} = \frac{1}{4a^2} \]

using (3.62) and (A) follows. Let now
\[ \frac{4d - 1}{4a} \leq \beta \leq \frac{\psi(a - 1)}{4a} \quad (3.63) \]

We want to find \( y = y_o \pmod{l} \) such that
\[ 0 < (y + fz + vt)^2 - (\beta - \frac{4d - 1}{4a}) < \frac{d}{a} \quad (3.64) \]

Now, since \( 1 < a \leq (\frac{9}{64} d^4)^{1/3} \) and \( d \leq 2 \), we have
\[ \frac{1}{8} < \frac{a^3}{d^3} \leq \frac{2d}{64} \leq \frac{9}{32} < 1. \]

Therefore
\[ 1 < \frac{d}{a} < 2, \]
so that \( \lfloor \frac{d}{a} \rfloor = 1, \frac{d}{a} \) is not an integer. \( \quad (3.65) \)

By Lemma 3, Chapter III, (3.64) will be satisfied if
\[ \beta - \frac{4d - 1}{4a} < \frac{1}{4l}, \]

which will be satisfied if
\[
\frac{\nu^2(a - 1)}{4a} < \frac{4d + a - 1}{a - 1} \quad \text{or} \\
\nu^2 < \frac{4d + a - 1}{a - 1} = \nu_0^2
\] (3.66)

Since \( f'(\nu) = 4(\nu - 1)^2(\nu + 2) > 0 \), \( f(\nu) \) is an increasing function of \( \nu \). Therefore it suffices to show that

\[ f(\nu_0) > 0 \quad \text{or} \quad (3.67) \]

\[ (4d + a - 1)^2 - 6(a - 1)(4d + a - 1) + (8\nu_0 - 3)(a - 1)^2 - \frac{4d^4}{a} > 0 \quad \text{or} \]

\[ 8(a - 1)^2(\nu_0 - 1) > 4d^3 - 4ad + 4a(a - 1) \] (3.68)

Since \( a > 1 \), \( \nu_0 > 1 \), (3.68) is clearly satisfied if we have

\[ g(a, d) = d^3 - 4ad + 4a(a - 1) \leq 0 \] (3.69)

Since

\[
\frac{\partial g}{\partial a} = 4\left\{2a - d - 1\right\} \\
\leq 4\left\{2\left(\frac{9}{64} d^4\right)^{1/3} - d - 1\right\} \\
= 4\left\{\frac{3}{2} d\left(\frac{d}{3}\right)^{1/3} - d - 1\right\} \\
< 4\left\{\frac{3}{2} d - d - 1\right\} \quad \text{(since} \ d \leq 2) \\
\leq 0
\]

Therefore for \( 1 < a \leq \left(\frac{9}{64} d^4\right)^{1/3} \) and \( \frac{4}{3} \leq d \leq 2 \), we have

\[ g(a, d) < g(1, a) = d^3 - 4d = d(d^2 - 4) \leq 0 \]
Hence (3.69) is satisfied with strict inequality and the result follows as before. This completes the proof of the lemma.

**Lemma 14:** If \( \frac{1}{4} \leq a < 1, \frac{4}{3} \leq d < 2 \), then again the theorem is true with strict inequality.

**Proof:** Let

\[
\mu = -5 + \left\{ \frac{16 + \frac{4d^4}{a(1-a)^2}}{2} \right\}^{1/2} \tag{3.70}
\]

be a root of

\[
\frac{\Delta}{\left\{ (1+\mu)(5+\mu) \right\}^{1/2}} = \frac{1-a}{4a}. \tag{3.71}
\]

We have \( \mu \geq 3 \), if

\[
\frac{4d^4}{a(1-a)^2} \geq 48 \quad \text{or} \quad a(1-a)^2 \leq \frac{d^4}{12} \tag{3.72}
\]

Now \( a(1-a)^2 \leq \frac{1}{3}(1 - \frac{1}{3})^2 = \frac{4}{27} < \frac{d^4}{12} \) since \( d \geq \frac{4}{3} \). Thus (3.72) is true and we have

\[
\mu \geq 3 \tag{3.73}
\]

By Lemma 3, Chapter II, we can find \((z,t) = (z_0,t_0) \pmod{1}\) such that

\[
\left\{ (1+\mu)(5+\mu) \right\}^{1/2} \leq \varphi(z,t) < \frac{\Delta}{\left\{ (1+\mu)(5+\mu) \right\}^{1/2}} \tag{3.74}
\]
or

$$\frac{-\mu(1-a)_{4a}^2}{4a} < \psi(z,t) < \frac{1}{4a} - \frac{1}{4}$$  \hspace{1cm} (3.75)$$

If

$$\frac{-\frac{4d - 1}{4a}}{4a} < \phi(y, z, t) < \frac{1}{4a} - \frac{1}{4}$$  \hspace{1cm} (3.76)$$

Choose \( y = y_0 \pmod{l} \) such that \(|y + fz + vt| \leq \frac{1}{2} \), so that

$$\frac{-\frac{4d - 1}{4a}}{4a} < \phi(y, z, t) = (y + fz + vt)^2 + (z, t) < \frac{1}{4a} - \frac{1}{4} = \frac{1}{4a}.$$  

Therefore (A) is satisfied and the result follows. Let now

$$\frac{4d - 1}{4a} \leq \beta = \gamma(z, t) \leq \frac{\mu(1 - a)_{4a}^2}{4a}$$  \hspace{1cm} (3.77)$$

We want to choose \( y = y_0 \pmod{l} \) so as to satisfy (A) which is equivalent to

$$0 < (y + fz + vt)^2 - (\beta - \frac{4d - 1}{4a}) \leq \frac{d}{a}$$  \hspace{1cm} (3.78)$$

By Lemma 3, Chapter III, (3.78) will be satisfied if we have

$$\beta - \frac{4d - 1}{4a} < \frac{(d - a)^2}{2a}$$

This will be satisfied if

$$\frac{\mu(1 - a)_{4a}^2}{4a} - \frac{4d - 1}{4a} < \frac{(d - a)^2}{2a} \quad \text{or}$$

$$a[-5 + 5a + 1 + \frac{16(1 - a)^2}{a} + \frac{4d}{a} 
\frac{1/2}] < (d + a)^2$$
or
\[ a^2 [16(1 - a)^2 + \frac{4}{a} - \frac{4}{a^3}] < (a^2 + 2ad - 4a^2 + 4a)^2 \]

or
\[ f(a,d) = 16a^3 - 4a^2(4 - d) - 4ad(2 - d)(1 + d) - d^3 < 0, \quad (3.79) \]

for \( \frac{1}{4} \leq a < 1 \).

By rule of signs, for \( \frac{4}{3} \leq d \leq 2 \), \( f(a,d) \) has at most one positive root, since \( f(\infty, d) > 0 \) and

\[ f(1,d) = 16 - 4(4 - d) - 4d(2 - d)(1 + d) - d^3 \]

\[ = 3d^3 - 4d^2 - 4d \]

\[ = d(3d + 2)(d - 2) \]

\[ \leq 0, \quad \text{for} \quad \frac{4}{3} \leq d \leq 2. \quad (3.80) \]

Therefore for \( \frac{1}{4} \leq a < 1, \frac{4}{3} \leq d \leq 2 \) we have

\[ f(a,d) < f(1,d) \leq 0, \]

using (3.80). The result then follows from Lemma 7.

Lemma 15: If \( a = 1 \) and \( \frac{4}{3} \leq d \leq 1 \), the result is true perhaps with equality.

Proof: By Lemma 5, Chapter II, with \( \mu = \infty \), we can find \((z,t)\)

\( \equiv (z_0, t_0) \) (mod 1) such that

\[ -\frac{d^2}{2} = -\Delta \leq -\beta = \gamma(z, t) < 0 \quad (3.81) \]

using (3.42).
We want to find \( y \equiv y_0 \pmod{l} \) to satisfy (A), i.e.

\[-\frac{4d - 1}{4} < (y + fz + vt)^2 - \beta \leq \frac{1}{4}\]  

(3.82)

If \( 0 < \beta < \frac{4d - 1}{4} \), then the result follows as in (3.76), by choosing \( y \equiv y_0 \pmod{l} \) with \( |y + fz + vt| \leq \frac{1}{2} \). Let now

\[\frac{4d - 1}{4} \leq \beta \leq \frac{d^2}{2}\]  

(3.83)

(3.82) is equivalent to

\[0 < (y + fz + vt)^2 - (\beta - \frac{4d - 1}{4}) \leq d\]  

(3.84)

By Lemma 3, Chapter III, (3.84) will be satisfied if we have

\[\beta - \frac{4d - 1}{4} \leq (\frac{d^2}{2} - \frac{1}{2})^2\]  

(3.85)

This will be satisfied if

\[\frac{d^2}{2} - \frac{4d - 1}{4} \leq \frac{d^2 - 2d + 1}{4}\]

i.e.

\[d \leq 2,\]

which is true. Hence the result follows from Lemma 7. Lemma 12 then follows from Lemmas 13, 14, and 15.

4. Case of equality.

Lemma 16: Equality occurs if and only if \( Q = Q_2 \) or \( Q_3 \).

Proof: From Lemma 15, it follows that equality can occur only if
Also we must have equality in Lemma 5, Chapter II, so that either

\[ \gamma(z,t) \sim c_1(z^2 - t^2); \quad (z_o, t_o) \equiv (1,1) \pmod{2} \]  

or

\[ \gamma(z,t) \sim c_2 z t; \quad (z_o, t_o) \equiv (0,0) \pmod{2}. \]

c_1, c_2 > 0. Since \( \Delta^2 = 4 \) we have \( c_1 = 1, c_2 = 2 \). Without loss of generality we can suppose that either

\[ \gamma(z,t) = z^2 - t^2, \quad (z_o, t_o) \equiv (1,1) \pmod{2} \]  

or

\[ \gamma(z,t) = 2 z t, \quad (z_o, t_o) \equiv (0,0) \pmod{2}. \]

We now discuss the two cases separately.

Case (i) \( \gamma(z,t) = z^2 - t^2; \quad (z_o, t_o) \equiv (1,1) \pmod{2} \).

If equality is to occur in (A), then the inequalities

\[ -\frac{7}{4} = -\frac{4d - 1}{4a} < F(y, z, t) = (y + fz + vt + y_o + \frac{f}{2} + \frac{v}{2})^2 \]

\[ + (z + \frac{1}{2})^2 - (t + \frac{1}{2})^2 < \frac{1}{4a} = \frac{1}{4} \]

should have no solutions in integers \( y, z, t \).

\[ -\frac{7}{4} < F(y,0,0) = (y + y_o + \frac{f}{2} + \frac{v}{2})^2 < \frac{1}{4} \]

is solvable for integer \( y \) unless

\[ y_o + \frac{f}{2} + \frac{v}{2} \equiv \frac{1}{2} \pmod{1} \]  

(4.5)
Similarly by considering \( F(y,-1,0) \) and \( F(y,0,-1) \) we find that if equality is to occur we must have

\[
y_o - \frac{f}{2} + \frac{v}{2} \equiv \frac{1}{2} \pmod{1} \quad \text{and} \quad (4.6)
\]

\[
y_o + \frac{f}{2} - \frac{v}{2} \equiv \frac{1}{2} \pmod{1} \quad (4.7)
\]

From (4.5), (4.6) and (4.7) we have

\[
f \equiv v \equiv 0 \pmod{1}; \quad y_o \equiv \frac{1}{2} \pmod{1} \quad (4.8)
\]

Using (3.), we get

\[
f = v = 0; \quad y_o \equiv \frac{1}{2} \pmod{1} \quad (4.9)
\]

Therefore, if equality is to occur we must have

\[
\varphi(y,z,t) = y^2 + z^2 - t^2; \quad (y_o,z_o,t_o) \equiv \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \pmod{1} \quad (4.10)
\]

Again, if equality is to occur, the inequalities

\[
0 < G(x,y,z,t) = (x + hy + gz + ut + \frac{h}{2}, \frac{g}{2} + \frac{u}{2} + x_o)^2 - (y + \frac{1}{2})^2 - (z + \frac{1}{2})^2 + (t + \frac{1}{2})^2 < 2 \quad (4.11)
\]

should have no solutions in integers \( x, y, z, t \).

\[
0 < G(x,0,0,0) = (x + \frac{h}{2} + \frac{g}{2} + \frac{u}{2} + x_o)^2 - \frac{1}{4} < 2
\]

is solvable for integer \( x \) unless

\[
\frac{h}{2} + \frac{g}{2} + \frac{u}{2} + x_o \equiv \frac{1}{2} \pmod{1} \quad (4.12)
\]
Similarly by considering $G(x,0,0,0), G(x_0,0,-1,0)$ and $G(x,-1,0,0)$ we find that if equality is to occur we must have

$$\frac{h}{2} + \frac{g}{2} - \frac{u}{2} + x_0 = \frac{1}{2} \pmod{1} \quad (4.13)$$

$$\frac{h}{2} - \frac{g}{2} + \frac{u}{2} + x_0 = \frac{1}{2} \pmod{1} \quad (4.14)$$

$$-\frac{h}{2} + \frac{g}{2} + \frac{u}{2} + x_0 = \frac{1}{2} \pmod{1} \quad (4.15)$$

From (4.12), (4.13), (4.14) and (4.15) we get

$$h = g = u = 0 \pmod{1}, \quad x_0 = \frac{1}{2} \pmod{1}.$$  

Using (3.36) we have

$$h = g = u = 0, \quad x_0 = \frac{1}{2} \pmod{1}. \quad (4.16)$$

Thus in case (i) equality can occur only if

$$Q = x^2 - y^2 - z^2 + t^2 = q_2; \quad (x_0, y_0, z_0, t_0) \equiv (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \pmod{1} \quad (4.17)$$

We next show that equality is needed for this form. For this it suffices to show that for integers $x, y, z, t$ we have either

$$(x + \frac{1}{2})^2 - (y + \frac{1}{2})^2 - (z + \frac{1}{2})^2 + (t + \frac{1}{2})^2 \leq 0$$

or

$$(x + \frac{1}{2})^2 - (y + \frac{1}{2})^2 - (z + \frac{1}{2})^2 + (t + \frac{1}{2})^2 \geq 2. \quad (4.18)$$

This is equivalent to proving that
\[ x^2 - y^2 - z^2 + T^2 \leq 0 \text{ or } \geq 8 \text{ for odd integers } X, Y, Z, T. \]

(4.19)

This is true, since

\[ x^2 - y^2 - z^2 + T^2 \equiv 1 - 1 - 1 + 1 \equiv 0 \pmod{8} \]

for odd integers \( X, Y, Z, T \).

This completes the proof of the lemma in case (i).

Case (ii) \( \gamma(z, t) = 2zt; \ (z_0, t_0) \equiv (0, 0) \pmod{1} \)  

(4.20)

If equality is to occur in (A), then the inequalities

\[ -\frac{7}{4} < F(y, z, t) = (y + fz + vt + y_0)^2 + 2zt < \frac{1}{4} \]  

(4.21)

should have no solutions in integers \( y, z, t \).

By considering \( F(y, 0, 0); F(y, 1, 0) \) and \( F(y, 0, 1) \) we see that if equality is to occur we must have

\[ y_0 \equiv \frac{1}{2} \pmod{1} \]  

(4.22)

\[ y_0 + f \equiv \frac{1}{2} \pmod{1} \]  

(4.23)

\[ y_0 + v \equiv \frac{1}{2} \pmod{1} \]  

(4.24)

From (4.22), (4.23) (4.24) and (3.36) we get

\[ f = v = 0, \ y_0 \equiv \frac{1}{2} \pmod{1} \]  

(4.25)

Thus if equality is to occur we must have

\[ \phi(y, z, t) = y^2 + 2zt; \ (y_0, z_0, t_0) \equiv (\frac{1}{2}, 0, 0) \pmod{1} \]  

(4.26)

Again, for equality the inequalities
0 < G(x,y,z,t) = (x + hy + gz + ut + \frac{h}{2} + x_o)^2 - (y + \frac{1}{2})^2 - 2zt < 2

(4.27)

should have no solutions in integers x,y,z,t.

By considering \( G(x,0,0,0) \), \( G(x,0,0,1) \), \( G(x,0,1,0) \) and \( G(x,-1,0,0) \) we see that if equality is to occur we must have

\[
x_o + \frac{h}{2} \equiv \frac{1}{2} \pmod{1}
\]

(4.28)

\[
x_o - \frac{h}{2} \equiv \frac{1}{2} \pmod{1}
\]

(4.29)

\[
x_o + \frac{h}{2} + g \equiv \frac{1}{2} \pmod{1}
\]

(4.30)

\[
x_o + \frac{h}{2} + u \equiv \frac{1}{2} \pmod{1}
\]

(4.31)

From (4.28), (4.29), (4.30), (4.31) and (3.36) we have

\[
h = g = u = 0, \quad x_o \equiv \frac{1}{2} \pmod{1}
\]

(4.32)

Thus equality can occur only if

\[
Q(x,y,z,t) = x^2 - y^2 - 2zt = q_2, \quad (x_o, y_o, z_o, t_o) \equiv (\frac{1}{2}, \frac{1}{2}, 0, 0) \pmod{1}
\]

(4.33)

We next show that equality is needed for this form. For this it suffices to show that for integers x,y,z,t we have

\[
(x + \frac{1}{2})^2 - (y + \frac{1}{2})^2 - 2zt \leq 0 \quad \text{or} \quad \geq 2 \quad \text{i.e.}
\]

\[
(2x + 1)^2 - (2y + 1)^2 - 8zt \leq 0 \quad \text{or} \quad \geq 8
\]

(4.34)

This is clearly true since left hand side \equiv 0 \pmod{8} for integers x,y,z,t. This completes the proof of the lemma and the theorem follows.
§1. In this chapter we prove the following result:

Theorem: Let \( K \) be a sphere in three dimensional space \( \mathbb{R}^3 \), then

\[
\frac{\sqrt{3}}{2} \pi \leq \mathcal{D}_2(K) \leq \frac{5\sqrt{5}}{12} \pi
\]  

(1)

where \( \mathcal{D}_2(K) \) denotes the double lattice covering density of \( \mathbb{R}^3 \) by \( K \). (For definitions see Introduction, §2.1.)

Proof: Without loss of generality we can take \( K \) to be the unit sphere \( x^2 + y^2 + z^2 = 1 \). Let \( \Lambda \) be a lattice in \( \mathbb{R}^3 \) which provides the thinnest single covering of \( \mathbb{R}^3 \) by \( K \). Let \( P, Q, R \) be a basis of \( \Lambda \). Let \( \Lambda_1 \) be the lattice generated by \( P_1 = \frac{1}{2}P, Q, R \). Then \( \Lambda_1 \) provides double covering of the space by \( K \). For this, it suffices to show that every point of the fundamental parallelepiped with edges \( GP_1, GQ, GR \) is covered at least twice by the sets \( K + \Lambda, \Lambda \in \Lambda_1 \).

This is so since every point of the parallelepiped is covered at least once by the translates of \( K \) by the grids \( \Lambda \) and \( \Lambda + P_1 \) respectively. These grids are mutually disjoint and

\[
\Lambda_1 = \Lambda \cup (\Lambda + P_1)
\]

\[
a(\Lambda_1) = \frac{1}{2} a(\Lambda).
\]

Therefore,
\[ \mathcal{Q}_2(K) \leq \frac{V(K)}{d(\Lambda)} = 2 \cdot \frac{2V(K)}{d(\Lambda)} = 2 \cdot 2 \mathcal{Q}_1(K) \]
\[ = 2 \cdot \frac{5\sqrt{5}}{24} \pi \]
\[ = \frac{5\sqrt{5}}{12} \pi , \]

since \( \mathcal{Q}_1(K) = \frac{5\sqrt{5}}{24} \pi \) (see Bambah [3]).

For the inequality on the other side, we need consider only the lattices \( \Lambda \), with

\[ d(\Lambda) \geq V(K) \cdot \frac{12}{5\sqrt{5} \pi} = \frac{16}{5\sqrt{5}} ; \]  \hspace{1cm} (2)

which provide a double covering of the space by \( K \). We choose a reduced basis \( P, Q, R \) of \( \Lambda \) in the sense of Gauss and Seeber (see e.g. Dickson [16]).

With a suitable choice of axes we can suppose

\[ P = (a,0,0), \quad a > 0 \]
\[ Q = (h,b,0), \quad b > 0, \quad 0 \leq h \leq \frac{a}{2} \]  \hspace{1cm} (3)
\[ R = (g,f,c), \quad |g| \leq \frac{a}{2}, \quad 0 \leq f \leq \frac{b}{2}, \quad c > 0 \quad \text{and} \]
\[ 0 < |P| \leq |Q| \leq |R| \]  \hspace{1cm} (4)

Since the point 0 must be covered by the sets \( K + A, A \in \Lambda \) at least twice, there exists \( A \in \Lambda, A \neq 0 \) such that \( |A| \leq 1 \).

Therefore,

\[ 0 < a = |P| \leq |A| \leq 1 \]  \hspace{1cm} (5)

The two spheres \( K + P \) and \( K - P \) intersect in the circle.
Consider the point \( S(0,0,\sqrt{1 - a^2}) \). For all \( A \in \Lambda \), \( A \neq 0 \), in the plane \( z = 0 \), we have

\[
|SA|^2 = 1 - a^2 + |A|^2 \geq 1 - a^2 + a^2 = 1
\]

Therefore, the only sets \( K + A \), such that \( A \) lies in the section of \( \Lambda \) by the \( x-y \) plane which cover the points \( S = (0,0,\sqrt{1 - a^2} + \xi) \), where \( \xi > 0 \) is arbitrarily small, is \( K \). Thus there exists a point \( A \in \Lambda \), \( A \) not in the \( x-y \) plane such that \( |SA| < 1 \). Therefore,

\[
|A| \leq |S| + |SA| \leq 1 + \sqrt{1 - a^2}
\]

Now we have

\[
|Q| \leq |R| \leq |A| \leq 1 + \sqrt{1 - a^2}
\]

From (8) we have

\[
b \leq |Q| \leq 1 + \sqrt{1 - a^2} \quad \text{and} \quad c \leq |R| \leq 1 + \sqrt{1 - a^2}
\]

If \( b \leq 1 \) or \( c \leq 1 \), then

\[
d(\Lambda) = abc \leq a(1 + \sqrt{1 - a^2})
\]

\[
\leq \sqrt{\frac{3}{2}}(1 + \sqrt{1 - \frac{3}{4}}) = \frac{3\sqrt{3}}{4} < \frac{16}{5\sqrt{5}},
\]

contrary to (2). Therefore
In particular, every point in the plane $z = 0$ must be covered by sets $K + A$, where $A$ is in the plane $z = 0$. Consider the point $T(0, \sqrt{1 - a^2}, 0)$. For all $A \in A$, $A$ on the x-axis, $A \neq 0$,

$$|AT|^2 \geq 1 - a^2 + a^2 = 1$$

By considering the double covering of points $T(0, \sqrt{1 - a^2} + \varepsilon, 0)$, $\varepsilon > 0$ sufficiently small, we find that there exists $A \neq 0$ of $A$ in the x-y plane, but not on the x-axis, such that $|AT| \leq 1$. Since $b > 1$, this point $A$ must be on the line through $Q$ parallel to the x-axis. Then by the definition of $Q'$ it is clear that

$$|QT| \leq |AT| \leq 1$$

(12)

Also, $|TP| = |(-T)P| = 1$, so that the triangle $\Delta$ with vertices $P$, $Q$ and $-P$ is contained in a circle of radius 1 with center at $T$. We next observe that the triangle $\Delta$ is an acute angled triangle. Since $|h| \leq \frac{a}{2}$, $\xi P$ and $\xi -P$ are clearly acute. Also

$$|OQ| \geq b > 1 \geq a = |OP| = |O(-P)|.$$

Therefore,

$$\xi QOP < \xi OPQ,$$

$$\xi OQ(-P) < \xi O(-P)Q,$$

so that
\[ \angle Q = \angle OQP + \angle OQ - P < \angle OPQ + \angle O(-P) = \pi - \angle Q \quad \text{or} \]
\[ \angle Q < \frac{\pi}{2}. \]

Let \( S \) be the circumcenter of \( \Delta \). Since the triangle is acute angled, \( S \) lies within it. Let \( r \) be its circumradius.

Now we have that

\[ |TP| = |T(-P)| = 1, \quad |TQ| \leq 1, \]

where \( T = (0, \sqrt{1 - a^2}, 0) \), \( S = (0, \sqrt{r^2 - a^2}, 0) \).

\( S \) and \( T \) both lie on the y-axis. If \( S \) is below \( T \) then \( r = |SP| = |S(-P)| \leq |TP| = |T(-P)| = 1 \). If \( S \) is above \( T \), then \( r = |SQ| \leq |TQ| \leq 1 \), since then \( \angle QST \) is obtuse. So that in either case we have

\[ r \leq 1 \quad (13) \]

Let \( H \) be the point \((0, \sqrt{r^2 - a^2}, \sqrt{1 - r^2}) \).

For any point \( A \in \mathbb{A}, A \neq 0 \) in the x-y plane, we have

\[ |SA| \geq |SP| = |S(-P)| = |SQ| = r. \quad (14) \]

Since if \( A \) does not lie on the x-axis or on the line through \( Q \) parallel to the x-axis, \( |SA| \geq b > 1 \geq r \). If \( A \) lies on the x-axis \( A \neq 0 \), then \( A = kP, |k| \geq 1, \) and \( |SA| \geq |SP| = |S(-P)| = r \).

If \( A \) lies on the line through \( Q \) parallel to the x-axis, \( |SA| \geq |SQ| = r; \) since \( Q \) is the point nearest to the y-axis on this line.
Thus, 

$$|AH|^2 = 1 - r^2 + |SA|^2 \geq 1 - r^2 + r^2 = 1.$$ 

Therefore the point $H$ must be covered by a set $K + B$ where $B = (\xi, \eta, \zeta)$, $\zeta \neq 0$, $B \in \Lambda$. If $\zeta < 0$ then $\zeta \leq -c < -1$ so that $|BH| \geq \sqrt{1 - r^2} - \zeta > 1$, and $K + B$ does not contain $H$. Therefore $\zeta > 0$, so that $\zeta = kc$, $k \geq 1$. If $k > 1$, then again $|BH| \geq |2c - \sqrt{1 - r^2}| \geq 2c - 1 > 1$, since $c > 1$. Thus we must have $\zeta = c$.

Also since $|BH| \leq 1$, we have

$$\left(c - \sqrt{1 - r^2}\right)^2 \leq |BH|^2 \leq 1 \quad \text{or} \quad c \leq 1 + \sqrt{1 - r^2} \quad (14)$$

Also since $SQ = r$, we have

$$\left(b - \sqrt{r^2 - a^2}\right)^2 + h^2 = r^2 \quad (15)$$

Therefore,

$$d(\Lambda) = abc \leq a(\sqrt{r^2 - h^2} + \sqrt{r^2 - a^2})(1 + \sqrt{1 - r^2})$$

$$\leq a(r + \sqrt{r^2 - a^2})(1 + \sqrt{1 - r^2})$$

$$= \frac{1}{\sqrt{3}} \left\{ 2\sqrt{a^2 + \frac{3}{4} r^2} + \sqrt{a^2 + 3(r^2 - a^2)} \right\} (1 + \sqrt{1 - r^2})$$

$$\leq \frac{1}{\sqrt{3}} \left\{ a^2 + \frac{3}{4} r^2 + \frac{a^2 + 3(r^2 - a^2)}{2} \right\} (1 + \sqrt{1 - r^2}) ,$$

by the inequality of arithmetic and the geometric mean. Thus
This completes the proof of the theorem.

\[ \frac{\sigma}{E} = \frac{G}{E} \cdot \frac{E}{\mu} = \frac{G}{(E)A} > (x)^{2} > \]

and

\[ \frac{\sigma}{E} = \]

\[ \left( \frac{6 - \tau}{8} + \tau \right) \frac{6}{2} \cdot \frac{1}{\gamma} = \]

\[ \left( \frac{6}{8} \right) \gamma > \]

(\text{see}) \quad (x) \left( \frac{x - \tau}{2} + \tau \right) = \frac{\gamma}{\gamma} > (\forall) \]
§1. In this chapter we prove the following theorem:

Theorem: Let \( f(x,y) = ax^2 + bxy + cy^2 \) be an indefinite binary quadratic form with discriminant \( d = b^2 - 4ac > 0 \). Let

\[
M_+^+(f) = \inf f(u,v),
\]

where the infimum is taken over all integers \( u,v \) such that \( f(u,v) > 0 \).

Then either

\[
M_+^+(f) \leq \frac{\sqrt{d}}{2} \quad \text{or} \quad M_+^+(f) \geq \frac{5}{4} \sqrt{\frac{\sqrt{d}}{6}}.
\]

Equality is necessary at both the places. There is no isolation in the first case.

Proof: If \( M_+^+(f) = 0 \), the result is trivially true. Therefore suppose

\[
M_+^+(f) > 0
\]

Since the ratio \( \frac{M_+^+(f)}{\sqrt{d}} \) is not affected by replacing \( f(x,y) \) by \( \zeta f(x,y) \) where \( \zeta > 0 \); by replacing \( f(x,y) \) by \( \frac{1}{M_+^+(f)} f(x,y) \) we can suppose that

\[
M_+^+(f) = 1
\]

Then it suffices to prove the following:

Theorem A: Let \( f(x,y) = ax^2 + bxy + cy^2 \) be an indefinite binary quadratic form with discriminant \( d = b^2 - 4ac > 0 \). Let \( M_+^+(f) = 1 \).
Then either \( d \geq 4 \) or \( d < \frac{96}{29} \). Equality is needed at both the places. There is no isolation at \( d = 4 \).

Proof of Theorem A: Since \( M_+(f) = 1 \), given \( 0 < \varepsilon_0 < 1 \), there exist integers \( u, v \) such that

\[
1 \leq f(u, v) = 1 + \varepsilon < 1 + \varepsilon_0
\]  

Since \( 1 + \varepsilon < 2 \), we must have \((u, v) = 1\). Since the result is not affected by replacing \( f(x, y) \) by an equivalent form we can suppose that \( f(1, 0) = 1 + \varepsilon \). Then \( f(x, y) \) can be written as

\[
f(x, y) = (1 + \varepsilon)(x + ay)^2 - \frac{\Delta}{1 + \varepsilon} y^2
\]

where

\[
4 \Delta = d.
\]

By a transformation of the type \( x = \pm X + mY, y = Y \), where \( m \) is a suitable integer we can suppose that

\[
0 \leq \alpha \leq \frac{1}{2}.
\]

Since \( M_+(f) = 1 \), for all integers \( u, v \) exactly one of the two holds

\[
P(u, v) : f(u, v) \geq 1
\]

\[
N(u, v) : f(u, v) \leq 0
\]

Take \((u, v) = (1,-1)\). If \( P(1,-1) \) holds, then
\[(1 + \xi)(1 - \alpha)^2 - \frac{\Delta}{1 + \xi} \geq 1 \quad \text{or} \]

\[
\frac{\Delta}{1 + \xi} \leq (1 + \xi)(1 - \alpha)^2 - 1 \\
\leq 1 + \xi - 1 = \xi, \quad \text{(by (8))}, \quad \text{or} \\
d = \frac{4}{\Delta} \leq \frac{4\xi(1 + \xi)}{4\xi(1 + \xi_0)} \\
(11)
\]

Since \(d\) is independent of \(\xi_0\) and positive, (11) cannot hold if \(\xi_0\) is taken small enough.

Therefore \(P(1,-l)\) cannot hold if \(\xi_0\) is taken sufficiently small, and we must have \(N(1,-l)\). Take \((u,v) = (1,1)\). If \(N(1,1)\) holds, then

\[
(1 + \xi)(1 + \alpha)^2 - \frac{\Delta}{1 + \xi} \leq 0, \quad \text{or} \\
d = 4\Delta \geq 4(1 + \xi)^2(1 + \alpha)^2 \\
\geq 4, \quad \text{since} \; \xi \geq 0, \; \alpha \geq 0.
\]

Hence the theorem is true if \(N(1,1)\) holds.

We can now suppose that \(P(1,1)\) holds. Take \((u,v) = (-3,2)\).

If \(N(-3,2)\) holds, then

\[
(1 + \xi)(3 - 2\alpha)^2 - \frac{4\Delta}{1 + \xi} \leq 0, \quad \text{or} \\
d = 4\Delta \geq (1 + \xi)^2(3 - 2\alpha)^2 \\
\geq 4, \quad \text{since} \; \xi \geq 0, \; 0 \leq \alpha \leq \frac{1}{2},
\]

and the result is true in this case also.
Thus we have either $d > \frac{K}{2}$, and the theorem is true; or

$$N(1,-1), P(1,1) \text{ and } P(-3,2) \text{ hold.} \quad (12)$$

We shall now show that if (12) holds, then $d \leq \frac{96}{25}$.

By (12) we have

$$\frac{(1 + \varepsilon)(1 - \alpha)^2}{1 + \varepsilon} \leq 0 \quad (13)$$

$$\frac{(1 + \varepsilon)(1 + \alpha)^2}{1 + \varepsilon} \geq 1 \quad (14)$$

$$\frac{(1 + \varepsilon)(3 - 2\alpha)^2}{1 + \varepsilon} \geq 1 \quad (15)$$

On substituting $2\alpha = A$, $\alpha^2 = \frac{\Delta}{(1 + \varepsilon)^2} = B$, (7), (8), (13), (14) and (15) reduce to

$$d = (1 + \varepsilon)^2(A^2 - 4B) \quad (16)$$

$$0 \leq A \leq 1 \quad (17)$$

$$1 - A + B \leq 0 \quad (18)$$

$$1 + A + B \geq \frac{1}{1 + \varepsilon} \quad (19)$$

$$9 - 6A + 4B \geq \frac{1}{1 + \varepsilon} \quad (20)$$

From (17) and (18) we have

$$B \leq A - 1 \leq 0 \quad (21)$$

From (19) and (20) we have

$$15 + 10B \geq \frac{7}{1 + \varepsilon} \quad \text{or} \quad B \geq \frac{(8 + 5\varepsilon)}{10(1 + \varepsilon)} \quad (22)$$
From (17) and (20) we have

$$0 \leq A \leq \frac{9 + 4B - \frac{1}{1+\varepsilon}}{6}$$  \hspace{1cm} (23)

Using (23), from (16) we get

$$d = \left(1 + \varepsilon\right)^2 \left(A^2 - 4B\right)$$

$$\leq \left(1 + \varepsilon\right)^2 \left\{ \left(9 + 4B - \frac{1}{1+\varepsilon}\right)^2 \cdot \frac{36}{9} - 4B \right\}$$

$$= \frac{(8 + 4B(1 + \varepsilon) + 9\varepsilon)^2}{36} - 4B(1 + \varepsilon)^2$$

$$= \frac{(8 + 4B)^2}{36} - 4B + 0(\varepsilon)$$

$$= \frac{4}{9}(8^2 - 5B + 4) + 0(\varepsilon)$$

$$\leq \frac{4}{9} \left\{ \left(\frac{8 + 15\varepsilon}{10(1 + \varepsilon)}\right)^2 + \frac{5(8 + 15\varepsilon)^2}{10(1 + \varepsilon)} + 4 \right\} + 0(\varepsilon),$$

by using (21) and (22). Thus

$$d \leq \frac{4}{9} \left\{ \frac{16}{25} + 4 + 4 \right\} + 0(\varepsilon)$$

$$= \frac{96}{25} + 0(\varepsilon_0)$$  \hspace{1cm} (24)

Since $d$ is independent of $\varepsilon_0$, we must have

$$d \leq \frac{96}{25}.$$

We next show that equality is needed at both the places.
(i) At \( d = 4 \). Consider

\[ f(x, y) = x^2 - y^2 \]  
\[ d(f) = 4 \]  

Since \( f(u, v) \) is an integer for integral values of \( u, v \), we have \( f(u, v) \geq 1 \) or \( f(u, v) \leq 0 \). Also \( f(1, 0) = 1 \). Therefore \( M_+(f) = 1 \). Hence equality is needed for this form.

(ii) \( d = \frac{96}{25} \). Consider

\[ f(x, y) = x^2 + \frac{4}{5} xy - \frac{4}{5} y^2 \]  
\[ d(f) = \frac{96}{25}, f(1, 0) = 1 \]. To prove that \( M_+(f) = 1 \), it suffices to show that for integers \( u, v \) we cannot have

\[ 5u^2 + 4uv - 4v^2 = 1, 2, 3 \text{ or } 4 \]  

If \( 5u^2 + 4uv - 4v^2 = 2, 3 \); we have

\[ u^2 \equiv 2, 3 \pmod{4}, \]

which is not possible.

If \( 5u^2 + 4uv - 4v^2 = 1, 4 \); we have

\[ 6u^2 - (u - 2v)^2 = 1, 4 \]

or

\[ (u - 2v)^2 \equiv 2 \pmod{3}, \]

which is not possible and the assertion is proved.

Now we prove that there is no isolation at \( d = 4 \). Consider the forms
where \( k \) is a positive integer. Then

\[
d(f_k) = 4(1 + \frac{2}{k})
\]

(29)
tends to 4 as \( k \) tends to infinity. To prove the result it suffices to prove that

\[
M_+(f_k) = 1
\]

(30)
for all \( k \). Since \( f_k(1,0) = 1 \), it suffices to show that if

\[
g_k(x,y) = kx^2 - (k + 2)y^2
\]

(31)
we cannot have integers \( u,v \) satisfying

\[
g_k(u,v) = k_1, \quad 0 < k_1 < k.
\]

(32)
Let if possible

\[
g_k(u,v) = ku^2 - (k + 2)v^2 = k_1.
\]

(33)
where \( 0 < k_1 < k \).

Among the solutions \((u,v)\) of (33) choose one for which \(|u|\) is least possible. Let it be \((u_1,v_1)\). Then \( u_1 \neq 0 \), since

\[
g_k(0,v_1) = -(k + 2)v_1^2 \leq 0.
\]

Also if \((u_1,v_1)\) satisfies (33), then so do \((-u_1,v_1)\).

Therefore we can suppose that
Thus we have

\[ g_k(u_1, v_1) = ku_1^2 - (k + 2)v_1^2 = k_1 > 0 \quad \text{and} \quad (35) \]

\[ g_k(u, v) = k_1 \quad \text{implies} \quad |u| \geq u_1 \quad (36) \]

It can be easily verified that an automorph of \( g_k(x, y) \) is given by

\[ g_k(x, y) = g_k((k + 1)x - (k + 2)y, kx - (k + 1)y) \quad (37) \]

Therefore

\[ g_k((k + 1)u_1 - (k + 2)v_1, ku_1 - (k + 1)v_1) = g_k(u_1, v_1) = k_1. \]

From (36), we have

\[ |(k + 1)u_1 - (k + 2)v_1| \geq u_1 \quad (38) \]

From (35) we have

\[ k(k + 2)u_1^2 - (k + 2)v_1^2 = k_1(k + 2) > 0 \quad \text{or} \]

\[ (k + 1)^2u_1^2 - (k + 2)^2v_1^2 > u_1^2 > 0 \quad \text{or} \]

\[ \{(k + 1)u_1 + (k + 2)v_1\} \{(k + 1)u_1 - (k + 2)v_1\} > 0. \]

Therefore,

\[ (k + 1)u_1 - (k + 2)v_1 > 0, \]
and (38) gives

\[(k + 1)u_1 - (k + 2)v_1 \geq u_1 \quad \text{or} \]
\[ku_1 \geq (k + 2)v_1 \quad \text{or} \quad (39)\]

\[u_1 \geq (1 + \frac{2}{k})v_1 > v_1 \geq 0 \quad \text{or} \]
\[u_1 > v_1 \geq 0 \quad (40)\]

Therefore, using (39) and (40) we get

\[k_1 = g_k(u_1, v_1) = ku_1^2 - (k + 2)v_1^2 \]
\[\geq ku_1^2 - ku_1v_1 \]
\[= ku_1(u_1 - v_1) \]
\[\geq k.\]

This contradicts the supposition that \(0 < k_1 < k\). Hence \(M_k(p_k) = 1\), and the proof of the theorem is completed.
CHAPTER VII

AN INHOMOGENEOUS MINIMUM OF A CLASS OF FUNCTIONS

1. Several authors (Barnes [5], Bambah [1], [4], Chalk [12], Mordell [17] and K Rogers [4], [18]) have proved theorems of the following type:

Theorem A. Let \( f(x,y) \) be a function of a given form. Then for given real \( x_0, y_0 \), there exist integers \( x, y \) such that

\[
|f(x + x_0, y + y_0)| \leq \max \{|f(\frac{1}{2},0)|, |f(0,\frac{1}{2})|, |f(\frac{1}{2},\frac{1}{2})|, |f(\frac{1}{2},-\frac{1}{2})|\}. \tag{1.1}
\]

In particular, Bambah [1] proved the result for all binary cubic forms with real coefficients and positive discriminants, and Bambah and K. Rogers [4] proved it for function corresponding to the "regions with hexagonal symmetry." They remarked that their theorem does not have an analogue for "general" regions with four or more asymptotes. Their example (pp. 345) shows that this is true if the right hand side of (1.1) is unchanged.

However, we can interpret the above theorem as stating that

\[
\sup_{x_0, y_0} \inf_{(x,y) \text{ integers}} |f(x + x_0, y + y_0)|
\]

is bounded above by the maximum of values of \( f(x,y) \) at a finite number of points, which can be written down as soon as \( f(x,y) \) is given. With this interpretation we prove that the results of theorem
A can be generalized to regions with four asymptotes with "octagonal symmetry."

Let \( L_1, L_2, L_3, L_4 \) be the lines \( y = 0, y = x, x = 0, y = -x \) respectively. We shall say that a region \( S \) in the \( x - y \) plane has "octagonal symmetry" if it satisfies the following conditions:

(i) \( S \) is symmetric with respect to the lines \( L_{i+4} \) (i.e. the lines bisecting the angles between them).

(ii) The boundary of \( S \) either has the lines \( L_{i+4} \) as asymptotes or terminates in them.

(iii) The region external to \( S \) and lying between \( O_1 \) and \( O_2 \) is convex.

(iv) Each of the eight branches of \( S \) is a continuous curve.

The automorphisms of \( S \) are generated by

1) Reflections in the axes.
2) Reflections in \( y = \pm x \).
3) Rotations through \( \pi/4 \).

We prove

**Theorem 1.** Let \( S : f(x,y) \leq 1 \) be a star region with "octagonal symmetry." Let \( \Lambda \) be a lattice. Let \( \omega \) be an automorphism of \( S \) (the existence of such automorphism is obvious) such that \( \omega \Lambda \) has a base \( A = (\alpha, \gamma)B = (\beta, \delta) \) satisfying
(i) \( 0 < \gamma < \alpha \)

(ii) \( OA \leq OB \)

(iii) \( \Delta AOB \leq \pi/2 \).

(iv) The direction of rotation from \( OA \) to \( OB \) is anti-clockwise. Then given any point \( (x_o,y_o) \) we can find \( (x,y) \equiv (x_o,y_o) \) (mod \( \Lambda \)) such that

\[
f(x,y) \leq \max \left[ f\left(\frac{\alpha + \beta}{2}, \frac{\gamma + \delta}{2}\right), f\left(\frac{\alpha - \beta}{2}, \frac{\gamma - \delta}{2}\right), f\left(\frac{\alpha}{2}, \frac{\alpha}{2}\right), f\left(\frac{\beta}{2}, \frac{\beta}{2}\right)\right]
\]

where \( (a, ka) \) is the point where the line \( x = a \) touches the curve \( f(x,y) = f(a, ka) \).

An immediate consequence of theorem 1 is

Theorem 2. Let \( \varphi(u,v) \) be a function such that a non-singular linear transformation \( x = \alpha u + \beta v, y = \gamma u + \delta v \) transforms the region \( \varphi(u,v) \leq 1 \) into a star region \( S : f(x,y) \leq 1 \) with "octagonal symmetry" (i.e. \( \varphi(u,v) = f(\alpha u + \beta v, \gamma u + \delta v) \)). Because of the automorphisms of \( S \), we can choose \( \alpha, \beta, \gamma, \delta \), satisfying the conditions of theorem 1. Then given \( (u_o, v_o) \) we can find integers \( u, v \) such that

\[
\varphi(u + u_o, v + v_o) \leq \max\left[ f\left(\frac{\alpha + \beta}{2}, \frac{\gamma + \delta}{2}\right), f\left(\frac{\alpha - \beta}{2}, \frac{\gamma - \delta}{2}\right), \right.
\]

\[
f\left(\frac{\alpha}{2}, \frac{\alpha}{2}\right), f\left(\frac{\beta}{2}, \frac{\beta}{2}\right), f\left(\frac{\alpha + \gamma}{2}, \frac{\alpha + \gamma}{2}\right), f\left(\frac{\beta - \delta}{2}, \frac{\beta - \delta}{2}\right)\]

where \( k \) is defined in theorem 1.
If we take \( f(x, y) = |xy(x^2 - y^2)| \)
then theorem 1 reduces to

**Theorem 3.** Let \( f(x, y) = |xy(x^2 - y^2)| \) and \( \Lambda \) be a lattice. We can find an automorphism \( \omega \) of \( f(x, y) \) such that the lattice \( \omega \Lambda \)
has a basis \( A = (\alpha, \gamma), B = (\beta, \delta) \) satisfying the conditions of theorem 1. Then given reals \((x_0, y_0)\) we can find \((x, y) = (x_0, y_0) \pmod{\Lambda})\) such that

\[
 f(x, y) \leq \max \{ f\left(\frac{\alpha + \beta}{2}, \frac{\gamma + \delta}{2}\right), f\left(\frac{\alpha - \beta}{2}, \frac{\gamma - \delta}{2}\right), f\left(\frac{\alpha}{2\sqrt{3}}, \frac{\gamma}{2\sqrt{3}}\right), \\
 f\left(\frac{\delta}{2\sqrt{3}}, \frac{\delta}{2\sqrt{3}}\right), f\left(\frac{\alpha + \gamma}{2}, \frac{\alpha + \gamma}{2}\right), f\left(\frac{\beta - \delta}{2\sqrt{6}}, \frac{\beta - \delta}{2\sqrt{6}}\right) \}
\]

and theorem 2 reduces to

**Theorem 4.** Let \( \Phi(u, v) = au^4 + 4bu^3v + 6cu^2v^2 + 4duv^3 + ev^4 \) be a binary quartic form which can be transformed to \( f(x, y) = |xy(x^2 - y^2)| \) by means of a non-singular linear transformation. By applying a suitable automorphism of \( f(x, y) \) we can suppose that the linear transformation is given by \( x = \alpha u + \beta v, y = \gamma u + \delta v \) such that \( A = (\alpha, \gamma), B = (\beta, \delta) \) satisfy the conditions of theorem 1. Then given any \((u_0, v_0)\), we can find integers \( u, v \) such that

\[
 |\Phi(u + u_0, v + v_0)| \leq \max\{ f\left(\frac{\alpha + \beta}{2}, \frac{\gamma + \delta}{2}\right), f\left(\frac{\alpha - \beta}{2}, \frac{\gamma - \delta}{2}\right), \\
 f\left(\frac{\alpha}{2\sqrt{3}}, \frac{\gamma}{2\sqrt{3}}\right), f\left(\frac{\delta}{2\sqrt{3}}, \frac{\delta}{2\sqrt{3}}\right), f\left(\frac{\alpha + \gamma}{2}, \frac{\alpha + \gamma}{2}\right), f\left(\frac{\beta - \delta}{2\sqrt{6}}, \frac{\beta - \delta}{2\sqrt{6}}\right) \}.
\]

**Theorem 5.** The results of theorems 1 to 4 are best possible in the sense that equality is necessary for certain functions.
2. Since the proofs of theorem 1 and theorem 3 are the same, we shall prove theorem 3 here. It is enough to prove

Theorem 3'. Let $f(x, y) = |xy(x^2 - y^2)|$ and $\Lambda$ be a lattice generated by the points $A = (a, \gamma)$, $B = (\beta, \delta)$ satisfying the conditions (i)-(iv) stated in theorem 1. Then given $(x_0, y_0)$ we can find $(x, y) \equiv (x_0, y_0) \pmod{\Lambda}$ such that

$$f(x, y) \leq \max \left[ f\left(\frac{a + \beta}{2}, \frac{\gamma + \delta}{2}\right), f\left(\frac{a - \beta}{2}, \frac{\gamma - \delta}{2}\right), f\left(\frac{a}{2}, \frac{\alpha}{2}\right), \right.$$  

$$\left. f\left(\frac{\beta + \gamma}{2}, \frac{\alpha + \gamma}{2}\right), f\left(\frac{\beta - \delta}{2}, \frac{\beta - \delta}{2}\right) \right].$$

To prove theorem 3', it is enough to prove

Theorem B: Let $\Lambda$ be the lattice generated by $A, B$ satisfying the conditions of theorem 1. Then the parallelogram $OACB$ where $C = A + B$ and hence the whole plane, is covered by the sets $S + P, P \in \Lambda$ where

$$S : f(x, y) \leq \max \left[ f\left(\frac{a + \beta}{2}, \frac{\gamma + \delta}{2}\right), f\left(\frac{a - \beta}{2}, \frac{\gamma - \delta}{2}\right), f\left(\frac{a}{2}, \frac{\alpha}{2}\right), \right.$$  

$$\left. f\left(\frac{\beta + \gamma}{2}, \frac{\alpha + \gamma}{2}\right), f\left(\frac{\beta - \delta}{2}, \frac{\beta - \delta}{2}\right) \right].$$

i.e. $S : f(x, y) \leq \mathcal{C}$ has $\Lambda$ as a covering lattice if $S$ contains the points

$$\left(\frac{a + \beta}{2}, \frac{\gamma + \delta}{2}\right), \left(\frac{a}{2}, \frac{\alpha}{2}\right), \left(\frac{\beta + \gamma}{2}, \frac{\alpha + \gamma}{2}\right),$$

and $$\left(\frac{\beta - \delta}{2}, \frac{\beta - \delta}{2}\right).$$

3. Some Lemmas:

Lemma 1. Let PQRS be a parallelogram with PQ, QS parallel to two consecutive asymptotes of $S$. Define $S' : f(x, y) \leq f(a, b)$, where
Then PQRS is covered by S' + P and S' + R.

Proof: is the same as that of Lemma (a) of Bambah and K. Rogers [4].

Lemma 2. Let PQR be a right angled triangle with PQ, PR parallel to two consecutive asymptotes, \( \angle PQR = \frac{\pi}{2} \) and PQ = \( \gamma \). Then the triangle PQR is covered by \( \{ f(x, y) \leq f(\gamma, \frac{\gamma}{\sqrt{3}}) \} + P \).

Proof. By symmetry it is enough to consider the case when PQ is parallel to \( y = 0 \) and PR is parallel to \( y = x \).

Let \( S' : f(x, y) \leq f(\gamma, \frac{\gamma}{\sqrt{3}}) \).

Then PQ, PR are consecutive asymptotes to the boundary of \( S' + P \). The point \( T = (\gamma, \frac{\gamma}{\sqrt{3}}) + P \) lies on its boundary and has QR as the tac line through it, so that PQR is covered by \( S' + P \) (Fig. 1).

\[ \text{Case I (Fig. 2).} \]

By lemma 1 the parallelogram bounded by the lines \( OL_1 \), \( OL_2 \) and the lines through \( C \) parallel to them is covered by \( S \) and \( S + C \).

[Since \( \frac{C}{2} \in S \)] and hence the parallelogram OACB is also covered.
Case II.

Draw $AG \parallel OL_2$, $BF \parallel OL_5$, $AH \parallel OL_4$, $BK \parallel OL_3$, $AM \parallel OL_3$ to meet $OL_2$ in $G, F, H, K, M$ respectively. Let $BK$ meet $AG$ in $D$. Then we have the following subcases to consider depending upon the relative positions of $C, D, F, G, H, K, M$.

(i) $D$ on $GA$ produced, $C$ in the first sector, $G$ below $F$ and $F$ below $H$ (Fig. 3).

(ii) $D$ on $GA$ produced, $C$ in the second sector $F$ below $H$
   (a) $G$ below $F$ (Fig. 4)
   (b) $F$ below $G$ (Fig. 5)

(iii) $D$ on $GA$ produced, $C$ in the first sector, $F$ above $H$
   (a) $F$ below $M$ (Fig. 6)
   (b) $F$ above $M$ (Fig. 7)

(iv) $D$ on $GA$ produced, $C$ in the second sector, $F$ above $H$
   (a) $F$ below $M$ (Fig. 8)
   (b) $F$ above $M$ (Fig. 9)

(v) $D$ in $AG$, $C$ in the first sector, $G$ below $F$ (Fig. 10)

(vi) $D$ in $AG$, $C$ in the second sector, $F$ below $G$ (Fig. 11)

(vii) $D$ on $AG$ produced, $C$ in the second sector (Fig. 12).

The following subcases cannot arise:

1. When $C$ is in the first sector and $F$ is below $G$. For, if this were so then $\beta < \gamma$ and $\gamma + \delta < \alpha + \beta$ so $\delta < \alpha$ and we get $OB^2 = \beta^2 + \delta^2 < \alpha^2 + \gamma^2 = OA^2$ which is contrary to the hypothesis.

2. When $C$ is in the second sector, $D$ is in $AG$ and $G$ is below $F$. For, if it were so then $\gamma < \beta$, $\alpha + \beta < \gamma + \delta$ and $\beta + \delta - \gamma < \alpha$. Then $\beta < \gamma$ from the last two which contradicts the first.
The shaded parallelograms are covered by $S + A$ and $S + B$ using lemma 1 since $\frac{A - B}{2} \in S$.

In the cases (i); (ii)(a); (ii)(b); (v); (vi)(b) it is enough to cover the triangles $OAH$ and $O BK$.

The curves

$$f(x, y) = f\left(\frac{\alpha}{2}, \frac{\alpha}{2\sqrt{3}}\right)$$

and

$$\left\{ f(x, y) = f\left(\frac{\alpha}{2}, \frac{\alpha}{2\sqrt{3}}\right) \right\} + A$$

touch the line $x = \frac{\alpha}{2}$ and the touching arcs lie on the opposite sides of it. The portion of the triangle $OAH$ on the side of $x = \frac{\alpha}{2}$ containing $0$ is covered by

$$f(x, y) \leq f\left(\frac{\alpha}{2}, \frac{\alpha}{2\sqrt{3}}\right)$$

and the portion on the side opposite to it is covered by

$$\left\{ f(x, y) \leq f\left(\frac{\alpha}{2}, \frac{\alpha}{2\sqrt{3}}\right) \right\} + A$$

and hence the triangle $OAH$ is covered by $S$ and $S + A$.

In fact, if we draw the line $x = x_1, 0 < x_1 < \alpha$ then the portion of the triangle $OAH$ to the right of this line is covered by

$$\left\{ f(x, y) \leq f\left(\alpha - x_1, \frac{\alpha - x_1}{\sqrt{3}}\right) \right\} + A$$

and the portion to the left is covered by $f(x, y) \leq f(x_1, \frac{x_1}{\sqrt{3}})$ by lemma 2 and the constant involved is the same if we take $x_1 = \frac{\alpha}{2}$ [Fig 13].
Similarly we can cover the triangle OBK by \( f(x,y) \leq f\left(\frac{5}{2}, \frac{5}{2\sqrt{3}}\right) \)
and \( \left\{ f(x,y) \leq f\left(\frac{5}{2}, \frac{5}{2\sqrt{3}}\right) \right\} + B \) and hence by \( S \) and \( S + B \).

In the cases iii(a); iii(b); iv(a); iv(b) the shaded parallelograms are covered by \( S + A \) and \( S + B \); \( S \) and \( S + C \) using lemma 1.

The triangles AOH and BOK are covered as before by \( S \), \( S + A \) and \( S + B \) respectively.

It is enough to cover triangle ACL but this triangle is congruent to the triangle OBK and hence is covered by \( S + A \) and \( S + C \).

Similarly in case(vii) it is enough to cover triangle DKG which can be covered by \( S, S + C \) and \( S + B \).

Thus case II has been dealt with completely.

Case III.

Now B is in the third sector. C cannot lie in the first sector, for, if this were so then \( \angle AOC < \pi/4, \angle COB \geq \pi/4 \) and so \( OB = AC < OA \) contrary to the hypothesis.

So we have the following cases to consider.

(i) D lies on GA produced

(a) C is in the second sector (Fig. 14)

(b) C is in the third sector (Fig. 15)

(ii) D lies in AG

(a) C is in the second sector (Fig. 16)

(b) C is in the third sector (Fig. 17)

(iii) D lies on AG produced, D is in the second sector

(a) C is in the second sector (Fig. 18)

(b) C is in the third sector (Fig. 19)
(iv) D lies on AG produced, D is in the third sector

(a) C is in the second sector (fig. 20)

(b) C is in the third sector (fig. 21)

In cases (i)(a); (i)(b); (ii)(a); (ii)(b), it is enough to
cover triangles OAH and OBK which can be covered by \( f(x, y) \leq f\left(\frac{\alpha + \gamma}{2}, \frac{\alpha + \gamma}{2\sqrt{3}}\right) \)
and \( \left\{ f(x, y) \leq f\left(\frac{\alpha + \gamma}{2}, \frac{\alpha + \gamma}{2\sqrt{3}}\right) \right\} + A \); and by \( f(x, y) \leq f\left(\frac{\beta - \delta}{2}, \frac{\beta - \delta}{2\sqrt{3}}\right) \) and
\( \left\{ f(x, y) \leq f\left(\frac{\beta - \delta}{2}, \frac{\beta - \delta}{2\sqrt{3}}\right) \right\} + B \) respectively as in case II and hence by \( S, S + A \)
and \( S, S + B \) respectively.

In cases iii(a) and iii(b) triangles OAH and OBK can be
covered as before and therefore it is enough to cover triangle DKG.

If we draw the line \( x + y = \frac{\alpha + \gamma}{2} \), the portion of the triangle
on the side of the line \( x + y = \frac{\alpha + \gamma}{2} \) in which \( O \) lies is covered by
\( f(x, y) \leq f\left(\frac{\alpha + \gamma}{2}, \frac{\alpha + \gamma}{2\sqrt{3}}\right) \) and the other portion is covered by
\( \left\{ f(x, y) \leq f\left(\frac{\alpha + \gamma}{2}, \frac{\alpha + \gamma}{2\sqrt{3}}\right) \right\} + A \) and hence by \( S \) and \( S + A \).

In cases (iv)(a) and (iv)(b), again it is enough to cover
triangle DKG. The part of the triangle in the second sector is
covered by \( f(x, y) \leq f\left(\frac{\alpha + \gamma}{2}, \frac{\alpha + \gamma}{2\sqrt{3}}\right) \) and \( \left\{ f(x, y) \leq f\left(\frac{\alpha + \gamma}{2}, \frac{\alpha + \gamma}{2\sqrt{3}}\right) \right\} + A \)
and hence by \( S \) and \( S + A \). The part in the third sector is covered
by \( f(x, y) \leq f\left(\frac{\beta - \delta}{2}, \frac{\beta - \delta}{2\sqrt{3}}\right) \) and \( \left\{ f(x, y) \leq f\left(\frac{\beta - \delta}{2}, \frac{\beta - \delta}{2\sqrt{3}}\right) \right\} + B \) and
hence by \( S \) and \( S + B \).

Hence the third case has also been disposed of and the theorem
B is proved.

5. **Proof of theorem 5.**

The results are best possible in the sense that equality is
needed for \( \varphi(u, v) = uv(u^2 - v^2) \) and the fundamental lattice \( \Lambda_o \).
We shall show that the point \( \left( \frac{1}{2}, \frac{-1}{2\sqrt{3}} \right) \) is just covered by the sets \( S + \Lambda_0 \).

It is enough to prove that

\[
\inf_{u, v \text{ integers}} \left| (u + \frac{1}{2})(v + \frac{1}{2\sqrt{3}})(u + v + \frac{1}{2} + \frac{1}{2\sqrt{3}})(u - v + \frac{1}{2} - \frac{1}{2\sqrt{3}}) \right| = \frac{1}{24\sqrt{3}}.
\]

Let

\[
\mathcal{K}(u, v) = \left| (u + \frac{1}{2})(v + \frac{1}{2\sqrt{3}})(u + v + \frac{1}{2} + \frac{1}{2\sqrt{3}})(u - v + \frac{1}{2} - \frac{1}{2\sqrt{3}}) \right|.
\]

Then \( \mathcal{K}(0,0) = \frac{1}{24\sqrt{3}} \).

If \( u = v \), then \( \mathcal{K}(u,v) = \left| (u + \frac{1}{2})(2u + \frac{1}{2} + \frac{1}{2\sqrt{3}}) \left( \frac{1}{2} - \frac{1}{2\sqrt{3}} \right) \right| \)

for integers \( u \), \( |u + \frac{1}{2}| \leq \frac{1}{2}, |u + \frac{1}{2\sqrt{3}}| \geq \frac{1}{2\sqrt{3}}, |2u + \frac{1}{2} + \frac{1}{2\sqrt{3}}| \geq \frac{1}{2} + \frac{1}{2\sqrt{3}} \).

Therefore

\[
\mathcal{K}(u,v) \geq \frac{1}{2} \cdot \frac{1}{2\sqrt{3}} \left( \frac{1}{2} + \frac{1}{2\sqrt{3}} \right) = \frac{1}{24\sqrt{3}}.
\]

If \( u \neq v \), \( u, v \) integers, then \( |u - v + \frac{1}{2} - \frac{1}{2\sqrt{3}}| \geq \frac{1}{2} + \frac{1}{2\sqrt{3}} \).

Therefore

\[
\mathcal{K}(u,v) \geq \frac{1}{2} \cdot \frac{1}{2\sqrt{3}} \left( \frac{1}{2} + \frac{1}{2\sqrt{3}} \right) = \frac{1}{24\sqrt{3}}.
\]

and the theorem is proved.
BIBLIOGRAPHY


