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By

Li-jen Du, B.S., M.Sc.

The Ohio State University
1965

Approved by

J. H. Richmond
Adviser
Department of
Electrical Engineering
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VITA

September 27, 1935  Born — Harbin, China

1958  B. Sc., National Taiwan University
      Taiwan, China

1962  M. Sc., The Ohio State University
      Columbus, Ohio

1962-1964  Research Assistant, Antenna Laboratory
          Department of Electrical Engineering
          The Ohio State University

1964-1965  Research Associate, Antenna Laboratory
          Department of Electrical Engineering
          The Ohio State University
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CHAPTER I
INTRODUCTION

The extension of Maxwell's theory from media at rest to those in motion was originally studied in the latter part of the nineteenth century. The covariance of the equations of electrodynamics under the Lorentz transformation was first proved by Lorentz [1] and Poincare. [2] Einstein [3] formulated the special theory of relativity in 1905. However, their work was confined to the question of the isolated electron and did not cover the case of ponderable bodies in general.

The problem of the electrodynamics of moving media was first formulated correctly in 1908 by Minkowski. [4] Despite the fact that his work was done almost sixty years ago, this subject has received very little attention; recently, however, there has been a revival of interest in this topic, principally as a result of the work of C. T. Tai, [5] Two other works on this subject which should be mentioned are those of Fano, Chu, and Adler, [6] and Boffi, [7] These authors have each presented a formulation of the electrodynamics of moving media that is apparently different from that of Minkowski. It has been shown by Tai, [8] however, that all
three of these formulations, as well as some other possible ones, are mathematically equivalent. For the case in which the velocity of the medium is small compared with the speed of light, the Maxwell-Minkowski equations can be simplified. Several studies were made under this assumption. Compton and Tai[9] have derived the dyadic Green's function for an infinite moving medium. Collier and Tai[10] discussed guided waves in moving media. A problem dealing with the reflection and refraction of a plane wave at the boundary of a semi-infinite moving medium was also investigated. [11]

The exact theory with no restriction upon the magnitude of the velocity was developed by Tai[12] in connection with the radiation problem in a moving isotropic medium. By transforming the wave equation into a conventional form and then solving it by means of an operational method due to Levine and Schwinger, a compact result was obtained. Another exact formulation for the same problem has been developed independently by Lee and Papas. [13] They derived the differential equations for potentials in the moving media from those which are well known in the stationary case through the Lorentz transformation, and then solved the transformed wave equations using the Green's function technique. Their method
was also extended to the case in which the moving medium is dispersive. [14]

In this study, we consider some additional problems in the electrodynamics of moving media, for the case in which there is no restriction on the velocity of the medium. In Chapter II, the Maxwell-Minkowski equations for the electromagnetic fields are presented and some suitable potential functions are introduced in a way analogous to that commonly used for stationary media. In Chapter III the work of Collier and Tai on the propagation of guided waves in moving media is extended to the case of arbitrary velocity. Chapter IV presents the theory of the radiation of a line source over a moving dielectric half-space. Four cases are considered: an electric line source and a magnetic line source each in two orientations, parallel to and perpendicular to the direction of the velocity of the medium. The moving medium is assumed to be lossless, isotropic, and to have an index of refraction greater than unity. A solution for the Maxwell-Minkowski equations is constructed in the form of a Fourier integral. The integral is evaluated for the far field by deforming the original contour into the steepest descent path. In the process, an additional branch cut integral is encountered, but this is found to give a negligible contribution to the far fields.
CHAPTER II
MAXWELL'S EQUATIONS AND WAVE EQUATIONS
ASSOCIATED WITH MOVING MEDIUM

In this chapter the wave equations as well as the equations
satisfied by the potential functions in the moving isotropic medium
will be derived which reduce to those of the stationary medium as
a special case.

Maxwell-Minkowski Equations and the
Transformation of the Field
Vectors

According to the special theory of relativity, the Maxwell's
equations must be covariant under the Lorentz transformation. In
other words, the Maxwell equations have the same form in all
inertial coordinate frames. Hence, for any medium moving or
stationary we have

\begin{align*}
(1) \quad \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
(2) \quad \nabla \times \vec{H} &= \vec{J} + \frac{\partial \vec{D}}{\partial t} \\
(3) \quad \nabla \cdot \vec{D} &= \rho \\
(4) \quad \nabla \cdot \vec{B} &= 0.
\end{align*}
We shall use primed quantities to denote field variables which are measured in an initial frame \( K' \) and unprimed quantities to denote the field variables in an initial frame \( K \). In particular, we assume all the electromagnetic sources and the observers to be stationary in the \( K \) frame and the medium which is moving at a uniform velocity \( \bar{v} \) with respect to the source and the observers to be stationary with respect to the \( K' \) frame. If we assume that the two inertial frames \( K \) and \( K' \) have the same orientation (i.e., the \( x, y, z \) axes are respectively parallel to the \( x', y', z' \) axes) and are coincident at \( t = t' = 0 \), then the field variables defined in the two frames transform according to the following relations:[15]

\[
(5a) \quad \bar{E}' = \gamma(\bar{E} + \frac{\bar{v}}{c^2} \times \bar{B}) + (1 - \gamma) \frac{\bar{E} \cdot \bar{v}}{v^2} \frac{\bar{v}}{v}
\]

\[
(6a) \quad \bar{B}' = \gamma(\bar{B} - \frac{1}{c^2} \frac{\bar{v}}{v} \times \bar{E}) + (1 - \gamma) \frac{\bar{B} \cdot \bar{v}}{v^2} \frac{\bar{v}}{v}
\]

\[
(7a) \quad \bar{H}' = \gamma(\bar{H} - \frac{\bar{v}}{c^2} \times \bar{D}) + (1 - \gamma) \frac{\bar{H} \cdot \bar{v}}{v^2} \frac{\bar{v}}{v}
\]

\[
(8a) \quad \bar{D}' = \gamma(\bar{D} + \frac{1}{c^2} \frac{\bar{v}}{v} \times \bar{H}) + (1 - \gamma) \frac{\bar{D} \cdot \bar{v}}{v^2} \frac{\bar{v}}{v}
\]

where

\[
\gamma = \frac{1}{\sqrt{1 - \beta^2}}
\]
\[ \beta = \frac{v}{c} \]

\[ c = \frac{1}{\sqrt{\mu_0\epsilon_0}} = \text{velocity of light in free space}. \]

If the velocity \( \overrightarrow{v} \) is directed in the \( y \)-direction Eqs. (5a), (6a), (7a), and (8a) can be written in a more compact form as follows:[16]

\[ (5b) \quad \overrightarrow{E}' = \overrightarrow{\Xi} \cdot (\overrightarrow{E} + \overrightarrow{v} \times \overrightarrow{B}) \]

\[ (6b) \quad \overrightarrow{B}' = \overrightarrow{\Xi} \cdot \left( \overrightarrow{B} - \frac{1}{c^2} \overrightarrow{v} \times \overrightarrow{E} \right) \]

\[ (7b) \quad \overrightarrow{H}' = \overrightarrow{\Xi} \cdot (\overrightarrow{H} + \overrightarrow{v} \times \overrightarrow{D}) \]

\[ (8b) \quad \overrightarrow{D}' = \overrightarrow{\Xi} \cdot \left( \overrightarrow{D} + \frac{1}{c^2} \overrightarrow{v} \times \overrightarrow{H} \right) \]

where the tensor \( \overrightarrow{\Xi} \) is defined as

\[ \overrightarrow{\Xi} = \begin{bmatrix} \epsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu \end{bmatrix} \]

For an isotropic linear medium which is stationary with respect to the \( K' \) frame the constitutive relations between the primed field vectors are then given by

\[ (9) \quad \overrightarrow{D}' = \epsilon \overrightarrow{E}' \]

\[ (10) \quad \overrightarrow{B}' = \mu \overrightarrow{H}' \]

where \( \epsilon \) and \( \mu \) denote, respectively, the permittivity and permeability.
of the medium when it is stationary. We assume the medium to be lossless. Expressing $\bar{E}'$, $\bar{D}'$, $\bar{H}'$ and $\bar{B}'$ in terms of $\bar{E}$, $\bar{D}$, $\bar{H}$ and $\bar{B}$ we find with the aid of Eqs. (5b), (6b), (7b), and (8b) the constitutive relations in the K frame which are

\begin{align}
(11) \quad \bar{D} + \frac{1}{c^2} \nabla \times \bar{H} &= \epsilon (\bar{E} + \nabla \times \bar{B}) \\
(12) \quad \bar{B} - \frac{1}{c^2} \nabla \times \bar{E} &= \mu (\bar{H} - \nabla \times \bar{D}) .
\end{align}

By solving for $\bar{B}$ and $\bar{D}$ in terms of $\bar{E}$ and $\bar{H}$ with $\nabla = v \hat{y}$, one finds [17]

\begin{align}
(13) \quad \bar{D} &= \epsilon \bar{\alpha} \cdot \bar{E} + \bar{\Omega} \times \bar{H} \\
(14) \quad \bar{B} &= \mu \bar{\alpha} \cdot \bar{H} - \bar{\Omega} \times \bar{E}
\end{align}

where

\[
\bar{\Omega} = \frac{(n^2 - i) \beta}{(1 - n^2 \beta^2) c} \hat{y}
\]

\[
n = \sqrt{\frac{\mu \varepsilon}{\mu_c \varepsilon_0}}
\]

\[
\bar{\alpha} = \begin{bmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{bmatrix}
\]

\[
a = \frac{1 - \beta^2}{1 - n^2 \beta^2}
\]

* This useful notation is due to Prof. C. T. Tai of the University of Michigan.
Substitution of (13) and (14) into (1) and (2) yields the Maxwell-Minkowski equations for moving isotropic medium. They are

\[(15)\quad \nabla \times \vec{E} = -\frac{\partial}{\partial t} \left[ \mu \vec{\sigma} \cdot \vec{H} - \vec{\Omega} \times \vec{E} \right]\]

\[(16)\quad \nabla \times \vec{H} = \vec{J} + \frac{\partial}{\partial t} \left[ \varepsilon \vec{\sigma} \cdot \vec{E} + \vec{\Omega} \times \vec{H} \right].\]

For harmonically oscillating fields with a time convention $e^{-i\omega t}$ (15) and (16) may be converted into

\[(17)\quad (\nabla + i\omega \vec{\Omega}) \times \vec{E} = i\omega \mu \vec{\sigma} \cdot \vec{H}\]

\[(18)\quad (\nabla + i\omega \vec{\Omega}) \times \vec{H} = \vec{J} - i\omega \varepsilon \vec{\sigma} \cdot \vec{E}\]

The time factor $e^{-i\omega t}$ is understood and $\vec{J}$ is considered
to be the current source term in a lossless moving medium.

The Wave Equations

The wave equations for $\vec{E}$ and $\vec{H}$ can be obtained by eliminating respectively $\vec{E}$ or $\vec{H}$ between (17) and (18). They are

\[(19)\quad (\nabla + i\omega \vec{\Omega}) \times \left[ \vec{\alpha}^{-1} \cdot (\nabla + i\omega \vec{\Omega}) \times \vec{E} \right] - k^2 \vec{\sigma} \cdot \vec{E} = i\omega \mu \vec{J}\]

\[(20)\quad (\nabla + i\omega \vec{\Omega}) \times \left[ \vec{\alpha}^{-1} \cdot (\nabla + i\omega \vec{\Omega}) \times \vec{H} \right] - k^2 \vec{\sigma} \cdot \vec{H} = (\nabla + i\omega \vec{\Omega}) \times (\vec{\alpha}^{-1} \cdot \vec{J})\]
where $\overline{\alpha}^{-1}$ denotes the inverse of $\overline{\alpha}$ and $k^2 = \omega^2 \mu \epsilon$. By virtue of (13) and (14), Maxwell's equations can be written as

\begin{align*}
(17) & \quad (\nabla + i\omega \overline{\Omega}) \times \overline{E} = i\omega \mu \overline{\alpha} \cdot \overline{H} \\
(18) & \quad (\nabla + i\omega \overline{\Omega}) \times \overline{H} = -i\omega \epsilon \overline{\alpha} \cdot \overline{E} + \overline{J} \\
(21) & \quad \nabla \cdot (\epsilon \overline{\alpha} \cdot \overline{E} + \overline{\Omega} \times \overline{H}) = \rho \\
(22) & \quad \nabla \cdot (\mu \overline{\alpha} \cdot \overline{H} - \overline{\Omega} \times \overline{E}) = 0 .
\end{align*}

Expanding (17) and (21) and eliminating the term including $\overline{H}$ we have instead of (21) the following relation

\begin{align*}
(23) & \quad (\nabla + i\omega \overline{\Omega}) \cdot (\epsilon \overline{\alpha} \cdot \overline{E}) = \overline{\Omega} \cdot \overline{J} + \rho .
\end{align*}

Similarly,

\begin{align*}
(24) & \quad (\nabla + i\omega \overline{\Omega}) \cdot (\mu \overline{\alpha} \cdot \overline{H}) = 0 .
\end{align*}

These equations are seen to be similar to those for the stationary medium, except for the substitution of the operator

\begin{align*}
(25) & \quad D_1 = \nabla + i\omega \overline{\Omega}
\end{align*}

for the nabla operator $\nabla$. If the vector field functions $\overline{E}_1$ and $\overline{H}_1$ are defined such that

\begin{align*}
(26) & \quad \overline{E}_1 = \overline{\alpha} \cdot \overline{E}
\end{align*}

and
\( (27) \quad \vec{H}_1 = \vec{\alpha} \cdot \vec{H} \)
	hen \( \vec{E}_1 \) and \( \vec{H}_1 \) satisfy the equation

\( (28a) \quad D_1 \times [\vec{\alpha}^{-1} \cdot D_1 \times (\vec{\alpha}^{-1} \cdot \vec{E}_1)] - k^2 \vec{E}_1 = \imath \omega \vec{J} \)

\( (28b) \quad D_1 \times [\vec{\alpha}^{-1} \cdot D_1 \times (\vec{\alpha}^{-1} \cdot \vec{H}_1)] - k^2 \vec{H}_1 = D_1 \times (\vec{\alpha}^{-1} \cdot \vec{J}) \).

It is not difficult to show by writing out the operators in cartesian coordinates that

\( (29) \quad D_1 \times [\vec{\alpha}^{-1} \cdot D_1 \times (\vec{\alpha}^{-1} \cdot \vec{E}_1)] = \frac{1}{a} [D_a(D_1 \cdot \vec{E}_1) - (D_a \cdot D_1) \vec{E}_1] \)

where the operator \( \vec{\nabla}_a \) and \( D_a \) are defined by

\( (30) \quad \vec{\nabla}_a = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{1}{a} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} = \frac{1}{a} \vec{\alpha} \cdot \vec{\nabla} \)

\( (31) \quad D_a = \vec{\nabla}_a + \frac{\imath \omega \vec{\Omega}}{a} \).

In view of Eqs. (28), (29), (23) and (24) the differential equations for \( \vec{E}_1 \) and \( \vec{H}_1 \) are then

\( (32) \quad (D_a \cdot D_1) \vec{E}_1 + k^2 a \vec{E}_1 = -\imath \omega a \vec{J} + \frac{D_a(\vec{\Omega} \cdot \vec{J} + \rho)}{\epsilon} \)

\( (32b) \quad (D_a \cdot D_1) \vec{H}_1 + k^2 a \vec{H}_1 = -a D_1 \times (\vec{\alpha}^{-1} \cdot \vec{J}) \)
Potential Functions

The equations satisfied by the electric and magnetic vector and scalar potentials are found by proceeding in the same way as for a stationary medium. \[18\] Since from (24) and (25)

\[(24a) \quad D_1 \cdot (\alpha \cdot H) = 0 \]

we may write (taking advantage of the fact that \(D_1 \cdot D_1 \times \bar{W} \equiv 0\) for any vector \(\bar{W}\))

\[(33a) \quad \mu \alpha \cdot \bar{H}^e = D_1 \times \bar{A} \]

where the superscript \(e\) denotes that \(\bar{H}^e\) is associated with fields of the electric type (transverse magnetic TM). Substituting (33a) into (17) we obtain

\[(34a) \quad D_1 \times (\bar{E}^e - i\omega \bar{A}) = 0 \]

In view of the above equation we introduce the electric scalar potential function \(U\). Since \(D_1 \times D_1 U \equiv 0\), we set

\[(35a) \quad \bar{E}^e - i\omega \bar{A} = -D_1 U \]

Applying the operator "\(D_1 \times\)" to (33a) we have

\[(36a) \quad D_1 \times (\mu \bar{H}^e) = D_1 \times [\bar{a}^{-1} \cdot D_1 \times \bar{A}] \]

Substituting (36a) in (18) and making use of (35a) gives
If we define another vector function $\overline{A}_1$ such that

$$(38a) \quad \overline{A}_1 = \overline{\alpha} \cdot \overline{A}$$

then $\overline{A}_1$ satisfies the following equation

$$(39a) \quad D_1 \times \left[ \overline{\alpha}^{-1} \cdot D_1 \times \overline{A} \right] = k \overline{\alpha} \cdot \overline{A} + i \omega \mu \epsilon \overline{\alpha} \cdot D_1 U + \mu \overline{J}.$$ 

We may now impose upon $\overline{A}_1$ and $U$ the supplementary condition

$$(40a) \quad D_1 \cdot \overline{A}_1 = i \omega \mu \epsilon a^2 U.$$ 

As a result of (29) and (40a), (39a) becomes

$$(41a) \quad (D_a \cdot D_1) \overline{A}_1 + k^2 a \overline{A}_1 = -a \mu \overline{J}.$$ 

Substituting (35a) in (23) and making use of (40a) one obtains the differential equation for $U$ as

$$(42a) \quad D_1 \cdot (\overline{\alpha} \cdot D_1 U) + k^2 a^2 U = -\frac{2i \omega \Omega}{\epsilon} \frac{\partial \overline{\tilde{J}}}{\partial \tilde{\rho}}.$$ 

The explicit expressions of (41a) and (42a) are

$$(43a) \quad \left[ \frac{\partial^2}{\partial x^2} + \frac{1}{a} \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{2i \omega \Omega}{a} \frac{\partial}{\partial y} - \frac{\omega \Omega^2}{a} \right] \overline{A}_1 = -a \mu \overline{J}.$$
Since $E^e$ and $H^e$ are derivable from $A$ and $U$, we have, from (33a), (35a) and (38a)

\[(45a) \quad E^e = i \omega A - D_1 U = i \omega \alpha^{-1} \cdot \bar{A}_1 - \frac{1}{i \omega \mu \epsilon} \frac{a}{a^2} D_1 (D_1 \cdot \bar{A}_1) \]

\[(46a) \quad H^e = \frac{1}{\mu} \alpha^{-1} \cdot D_1 \times \bar{A} = \frac{1}{\mu} \alpha^{-1} \cdot D_1 \times \alpha^{-1} \cdot \bar{A}_1. \]

Assuming $J = 0$ and $\rho = 0$ and there is a magnetic current source $\bar{M}$ in (17) becomes

\[(17a) \quad (\nabla + i \omega \Omega) \times \bar{E} = i \omega \mu \alpha \cdot \bar{H} - \bar{M}. \]

A similar procedure is followed to find the equations satisfied by the potential functions $\bar{F}$ and $V$ associated with the fields $E^m$ and $H^m$ of the magnetic type (transverse electric, TE). A summary of these results is given below. The equation numbers correspond to those in deriving the electric type fields as shown above. $E^m$ and $H^m$ are evaluated in terms of $\bar{F}$ and $V$ as

\[(45b) \quad E^m = -\frac{1}{\epsilon} \bar{\alpha}^{-1} \cdot (D_1 \times \bar{F}) = -\frac{1}{\epsilon} \bar{\alpha}^{-1} \cdot [D_1 \times (\bar{\alpha}^{-1} \cdot \bar{F}_1)] \]

\[(46b) \quad H^m = i \omega \bar{F} - D_1 V = i \omega \bar{\alpha}^{-1} \cdot \bar{F}_1 - D_1 V \]

$\bar{F}$ and $\bar{F}_1$ are related by

\[(38b) \quad \bar{F}_1 = \bar{\alpha} \cdot \bar{F}. \]
The supplementary condition imposed on $F_1$ and $V$ is

\[(40b)\quad D_1 \cdot \overline{F}_1 = i\omega \mu e a^2 V\]

$F_1$ and $V$ satisfy the following differential equations

\[(41b)\quad (D_a \cdot D_1) \overline{F}_1 + k a^2 \overline{F}_1 = -a\varepsilon M\]

and

\[(42b)\quad D_1 \cdot (\overline{\sigma} \cdot D_1 V) + k a^2 V = -\frac{\overline{\Omega} \cdot \overline{M}}{\mu}\]

The explicit expressions for (41b) and (42b) are

\[(43b)\quad \left[ \frac{\partial^2}{\partial x^2} + \frac{1}{a} \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{2i\omega \Omega}{a} \frac{\partial}{\partial y} - \frac{\omega^2 \Omega^2}{a} + k^2 a \right] \overline{F}_1 = -a\varepsilon M\]

and

\[(44b)\quad \left[ \frac{\partial^2}{\partial x^2} + \frac{1}{a} \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{2i\omega \Omega}{a} \frac{\partial}{\partial y} - \frac{\omega^2 \Omega^2}{a} + k^2 a \right] V = -\frac{\overline{\Omega} \cdot \overline{M}}{a\mu}\]

All four of the potential functions $\overline{A}_1$, $\overline{F}_1$, $U$ and $V$ satisfy the same equation as that of $\overline{E}_1$ and $\overline{H}_1$ in the source-free region. By combining (45) and (46) one finds the total $\overline{E}$ and $\overline{H}$ fields in terms of the four potentials. These are

\[(45)\quad E = i\omega A - D_1 U - \frac{1}{\epsilon} \overline{\alpha}^{-1} \cdot (D_1 \times F)
= i\omega \alpha^{-1} \cdot \overline{A}_1 - \frac{1}{i\omega \mu e a^2} D_1 (D_1 \cdot \overline{A}_1) \frac{1}{\epsilon} \overline{\alpha}^{-1}
\cdot [D_1 \times (\overline{\alpha}^{-1} \cdot \overline{F}_1)]\]
\( H = \frac{1}{\mu} A^{-1} \cdot D_1 \times \bar{A} + i\omega \bar{F} - D_1 V \)

\[
= \frac{1}{\mu} A^{-1} \cdot D_1 \times (\bar{a}^{-1} \cdot \bar{A}_1) + i\omega A^{-1} \cdot \bar{F}_1 \\
- \frac{1}{i\omega \mu e a^2} D_1(D_1 \cdot \bar{F}_1).
\]
CHAPTER III
PROPAGATION IN WAVEGUIDES

In this chapter we shall consider the problems of electromagnetic waves in the interior of a cylindrical or rectangular waveguide which is filled with a homogeneous, isotropic and lossless medium with constitutive parameters $\mu$ and $\epsilon$ moving at uniform velocity $\vec{v} = z\vec{v}$ along the axis of the guide. The waveguide is assumed to have perfectly conducting walls and to be infinitely long. The field solution in the guide can be divided into two basic modes, TE and TM. TM modes have no axial component of magnetic field, and the field components can be derived from a vector potential $\vec{A} = 2\vec{A}$. TE modes have no axial component of electric field and the fields may be derived from a vector potential $\vec{F} = 2\vec{F}$.

Equation (43) which the potentials $\vec{A}$ and $\vec{F}$ just mentioned have to satisfy can be written for the problems discussed in this chapter as

\[
(47) \quad \left[ \nabla_\perp^2 + \frac{1}{a} \frac{\partial^2}{\partial z^2} + \frac{2i\omega}{a} \frac{\partial}{\partial z} - \frac{\omega^2 a^2}{4} + k^2 a \right] \begin{pmatrix} A_1 \\ F_1 \end{pmatrix} = 0
\]

where

$\vec{A}_1 = zA_1 = \vec{a} \cdot \vec{A} = \vec{A} = zA$ and $\vec{F}_1 = zF_1 = \vec{a} \cdot \vec{F} = \vec{F} = zF$. 

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The operator $\nabla_t^2$ is the transverse part of the $\nabla^2$ operator. In the $u_1$, $u_2$, $z$ coordinate system with scale factors $h_1$, $h_2$, and unity, Eq. (47) becomes

(48) \[
\left[ \frac{1}{h_1 h_2} \frac{\partial}{\partial u_1} \frac{h_2}{h_1} \frac{\partial}{\partial u_1} + \frac{1}{h_1 h_2} \frac{\partial}{\partial u_2} \frac{h_1}{h_2} \frac{\partial}{\partial u_2} + \frac{1}{a} \frac{\partial^2}{\partial z^2} + \frac{2i\omega \Omega}{a} \frac{\partial}{\partial z} - \frac{\omega_2^2}{a} + k^2 \right] \begin{bmatrix} A_1 \\ F_1 \end{bmatrix} = 0.
\]

Since $\bar{A}_1 = \bar{A}$ and $\bar{F}_1 = \bar{F}$ we will formulate the problem in terms of $\bar{A}$ and $\bar{F}$ directly.

The Rectangular Waveguide

The appropriate solution of $A$ and $F$ which satisfies the boundary conditions for the waveguide configuration shown in Fig. 1 is

(49) \[
\bar{A} = \hat{A}A = \hat{A}A_0 \sin k_x x \sin k_y y e^{ihz} \quad m = 1, 2, 3, \ldots
\]

(50) \[
\bar{F} = \hat{F}F = \hat{F}F_0 \cos k_x x \cos k_y y e^{ihz} \quad m = 0, 1, 2, \ldots
\]

Substituting (49) or (50) into (47) or (48) gives the following equation relating the propagation constants

(51) \[
-(k_x^2 + k_y^2) - \frac{h_1^2}{h_2} - \frac{2i\omega \Omega}{a} \frac{\partial^2}{\partial z^2} + \frac{\omega_2^2}{a} + k^2 a = 0.
\]

The above equation can be solved to give
where \( k_c^2 = k_x^2 + k_y^2 = \left( \frac{m\pi}{x_0} \right)^2 + \left( \frac{l\pi}{y_0} \right)^2 \). Each set of integers \( m \) and \( l \) corresponds to a given mode which will be designated as the \( \text{TM}_{m,l} \) (or \( \text{TE}_{m,l} \)) modes.

The TM modes may be obtained from \( A \) by means of Eq. (45a) and (46a) where we have to interchange the \( y \) and \( z \) coordinates because the medium here is moving in the \( +z \) direction. Thus

\[
E = i\omega A - (\nabla + i\omega) \left( \frac{\nabla + i\omega A}{\omega \mu \epsilon a^2} \right) \cdot \nabla = -x \frac{A_0(h+\omega)}{\omega \mu \epsilon a^2} \cos k_x x \sin k_y y e^{ihz}
\]

\[
-\hat{y} \frac{A_0(h+\omega)}{\omega \mu \epsilon a^2} \sin k_x x \cos k_y y e^{ihz}
\]

\[
+ \hat{z} A_0 \left[ i\omega + \frac{(h+\omega)^2}{\omega \mu \epsilon a^2} \right] \sin k_x x \sin k_y y e^{ikz}
\]
\[ (53b) \quad \overline{H} = \frac{1}{\mu} \overline{a}^{-1} \left[ (\nabla + i\omega \overline{\mu}) \times \overline{A} \right] \]
\[ = \frac{A_0 k_y}{\mu a} \sin k_x x \cos k_y y e^{ihz} \]
\[ - \frac{-\overline{A_0 k_x}}{\mu a} \cos k_x x \sin k_y y e^{ihz} \]

In a similar way the TE modes may be obtained from \( \overline{F} \) by means of Eqs. (45b) and (46b) giving

\[ (54a) \quad \overline{E} = \frac{-1}{\epsilon} \overline{a}^{-1} \left[ (\nabla + i\omega \overline{\epsilon}) \times \overline{F} \right] \]
\[ = \frac{F_0 k_y}{\epsilon a} \cos k_x x \sin k_y y e^{ihz} \]
\[ - \overline{\frac{-F_0 k_x}{\epsilon a}} \sin k_x x \cos k_y y e^{ihz} \]

\[ (54b) \quad \overline{H} = i\omega \overline{F} - (\nabla + i\omega \overline{\epsilon}) \left( \nabla + i\omega \overline{\mu} \right) \frac{\overline{F}}{i\omega \mu a^2} \]
\[ = \frac{F_0 k_x (h + \omega \overline{\mu})}{\omega \mu a^2} \sin k_x x \cos k_y y e^{ihz} \]
\[ + \frac{F_0 k_y (h + \omega \overline{\epsilon})}{\omega \mu e a^2} \cos k_x x \sin k_y y e^{ihz} \]
\[ + \frac{2}{\mu} F_0 \left[ i\omega + \frac{(h + \omega \overline{\epsilon})}{i\omega \mu e a^2} \right] \cos k_x x \cos k_y y e^{ihz} \].
Cylindrical Waveguides

The proper form of $\vec{A}$ and $\vec{F}$ for the cylindrical waveguide shown in Fig. 2 may be written as

\begin{align}
\vec{A} &= \hat{z} A = \hat{z} A_0 J_m(k_c r) \cos \theta \sin \phi e^{ihz} \\
\vec{F} &= \hat{z} F = \hat{z} F_0 J_m(k_c r) \cos \theta \sin \phi e^{ihz}
\end{align}

where $J_m(k_c r)$ is the Bessel function of integer order $m$. $A$ and $F$ satisfy (48) which for this problem is

\begin{align}
\left[ \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial A}{\partial r}) + \frac{1}{r^2} \frac{\partial^2 A}{\partial \phi^2} + \frac{1}{a} \frac{\partial^2 A}{\partial z^2} + \frac{2i \omega \Omega}{a} \frac{\partial}{\partial z} - \frac{\omega^2 \Omega^2}{a} + k_c^2 \right] A = 0
\end{align}

\begin{align}
\frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial F}{\partial r}) + \frac{1}{r^2} \frac{\partial^2 F}{\partial \phi^2} + \frac{1}{a} \frac{\partial^2 F}{\partial z^2} + \frac{2i \omega \Omega}{a} \frac{\partial}{\partial z} - \frac{\omega^2 \Omega^2}{a} + k_c^2 \right] F = 0
\end{align}

Fig. 2--The cylindrical waveguide

From (55) (or (56)) and (57) we obtain the relation

\begin{align}
k_c^2 = k^2 a - \omega^2 \Omega^2 - \frac{2 \omega \Omega}{a} - \frac{h^2}{a^2}
\end{align}
The TM modes may be derived from $A$ by means of Eqs. (45a) and (46a) in the form

\begin{equation}
\vec{E} = \omega \vec{A} - (\vec{\nabla} + i\omega \vec{\Omega}) \frac{\vec{\nabla} \cdot \vec{A}}{i\omega \mu \epsilon a^2} - \frac{A_\Omega(h + i\omega \Omega)}{\omega \mu \epsilon a^2} \frac{dJ_m(k_c r)}{dr} \cos m\phi e^{ihz} - \frac{\hat{A}}{r} \frac{m A_\Omega(h + i\omega \Omega)}{\omega \mu \epsilon a^2} \times \frac{1}{r} J_m(k_c r) \frac{-\sin m\phi e^{ihz}}{\cos m\phi e^{ihz}} + \hat{A} \frac{2 A_\Omega}{r} \left[ i\omega + \frac{(h + i\omega \Omega)^2}{i\omega \mu \epsilon a^2} \right] J_m(k_c r) \frac{\cos m\phi e^{ihz}}{\sin m\phi e^{ihz}}.
\end{equation}

\begin{equation}
\vec{H} = \frac{\hat{A}}{\mu a} \frac{m A_\Omega}{r} \times \frac{1}{r} J_m(k_c r) \frac{-\sin m\phi e^{ihz}}{\cos m\phi e^{ihz}} - \frac{\hat{A}}{\mu a} \frac{dJ_m(k_c r)}{dr} \frac{\cos m\phi e^{ihz}}{\sin m\phi e^{ihz}}.
\end{equation}

The boundary condition at $r = r_o$ requires

\begin{equation}
J_m(k_c r_o) = 0
\end{equation}

which determines the allowed values of $k_c$. There are an infinite number of solutions which will be enumerated as $\rho m \ell$. Hence $k_c$ can assume only those values

\begin{equation}
k_c, m\ell = \frac{\rho_m \ell}{r_o}
\end{equation}

and the corresponding modes will be labeled $TM_{m\ell}$ where the first subscript refers to the number of cyclic variations with $\phi$, and the second subscript refers to the $\ell$th root of the Bessel function.
The TE modes may be derived by means of Eqs. (45b) and (46b) as

\[ E = - \frac{1}{r} \frac{m F_o}{\epsilon a} J_m(k_c r) \sin \frac{m \phi}{e} \cos \frac{ihz}{e} + \phi \frac{F_o}{\epsilon a} \frac{d}{dr} J_m(k_c r) \cos \frac{m \phi}{e} \sin \frac{ihz}{e} \]

\[ H = - \frac{1}{r} \frac{m F_o(h + \omega \Omega)}{\omega \mu e a^2} \frac{d}{dr} J_m(k_c r) \cos \frac{m \phi}{e} \sin \frac{ihz}{e} \]

\[ = \phi \frac{m F_o(h + \omega \Omega)}{\omega \mu e a^2} \times \frac{1}{r} \frac{1}{r} J_m(k_c r) \sin \frac{m \phi}{e} \cos \frac{ihz}{e} \]

\[ + \phi F_o \left[ i \omega + \frac{(h + \omega \Omega)^2}{i \omega \mu e a^2} \right] J_m(k_c r) \cos \frac{m \phi}{e} \sin \frac{ihz}{e} \]

The boundary condition at \( r = r_o \) requires

\[ \frac{d}{dr} J_m(k_c r) \bigg|_{r=r_o} = 0 \]

There are also infinitely many solutions for (63) which will be designated by \( \rho_{m\ell} \) and \( k_c \) is given by

\[ k_c, m\ell = \frac{\rho_{m\ell}}{r_o} \]

The corresponding modes will be labeled TE\(_{m\ell}\). Equation (58) can be used to solve for \( h \) to give

\[ h = -\omega \Omega + \sqrt{k^2 a^2 - k_c^2 a} \]
where
\[ k_c = \frac{\rho_m l}{r_o} \quad \text{(TM modes)} \]
\[ k_c = \frac{\rho_m l}{r_o} \quad \text{(TE modes)} \]

Waveguide Parameters

The formula and the conclusions given in this section apply to both rectangular and cylindrical waveguides with few exceptions which will be specified individually. \( k_c \) will assume the value given in (52) for the rectangular waveguides and the value given in (61) or (64) for cylindrical waveguides.

The propagation constant is given in (52) and (65) as
\[ h = -\omega \Omega + \sqrt{k^2 a^2 - k_c^2} \]

When \( n \beta < 1 \) the cut off occurs for
\[ k^2 a \leq k_c^2 \]
i.e.,
\[ k_o \sqrt{n \frac{2 (1-\beta^2)}{1-n^2 \beta^2}} \leq k_c \]

where \( k_o = \omega \sqrt{\mu_o \varepsilon_o} \quad n = \sqrt{\frac{\mu_o \varepsilon_o}{\varepsilon_o}} \), and the cut off frequency is
\[ f_c = \frac{k_c}{2\pi \sqrt{\mu_o \varepsilon_o} \sqrt{n \frac{2 (1-\beta^2)}{1-n^2 \beta^2}}} \]
When \( f \) is less than \( f_c \) wave may propagate in the \(-z\) direction with phase velocity \( v_p = 1/\Omega \) and an exponentially varying magnitude. When \( f \) is slightly greater than \( f_c \) there is no attenuation, but waves can propagate in the \(-z\) direction only (two waves with different phase velocities) unless \( f \) is large enough such that the following relation is satisfied

\[
(68) \quad k^2a^2 - k_c^2a \geq \omega^2\Omega^2
\]

Equation (68) can be manipulated to the form

\[
(69) \quad f \geq f_+ = \frac{k_c}{2\pi\sqrt{\mu_0\varepsilon_0}} \sqrt{\frac{\pi^2 - \beta^2}{1 - \beta^2}}
\]

For frequencies greater than \( f_+ \) waves can propagate in either direction without attenuation but with different phase velocities.

If \( v = 0 \) then \( \beta = 0 \) and we have

\[
(70) \quad f_+ = f_c = \frac{k_c}{2\pi\sqrt{\mu\varepsilon}}
\]

which is the usual cut off frequency in the stationary case. When \( n\beta > 1 \) \( \alpha \) will be negative while \(-\Omega\) is positive. In this case there is no cut off phenomenon at all. Propagation (with two different phase velocities) will be possible in the \( +z \) direction only, unless the following relation is true.
Equation (71) can be simplified to give

\[ f \leq f_- = \frac{k_c}{2\pi \sqrt{\mu_0 \varepsilon_0} \sqrt{\frac{n^2 - \beta^2}{1 - \beta^2}}} \]

The summary of these results is exhibited in Fig. 3.

---

Fig. 3--Frequency ranges for wave propagation in the waveguide with the medium in it moving in +z direction

There are an infinite number of modes which can exist in the waveguide but for a given frequency only a finite number of them can propagate freely if the velocity \( \overline{v} = \hat{\nu} \) is small such that \( n\beta < 1 \).
Therefore, when \( n\beta < 1 \) several parameters can be expressed in terms of the cut off frequency. These are

\[
\begin{align*}
(73) & \quad k_c = 2\pi f_c \sqrt{\mu \varepsilon a} \\
(74a) & \quad h = -\omega \Omega^+ a\omega \sqrt{\mu \varepsilon} \left[ \frac{\omega}{1-n^2\beta^2} \right] \left[ 1 - \left( \frac{f_c}{f} \right)^2 \right]^{\frac{1}{2}} \\
& \quad = \frac{\omega}{(1-n^2\beta^2) c} \left\{ (1-n^2) \beta^+ n(1-\beta^2) \left[ 1 - \left( \frac{f_c}{f} \right)^2 \right]^{\frac{1}{2}} \right\} (f > f_c) \\
(74b) & \quad h = -\omega \Omega^+ i\omega c \sqrt{\mu \varepsilon} \left[ \frac{\omega}{1-n^2\beta^2} \right] \left[ 1 - \left( \frac{f_c}{f} \right)^2 \right]^{\frac{1}{2}} \\
& \quad = \frac{1}{(1-n^2\beta^2) c} \left\{ \omega(1-n^2) \beta^+ i\omega c n(1-\beta^2) \left[ 1 - \left( \frac{f_c}{f} \right)^2 \right]^{\frac{1}{2}} \right\} (f < f_c)
\end{align*}
\]

where \( \omega_c = 2\pi f_c \). The guide phase velocity and guide wavelength are respectively

\[
\begin{align*}
(75) & \quad v_g = \omega/h = (1-n^2\beta^2) c/\left\{ (1-n^2) \beta^+ n(1-\beta^2) \left[ 1 - \left( \frac{f_c}{f} \right)^2 \right]^{\frac{1}{2}} \right\} \\
(76) & \quad \lambda_g = 2\pi/h = (1-n^2\beta^2) \lambda_o/\left\{ (1-n^2) \beta^+ n(1-\beta^2) \left[ 1 - \left( \frac{f_c}{f} \right)^2 \right]^{\frac{1}{2}} \right\}
\end{align*}
\]

where \( c = 1/\sqrt{\mu_o \varepsilon_o} \) and \( \lambda_o \) is the free space wavelength. The \( \text{TM}_{m\ell} \) characteristic wave impedance is

\[
\begin{align*}
(77) & \quad Z_{m\ell}^{\text{TM}} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} \quad \text{(rectangular waveguides)} \\
& \quad = \frac{E_r}{H_\phi} = -\frac{E_\phi}{H_r} \quad \text{(cylindrical waveguides)}
\end{align*}
\]
\[
Z_{m\ell} = \frac{h + \omega \Omega}{\omega \epsilon a} + \frac{\sqrt{k_a^2 a^2 - k_c^2 a^2}}{\omega \epsilon a} \quad (n\beta > 1 \text{ or } n\beta < 1)
\]
\[
= + \frac{1}{\omega} \eta \left[ 1 - \left( \frac{f_c}{f} \right)^2 \right]^{1/2} \quad (n\beta < 1 \text{ and } f > f_c)
\]
\[
= + i \frac{\omega c}{\omega} \eta \left[ 1 - \left( \frac{f}{f_c} \right)^2 \right]^{1/2} \quad (n\beta < 1 \text{ and } f < f_c)
\]

where \( \eta = \sqrt{\frac{\mu}{\epsilon}} \) is the intrinsic impedance of the medium. Similarly, the \( TE_{m\ell} \) characteristic wave impedance is

\[
Z_{m\ell}^{TE} = \frac{E_x}{H_y} = - \frac{E_y}{H_x} \quad \text{(rectangular waveguides)}
\]
\[
= \frac{E_r}{H_\phi} = - \frac{E_\phi}{H_r} \quad \text{(cylindrical waveguides)}
\]
\[
= \frac{\omega \mu a}{h + \omega \Omega} \quad (n\beta > 1 \text{ or } n\beta < 1)
\]
\[
= + \frac{\omega \mu a}{\sqrt{k_a^2 a^2 - k_c^2 a^2}} \quad (n\beta < 1 \text{ and } f > f_c)
\]
\[
= + i \frac{\omega}{\omega_c} \eta \left[ 1 - \left( \frac{f}{f_c} \right)^2 \right]^{1/2} \quad (n\beta < 1 \text{ and } f < f_c).
\]

It is interesting to note that the product \( Z_{m\ell}^{TM} Z_{m\ell}^{TE} = \eta^2 = \mu/\epsilon \) at all frequencies and all velocities of motion of the medium.
\( Z_{m\ell}^{TE} \) as given in (77) and (78) when \( n\beta < 1 \) are of the same form as those when the medium in the waveguide is not moving. The power flow in the rectangular waveguide for TM modes is

\[
P = \frac{1}{2} \text{Re} \int_0^{x_0} \int_0^{y_0} \overline{E} \times \overline{H}^* \cdot d\mathbf{S}
\]

\[
= \pm \frac{x_0y_0 |A_0|^2 k_c^2}{2\epsilon_{om}\epsilon_{o\ell}} \sqrt{\frac{k_c^2 - k_r^2}{\omega \mu^2 \varepsilon_a^3}} \quad (n\beta > 1 \text{ or } n\beta < 1)
\]

\[
= \pm \frac{x_0y_0 |A_0|^2 k_c^2}{2\epsilon_{om}\epsilon_{o\ell}} \times \left[ \frac{1-(f_c/f)^2}{f_c^2 \epsilon} \right]^{1/2} \quad (n\beta < 1, f > f_c)
\]

where \( \epsilon_{o\ell} \) is defined as equal to 1 when \( \ell = 0 \) and equal to 2 when \( \ell > 0 \). For TE modes it is

\[
P = \frac{x_0y_0 |F_0|^2 k_c^2}{2\epsilon_{o\ell} \epsilon_{om}} \sqrt{\frac{k_c^2 - k_r^2}{\omega \mu^2 \varepsilon_a^3}} \quad (n\beta > 1 \text{ or } n\beta < 1)
\]

\[
= \frac{x_0y_0 |F_0|^2 k_c^2}{2\epsilon_{o\ell} \epsilon_{om}} \times \left[ \frac{1-(f_c/f)^2}{f_c^2 \epsilon} \right]^{1/2} \quad (n\beta < 1, f > f_c)
\]

In cylindrical waveguides the corresponding expressions are

\[
P = \pm \frac{\pi |A_0|^2 r_o^2}{2\epsilon_{om}} \left[ \frac{d}{dr} J_m(k_r r) \right]_{r=r_0}^2 \sqrt{\frac{k_c^2 - k_r^2}{\omega \mu^2 \varepsilon_a^3}} \quad (n\beta > 1 \text{ or } n\beta < 1)
\]
for TM modes and

\[
P = \pm \frac{\pi |A_0|^2 r_o ^2 k_c ^2}{2\varepsilon_0 m} \left(1 - \frac{m^2}{k_c r_o ^2}\right) J_m^2(k_c r_o) \sqrt{\frac{k_a ^2 - k_c ^2}{\omega \mu \varepsilon a ^3}}
\]

\[(n\beta > 1 \text{ or } n\beta < 1)\]

for TE modes. Although the phase propagates in the way shown in Fig. 3, the power flow for the two waves in the guide are of the same magnitude and in opposite directions.

When the velocity \( v \) approaches zero or when the constitutive parameters of the medium are equal to those of free space, \( \Omega \) will approach zero and \( a \) will approach one; the expressions and results obtained reduce to the familiar ones for media at rest.
CHAPTER IV
ELECTRIC AND MAGNETIC LINE SOURCES LOCATED OVER A SEMI-INFINITE MOVING HALF-SPACE

The geometry of the problem which will be considered in this chapter is illustrated in Fig. 4. \(J\) (or \(M\)) is a line source of electric (or magnetic) current located at \(x = d\) above the plane \(x = 0\). The line source is either parallel to the \(y\)-axis or the \(z\)-axis. The half-space \(x > 0\) is assumed to be free-space. The half-space \(x < 0\) is assumed to be a dielectric with permittivity \(\epsilon\) and permeability \(\mu\) and it moves at constant velocity \(v\) in the positive \(y\)-direction.

The purpose of this study is to find out the effect of the motion of this...
region on the radiation pattern. Analytically the problem is quite similar to that of a line source above a grounded dielectric slab as given by Tai, [19] Barone, [20] and Whitmer, [21] The Fourier transform method has been used to construct a solution for the field in integral form. The resultant integral is solved by the saddle point method for the far field. A branch cut integration must be included in several cases which will be discussed later on. The time dependence is assumed to be of the form $e^{-i\omega t}$.

Electric Line Source
Parallel to Z-Axis

An electric line source located at $x = d$, $y = 0$ and parallel to the $z$-axis can be represented as

\begin{equation}
J = \hat{A} I_0 \delta (x - d) \delta (y) .
\end{equation}

Since the current has no variation in the $z$-direction, the radiated field is also independent of $z$. The $z$-component of the magnetic field is zero, and hence the electromagnetic fields may be derived from vector potentials $\vec{A}_T$ in region I and $\vec{A}_B$ in region II having only $z$-components. The electric and magnetic fields in region I are given by

\begin{equation}
\begin{align*}
\vec{E} &= i\omega \vec{A}_T - \frac{\nabla \nabla \cdot \vec{A}_T}{i\omega \mu_0 \epsilon_0} \\
\vec{H} &= \frac{1}{\mu_0} \nabla \times \vec{A}_T
\end{align*}
\end{equation}
or

\[(84b) \quad E_x = 0 \quad H_x = \frac{1}{\mu_0} \frac{\partial A_T}{\partial y} \]

\[E_y = 0 \quad H_y = -\frac{1}{\mu_0} \frac{\partial A_T}{\partial x} \]

\[E_z = i\omega A_T \quad H_z = 0 \]

where \(A_T = A_T \hat{z}\) and \(A_T\) satisfies the wave equation

\[(85) \quad \nabla^2 A_T + k_o^2 A_T = -\mu_0 I_0 \delta(x-d) \delta(y) \]

and suitable boundary conditions as discussed below.

Let the Fourier transform of \(A_T\) be denoted by \(g_1\)

\[(86) \quad g_1(x, k_y) = \int_{-\infty}^{\infty} A_T(x, y) \, e^{-ik_y y} \, dk_y \]

then the Fourier integral

\[(87) \quad A_T(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g_1(x, k_y) \, e^{ik_y y} \, dk_y \]

is a solution of Eq. (85) provided that \(g_1\) satisfies the equation

\[(88) \quad \frac{\partial^2 g_1}{\partial x^2} + (k_o^2 - k_y^2) g_1 = -\mu_0 I_0 \delta(x-d). \]

A suitable form for the function \(g_1\) is

\[(89a) \quad g_{11}(x, k_y) = c_1 e^{i\ell(x-d)} \quad (x \geq d) \]

\[(89b) \quad g_{12}(x, k_y) = c_2 [e^{-i\ell(x-d)} + R e^{i\ell(x+d)}] \quad (0 \leq x \leq d) \]
where $I^2 = k_o^2 - k_y^2$ and $c_1$, $c_2$ and $R$ are constants to be determined.

At $x = d$, $g_1$ is continuous but $\frac{dg_1}{dx}$ is discontinuous. Integrating (88) from $x = d_-$ to $x = d_+$ gives

$$\left. \frac{dg_1}{dx} \right|_{d_-}^{d_+} = -\mu_o I_o$$

and this specifies the discontinuity of the derivative at the source.

The boundary conditions at $x = d$ are now readily found to give

(90a) $$g_{11} = \left( \frac{-\mu_o I_o}{2il} \right) \left[ 1 + R e^{i2ld} \right] e^{i(x-d)} \quad (x \geq d)$$

(90b) $$g_{12} = \left( \frac{-\mu_o I_o}{2il} \right) \left[ e^{-il(x-d)} + R e^{il(x+d)} \right] \quad (0 \leq x \leq d)$$

In region II let

(91) $$\begin{align*}
\bar{v} &= \hat{y} \bar{v} \\
\bar{\Omega} &= \hat{y} \bar{\Omega} \\
\bar{A}_B &= \hat{z} A_B
\end{align*}$$

where $\bar{A}_B$ is the vector potential and $\bar{\Omega}$ is defined as before. Introducing the vector function $\bar{A}_{B1}$ such that,

(92) $$\bar{A}_{B1} = \hat{z} A_{B1} = \bar{a} \cdot \bar{A}_B = a A_B = \hat{z} a A_B$$

then $A_{B1}$ has to satisfy Eq. (43a) which in this case is

(93) $$\left[ \frac{\partial^2}{\partial x^2} + \frac{1}{a} \frac{\partial^2}{\partial y^2} + \frac{2i\omega \Omega}{a} \frac{\partial}{\partial y} - \frac{\omega^2 \Omega^2}{a} + k_a^2 \right] A_{B1} = 0.$$
The Fourier integral

\[ A_{B_1} = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(k_y) e^{ik_x x + ik_y y} \, dk_y \]

is a solution of (93) where \( k_x \) is given by

\[ k_x^2 = k^2 - \frac{\omega^2 \ell^2}{a} - \frac{2\omega\Omega}{a} k_y - \frac{1}{a} k_y^2 \]

and \( f_1 \) has to be determined from the boundary condition at \( x = 0 \).

For transverse wave numbers \( \ell \) and \( k_x \), the branches that lead to attenuated waves so the Fourier integrals converge must be chosen. This requires that we choose the roots such that,

\[(96a) \quad \text{Im} \ell \geq 0 \]

and

\[(96b) \quad \text{Im} k_x \leq 0 \]

also the conditions \( \text{Re} \ell \geq 0 \) and \( \text{Re} k_x \leq 0 \) correspond to outward propagating waves.

The fields in region II are given from Eqs. (45a) and (46a) by

\[(97a) \quad \begin{align*}
\bar{E} &= i\omega \bar{A}_B - D_1 U = \bar{\omega} \bar{A}_B - D_1 U \\
&= \frac{i\omega \bar{A}_{B_1}}{a} - \frac{(\nabla + i\omega \bar{\Omega}) \cdot (\nabla + i\omega \bar{\Omega}) \cdot \bar{A}_{B_1}}{i\omega \mu \varepsilon a^2}
\end{align*} \]
\( \vec{H} = \frac{1}{\mu} \alpha^{-1} \cdot (\nabla + i\omega \vec{d}) \times \vec{A}_B \)

\[ = \frac{1}{\mu} \alpha^{-1} \cdot (\nabla + i\omega \vec{d}) \times (\alpha^{-1} \cdot \vec{A}_B) \]

or

\[
E_x = 0 \quad H_x = \frac{1}{\mu a^2} \left[ \frac{\partial A_{B1}}{\partial y} + i\omega A_{B1} \right] \\
E_y = 0 \quad H_y = \frac{-1}{\mu a} \frac{\partial A_{B1}}{\partial x} \\
E_z = \frac{i\omega A_{B1}}{a} \quad H_z = 0 \]

The boundary conditions of continuous tangential electric and magnetic fields at \( x = 0 \) demand the following relations:

\[
(98a) \quad f_1 = \frac{-\mu_0 a I_0}{2i\ell} e^{i\ell d} (1+R) \\
(98b) \quad f_1 = \frac{\mu a I_0}{2ik_x} e^{i\ell d} (1-R)
\]

which gives the solutions

\[
(99a) \quad R = \frac{\mu \ell + \mu_0 k_x}{\mu \ell - \mu_0 k_x} \\
(99b) \quad f_1 = \frac{iI_0 a \mu \mu_0 e^{i\ell d}}{\mu \ell - \mu_0 k_x}
\]

Therefore the final expressions for the field components in the region \( x \geq d \) are given by the integrals of the form,
\( \bar{E} = \frac{\hat{z}}{z} E_z = \frac{\hat{z}}{z} \omega A_T = \frac{\omega}{4\pi} \int_{-\infty}^{\infty} \frac{1}{\ell} \left[ 1 + \frac{\mu l + \mu_0 k_x}{\mu l - \mu_0 k_x} e^{2i\ell d} \right] \cdot e^{i \ell (x-d) + ik_y y} \ dk_y \)

\[
H_x = \frac{-I_0}{4\pi} \int_{-\infty}^{\infty} \frac{k_y}{\ell} \left[ 1 + \frac{\mu l + \mu_0 k_x}{\mu l - \mu_0 k_x} e^{i2\ell d} \right] \cdot e^{i \ell (x-d) + ik_y y} \ dk_y
\]

\[
H_y = \frac{I_0}{4\pi} \int_{-\infty}^{\infty} \left[ 1 + \frac{\mu l + \mu_0 k_x}{\mu l - \mu_0 k_x} e^{i2\ell d} \right] \cdot e^{i \ell (x-d) + ik_y y} \ dk_y.
\]

The branch points in the \( k_y \)-plane can be located by solving the equations.

\[
(101a) \quad k^2 - k_y^2 = 0
\]

\[
(101b) \quad k_a^2 - \frac{2\omega}{a} k_y - \frac{1}{a} k_y^2 = 0
\]

The results are

\[
(102a) \quad k_y = \pm k_o
\]

\[
(102b) \quad k_{y2} = -\omega \Omega \pm ka = k_o \frac{n + \beta}{1 + n\beta} \quad \text{and} \quad -k_o \frac{n - \beta}{1 - n\beta}
\]

When \( n\beta < 1 \) one of the branch point shown in (102b) is on the right side of point \( k_o \) while the other one on the left side of point \( -k_o \).
When \( n \beta > 1 \) both are located on the right side of point \( k_0 \). When \( n = 1 \) the branch points given in (102b) are the same as those given in (102a). This is shown in Fig. 8.

The proper positions of the branch cuts can be found as follows.

Assuming that there is small loss in the free space the wave numbers \( k_0, \ell \) and \( k_x \) become

\begin{equation}
(103) \quad k_0 = k_0' + ik_0''
\end{equation}

\begin{equation}
(104) \quad \ell^2 = (k_0 + ik_0')^2 - (k_y + ik_y')^2 = (k_0'^2 - k_0''^2 - k_y^2 + k_y'^2) + i2(k_0'k_0'' - k_y'k_y')
\end{equation}

\begin{equation}
(105) \quad k_x^2 = (k_0 + ik_0')^2 \frac{n^2 - \beta^2}{1 - \beta^2} - 2(k_0 + ik_0') \frac{n^2 - 1}{1 - \beta^2} \frac{\beta}{(k_y + ik_y')^2}
\end{equation}
The branch cuts that separate the proper and improper branches occur along the curves \( \text{Im } \ell = 0 \) and \( \text{Im } k_x = 0 \) which are portions of the hyperbolas determined by the equations.

\[
(106) \quad k_y^{\prime\prime} - k_0^{\prime\prime} = 0
\]

and

\[
(107a) \quad k_0^{\prime\prime} k_0^{\prime\prime} \frac{n^2 - \beta^2}{1 - \beta^2} - \frac{(n^2 - 1)}{1 - \beta^2} (k_0^{\prime\prime} k_y^{\prime} + k_0^{\prime\prime} k_y^{\prime})
\]

\[
- \frac{(1 - n^2 \beta^2)}{1 - \beta^2} k_y^{\prime\prime} k_y = 0
\]

i.e.,

\[
(107b) \quad \left[ k_y^{\prime} + \frac{(n^2 - 1) \beta}{1 - n^2 \beta^2} k_0^{\prime} \right] \left[ k_y^{\prime\prime} + \frac{(n^2 - 1) \beta}{1 - n^2 \beta^2} k_0^{\prime\prime} \right] = k_0^{\prime\prime} \frac{n^2 (1 - \beta^2)^2}{(1 - n^2 \beta^2)^2} k_0^{\prime\prime}.
\]

The branch cuts due to transverse wave number \( \ell \) are shown in Fig. 5. They run from the branch points toward the imaginary axis. In the limit as \( k_0^{\prime\prime} \to 0 \) the cut becomes part of the real axis between \(-k_0\) and \(k_0\) and the imaginary axis. In the crosshatched region both the real and imaginary parts are positive provided we choose the positive root or branch for \( \ell \). The original contour \( c_0 \) must lie entirely in this region to represent outgoing waves.
Fig. 5—Proper branch cut due to $\ell$

The branch cuts due to the transverse wave number $k_x$ are shown in Fig. 6 when $n\beta < 1$ and in Fig. 7 when $n\beta > 1$. The centers of the hyperbola are in the third quadrant for $n\beta < 1$ and in the first quadrant when $n\beta > 1$. In the limit when $k_0''$ reduces to
Fig. 6—Proper branch cut due to $k_x$ when $n\beta < 1$
Fig. 7—Proper branch cut due to $k_x$ when $n\beta > 1$
zero the cut becomes the section of the real axis from \(-\frac{n-\beta}{1-n\beta}\) to \(\frac{n+\beta}{1+n\beta}\) and the line \(k_y = -\frac{(n^2-1)\beta}{1-n^2\beta^2}\) when \(n\beta < 1\). This line moves further toward left as \(n\beta\) approaches 1. When \(n\beta > 1\) the branch cuts become the sections of the real axis from \(\frac{n-\beta}{n\beta - 1}\) to \(+\infty\) and from \(\frac{n+\beta}{1+n\beta}\) to \(-\infty\). In the crosshatched regions the real and imaginary parts of \(k_x\) are both negative provided we choose the negative root or branch. The entire original contour \(c_o\) must lie in this region also. As the branch cuts may be chosen quite arbitrarily, as long as they do not intersect the contour \(c_o\), we fix their positions in the way shown in Fig. 8. This choice will be convenient for estimating the branch cut integrals, as discussed later on.

The location of the poles are determined by requiring

\[(108a) \quad \mu l = \mu_0 k_x.\]

As \(l^2 = k_o^2 - k_y^2\) and \(k_x = k^2a - \frac{\omega^2\Omega^2}{a} - \frac{2\omega\Omega}{a} k_y - \frac{1}{a} k_y^2\). Equation (108a) can be solved by first squaring both sides. The roots obtained have to be examined to find out on which sheet of the Riemann surface they are located. Thus,

\[\mu^2 l^2 = \mu_0^2 k_x^2\]

i.e.,
\[
 n\beta < 1 \\
 \frac{n - \beta}{1 - n\beta} k_0 \\
 n\beta > 1 \\
 \frac{n + \beta}{1 + n\beta} k_0 \\
 \frac{n - \beta}{n\beta - 1} k_0 
\]

Fig. 8--Relative positions of branch points and branch cuts

(108b) \[
\left(\mu^2 - \frac{\mu_0^2}{a}\right) k_y^2 - \frac{2b k_0 \mu_0^2}{a} k_y + \mu_0^2 n^2 k_o^2 a - \frac{\mu_0^2 b^2 k_o^2}{a} - \mu^2 k_o^2 = 0
\]

where \( b = \frac{(n^2 - 1) \beta}{1 - n^2 \beta^2} = \omega \). The solutions are,
\[ (109a) \quad k_y = k_0 \left[ \frac{b\mu_o^2}{a} + A \right] \left( \mu^2 - \frac{\mu_o^2}{a} \right) \]

where

\[ A = \left[ \frac{\mu^2}{a} \left( b^2 - 1 \right) + \mu_o^2 \mu^2 \right] \left( \mu^2 - \mu_o^2 a \right) + \mu^4 \].

For the special case where \( \mu = \mu_o \), Eq. (109a) reduces to

\[ (109b) \quad k_y = \frac{k_0}{\beta} \geq k_0 \]

and the pole is of order 2.

On the proper sheet of the Riemann surface we require that \( \text{Im} \; \ell \geq 0 \) and \( \text{Im} \; k_x \leq 0 \). So the poles which are solutions of Eq. (108a) are on the sheets either proper for \( \ell \) and improper for \( k_x \) or vice versa.

When \( \nu = 0 \), Eqs. (100) reduce to that of reflection from a semi-infinite dielectric medium. When \( \mu = \mu_o \) and \( \epsilon = \epsilon_o \) it reduces to zero and Eqs. (100) represents the field radiated from a line source in free-space. Hence the motion of the bottom region which has the same constitutive parameters as the upper region will not produce any effect on the radiation field.
Evaluation of the Contour Integral

The integrals shown in Eqs. (100) are too complicated to be evaluated rigorously and approximate method must be used. The saddle point method is useful for an asymptotic estimation of integrals of this type when there is a large parameter involved.

To apply this method of integration it is convenient to transform the integral into a complex $\phi$-plane defined as

\begin{align}
(110a) \quad & k_y = k_o \sin \phi \\
(110b) \quad & dk_y = k_o \cos \phi \, d\phi \\
(110c) \quad & \ell = \sqrt[k_y^2 - k_o^2} = k_o \cos \phi \\
(110d) \quad & \phi = \sigma + i\eta \\
(110e) \quad & y = r \sin \theta \\
(110f) \quad & x - d = r \cos \theta.
\end{align}

The integral (100a) becomes

\begin{equation}
E_z = \frac{-\omega \mu_o I_o}{4\pi} \int_{C_0} P(\phi) \, e^{ik_or \cos(\phi - \theta)} \, d\phi
\end{equation}

where
Fig. 9a—Path of integration in $k_y$-plane and $\phi$-plane when $n\beta < 1$
Fig. 9b--Path of integration in $k_y$-plane and $\phi$-plane when $n\beta > 1$
Equation (110) represents a mapping of the complex $k_y$-plane into a stripe of the complex $\phi$-plane. The transformed path of integration $c_o$ and the branch cuts in $\phi$-plane are shown in Fig. 9. The branch cuts which separate the sheet for which $\text{Im} \ell \geq 0$ from the sheet for which $\text{Im} \ell \leq 0$ are no longer cuts in $\phi$-plane. Because of the choice of positive sign in (110c) the hatched region in Fig. 9 corresponds to the proper sheet for $\ell$. The branch points associated with $k_x$ in the $\phi$-plane are transformed from (102b) as the solution of the equation

\[
\sin(\sigma + i\eta) = \frac{n + \beta}{1 + n\beta} \quad \text{and} \quad \frac{n - \beta}{1 - n\beta}
\]

and the poles from (109) as

\[
\sin(\sigma_p + \eta_p) = \left[ \frac{b\mu_o^2}{a} + \frac{1}{A^2} \right]^{\frac{1}{2}} \left( \mu^2 - \frac{\mu_o^2}{a} \right)
\]

\[
= 1/\beta \quad \text{(if} \quad \mu = \mu_o)\]

Their positions on the $\phi$-plane as relative to those branch points are shown in Fig. 10 when $\mu = \mu_o$. As was mentioned earlier, those poles given by (109) and (112b, c) are located on sheets which are
proper for one of the transverse wave number and improper for the other one, so the poles in the hatched region of $\phi$-plane are on the bottom sheet while the poles in the non-hatched region are on the top sheet.

The saddle point of the exponential term occurring in Eq. (III) is found by setting
\[
\frac{d}{d\phi} \cos(\phi - \theta) = 0
\]

which gives \( \phi = \theta \). The steepest descent contour is determined by \( \cos(\sigma - \theta) \cosh \eta = 1 \) which is denoted by \( c_s \) in Fig. 9 where \( c_{s_1} \) is the contour corresponding to positive observation angle \( \theta_1 \) and \( c_{s_2} \) corresponding to a negative observation angle \( \theta_2 \).

The main task now is to deform the original path \( c_0 \) into the steepest descent contour \( c_s \) and then perform the integration along \( c_s \).

In deforming the contour \( c_0 \) into contour \( c_s \) passing through the saddle point \( \phi = \theta \) some of the poles of \( P(\phi) \) given in (109) or (112b, c) may be encountered and the contour may be intercepted by the branch cut. So (111) may be written as

\[
E_z = -\frac{\omega \mu_0 I_0}{4\pi} \int_{c_s} P(\phi) e^{ik_0r \cos(\phi - \theta)} d\phi
\]

\[
-\frac{i\omega \mu_0 I_0}{2} \sum_{\mathcal{P}} F(\phi_p) e^{ik_0r \cos(\phi_p - \theta)} + K
\]

where \( \phi_p \) is the value of \( \phi \) at the pole as given by (112b, c), \( F(\phi_p) \) has a magnitude equal to the residue of \( P(\phi) \) at the pole and a sign chosen to correspond to the position of the pole. \( K \) is the branch cut integral.
If $P(\phi)$ does not have a pole in the vicinity of the saddle point
the integral along the steepest descent contour is readily evaluated
by expanding $P(\phi)$ in a Taylor series about $\theta$ as

\begin{equation}
P(\phi) = P(\theta) + \sum_{m=1}^{\infty} P^m(\theta) \frac{(\phi-\theta)^m}{m!}
\end{equation}

where

$$P^m(\phi) = \left. \frac{d^m P(\phi)}{d\phi^m} \right|_{\phi=\theta}$$

Along the steepest descent contour $c_s$ near $\theta$, let $\phi-\theta = \rho e^{iw}$. Then

$$f(\phi) = ik_0 r \cos(\phi-\theta) = f(\theta) + \frac{\partial f}{\partial \phi} \bigg|_{\phi=\theta} (\phi-\theta) + \frac{1}{2} \frac{\partial^2 f}{\partial \phi^2} \bigg|_{\phi=\theta} (\phi-\theta)^2$$

$$+ \cdots$$

$$\simeq f(\theta) + \frac{1}{2} \frac{\partial^2 f}{\partial \phi^2} \bigg|_{\phi=\theta} (\phi-\theta)^2$$

$$= ik_0 r - \frac{1}{2} ik_0 r \rho^2 \cos 2w + \frac{1}{2} k_0 r \rho^2 \sin 2w$$

$$= ik_0 r - \frac{1}{2} k_0 r \rho^2$$

and

$$w = \pi - \frac{\pi}{4} \quad \phi = \theta + \rho e^{i\pi/4} \quad d\phi = e^{i\pi/4} d\rho \text{ in the 2nd quadrant}$$

$$w = -\frac{\pi}{4} \quad \phi = \theta + \rho e^{-i\pi/4} \quad d\phi = e^{-i\pi/4} d\rho \text{ in the 4th quadrant.}$$
Hence the integral term in (113) becomes

\[
\frac{-\omega \mu_0 I_0}{4\pi} \sum_{s=2}^{\infty} J_s(\rho) e^{ik_0r} \cos(\phi - \theta) d\phi
\]

\[
= \frac{-\omega \mu_0 I_0}{4\pi} \sum_{m=0}^{\infty} \left[ \frac{P_m(\theta)}{m!} \frac{(\phi - \theta)^m}{m!} e^{ik_0r \cos(\phi - \theta)} d\phi \right]
\]

\[
= \frac{-\omega \mu_0 I_0}{4\pi} \sum_{m=0}^{\infty} \left[ \int_0^{\rho_1} \rho^m e^{\text{im}} e^{i\pi} e^{i\left(\frac{3\pi}{4}\right) k_0 r - \frac{k_0 r}{2} \rho^2} d\rho \right] + \int_0^{\rho_1} \rho^m e^{\text{im}} e^{i\pi} e^{i\left(\frac{3\pi}{4}\right) k_0 r - \frac{k_0 r}{2} \rho^2} d\rho
\]

The major contribution to the integral comes from a small range

\( 0 < \rho < \rho_1 \) along the contour \( c_s \) provided \( k_0 r \) is sufficiently large.

Combining the two terms in (115) gives

\[
-\omega \mu_0 I_0 \sum_{m=0}^{\infty} \frac{P_m(\theta)}{m!} \left[ 1 + (-1)^m \right] \int_0^{\rho_1} \rho^m e^{-\frac{1}{2} k_0 r \rho^2} d\rho
\]

Because of the rapid decay of the exponential, the integral from 0 to \( \rho_1 \) does not differ much from

\[
\int_0^\infty \rho^m e^{-\frac{1}{2} k_0 r \rho^2} d\rho = \frac{\Gamma\left(\frac{m+1}{2}\right)}{2^{(m+1)/2} k_0 r^{m+1/2}} \quad m > -1
\]
where $\Gamma$ is the gamma function. The leading term of (116a) is therefore

\begin{equation}
E_z \sim - \frac{\omega \mu_o I_o}{2 \sqrt{2} \pi (k_o r)^{\frac{1}{2}}} e^{i k_o r - i \frac{\pi}{4}} P(\theta).
\end{equation}

The residue of $P(\phi)$ at the pole $\phi_p$ is given by

\begin{equation}
F(\phi_p) = \frac{\frac{\mu}{a} \cos \phi + \mu_o (n^2 a - \frac{b^2}{a} - \frac{2b}{a} \sin \phi - \frac{1}{a} \sin^2 \phi)^{\frac{1}{2}}}{\frac{d}{d\phi} \left[ \mu \cos \phi + \mu_o (n^2 a - \frac{b^2}{a} - \frac{2b}{a} \sin \phi - \frac{1}{a} \sin^2 \phi)^{\frac{1}{2}} \right]} |_{\phi = \phi_p}
\end{equation}

\begin{equation}
+ \frac{\frac{d}{d\phi} \left[ \mu \cos \phi + \mu_o (n^2 a - \frac{b^2}{a} - \frac{2b}{a} \sin \phi - \frac{1}{a} \sin^2 \phi)^{\frac{1}{2}} \right]}{\frac{d}{d\phi} \left[ \mu \cos \phi + \mu_o (n^2 a - \frac{b^2}{a} - \frac{2b}{a} \sin \phi - \frac{1}{a} \sin^2 \phi)^{\frac{1}{2}} \right]} |_{\phi = \phi_p}
\end{equation}

\begin{equation}
= \frac{2 \mu^2 \cos \phi_p e^{i 2d k_o \cos \phi_p}}{-\mu^2 \sin \phi_p + \frac{\mu_o}{a} (\beta + \sin \phi_p)}
\end{equation}

\begin{equation}
= \frac{4a}{(a-1)} \tan \phi_p (1 + i 2d k_o \cos \phi_p) e^{i 2d k_o \cos \phi_p} \text{ (if } \mu = \mu_o).}
\end{equation}

If a pole $\phi_p$ happens to be near the saddle point $\phi = 0$ the Taylor series expansion of $P(\phi)$ is no longer valid in a sufficiently large enough region around the point $\theta$. Then a Laurent series must be used. When $\mu = \mu_o$ the pole is of order two and unless the observation angle $\theta$ is near 90 and $\beta$ is nearly equal one this pole of order two will not be in the vicinity of the saddle point. In all other cases
the pole is a simple one. We have

\begin{equation}
(119) \quad P(\phi) = \frac{F(\phi_p)}{\phi - \phi_p} + P_1(\phi)
\end{equation}

where $P_1(\phi)$ is analytic in the region around $\theta$ and may be developed into a Taylor series. $F(\phi_p)$ is the residue of $P(\phi)$ at $\phi = \phi_p$ and $P_1(\phi)$ can be expanded in Taylor's series as,

\begin{equation}
P_1(\phi) = \sum_{m=0}^{\infty} a_m(\phi - \theta)^m
\end{equation}

\begin{equation}
a_m = \frac{1}{m!} \frac{d^m}{d\phi^m} \left[ P(\phi) - \frac{F(\phi_p)}{\phi - \phi_p} \right].
\end{equation}

The portion of the integral for the field involving $P_1(\phi)$ leads to the same form discussed before. The remaining integral to be evaluated is,

\begin{equation}
(120) \quad \frac{-\omega \mu_0 I_0}{4\pi} \int_{c_s} \frac{F(\phi_p)}{\phi - \phi_p} e^{ik_0 r \cos(\phi - \theta)} e^{i2\omega \phi} d\phi.
\end{equation}

Near the saddle point and along the contour $c_s$

\begin{align*}
\cos(\phi - \theta) &\approx 1 - (\phi - \theta)^2/2 = 1 - \frac{1}{2} \rho^2 e^{i2\omega} \\
\phi - \phi_p &= \phi - \theta - (\phi_p - \theta) \\
\phi - \phi_p &= \rho e^{i\frac{3\pi}{4}} - (\phi_p - \theta) = -\rho e^{-i\frac{\pi}{4}} - (\phi_p - \theta) \\
d\phi &= e^{i\frac{3\pi}{4}} d\rho \text{ in 2nd quadrant}
\end{align*}
\[
\phi - \phi_p = \rho e^{-\frac{i\pi}{4} - (\phi_p - \theta)} \quad \text{d}\phi = e^{-\frac{i\pi}{4}} \, d\rho \text{ in 4th quadrant}
\]

We get in place of expression (120),

\[
(121) \quad -\frac{\omega \mu_0 I_0}{4\pi} F(\phi_p) e^{i k_0 r} \left[ \int_{\rho_1}^{1} e^{-\frac{k_0 r \rho^2}{2}} e^{-i\frac{3\pi}{4} - (\phi_p - \theta)} \, d\rho \right]
\]

\[
+ \int_{\rho_1}^{1} e^{-i\frac{\pi}{4} - (\phi_p - \theta)} \, d\rho
\]

\[
\approx -\frac{\omega \mu_0 I_0}{2\pi} F(\phi_p) e^{i k_0 r + i\frac{\pi}{4}} (\phi_p - \theta) \int_{0}^{\infty} \frac{e^{-\frac{k_0 r \rho^2}{2}}}{\rho^2 - i(\phi_p - \theta)^2} \, d\rho.
\]

Making the change of variable \( t = \rho^2 \) in the integral

\[
I = \int_{0}^{\infty} \frac{e^{-\frac{k_0 r \rho^2}{2}}}{\rho^2 - i(\phi_p - \theta)^2} \, d\rho
\]

and following the steps as shown by Oberhettinger[23] we have

\[
(122) \quad I = \frac{1}{2} \int_{0}^{\infty} e^{-\frac{k_0 r t}{2}} \frac{1}{t^2 - i[(\phi_p - \theta)^2]} \, dt
\]

\[
= \frac{\pi}{2} e^{-\frac{i k_0 r}{2} (\phi_p - \theta)^2} \left[ -i(\phi_p - \theta)^2 \right]^{-\frac{1}{2}} \text{erfc} \left[ -\frac{i k_0 r}{2} (\phi_p - \theta)^2 \right]^{\frac{1}{2}}
\]

where

\[
\text{erfc} (x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-t^2} \, dt.
\]
The error function may be expanded asymptotically for $x \to \infty$ as \[24\]

\[
\text{erfc}(x) = \frac{e^{-x^2}}{x\sqrt{\pi}} \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})} x^{-2m}
\]

provided the phase of $x$ is in the range between $-\frac{3\pi}{4}$ and $\frac{3\pi}{4}$. $\text{erf}(x)$ is an odd function of $x$ so if the phase of $x$ is not within this range the above asymptotic expansion can still be used through the use of the following relation, \[124\] \[
\text{erfc}(x) = 1 - \text{erf}(x) = 1 - \text{erf}(-x) = 2 - \text{erfc}(-x).
\]

If the phase angle of $\left[-\frac{ik_Or}{2} (\phi_p - \theta)^2\right]^{1/2}$ is in the range between $-\frac{3\pi}{4}$ and $\frac{3\pi}{4}$ integral (120) becomes

\[
\frac{-\omega \mu_0 I_0}{2\pi} F(\phi_p) e^{ik_Or + \frac{i\pi}{4}} (\phi_p - \theta) \int_0^{\infty} \frac{e^{-\frac{2}{\rho^2 - i(\phi_p - \theta)^2}}}{\rho^2 - i(\phi_p - \theta)^2} d\rho
\]

\[
= \frac{-\omega \mu_0 I_0}{4\sqrt{\pi}} F(\phi_p) (\phi_p - \theta) e^{ik_Or + \frac{i\pi}{4}} e^{-\frac{k_Or}{2} (\phi_p - \theta)^2} \left[\frac{\frac{1}{2} - i(\phi_p - \theta)^2}{\left[-\frac{ik_Or}{2} (\phi_p - \theta)^2\right]^{1/2}} \right]^{1/2}
\]

\[
\sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})} \left[-\frac{ik_Or}{2} (\phi_p - \theta)^2\right]^{-m}
\]
This expression shows a significant modification of the radiation field when $\phi_p$ is located close to the saddle point $\theta$.

When $n\beta < 1$ and if the angle of observation $\theta$ extends over one of the two limits (one on the right side and another on the left side of the $x$-axis given by $\theta_1 = \sin^{-1} \left( \frac{1+n\beta}{n+\beta} \right)$ and $\theta_2 = \sin^{-1} \left( \frac{1-n\beta}{n-\beta} \right)$) the path of integration will be intercepted by a branch cut. To evaluate the integral in this case a branch cut integration must be included. Following Ott's [25] method, a complete contour can be constructed as shown by $c_{sb1}$ or $c_{sb2}$ in Fig. 11a where the dotted line denotes the part of the path which is traveled on the bottom sheet of the two Riemann surfaces. The corresponding path of integration in the $k_y$-plane is shown in the same figure.

When $n\beta > 1$ and if the angle observation $\theta$ is in the range between $\theta = \sin^{-1} \left( \frac{1+n\beta}{n+\beta} \right)$ and $\theta = \sin^{-1} \left( \frac{n\beta-1}{n-\beta} \right)$ a branch cut integral must also be involved. The path of integration is shown in Fig. 11b. The branch cut integration which is now part of the total integration can
Fig. 11a -- Path of integration when the steepest descent path is intercepted by the branch cut and $n\beta < 1$
Fig. 11b--Path of integration when the steepest descent path $C_s$ is intercepted by the branch cut, $n\beta > 1$ and
\[
\sin^{-1}\left(\frac{n\beta - 1}{n - \beta}\right) < \theta < \sin^{-1}\left(\frac{n\beta + 1}{n + \beta}\right)
\]
Fig. 11c--Path of integration when the steepest descent path $C_s$ is intercepted by the branch cut, $n\beta > 1$ and $\theta > \sin^{-1}\left[\frac{n\beta + 1}{n + \beta}\right]$.
be performed in the complex $k_y$-plane as follows:

Let a change of variable be made such that

$$k_y = k_{yo} + \frac{it}{y}$$

where $k_{yo}$ is one of the branch points given in (102b). In terms of the new variable $t$, the following approximations are permissible for large values of $y$.

$$f^2 = k_o^2 - k_y^2 = k_o^2 - (k_{yo} + \frac{it}{y})^2 = k_o^2 - k_{yo}^2 + \frac{t^2}{y^2} - 2ik_{yo}t/y$$

$$\sim k_o^2 - k_{yo}^2$$

$$k_x^2 = n k_o^2 - \frac{b^2 k_o^2}{a} - \frac{2bk_o}{a} \left(k_{yo} + \frac{i}{y}\right) - \frac{1}{a} \left(k_{yo} + \frac{i}{y}\right)^2$$

$$= -\frac{2bk_o}{a} \frac{it}{y} - \frac{2k_{yo}}{a} \frac{it}{y} + \frac{1}{a} \frac{t^2}{y^2}$$

$$\sim -\frac{2}{a} (bk_o + k_{yo}) \left(\frac{it}{y}\right)$$

$$\frac{1}{f} \left[1 + \frac{\mu f + \mu_o k_x}{\mu f - \mu_o k_x} e^{i2fd}\right] = \frac{1}{f} \left[1 + \frac{1 + \frac{\mu o k_x}{\mu f}}{1 - \frac{\mu o k_x}{\mu f}} e^{i2fd}\right]$$

$$\sim \frac{1}{f} \left[1 + \left(1 + 2 \frac{\mu_o k_x}{\mu f}\right) e^{i2fd}\right].$$

Equation (100a) reduces to
The above result is obtained by making use of the contour representation of the gamma function. [26] The branch cut integral given above contains an exponential damping factor and it is inversely proportional to the three half power of \( y \). Hence it is always negligible as compared with those given by (116) or (125).

Those residue terms which have to be considered as shown in (113) when the contour was deformed contain an exponential factor

\[
e^{ik_0 r \cos(\phi_p - \theta)} = e^{ik_0 r \cos(\sigma_p - \theta)} \cosh \eta_p + k_0 r \sin(\sigma_p - \theta) \sinh \eta_p
\]

and the real part of the exponent is always negative. Except when \( \theta = 90^\circ \) the residue wave is also negligible as compared to the space wave given by (116).
The same procedure can be used in evaluating the field components \( H_x \) and \( H_y \) as given in (100b) and (100c). The leading terms from the saddle point method of integration are

\[
H_x \sim -\frac{k_0 I_0}{4} \sqrt{\frac{2}{\pi}} e^{ik_0 r - i \frac{\pi}{4}} \frac{1}{(k_0 r)^{\frac{1}{2}}} \sin \theta \ P(\theta)
\]

\[
H_y \sim \frac{k_0 I_0}{4} \sqrt{\frac{2}{\pi}} e^{ik_0 r - i \frac{\pi}{4}} \frac{1}{(k_0 r)^{\frac{1}{2}}} \cos \theta \ P(\theta)
\]

where \( P(\theta) \) is given in (111a). Converting into cylindrical coordinates we have

\[
H_r \sim 0
\]

\[
H_\theta \sim H_y \cos \theta - H_x \sin \theta = \frac{k_0 I_0}{2\sqrt{2\pi}} e^{ik_0 r - i \frac{\pi}{4}} P(\theta)
\]

\[
E_z \sim \frac{-\omega \mu_0 I_0}{2\sqrt{2\pi} (k_0 r)^{\frac{1}{2}}} e^{ik_0 r - i \frac{\pi}{4}} P(\theta)
\]

and the Poynting vector is

\[
\vec{S} = \frac{1}{2} \vec{E} \times \vec{H}^* = \frac{\omega \mu_0 I_0^2}{16 \pi r} \ |P(\theta)|^2
\]

which has only a radial component.
Magnetic Line Source Parallel to Z-Axis

For a magnetic line current source located at \( x = d, \ y = 0 \) and parallel to z-axis we write

\[
\bar{M} = \frac{\delta}{2} \bar{M}_0 \delta(x-d) \ \delta(y) .
\]

The radiated field has no variation with respect to z. As the z-component of the electric field is zero the total fields may be derived from electric vector potentials \( \bar{F}_T \) in region I and \( \bar{F}_B \) in region II having only z-components. Similar procedures are used in solving this problem as that for the electric line current source case given in the previous section. Only those important steps and results are given below.

The field components in upper region are derived from \( \bar{F}_T \) as

\[
\begin{align*}
(129a) \quad \bar{E} &= \frac{-1}{\epsilon_0} \nabla \times \bar{F}_T \\
(129b) \quad \bar{H} &= i\omega \bar{F}_T - \frac{\nabla \nabla \cdot \bar{F}_T}{i\omega \mu_0 \epsilon_0}
\end{align*}
\]

or

\[
\begin{align*}
E_x &= \frac{-1}{\epsilon_0} \frac{\partial F_T}{\partial y} \quad H_x = 0 \\
E_y &= \frac{1}{\epsilon_0} \frac{\partial F_T}{\partial x} \quad H_y = 0 \\
E_z &= 0 \quad H_z = i\omega F_T
\end{align*}
\]
where $\overline{F_T} = \hat{a} F_T$ and $F_T$ satisfy the wave equation

$\nabla^2 F_T + k_o^2 F_T = -\varepsilon_0 M_0 \delta(x-d) \delta(y)$ \hspace{1cm} (130)

and the suitable boundary conditions. The solution for (130) is

$F_T = \frac{1}{2\pi} \int_{-\infty}^{\infty} g_2(x, k_y) e^{ik_y y} dk_y \hspace{1cm} (131)$

where $g_2$ is given by

$(132a) \hspace{1cm} g_{21}(x, k_y) = \left(\frac{-\varepsilon_0 M_0}{2i\ell}\right) [1 + Re^{i2\ell d}] \hspace{1cm} (x \geq d)$

$(132b) \hspace{1cm} g_{22}(x, k_y) = \left(\frac{-\varepsilon_0 M_0}{2i\ell}\right)[e^{-i\ell(x-d)} + Re^{i\ell(x+d)}](0 \leq x \leq d)$

with $\ell^2 = k_o^2 - k_y^2$.

In the bottom region let

$\overline{\nu} = \hat{y} \nu \hspace{1cm} \overline{\Omega} = \hat{y} \Omega \hspace{1cm} (133)$

$\overline{F_B} = \hat{a} F_B \hspace{1cm} (134)$

and introduce the vector function such that

$F_{B1} = a \cdot F_B = \overline{F_B} = \overline{F_B} = \hat{a} F_B \hspace{1cm} (135)$

where $F_{B1}$ satisfies Eq. (43b) which reduces here to

$F_{B1} = 0 \hspace{1cm} (135)$
The field components in this region are derived from \( \overline{F_B} \) by Eqs. (45b) and (46b)

\[
(136a) \quad \overline{E} = \frac{-1}{\varepsilon} \alpha^{-1} \cdot [ (\nabla + i\omega) \times \overline{F_B} ] = \frac{-1}{\varepsilon} \alpha^{-1} 
\]

\[
\cdot [ (\nabla + i\omega) \times (\alpha^{-1} \cdot \overline{F_{B1}}) ]
\]

\[
(136b) \quad \overline{H} = i\omega \overline{F_B} - D_1 V = i\omega \alpha^{-1} \cdot \overline{F_{B1}} - \frac{(\nabla + i\omega) \cdot (\nabla + i\omega) \cdot \overline{F_{B1}}}{i\omega \mu \varepsilon a^2}
\]

or

\[
E_x = \frac{-1}{\varepsilon a^2} \left[ \frac{\partial F_{B1}}{\partial y} + i\omega \Omega F_{B1} \right] \quad H_x = 0
\]

\[
E_y = \frac{1}{\varepsilon a} \frac{\partial F_{B1}}{\partial x} \quad H_y = 0
\]

\[
E_z = 0 \quad H_z = \frac{i\omega}{a} F_{B1}.
\]

The Fourier integral

\[
(137) \quad F_{B1} = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_2(k_y) e^{ik_x x + ik_y y} \, dk_y
\]

is a solution of (135) provided \( k_x \) is given as shown in Eq. (95).

By matching boundary conditions at \( x = 0 \) we get

\[
(138a) \quad R = \frac{\varepsilon \ell + \varepsilon_0 k_x}{\varepsilon \ell - \varepsilon_0 k_x}
\]
(138b) \[ f_z = \frac{ia_0 e M_0 e^{i\ell d}}{\varepsilon \ell - \varepsilon_0 k_x} \, . \]

The choice of the branches of \( \ell \) and \( k_x \) are decided in the same way as for the electric line source case. The field components in the region \( x \geq d \) are given by integrals of the form,

(139a) \[ H_z = -\frac{\omega \varepsilon_0 M_0}{4\pi} \int_{-\infty}^{\infty} \frac{1}{\ell} \left[ 1 + \text{Re} e^{i2\ell d} \right] e^{i\ell(x-d)+ik_y y} \, dk_y \]

(139b) \[ E_x = \frac{M_0}{4\pi} \int_{-\infty}^{\infty} \frac{k_y}{\ell} \left[ 1 + \text{Re} e^{i2\ell d} \right] e^{i\ell(x-d)+ik_y y} \, dk_y \]

(139c) \[ E_y = -\frac{M_0}{4\pi} \int_{-\infty}^{\infty} \left[ 1 + \text{Re} e^{i2\ell d} \right] e^{i\ell(x-d)+ik_y y} \, dk_y \, . \]

The branch points are given in Eq. (102) and their relative positions shown in Fig. 8. The poles are found by solving the equation,

(140) \[ \varepsilon \ell = \varepsilon_0 k_x \, . \]

Squaring both sides of (140) we have the following results

(141a) \[ k_y = k_0 \left[ \frac{b^2 \varepsilon_0}{a} + \frac{1}{B^2} \right] \sqrt{\left( \frac{\varepsilon^2 - \varepsilon_0^2}{a} \right)} \]

where

\[ B = \left[ \frac{\varepsilon^2 \varepsilon_0^2}{a} (b^2 - 1) + \varepsilon_0^2 n^2 (\varepsilon_0^2 - \varepsilon^2 a) + \varepsilon^4 \right] \]

and
(141b) \[ k_y = \frac{k_o}{\beta} \geq k_o \quad (\text{if } \epsilon = \epsilon_0) \]

Applying the transformation shown in (110) Eq. (139a) changes to

\[ H_z = \frac{-\omega \epsilon_0 M_0}{4\pi} \int_{c_o} Q(\phi) e^{i k_o r \cos(\phi-\theta)} d\phi \]

where

\[ Q(\phi) = 1 + \frac{\epsilon \cos \phi - \epsilon_0 \left( n^2 a - \frac{b^2}{a} - \frac{2b}{a} \sin \phi - \frac{1}{a} \sin^2 \phi \right)^{1/2}}{\epsilon \cos \phi + \epsilon_0 \left( n^2 a - \frac{b^2}{a} - \frac{2b}{a} \sin \phi - \frac{1}{a} \sin^2 \phi \right)^{1/2}} e^{i2dk_o \cos \phi} \]

The transformed path of integration \( c_o \), the path of steepest descent, the saddle point and the branch points due to \( k_x \) are the same as those shown in Fig. 9. The poles are transformed from (141) as

\[ \sin \phi_p = \sin(\sigma_p + i\eta_p) = \left[ \frac{b \epsilon_o}{a} \pm B^{1/2} \right] / \left( \epsilon^2 - \frac{\epsilon_o^2}{a} \right) \]

\[ = \frac{1}{\beta} \quad (\text{if } \epsilon = \epsilon_0) \]

Because of the deformation of the path of integration the integral in (142) becomes

\[ H_z = -\frac{\omega \epsilon_0 M_0}{4\pi} \int_{c_s} Q(\phi) e^{i k_o r \cos(\phi-\theta)} d\phi \]

\[ - \frac{i\omega \epsilon_0 M_0}{2} \sum_p G(\phi_p) e^{i k_o r \cos(\phi_p - \theta)} + K \]
where $\phi_p$ is given by (143), $G(\phi_p)$ is the residue of $Q(\phi)$ at $\phi_p$ and $K$ is the branch cut integral. If there is no pole near the saddle point $\theta$ the first term in (144) can be evaluated approximately for far field as

\begin{equation}
H_z = \frac{\omega \epsilon_0 M_0}{4\pi} e^{ik_0r - i\frac{\pi}{4}} \sum_{m=0}^{\infty} \frac{Q^m(\theta)}{m!} e^{-im\frac{\pi}{4}} \cdot \left[ 1 + (-1)^m \right] \frac{\Gamma\left(\frac{m+1}{2}\right)}{2\left(\frac{k_0r}{2}\right)^{m+1}}
\end{equation}

where

\[ Q^m(\theta) = \frac{d^m}{d\phi^m} Q(\phi) \bigg|_{\phi=\theta}. \]

The leading term of (145a) is

\begin{equation}
(145b) \quad \frac{-\omega \epsilon_0 M_0}{2\sqrt{2\pi(k_0r)^{\frac{1}{2}}}} e^{ik_0r - i\frac{\pi}{4}} Q(\theta).
\end{equation}

The residue of $Q(\phi)$ at pole $\phi_p$ is

\begin{equation}
G(\phi_p) = \frac{2\epsilon^2 \cos \phi_p e^{i2d k_0 \cos \phi_p}}{-\epsilon^2 \sin \phi_p + \frac{\epsilon_0}{2} (b + \sin \phi_p)}
\end{equation}

\begin{equation}
(146b) \quad = \left( \frac{4a}{a-1} \right) \tan \phi_p (1 + i2d k_0 \cos \phi_p) e^{i2d k_0 \cos \phi_p} \text{ (if } \epsilon = \epsilon_0). \]
The branch cut integral $K$ is given approximately by

$$H_z(\text{branch}) = \frac{\omega \varepsilon_0 M_0}{2 \sqrt{\pi \varepsilon} (k_o^2 - k_{yo}^2) y^3} \left[ \frac{i2}{a} (bk_o + k_{yo}) \right]^{\frac{1}{2}} \cdot e^{-\sqrt{\frac{2}{k_{yo} - k_o^2} (x+d) + ik_{yo}y}}$$

where $k_{yo}$ is the branch point given in (102b). The leading terms for $E_x$ and $E_y$ from saddle point method of integration are

$$E_x \approx \frac{M_0 k_o}{2 \sqrt{2\pi}} e^{\frac{i}{(k_o r)^\frac{1}{2}}} \sin \theta Q(\theta)$$

$$E_y \approx \frac{M_0 k_o}{2 \sqrt{2\pi}} e^{\frac{i}{(k_o r)^\frac{1}{2}}} \cos \theta Q(\theta)$$

where $Q(\theta)$ is given by (142a). In terms of cylindrical coordinates the field components are

$$E_r \approx 0$$

$$E_\theta \approx \frac{M_0 k_o}{2 \sqrt{2\pi}} e^{\frac{i}{(k_o r)^\frac{1}{2}}} Q(\theta)$$

$$H_y \approx \frac{-\omega \varepsilon_0 M_0}{2 \sqrt{2\pi (k_o r)^\frac{1}{2}}} e^{\frac{i}{(k_o r)^\frac{1}{2}}} Q(\theta).$$

The Poynting vector is

$$\mathbf{S} \approx \frac{1}{2} \mathbf{E} \times \mathbf{H}^* = \hat{r} \frac{\omega \varepsilon_0 M_0^2}{16 \pi r} |Q(\theta)|^2.$$
Electric Line Source Parallel to Y-Axis

The electric line source located at \( x = d, \ z = 0 \) and parallel to \( y \)-axis can be represented by

\[
\overline{J} = \hat{y} J_0 \, \delta(x-d) \, \delta(z)
\]

As the current has no variation in \( y \)-direction the radiated field will be independent of \( y \). The method of potentials is used in constructing the total electromagnetic fields. The fields will be expressed in terms of the magnetic and electric vector potentials \( \overline{A}_T \) and \( \overline{F}_T \) in the upper region and \( \overline{A}_B \) and \( \overline{F}_B \) in the lower region having only \( y \)-components. The field components in the upper region are given as

\[
\begin{align*}
(147a) \quad \overline{E} &= i \omega \overline{A}_T - \frac{\nabla \nabla \cdot \overline{A}_T}{i \omega \mu_0 \epsilon_0} - \frac{1}{\epsilon_0} \nabla \times \overline{F}_T \\
(147b) \quad \overline{H} &= \frac{1}{\mu_0} \nabla \times \overline{A}_T + i \omega \overline{F}_T - \frac{\nabla \nabla \cdot \overline{F}_T}{i \omega \mu_0 \epsilon_0}
\end{align*}
\]

or

\[
\begin{align*}
E_x &= \frac{1}{\epsilon_0} \frac{\partial F_T}{\partial z} \\
E_y &= i \omega A_T \\
E_z &= -\frac{1}{\epsilon_0} \frac{\partial F_T}{\partial x} \\
H_x &= -\frac{1}{\mu_0} \frac{\partial A_T}{\partial z} \\
H_y &= i \omega F_T \\
H_z &= \frac{1}{\mu_0} \frac{\partial A_T}{\partial x}
\end{align*}
\]
where $A_T = \hat{y} A_T$ and $F_T = \hat{y} F_T$. $A_T$ and $F_T$ satisfy respectively the wave equations

(148a) \[ \nabla^2 A_T + k_o^2 A_T = -\mu_o I_o \delta(x-d) \delta(z) \]

and

(148b) \[ \nabla^2 F_T + k_o^2 F_T = 0 \]

The solutions are

(149a) \[ A_T = \frac{1}{2\pi} \int_{-\infty}^{\infty} g_1(x,k_z) e^{ik_z z} dk_z \]

(149b) \[ F_T = \frac{1}{2\pi} \int_{-\infty}^{\infty} g_2(k_z) e^{i[k_x+i k_z z]} dk_z \]

where $I^2 = k_o^2 - k_z^2$ and $g_1$ is the same as that shown in (90). $g_2$ and $R$ have to be determined by the boundary conditions at $x = 0$ which will be discussed later on.

In the bottom region we introduce the vector functions

(150a) \[ \bar{A}_{B1} = \hat{y} A_{B1} = \alpha \cdot \bar{A}_B = \bar{A}_B = \hat{y} A_B \]

and

(150b) \[ \bar{F}_{B1} = \hat{y} F_{B1} = \alpha \cdot \bar{F}_B = \bar{F}_B = \hat{y} F_B \]
\( A_{B1} \) and \( F_{B1} \) have to satisfy the equation (43) which in this problem is

\[
\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} - \frac{\omega^2 \Omega^2}{a} + k^2a \right] \begin{Bmatrix}
A_{B1} \\
F_{B1}
\end{Bmatrix} = 0.
\]

The solutions are

\[(152a) \quad A_{B1} = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(k_z) e^{ik_x x + ik_z z} dk_z \]

and

\[(152b) \quad F_{B1} = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_2(k_z) e^{ik_x x + ik_z z} dk_z \]

where \( k_x^2 = k^2 - \frac{\omega^2 \Omega^2}{a} - k_z^2 \). \( f_1 \) and \( f_2 \) are determined by the boundary conditions at \( x = 0 \). The roots of \( k \) and \( k_x \) will be chosen in the same way as shown in (96). Using Eqs. (45) and (46) we get the field components in the bottom region

\[(153a) \quad \overline{E} = i\omega \overline{A}_{B1} - D_1 \frac{\overline{A}_{B1}}{i\omega \mu \varepsilon a^2} - \frac{1}{\varepsilon} \frac{\Omega}{\alpha^{-1}} \cdot (D_1 \times \overline{F}_{B1}) \]

\[= i\omega \overline{A}_{B1} - (\nabla + i\omega \Omega) \frac{[\nabla + i\omega \Omega] \cdot \overline{A}_{B1}}{i\omega \mu \varepsilon a^2} - \frac{1}{\varepsilon} \frac{\Omega}{\alpha^{-1}} \]

\[\cdot [ (\nabla + i\omega \Omega) \times \overline{F}_{B1} ] \]
The tangential components of electric and magnetic field components at the boundary \( x = 0 \) are required to be continuous. As a consequence we find

\begin{align*}
\text{(154a)} & \quad \left( \frac{-\mu_0 \omega}{2i\ell} \right) (1 + R) e^{i \omega d} = \left( 1 - \frac{\Omega^2}{\mu \varepsilon a^2} \right) f_1 \\
\text{(154b)} & \quad \frac{k}{\varepsilon_0} \left( \frac{k_z}{\mu \varepsilon a^2} \right) f_1 + \left( \frac{k_x}{\varepsilon a} \right) f_2 \\
\text{(154c)} & \quad g_2 = \left( 1 - \frac{\Omega^2}{\mu \varepsilon a^2} \right) f_2 \\
\text{(154d)} & \quad \frac{I_0}{2} (1 - R) e^{i \omega d} = \frac{i k_x}{\mu a} f_1 - \frac{i \omega k_z}{\mu \varepsilon a^2} f_2 .
\end{align*}
Solving these simultaneous equations the results are

\[(155) \quad R = \frac{\varepsilon a(1-\alpha) \ell - \varepsilon_0 k_x}{\varepsilon a(1-\alpha) \ell - \varepsilon_0 k_x} \frac{[\mu a(1-\alpha) \ell + \mu_0 k_x] - \alpha \mu_0 \varepsilon_0 k_z^2}{[\mu a(1-\alpha) \ell - \mu_0 k_x] + \alpha \mu_0 \varepsilon_0 k_z^2} \]

\[(156) \quad E_z = \frac{i \mu_0 \varepsilon_0 \ell_0 (1-\alpha) \mu k_z^2}{[\varepsilon a(1-\alpha) \ell - \varepsilon_0 k_x] [\mu a(1-\alpha) \ell - \mu_0 k_x] + \alpha \mu_0 \varepsilon_0 k_z^2} \]

where

\[\alpha = \frac{\Omega}{\mu \varepsilon a^2} \]

Hence, the final expressions for the field components in the region \( x \geq d \) are given in the following integral form.

\[(157a) \quad E_x = - \frac{\mu_0 I_0 (1-\alpha) \Omega}{2\pi} \int_{-\infty}^{\infty} \frac{2 i \ell (x+d) + i k_z z}{k_z e} \frac{R_d}{R} \, dk_z \]

\[(157b) \quad E_y = - \frac{\omega \mu_0 I_0}{4\pi} \int_{-\infty}^{\infty} \frac{1}{\ell} \left[ 1 + \frac{R_n}{R_d} \right] e^{\frac{i \ell (x-d) + i k_z z}{k_z e}} \, dk_z \]

\[(157c) \quad E_z = \frac{\mu_0 I_0 (1-\alpha) \Omega}{2\pi} \int_{-\infty}^{\infty} \frac{\ell k_z e}{R_d} \, dk_z \]

\[(157d) \quad H_x = \frac{I_0}{4\pi} \int_{-\infty}^{\infty} \frac{k_z}{\ell} \left[ 1 + \frac{R_n}{R_d} \right] e^{\frac{i \ell (x-d) + i k_z z}{k_z e}} \, dk_z \]

\[(157e) \quad H_y = - \frac{\omega \mu_0 \ell_0 I_0 (1-\alpha) \Omega}{2\pi} \int_{-\infty}^{\infty} \frac{k_z e}{R_d} \, dk_z \]
(157f) \[ H_z = -\frac{I_0}{4\pi} \int_{-\infty}^{\infty} \left[ 1 + \frac{R_n}{R_d} e^{i2\ell d} \right] e^{i\ell(x-d) + ik_z z} \, dk_z \]

where

\[
R_n = [\varepsilon a(1-\alpha) \ell - \varepsilon_0 k_x] [\mu a(1-\alpha) \ell + \mu_0 k_x] - \sigma \mu_0 \varepsilon_0 k_z^2
\]

\[
R_d = [\varepsilon a(1-\alpha) \ell - \varepsilon_0 k_x] [\mu a(1-\alpha) \ell - \mu_0 k_x] + \sigma \mu_0 \varepsilon_0 k_z^2.
\]

Two of the branch points are given in (102a) while the other two are

(102c) \[ k_{zz} = \pm k_0 \sqrt{\frac{n^2 - \beta^2}{1 - \beta^2}}. \]

One of these two branch points is on the right side of \( k_0 \) and the other one on the left side of \(-k_0\) in the \( k_z \)-plane. This is shown in Fig. 12. The location of the poles are found by solving

(158a) \[ [\varepsilon a(1-\alpha) \ell - \varepsilon_0 k_x] [\mu a(1-\alpha) \ell - \mu_0 k_x] + \sigma \mu_0 \varepsilon_0 k_z^2 = 0 \]

i.e.,

(158b) \[ [\sigma \mu_0 \varepsilon_0 - \mu_0 \varepsilon_0 - \mu \varepsilon_0 k_z^2 + \mu_0 \varepsilon_0 k_0 \left( n^2 \varepsilon_a - \frac{b^2}{a} \right) + \mu \varepsilon_0 k_0 (1-\alpha) k_z^2 \]

\[ = a(1-\alpha) (\mu \varepsilon_0 + \mu_0 \varepsilon_0) \ell k_x \]
Fig. 12--Relative positions of branch points, branch cuts and the path of integration in $k_z$-plane and $\phi$-plane
Squaring both sides of (158b) we get the solutions

\begin{equation}
(159a) \quad k_z = \pm k_0 \left( A \pm \sqrt{B} \right) / C \right)^{1/2}
\end{equation}

where

\[
A = - \left[ 2(n^4 - 1) \beta^2 + (\mu_x + \epsilon_x)^2 \left( 1 + m^2 - 2\beta^2 \right) \right]
\]

\[
B = (\mu_x + \epsilon_x)^2 \left[ (n^2 - 1)^2 \left( \mu_x + \epsilon_x \right)^2 + 4(m^2 - 2\beta^2) \right] + 16(n^2 - \beta^2)^2 \}
\]

\[
C = 2(1 - \beta^2) \left[ (1 + n^2)^2 - (\mu_x + \epsilon_x)^2 \right]
\]

\[
\mu_x = \mu / \mu_0 \quad \epsilon_x = \epsilon / \epsilon_0
\]

or

\begin{equation}
(159b) \quad k_z = \pm k_0 \left( D / E \right) \frac{1}{2} \quad \text{(if } \mu = \mu_0 \text{ or } \epsilon = \epsilon_0 \text{)}
\end{equation}

where

\[
D = (n^2 + 1)^2 \left[ (n^2 + 1) \beta^2 - n^2 \right] - 4m^2 \beta^4
\]

\[
E = (1 - \beta^2) (n^2 + 1) \left( n^2 + 1 - 2\beta \right) (n^2 + 1 + 2\beta) \cdot
\]

The location of these poles must be examined to see which Riemann sheet they are on.

When \( v = 0 \) \( \Omega \) as well as \( \sigma \) and \( g_2 \) equal zero and the \( R \) shown in (155) reduces to that given in (99a) which eventually becomes the reflection coefficient from a semi-infinite stationary medium.[27]

When \( \mu = \mu_0 \) and \( \epsilon = \epsilon_0 \) both \( R \) and \( g_2 \) reduce to zero we have the
same conclusion as the one given in Section IV-1 that the motion of the bottom region have no effect upon the whole radiation field. The motion of the bottom region creates those field components derived from the potential functions \( F_T \) and \( F_B \) which do not exist when the bottom region is stationary or moving in the direction perpendicular to the line source.

Applying the transformation given in (110) with \( k_y \) replaced by \( k_z \) Eq. (157b) becomes

\[
E_y = -\frac{\omega \mu_0 I_0}{4\pi} \int_0^{\pi} P(\phi) e^{ik_o r \cos(\phi - \theta)} d\phi \quad \text{C}_o
\]

where

\[
P(\phi) = 1 + \frac{\left[ \left( \epsilon_r a (1-\sigma) \cos \phi + (n^2 a - \frac{b^2}{a} + \sin^2 \sigma) \right) \left[ \mu_r a (1-\sigma) \cos \phi - (n^2 a - \frac{b^2}{a} - \sin^2 \sigma) \right] - \sigma \sin \phi \right]}{\left[ \left( \epsilon_r a (1-\sigma) \cos \phi + (n^2 a - \frac{b^2}{a} - \sin^2 \sigma) \right) \left[ \mu_r a (1-\sigma) \cos \phi + (n^2 a - \frac{b^2}{a} + \sin^2 \sigma) \right] + \sigma \sin \phi \right]}
\]

\[
e^{i2\sigma k_o \cos \sigma}
\]

The transformed path of integration \( C_o \), the path of the steepest descent \( C_s \) which is determined by \( \cos(\sigma - \theta) \cosh \eta = 1 \), the saddle point \( \theta \) and the branch cuts in the \( \phi \)-plane are shown in Fig. 12. The hatched region corresponding to the proper branch for \( l \).

The branch points due to \( k_x \) in the \( \phi \)-plane are transformed from (102c) as the solution of
(161a) \[ \sin \phi = \pm \sqrt{n^2 - \beta^2 \over 1 - \beta^2} \]

and the pole from (159) as

(161b) \[ \sin \phi_p = \frac{\pm}{2} \left[ (A + \sqrt{B})/C \right] \]

(161c) \[ = \frac{1}{2} \sqrt{D/E} \] (if \( m = \mu_0 \) or \( \epsilon = \epsilon_0 \)).

Deforming the path of integration from \( C_0 \) to \( C_s \) the integral given in (160) changes to

(162) \[ E_y = \frac{-\omega \mu_0 I_0}{4\pi} \int_{C_S} P(\phi) e^{i k_0 r \cos(\phi - \theta)} d\phi \]

\[ - \frac{i \omega \mu_0 I_0}{2} \sum_P F(\phi_p) e^{i k_0 r \cos(\phi_p - \theta)} + K \]

where \( \phi_p \) is given by (161b,c), \( F(\phi_p) \) is the residue of \( P(\phi) \) at \( \phi_p \) and \( K \) is the branch cut integral. If there are no poles of the integrand near the saddle point the first term in (162) can be evaluated asymptotically as

(163a) \[ E_y \approx \frac{-\omega \mu_0 I_0}{4\pi} e^{i k_0 r - i \pi/4} \sum_{m=0}^{\infty} \frac{P_m(\theta)}{m!} e^{-i m \pi/4} \]

\[ \cdot \left[ 1 + (-1)^m \right] \frac{\Gamma \left( \frac{m+1}{2} \right)}{2 \left( \frac{k_0 r}{2} \right)^{m+1}} \]
where $P^m(\theta) = \left. \frac{d}{d\phi} P(\phi) \right|_{\phi=0}^m$ and $\Gamma$ is gamma function. The leading term is

$$E_y \sim -\frac{\omega \mu_o I_o}{2\sqrt{2\pi(k_o r)^2}} e^{ik_or - i \frac{\pi}{4}} P(0).$$

Since these poles are simple ones the residue of $P(\phi)$ at $\phi_p$ is

$$F(\phi_p) = \frac{F_n}{F_d}$$

$$F_n = 2\mu_\tau a(1-\alpha) \cos \phi_p \left[ \epsilon_\tau a(1-\alpha) \cos \phi_p + \left( n^2 a - \frac{b^2}{a} - \sin^2 \phi_p \right) \right]^{1/2}$$

$$F_d = 2 \alpha \sin \phi_p \cos \phi_p - \sin \phi_p \left[ \epsilon_\tau a(1-\alpha) \cos \phi_p + \left( n^2 a - \frac{b^2}{a} - \sin^2 \phi_p \right) \right]^{1/2}$$

If the pole happens to be in the near vicinity of the saddlepoint the first term in (162) can be evaluated asymptotically in the same manner as the one shown in Section IV-1.
The radiation patterns are symmetric with respect to x-axis.

A branch cut integral must be included if the path $C_s$ is intercepted by the branch cut when the angle of observation $\theta$ extends over the limit given by

\[(165)\quad \theta = \sin^{-1}\left[\sqrt{\frac{1 - \beta^2}{n^2 - \beta^2}}\right].\]

The complete contour is constructed in the same way as that shown in Fig. 11a. The branch cut integral can be performed approximately as follows.

Making the transform of variable such that

\[(166)\quad k_z = k_{z0} + \frac{it}{z}\]

where $k_{z0}$ is one of the branch points shown in (102c). In terms of the new variable $t$, the following approximations are permissible for large values of $z$.

\[
k^2 = k_0^2 - k_z^2 = k_0^2 - (k_{z0} + \frac{it}{z})^2 \approx k_0^2 - k_{z0}^2
\]

\[
k_x^2 = k_0^2\left(\frac{z}{a} - \frac{b^2}{a}\right) - k_z^2 = k_0^2\left(\frac{z}{a} - \frac{b^2}{a}\right) - (k_{z0} + \frac{it}{z})^2
\]

\[= k_0^2\left(\frac{z}{a} - \frac{b^2}{a}\right) - k_{z0}^2 - 2ik_{z0}t/z + t^2/z^2
\]

\[\approx -2ik_{z0}t/z .
\]

Equation (157b) reduces to
(167) \[ E_y = -\frac{\omega \mu_0 I_0}{4\pi} \int_{\infty}^{(+0)} \frac{1}{l} \left( \begin{array}{c} 1 + \frac{\epsilon_r \mu_r a^2(1-\alpha)^2 l^2 - (\mu_r - \epsilon_r) \alpha l k_x - k_x^2 \alpha k_{zo}^2}{\epsilon_r \mu_r a^2(1-\alpha)^2 l^2 - (\mu_r + \epsilon_r) \alpha l k_x - k_x^2 + \alpha k_{zo}^2} \cdot i2\ell d \right. \\
\left. \cdot e^{\frac{ik_{zo}z - t}{2z}} \left( \begin{array}{c} \frac{1}{z} \end{array} \right) dt \right) \]

\[ \propto \frac{i\omega \mu_0 I_0 a(1-\alpha)\sqrt{2k_{zo}}}{4\sqrt{\pi}} \left( \frac{1}{z} \right)^{3/2} \left( \frac{\epsilon_r \mu_r a^2(1-\alpha)^2(k_0^2 - k_{zo}^2) - \alpha k_{zo}^2}{\epsilon_r \mu_r a^2(1-\alpha)^2(k_0^2 - k_{zo}^2) + \alpha k_{zo}^2} \right) \]

\[ \cdot \left( \begin{array}{c} \mu_r + \epsilon_r \\
\mu_r - \epsilon_r \end{array} \right) - \left( \begin{array}{c} \epsilon_r \mu_r a^2(1-\alpha)^2(k_0^2 - k_{zo}^2) - \alpha k_{zo}^2 \\
\epsilon_r \mu_r a^2(1-\alpha)^2(k_0^2 - k_{zo}^2) + \alpha k_{zo}^2 \end{array} \right) \]

\[ \cdot e^{\frac{ik_{zo}z - \sqrt{2}}{\sqrt{k_{zo}^2 - k_0^2(x+d)}}} \cdot e^{-k_0(x+d)} \]

It is clear that both the branch cut integral shown in (167) and the residue wave term shown in (162) are negligible as compared with the space wave term given in (163) when only the far field is concerned. The same procedures shown above can be used in evaluating other field components given in (157). The leading terms from the saddle point method of integration are,

(163c) \[ E_x = -\frac{I_0(1-\alpha)\Omega k_0}{\epsilon_0 \sqrt{2\pi(k_0 r)^2}} e^\frac{ik_0 r - i \frac{\pi}{4}}{4} \sin \theta P_t(\theta) \]
where $P(\phi)$ is given in (160a) and

$$P_1(\phi) = \frac{\sin \phi \cos \phi e^{i2d k_0 \cos \phi}}{[\epsilon_a (1-a) \cos \phi + (n^2 a - \frac{b^2}{a} - \sin^2 \phi)^{\frac{1}{2}}] [\mu_a (1-a) \cos \phi + (n^2 a - \frac{b^2}{a} - \sin^2 \phi)^{\frac{1}{2}}] + \alpha \sin^2 \phi}.$$
\[(163j) \quad \overrightarrow{S} = \frac{1}{2} \overrightarrow{E} \times \overrightarrow{H}^* = \frac{t}{r} \frac{\omega \mu_0 I_0^2}{16\pi r} \left| P(\theta) \right|^2 \]

and those due to potential \( F_T \) are

\[(163k) \quad E_r \sim 0 \]

\[(163l) \quad E_\theta \sim \frac{I_0(1-\alpha)\Omega k_\Omega}{\varepsilon_0 \sqrt{2\pi(k_\Omega r)^3}} e^{ik_\Omega r - i\frac{\pi}{4}} P_1(\theta) \]

\[(163f) \quad H_y \sim -\frac{\omega I_0(1-\alpha)\Omega}{\sqrt{2\pi(k_\Omega r)^3}} e^{ik_\Omega r - i\frac{\pi}{4}} P_1(\theta) \]

\[(163m) \quad \overrightarrow{S} \sim \frac{1}{2} \overrightarrow{E} \times \overrightarrow{H}^* = \frac{t}{r} \frac{\omega (1-\alpha)^2 I_0^2 \Omega^2}{4\pi \varepsilon_0 r} \left| P_1(\theta) \right|^2 . \]

**Magnetic Line Source Parallel to Y-Axis**

For a magnetic line source located at \( x = d, z = 0 \) and parallel to \( y \)-axis we write it as

\[(169) \quad \overrightarrow{M} = \overrightarrow{y} M_0 \delta(x-d) \delta(z) . \]

The radiated field has no variation with respect to \( y \). The formulation of the problem in this section is similar to the one in the previous section with minor differences. The field components in the upper region are given by (147) but \( A_T \) and \( F_T \) satisfy respectively the following wave equations.
\[(170a) \quad \nabla^2 A_T + k_0^2 A_T = 0 \]

and

\[(170b) \quad \nabla^2 F_T + k_0^2 F_T = -\epsilon_0 M_0 \delta(x-d) \delta(z). \]

The solutions are

\[(171a) \quad A_T = \frac{1}{2\pi} \int_{-\infty}^{\infty} g_1(k_z) e^{ikx+ikz} dk_z \]

\[(171b) \quad F_T = \frac{1}{2\pi} \int_{-\infty}^{\infty} g_2(x, k_z) e^{ikz} dk_z \]

where \(i^2 = k_0^2 - k_z^2\) and \(g_2\) is the same as that shown in (132). \(g_1\) and \(R\) are determined later on by the boundary conditions at \(x = 0\).

In the bottom region we use the same formulation from (150) through (153). Matching the tangential electric and magnetic field components of both sides at the boundary \(x = 0\) gives the following simultaneous equations.

\[(172a) \quad g_1 = \left(1 - \frac{\Omega^2}{\mu \epsilon a^2}\right)f_1 \]

\[(172b) \quad \frac{M_0}{2} (1-R) e^{ild} = \frac{i\Omega k_z}{\mu \epsilon a^2} f_1 + \frac{ik_x}{\epsilon a} f_2 \]

\[(172c) \quad \frac{-\epsilon_0 M_0}{2il} (1+R) e^{ild} = \left(1 - \frac{\Omega^2}{\mu \epsilon a^2}\right)f_2 \]
\[ g_1 = \frac{k_x}{\mu a} f_1 - \frac{\Omega k_z}{\mu \varepsilon a^2} f_2. \]

From (172) we have
\[ R = \frac{[\mu a (1-\alpha) \ell - \mu_0 k_x]}{[\mu a (1-\alpha) \ell - \mu_0 k_x]} \left[ \frac{\varepsilon a (1-\alpha) \ell + \varepsilon_0 k_x}{\varepsilon a (1-\alpha) \ell - \varepsilon_0 k_x} \right] - \alpha \mu_0 \varepsilon_0 k_z^2. \]

\[ g_1 = \frac{-i \mu_0 \varepsilon_0 (1-\alpha) \Omega M_0 k_z e^{ifd}}{[\mu a (1-\alpha) \ell - \mu_0 k_x]} \left[ \frac{\varepsilon a (1-\alpha) \ell - \varepsilon_0 k_x}{\varepsilon a (1-\alpha) \ell - \varepsilon_0 k_x} \right] + \alpha \mu_0 \varepsilon_0 k_z^2. \]

The expressions in integral form for the field components in the region \( x \geq d \) are
\[ E_x = \frac{-M_0}{4\pi} \int_{-\infty}^{\infty} \frac{k_z}{\ell} \left[ 1 + \frac{R_n}{R_d} e^{i2\ell d} \right] e^{i\ell(x-d) + ik_z z} \, dk_z. \]

\[ E_y = \frac{\omega \mu_0 \varepsilon_0 (1-\alpha) \Omega M_0}{2\pi} \int_{-\infty}^{\infty} \frac{k_z}{R_d} \left[ 1 + \frac{R_n}{R_d} e^{i2\ell d} \right] e^{i\ell(x+d) + ik_z z} \, dk_z. \]

\[ E_z = \frac{M_0}{4\pi} \int_{-\infty}^{\infty} \left[ 1 + \frac{R_n}{R_d} e^{i2\ell d} \right] e^{i\ell(x-d) + ik_z z} \, dk_z. \]

\[ H_x = \frac{-\varepsilon_0 (1-\alpha) \Omega M_0}{2\pi} \int_{-\infty}^{\infty} \frac{k_z e^{i2\ell d}}{R_d} \, dk_z. \]
(175e) \[ H_y = \frac{-\omega \epsilon_0 M_0}{4\pi} \int_{-\infty}^{\infty} \frac{1}{l} \left[ 1 + \frac{R_n}{R_d} e^{i2\ell d} \right] e^{il(x-d) + ik_z z} \] \[ dk_z \]

(175f) \[ H_z = \frac{\epsilon_0 (1-\alpha) \Omega M_0}{2\pi} \int_{-\infty}^{\infty} \frac{k_z l}{R_d} e^{il(x+d) + ik_z z} \] \[ dk_z \]

where

\[ R_n = [\mu a(1-\alpha) l - \mu_0 k_x] [\epsilon a(1-\alpha) l + \epsilon_0 k_x] - \alpha \mu_0 \epsilon_0 k_z^2 \]

\[ R_d = [\mu a(1-\alpha) l - \mu_0 k_x] [\epsilon a(1-\alpha) l - \epsilon_0 k_x] + \alpha \mu_0 \epsilon_0 k_z^2 . \]

Branch points are given in (102a, c) and shown in Fig. 12. Poles can be found from (159). Using the transformation (110) where \( k_y \) is replaced by \( k_z \) (175e) becomes

(176) \[ H_y = -\frac{\omega \epsilon_0 M_0}{4\pi} \int_{-\infty}^{\infty} Q(\phi) e^{ik_0 r \cos(\phi - \theta)} d\phi \]

where

(176a) \[ Q(\phi) = 1 + \]

\[ \frac{[\mu r a(1-\alpha) \cos \phi + (n^2 a - \frac{b^2}{a} - \sin^2 \phi)^{\frac{1}{2}}] [\epsilon r a(1-\alpha) \cos \phi - (n^2 a - \frac{b^2}{a} - \sin^2 \phi)^{\frac{1}{2}}] - \alpha \sin^2 \phi}{[\mu r a(1-\alpha) \cos \phi + (n^2 a - \frac{b^2}{a} - \sin^2 \phi)^{\frac{1}{2}}] [\epsilon r a(1-\alpha) \cos \phi + (n^2 a - \frac{b^2}{a} - \sin^2 \phi)^{\frac{1}{2}}] + \alpha \sin^2 \phi} \]

\[ \cdot e^{i2d k_0 \cos \phi} . \]
The transformed path of integration $C_o$ and the steepest descent path $C_s$ are also shown in Fig. 12. The branch points and poles in $\phi$-plane are given by (161). Shifting the path of integration from $C_o$ to $C_s$ the integral in (176e) can be written as

\[
H_y = -\frac{\omega \varepsilon_o M_o}{4\pi} \int_{C_s} Q(\phi) e^{ik_o r \cos(\phi - \theta)} d\phi
\]

\[
-\frac{i\omega \varepsilon_o M_o}{2} \sum_p G(\phi_p) e^{ik_o r \cos(\phi_p - \theta)} + K
\]

where $G(\phi_p)$ is the residue of $Q(\phi)$ at the pole $\phi_p$ which is given in (161b,c). If there are no poles located near the saddle point the first term in (177) can be evaluated approximately for the far field as

\[
H_y = -\frac{\omega \varepsilon_o M_o}{4\pi} e^{i k_o r - i \frac{\pi}{4}} \sum_{m=0}^{\infty} \frac{Q_m(\theta)}{m!} e^{-im \frac{\pi}{4}} \left[ 1 + (-1)^m \right]
\]

\[
\frac{\Gamma \left( \frac{m+1}{2} \right)}{2 \left( \frac{k_o r}{2} \right)^{m+1} \left( \frac{1}{2} \right)^{m+1}}
\]

where the leading term is

\[
H_y \approx -\frac{\omega \varepsilon_o M_o}{2 \sqrt{2\pi (k_o r)}} \frac{e^{i k_o r - i \frac{\pi}{4}}}{\sqrt{2\pi (k_o r)}} Q(\theta)
\]
The residue of $Q(\phi)$ at pole $\phi_p$ is

$G(\phi_p) = G_n/G_d$

$G_n = 2\varepsilon r a (1-\alpha) \cos \phi_p \left[ \mu_r a (1-\alpha) \cos \phi_p + \left( n^2 a - \frac{b^2}{a} - \sin^2 \phi_p \right)^{1/2} \right]$

$* \quad e^{i2d k_0 \cos \phi_p}$

$G_d = 2\varepsilon \sin \phi_p \cos \phi_p \sin \phi_p \left\{ \left[ \varepsilon r a (1-\alpha) \cos \phi_p + \left( n^2 a - \frac{b^2}{a} - \sin^2 \phi_p \right)^{1/2} \right] \right.$

$* \quad \left[ \mu_r a (1-\alpha) + \frac{\cos \phi_p}{\left( n^2 a - \frac{b^2}{a} - \sin^2 \phi_p \right)^{1/2}} \right]$

$+ \left[ \varepsilon r a (1-\alpha) + \frac{\cos \phi_p}{\left( n^2 a - \frac{b^2}{a} - \sin^2 \phi_p \right)^{1/2}} \right]$

$\left[ \mu_r a (1-\alpha) \cos \phi_p + \left( n^2 a - \frac{b^2}{a} - \sin^2 \phi_p \right)^{1/2} \right]$

$* \quad \left[ \mu_r a (1-\alpha) \cos \phi_p + \left( n^2 a - \frac{b^2}{a} - \sin^2 \phi_p \right)^{1/2} \right]$

A branch cut integral has to be included if the angle of observation $\theta$ is beyond the limit given in (165). Its approximate value is

$H_Y^{(branch)} \approx \frac{i\omega \varepsilon_a M_0 (1-\alpha) \sqrt{2k_{zo}}}{4\pi} \left( \frac{i}{z} \right)^{3/2}$

$* \quad \left[ \varepsilon r H_{r z} \frac{a^2 (1-\alpha)}{2(k_{zo}^2 - k_{zo}^2) + \alpha k_{zo}^2} \right]$

$\varepsilon r H_{r z} \frac{a^2 (1-\alpha) (k_{zo}^2 - k_{zo}^2) - \alpha k_{zo}^2}{\varepsilon r H_{r z} a^2 (1-\alpha) (k_{zo}^2 - k_{zo}^2) + \alpha k_{zo}^2}$
where \( k_{z_0} \) is given in (102c). The leading term from the saddle point method of integration for the other field components in (175) are

\[
\begin{align*}
(178c) \quad E_x & \approx \frac{-M_0 k_0}{2 \sqrt{2\pi (k_0 r)^2}} e^{i k_0 r - i \frac{\pi}{4}} \sin \theta Q(\theta) \\
(178d) \quad E_y & \approx \frac{\omega (1-\alpha) \Omega M_o}{\sqrt{2\pi (k_0 r)^2}} e^{i k_0 r - i \frac{\pi}{4}} P_1(\theta) \\
(178e) \quad E_z & \approx \frac{M_0 k_0}{2 \sqrt{2\pi (k_0 r)^2}} e^{i k_0 r - i \frac{\pi}{4}} \cos \theta Q(\theta) \\
(178f) \quad H_x & \approx \frac{-(1-\alpha) \Omega M_0 k_0}{\mu_0 \sqrt{2\pi (k_0 r)^2}} e^{i k_0 r - i \frac{\pi}{4}} \sin \theta P_1(\theta) \\
(178g) \quad H_z & \approx \frac{(1-\alpha) \Omega M_0 k_0}{\mu_0 \sqrt{2\pi (k_0 r)^2}} e^{i k_0 r - i \frac{\pi}{4}} \cos \theta P_1(\theta)
\end{align*}
\]

where \( Q(\phi) \) is given in (176a) and \( P_1(\phi) \) in (168). In cylindrical coordinates the field components and Poynting vector due to potential \( F_T \) are
(178j) \[ E_T \sim 0 \]

(178i) \[ E_{t0} \sim \frac{M_0 k_0}{2 \sqrt{2\pi (k_0 r)^2}} e^{ik_0 r - \frac{i\pi}{4}} Q(\theta) \]

(178b) \[ H_y \sim \frac{-\omega \epsilon_0 M_0}{2 \sqrt{2\pi (k_0 r)^2}} e^{ik_0 r - \frac{i\pi}{4}} Q(\theta) \]

(178j) \[ S \sim \frac{\omega \epsilon_0 M_0^2}{16\pi r} |Q(\theta)|^2 \]

and those due to \( A_T \) are

(178k) \[ H_r \sim 0 \]

(178l) \[ H_0 \sim \frac{M_0 (1-\alpha) \Omega k_0}{\mu_0 \sqrt{2\pi (k_0 r)^2}} e^{ik_0 r - \frac{i\pi}{4}} P_1(\theta) \]

(178d) \[ E_y \sim \frac{\omega (1-\alpha) \Omega M_0}{\sqrt{2\pi (k_0 r)^2}} e^{ik_0 r - \frac{i\pi}{4}} P_1(\theta) \]

(178m) \[ S \sim \frac{\omega (1-\alpha)^2 \Omega^2 M_0^2}{4\pi \mu_0 r} |P_1(\theta)|^2 \]
CHAPTER V

CONCLUSIONS

The definite form of Maxwell's equations for a moving isotropic and lossless medium has been derived by making use of the constitutive relations as found in Minkowski's theory. The wave equations satisfied by the field vectors were given. The integration of these equations is performed by the introduction of the vector and scalar potential functions following a method similar to that used for the stationary medium. The results are then applied to treat two problems. The first is the problem of rectangular and cylindrical waveguides which are filled with isotropic and lossless media moving uniformly in the direction parallel to the axis of the guide. The second problem deals with the effect of uniform motion of a dielectric half-space on the radiation from a line source located above and parallel to the half-space. The sources considered are electric and magnetic line sources, perpendicular or parallel to the direction of motion.

The results thus obtained reduce, as expected, to the known solutions for a medium at rest when the medium becomes stationary or when the constitutive parameters of the medium are set equal to those of free space.
In the waveguide problem it is found that when the velocity of the medium is small such that $n\beta < 1$ there is another frequency limit denoted by $f_+$ which is larger than the cut off frequency $f_c$. Two waves with different phase velocity can propagate freely and the faster one travels in the direction opposite to that of the moving medium while the other one travels in the direction of the released direction depending on whether the frequency is below or above $f_+$. $f_+$ reduces to $f_c$ when $\beta$ approaches zero or $n$ approaches one. As the velocity of the medium becomes higher and $n\beta > 1$, the cut off phenomenon disappears but there is still another frequency limit denoted by $f_-$ given by the same formula as that for $f_+$. It is also possible to have two waves of unequal phase velocity propagating without attenuation. Both will travel in the same direction as that of the medium unless the frequency is lower than $f_-$ in which case the slower one travels in the opposite direction.

In the case of a moving half-space when the line source is oriented perpendicular to the moving direction, the field expressions are similar to those in the stationary case and the far field is linearly polarized. When the line source is oriented parallel to the moving direction another wave component arises and its magnitude varies
mainly with the velocity. Then the radiated field is in general elliptically polarized. However, the direction of power flow for each individual wave component is found to be radial.

The proper location of the branch cuts depends upon the magnitude of the index of refraction and the velocity of the moving medium. Stronger variations are observed when the line source is oriented perpendicular to the moving direction.
REFERENCES


