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DISSERTATION

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By

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The Ohio State University
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INTRODUCTION

Let $K$ be a field, $K[t]$ be the ring of polynomials over $K$ and $K(t)$ be the corresponding field of rational functions.

Let $x$ be a typical non-zero element of $K(t)$. Then

$$x = \frac{a_m t^m + \ldots + a_0}{b_n t^n + \ldots + b_0}$$

where $a_m \neq 0 \neq b_n$ and $a's$, $b's$ belong to $K$. We define the usual valuation in $K(t)$ by $|x| = e^m - n$, $|0| = 0$ ($e > 1$). The completion $K \{t\}$ of $K(t)$ with respect to this valuation is then the field of formal Laurent power series, a typical element of which is

$$x = \alpha_m t^m + \alpha_{m-1} t^{m-1} + \ldots \ (upto - \infty).$$

We define

$$\|x\| = \begin{cases} |x| & \text{if } m < 0 \\ |\alpha_{-1} t^{-1} + \alpha_{-2} t^{-2} + \ldots| & \text{if } m \geq 0. \end{cases}$$

It is well known that the ring $K[t]$ and the field $K(t)$ and $K \{t\}$ can be treated as analogues of the ring of integers and the fields of rational and real numbers. In this thesis, we have proved in this setup various theorems on Diophantine approximations and Diophantine inequalities.
In Chapter I, we give a few definitions and results due to Mahler. In Chapter II, we prove theorems on simultaneous approximations of homogeneous linear forms by elements of $K[t]$. In Chapter III we prove analogues of theorems of Kronecker and Khintchine on approximations of non-homogeneous linear forms. In Chapter IV we prove analogues of some transference theorems of Mahler and Hlawka.

Let $L_1, \ldots, L_{r+2s}$ be linear forms in $r + 2s = n$ variables such that $L_1, \ldots, L_r$ are real, $L_{r+1}, \ldots, L_{r+s}$ have complex coefficients and $L_{r+s+1}, \ldots, L_{r+2s}$ are complex conjugates to $L_{r+1}, \ldots, L_r$ respectively. Let $\Delta \neq 0$ be the determinant of these forms. In the cases $r = 2, s = 0$; $r = 1, s = 1$; $r = 0, s = 2$ Davenport proved that there exist real numbers $\xi_1, \ldots, \xi_n$ such that for all $(x_1, \ldots, x_n) \equiv (\xi_1, \ldots, \xi_n) \pmod{1}$

$$|L_1 \ldots L_n| \geq C_{r, s} |\Delta|$$

where $C_{r, s}$ are positive constants independent of $L_1, \ldots, L_n$. In Chapter V we prove analogues of these results with best possible values of $C_{r, s}$. It may be remarked that the best possible values of $C_{r, s}$ in the case of real and complex forms are not known.

For the sake of completeness, full details have been given in all parts of this thesis.
CHAPTER I
PRELIMINARIES

In this chapter we explain the notation, give a few definitions, and state some results, which will be used in the subsequent chapters.

1.1 Let $K$ be a field. A function

$$| | : K \rightarrow \text{non-negative reals}$$

is said to be a valuation, if for arbitrary elements $x, y \in K$;

(a) $|x| = 0 \iff x = 0$,

(b) $|xy| = |x| \cdot |y|$,

(c) $|x + y| \leq |x| + |y|$.

If further

(c') $|x + y| \leq \max(|x|, |y|)$,

we say the valuation is Non-Archimedean.

In what follows, $K$ will denote an arbitrary field (unless some restrictions are imposed and stated explicitly), $K[t]$ the ring of polynomials in $t$ over $K$, and $K(t)$ the field of rational functions of $t$ in which the integral domain $K[t]$ is imbedded. A typical (non-zero) element of $K(t)$ is of the form

$$x = \frac{a_m t^m + \ldots + a_0}{b_n t^n + \ldots + b_0}$$

where $a_m \neq 0 \neq b_n$ and $a$'s, $b$'s belong to $K$. Define a function
in $K(t)$ by $|x| = e^{m-n}$, $|0| = 0$. It can be verified easily that this function is a non-Archimedean valuation.

It is well known that the field $K\{t\}$, the elements of which are of the form

$$x = \alpha_1t^f + \alpha_{f-1}t^{f-1} + \ldots \text{ (upto } -\infty)$$

(1.1.1)

where $f$ is any integer, $\alpha_i \in K$, is the completion of $K(t)$, with respect to this valuation. The induced valuation in $K\{t\}$ is given as follows:

Let $x \in K\{t\}$. If $x = 0$, define $|x| = 0$. If $x \neq 0$, let

$$x = \alpha_1t^f + \alpha_{f-1}t^{f-1} + \ldots \alpha_f \neq 0, \alpha_i \in K, \alpha_f = 0$$

then $|x| = e^f$. Trivially for all non-zero $x \in K[t]$, $|x| \geq 1$.

Also $|x| = |-x|$.

1.2 Definition 1.1: Let $x \in K\{t\}$. Then we define

$$||x|| = \begin{cases} 
|x| & \text{if } x \leq 1 \\
|\alpha_1t^{-1} + \alpha_2t^{-2} + \ldots| & \text{if } |x| \geq 1
\end{cases}$$

and $x = \alpha_1t^f + \alpha_{f-1}t^{f-1} + \ldots \alpha_f \neq 0, \alpha_i \in K$.

Clearly $||x|| = ||-x||$. 
Properties of $\|x\|$ 

Let $x$, $y$ denote arbitrary elements of $K\{t\}$. Then

(a) $\|x\| \leq \min (\|x\|, e^{-1})$

(b) $\|x\| = \|x - A\| \leq \|x - A\|$ for all $A \in K[t]$

(c) $\|x + y\| \leq \max (\|x\|, \|y\|)$

(d) $\|x\| = \|x + y - y\| \leq \max (\|x + y\|, \|y\|)$

(e) $\|A x\| \leq |A| \cdot \|x\|$ for all $A \in K[t]$.

Notation: We shall denote by $P_n$ the set $(K[t])^n$ i.e. the set of elements $X = (x_1, \ldots, x_n)$ where $x_i \in K[t] (1 \leq i \leq n)$ and by $R_n$ the set $(K\{t\})^n$ i.e. the set of elements $X = (x_1, \ldots, x_n)$, $x_i \in K\{t\} (1 \leq i \leq n)$. Usually an element of $R_n$ will be called a vector. We denote the vector $(0, 0, \ldots, 0)$ by $0$.

Remark 1.1: The ring $K[t]$, the fields $K(t)$ and $K\{t\}$ are analogues in a natural way to the ring of integers and fields of rational numbers, real numbers respectively. $\|x\|$ corresponds to the fractional part of a real number $x$, $P_n$ corresponds to $n$-dimensional fundamental lattice and $R_n$ corresponds to $n$-dimensional Euclidean space.

1.3 Definition 1.2: A function $F: R_n \rightarrow \{0\} \cup \{e^n; n \in \mathbb{Z}\}$ (rational integers) will be called a distance function, if it has the following properties:

(A) $F(X) = 0 \iff X = 0$.

(B) $F(aX) = |a| F(X)$ for all $a \in K\{t\}, X \in R_n$. 

(C) \( F(X + Y) \leq \max (F(X), F(Y)) \) for all \( X, Y \in \mathbb{R}^n \).

Definition 1.3: We define the length \( |x| \) of a vector \( X = (x_1, \ldots, x_n) \in \mathbb{R}^n \) by \(|X| = \max (|x_1|, \ldots, |x_n|)\).

Remark 1.2: It can be seen easily that \(|X|\) is a distance function.

Definition 1.4: Let \( k \) be an integer and \( F(X) \) be a distance function. Then the set

\[ C(k) = \{X \in \mathbb{R}^n ; F(X) \leq e^k \} \]

is called a convex body in \( \mathbb{R}^n \).

Definition 1.5: Let

\[ L(k) = \{X \in \mathbb{P}_n ; F(X) \leq e^k \} \]

Then it can be shown that \( L(k) \) is a finite dimensional vector space over \( K \) (See Mahler [20] p. 504). Suppose the dimension of \( L(k) \) over \( K \) is \( M(k) \), and suppose the dimension of

\[ L_0(k) = \{X \in \mathbb{P}_n ; |X| \leq e^k \} \]

is \( M_0(k) \). Then we define the volume \( V \) of the convex body \( C(0) \) by

\[ V = \lim_{k \to \infty} \exp \left[ M(k) - M_0(k) \right]. \]
This volume always exists and is finite. [Mahler p. 505]

Remark 1.3: Obviously the volume of $|X| \leq 1$ is 1.

Definition 1.6: Let $X_1, \ldots, X_m \in \mathbb{R}$. Then we denote the space spanned by these over $K\{t\}$ by $<X_1, \ldots, X_m>$. 

The following theorems are particularly useful in our further investigation.

Theorem A: (Armitage [1]).

Let $L_i(U) = L_i(u_1, \ldots, u_n) = \sum_{j=1}^{n} a_{ij} u_j$, $a_{ij} \in K\{t\}$ 

(1\leq i \leq n) be a system of n linear forms in n variables $u_1, \ldots, u_n$ with det. $D = |a_{ij}|$. Let $|D| = e^d$. Let $r_1, \ldots, r_n$ be any set of integers such that

$$r_1 + \ldots + r_n \geq d - (n - 1).$$

Then there exists $U \neq 0$ in $P_n$ such that

$$|L_i(U)| \leq e^{r_i} (i = 1, \ldots, n).$$

Theorem B: (Mahler [20], p. 491)

Let $F(X)$ be a distance function defined on $R_n$. Then there exists a positive constant $\gamma$ such that for all vectors $X \in P_n$

$$F(X) \geq \gamma |X|.$$
Theorem C: (Mahler [20], p. 505)

Let \( T = (a_{nk})_{n, k=1, \ldots, n} \) where \( a_{nk} \in K \{t\} \), be a non-singular matrix. Then the linear transformation \( Y = TX \) changes the distance function \( F(X) \) into a new distance function \( F'(Y) = F(T^{-1}Y) \). Let \( V, V' \) be the volumes of \( \{X \in \mathbb{R}^n; F(X) < 1\} \) and \( \{Y \in \mathbb{R}^n; F'(Y) < 1\} \) respectively. Then

\[
V' = |\det T| \cdot V.
\]

Theorem D: (Mahler [20], p. 508)

To every distance function \( F(X) \), there correspond linearly independent vectors \( X^{(1)}, \ldots, X^{(n)} \in P_n \) with the following property.

\[
\sigma^{(1)} = F(X^{(1)}) = \inf \left\{ F(X); X \in P_n, X \neq 0 \right\},
\]
\[
\sigma^{(2)} = F(X^{(2)}) = \inf \left\{ F(X); X \in P_n, X \neq X^{(1)} \right\},
\]

\[\vdots\]

\[
\sigma^{(n)} = F(X^{(n)}) = \inf \left\{ F(X); X \in P_n, X \neq X^{(1)}, \ldots, X^{(n-1)} \right\},
\]

and

\[
|D| = |\det (x^{(k)}_{hk})_{1 \leq h, k \leq n}| = 1 \quad \text{and}
\]

\[
\sigma^{(1)} \sigma^{(2)} \ldots \sigma^{(n)} = \frac{1}{V} \quad (1.3.1)
\]
where \( V \) is the volume of the convex body \( \{ x \in \mathbb{R}^n ; F(x) \leq 1 \} \).

Remark 1.4: Obviously we have

\[
\sigma^{(1)} \leq \sigma^{(2)} \leq \cdots \leq \sigma^{(n)}
\]

from (1.3.1) we get \( (\sigma^{(1)})^n \leq \frac{1}{V} \) or \( \sigma^{(1)} \leq \frac{1}{\sqrt[n]{V}} \) i.e. to every distance function \( F(X) \), there is \( X \neq 0 \) in \( P_n \) such that

\[
F(X) \leq \frac{1}{\sqrt[n]{V}}.
\]

Lemma 1.1: Let \( K \) be a finite field. Let \( L_1(U), \ldots, L_n(U) \) be \( n \) linear forms over \( K\{t\} \) in \( n \) variables \( u_1, \ldots, u_n \) with determinant not equal to zero. Then for every constant \( C > 0 \), there exist only finitely many elements \( U \) of \( P_n \) satisfying

\[
\max (|L_1(U)|, \ldots, |L_n(U)|) \leq C \tag{1.3.2}
\]

Proof: Consider the function

\[
F(U) = \max (C^{-1} |L_1(U)|, \ldots, C^{-1} |L_n(U)|).
\]

It can be verified that this is a distance function. Let

\[
C(1) = \{ U \in \mathbb{R}_n^* : F(U) \leq 1 \}.
\]

By theorem B there exists a positive constant
\[ y \text{ such that for all } U \in P_n; \]
\[ F(U) \geq y |U| . \]

Therefore for all elements \( U \) of \( P_n \) in \( C(1) \),
\[ y |U| \leq 1 \text{ or } |U| \leq y^{-1} \]

i.e. \( U \) is a polynomial in \( t \) with a bounded degree. Hence the lemma follows.

1.4: Let \( K \) be a field, in which \( x^2 + 1 = 0 \) is not soluble. Then we can easily see that \( x^2 + 1 = 0 \) is not soluble in \( K \{t\} \) also. Adjoin \( i \), the root of \( x^2 + 1 = 0 \) to the field \( K \{t\} \) and we get the field \( K \{t\} (i) \) (to be denoted by \( K^C \{t\} \) hereafter).

The elements of \( K^C \{t\} \) are of the form \( Z = x + iy; \) where \( x, y \in K \{t\} \). We define a function \( f: K^C \{t\} \rightarrow \) The set of non-negative reals by
\[ f(Z) = f(x + iy) = \max (|x|, |y|). \]

First we check that this function is a valuation in \( K^C \{t\} \). For this, we observe the following statements, which are obvious.

(a) \( f(Z) = 0 \iff Z = 0 \)
(b) For all $Z_1, Z_2 \in K^c \{t\}$,

$$f(Z_1 + Z_2) \leq \max(f(Z_1), f(Z_2)).$$

Next we assert that

(c) $f(Z_1 Z_2) = f(Z_1) \cdot f(Z_2)$ for all $Z_1, Z_2 \in K^c \{t\}$.

If either $Z_1 = 0$ or $Z_2 = 0$, (c) is obvious. So we assume $Z_1 \neq 0 \neq Z_2$. Let

$$Z_1 = x_1 + iy_1; x_1, y_1 \in K \{t\},$$

$$Z_2 = x_2 + iy_2; x_2, y_2 \in K \{t\}.$$

Now we distinguish the following cases.

Case I: One of $x_1, y_1$, and one of $x_2, y_2$ is zero.

Case II: Exactly one of $x_1, y_1, x_2, y_2$ is zero.

Case III: None of $x_1, y_1, x_2, y_2$ is zero.

Case I: One of $x_1, y_1$, and one of $x_2, y_2$ is equal to zero.

Then $Z_1 Z_2 = x_1 x_2$ or $i x_1 y_2$ or $iy_1 x_2$ or $-y_1 y_2$ and in all cases

$$f(Z_1 Z_2) = f(Z_1) \cdot f(Z_2).$$

Case II: Exactly one of $x_1, y_1, x_2, y_2$ is zero.

Without loss of generality assume $y_1 = 0$. 


Then

\[ z_1 z_2 = x_1(x_2 + iy_2) = x_1x_2 + ix_1y_2. \]

So

\[ f(z_1z_2) = \max(x_1x_2, x_1y_2) \]
\[ = |x_1| \max(|x_2|, |y_2|) \]
\[ = f(z_1) \cdot f(z_2). \]

Case III: None of \( x_1, x_2, y_1, y_2 \) is equal to zero.

Let

\[ x_1 = a_{11}t^{h_1} + a_{12}t^{h_1-1} + \ldots \]
\[ y_1 = b_{11}t^{k_1} + b_{12}t^{k_1-1} + \ldots \]
\[ x_2 = a_{21}t^{h_2} + a_{22}t^{h_2-1} + \ldots \]
\[ y_2 = b_{21}t^{k_2} + b_{22}t^{k_2-1} + \ldots \]

where \( a's, b's \) belong to \( K \), \( a_{11} \neq 0 \neq b_{11}, a_{21} \neq 0 \neq b_{21} \).
Now

\[ Z_1Z_2 = (x_1 + iy_1)(x_2 + iy_2) \]
\[ = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1). \]

Suppose first \( h_1 > k_1 \) and \( h_2 \geq k_2 \). Now consider the degree of \( x_1x_2 \).

\[ \deg(x_1x_2) > \deg(y_1y_2), \deg(y_1x_2) \text{ and } \]
\[ \deg(x_1x_2) \geq \deg(x_1y_2). \]

Therefore \( x_1x_2 - y_1y_2 \neq 0 \) and \( \deg(x_1x_2 - y_1y_2) \geq \deg(x_1y_2 + x_2y_1) \).

So \( f(Z_1Z_2) = \max(|x_1x_2 - y_1y_2|, |x_1y_2 + x_2y_1|) \)
\[ = |x_1x_2 - y_1y_2| = |x_1x_2| \]
\[ = |x_1| \cdot |x_2| = f(Z_1) \cdot f(Z_2). \]

Similarly we can see that when \( h_1 > k_1, k_2 > h_2 \) or \( h_1 \leq k_1, h_2 > k_2 \) or \( h_1 \leq k_1, h_2 \geq k_2 \) or \( h_1 \leq k_1, k_2 > h_2 \), we have \( f(Z_1Z_2) = f(Z_1) \cdot f(Z_2) \).

So assume \( h_1 = k_1, h_2 = k_2 \).

Coefficient of \( t^{h_1+h_2} \) in \( x_1x_2 - y_1y_2 = a_{11}a_{21} - b_{11}b_{21} \)

Coefficient of \( t^{h_1+h_2} \) in \( x_1y_2 - y_1x_2 = a_{11}b_{21} + a_{21}b_{11} \)

If either of \( a_{11}a_{21} - b_{11}b_{21} \) or \( a_{11}b_{21} + a_{21}b_{11} \) is not zero, then (c) follows at once.

Therefore, assume

\[ a_{11}a_{21} - b_{11}b_{21} = 0 \]
\[ a_{11}b_{21} + a_{21}b_{11} = 0. \]
So \[ \frac{a_{11}}{b_{11}} = \frac{-a_{21}}{b_{21}} = \frac{b_{21}}{a_{21}} (a_{21} \neq 0, b_{11} \neq 0 \neq b_{21}) \]

and \[ \left( \frac{a_{21}}{b_{21}} \right)^2 + 1 = 0 \]

which is not possible, since \( x^2 + 1 = 0 \) is not soluble in \( K \).

Obviously \( f(Z) \), when restricted to \( K \{t\} \) is the original valuation. So without ambiguity, we shall write

\[ f(Z) = |Z|. \]

Definition 1.7: Let \( Z = x + iy \in K^C \{t\} \). Then we define

\[ ||Z|| = \max( ||x||, ||y|| ) . \]

Definition 1.8: The elements of \( K^C \{t\} \) will be called complex quantities.

Definition 1.9: We shall call the elements \( Z_1 = x_1 + iy_1 \), \( Z_2 = x_1 - iy_1 \) of \( K^C \{t\} \), "complex conjugates" of each other and we shall write

\[ Z_1 = \overline{Z}_2 \text{ or } Z_2 = \overline{Z}_1 . \]

Remark 1.5: If \( Z_1 = \overline{Z}_2 \), then \( |Z_1| = |Z_2| \).
CHAPTER II
HOMOGENEOUS APPROXIMATION

This chapter is divided into three sections. In section 1, we study the problem of simultaneously making the fractional part of a system of linear forms $L_1(U)$ over $K\{t\}$, small for $U \in \mathbb{P}_n$.

In section 2, we approximate an element of $K^c\{t\}$ by elements of the form $x + iy$, $(x,y) \in \mathbb{P}_2$. In section 3, our main aim is to prove that the inequality

$$\left( \max_{i=1, \ldots, m} |L_1(U)|^m \right) |U|^{n-m} \leq \exp(n-m-5)$$

admits an infinity of solutions $U \in \mathbb{P}_n$ under certain conditions.

2.1 Fenna [10] has given a method of simultaneously approximating $n$ elements $\theta_1, \ldots, \theta_n$ of $K\{t\}$ by elements of $K(t)$ with the same denominator. We give here another proof of a part of his result.

Theorem 1: Let $\theta_1, \ldots, \theta_n$ be $n$ elements of $K\{t\}$ and $V$ be another element of $K\{t\}$ with $|V| > 1$. Then there is an element $v$ of $K[t]$ satisfying

$$0 < |v| \leq |V|^n \quad \text{and} \quad \|v\theta_i\| < |V|^{-1} \quad (1 \leq i \leq n) \quad (2.1.1)$$
Proof: Consider the inequalities:
\[
|v\theta_i - u_i| \leq e^{-l} |v|^{-l} \quad (i = 1, \ldots, n)
\]
\[
|v| \leq |v|^n
\]

The sum of the exponents on the right hand side is \(-n = -(n + 1) + 1\) and the determinant of linear forms has valuation 1, so by theorem A of chapter I, there exist elements \(u_1, \ldots, u_n, v\) of \(K[t]\), not all zero satisfying (2.1.2). Now suppose \(v = 0\). Then from the first \(n\) inequalities in (2.1.2) we get
\[
|u_i| < 1 \quad (n = 1, \ldots, n).
\]

But \(u_i\)'s are elements of \(K[t]\), so we must have \(u_1 = \ldots = u_n = 0\), which contradicts that \((u_1, \ldots, u_n, v) \neq 0\) and this proves the theorem.

Corollary: If \(\theta_1, \ldots, \theta_n\) are \(n\) elements of \(K[t]\), and at least one of them does not belong to \(K(t)\), then there are infinitely many elements \((v, u_1, \ldots, u_n) \in \mathbb{P}_{n+1}\) satisfying
\[
|v(v\theta_i - u_i)^n| \leq e^{-n} \quad (n = 1, \ldots, n), \ v \neq 0. \quad (2.1.3)
\]

Proof: Without loss of generality assume \(\theta_1 \notin K(t)\). If possible suppose (2.1.3) has only a finite number of solutions. Consider
\[
\min |v\theta - u|
\]
minimum being taken over all \((u,v) \in \mathbb{Q}^2\) satisfying (2.1.3). Since, by assumption, there are only a finite number of solutions of (2.1.3), minimum is attained. Let this minimum be \(e^{-r}\). From theorem 1,
we can see that we must have \( r \geq 1 \). Then by theorem 1 there exists a non-zero element \( v \) of \( K[t] \) satisfying

\[
\|v\theta_k\| < e^{-r} \quad \text{and} \quad 0 < |v| \leq e^{nr} \quad (i = 1, \ldots, n)
\]

i.e. there are \( v, u_1, \ldots, u_n \in K[t] \), \( v \neq 0 \) satisfying

\[
|v\theta_i - u_i| \leq e^{-r-1}, \quad 0 < |v| \leq e^{nr}
\]

and hence we have

\[
|v(v\theta_i - u_i)^n| \leq e^{-n}
\]

and this contradicts the minimal nature of \( r \) and hence the corollary follows.

Now we extend the result to linear forms. The analogous result in the real numbers is essentially due to Lejeune Dirichlet [18]. However a convenient form has been given by Kronecker (See Koksma [17], p. 5).

**Theorem 2:** Let \( L_h(U) = L_h(u_1, \ldots, u_n) \) \((1 \leq h \leq m)\) be \( m \) linear forms over \( K\{t\} \) in \( n \) variables \( u_1, \ldots, u_n \). To every integer \( \lambda \geq 0 \), there corresponds a non-zero element \( U \in P_n \) such that

\[
\|L_h(U)\| \leq e^\gamma, \quad |U| \leq e^{\lambda} \quad (1 \leq h \leq m)
\]

where

\[
-\gamma = \left[ \frac{m + n + n\lambda - 1}{m} \right].
\]
Proof: Consider the inequalities

\[
\begin{align*}
|L_h(u) - v_h| &\leq e^\gamma (1 \leq h \leq m) \\
|u_k| &\leq e^\lambda (1 \leq k \leq n)
\end{align*}
\]  

(2.1.6)

The determinant of the linear forms on the left hand side is of valuation 1, and the sum of exponents on the right hand side of (2.1.6) is \(m\gamma + n\lambda\).

But by definition

\[-\gamma \leq \frac{m + n + n\lambda - 1}{m} < -\gamma + 1\]

or \(\gamma \geq \frac{-m - n - n\lambda + 1}{m} > \gamma - 1\)

So \(m\gamma + n\lambda \geq -m - n - n\lambda + 1 + n\lambda = -m + n + 1\).

Therefore, by theorem A of chapter 1, there exist \((u_1, \ldots, u_n, v_1, \ldots, v_m) \neq 0\) in \(\mathbb{P}_{m+n}\) satisfying (2.1.6).

We observe here

\[
\frac{m + n + n\lambda - 1}{m} \geq + 1
\]

By definition of \(\gamma\), \(-\gamma \geq + 1\) or \(+ \gamma \leq -1\).

Now we assert that \(u_1, \ldots, u_n\) are not all zero.

Otherwise, from (2.1.6) we have

\[
|v_h| \leq e^\gamma \leq e^{-1} (1 \leq h \leq m)
\]
and hence $v_n = 0 (1 \leq h \leq m)$, which contradicts the hypothesis that $(u_1, \ldots, u_n, v_1, \ldots, v_m) \neq 0$. Thus $(u_1, \ldots, u_n) \neq 0$ and the theorem follows. We next prove

Theorem 3: Let $m, n$ be two positive integers and $K$ be a field having at least $(m + n - 1)$ distinct elements. Then there exists a constant $\lambda > 0$ depending on $K$ only and $m$ linear forms $L_p(U) = L_p(u_1, \ldots, u_n) (1 \leq p \leq m)$ in $n$ variables $u_1, \ldots, u_n$ with coefficients from $K\{t\}$, such that

$$|U|^n \cdot \left( \max_p \left| L_p(U) \right| \right)^m \geq \lambda$$

(2.1.7)

for all $U \neq 0$ in $P_n$.

Proof: Since $K$ has at least $(m + n - 1)$ distinct elements, there exist in $K\{t\} (m + n)$ conjugate algebraic integers $\phi_1, \ldots, \phi_{m+n}$ over $K[t]$ {See Armitage [1]} i.e. they are roots of an irreducible monic polynomial of degree $m + n$ over $K[t]$.

Let $r = m + n$. Set

$$Q_k(U, V) = \sum_{p=1}^{m} \phi_k^{p-1} v_p + \sum_{q=1}^{n} \phi_k^{m+q-1} u_q (1 \leq k \leq r)$$

Now we assert that for elements $0 \neq (u_1, \ldots, u_n, v_1, \ldots, v_m)$ in $P_{m+n}$, $Q_k(U, V) \neq 0$ for any $k (1 \leq k \leq r)$. If not, suppose $Q_k(U, V) = 0$ for some $k$. Then $Q_k(U, V) = 0$ for all $k$ in $1 \leq k \leq r$ because $Q_k(U, V)$ are all conjugate to each other. But
the coefficients of $Q_k(U, V)$ form a Van-de-Monde* matrix. This has, therefore, a non-zero determinant, and therefore, the only solution of

$$Q_k(U, V) = 0 (1 \leq k \leq r)$$

is $(U, V) = (0, 0)$.

Thus $Q_k(U, V) \neq 0$ for any $k (1 \leq k \leq r)$ for $(U, V) \neq (0, 0)$. Therefore, $\prod_k Q_k(U, V) \neq 0$ for $(U, V) \neq (0, 0)$. Therefore, $\prod Q_k(U, V)$ is an element of $K[t]$. So

$$| \prod_k Q_k(U, V) | \geq 1 \text{ or } | \prod_k Q_k(U, V) | \geq 1.$$  \hspace{1cm} (2.1.8)

Now consider the equations

$$\sum_{p=1}^{m} \varphi_k^{p-1} v_p = - \sum_{q=1}^{n} \varphi_k^{m+q-1} u_q (1 \leq k \leq m)$$  \hspace{1cm} (2.1.9)

in variables $v_1, \ldots, v_m$. The matrix of this equation is a Van-der-Mond's matrix and hence non-singular. So solving these equations by Cramer's rule we get

$$v_p = L_p(U) \quad (1 \leq p \leq m)$$

where $L_p(U)$ are $m$ linear forms in $n$ variables $u_1, \ldots, u_n$.

The coefficients of $L_p(U)$ depend only on $\varphi$'s.

Now we write $Q_k(U, V) (m + 1 \leq k \leq r)$ in the form
\[ Q_k(U, V) = \sum_{p=1}^{m} \phi_k^{p-1} (v_p - L_p(U)) + \sum_{q=1}^{n} y_{kq} u_q (m+1 \leq k \leq r) \quad (2.1.10) \]

where \[ y_{kq} (m + 1 \leq k \leq r, 1 \leq q \leq n) \text{ belong to } K \{ t \} \text{ depending upon } \phi \text{'s.} \]

Also \[ v_p = L_p(U) \ (1 \leq p \leq m) \text{ are solutions of } (2.1.9). \]

So we get

\[ \sum_{p=1}^{m} \phi_k^{p-1} L_p(U) = - \sum_{q=1}^{n} \phi_k^{m+q-1} u_q \ (1 \leq k \leq m). \]

Therefore, we can write \[ Q_k(U, V) \text{ for } 1 \leq k \leq m \] in the form

\[ Q_k(U, V) = \sum_{p=1}^{m} \phi_k^{p-1} (v_p - L_p(U)) \ (1 \leq k \leq m) \quad (2.1.11) \]

Let \[ U = (u_1, \ldots, u_n) \text{ be any vector of } P_n \text{ not equal to } 0. \text{ Let} \]

\[ e^\alpha = \max_{1 \leq q \leq n} |u_q| = |U|, \quad (2.1.12) \]

\[ e^\beta = \max_{1 \leq p \leq m} ||L_p(U)||. \quad (2.1.13) \]

Then

\[ e^\beta \leq e^{-1} < e^\alpha \text{ i.e. } \beta < \alpha. \quad (2.1.14) \]

Let \[ v_1, \ldots, v_m \text{ be elements of } K[t] \text{ such that} \]
\[ \| L_p(U) \| = \| L_p(U) - v_p \| \quad (1 \leq p \leq \mathbf{m}) \quad (2.1.15) \]

Let \( v_1, \ldots, v_m \) be as in (2.1.15) and \( u_1, \ldots, u_n \) as in (2.1.12) and (2.1.13). Then we have from (2.1.11),
\[ |Q_k(U, V)| \leq e^{\gamma_1 + \beta} \quad \text{for} \quad 1 \leq k \leq \mathbf{m} \quad (2.1.16) \]
where \( \gamma_1 \) depends only on \( \emptyset \)'s.

Also from (2.1.10), (2.1.12), (2.1.13) and (2.1.14) we get
\[ |Q_k(U, V)| \leq \max \left( e^{\beta + \gamma_2}, e^{\alpha + \gamma_3} \right) \]
\[ \leq e^{\alpha + \gamma_4} \quad \text{for} \quad (m + 1 \leq k \leq \mathbf{r}) \quad (2.1.17) \]

The \( \gamma \)'s occurring in (2.1.17) depend only on \( \emptyset \)'s and hence on \( K \).

From (2.1.8), (2.1.16) and (2.1.17) we get
\[ 1 \leq \frac{\gamma_1}{\gamma} \quad Q_k(U, V) \leq e^{m \gamma_1 + m\beta + n\alpha + n \gamma_4} \]
or
\[ e^{-m \gamma_1 - n \gamma_4} \leq e^{m\beta + n\alpha} = |U|^n \cdot (\max_p \|L_p(U)\|)^m \]
and this proves the theorem.

2.2: In this section, we aim at giving an approximation of complex quantities, by elements of the form \( u + iv \) where \( u \in K[t], v \in K[t] \).

Lemma 2.1: Let \( \alpha \in K[\{t\}], \) and \( k \) be a natural number. Then there exists at least one \( Z = x + iy \); \( x, y \in K[t] \), satisfying
\[ \|\alpha Z\| \leq e^{-k}, \quad 0 < |Z| < e^k. \tag{2.2.1} \]

Proof: Let \( \alpha = a + ib \). Now

\[
\|\alpha Z\| = \| (a + ib)(x + iy) \| = \| ax - by + i(bx + ay) \|
\]

\[
= \max (\|ax - by\|, \|bx + ay\|) \quad \text{(By defn.)}
\]

\[
\leq \max (|ax - by + u|, |bx + ay + v|)
\]

for every fixed \((u, v) \in P_2\).

Now consider the inequalities

\[ |x| \leq e^{k-1} \tag{2.2.2_1} \]

\[ |y| \leq e^{k-1} \tag{2.2.2_2} \]

\[ |ax - by + u| \leq e^{-k} \tag{2.2.2_3} \]

\[ |bx + ay + v| \leq e^{-k} \tag{2.2.2_4} \]

The linear forms on the left hand side have the determinant equal to 1, and the sum of the exponents on right hand side is \(-2\).

Therefore, by theorem A of chapter 1, there exists a non-zero vector \( A = (x, y, u, v) \in P_4 \) satisfying the above inequalities. Further we claim that not both of \( x, y \) can be zero. For otherwise, \((2.2.2_3)\) and \((2.2.2_4)\) give \( u = v = 0 \) and this implies that \( A = 0 \).

Hence the lemma is proved.

\textbf{Theorem 4}: Let \( \alpha = a + ib \in K^c \{t\} \) be such that at least one of \( a, b \) does not belong to \( K(t) \). Then there exists infinitely many
Let $Z = x + iy$, $W = u + iv$; $x, y, u, v \in K[t]$ with $\gcd(Z, W) = 1$ satisfying

\[ |\alpha + \frac{W}{Z} | < \frac{1}{|Z|^2} \quad (Z \neq 0) \quad (2.2.3). \]

Proof. Let $k \geq 1$ be a natural number.

Consider a solution $Z = x + iy$; $x, y \in K[t]$, given by Lemma 2.1. Then we have

\[ |\alpha Z| \leq e^{-k} < \frac{1}{|Z|}, \]

or $|\alpha Z + W| < \frac{1}{|Z|}$ for some $W = u + iv$; $(u, v) \in P_2$,

or $|\alpha + \frac{W}{Z}| < \frac{1}{|Z|^2}$.

Without loss of generality, we can assume $\gcd(W, Z) = 1$, so that there exists at least one pair $(W, Z)$ with $\gcd(W, Z) = 1$ satisfying (2.2.3). Now suppose there exists only a finite number of $Z = x + iy$, $W = u + iv$; $x, y, u, v \in K[t]$; $Z \neq 0$ with $\gcd(W, Z) = 1$ satisfying

\[ |\alpha + \frac{W}{Z}| < \frac{1}{|Z|^2} \]

or $||\alpha Z + W|| < \frac{1}{|Z|}$.

For this finite number of solutions, consider

\[ \min |\alpha Z + W|. \]
First we assert that this minimum is not zero. For this it is enough to prove that $ax - by$ and $bx + ay$ cannot simultaneously belong to $K[t]$, provided $(x, y) \neq (0, 0)$. If exactly one of $a, b$ does not belong to $K(t)$, there is nothing to prove. So assume both $a, b$ do not belong to $K(t)$, and if possible, suppose

$$ax - by = u, \quad bx + ay = v$$

for some $x, y, u, v \in K[t]$ with $(x, y) \neq (0, 0)$. Then

$$a(x^2 + y^2) = ux + vy.$$

But $x^2 + y^2 \neq 0$ [Since $x^2 + 1 = 0$ is not soluble in $K\{t\}$.]

So

$$a = \frac{ux + vy}{x^2 + y^2} \notin K(t),$$

which is a contradiction. Hence $\min |\alpha Z + W| \neq 0$. Let this minimum be $e^{-r}$. Lemma 2.1 ensures that $r \geq 1$. Take $k > r \geq 1$.

Then by lemma 2.1, there exists a non-zero $Z = x + iy$, $(x, y) \in P_2$ satisfying

$$||\alpha Z|| \leq e^{-k}, \quad 0 < |Z| < e^k.$$

Then there exists $W = u + iv$, $(u, v) \in P_2$ such that

$$|\alpha Z + W| \leq e^{-k} < e^{-r}.$$

Without loss of generality, we can assume $\gcd(W, Z) = 1$. 
Also

$$|\alpha z + w| < \frac{1}{|z|^2}$$

and this contradicts the minimal nature of \( r \). Hence the theorem is proved.

Remark 2.1: It is not known, whether this result is best possible. The best possible result in complex number case is due to Ford [6], who states the following:

Let \( \alpha = a + ib \) be a complex number, such that at least one of \( a, b \) is irrational. Then there exist infinitely many non-zero Gaussian integers \( Z = x + iy \), \( W = u + iv \) satisfying

$$|\alpha + \frac{W}{Z}| < \frac{1}{\sqrt{3} |Z|^2}.$$ 

The corresponding result for real numbers is due to Hurwitz; e.g. see Niven [23], p. 6, which states the following:

Given any irrational \( \alpha \), there exist infinitely many integers \( u, v; v > 0 \) satisfying

$$|\alpha + \frac{u}{v}| < \frac{1}{\sqrt{5} v^2}$$

i.e. the constant in the complex number case is greater than the constant in the real number case.
2.3: In this section, we prove a few results, the analogues of which in p-adic fields, have been proved by Lutz [19], [pp. 20-22, 32-33].

Notation: All the linear forms which occur in this section and hereafter will have the coefficients in $K \{t\}$, unless stated otherwise.

Now we give a few definitions.

Definition 2.1: Let $\Lambda$ be a set of linear forms

$$L_p(U) = L_p(u_1, \ldots , u_n)(1 \leq p \leq m).$$

Then we shall say that $\Lambda$ is a linear system and write it as $\Lambda(L_p(U) = L_p(u_1, \ldots , u_n), 1 \leq p \leq m)$.

Definition 2.2: A linear system $\Lambda(L_p(U) = L_p(u_1, \ldots , u_n); 1 \leq p \leq m)$ is said to be linearly independent over $K\{t\}$, if

$$\sum_{p=1}^{m} \lambda_p L_p(U) = 0, \quad \lambda_p \in K\{t\} \Rightarrow \lambda_p = 0 (1 \leq p \leq m).$$

Remark 2.2: $L_p(U) = L_p(u_1, \ldots , u_n)(1 \leq p \leq m)$ are linearly independent, implies that $m \leq n$.

Definition 2.3: A linear system $\Lambda(L_p(U) = L_p(u_1, \ldots , u_n), 1 \leq p \leq m)$ is called of signature $(m, n)$ if linear forms

$$L_p(U) = L_p(u_1, \ldots , u_n)(1 \leq p \leq m)$$

are linearly independent over $K\{t\}$.

Definition 2.4: Let $\Lambda$ be a linear system of signature $(m, n)$. Let $A$ be the matrix of the coefficients. We define rational integers $\delta = \delta(\Lambda)$ and $\rho = \rho(\Lambda)$ as follows:

$$e^{-\delta} = \max_D \left( |\det D| \right),$$
where \( D \) runs over the minors of rank \( m \) from the matrix \( A \).

And

\[
e^a = \max_D \left( \min_{D^*} \frac{\det D}{\det D^*} \right)
\]

where \( D \) runs over all the non-singular minors of the matrix \( A \) of rank \( \leq m - 1 \) and \( D^* \) runs over non-singular minors of \( A \) containing \( D \) by adding one row and one column.

[Since the linear forms are linearly independent, such a \( D \) always exists].

Definition 2.5: A linear system \( \bigwedge (L_p(U) = L_p(u_1, \ldots, u_n), 1 \leq p \leq m) \) is called annulable, if there exists non-zero vector \( U \in \mathbb{P}_n \) such that \( L_p(U) = 0 \) \( (1 \leq p \leq m) \).

Theorem 5: Let \( L_p(U) = L_p(u_1, \ldots, u_n)\) \( (1 \leq p \leq m) \) be a system of \( m \) linear forms in variables \( u_1, \ldots, u_n \). Let \( m \geq n \).

Suppose the matrix of the coefficients has rank \( n \). Then the volume \( V \) of the set

\[
\left\{ \begin{array}{l}
U \in \mathbb{P}_n ; \quad \max_{1 \leq p \leq m} |L_p(U)| \leq 1 \\
\end{array} \right\}
\]

equals \( \min_D |\det D^{-1}| \),

where \( D \) denotes an arbitrary minor of rank \( n \), taken from the coefficients of the forms \( L_p(U) \) \( (1 \leq p \leq m) \).

For the proof, we use the following lemma of Mahler:

Lemma 2.2: Let \( L_p(U) = L_p(u_1, \ldots, u_n) \) \( (1 \leq p \leq n) \) be \( n \)
linear forms with determinant $D \neq 0$. Then the volume $V$ of the set

$$S = \left\{ U \in \mathbb{R}^n ; |L_p(U)| \leq 1 \quad (1 \leq p \leq n) \right\}$$

equals $|D|^{-1}$.

Remark 2.3: Lemma 2.2 is a particular case of theorem 5 with $m = n$.

Proof of theorem 5: Take all the possible combinations of $n$ forms among $L_p(U) = L_p(u_1, \ldots, u_n) (1 \leq p \leq m)$. Let $D$ denote the matrix of an arbitrary set of $n$ forms. Consider $\max_p |\det D|$.

Since the matrix formed by the coefficients has rank $n$, $\max |\det D| \neq 0$. Without loss of generality, we assume that the forms associated to $\max \left| \det D \right|$ are $L_1, \ldots, L_n$. Then the determinant $\Delta$ of the forms $L_1, \ldots, L_n$, is not zero.

Let

$$L_p(U) = \sum_{q=1}^{n} \alpha_{pq} u_q \quad (p = 1, \ldots, m).$$

Then we can solve

$$L_{n+k}(U) = \sum_{q=1}^{n} \frac{L_q(U)}{\Delta} y_{kq} \quad (k = 1, \ldots, m-n).$$

For $y_{kq}$ and by Cramer's rule we get that

$$y_{kq} = \Delta_{kq} \quad (k = 1, \ldots, m-n ; q = 1, \ldots, n),$$

where $\Delta_{kq}$ is a certain $n \times n$ determinant taken from the matrix of $(n+1)$
forms $L_1(U), \ldots, L_n(U), L_{n+k}(U)$. Further by our choice of $\Delta$

\[ \sum_{q=1}^{n} \frac{y_{kq}}{\Delta} L_p(U) \leq 1 \quad (k = 1, \ldots, m - n). \]
Thus the two sets

\[ \{ U \in \mathbb{R}^n ; \max_{1 \leq p \leq n} |L_p(U)| \leq 1 \} \quad \text{and} \quad \{ U \in \mathbb{R}^n ; \max_{1 \leq p \leq m} |L_p(U)| \leq 1 \} \]

are identical. However, by lemma 2.2, the volume of the latter is $|\Delta|^{-1}$. Hence the theorem follows.

**Theorem 6:** Let $\Lambda(L_p(U) = L_p(u_1, \ldots, u_n)(1 \leq p \leq m))$ be a linear system of signature $(m, n)$. Let $\delta$ and $\rho$ be as in definition 2.5. Let $\lambda_1, \ldots, \lambda_m$ be rational integers $\geq \rho$.

Then

\[ \text{Vol} \left\{ U \in \mathbb{R}^n ; |U| \leq 1, |L_p(U)| \leq e^{-\lambda_p}; p = 1, \ldots, m \right\} \]

\[ = \exp \left( \delta - \sum_{p=1}^{m} \lambda_p \right). \]

**Proof:** Consider the following system of $m + n$ linear forms

\[ \lambda^p L_p(U) \quad (p = 1, \ldots, m) \]

\[ u_q \quad (q = 1, \ldots, n). \]

Out of these $m + n$ linear forms, take an arbitrary set of $n$ linearly independent forms. Let $s(0 \leq s \leq m)$ be the number of forms $\lambda^p L_p(U)$ among these and $(n - s)$ be the number of the
forms \( u_q \). The valuation of such a system is then

\[
\exp \left( \sum_{p \in S} \lambda_p \right) |\det D|
\]

where \( S \) denotes the set of indices of the forms \( \ell^p L_p(U) \)
considered above, and \( D \) is a certain minor of \( A \) with \( s \) rows
and \( s \) columns. If we denote by \( \mathcal{S} \) the set

\[
\mathcal{S} = \left\{ U \in \mathbb{R}^n; |U| \leq 1, |\ell^p L_p(U)| \leq 1, p = 1, \ldots, m \right\}
\]

we have by theorem 5

\[
\text{Vol } \mathcal{S} = \min \left( \exp \left( - \sum_{p \in S} \lambda_p \right) |\det D|^{-1} \right)
\]

where minimum has been taken over all the possible combinations of
\( n \) linearly independent forms.

Consider a value of \( s < m \), and adjoin to the forms
\( \ell^p L_p(U) (p \in S) \) already considered an \((s + 1)\)th form of the above
type. Since \( \wedge \) is of signature \((m, n)\); these \( s + 1 \) forms are
linearly independent. This \((s + 1)\)th form is to be suitably
chosen subsequently. For the time being call it \( \ell^* L^* (U) \).

With these \((s + 1)\) forms, we take \((n - s - 1)\) forms among the \( u_q \)
already considered. The value of the new determinant is then

\[
\left[ \exp \left( \sum_{p \in S} \lambda_p + \lambda_* \right) \right] |\det D^*|,
\]
where $D^*$ is a certain minor of $A$ with rank $(s + 1)$, and can be obtained from $D$ by adding to it one row and one column. But by definition

$$e^\xi = \max_{D^*} \min_D \left( \left| \frac{\det D}{\det D^*} \right| \right)$$

So $e^\xi \geq \min_{D^*} \left| \frac{\det D}{\det D^*} \right|$ for all $D$.

or $e^\xi \geq \left| \frac{\det D}{\det D^*} \right|$ for all $D$, and $D$ being given for some $D^*$ containing $D$.

Now we choose that form $\lambda^* \in L^*(U)$, for which

$$e^\xi \geq \left| \frac{\det D}{\det D^*} \right|.$$ 

But $\lambda_p \geq \xi$ for all $p$.

Therefore, $e^\lambda^* \geq e^\xi \geq \left| \frac{\det D}{\det D^*} \right|$.

Hence the valuation of the new determinant is greater than or equal to the valuation of the old determinant. Hence the set $S$ for which $V(\delta)$ is attained must contain all the $m$ indices.

Thus by definition of $\delta$, we must have

$$V(\delta) = \exp(\delta - \sum_{p=1}^{m} \lambda_p)$$

and this proves the theorem.
Corollary 6.1: \[ \text{Vol} \left\{ U \in \mathbb{R}^n; |U| \leq e^k, |L_p(U)| \leq e^{-\lambda_p} \right\} = \exp \left( \delta + (n - m) k - \sum_{p=1}^{m} \lambda_p \right) \]

where \( k \geq 0 \) is an integer.

Proof: Let \[ \mathcal{B} = \left\{ U \in \mathbb{R}^n; |U| \leq e^k, |L_p(U)| \leq e^{-\lambda_p}, p = 1, \ldots, m \right\} \]

\[ = \left\{ U \in \mathbb{R}^n; |t^{-k}U| \leq 1, |t^{\lambda_p}L_p(U)| \leq 1, p = 1, \ldots, m \right\}. \]

Let \[ F(U) = \max(|t^{-k}U|, |t^{\lambda_p}L_p(U)|, p = 1, \ldots, m). \]

Then \( F(U) \) is a distance function and \( \mathcal{B} \) is the convex body \( F(U) \leq 1 \). Consider the transformation \( T \) defined by

\[ V = t^{-k}U = TU \]

This transformation maps \( F(U) \leq 1 \) into \( \mathcal{B}': F'(V) \leq 1 \) where

\[ F'(V) = \max(|V|, |t^{k+\lambda_p}L_p(V)|, p = 1, \ldots, m) \]

i.e.

\[ \mathcal{B}' = \left\{ V \in \mathbb{R}^n; |V| \leq 1, |t^{k+\lambda_p}L_p(V)| \leq 1, p = 1, \ldots, m \right\}. \]

Using theorem 6, we have

\[ \text{Vol} \mathcal{B}' = \exp \left( \delta - mk - \sum_{p=1}^{m} \lambda_p \right). \]
But by using theorem C of chapter 1, we get

\[ V(\hat{b}) = |\det Tr^2 V(\hat{b}')| = \exp (\delta + (n-m)k - \sum_{p=1}^{m} \lambda_p) \]

and this proves the corollary.

**Theorem 7:** Let \( \Lambda \) and \( \lambda \)'s be as in previous theorem. Let \( k \geq 0 \) be a rational integer. Then there exists \( U \in P_n \) satisfying

\[ |L_p(U)| \leq e^{-\lambda_p} \] for \( p = 1, \ldots, m \), \( 0 < \| U \| \leq e^k \),

if

\[(n - m)k \geq \sum_{p=1}^{m} \lambda_p - \delta(\Lambda).\]

**Proof:** Let

\[ \hat{b} = \left\{ U \in R_n; \| U \| \leq e^k, |L_p(U)| \leq e^{-\lambda_p}, p = 1, \ldots, m \right\}. \]

By Corollary 6.1, we get

\[ V(\hat{b}) = \exp(\delta + (n - m)k - \sum_{p=1}^{m} \lambda_p). \]

By remark 1.4, we see that if \( V(\hat{b}) \geq 1 \), then it must contain a non-zero \( U \in P_n \). Thus if

\[ \delta + (n - m)k - \sum_{p=1}^{m} \lambda_p \geq 0, \]

\( \hat{b} \) contains a non-zero \( U \in P_n \), and this proves the theorem.
By taking $\lambda_p = \lambda (p = 1, \ldots, m)$ in theorem 7, we get

Corollary 7.1: For each rational integer $\lambda \geq \rho(\Lambda)$, there exists $U \in \mathbb{P}_n$ satisfying

$$|L_i(U)| \leq e^{-\lambda}, \ 0 < |U| \leq e^\gamma$$

where $\gamma$ is a positive integer such that

$$\gamma \geq \frac{m \lambda - 5}{n - m}.$$  

Definition 2.6: A point $(u_1, \ldots, u_n) = U \in \mathbb{P}_n$ is called a primitive point if

$$\gcd(u_1, \ldots, u_n) = 1$$

Remark 2.4: In theorem 7 and its corollary, obviously we can take $U$ to be primitive.

Theorem 8: Suppose $\Lambda(L_p(U) = L_p(u_1, \ldots, u_n) (1 \leq p \leq m))$ is a linear system of signature $(m, n)$ with $1 \leq m < n$ and suppose $\Lambda$ is not annulable. Then the inequality

$$(\max_{p=1, \ldots, m} |L_p(U)|)^m |U|^{n-m} \leq \exp(n - m - 5)$$  \hspace{1cm} (2.3.1)

admits infinity of solutions $U \in \mathbb{P}_n$.

Proof: By corollary 7.1, we know that for each rational integer $\lambda \geq \rho$, there exists $U \in \mathbb{P}_n$ satisfying

$$|L_p(U)| \leq e^{-\lambda} (p = 1, \ldots, m), \ 0 < |U| \leq e^\gamma$$
where $\gamma \geq \frac{m-\delta}{n-m}$ is a positive integer. If $\lambda$ is large, we can choose integer $\gamma \geq 0$ such that

$$\gamma - 1 < \frac{m\lambda - \delta}{n-m} \leq \gamma.$$ 

Thus for large values of $\lambda$, there exists $U \in \mathbb{P}_n$ satisfying

$$|L_p(U)| \leq e^{-\lambda}(p = 1, \ldots, m), 0 < |U| \leq \frac{m\lambda - \delta}{e^{n-m}} + 1$$

and we get a solution $U \in \mathbb{P}_n$ of (2.3.1). As the value of $\lambda$ increases, $e^{-\lambda}$ and hence $\max_{p=1, \ldots, m} |L_p(U)|$ decreases, and we get infinitely many distinct solutions of (2.3.1), unless

$$\max_{p=1, \ldots, m} |L_p(U)| = 0$$

for some non-zero $U \in \mathbb{P}_n$, which is denied by the hypothesis. Hence the theorem follows.
CHAPTER III

INHOMOGENEOUS APPROXIMATION

The main aim of this chapter is to prove the analogues of two results by Khintchine (see [15] and [16]).

3.1: In this section, we shall prove

Theorem 9: Let \( K \) be a finite field. Let \( m, n \) be any positive integers. Let \( L_i(U) = L_i(u_1, ..., u_n) \) \((1 \leq i \leq m)\) be a system of linear forms over \( K \{t\} \). Then there exist \( C = (c_1, ..., c_m) \in \mathbb{R}_m \) such that

\[
\left( \max_{1 \leq i \leq m} \|L_i(U) + c_i\| \right)^m \|U\|^n \geq e^{-m-2n} \quad (3.1.1)
\]

for all non-zero vectors \( U \in \mathbb{P}_n \).

To prove the above theorem, we shall prove a few lemmas.

Lemma 3.1: Let \( X_r = (x_{r1}, ..., x_{rm}) \neq 0 \) for \( r = 1, 2, ... \) be a finite or infinite sequence of elements of \( \mathbb{P}_m \). Define

\[
e^r = \max (|x_{r1}|, ..., |x_{rm}|).
\]

Suppose

\[
e^{r+1} > e^r \quad (3.1.2)
\]

for all \( r \). Then there exists \( C = (c_1, ..., c_m) \in \mathbb{R}_m \) such that

\[
\|x_r^C\| = \|x_{r1}c_1 + ... + x_{rm}c_m\| = e^{-1} \quad (3.1.3)
\]
for all $r$.

The above lemma amounts to proving the following:

Lemma 3.1': Let $X_r = (x_{r1}, \ldots, x_{rm}) \neq 0$ for $r = 1, 2, \ldots$ be a finite or infinite sequence of $m$-tuples of polynomials in $t$ over $K$. Define

$$\sigma_r = \max_{1 \leq i \leq m} \deg (x_{ri}) \quad r = 1, 2, \ldots$$

Suppose $\sigma_{r+1} \geq \sigma_r + 1$.

Then there are $m$ elements $c_i (1 \leq i \leq m)$ of $K \{t\}$, such that for every $r$, the coefficient of $t^{-1}$ in $x_{r1} c_1 + \ldots + x_{rm} c_m$ is not zero.

Remark 3.1: We shall prove the theorem for $m = 2$. The proof for general $m$ is similar. For convenience of notation, we shall restate our lemma for $m = 2$.

Lemma 3.1'': Let $X_r = (y_r, z_r) \neq 0$ for $r = 1, 2, \ldots$ be a finite or infinite sequence of pairs of polynomials over $K$. Let

$$\sigma_r = \max (\deg y_r, \deg z_r).$$

Suppose $\sigma_{r+1} \geq \sigma_r + 1$.

Then there is a pair $(\alpha, \beta)$ of elements of $K \{t\}$, such that for each $r$ the coefficient of $t^{-1}$ in $y_r \alpha + z_r \beta$ is not zero.
Proof of lemma 3.1! Let
\[ y_r = a_r, \sigma_r t^r + \ldots + a_r, 0 \] \quad r = 1, 2, \ldots
\[ z_r = b_r, \sigma_r t^r + \ldots + b_r, 0 \]

Then either \( a_r, \sigma_r \) or \( b_r, \sigma_r \) is not equal to zero.

We shall show that we can inductively choose \( \alpha_1, \alpha_2, \ldots \)
and \( \beta_1, \beta_2, \ldots \) such that
\[ \alpha = \alpha_1 t^{-1} + \alpha_2 t^{-2} + \ldots \]
\[ \beta = \beta_1 t^{-1} + \beta_2 t^{-2} + \ldots \]

have the required property.

The coefficient of \( t^{-1} \) in \( y_1 \alpha + z_1 \beta \) is
\[ = a_1, 0 \alpha_1 + a_1, 1 \alpha_1 + \ldots + a_1, \sigma_1 \sigma_1^{-1} + 1 \]
\[ + b_1, 0 \beta_1 + b_1, 1 \beta_1 + \ldots + b_1, \sigma_1 \sigma_1^{-1} + 1 \]

We choose \( \alpha_1, \ldots, \alpha_{\sigma_1^{-1}}, \beta_1, \ldots, \beta_{\sigma_1^{-1}} \) arbitrarily.

Since at least one of \( a_1, \sigma_1 \) and \( b_1, \sigma_1 \) is not zero, we can
select \( \alpha_{\sigma_1^{-1}} + 1, \beta_{\sigma_1^{-1}} \) suitably such that the coefficient of \( t^{-1} \)
in \( y_1 \alpha + z_1 \beta \) is not zero. Suppose we have selected
\[ \alpha_1, \ldots, \alpha_{\sigma_k^{-1}} + 1, \beta_1, \ldots, \beta_{\sigma_k^{-1}} \] such that the coefficient
of $t^{-1}$ in $y_r \alpha + z_r \beta$ for $r = k$ is not zero. Then we have
to choose $\alpha_{\sigma_k + 2'} \cdot \cdot \cdot , \alpha_{\sigma_{k+1} + 1'} \beta_{\sigma_k + 2'} \cdot \cdot \cdot \beta_{\sigma_{k+1} + 1}$ such
that the coefficient of $t^{-1}$ in $y_r \alpha + z_r \beta$ for $r = k + 1$ is
not zero.
The coefficient of $t^{-1}$ in $y_{k+1} \alpha + z_{k+1} \beta$ is

\[= a_{k+1, \sigma_{k+1} + 1} + \cdot \cdot \cdot + a_{k+1, 0} \alpha_{1} + b_{k+1, \sigma_{k+1} + 1} + \cdot \cdot \cdot + b_{k+1, 0} \beta_{1}.\]

Since $\sigma_{k+1} \geq \sigma_k + 1$ and at least one of

$a_{k+1, \sigma_{k+1}}, b_{k+1, \sigma_{k+1}}$ is not zero, we choose

$\alpha_{\sigma_k + 2'} \cdot \cdot \cdot , \alpha_{\sigma_{k+1} + 1'} \beta_{\sigma_k + 2'} \cdot \cdot \cdot \beta_{\sigma_{k+1} + 1}$ arbitrarily and

$\alpha_{\sigma_{k+1} + 1'} + \beta_{\sigma_{k+1} + 1}$ in such a way that the right hand side of

(3.1.4) is not zero, i.e. we have been able to select

$\alpha_{\sigma_k + 2'} \cdot \cdot \cdot , \alpha_{\sigma_{k+1} + 1'} \beta_{\sigma_k + 2'} \cdot \cdot \cdot \beta_{\sigma_{k+1} + 1}$ such that

the coefficient of $t^{-1}$ in $y_{k+1} \alpha + z_{k+1} \beta$ is not zero.

This choice will give $\alpha$, $\beta$ with the required properties.

Definition 3.1: Let

\[L_i(U) = \sum_{j=1}^{n} \theta_{j1} u_j \quad (1 \leq i \leq m)\]

be a system of $m$ linear forms in variables $u_1, \cdot \cdot \cdot , u_n$. 
Then the system of \( n \) linear forms
\[
M_j(X) = \sum_{i=1}^{m} \theta_{ji} x_i \quad (1 \leq j \leq n)
\]
in \( m \) variables \( x_1, \ldots, x_m \) is said to be the transposed set of the linear forms \( L_i(U) \) \((1 \leq i \leq m)\).

**Definition 3.2:** Let \( M_j(X) = M_j(x_1, \ldots, x_m) \) \((1 \leq j \leq n)\) be the set of linear forms transposed to the given forms \( L_i(U) \) \((1 \leq i \leq m)\). For all non-negative integers \( k \), define \( \eta(k) \) to be the minimum of
\[
\max_{1 \leq j \leq n} \| M_j(X) \|
\]
over all \( X \in \mathbb{P}_m \) satisfying \( 0 < |X| \leq e^k \).

**Lemma 3.2:** (1) \( \eta(k) \) does not increase, if \( k \) increases.
(2) \( (\eta(k))^n e^{mk} \leq e^{-m} \) for all \( k \).

**Proof:** (1) It is obvious from definition of \( \eta(k) \).
(2) Consider
\[
\begin{align*}
| M_j(X) + y_j | & \leq e^D \quad (1 \leq j \leq n) \\
| x_i | & \leq e^k \quad (1 \leq i \leq m)
\end{align*}
\]
where \( D \) is chosen to satisfy
\[
-m - n + 1 \leq nD + mk \leq -m.
\]
Linear forms on the left hand side of (3.1.5) have determinant
± 1. By the left hand side of (3.1.6) and by theorem A of chapter 1, there exists a non-zero vector \((x_1, \ldots, x_m, y_1, \ldots, y_n) \in P_{m+n}^m+n\) satisfying (3.1.5). Now we assert that \((x_1, \ldots, x_m) \neq 0\).

For, otherwise, from (3.1.5) we get that \(|y_j| \leq e^D (1 \leq j \leq n)\).

But (3.1.6) gives us that \(D < 0\), so that \(|y_j| < 1 (1 \leq j \leq n)\).

Since \(y_j (1 \leq j \leq n)\) are elements in \(K[t] \), \(y_j = 0 (1 \leq j \leq n)\) and \((x, y) = 0\), which is a contradiction. Thus there exists a non-zero \(X = (x_1, \ldots, x_m) \in P_m\) satisfying (3.1.6). From (3.1.6) we get

\[
|\| M_j(X) \| | \leq |M_j(X) + y_j | \leq e^D.
\]

Therefore, by definition, \(\eta(k) \leq e^D\).

\[
(\eta(k))^n e^{mk} \leq e^{nD + mk} \leq e^{-m} \quad \text{(from (3.1.6)}
\]

and this proves (2).

If in (2), we allow \(k \to \infty\) we get

Corollary 9.1: \(\eta(k) \to 0 \text{ as } k \to \infty\).

Lemma 3.3: We can find a sequence of non-zero vectors \(X_r = (x_{r1}, \ldots, x_{rm}) \in P_m\) with the following properties.

Let

\[
e^{\xi_r} = \max \{|x_{r1}|, \ldots, |x_{rm}|\}, \quad (3.1.7)
\]
Then

\[ \xi_1 = 0, \quad (3.1.8) \]

\[ \xi_r + 1 \geq \xi_r + 1, \quad (3.1.9) \]

and

\[ \max_{1 \leq j \leq n} \| M_j(X_r) \| = \eta(\xi_r + 1 - 1) \quad (3.1.10) \]

The sequence is infinite, unless there is a non-zero vector

\[ X \in P^m \text{ with } \| M_j(X) \| = 0 \quad (1 \leq j \leq n). \]

If there is such an \( X \),

the sequence terminates with an \( X_R \) such that

\[ \max_{1 \leq j \leq n} \| M_j(X_R) \| = 0, \text{ but } \max_{1 \leq j \leq n} \| M_j(X_r) \| \neq 0 \text{ for } r < R. \]

Proof: Suppose there is a non-zero vector \( X \in P^m \) with

\[ \| M_j(X) \| = 0 \quad (1 \leq j \leq n). \]

We first construct a sequence \( Y_r \) of elements of \( P^m \) and integers \( \xi_r \) defined by

\[ e^{\xi_r} = \max(|y_{r1}|, \ldots, |y_{rm}|), \]

as follows: (i) \( Y_1 \neq 0 \) is a vector from \( P^m \) with

\[ \| M_j(Y_1) \| = 0 \quad (1 \leq j \leq n) \]

for which \( \xi_1 \) is as small as possible.

(ii) If \( Y_1, \ldots, Y_R \) have been constructed for some \( R \) and \( \xi_R = 0 \), we stop with \( Y_R \).

(iii) If \( Y_1, \ldots, Y_R \) have been constructed and

\[ \xi_r \geq 1, \]

then choose a vector \( Y_{r+1} \neq 0 \) of \( P^m \).
with
\[ \sigma_{r+1} \leq \sigma_r - 1, \quad \max ||M_j(Y_{r+1})|| = \eta(\sigma_r - 1), \]

which exists by definition of \( \eta(k) \).

Since \( \sigma_{r+1} \leq \sigma_r - 1 \), the sequence does terminate with a \( Y_R \). Then \( x_r = y_{R+1-r} \) and \( \sigma_r = \sigma_{R+1-r} \) clearly satisfy (3.1.7), (3.1.8), (3.1.9) and (3.1.10).

\( i' \) Now assume there is no non-zero vector \( x \in P_m \) with
\[ ||M_j(x)|| = 0 \quad (1 \leq j \leq n). \]

Let \( \lambda > 0 \) be arbitrarily large. By theorem 2, we find a non-zero vector \( Y(1, \lambda) \in P_m \) satisfying
\[ \max_{1 \leq j \leq n} ||M_j(Y(1, \lambda))|| \leq e^{-\lambda} \]

and we construct the sequence \( Y(1, \lambda), Y(2, \lambda), \ldots, Y(R, \lambda) \) as indicated in (ii) and (iii) above. Then \( x(1, \lambda) = y_{R+1-r, \lambda} \) satisfy (3.1.8), (3.1.9) and (3.1.10).

Since the field is finite, there are only a finite number of \( x(1, \lambda) \) satisfying (3.1.8). So one of them, say \( x_1 \), must occur infinitely often for arbitrarily large \( \lambda \). Since, by hypothesis \( \max ||M_j(x_1)|| = 0 \), we must have \( R(\lambda) \geq 2 \) for large \( \lambda \). Since by corollary 9.1, \( \eta(k) \to 0 \) as \( k \to \infty \) and the field is finite, there are at most a finite number of vectors \( x(2, \lambda) \) which satisfy (3.1.10) with \( r = 1 \), and the chosen \( x(1, \lambda) = x_1 \). We choose \( x_2 \) such that \( x(1, \lambda) = x_1 \) and \( x(2, \lambda) = x_2 \) occur together for
arbitrarily large \( \lambda \). Suppose \( X_1, \ldots, X_r \) have already been chosen so that

\[
X(1, \lambda) = X_1, \ldots, X(r, \lambda) = X_r
\]

occur simultaneously for arbitrarily large \( \lambda \). Since

\[
\max \| M_j(X_r) \| \neq 0,
\]

by hypotheses, we must have \( R(\lambda) \geq r + 1 \) if \( \lambda \) is large enough and (3.1.11) holds. As before (3.1.10) shows that there are only a finite number of choices for \( X(r+1, \lambda) \) compatible with (3.1.11), so one of them says \( X_{r+1} \) must occur for arbitrarily large \( \lambda \). Then \( X_1, \ldots, X_r, \ldots \) constructed in this way clearly have all the required properties.

Proof of theorem 9: Let \( \{X_r\} \) be the sequence of vectors constructed in lemma 3.3. Choose \( C = (c_1, \ldots, c_m) \in \mathbb{R}_m \) by lemma 3.1, so that

\[
\| X_r C \| = e^{-1}
\]

for \( 1 \leq r \leq R \) (or \( 1 \leq r < \infty \)) as the case may be. Let a non-zero vector \( U \in \mathbb{P}_n \) be given. Put

\[
\max_{1 \leq i \leq m} \| L_i(U) + c_i \| = e^\lambda, \quad \max_{1 \leq j \leq n} | u_j | = e^k
\]

Then

\[
e^{-1} = \| X_r C \| = \| x_1 c_1 + \ldots + x_m c_m \|
\]
\[
\left\| \sum_{i=1}^{m} x_i (L_i(U) + c_i) - \sum_{i=1}^{m} x_i L_i(U) \right\| \\
\leq \max \left( \left\| \sum_{i=1}^{m} x_i (L_i(U) + c_i) \right\|, \left\| \sum_{i=1}^{m} x_i L_i(U) \right\| \right)
\]

But \( \sum_{i=1}^{m} x_i L_i(U) = \sum_{j=1}^{n} u_j M_j(x_r) \).

So \( e^{-1} = \| x_r c \| \leq \max \left( \left\| \sum_{i=1}^{m} x_i (L_i(U) + c_i) \right\|, \left\| \sum_{j=1}^{n} u_j M_j(x_r) \right\| \right) \)

\[
\leq \max \left( \| x_r (L_i(U) + c_i) \|, \ldots, \| x_r (L_m(U) + c_m) \|, \| u_1 M_1(x_r) \|, \ldots, \| u_n M_n(x_r) \| \right)
\]

\[
\leq \max \left( \| x_r \| \cdot \| L_i(U) + c_i \|, \ldots, \| x_r \| \cdot \| L_m(U) + c_m \|, \| u_1 \| \cdot \| M_1(x_r) \|, \ldots, \| u_n \| \cdot \| M_n(x_r) \| \right)
\]

\[
\leq \max \left( e^{\epsilon R + \lambda}, e^{-r} \right) \quad \text{(3.1.13)}
\]

where

\[
e^{-r} = \begin{cases} 
\eta(\zeta R - 1) & \text{when } r \neq R \\
0 & \text{if } r = R
\end{cases} \quad \text{(3.1.14)}
\]

Suppose, first that we can choose an integer \( r \) such that

\[
k + D_r - 1 \geq -2 \geq k + D_r. \quad \text{(3.1.15)}
\]

From (3.1.13) we have

\[
e^{\epsilon R + \lambda} \geq e^{-1},
\]

or \( e^{\lambda} \geq e^{-1 - \epsilon R} \). \quad \text{(3.1.16)}
Since $e^x$ is an increasing function of $x$, from (3.1.15) we have

$$e^{k+D_r} \geq e^{-2}$$

or

$$e^k \geq e^{-2-D_r-1}.$$ 

So

$$e^{m\lambda+nk} \geq e^{m(-1-\zeta r)} \cdot e^{n(-2-D_r-1)} \quad \text{from 3.1.16}$$

$$= e^{-(m+2n)} \cdot e^{-(m \zeta r + nD_r - 1)}$$

$$= e^{-(m+2n)} \cdot e^{-m \zeta r} \cdot (\eta(\zeta r - 1))^{-n} \quad \text{from 3.1.14}$$

$$= e^{-(2m+2n)} \cdot e^{-m(\zeta r - 1)} \cdot (\eta(\zeta r - 1))^{-n}$$

$$\geq e^{-(2m+2n)} \cdot e^m \quad \text{[From (2) of lemma 3.2]}$$

$$= e^{-(m+2n)}$$

And this proves the theorem in this case.

Since $D_r \rightarrow \infty$ as $r \rightarrow \infty$, the choice of $r$ satisfying (3.1.15) is always possible, unless

$$k + D_1 \leq -2.$$ 

Therefore, for $r = 1$, from (3.1.13) we get

$$e^{\zeta_1 + \lambda} \geq e^{-1}$$

or

$$\zeta_1 + \lambda \geq -1 \quad \text{or} \quad \lambda \geq -\zeta_1 - 1 = -1 \quad \text{(By (3.1.8))}$$
Since $u \in P_n$ is a non-zero vector and $k \geq 0$, we have

$$nk + m \lambda \geq m \lambda \geq -m.$$ 

So $\max(\|L_1(u) + C_1\|)^m \cdot |u|^n = e^{nk + m \lambda}$ (From (3.1.12))

$$\geq e^{-m} \geq e^{-m - 2n}$$

And this completes the proof of the theorem.

**Corollary 9.2:** Taking $m = n = 1$ in the theorem, we get the following analogue of a theorem of Khintchine. If $K$ is finite and $\theta \in K\{t\}$, then there exists $C \in K\{t\}$ such that

$$\|\theta u + C\| \cdot |u| \geq e^{-3}$$

for all non-zero $u \in K[t]$.

We shall improve the above result to $e^{-2}$ in the next section.

**3.2 Theorem 10:** Let $K$ be a finite field. Then to every

$\theta \in K\{t\}$, there exists $C \in K\{t\}$ such that

$$|u| \cdot \|\theta u + C\| \geq e^{-2}$$

for all non-zero $u \in K[t]$.

**Remark 3.2:** For the sake of convenience let us restate lemmas of the last section for $m = n = 1$.

**Lemma 3.1':** Let $x_r (r = 1, 2, \ldots)$ be a finite or infinite sequence of non-zero elements of $K[t]$. Define $\phi_r$ by

$$e^{\phi_r} = |x_r|.$$
Suppose

\[ \phi_{r+1} \geq \phi_r + 1 \quad \text{for all } r. \]

Then there exists \( C \in K\{t\} \) such that for every \( r \)

\[ \| x_r C \| = e^{-1}. \]

Lemma 3.2': Define \( \eta(k) \) by

\[
\eta(k) = \min_{x \in K[t]} \| \Theta x \|.
\]

Then we have

(a) \( \eta(k) \) does not increase, if \( k \) increases.

(b) \( e^k \cdot \eta(k) \leq e^{-1} \) for all \( k \).

and hence \( \eta(k) \to 0 \) as \( k \to \infty \).

Lemma 3.3': If a field \( K \) is finite, we can find a sequence of

non-zero elements of \( K[t] \) with \( \phi_r \) defined by

\[
e^{\phi_r} = |x_r|
\]

such that

\[
\eta_1 = 0
\]

\[ \phi_{r+1} \geq \phi_r + 1 \]

\[ \| \Theta x \| = \eta(\phi_{r+1} - 1) \]
The sequence is finite if and only if $\theta \in K(t)$.

Proof of the theorem: Let $x_r$ be the sequence determined by lemma 3.3' and $C$ be given by lemma 3.1'''. Then we have

$$\|x_rC\| = e^{-1} \text{ for all } r.$$ 

Now for any non-zero $u \in K[t]$, we have

$$e^{-1} = \|x_rC\| = \|x_r(C + \theta u) - ux_r\theta\|$$

$$\leq \max (\|x_r(\theta u + C)\|, \|ux_r\theta\|) \quad (3.2.6)$$

$$\leq \max (\|x_r\| \cdot \|\theta u + C\|, |u| \cdot \|x_r\|)$$

Suppose there exists an $r$ such that

$$|u| \cdot \|x_r\theta\| \geq (|u| \cdot \|\theta u + C\|)^{\frac{1}{2}} \geq |u| \cdot \|x_{r+1}\theta\| \quad (3.2.7)$$

From (3.2.1), (3.2.2) and (3.2.5) we get

$$\|x_{r+1}\| \cdot \|x_r\theta\| = e^{r+1} \eta(e^{r+1} - 1) \leq e \cdot e^{-1} = 1 \quad (3.2.8)$$

From (3.2.7) we get

$$\left( |u| \cdot \|\theta u + C\| \right)^{\frac{1}{2}} \leq |u| \cdot \|x_r\theta\|$$

Multiplying both sides by $\frac{1}{|u|^{\frac{1}{2}}}$ and by using
(3.2.8) we get

\[ |x_{r+1}| \cdot ||\theta u + c|| \leq (|u| \cdot ||\theta u + c||)^{1/2} \frac{1}{2} |x_{r+1}| \cdot ||x_r\theta|| \]

\[ \leq (|u| \cdot ||\theta u + c||)^{1/2} \]

(3.2.9)

Let

\[ m = |u| \cdot ||x_{r+1}\theta||, n = |x_{r+1}| \cdot ||\theta u + c|| \]

\[ p^2 = |u| \cdot ||\theta u + c||. \]

Then

\[ m \leq p, \quad (\text{By (3.2.7)}) \]

\[ n \leq p. \quad (\text{By (3.2.9)}) \]

And from (3.2.6) we get

\[ e^{-\frac{1}{2}}||x_{r+1}c|| \leq \max (|x_{r+1}| \cdot ||\theta u + c||, |u| \cdot ||x_{r+1}\theta||) \]

\[ = \max (m, n) \leq p. \]

Therefore

\[ p = (|u| \cdot ||\theta u + c||)^{1/2} \geq e^{-1} \]

or \[ |u| \cdot ||\theta u + c|| \geq e^{-2} \]

and this proves the theorem in this case.

Since \( \eta(k) \to 0 \) as \( k \to \infty \), (3.2.7) is always satisfied, except when

\[ \frac{1}{2} (|u| \cdot ||\theta u + c||)^2 \geq |u| \cdot ||x_l\theta|| \]
If also \((|u| \cdot ||\theta u + C||)^{\frac{1}{2}} \geq |x_1| \cdot ||\theta u + C||\), we can apply the previous argument by taking \(r + 1 = 1\), otherwise we have

\[ (x_1 \cdot ||\theta u + C||)^{\frac{1}{2}} \leq |x_1| \cdot ||\theta u + C|| \]

But \(|x_1| = e^{\Theta} = 1 \leq |u|\) for all non-zero \(u \in K[t]\).

Hence \((|u| \cdot ||\theta u + C||)^{\frac{1}{2}} \leq |u| \cdot ||\theta u + C||\)

or \(|u| \cdot ||\theta u + C|| \geq 1 \geq e^{-2}\)

and the theorem is true in this case also.

3.3: Kronecker's theorem in real number field states:

Let

\[ L_i(U) = L_i(u_1, \ldots, u_n) \quad (1 \leq i \leq m) \]

be \(m\) real homogeneous linear forms in variables \(u_1, \ldots, u_n\).

Then the following two statements about a real vector \(C = (c_1, \ldots, c_m)\) are equivalent.

(a) For each \(\varepsilon > 0\), there is an integral vector \(A = (a_1, \ldots, a_n)\) such that

\[ ||L_j(A) + c_j|| < \varepsilon \quad (1 \leq j \leq n). \]

(b) If \(X = (x_1, \ldots, x_m)\) is any integral vector such that

\[ x_1L_1(U) + \ldots + x_mL_m(U) \]
has integral coefficients, considered as a form in variables
\[ u_j (1 \leq j \leq n) , \text{ then} \]
\[ x_1 c_1 + \ldots + x_m c_m \]
is an integer.

In analogy to the above, we shall prove the following theorems.

**Theorem 11:** Let \( K \) be a finite field. Let
\[ L_i (U) = \sum_{j=1}^{n} \theta_{ij} u_j \ (1 \leq i \leq m) \]  
(3.3.1)
be \( m \) linear forms in variables \( u_1, \ldots, u_n \). Then the following

two statements about \( C = (c_1, \ldots, c_m) \in R_m \) are equivalent:

1. For each integer \( \lambda > 0 \), there is a vector
\[ U = (u_1, \ldots, u_n) \in P_n \] such that simultaneously
\[ || L_i (U) + c_i || \leq e^{-\lambda} \ (1 \leq i \leq m) \]
holds.

2. If \( X = (x_1, \ldots, x_m) \in P_m \) is any vector such that
\[ x_1 L_1 (U) + \ldots + x_m L_m (U) \]
has coefficients in \( K[t] \), considered as a form in variables \( u_1, \ldots, u_n \), then
\[ x_1 c_1 + \ldots + x_m c_m \]
is an element of \( K[t] \).

Condition (2) is equivalent to the following.

\( (2') \): Let \( M_j(X) (1 \leq j \leq n) \) be the system of linear forms transposed to \( L_1(U) (1 \leq i \leq m) \).

If there exists \( X = (x_1, \ldots, x_m) \in P_m \) such that

\[
\| M_j(X) \| = 0 \quad (1 \leq j \leq n),
\]

then

\[
\| XC \| = \| x_1 c_1 + \ldots + x_m c_m \| = 0.
\]

We shall deduce theorem 11 from the following.

Theorem 12: Let \( F_i(U) \) and \( G_i(X) (1 \leq i \leq n) \) be linear forms in variables \( (u_1, \ldots, u_n) = U \) and \( (x_1, \ldots, x_n) = X \) respectively.

Suppose that

\[
\sum_{i=1}^{n} F_i G_i = \sum_{i=1}^{n} u_i x_i
\]  \( (3.3.4) \)

Let \( C = (c_1, \ldots, c_n) \in R_n \). A necessary and sufficient condition that

\[
| C_i + F_i(U) | \leq 1 \quad (1 \leq i \leq n)
\]

for some element \( U \in P_n \) is that
\[ \| \sum_i c_i G_i(X) \| \leq \max_i |G_i(X)| \]  \hspace{1cm} (3.3.6) 

for all elements \( X \in P_n \).

Proof of theorem 12: First we assume that (3.3.5) holds for some \( U \in P_n \). Also

\[ \sum_{i=1}^{n} F_i(U) G_i(X) = \sum_{i=1}^{n} u_i x_i \]

is an element of \( K[t] \) if \( U \in P_n \) and \( X \in P_n \).

Take any element \( X \in P_n \). For this \( X \in P_n \) and \( U \in P_n \) satisfying (3.3.5), we have

\[ \| \sum c_i G_i(X) \| = \| \sum_i G_i(X) (c_i + F_i(U)) \| \]

\[ \leq \max_i \| G_i(X) (c_i + F_i(U)) \| \]

\[ \leq \max_i \left( |G_i(X)| \cdot |c_i + F_i(U)| \right) \]

\[ = \max_i \left( |G_i(X)| \cdot |c_i + F_i(U)| \right) \]

\[ \leq \max_{1 \leq i \leq n} |G_i(X)| \]

and this proves (3.3.6) holds for all \( X \in P_n \).

Now we assume that (3.3.6) holds for all \( X \in P_n \) and prove the existence of \( U \in P_n \) satisfying (3.3.5). We regard

\[ X = (x_1, \ldots, x_n) \]

as a row matrix, \( U = (u_1, \ldots, u_n) \) and
and \( C = (c_1, \ldots, c_n) \) as column matrices.

Let \( G \) be an \( n \times n \) matrix, whose \( i \)-th column consists of the coefficients of \( G_i \) and \( F \) be another \( n \times n \) matrix, whose \( j \)-th row consists of the coefficients of \( F_j \). Then by (3.3.4), we have

\[
F^{-1} = G
\]

so that \( F \) and \( G \) are non-singular matrices. By lemma 1.1, the set

\[
\{ x : |G_i(x)| \leq 1 \ (1 \leq i \leq n) \}
\]

has volume \(|\det G|^{-1}\). Its distance function is

\[
D(X) = \max \{|G_1(X)|, \ldots, |G_n(X)|\}.
\]

By theorem D of chapter 1, there exist elements \( x^{(1)}, \ldots, x^{(n)} \) of \( P_n \) such that

\[
\max \left| G_i(x^{(j)}) \right| = \sigma^{(j)} \text{ (say)} \ (1 \leq j \leq n)
\]

and

\[
\sigma^{(1)} \sigma^{(2)} \ldots \sigma^{(n)} = |\det G|.
\]

Further, if \( M = (x^{(i)}_j)_{i,j} \), \( j = 1, \ldots, n \), then we know

\[
|\det M| = 1. \text{ Consider MGC. This is a column matrix with j-th element} \sum_{i=1}^{n} c_i G_i(x^{(j)}).
\]
Now we write

\[ MGC = Z^{(1)} + R \]  

(3.3.10)

where \( Z^{(1)} \) is a matrix obtained from \( MGC \), by omitting the powers of \( t^{-1}, t^{-2} \) from each entry. Denote by \( r_j \), the \( j \)-th element of the column matrix \( R \). Then,

\[ |r_j| = || \sum_i c_i G_i(x^{(j)}) || \leq \max |G_i(x^{(j)})| \text{ (by hypothesis)} \]

\[ = \sigma^{(j)} \text{ (from (3.3.8))} \]  

(3.3.11).

From (3.3.10) we get

\[ C = FZ^{(2)} + S \]  

(3.3.12)

where

\[ Z^{(2)} = -M^{-1} Z^{(1)}, R = MGS \]  

(3.3.13)

Since \( |M| = 1 \) and entries of \( M \) are from \( K[t] \), \( Z^{(2)} \) is also a matrix with coefficients from \( K[t] \). Now we can solve (3.3.13) for \( s_j \) (i.e. \( j \)-th element of \( S \)) by Cramer's rule. By expanding the determinant so obtained in the numerator and using (3.3.8), (3.3.11), we get

\[ |s_j| \leq \max_j \frac{\sigma^{(1)} \cdots \sigma^{(j-1)} |r_j| \sigma^{(j+1)} \cdots \sigma^{(n)}}{|\det G|} \]

\[ \leq \frac{\sigma^{(1)} \cdots \sigma^{(n)}}{|\det G|} = 1 \text{ (by (3.3.9))} \]  

(3.3.14)
Now theorem follows from (3.3.12) and (3.3.14).

Corollary 12.1: Let

\[ L_i(U) = \sum_{j=1}^{n} \Theta_{ij} u_j \quad (1 \leq i \leq m) \]

and

\[ M_j(X) = \sum_{i=1}^{m} \Theta_{ij} x_i \quad (1 \leq j \leq n) \]

be two sets of linear forms. Let integers \( \lambda \geq 1 \), \( k \geq 0 \) and \( B = (b_1, \ldots, b_m) \in \mathbb{R}_m \) be given. A necessary and sufficient condition that

\[ ||L_i(U) + b_i|| \leq e^{-\lambda}, \quad |u_j| \leq e^k (1 \leq i \leq m, 1 \leq j \leq n) \]

(3.3.15)

for some element \( U \in \mathbb{P}_n \) is that

\[ ||XB|| \leq \max_j (e^k \max_i |M_j(X)|), \quad e^{-\lambda} \max_i |x_i| \]

(3.3.16)

holds for all elements \( X \in \mathbb{P}_m \).

Proof: The corollary is a special case of theorem 12 with

\[
F_i(U, V) = \begin{cases} 
    t^\lambda (L_i(U) + v_i) & (1 \leq i \leq m) \\
    t^{-k} u_i - n & (m + 1 \leq i \leq m + n) 
\end{cases}
\]

and

\[
G_j(X, Y) = \begin{cases} 
    t^{-\lambda} x_j & (1 \leq j \leq m) \\
    t^k (y_{j-n} - M_{j-n}(X)) & (m + 1 \leq j \leq m + n) 
\end{cases}
\]
and $C = (t^λ b_1, \ldots, t^λ b_m, 0, \ldots, 0)$.

Proof of theorem 11: Suppose condition (1) of the theorem holds. Let $X \in P_m$ satisfy (3.3.2). Suppose $\max_{1 \leq i \leq m} |x_i| = e^p$.

Then from condition (1), we get $U \in P_n$ satisfying

$$\|L_i(U) + c_i\| \leq e^{-\lambda - p}, \quad (1 \leq i \leq m)$$

for any arbitrary integer $\lambda \geq 1$. Then from corollary 12.1, we get for some integer $k$,

$$\|XC\| \leq \max \left( e^k \max_{1 \leq j \leq n} \|M_j(X)\|, e^{-\lambda - p} \max_{1 \leq i \leq m} |x_i| \right)$$

If $\max_{j} \|M_j(X)\| = 0$, we have

$$\|XC\| \leq e^{-\lambda - p} \max_{1} |x_i| = e^{-\lambda}$$

for any arbitrary integer $\lambda \geq 1$.

But the left hand side is independent of $\lambda$ and hence

$$\|XC\| = 0.$$ 

This proves that condition (2') holds.

Now suppose condition (2') holds. Since $\|y\| \leq e^{-1}$ for all $y \in K \{t\}$, (3.3.16) trivially holds for all $X \in P_m$ with

$$\max_{i} |x_i| \geq e^{\lambda - 1}.$$ 

We, therefore, need consider only those $X$'s
for which \( \max_i |x_i| \leq e^{\lambda - 2} \).

Since the field \( K \) is finite, there are only a finite number of such \( x_i \)'s belonging to \( K[t] \) and hence only a finite number of \( X \)'s belonging to \( P_m \). For each of these finite number of \( X \)'s, we consider \( \max_j ||M_j(X)|| \).

If \( ||M_j(X)|| = 0 \ (1 \leq j \leq n) \), then by condition (2'), \( ||XC|| = 0 \) and (3.3.16) holds for such \( X \)'s.

If \( ||M_j(X)|| \neq 0 \), we consider minimum of all those such that \( \max_j ||M_j(X)|| \neq 0 \), and choose \( k \) large enough to satisfy

\[
e^k \cdot \max_j ||M_j(X)|| \geq e^{-1}.
\]

Then (3.3.16) is again satisfied for all \( X \in P_m \). Therefore (3.3.15) holds for some \( U \in P_n \) i.e. condition (1) is satisfied. This completes the proof.

Remark 3.3: For the work in real number field, see Khintchine [16].
CHAPTER IV

TRANSFEREN ACE THEOREMS

In this chapter, we propose to show, how information about a problem for a given set of linear forms sometimes gives information about another for a related set of linear forms.

4.1: The results in this section are analogous to those in the real number case proved by Mahler [21] (See also [22]).

Theorem 13: Let $L_1, \ldots, L_n$ and $M_1, \ldots, M_n$ be two sets of $n(>1)$ linearly independent linear forms in variables $(u_1, \ldots, u_n) = U$ and $(x_1, x_2, \ldots, x_n) = X$ respectively.

Let $D$ be the determinant of $M_1, \ldots, M_n$ with $|D| = e^d$.

Let the coefficients of $u_i x_j \ (1 \leq i, j \leq n)$ in

$$\Phi(U, X) = \sum_{i=1}^{n} L_i(U) M_i(X) \quad (4.1.1)$$

be in $K[t]$. Suppose the inequalities

$$|L_i(U)| \leq e^\lambda \ (1 \leq i \leq n) \quad (4.1.2)$$

are solvable with a non-zero element $U \in P_n$ for some fixed integer $\lambda$.

Then we can solve for non-zero $X \in P_n$, the inequalities

$$|M_i(X)| \leq e^\gamma \ (1 \leq i \leq n) \quad (4.1.3)$$
where \( \gamma \) is an integer defined by the inequalities

\[
\gamma \geq \frac{-n + 2 + \delta}{n-1} > \gamma - 1 \quad (4.1.4)
\]

Proof: Since \( L_1, \ldots, L_n \) are linearly independent, the determinant of these forms is not zero and hence, the non-zero vector which satisfies (4.1.2), also satisfies the inequalities

\[
0 < \max_{1 \leq i \leq n} |L_i(U)| \leq e^\lambda
\]

Let \( U \) be one such element of \( P_n \). Interchanging \( L_1, \ldots, L_n \) if necessary, we can assume that

\[
|L_n(U)| = \max_{1 \leq i \leq n} |L_i(U)| = e^{\lambda} \quad (say) \quad (4.1.5)
\]

Clearly \( \lambda \leq \lambda \). \( \quad (4.1.6) \)

Consider now \( n \) linear forms

\[
\overline{\Phi}(U, X), M_1(X), \ldots, M_{n-1}(X)
\]

in variables \( (x_1, \ldots, x_n) = X \). Suppose these forms have determinant \( D' \). Then

\[
D' = \det (M_1, \ldots, M_{n-1}, \overline{\Phi})
= \det (M_1, \ldots, M_{n-1}, \sum_{i=1}^{n} L_i(U) M_i)
= \det (M_1, \ldots, M_{n-1}, L_n(U) M_n)
\]
So

\[ |D'| = | \det(M_1, \ldots, M_{n-1}, L_n(U) M_n) | \]

\[ = |L_n(U)| \cdot | \det(M_1, \ldots, M_n) | = e^{\lambda} \cdot |D| = e^{\lambda + d}. \]

Consider the inequalities

\[
\begin{align*}
|\Phi(u, X)| & \leq e^{-1} \\
|M_i(X)| & \leq e^\gamma (1 \leq i \leq n - 1)
\end{align*}
\]

where \( \gamma \) is defined by (4.1.4).

The sum of the exponents on the right hand side of (4.1.7) is

\[-1 + (n - 1) \gamma \geq -1 + \lambda - n + 2 + d = -n + 1 + \lambda + d \geq -n + 1 + \lambda + d \text{ (by (4.1.6))}\]

i.e.

\[ e^{(n - 1) \gamma} - 1 \geq e^{-(n-1)d}. \]

Hence by theorem A of chapter 1, the inequalities in (4.1.7) have a non-zero solution \( X \in P_n \). Now we assume that \( U \in P_n \) and \( X \in P_n \) chosen above to satisfy (4.1.5) and (4.1.7) respectively are fixed.

Since the coefficients of \( x_i u_j (1 \leq i, j \leq n) \) in \( \Phi(u, X) \) belong to \( K[t] \), we must have

\[ \Phi(u, X) = 0 \]

So

\[ L_n(U) M_n(X) = - \sum_{i=1}^{n-1} L_i(U) M_i(X), \]
or

\[ |L_n(U)M_n(X)| = \left| \sum_{i=1}^{n-1} L_i(U)M_i(X) \right| \leq \max_{1 \leq i \leq n-1} |L_i(U)M_i(X)|, \]

or

\[ e^{\mu} \cdot |M_n(X)| \leq \max_{1 \leq i \leq n-1} |L_i(U)| \cdot |M_i(X)| \leq e^\mu \cdot e^\gamma \quad \text{(By (4.1.5) and (4.1.2))}, \]

or

\[ |M_n(X)| \leq e^\gamma, \quad (4.1.8) \]

i.e. the non-zero vector \( X \in \mathbb{P}_n \) satisfying (4.1.7) also satisfies (4.1.8). This proves the theorem.

Remark 4.1: The result of theorem 13 is best possible. For example take \( L_i(U) = u_i, M_i(x) = x_i \) (\( i = 1, \ldots, n \)), \( \lambda = 0 \). Then we get \( \gamma = 0 \), which is obviously best possible.

Theorem 14: Let

\[ L_i(U) = \sum_{j=1}^{m} \theta_{ij}u_j \quad (1 \leq i \leq m) \]

and

\[ M_j(x) = \sum_{i=1}^{m} \theta_{ji}x_i \quad (1 \leq j \leq n) \]

be two sets of linear forms. Suppose

\[ ||L_i(U)|| \leq e^{-\lambda}, \quad |u_j| \leq e^k \quad (1 \leq i \leq m, 1 \leq j \leq n) (4.1.9) \]

are solvable in non-zero elements \( U \in \mathbb{P}_n \) for some integers \( \lambda \geq 1, k \geq 0 \). Then there exist non-zero vectors \( X \in \mathbb{P}_n \) such that

\[ ||M_j(X)|| \leq e^{\gamma-k}, \quad |x_i| \leq e^{\gamma+\lambda} \quad (1 \leq i \leq m, 1 \leq j \leq n) \quad (4.1.10) \]
where $\gamma$ is an integer defined by the inequalities

$$\gamma \geq \frac{n(k - 1) + 2 - m(\lambda + 1)}{m + n - 1} \geq \gamma - 1 \quad (4.1.11)$$

Proof: We introduce the new variables

$$V = (v_1, \ldots, v_m) \quad \text{and} \quad Y = (y_1, \ldots, y_n).$$

Consider the forms

$$L_i'(U, V) = \begin{cases} t^\lambda (L_i(U) + v_i) & (1 \leq i \leq m) \\ t^{-k} u_{i-m} & (m + 1 \leq i \leq m + n), \end{cases}$$

and

$$M_j'(X, Y) = \begin{cases} t^\lambda x_j & (1 \leq j \leq m) \\ t^{-k} (-M_j(X) + y_{j-m}) & (m + 1 \leq j \leq m + n). \end{cases}$$

Obviously each of the systems $L_i'(U, V) \quad (i=1, \ldots, m+n)$ and $M_j'(X, Y) \quad (j=1, \ldots, m+n)$ is linearly independent. Let $D$ be the determinant of $M_j'(X, Y) \quad (1 \leq j \leq m+n)$. Then $|D| = e^{nk-m\lambda}$.

Now

$$\sum_{h=1}^{m+n} L_h' M_h' = \sum_{i=1}^{m} v_i x_i + \sum_{j=1}^{n} u_j y_j$$

has coefficients in $K[t]$. Also by hypothesis

$$|L_i'(U, V)| \leq 1$$
can be solved in non-zero elements \((U, V) \in P_{m+n}\) by theorem 13 with \(\lambda = 0\), \(d = nk - m\lambda\) and \(m + n\). For \(n\), we find a non-zero solution \((X, Y) \in P_{m+n}\) of the inequalities
\[
|M_j' (X, Y)| \leq e^\gamma \quad (1 \leq j \leq m + n)
\]
(4.1.12)
where \(\gamma\) is defined by (4.1.11). This means that
\[
|x_1| \leq e^{\gamma + \lambda}, \quad |M_j(X) + y_j| \leq e^{\gamma - k} (1 \leq i \leq m, 1 \leq j \leq n)
\]
can be solved in non-zero \(X \in P_{m+n}\).

If \(\gamma - k < 0\) and \(X = 0\), we must have \((X, Y) = (0, 0)\), which contradicts the fact that \((X, Y) = (0, 0)\). Hence either \(X \neq 0\) or \(\gamma - k \geq 0\). Since \(X \neq 0\) implies the theorem, we can suppose \(\gamma - k \geq 0\). Now,
\[
\gamma + \lambda + 1 \geq \frac{n(x-1) + 2 - m(\lambda+1)}{m + n - 1} + 1
\]
\[
= \frac{n(x-1) + 2 - m(\lambda+1) + m(\lambda+1) + (n-1)\lambda + n - 1}{m + n - 1}
\]
\[
= \frac{n \lambda + 1 + (n-1)}{n + n - 1} > 0.
\]
Also \(\gamma\) and \(\lambda\) are integers, so that \(\gamma + \lambda + 1 \geq 1\) and \(\gamma + \lambda \geq 0\). Every \(x\) with \(|x_1| \leq 1\) is a solution of (4.1.10), since \(\gamma - k \geq 0\), \(\gamma + \lambda \geq 0\).

Corollary 14.1: A necessary and sufficient condition that there exists an integer \(\alpha\) such that
\[(\max ||L_1(U)||)^m \cdot (\max |u_j|)^n \geq e^\alpha \quad (4.1.13)\]

for all non-zero elements \( U \in P_n \) is that there exists an integer \( \beta \) such that

\[(\max ||M_j(X)||)^n \cdot (\max |x_i|)^m \geq e^\beta \quad (4.1.14)\]

for all non-zero elements \( X \in P_m \).

Proof: We shall prove that if there exists an integer \( \beta \) satisfying (4.1.14) for all non-zero \( X \in P_m \), then there exists an integer \( \alpha \) satisfying (4.1.13) for all non-zero \( U \in P_n \). The other part is similar.

Let \( U \in P_n \) be any non-zero vector. Let

\[e^k = \max_j |u_j|, \quad e^{-\lambda} \max_i |L_i(U)|,\]

Then \( k \geq 0 \) and we can assume \( \lambda \geq 1 \). Thus

\[\max_{1 \leq j \leq n} |u_j| \leq e^k, \quad \max_{1 \leq i \leq m} ||L_i(U)|| \leq e^{-\lambda}\]

has a solution in non-zero \( U \in P_n \). Using theorem 14, we get a non-zero \( X \in P_m \) satisfying (4.1.10), where \( \gamma \) is an integer defined by (4.1.11).

Since (4.1.14) is true for every non-zero \( X \in P_m \), we have for non-zero \( X \in P_m \) chosen above
\[(e^{\gamma-k})^n \cdot (e^{\gamma+\lambda})^m \geq (\max ||M_j(x)||)^n \cdot (\max |x_1|)^m \geq e^\beta \]

or \[n(\gamma-k) + m(\gamma+\lambda) \geq \beta \]

or \[\gamma + (m+n-1)\gamma - nk + m\lambda \geq \beta \]

or \[\frac{n k - m\lambda + 1}{m + n - 1} + n k - m\lambda + 1 - nk + m\lambda \geq \beta , \]

since by (4.1.11) we have \[\gamma < (\frac{nk - m\lambda + 1}{m + n - 1}) \]

or \[nk - m\lambda + 1 \geq (\beta - 1)(m + n - 1) \]

or \[nk - m\lambda \geq (\beta - 1)(m + n - 1) - 1 = \alpha \text{ (say)}. \]

With this choice of \(\alpha\), our assertion follows.

Theorem 15: Let \(\theta_1, \ldots, \theta_m\) be any \(m\) numbers in an algebraic field of degree \((m+1)\) over \(K(t)\), such that \(K(\theta_1, \ldots, \theta_m)\) is a subfield of \(K\{t\}\). Suppose that \(\theta_1, \ldots, \theta_m\) are linearly independent over \(K(t)\). Then there is an integer \(\alpha\) depending only on \(\theta_1, \ldots, \theta_m\) such that

\[(\max_{1 \leq i \leq m} ||u\theta_i||)^m \cdot |u| \geq e^\alpha \quad (4.1.15)\]

for all non-zero elements \(u\) of \(K[t]\).

Proof: By corollary 4.1, it is enough to show the existence of an integer \(\beta\) such that

\[||x_1\theta_1 + \ldots + x_m\theta_m|| \cdot (\max_{1 \leq i \leq m} |x_i|)^m \geq e^\beta \]
\[ ||x_1^{\theta_1} + \cdots + x_m^{\theta_m}|| \geq e^\beta \cdot (\max_{1 \leq i \leq m} |x_i|)^{-m} \quad (4.1.16) \]

for all non-zero elements \( x \in P_m \).

Take a non-zero \( x \in P_m \). We can write left hand side of (4.1.16) as

\[ |x_1^{\theta_1} + \cdots + x_m^{\theta_m} + y| \]

for some element \( y \in K[t] \), so that

\[ |x_1^{\theta_1} + \cdots + x_m^{\theta_m} + y| \leq e^{-1}. \quad (4.1.17) \]

Since \( \theta_1, \ldots, \theta_m \) belong to an algebraic field of finite degree, there exists \( q \in K[t] \), such that \( q \theta_1, \ldots, q \theta_m \) are algebraic integers over \( K(t) \). Let

\[ \phi = q y + q x_1^{\theta_1} + \cdots + q x_m^{\theta_m}. \quad (4.1.18) \]

Then \( \phi \) is an algebraic integer, non-zero by hypothesis. Let

\[ \phi' = q y + q x_1^{\theta_1'} + \cdots + q x_m^{\theta_m'} \quad (4.1.19) \]

be any one of the other \( m \) algebraic conjugates of \( \phi \).

Then

\[ |\phi'| \leq \max (|\phi|, |\phi' - \phi|) \]

\[ \leq \max (e^{-1} \cdot |q_1|, |q_1(\phi_1 - \phi_1')|, \ldots, |q_m(\theta_m - \theta_m')|) \]

(By (4.1.17), (4.1.18) and (4.1.19))

\[ \leq c(\max |x_i|), \]
where \( c \) is independent of \( x_i \) and depends on \( \theta_1, \ldots, \theta_m \) only. On the other hand, the product of \( \phi \) with its other conjugates is a non-zero element of \( K[t] \) and hence valuation must be \( \geq 1 \).

So

\[
|\phi| \cdot (c \max |x_i|)^m \geq 1,
\]

or

\[
|q| \cdot |x_1 \theta_1 + \cdots + x_m \theta_m + q| \cdot c^m \cdot (\max |x_i|)^m \geq 1,
\]

or

\[
|\|x_1 \theta_1 + \cdots + x_m \theta_m\|| \cdot (\max |x_i|)^m \geq c^{-m} |q|^{-1}.
\]

This proves the theorem with any \( \beta \) such that \( e^\beta \leq c^{-m} |q|^{-1} \).

Remark: Such fields do exist.

4.2 In this section, we shall prove a few results, in which certain information about a set of homogeneous linear forms is employed to get information about a related set of inhomogeneous linear forms. The corresponding results in real number field were proved by Hlawka [12].

Theorem 16: Let \( L_i(U) = L_i(u_1, \ldots, u_n) (1 \leq i \leq n) \) be a set of linear forms with determinant \( D \neq 0 \). Let \( |D|= e^d \). Let \( r_1, \ldots, r_n \) be any set of integers such that

\[
r_1 + \cdots + r_n = d - n \quad (4.2.1)
\]

Suppose

\[
|L_i(U)| < e^{r_i+1} \quad (i = 1, \ldots, n) \quad (4.2.2)
\]

have no solution in non-zero elements of \( P_n \). Then for all elements
(c_1, \ldots, c_n) \in \mathbb{R}_n$, there is a solution of

\[ |L_i(U) + c_i| \leq e^{r_i} \] (i = 1, \ldots, n) \quad (4.2.3)

from elements of \( P_n \).

Proof: Consider \( n + 1 \) linear forms

\[ L_i(U) + c_iZ, \quad Z (i = 1, \ldots, n) \]

in \((n + 1)\) variables \( u_1, \ldots, u_n, Z \) where \( Z \) is a new variable. These forms have determinant \( D \) and \( |D| = e^d \). By theorem A of chapter 1, we must have a solution \((u_1, \ldots, u_n, Z) \neq 0\) from elements of \( P_{n+1} \) for the inequalities

\[ |L_i(U) + c_iZ| \leq e^{r_i}, \quad (1 \leq i \leq n) \] (4.2.4)

\[ |Z| \leq 1. \] (4.2.5)

Now we assert that \( Z \neq 0 \), for otherwise we shall get a non-zero solution \( U \in P_n \) for

\[ |L_i(U)| \leq e^{r_i} < e^{r_i + 1}, \]

contrary to hypothesis. Also from (4.2.5), we have \( Z = k \), where \( k \in K \). Now suppose \((u_1, \ldots, u_n, k)\) is a solution of (4.2.4) and (4.2.5) where \( u_i \in K[t] \) \( i = 1, \ldots, n \) and \( k \in K, k \neq 0 \).

Then \( (\frac{u_1}{k}, \ldots, \frac{u_n}{k}) \) is a solution of (4.2.3).
Corollary 16.1: Let $L_i(U) = L_i(u_1, \ldots, u_n)$ $(1 \leq i \leq m)$ be $m$ linear forms. Suppose that there is no non-zero element of $P_n$ satisfying

$$||L_i(U)|| < e^{\lambda_i + 1}, |u_j| < e^{k_j + 1}$$

(4.2.6)

where

$$\sum_{i=1}^{m} \lambda_i + \sum_{j=1}^{n} k_j = -m - n$$

(4.2.7)

Then for all elements $(c_1, \ldots, c_m) \notin R_m$, the inequalities

$$||L_i(U) + c_i|| \leq e^{\lambda_i}, |u_j| \leq e^{k_j} (1 \leq i \leq m, 1 \leq j \leq n),$$

(4.2.8)

have a solution from element of $P_n$.

Proof: We introduce new variables $v_1, \ldots, v_m$. Consider the inequalities

$$|L_i(U) - v_i| < e^{\lambda_i + 1}, |u_j| < e^{k_j + 1} (1 \leq i \leq m, 1 \leq j \leq n)$$

(4.2.9)

These are $m + n$ forms in $m + n$ variables $u_1, \ldots, u_n, v_1, \ldots, v_m$ and have the determinant $+1$, so that $|D| = 1$.

Also owing to the hypothesis, we see that the inequalities in (4.2.9) have no solution from elements of $P_{m+n}$ other than $(U, V) = 0$. Hence applying theorem 16, we get that

$$|L_i(U) - v_i + c_i| \leq e^{\lambda_i}, |u_j| \leq e^{k_j}$$
have a solution from elements of \( P_m + n \), so that (4.2.8) have a solution from elements of \( P_n \).

**Theorem 17:** Let

\[
L_i(U) = \sum_{j=1}^{n} \theta_{ji} u_j \quad (1 \leq i \leq m),
\]

\[
M_j(X) = \sum_{i=1}^{m} \theta_{ji} x_i \quad (1 \leq j \leq n)
\]

be two sets of linear forms in variables \((u_1, \ldots, u_n) = U\) and \((x_1, \ldots, x_m) = X\) respectively. For all elements \((c_1, \ldots, c_m) \in \mathbb{R}_m\), let there be elements \(U \in P_n\) satisfying the inequalities

\[
||L_i(U) + c_i|| \leq e^{\lambda_i}, \quad |u_j| < e^{\gamma_j} \quad (1 \leq i \leq m, 1 \leq j \leq n). \tag{4.2.10}
\]

Then there is no non-zero element \(X \in P_m\) satisfying the inequalities

\[
||M_j(X)|| \leq e^{-\gamma_j}, \quad |x_i| < e^{-\lambda_i - 1} \quad (1 \leq i \leq m, 1 \leq j \leq n) \tag{4.2.11}
\]

**Proof:** Suppose there exists a non-zero element \(X \in P_n\) satisfying (4.2.11). Then we can choose elements \((c_1, \ldots, c_m) \in \mathbb{R}_m\) such that

\[
\sum_{i=1}^{m} c_i x_i = t^{-1}. \tag{4.2.12}
\]

First we note that

\[
\sum_{j=1}^{n} u_j M_j(X) = \sum_{i=1}^{m} x_i L_i(U). \tag{4.2.13}
\]
For $c_1, \ldots, c_m$ selected above, let $U = (u_1, \ldots, u_n)$ be a solution of inequalities in (4.2.10). Now from (4.2.12) and (4.2.13) we have

$$e^{-1} = \left| \sum_{i=1}^{m} c_i x_i \right| \leq \max \left( \left| \sum_{i=1}^{m} x_i (c_i + L_i(U)) \right|, \left| \sum_{i=1}^{n} u_i M_j(X) \right| \right)$$

$$= \max \left( \left| \sum_{i=1}^{m} x_i (c_i + L_i(U)) \right|, \left| \sum_{j=1}^{n} u_j M_j(X) \right| \right)$$

$$\leq \max_{1 \leq i \leq m, 1 \leq j \leq n} \left( \left| x_i \right| \cdot \left| c_i + L_i(U) \right|, \left| u_j \right| \cdot \left| M_j(X) \right| \right) < e^{-1}$$

which is a contradiction and the theorem follows.
CHAPTER V
INHOMOGENEOUS MINIMA OF BINARY QUADRATIC,
TERNARY CUBIC AND QUATERNARY QUARTIC FORMS

Let \( L_1, L_2', \ldots, L_r + 2s \) be linear forms in \( r + 2s = n \) variables such that \( L_1, \ldots, L_r \) are real, \( L_r + 1', \ldots, L_r + s \) have complex coefficients and \( L_r + s + 1', \ldots, L_r + 2s \) are complex conjugates to \( L_r + 1', \ldots, L_r + 2s \), respectively. Let \( \Delta \neq 0 \) be the determinant of these forms. Then Davenport has proved the existence of positive constants \( c_r, s \) and reals \( \xi_1', \ldots, \xi_n \) such that

\[
|L_1L_2' \cdots L_n| \geq \frac{|\Delta|}{c_{r, s}}
\]

for all \( (x_1, \ldots, x_n) = (\xi_1', \ldots, \xi_n) \) (mod 1) in the cases \( r = 2, s = 0; r = 1, s = 1 \) and \( r = 0, s = 2 \). He proved the theorems with \( c_2, 0 = 128, c_{1, 1} = 8 \times 10^{13}, c_0, 2 = 10^{132} \).

Cassels gave an alternative proof with \( c_2, 0 = 48, c_{1, 1} = 420 \) and \( c_0, 2 = 5300 \). The best value of these \( c \)'s, however, are not known.

We prove the analogues of these results with the best possible values of the corresponding \( c \)'s. Our proof is based upon that of Cassels [2].
Earlier, Armitage gave a proof based on that of Davenport in the case of quadratic and cubic forms, but he was interested only in the cases where $L_1L_2 \ldots , L_n$ does not represent zero non-trivially i.e. $L_1L_2 \ldots , L_n$ is not zero for $x_1, \ldots , x_n \notin K[t]$ and not all zero. If the product $L_1L_2 \ldots , L_n$ has coefficients in $K(t)$, he proved that $\xi_1, \ldots , \xi_n$ can also be chosen in $K(t)$. We do not do so.

In this chapter $K$ will be a finite field, unless mentioned otherwise.

5.1: First we prove an analogue of Weierstrass' theorem in the field of Laurent Series (Shilling [24]).

Definition: A sequence $\{x_n\}$ from elements of $K\{t\}$ is said to be bounded if $\{|x_n|\}$ is bounded.

Theorem 18: Every infinite bounded sequence in $K\{t\}$ has a convergent subsequence.

Proof: Let $S$ be the given sequence. Since $S$ is bounded, there exists an integer $M$ such that

$$|x| \leq e^M$$

for all $x \in S$.

Therefore, all elements $x$ of $S$ are of the form

$$x = a_M t^M + a_{M-1} t^{M-1} + \ldots$$

where $a_i \in K$ ($i = M, M-1, \ldots$), $a_M$ may be zero.
Now \( \mathbb{F} \subset K \) and \( K \) is a finite field, so there are only a finite number of choices for \( a_{M} \). Consider an infinite subsequence \( S' \) of \( S \) with the same \( a_{M} \), say \( a_{M}^{-} \). Take \( x_{1} \) to be the first element of \( S' \). From \( S' \), consider an infinite subsequence \( S'' \) of \( S' \) with the same \( a_{M-1} \), say \( a_{M-1}^{-} \). Take \( x_{2} \) to be the first element of \( S'' \) and so on. Then

\[
a_{M}^{-} t^{M} + a_{M-1}^{-} t^{M-1} + \ldots \]

is the limit of the subsequence \( \{ x_{i} \} \) of \( S \).

**Theorem 19:** Let \( f(x, y) \) be a binary quadratic form defined by

\[
f(x, y) = a(x - \Theta y) (x - \Phi y), \quad a \neq 0
\]

where \( \Theta, \Phi \in K\{t\} \), \( \Theta \neq \Phi \). Then there exist \( x_{0}, y_{0} \in K\{t\} \), such that for all \( (x, y) \equiv (x_{0}, y_{0}) \pmod{P_{2}} \), we have

\[
|f(x, y)| \geq \frac{|a| \cdot |\Theta - \Phi|}{e^2}.
\]

**Remark:** Without loss of generality, we can assume \( a = 1 \). Then we have

\[
f(x, y) = (x - \Theta y) (x - \Phi y), \quad (5.1.1)
\]

and we have to prove that there exists \( (x_{0}, y_{0}) \) such that for all \( (x, y) \equiv (x_{0}, y_{0}) \pmod{P_{2}} \), we have

\[
|f(x, y)| \geq \frac{|\Theta - \Phi|}{e^2}. \quad (5.1.2)
\]
Lemma 5.1: Let \( \left\{ \lambda_n \right\} = \left\{ (\lambda_{n1}, \ldots, \lambda_{nr}) \right\} \) for \( n = 1, 2, \ldots \) be a finite or infinite sequence of non-zero vectors with \( \lambda_{ni} \in K \{ t \} \) \( (i = 1, 2, \ldots, r, n = 0, 1, 2, \ldots) \).

Define \( \xi_n \) by

\[
e^\xi_n = \max(|\lambda_{n1}|, \ldots, |\lambda_{nr}|).
\]

Suppose \( \xi_{n+1} > \xi_n \) for all \( n \). Let \( \mu_0, \mu_1, \ldots \) be an arbitrary sequence in \( K \{ t \} \). Then there exist \( \xi = (\xi_1, \ldots, \xi_r) \in R_r \) such that

\[
\| \lambda_{n1} \xi_1 + \ldots + \lambda_{nr} \xi_r + \mu_n \| = e^{-1}
\]

(5.1.3)

for all \( n \).

Proof of lemma 5.1: We shall prove the lemma for \( r = 2 \). The proof for general \( r \) is similar. Suppose

\[
\lambda_{n1} = a_n^{(n)} t^n + a_{n-1}^{(n)} t^{n-1} + \ldots, \quad n = 0, 1, 2, \ldots,
\]

\[
\lambda_{n2} = b_n^{(n)} t^n + b_{n-1}^{(n)} t^{n-1} + \ldots,
\]

where at least one of \( a_n^{(n)}, b_n^{(n)} \) is not zero, and \( a \)'s, \( b \)'s belong to \( K \).

Without loss of generality, we can take the degree of \( \mu_n \) to be at most \(-1\).
Let

\[ \mu_n = r_{-1}^{(n)} t^{-1} + r_{-2}^{(n)} t^{-2} + \ldots, \quad r_{-j}^{(n)} \in K; \quad j = 1, 2, \ldots. \]

We shall take

\[ \xi_1 = \alpha_{-\zeta_0-1} t^{-\zeta_0-1} + \alpha_{-\zeta_0-2} t^{-\zeta_0-2} + \ldots, \]
\[ \xi_2 = \beta_{-\zeta_0-1} t^{-\zeta_0-1} + \beta_{-\zeta_0-2} t^{-\zeta_0-2} + \ldots, \]

where \( \alpha \)'s and \( \beta \)'s are to be determined. We shall choose \( \alpha \)'s and \( \beta \)'s inductively.

Coefficient of \( t^{-1} \) in \( \lambda_{01} \xi_1 + \lambda_{02} \xi_2 + \mu_0 \)

\[ = a^{(0)}_{\zeta_0} \alpha_{-\zeta_0-1} + b^{(0)}_{\zeta_0} \beta_{-\zeta_0-1} + r^{(0)}_{-1} \quad (5.1.4) \]

Since at least one of \( a^{(0)}_{\zeta_0}, b^{(0)}_{\zeta_0} \) is not equal to zero,

we can choose \( \alpha_{-\zeta_0-1}, \beta_{-\zeta_0-1} \) such that the right hand side of \( (5.1.4) \) is not zero. Suppose we have selected already

\( \alpha_{-\zeta_0-1}, \ldots, \alpha_{-\zeta_k-1}, \beta_{-\zeta_0-1}, \beta_{-\zeta_k-1} \) so that coefficient of \( t^{-1} \) in \( \lambda_{k1} \xi_1 + \lambda_{k2} \xi_2 + \mu_k \) is not zero. Then to complete the induction, we have simply to choose \( \alpha_{-\zeta_k-2}, \ldots, \alpha_{-\zeta_k-1}, \ldots, \beta_{-\zeta_k-2}, \ldots, \beta_{-\zeta_{k+1}-1} \) is not equal to zero. Choose
\[ \alpha^2, \ldots, \alpha_{, k+1}, \beta^2, \ldots, \beta_{, k+1} \]

arbitrarily and choose \( \alpha_{, k+1}, \beta_{, k+1} \) in such a way that coefficient of \( t^{-1} \) in

\[
\lambda_{k+1, 1}^2 + \lambda_{k+1, 2}^2 + \mu_{k+1}
\]

\[
= a_{k+1}^2 - \alpha_{, k+1}^2 + a_{k+1} \alpha_{, k+1} + \cdots + a_0 \alpha_{, 0} - 1
\]

is not zero. Since at least one of \( a_{k+1}, b_{k+1} \) is not zero, such a selection is possible. Thus the induction is completed and the lemma is proved.

Now we distinguish the following three cases.

(i) \( \emptyset \in K(t), \emptyset \in K(t) \); (ii) \( \emptyset \in K(t), \emptyset \notin K(t) \)

(iii) \( \emptyset \notin K(t), \emptyset \notin K(t) \).

Lemma 5.2: Let \( M > 0 \) be a given integer. Then there exists a finite sequence \( \gamma_M \) of pairs \((u_0, v_0), (u_1, v_1), \ldots, (u_N, v_N)\) of relatively prime polynomials (i.e. elements of \( K[t] \)) such that

(i) \[ |u_n - v_n \theta| \cdot |u_n - v_n \phi| \leq e^{-1} |\theta - \phi| . \]

(ii) \[ |u_n - v_0 \theta| \leq e^{-1} |u_{n-1} - v_{n-1} \theta| \quad (n \geq 1) . \]

(iii) \[ |u_n - v_0 \phi| \cdot |u_{n-1} - v_{n-1} \phi| \leq |\theta - \phi| \quad (n \geq 1) . \]

(iv) \[ |u_0 - v_0 \phi| \leq e^{-M}, \quad |u_0 - v_0 \theta| \geq e^M . \]
(v) \[ |u_n - v_n \theta| \leq e^{-M} \]

(vi) \[ |u_n - v_n \phi| \geq |u_{n-1} - v_{n-1} \phi| \quad (n \geq 1). \]

Proof of lemma 5.2: Consider the solutions of

\[
\begin{align*}
|u - v \phi| &\leq e^{-M} \\
|u - v \theta| &\leq \max (e^M, e^M - 1 | \theta - \phi|).
\end{align*}
\]

(5.1.5)

By theorem A of chapter 1, these inequalities have a non-zero solution \((u, v) \in P_2\). Also by lemma 1.1, the number of these solutions is finite. Consider

\[
\min |u - v \phi|,
\]

where the minimum is taken over all \((u, v) \neq (0, 0)\) in \(P_2\) satisfying (5.1.5) this minimum is attained. Since \(\phi \notin K(t)\), this minimum is not zero. Let this minimum be \(e^{-T}\). Then \(T \geq M\).

Consider the solutions of

\[
\begin{align*}
|u - v \phi| &\leq e^{-T-1} \\
|u - v \theta| &\leq e^T | \theta - \phi|.
\end{align*}
\]

(5.1.6)

Again by theorem A of chapter 1, these inequalities have a non-zero solution from \(P_2\) and again as before the number of solutions is finite. Hence

\[
\min |u - v \phi|,
\]

where the minimum is taken over all \((u, v) \neq (0, 0)\) in \(P_2\).
satisfying (5.1.6) is attained, say at \((u_o, v_o)\). Then

\[
|u_o - v_o \phi| \leq e^{-T-1} < e^{-M}.
\]

Further we assert that we must have

\[
|u_o - v_o \Theta| \geq e^M.
\]

Otherwise, \((u_o, v_o)\) will satisfy (5.1.5) and then \(|u_o - v_o \phi|\)
will violate the minimality of \(T\). (i) is trivially satisfied for \(n = 0\). Now we claim that if there exist non-zero relatively positive polynomials \((u, v)\) satisfying

(1) \( |u - v\Theta| \cdot |u - v\phi| \leq e^{-1} |\Theta - \Phi| \)

(2) \( |u - v\Theta| \leq e^{-1} |u_o - v_o \Theta| \)

(3) \( |u - v\phi| \cdot |u_o - v_o \Theta| \leq |\Theta - \Phi| \), then we must have

(4) \( |u - v\phi| \geq |u_o - v_o \phi| \).

For, if not, we have

(5) \( |u - v\phi| < |u_o - v_o \phi| \),

and (2) and (5) show that this \((u, v)\) satisfies (5.1.6). But then (5) contradicts minimal nature of \(|u_o - v_o \phi|\) and hence (4) is satisfied. Thus if we get \((u_1, v_1)\) satisfying (i), (ii), (iii) of lemma 5.2, (vi) is automatically satisfied for \(n = 1\). Now suppose we have constructed \((u_{n-1}, v_{n-1})\).
Define \((u_n, v_n)\) by

\[
|u_n - v_n \theta| \leq e^{-1} |u_{n-1} - v_{n-1} \theta| \\
|u_n - v_n \phi| \leq \frac{|\theta - \phi|}{|u_{n-1} - v_{n-1} \theta|}
\]

and \(|u_n - v_n \phi| = \text{min} \ |u - v \phi|\), where minimum is taken over all \((u, v) \neq (0, 0)\) in \(P_2\) satisfying (5.1.7).

Since (5.1.7) has at least one non-zero solution belonging to \(P_2\) and the number of such solutions is finite, such a \((u_n, v_n)\) must exist. Then (vi) is automatically satisfied for \(n\). Otherwise

\[
|u_n - v_n \phi| < |u_{n-1} - v_{n-1} \phi|
\]

and we could have selected \((u_n, v_n)\) at the previous stage.

Since \(|u_n - v_n \theta| \leq e^{-1} |u_{n-1} - v_{n-1} \theta|\), after a certain stage onwards, we must have

\[
|u_n - v_n \theta| \leq e^{-M}
\]

and we stop at one such \(N\).

Remark: 1. Clearly \(\mathcal{J}_M\) is not unique.

2. If \(M_1 > M_2\), every \(\mathcal{J}_{M_1}\) is a \(\mathcal{J}_{M_2}\) also.

Definition 5.1: We now define a bilinear form

\[
F(x, y; u, v) = (x - y \theta) (u - v \phi) - (x - y \phi) (u - v \theta) \\
= (\theta - \phi) (xv - yu).
\]

(5.1.8)
Lemma 5.3: There exists \((x_0^{\text{(M)}}, y_0^{\text{(M)}}) \in \mathbb{R}_2\) such that

\[
\left| F(x, y; u_n, \nu_n) \right| \geq e^{-1} |\theta - \phi| 
\]

(5.1.9)

for all \((x, y) \equiv (x_0^{\text{(M)}}, y_0^{\text{(M)}}) \pmod{P_2}\) and all pairs \((u_n, \nu_n) (n = 0, 1, \ldots, N)\).

Proof: In lemma 5.1, take \(r = 1, \Lambda_n = \lambda_n = \frac{u_{N-n} - v_{N-n} \theta}{\theta - \phi}\), \(\mu_n = 0 (n = 0, 1, \ldots, N)\). Then by condition (ii) of lemma 5.2, the required condition of lemma 5.1 is satisfied and we get \(\xi \in K \{t\}\) such that

\[
\left| \frac{u_n - v_n \theta}{\theta - \phi} \xi \right| = e^{-1}, (n = 0, 1, \ldots, N).
\]

Now define \(x_0^{\text{(M)}}, y_0^{\text{(M)}}\) by the equations

\[
x_0^{\text{(M)}} - y_0^{\text{(M)}} \theta = 0, \quad x_0^{\text{(M)}} - y_0^{\text{(M)}} \phi = -\xi.
\]

Then

\[
F(x_0^{\text{(M)}}, y_0^{\text{(M)}}, u_n, \nu_n) = (u_n - v_n \theta) \xi.
\]

Further, if \((x, y) \equiv (x_0^{\text{(M)}}, y_0^{\text{(M)}}) \pmod{P_2}\), we have

\[
F(x, y; u_n, \nu_n) - F(x_0^{\text{(M)}}, y_0^{\text{(M)}}, u_n, \nu_n)
= (\theta - \phi) [v_n(x - x_0^{\text{(M)})} - u_n(y - y_0^{\text{(M)})}]
= (\theta - \phi) g_n(t),
\]
where \( g_n(t) \in K[t] \). Hence
\[
\frac{F(x, y; u_n, v_n)}{\theta - \phi} = \frac{F(x_0(M), y_0(M); u_n, v_n)}{\theta - \phi} = g_n(t),
\]
and
\[
\|F(x, y; u_n, v_n)\| = \|F(x_0(M), y_0(M); u_n, v_n)\| = \|\frac{(u_n - v_n \theta)\xi}{\theta - \phi}\| = e^{-1} \quad (n = 0, 1, \ldots, N),
\]
and
\[
|F(x, y; u_n, v_n)| \geq e^{-1} |\theta - \phi| (n = 0, 1, \ldots, N),
\]
So \(|x u_n + y u_n| \geq e^{-1} (n = 0, 1, \ldots, N)|.

This proves the lemma.

**Lemma 5.4:** There exists a doubly infinite sequence of pairs of relatively prime polynomials \((u_n, v_n) (-\infty < n < \infty)\), such that (i), (ii), (iii) and (vi) of lemma 5.2 hold and
\[
\lim_{n \to \infty} |u_n - v_n \theta| = 0 = \lim_{n \to -\infty} |u_n - v_n \phi|
\]
Also
\[
\lim_{n \to \infty} |u_n - v_n \phi| = \infty = \lim_{n \to -\infty} |u_n - v_n \theta|.
\]
Further, there is an \((x_0, y_0)\), such that (5.1.9) holds for all \((x, y) = (x_0, y_0) \mod P_2\) and all \(n\).

**Proof:** Let \(L\) be any given integer. First we assert there are only a finite number of \((u, v), (u', v') \in P_2\) satisfying
\[
\begin{align*}
(a) & \quad |u - v \theta| \cdot |u - v \phi| \leq e^{-1} |\theta - \phi| \\
(b) & \quad |u' - v' \phi| \cdot |u - v \theta| \leq |\theta - \phi| \\
(c) & \quad |u' - v' \theta| \leq e^{L}|u - v \theta|.
\end{align*}
\]
From (b) and (c) we get

\[ |u' - v' \theta| \leq e^L \]

and

\[ |u' - v' \phi| \leq |u - v \theta|^{-1} |\theta - \phi| \leq e^{-L} |\theta - \phi|; \]

By lemma 1.1, there can be only a finite number of choices for \((u', v') \in P_2\). Now given \((u', v')\) we assert these are only a finite number of choices for \((u, v) \in P_2\). From (a), (b), (c) we get

\[ |u - v \theta| \leq |u' - v' \phi|^{-1} |\theta - \phi| \text{ (Since } \phi \notin K(t), u' - v' \phi \neq 0) \]

\[ |u - v \phi| \leq e^{-1} |\theta - \phi| |u - v \theta|^{-1} \leq e^{-L-1} |\theta - \phi|. \]

and the assertion again follows from lemma 1.1.

Now suppose \(M > 0\) is a given integer. Construct a sequence \(\mathcal{J}_M\) satisfying the conditions of lemma 5.2. By conditions (iv), (v), and (ii) of that lemma, there exists a unique integer \(N_1\) such that \(N \geq N_1 \geq 1\) and

\[ |u'_{N_1} - v_{N_1} \theta| \leq 1 < |u_{N_1-1} - v_{N_1-1} \theta| \]

Let \(\mathcal{J}_M\) be the sequence of \((u'_n, v'_n)\) defined by

\[ (u'_n, v'_n) = (u_n + N_1, v_n + N_1) \text{ for } -N_1 \leq n \leq N_2 = N - N_1. \]

Then conditions (i), (ii), (iii) and (vi) of lemma 5.2 hold
for \((u'_n, v'_n)\) and
\[
|u'_o - v'_o \theta| \leq 1 < |u'_{-1} - v'_{-1} \theta|,
\]
\[
|u'_{N_2} - v'_{N_2} \theta| \leq e^{-M}, \quad |u'_{-N_1} - v'_{-N_1} \theta| \geq e^M.
\] (5.1.10)

We now construct \(\mathcal{J}\) by a diagonal process. By the observation already made, there are only a finite number of possibilities for \((u'_o, v'_o)\) and \((u'_{-1}, v'_{-1})\), and so at least one of these must occur infinitely often, as \(M\) runs through the set of natural numbers.

We fix one such possible pair, say \((\overline{u}_o, \overline{v}_o)\), \((\overline{u}_{-1}, \overline{v}_{-1})\). Let \(\mathcal{G}_1\) be the infinite sequence of those \(\mathcal{J}'_M\)'s to which \((\overline{u}_o, \overline{v}_o)\) and \((\overline{u}_{-1}, \overline{v}_{-1})\) belong. Now we consider only these \(\mathcal{J}'_M\)'s.

For these \(M\), condition (5.1.10) imply that for all \(M\) large enough \(N_2 > 0\), otherwise \(u'_o - v'_o \theta = 0\) and this contradicts the hypothesis that \(\theta \notin K(t)\). Therefore, for \(M\) large, the terms \((u'_{-1}, v'_{-1})\), \((u'_o, v'_o)\), \((u'_{-1}, v'_1)\) occur in \(\mathcal{J}'_M\).

Since for each \(\mathcal{J}'_M\),
\[
|u'_1 - v'_1 \theta| \leq e^{-1} |\overline{u}_o - \overline{v}_o \theta| < |\overline{u}_o - \overline{v}_o \theta|.
\]

There are only a finite number of possible choices for \((u'_1, v'_1) \in P_1\).

Then we can pick out, say \((\overline{u}_1, \overline{v}_1)\) with \(\gcd(\overline{u}_1, \overline{v}_1) = 1\) which occurs infinitely often. Let \(\mathcal{G}_2\) be the infinite subsequence of \(\mathcal{G}_1\) of those \(\mathcal{J}'_M\)'s to which \((\overline{u}_1, \overline{v}_1)\) belong. Now we consider only
these $\mathcal{J}_M$'s. For these $\mathcal{J}_M$'s, $N_2 \geq 2$, $N_1 \geq 2$, for $M$ large.

For each $\mathcal{J}_M$

$$\overline{u}_{-1} - \overline{v}_{-1} \leq \overline{u}_{-1} - \overline{v}_{-1} < u_{-2} - v_{-2} \quad ,$$

And there are only a finite number of choices for $(u_{-2}', v_{-2}') \in P_2$.

We pick out $(\overline{u}_{-2}', \overline{v}_{-2}')$, which occurs infinitely often. Let $\mathcal{G}_3$ be the infinite subsequence of $\mathcal{G}_2$ of those $\mathcal{J}_M$'s to which $(\overline{u}_{-2}', \overline{v}_{-2}')$ belong and we consider only these $\mathcal{J}_M$'s.

Now

$$|u_{-2}' - v_{-2}' \theta| \leq e^{-1} |\overline{u}_1 - \overline{v}_1 \theta| < |\overline{u}_1 - \overline{v}_1 \theta|$$

and pick out $(\overline{u}_2, \overline{v}_2)$ which occurs for infinitely many $\mathcal{J}_M$'s.

Let $\mathcal{G}$ be the infinite subsequence of $\mathcal{G}_3$ to which $(\overline{u}_2, \overline{v}_2)$ belong. Continuing this process, we get a doubly infinite sequence of $(u_n, v_n)$ ($-\infty < n < \infty$) which satisfies conditions (i), (ii), (iii), (vi) of lemma 5.2.

Obviously by construction

$$\lim_{n \to \infty} |\overline{u}_n - \overline{v}_n \theta| = 0, \lim_{n \to -\infty} |\overline{u}_n - \overline{v}_n \theta| = \infty \quad .$$

Due to condition (1) of the lemma, we must have

$$\lim_{n \to -\infty} |\overline{u}_n - \overline{v}_n \phi| = 0 \quad .$$
Also, \( \lim_{n \to \infty} |\bar{u}_n - \bar{v}_n \Phi| = \infty \), for otherwise there exists an integer \( k \) such that

\[
|\bar{u}_n - \bar{v}_n \Phi| \leq e^k
\]

for all \( n \). Then

\[
|u - v \Theta| \leq 1, |u - v \Phi| \leq e^k
\]

has infinitely many solutions \((u, v) \in P_2\), which is denied by lemma 1.1. In the remaining of this case of theorem, we suppress bar at the top of \((u_n, v_n)\). Thus this sequence has all the required properties.

During our construction of \((u_i, v_i)'s\, , we have used sub­sequences \( \mathcal{G}_1, \mathcal{G}_2, \ldots \) of the sequence of \( \bar{f}_n \)'s such that

(i) \( \mathcal{G}_{i+1} \) is a subsequence of \( \mathcal{G}_i \).

(ii) Given any finite set of \((u_i, v_i)\, , this is a subset of \( \mathcal{G}_i \) for \( i \) large.

Let \( \mathcal{f}_i^\times \) be the first member of \( \mathcal{G}_i \). Then the sequence \( \mathcal{f}_i^\times \) has the following properites.

(1) \( \mathcal{f}_i^\times = \mathcal{f}_m^\times \) for some \( m_i ; m_i \to \infty \) as \( i \to \infty \)

(2) Each finite set of \((u_i, v_i)\, is a subset of \( \mathcal{f}_i^\times \) for all \( i \) large enough.
By lemma 5.3, there exists a sequence $(x_{o_i}^{(m_i)}, y_{o_i}^{(m_i)})$

$i = 1, 2, \ldots$, and we may assume that

$$|x_{o_i}^{(m_i)}| < 1, |y_{o_i}^{(m_i)}| < 1.$$ 

By theorem 18, this has a subsequence tending to a limit $(x_o, y_o)$ (say). This choice of $(x_o, y_o)$ does what is required. For take $(x, y) = (x_o, y_o) \pmod{P_2}$. Let $(u_n, v_n)$ be any member of this double sequence. For all $i$ large enough, $(u_n, v_n) \in \mathcal{F}_i = \mathcal{F}_k$ say

Then

$$|x u_n + y v_n| = |(x - x_o + x_o^{(k)}) u_n + (y - y_o + y_o^{(k)}) v_n + (x_o - x_o^{(k)}) u_n + (y_o - y_o^{(k)}) v_n|.$$ 

Since

$$(x - x_o + x_o^{(k)}, y - y_o + y_o^{(k)}) \equiv (x_o^{(k)}, y_o^{(k)}) \pmod{P_2},$$

$$|(x - x_o + x_o^{(k)}) u_n + (y - y_o + y_o^{(k)}) v_n| \geq e^{-1}.$$ 

Also for $i$ large enough, $k$ is sufficiently large and

$$|(x_o - x_o^{(k)}) u_n + (y_o - y_o^{(k)}) v_n| < \frac{1}{e^2}.$$ 

Therefore,

$$|x u_n + y v_n| \geq e^{-1}.$$ 

And

$$|F(x, y; u_n, v_n)| \geq e^{-1} |\theta - \phi|.$$
Proof of theorem 19: (Case I): Consider the double infinite sequence 
\((u_n, v_n) (-\infty < n < \infty)\) and \((x_0, y_0) \in R_2\) given by lemma 5.4.

Let \((x, y) \equiv (x_0, y_0) \mod P_2\) be any element of \(R_2\). Now we claim that \(x - y \theta + 0, x - y \varphi \neq 0\).

Suppose first that \(x - y \theta = 0\). Then we have

\[
\lim_{n \to \infty} |F(x, y; u_n, v_n)| = \lim_{n \to \infty} |(u_n - v_n \theta) (x - y \varphi)|
\]

\[
= |x - y \varphi| \lim_{n \to \infty} |u_n - v_n \theta| = 0
\]

which contradicts the fact that \((5.1.9)\) holds for all 
\((x, y) \equiv (x_0, y_0) \mod P_2\) for all \(n\). Hence \(x - y \theta \neq 0\).

Similarly \(x - y \varphi \neq 0\), so that

\[
f(x, y) = (x - y \theta) (x - y \varphi) \neq 0.
\]

Since \(\lim_{n \to \infty} |u_n - v_n \varphi| = 0, \lim_{n \to \infty} |u_n - v_n \theta| = \infty\),

and \(|u_n - v_n \varphi| \geq |u_n - 1 - v_n - 1 \varphi|\) for all \(n\), we can choose

\((u_n, v_n)\) such that

\[
|u_n - v_n \varphi|^2 \leq (\theta - \varphi) \cdot \frac{|x - y \varphi|}{|x - y \theta|} \leq |u_n + 1 - v_n + 1 \varphi|^2 (5.1.11)
\]

Since condition (iii) of lemma 5.2 holds for all \(n\), we get

\[
|u_n - v_n \theta|^2 \leq \frac{|\theta - \varphi|^2}{|u_n + 1 - v_n + 1 \varphi|^2} \leq \frac{|\theta - \varphi| \cdot |x - y \theta|}{|x - y \varphi|} (5.1.12)
\]

Put

\[
l = |u_n - v_n \varphi| \cdot |x - y \theta|, m = |u_n - v_n \theta| \cdot |x - y \varphi|, \]

\[
p^2 = |\theta - \varphi| \cdot |f(x, y)|.
\]
From (5.1.11) and (5.1.12) we get; 1 ≤ p, m ≤ p. But

\[ |F(x, y; u_n, v_n)| = |(x - y \theta)(u_n - v_n \phi) - (x - y \phi)(u_n - v_n \theta)| \]

\[ \leq \max \left( |x - y \theta| \cdot |u_n - v_n \phi|, |x - y \phi| \cdot |u_n - v_n \theta| \right) \]

\[ = \max (1, m) \leq p. \]

So

\[ |F(x, y; u_n, v_n)|^2 \leq p^2. \]

But from lemma 5.4, we have

\[ \left| \frac{F(x, y; u_n, v_n)}{0 - \phi} \right| \geq e^{-1} \]

for all \((x, y) \equiv (x_0, y_0) \pmod{P_n}\) and all \(n\).

So

\[ p^2 \geq |F(x, y; u_n, v_n)|^2 \geq |\theta - \phi|^2 e^{-2}, \]

or

\[ |\theta - \phi| \cdot |f(x, y)| \geq \frac{|\theta - \phi|^2}{e^2}, \]

or

\[ |f(x, y)| \geq \frac{|\theta - \phi|}{e^2}. \]

This proves the theorem in this case.

Case II: \(\theta \in K(t), \phi \notin K(t)\)

In this case, we shall first prove the following lemma.

**Lemma 5.2':** For every given integer \(M \geq 0\), there exists a finite number \(s_M\) of pairs of relatively prime non-zero polynomials \((u_0, v_0), (u_1, v_1), \ldots, (u_N, v_N)\) satisfying conditions (i),

\[1 \leq p, m \leq p, \]

\[\max (1, m) \leq p, \]

\[|F(x, y; u_n, v_n)|^2 \leq p^2, \]

\[\left| \frac{F(x, y; u_n, v_n)}{0 - \phi} \right| \geq e^{-1}, \]

\[p^2 \geq |F(x, y; u_n, v_n)|^2 \geq |\theta - \phi|^2 e^{-2}, \]

or

\[|\theta - \phi| \cdot |f(x, y)| \geq \frac{|\theta - \phi|^2}{e^2}, \]

or

\[|f(x, y)| \geq \frac{|\theta - \phi|}{e^2}. \]
(ii), (iii), (iv), (vi) of Lemma 5.2 and

(v') \( u_n - v_N \theta = 0 \), \( u_n - v_n \theta \neq 0 \) for \( 0 \leq n < N \).

Proof of Lemma 5.2': As in lemma 5.2, we select \((u_0, v_0)\). Suppose \((u_o, v_o), (u_1, v_1), \ldots, (u_{n-1}, v_{n-1})\) have already been chosen and \(u_{n-1} - v_{n-1} \theta \neq 0\). Define \((u_n, v_n)\) by

\[
\begin{align*}
|u_n - v_n \theta| &\leq e^{-1} |u_{n-1} - v_{n-1} \theta| , \\
|v_n - v_n \phi| &\leq \frac{|\phi - \phi|}{|u_{n-1} - v_{n-1} \theta|} ,
\end{align*}
\]

(5.1.13)

And \( |u_n - v_n \phi| = \text{Min} |u - v \phi| \), where the minimum is taken over all \((u, v) \neq (0, 0)\) satisfying (5.1.13).

As in lemma 5.2, this sequence \((u_o, v_o), \ldots,(u_n,v_n), \ldots\) satisfies conditions (i), (ii), (iii), (iv) and (vi) of lemma 5.11.

Since \( \theta \in K(t) \), we can write \( \theta = \frac{r}{s} \) where \( r \in K[t], s \in K[t] \).

If \( u_n - v_n \theta \neq 0 \) for any \( n \), we must have \( |u_n - v_n \theta| \geq \frac{1}{|s|} \). But \( u_n - v_n \theta \) is constantly decreasing. So we must get an \( N \) such that

\[ u_N - v_N \theta = 0 \]

and at this point we stop.

**Lemma 5.3':** There exists an infinite sequence of pairs of relatively prime polynomials \((u_n, v_n)\) \((- \infty < n \leq P)\) satisfying conditions

(i), (ii), (iii), (vi) of lemma 5.2,
and

\[(iv') \colon \lim_{n \to \infty} |u_n - v_n \theta| = \infty, \lim_{n \to \infty} |u_n - v_n \phi| = 0\]

\[(v') \colon u_p - v_p \theta = 0, u_n - v_n \theta = 0 \text{ for } -\infty < n < P.\]

Proof: Let \(L\) be any given integer. First we assert there are only a finite number of \((u, v), (u', v') \in \mathbb{P}_2\) satisfying

(a) \(|u - v \theta| \cdot |u - v \phi| \leq e^{-1} |\theta - \phi|.

(b) \(|u' - v' \phi| \cdot |u - v \theta| \leq |\theta - \phi|.

(c) \(|u' - v' \theta| \leq e^L < |u - v \theta|.

The assertion follows as in lemma 5.4.

Now suppose \(M > 0\) is a given integer. Construct a sequence \(S_M\) satisfying the conditions of lemma 5.2. By conditions

\[|u_0 - v_0 \theta| \geq e^M, \quad |u_0 - v_0 \phi| \leq e^{-M}, \quad u_N - v_N \theta = 0\]

and condition (ii) of lemma 5.2, there exists a unique integer \(N_1\) such that \(N \geq N_1 > 1\) and

\[|u_{N_1} - v_{N_1} \theta| \leq 1 < |u_{N_1-1} - v_{N_1-1} \theta|.

Let \(S'_M\) be the sequence of \((u'_n, v'_n)\) defined by

\[(u'_n, v'_n) = (u_n + N_1, v_n + N_1) \text{ for } -N_1 \leq n \leq N_2 = N - N_1 .\]

Obviously \(N_2 \geq 0.\)
Then conditions (i), (ii), (iii) and (vi) of lemma 5.2 hold for \((u_n', v_n')\) and

\[ |u_o' - v_o'\theta| \leq 1 < |u_{-1}' - v_{-1}'\theta| \]

\[ u_{N_2} - v_{N_2}'\theta = 0, \quad |u_{-N_1}' - v_{-N_1}'\theta| \geq e^M. \]  

(5.1.14)

We now construct \(\mathcal{F}\) by a diagonal process. By the observation already made, there are only a finite number of possibilities for \((u_o', v_o')\) and \((u_{-1}', v_{-1}')\) and so at least one of these must occur infinitely often, as \(M\) runs through the set of natural numbers.

We fix one such possible pair, say \((\overline{u}_o, \overline{v}_o), (\overline{u}_{-1}, \overline{v}_{-1})\).

If \(\overline{u}_o - \overline{v}_o 0 = 0\), we stop in "positive direction". If not, since

\[ |u_{1}' - v_{1}'\theta| \leq e^{-1} |\overline{u}_o - \overline{v}_o\theta| < |\overline{u}_o - \overline{v}_o\theta|, \]

there are only a finite number of possible choices for \((u_1', v_1')\).

Therefore, as before we pick out one (say \((u_1, v_1)\)), which occurs infinitely often, and fix our attention on those \(\mathcal{C}_M'\)'s in which

\((\overline{u}_{-1}, \overline{v}_{-1}),(\overline{u}_o, \overline{v}_o)\) and \((\overline{u}_1, \overline{v}_1)\) occur.

In a similar way, we get \((\overline{u}_2, \overline{v}_2), \ldots, (\overline{u}_p, \overline{v}_p)\) such that

\[ \overline{u}_p - \overline{v}_p\theta = 0 \text{ and } \overline{u}_n - \overline{v}_n\theta \neq 0 \text{ for } n < p. \]

Since

\[ |\overline{u}_n - \overline{v}_n\theta| \leq e^{-1} |\overline{u}_{n-1}' - \overline{v}_{n-1}'\theta|, \]
as in lemma 5.2', we must get at \((\bar{u}_p, \bar{v}_p)\) such that 

\[ u_p - v_p \theta = 0. \]

But in "negative direction", we can continue as far as we please. Obviously 

\[ \lim_{n \to \infty} |\bar{u}_n - \bar{v}_n \theta| = \infty. \]

But since 

\[ |\bar{u}_n - \bar{v}_n \theta| \cdot |\bar{u}_n - \bar{v}_n \phi| \leq e^{-1} |\theta - \phi| \]

for all \(n\) we must have 

\[ \lim_{n \to \infty} |\bar{u}_n - \bar{v}_n \phi| = 0. \]

And this proves this lemma.

Define \(F(x, y; u_n, v_n)\) as in definition 5.2.

**Lemma 5.4':** There exists \((x_0, y_0) \in R_2\) such that 

\[ |F(x, y; u_n, v_n)| \geq e^{-1} |\theta - \phi| (-\infty < n \leq P) \]

for all \((x, y) = (x_0, y_0) \mod P_2).\)

**Proof:** Since \(\phi \notin K\{t\}, u_p - v_p \phi \notin 0\) and we can take a solution \(K\) of 

\[ ||(u_p - v_p \phi) K|| = e^{-1} \]

(5.1.15)

In fact, there are infinitely many solutions \(K\) of (5.1.15).

Now we apply lemma 5.1 with \(r = 1, \lambda_n = u_n - v_n \theta,\)

\[ \lambda_n = (u_n - v_n \phi) K (-P < n < \infty). \]

By condition (ii) of lemma 5.2', the required condition of lemma 5.1 is satisfied and we got \(\{\in K \{t\}\)
such that

\[ \| (u_n - v_n \theta) \xi - (u_n - v_n \phi) K \| = e^{-1} ( - \infty < n < P ) \]  (5.1.16)

Since \( u_P - v_P \theta = 0 \) and we have (5.1.15), so (5.1.16) must hold for \( n = P \). Now take \( x_0, y_0 \) to be the solution of

\[ x_0 - y_0 \phi = \xi (\theta - \phi) \]
\[ x_0 - y_0 \theta = K (\theta - \phi) . \]

Then

\[ F(x_0, y_0; u_n, v_n) = (x_0 - y_0 \theta) (u_n - v_n \phi) - (x_0 - y_0 \phi) (u_n - v_n \theta) \]
\[ = - [ \xi (\theta - \phi) (u_n - v_n \theta) - K (\theta - \phi) (u_n - v_n \phi) ] . \]

So

\[ F(x_0, y_0; u_n, v_n) \]
\[ \frac{\theta - \phi}{\theta - \phi} = \xi (u_n - v_n \theta) - K (u_n - v_n \phi) \]

so by (5.1.16) we get

\[ \| F(x_0, y_0; u_n, v_n) \| = e^{-1} . ( - \infty < n \leq P ) \]

Now suppose \( (x, y) \equiv (x_0, y_0) \) (mod \( P_2 \)) .

Then

\[ F(x, y; u_n, v_n) - F(x_0, y_0; u_n, v_n) \]
\[ = (\theta - \phi) [v_n (x - x_0) - u_n (y - y_0)] \]
\[ = (\theta - \phi) g_n (t) \]

where \( g_n (t) \in K[t] . \)
Hence
\[ \left| \frac{F(x, y; u_n, v_n)}{\theta - \phi} \right| = \left| \frac{F(x, y; u_n, v_n)}{\theta - \phi} \right| = e^{-1} . \]
So
\[ \left| \frac{F(x, y; u_n, v_n)}{\theta - \phi} \right| \geq \left| \frac{F(x, y; u_n, v_n)}{\theta - \phi} \right| = e^{-1} \left( -\infty < n \leq p \right). \]
And
\[ \left| F(x, y; u_n, v_n) \right| \geq e^{-1} |\theta - \phi| \left( -\infty < n \leq p \right) \]
for all \((x, y) \equiv (x_o, y_o) \pmod{P_2}\)

Proof of theorem 19 (Case II): Let \((u_n, v_n) \left( -\infty < n \leq p \right)\) be the infinite sequence given by lemma 5.3' and \((x_o, y_o)\) be given by lemma 5.4'. Take
\[(x, y) \equiv (x_o, y_o) \pmod{P_2} . \]
Then we assert that \(x - y \neq 0 \neq x - y \phi . \) Suppose \(x - y \theta = 0 . \)
Then
\[ F(x, y; u_n, v_n) = - (x - y \phi) (u_n - v_n \theta) \]
So
\[ F(x, y; u_p, v_p) = - (x - y \phi) (u_p - v_p \theta) = 0 \]
and this contradicts lemma 5.4'.

Suppose \(x - y \phi = 0 . \) Then
\[ F(x, y; u_n, v_n) = (x - y \theta) (u_n - v_n \phi) \]
Therefore
\[
\lim_{n \to \infty} |F(x, y; u_n, v_n)| = |x - y \theta| \cdot \lim_{n \to \infty} |u_n - v_n \phi| = 0
\]
and this again contradicts lemma 5.4'.

If as in case I, we can find \((u_n, v_n)\) satisfying (5.1.11), then the assertion follows as before. If we cannot do so, we must have
\[
|u_p - v_p \phi|^2 < |\theta - \phi| \cdot \left| \frac{x - y \phi}{x - y \theta} \right|^2.
\]
And then
\[
|F(x, y; u_p, v_p)| = |x - y \theta| \cdot |u_p - v_p \phi|
\]
\[
< |x - y \theta| \frac{1}{2} \cdot |\theta - \phi| \frac{1}{2} \cdot |x - y \phi| \frac{1}{2}
\]
\[
= |\theta - \phi| \frac{1}{2} \cdot |f(x, y)| \frac{1}{2},
\]
so that
\[
|\theta - \phi| \frac{1}{2} |f(x, y)| \frac{1}{2} \geq |F(x, y; u_p, v_p)| \geq |\theta - \phi| e^{-1}
\]
and
\[
|f(x, y)| \geq \frac{|\theta - \phi|}{e^2}.
\]
Thus the theorem is true in this case also.

Case III: \(\theta \in K(t), \phi \in K(t)\).

First we prove the following lemma.
Lemma 5.2": There exists a finite sequence $\mathcal{F}$ of relatively prime polynomials $(u_0, v_0), (u_1, v_1), \ldots, (u_N, v_N)$ such that conditions (i), (ii), (iii), (vi) of lemma 5.2 are satisfied and

(iv''): $u_o - v_o \phi = 0, \ u_n - v_n \phi \neq 0 \text{ for } 0 < n \leq N$.

(v''): $u_N - v_N \theta = 0, \ u_n - v_n \theta \neq 0 \text{ for } 0 \leq n < N$.

Proof: Since $\phi \in K(t)$, there exist relatively prime polynomials $(u_0, v_0)$, not both zero such that

$$u_o - v_o \phi = 0.$$ 

Then $u_o - v_o \theta = 0$; since $\theta \neq \phi$.

If $(u_1, v_1)$ is any pair of relatively prime polynomials, not both zero, then

$$|u_1 - v_1 \phi| \geq 0 = |u_o - v_o \phi|.$$ 

Suppose we have constructed $(u_0, v_0), \ldots, (u_{n-1}, v_{n-1})$ satisfying conditions (i), (ii), (iii), and (vi) of lemma 5.2, (iv'') of lemma 5.2" and $u_{n-1} - v_{n-1} \theta \neq 0$. Define $(u_n, v_n)$ by

$$|u_n - v_n \theta| \leq e^{-1} |u_{n-1} - v_{n-1} \theta|,$$

$$|u_n - v_n \phi| \leq \frac{\theta - \phi}{|u_{n-1} - v_{n-1} \theta|}, \quad (5.1.17)$$

and

$$|u_n - v_n \phi| = \text{Min} \ |u - v \phi|.$$
where minimum is taken over all non-zero \((u, v) \in P_2\) satisfying (5.1.17).

By theorem A of chapter 1, (5.1.17) has at least one non-zero solution \((u, v) \in P_2\); and by lemma 1.1, the number of such solutions is finite. Therefore, such a \((u_n, v_n)\) must exist. Then condition (vi) of lemma 5.2 is automatically satisfied for \(n\), for otherwise

\[
|u_n - v_n \phi| < |u_{n-1} - v_{n-1} \phi|
\]

and we could have selected \((u_n, v_n)\) at the previous stage. We can suppose all \(u_i\) are monic. If \(u_i = 0\), then \(v_i\) is. Also \((u, v)\) relatively prime, \((u, v) \neq (u_0, v_0) \Rightarrow u - v \phi \neq 0\). Thus condition (iv") is satisfied.

Since \(\theta \in K(t)\), we can write \(\theta = \frac{r}{s}\), where \(r \in K[t], s \in K[t]\).

If \(|u_n - v_n \theta| \neq 0\), then we must have \(|u_n - v_n \theta| \geq \frac{1}{|s|}\). But

\[
|u_n - v_n \theta|
\]

is strictly decreasing, so we must get at an \(N\) such that

\[
u_N - v_N \theta = 0
\]

and at this point, we stop.

Define \(F(x, y; u_n, v_n)\) as in definition 5.1.

**Lemma 5.4": There exists \((x_o, y_o) \in R_2\) such that

\[
|F(x, y; u_n, v_n)| \geq e^{-1} |\theta - \phi| (0 \leq n \leq N)
\]

Define \(F(x, y; u_n, v_n)\) as in definition 5.1.
for all \((x, y) \equiv (x_0, y_0) \pmod{P_2}\).

The proof of lemma 5.4" is exactly the same as that of lemma 5.4', except that we have a finite sequence, instead of an infinite one.

Proof of theorem 19 (Case III): Let \((u_n, v_n) (0 \leq n \leq N)\) be the sequence given by lemma (5.2") and \((x_0, y_0)\) be given by lemma 5.4". Then we assert neither \(x - y \theta = 0\), nor \(x - y \phi = 0\).

Suppose \(x - y \theta = 0\). Then

\[
F(x, y; u_n, v_n) = -(x - y \phi) (u - v 0).
\]

So \(F(x, y; u_p, v_p) = -(x - y \phi) (u_p - v_p \theta) = 0\), and this contradicts lemma 5.4".

Similarly if \(x - y \phi = 0\), \(F(x, y; u_o, v_o) = 0\), which is again a contradiction to lemma 5.4". Now we proceed exactly as in the proof of the theorem in case II.

Remark 5.2: The result of theorem 19 is best possible in the sense that there exist forms \(L_1, L_2\) such that for every \((x_0, y_0) \in R_2\), there exists \((x, y) \equiv (x_0, y_0) \pmod{P_2}\), and

\[
\left| L_1 L_2 \right| \leq \frac{|\Delta|}{e^2}.
\]

For example, take \(L_1 = x, L_2 = x + y\).
5.2: In this section, we shall prove the following theorem.

**Theorem 20:** Let $K$ be a finite field, in which $x^2 + 1 = 0$ is not soluble. Let $L_1, L_2, L_3$ be three linear forms in variables $x, y, z$ with determinant $\Delta \neq 0$. Then there exists a triplet $(x_0, y_0, z_0) \in R_3$ such that

$$|L_1(x) (L_2^2(x) + L_3^2(x))| \geq \frac{A}{e^3} \quad (5.2.1)$$

for all $X = (x, y, z) \equiv (x_0, y_0, z_0) \pmod{P_3}$.

**Remark 5.3:** This result is best possible in the sense that there exist forms $L_1, L_2, L_3$ such that for every $(x_0, y_0, z_0) \in R_3$,

there exists $(x, y, z) \equiv (x_0, y_0, z_0) \pmod{P_3}$,

and

$$|L_1(L_2^2 + L_3^2)| \leq \frac{A}{e^3}.$$ .

For example, take $L_1 = x, L_2 = y, L_3 = z$.

Before proving the actual theorem, we need a few preliminaries.

Write

$$L_j(x, y, z) = a_jx + b_jy + c_jz \quad (j = 1, 2, 3)$$

and $L_2' = L_2 + iL_3, L_3' = L_2 - iL_3 (i^2 = -1)$.

Then

$$\Delta' = \det(L_1, L_2', L_3') = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 + ia_3 & b_2 + ib_3 & c_2 + ic_3 \\ a_2 - ia_3 & b_2 - ib_3 & c_2 - ic_3 \end{vmatrix}$$
\[
\begin{vmatrix}
  a_1 & b_1 & c_1 \\
  2a_2 & 2b_2 & 2c_2 \\
  a_2 - ia_3 & b_2 - ib_3 & c_2 - ic_3
\end{vmatrix}
\]

\[
= -2i \begin{vmatrix}
  a_1 & b_1 & c_1 \\
  a_2 & b_2 & c_2 \\
  a_3 & b_3 & c_3
\end{vmatrix} = -2i \Delta.
\]

Since \( x^2 + 1 = 0 \) is not soluble in the field, characteristic of the field is not two and so \( 2 \neq 0 \).

So
\[
|\Delta'| = |-2i\Delta| = |\Delta|.
\]

Let \( M_1', M_2', M_3' \) be forms in variables \( u, v, w \) such that the coefficients of \( M_1', M_2', M_3' \) are cofactors of the corresponding coefficients of \( L_1, L_2', L_3' \) respectively. The determinant of \( M_1', M_2', M_3' \) is

\[
\begin{vmatrix}
  2i(b_3c_2' - b_2c_3') & 2i(c_3a_2' - c_2a_3') & 2i(a_3b_2' - a_2b_3') \\
  c_1b_2' - b_1c_2' + a_1c_2' - a_2c_1' + b_1a_2' - a_1b_2' + i(b_1c_3' - c_1b_3') + i(c_1a_3' - a_1c_3') + i(a_1b_3' - a_3b_1') \\
  b_1c_2' - c_1b_2' + a_2c_1' - a_1c_2' + a_1b_2' - b_1a_2' + i(b_1c_3' - c_1b_3') + i(c_1a_3' - a_1c_3') + i(a_1b_3' - a_3b_1')
\end{vmatrix}
\]

so that

\[
\det (M_1', M_2', M_3') = \Delta' \cdot 2
\]
Also \( L_1 M_1' + L_2 M_2' + L_3 M_3' = \Delta' (xu + yv + zw) \).

Define

\[
M_j = \frac{1}{i} M_j' \quad (j = 1, 2, 3)
\]

Then coefficients of \( M_1 \) are in \( K \{t\} \) and \( M_2 = \overline{M}_3 \). Also

\[
|\text{Det} (M_1, M_2, M_3)| = \left| \frac{\text{Det} (M_1', M_2', M_3')}{{i}^3} \right| = |\Delta'|^2 = |\Delta|^2
\]

and

\[
L_1 M_1 + L_2 M_2' + L_3 M_3' = \frac{\Delta'}{i} (xu + yv + zw)
\]

\[
= -i \Delta' (xu + yv + zw) \quad (5.2.2)
\]

Now we prove a few lemmas.

**Lemma 5.5:** If integers \( p, q \) are given such that

\[
e^p + 2q \geq e^{-2} |\Delta|^2,
\]

then there exists non-zero \( U = (u, v, w) \in P_3 \) such that

\[
|M_1(u, v, w)| \leq e^p, \quad |M_2(u, v, w)| = |M_3(u, v, w)| \leq e^q \quad (5.2.3)
\]

**Further there are only a finite number of non-zero vectors** \( U \in P_3 \) satisfying (5.2.3).
Proof: Since $M_2, M_3$ are complex conjugates, let

$$M_2 = N_2 + i N_3$$
$$M_3 = N_2 - i N_3$$

where $N_2, N_3$ are linear forms with coefficients from $K\{t\}$.

As remarked earlier, characteristic of the field is not equal to 2. As before we can see that

$$|\det (M_1, N_2, N_3)| = |\det (M_1, M_2, M_3)| = |\Delta|^2.$$  

Since $e^p + 2q \geq e^{-2|\Delta|^2}$, by theorem A of chapter 1, there exists a non-zero vector $U = (u, v, w) \in P_3$ satisfying the inequalities

$$|M_1(U)| \leq e^p, \quad |N_2(U)| \leq e^q, \quad |N_3(U)| \leq e^q \quad (5.2.4)$$

For any $U \in P_3$,

$$|M_2(U)| = |M_3(U)| = \max (|N_2(U)|, |N_3(U)|) \quad (5.2.5)$$

Hence the $U$, which satisfies (5.2.4) must also satisfy (5.2.3), and this proves the first part.

To prove that (5.2.3) has only a finite number of solutions, we first observe that any solution of (5.2.3) is also a solution of (5.2.4). But lemma 1.1, (5.2.4) has only a finite number of solutions. Hence our assertion follows.
Remark 5.4: Let $K$ be a field (not necessarily finite), in which $x^2 + 1 = 0$ is not soluble. Let $L_1, L_2, L_3$ be three linear forms in variables $(x, y, z) = X$ with determinant $\Delta \neq 0$. Then lemma 5.5 incidentally proves that there exist a non-zero vector $X \in P_3$ satisfying

$$|L_1(L_2^2 + L_3^2)| \leq e^{-2} |\Delta|.$$

Now we distinguish the following four cases.

(i) $M_1 \neq 0$, $M_2 \neq 0 \neq M_3$ for any non-zero $U \in P_3$.

(ii) $M_1 \neq 0$ for any non-zero $U \in P_3$, but $M_2 = 0 = M_3$ for some non-zero $U \in P_3$.

(iii) $M_1 = 0$ for some non-zero $U \in P_3$, but $M_2 \neq 0 \neq M_3$ for any non-zero vector $U \in P_3$.

(iv) $M_1 = 0$ for some non-zero $U \in P_3$ and $M_2 = 0 = M_3$ for some non-zero $V \in P_3$.

Since the determinant of the forms $M_1, M_2, M_3$ is not zero, $M_1$ and $M_2$ cannot vanish simultaneously for any non-zero vector $U \in R_3$.

Case I: $M_1 \neq 0$, $M_2 \neq 0 \neq M_3$ for any polynomials $u, v, w$.

Lemma 5.6: Let integer $M \geq 0$ be given. Then there is a finite set of values $\alpha_n, \beta_n, \gamma_n = \bar{\beta}_n (0 \leq n \leq N)$ of $M_1, M_2, M_3$. 

corresponding to the sequence \( \mathcal{S}_M \) of polynomials \((u_n, v_n, w_n)\) (not all zero) with \(\gcd(u_n, v_n, w_n) = 1\), such that

\[
\begin{align*}
(i) & \quad |\alpha_n| |\beta_n|^2 \leq e^{-2} |\Delta|^2. \\
(ii) & \quad |\alpha_n| \leq e^{-1} |\alpha_{n-1}|. \\
(iii) & \quad |\alpha_{n-1}| \cdot |\beta_n|^2 \leq |\Delta|^2 \quad (n \geq 1). \\
(iv) & \quad |\beta_0| \leq e^{-M}, \quad |\alpha_0| \geq e^M \quad (n \geq 1). \\
(v) & \quad |\alpha_N| \leq e^{-M} \\
(vi) & \quad |\beta_n| \geq |\beta_{n-1}| \quad (n \geq 1).
\end{align*}
\]

Proof: Consider the solutions of

\[
\begin{align*}
|M_2(u, v, w)| & \leq e^{-M}, \\
|M_3(u, v, w)| & \leq e^{-M}, \\
|M_1(u, v, w)| & \leq \max(e^{2M}, e^{2M} |\Delta|^2).
\end{align*}
\]

By lemma 5.5, these inequalities have a non-zero solution \(U = (u, v, w) \in P_3\) and the number of such solutions is finite.

Consider

\[
\text{Min } |M_2(u, v, w)|
\]

as \((u, v, w)\) runs over non-zero solutions from \(P_3\) of the above inequalities. Since the number of these solutions is finite, this minimum is attained. Since \(M_2 \neq 0\) for any non-zero \(U \in P_3\), this
minimum is not zero. Let this minimum be \( e^{-T} \). Then \( T \geq M \).

Consider the solutions of

\[
\begin{align*}
|M_2(u, v, w)| &\leq e^{-T - 1}, \\
|M_3(u, v, w)| &\leq e^{-T - 1}, \\
|M_1(u, v, w)| &\leq e^{2T} |\Delta|^2.
\end{align*}
\]

(5.2.7)

By lemma 5.5, the inequalities (5.2.7) have at least one non-zero solution \( U = (u, v, w) \in P_3 \). Further the number of such solutions is finite. Hence \( \min M_2(u, v, w) \) taken over non-zero vectors in \( P_3 \) satisfying (5.2.7) is attained, say at \((u_0, v_0, w_0)\).

Obviously we must have \( \gcd(u_0, v_0, w_0) = 1 \). Let

\[
\alpha_0 = M_1(u_0, v_0, w_0), \quad \beta_0 = M_2(u_0, v_0, w_0), \quad \gamma_0 = M_3(u_0, v_0, w_0).
\]

Then

\[
|\beta_0| = |\gamma_0| \leq e^{-T - 1} \leq e^{-M - 1}; \text{ since } T \geq M.
\]

Also for every non-zero vector \( U = (u, v, w) \in P_3 \), satisfying (5.2.7), we must have

\[
|M_1(u, v, w)| > e^{2M},
\]

Otherwise, we have

\[
|M_1(u, v, w)| \leq e^{2M}.
\]
and then this \((u, v, w)\) satisfies the system (5.2.6), which contradicts the minimal nature of \(T\). Hence, in particular

\[|\alpha_0| \geq e^{2M}\]

Since \((u_0, v_0, w_0)\) satisfies (5.2.7), \((u_o, v_o, w_o)\) also satisfies condition (i) of lemma 5.6.

Suppose there exists \((u, v, w)\) satisfying

\[|M_1(u, v, w)| \leq e^{-1}|\alpha_o|.

Then we must have

\[|M_2(u, v, w)| \geq \beta_o.

For, suppose we have

\[|M_2(u, v, w)| < \beta_o.

Then this \((u, v, w)\) also satisfies (5.2.7), and thus contradicts the minimal nature of \((u_o, v_o, w_o)\).

Now we shall construct a sequence \(\mathcal{M}\) satisfying the conditions of lemma 5.6. \((u_o, v_o, w_o)\) satisfies conditions (i) and (iv) by construction. The conditions (ii), (iii) and (vi) are vacuously satisfied. Suppose \((u_{n-1}, v_{n-1}, w_{n-1})\) have already been chosen. Then

\[\alpha_{n-1} = M_1(u_{n-1}, v_{n-1}, w_{n-1}) + 0.\]
Let
\[ |M_1(u_{n-1}, v_{n-1}, w_{n-1})| = e^k. \]

We distinguish the following cases,

(A) \( k \) odd.  
(B) \( k \) even.

Case A: Consider non-zero vectors from \( P_3 \), which satisfy

\[ |M_1(u, v, w)| \leq e^{-1} |\alpha_{n-1}|, \]

\[ |M_2(u, v, w)| \leq \left( \frac{e^{-1}}{|\alpha_{n-1}|} \right)^{\frac{1}{2}}, \]

\[ |M_3(u, v, w)| \leq \left( \frac{e^{-1}}{|\alpha_{n-1}|} \right)^{\frac{1}{2}}. \]

By lemma 5.5, such vectors exist and the number of these is finite.

Case B: Consider non-zero vectors from \( P_3 \) which satisfy

\[ |M_1(u, v, w)| \leq e^{-2} |\alpha_{n-1}|, \]

\[ |M_2(u, v, w)| \leq |\alpha_{n-1}|^{\frac{1}{2}} |\triangle|, \]

\[ |M_3(u, v, w)| \leq |\alpha_{n-1}|^{\frac{1}{2}} |\triangle|. \]

By lemma 5.5 again such vectors exist, and the number of such vectors is finite.
Thus in either case, we have a finite number of non-zero elements of $P_3$, which satisfy

(a) $|M_1(u, v, w) \cdot M_2(u, v, w)|^2 \leq e^{-2} |\Delta|^2$,

(b) $|M_1(u, v, w)| \leq e^{-1} |\alpha_{n-1}|$,

(c) $|\alpha_{n-1}| \cdot |M_2(u, v, w)|^2 \leq |\Delta|^2$.

We can now get a non-zero vector $(u_n, v_n, w_n)$ from $P_3$ satisfying (a), (b), (c) such that

$$|\beta_n| = |M_2(u_n, v_n, w_n)| = \min |M_2(u, v, w)|$$

where minimum has been taken over these finite number of solutions from $P_3$.

The sequence $(u_n, v_n, w_n)$ so chosen satisfies conditions (i), (ii), (iii) of our lemma. Condition (vi) is automatically satisfied because, otherwise, we would have selected $(u_n, v_n, w_n)$ at the previous stage. Because of condition (ii) of the lemma $|M_1(u_n, v_n, w_n)|$ is constantly decreasing, so we will have

$$|M_1(u_n, v_n, w_n)| \leq e^M$$

for all $n$ from some stage onwards.

We stop at such an $N$. 
Remark: 1. Clearly $\mathcal{F}_M$ is not unique.

2. If $M_1 > M_2$, then every $\mathcal{F}_{M_1}$ is an $\mathcal{F}_{M_2}$ also.

Lemma 5.7: There exists a triplet $(x_0(M), u_0(M), z_0(M)) \in R_3$ such that

$$|x u_n + y v_n + z w_n| \leq e^{-1}$$

for all $(x, y, z) = (x_0(M), y_0(M), z_0(M)) \pmod{p_3}$ and all triplets

$(u_n, v_n, w_n)$ for $n = 0, 1, \ldots, N$.

Proof: Consider

$$\wedge_n = \frac{-i \alpha_n - n}{\Delta} \quad (n = 0, 1, \ldots, N).$$

Since

$$\Delta' = -2i \Delta, \quad \frac{-i \alpha_n - n}{\Delta} \in K[t].$$

Now owing to condition (ii) of lemma 5.6,

$$|\wedge_n| \geq e^{1/|\wedge_n - 1|}.$$

We can, therefore, apply lemma 5.1 with $r = 1, n = 0, 1, \ldots, N$ and get $\xi \in K[t]$ such that

$$\left| \frac{-i \alpha_n}{\Delta} \xi \right| = e^{-1} \quad (n = 0, 1, \ldots, N) \quad (5.2.10)$$

Now we define $(x_0(M), y_0(M), z_0(M)) \in R_3$ by the equations

$$L_1(x_0(M), y_0(M), z_0(M)) = 0,$$

$$L_2(x_0(M), y_0(M), z_0(M)) = 0 = L_3(x_0(M), y_0(M), z_0(M)).$$
With this choice of \((x_o(M), y_o(M), z_o(M))\), we also have

\[ L_2 '(x_o(M), y_o(M), z_o(M)) = 0 = L_3 '(x_o(M), y_o(M), z_o(M)), \]

so that substituting \(x = x_o(M), y = y_o(M), z = z_o(M),\)

\(u = u_n, v = v_n, w = w_n\) in (5.2.2), we get

\[ \xi \alpha_n = -i \nabla \cdot (x_o(M) u_n + y_o(M) v_n + z_o(M) w_n) \]
or

\[ x_o(M) u_n + y_o(M) v_n + z_o(M) w_n = \frac{i \xi \alpha_n}{\Delta}, \]

and

\[ \| x_o(M) u_n + y_o(M) v_n + z_o(M) w_n \| = \| \frac{i \xi \alpha_n}{\Delta} \| = e^{-1} \]

\((n = 0, 1, \ldots, N)\)

Further, if \((x, y, z) \equiv (x_o(M), y_o(M), z_o(M)) \pmod{p_3}\), then

\[ = (x - x_o(M)) u_n + y - y_o(M) v_n + (z - z_o(M)) w_n \]

and the right hand side is an element of \(K[t]\). Hence

\[ \| x u_n + y v_n + z w_n \| = \| x_o(M) u_n + y_o(M) v_n + z_o(M) w_n \| \]

and \( |x u_n + y v_n + z w_n| \geq \| x u_n + y v_n + z w_n \| = e^{-1} \) \((n = 0, 1, \ldots, N)\)

This proves the lemma.
Lemma 5.8: There exists a doubly infinite sequence of triplets of polynomials \((u_n, v_n, w_n)\) \((-\infty < n < \infty)\) with \(\gcd(u_n, v_n, w_n) = 1\) such that conditions (i), (ii), (iii) and (vi) of lemma 5.6 hold and

\[
\lim_{n \to \infty} |\alpha_n| = 0 = \lim_{n \to -\infty} |\beta_n|.
\]

Also

\[
\lim_{n \to \infty} |\beta_n| = \infty = \lim_{n \to -\infty} |\alpha_n|.
\]

Further there is an \((x_0, y_0, z_0) \in \mathbb{R}_3\), such that (5.2.9) holds for all \((x, y, z) \equiv (x_0, y_0, z_0) \mod P_3\) and all \(n\).

Proof: Let \(L\) be any given integer. First we assert there are only a finite number of \((u_0, v_0, w_0)\) and \((u_0', v_0', w_0')\) \in \(P_3\) satisfying

(a) \(|M_1(u, v, w)| \cdot |M_2(u', v', w')|^2 \leq |\Delta|^2\).

(b) \(|M_1(u, v, w)| \cdot |M_2(u, v, w)|^2 \leq \varepsilon^{-2} |\Delta|^2\).

(c) \(|M_1(u', v', w')| \leq \varepsilon^L < |M_1(u, v, w)|\).

From (a) and (c) we get

\[
|M_1(u', v', w')| \leq e^L
\]

\[
|M_2(u', v', w')|^2 \leq |M_1(u, v, w)|^{-1} |\Delta|^2 = e^{-L} |\Delta|^2
\]

Therefore,

\[
|M_2(u', v', w')| \leq e^{\left[\frac{L}{2}\right]} |\Delta|.
\]
But for \((u', v', w') \in P_3\),

\[ |M_2 (u', v', w')| = |M_3 (u', v', w')| .\]

So

\[ |M_2 (u', v', w')| \leq e^{\left[\frac{L}{2}\right]} |\Delta| ,\]

and

\[ |M_3 (u', v', w')| \leq e^{\left[\frac{L}{2}\right]} |\Delta| . \]

Also

\[ |M_1 (u', v', w')| \leq e^L . \]

But by lemma 5.5, the number of \((u', v', w') \in P_3\) satisfying (5.2.11) is finite.

Now suppose \((u', v', w')\) is given to us. Then prove that there can exist only a finite number of \((u, v, w)\) satisfying (a), (b), (c). From (a) we get

\[ |M_1 (u, v, w)| \leq |M_2 (u', v', w')|^{-2} |\Delta|^2 (M_2 (u', v', w') \neq 0) \]

From (b) and (c) we have

\[ |M_2 (u, v, w)|^2 \leq |M_1 (u, v, w)|^{-1} \cdot e^{-2} |\Delta|^2 = e^{-L-2} |\Delta|^2 \]

or

\[ |M_2 (u, v, w)| \leq e^{\left[\frac{L}{2}\right] - 1} . \]

Then as before we can see that

\[ |M_3 (u, v, w)| \leq e^{\left[\frac{L}{2}\right] - 1} . \]
By using lemma 1.1, we can see that there are only a finite number of \((u, v, w) \in P_3\) satisfying these.

Now suppose \(M > 0\) is a given integer. Construct a sequence \(\mathcal{C}_M\) satisfying the conditions of lemma 5.6. By conditions (iv), (v) and (ii) of lemma 5.6, there exists a unique integer \(N_1\), such that \(N \geq N_1 \geq 1\) and

\[|\alpha_{N_1}| \leq 1 < |\alpha_{N_1-1}|.\]

Let \(\mathcal{C}'_M\) be the sequence of \((u_n', v_n', w_n')\) defined by

\[(u_n', v_n', w_n') = (u_n + N_1, v_n + N_1, w_n + N_1)\]

for \(-N_1 \leq n \leq N_2 = N - N_1\). Then conditions (i), (ii), (iii), (vi) of lemma 5.6 hold for \((u_n', v_n', w_n')\) and

\[|\alpha_o'| \leq 1 < |\alpha_{-1}'|\]

\[|\alpha_{N_2}'| \leq e^{-M}, |\alpha_{-N_1}'| \geq e^M\]  

(5.2.12)

We now construct \(\mathcal{C}\) by a diagonal process. By the observation already made, there are only a finite number of possibilities for \((u_o', v_o', w_o')\) and \((u_{-1}', v_{-1}', w_{-1}')\), and so at least one of these must occur infinitely often, as \(M\) runs through the set of natural numbers. We fix one such possible pair, say \((\overline{u}_o, \overline{v}_o, \overline{w}_o)\), \((\overline{u}_{-1}, \overline{v}_{-1}, \overline{w}_{-1})\). Let \(\mathcal{G}_1\) be the sequence of \(\mathcal{C}_M\)'s in which
\((\overline{u}_0, \overline{v}_0, \overline{w}_0)\) and \((\overline{u}_{-1}, \overline{v}_{-1}, \overline{w}_{-1})\) occur. Now we consider only these \(f'_M\)'s.

Denote by \(\alpha_n, \beta_n, \gamma_n\) the values assumed by \(M_1, M_2, M_3\) respectively at \((\overline{u}_n, \overline{v}_n, \overline{w}_n)\).

Conditions (5.2.12) and the fact that \(M_1\) does not assume the value zero for \((u, v, w) \in P_3\) implies that \(N_2 > 0\) for \(M\) large enough, so that the terms \((u_{-1}', v_{-1}', w_{-1}')\), \((u_0', v_0', w_0')\) and \((u_1', v_1', w_1')\) occur in \(f'_M\) for \(M\) large.

Now

\[|\alpha_1'| \leq e^{-1} |\alpha_0| < |\alpha_0| .\]

As remarked earlier, there are only finitely many choices for \((u_1', v_1', w_1')\). We pick out one say \((\overline{u}_1, \overline{v}_1, \overline{w}_1)\) which occurs infinitely often. Let \(\overline{C}_2\) be the infinite subsequence of \(\overline{C}_1\) of those \(f'_M\)'s to which \((\overline{u}_1, \overline{v}_1, \overline{w}_1)\) belong. Now we consider only these \(f'_M\)'s. For \(M\) large, \(N_1 > 2\).

Now

\[|\alpha_{-1}| \leq |\alpha_{-1}| < |\alpha_{-2}'| .\]

As before, there are only finitely many choices for \((u_2', v_2', w_2')\). Again we pick out \((\overline{u}_{-2}, \overline{v}_{-2}, \overline{w}_{-2})\) which occurs infinitely often.
Let $\tilde{\gamma}_3$ be the infinite subsequence of $\tilde{\gamma}_2$ of those $\gamma_M$'s to which $(\tilde{u}_2, \tilde{v}_2, \tilde{w}_2)$ belong. Now we consider only these $\gamma'_M$'s. For $M$ large $N_2 > 2$. Again

$$|\alpha_2'| \leq e^{-1} |\tilde{\alpha}_1| < |\tilde{\alpha}_1|.$$ 

As before, there are finitely many choices for $(u_2', v_2', w_2')$ and we pick $(\tilde{u}_2, \tilde{v}_2, \tilde{w}_2)$ and $\tilde{\gamma}_4$. Continuing in this way, we get a doubly infinite sequence $(\tilde{u}_n, \tilde{v}_n, \tilde{w}_n)(-\infty < n < \infty)$ of non-zero elements of $\mathbb{P}_3$ such that $\gcd(u_n, v_n, w_n) = 1$, and hold conditions (i), (ii), (iii) and (vi) of lemma 5.6. Also

$$\lim_{n \to \infty} |\tilde{\alpha}_n| = 0, \quad \lim_{n \to -\infty} |\tilde{\alpha}_n| = \infty.$$ 

Since

$$|\tilde{\alpha}_n \tilde{\beta}_n^2| \leq e^{-2} |\triangle|^2$$

for all $n$, we must have

$$\lim_{n \to -\infty} |\tilde{\beta}_n| = 0.$$ 

Also we must have

$$\lim_{n \to \infty} |\tilde{\beta}_n| = \infty.$$ 

For, otherwise, $|\tilde{\beta}_n|$ will become bounded.

But the inequalities of the type
\[ |M_1 (u, v, w)| \leq e^{c_1} \]
\[ |M_2 (u, v, w)| \leq e^{c_2} \]
\[ |M_3 (u, v, w)| \leq e^{c_2} \]

cannot have infinitely many solutions \((u, v, w) \in P_3\). Hence this sequence has the required properties.

In the rest of this theorem, we suppress bars at the top of \(u_n, v_n, w_n\) and \(\sim\) at the top of \(\alpha_n, \beta_n, \gamma_n\).

During our construction of \((u_j, v_j, w_j)\)'s we have used subsequences \(\mathcal{G}_1, \mathcal{G}_2, \ldots\) of the sequence of \(\mathcal{I}_M\)'s such that

1. \(\mathcal{G}_1 + 1\) is a subsequence of \(\mathcal{G}_1\).
2. Given any finite set of \((u_j, v_j, w_j)\)'s, this is a subset of each member of \(\mathcal{G}_j\) for \(j\) large.

Let \(\mathcal{J}_j\) be the first member of \(\mathcal{G}_j\). Then the sequence \(\{\sim_j\}\) has the following properties.

1. \(\mathcal{J}_j = \mathcal{J}_{m_j}\) for some \(m_j, m_j \to \infty\) as \(j \to \infty\).
2. Each finite set of \((u_j, v_j, w_j)\)'s is a subset of \(\mathcal{J}_j^*\) for all \(j\) large enough.

For all these \(\mathcal{J}_{m_j}\)'s, consider the sequence
\[
\{(x_0^{(m_j)}, y_0^{(m_j)}, z_0^{(m_j)})\}.
\]
Now \(|x_0^{(m_j)}| < 1, |y_0^{(m_j)}| < 1, |z_0^{(m_j)}| < 1\).
for all $m_j$. Therefore, by theorem 18, the sequence
\[
\left\{ (x^{(m_j)}, y^{(m_j)}, z^{(m_j)}) \right\}
\]
must have a convergent subsequence tending to a limit $(x_0, y_0, z_0)$.

This choice of $(x_0, y_0, z_0)$ does what is required. For take any $(u_n, v_n, w_n)$. Let $M$ be large enough, so that $(u_n, v_n, w_n)$ occurs in $\mathcal{F}_M = \mathcal{F}_k$ (say). Then by lemma 5.7,
\[
|x^{(k)}(u_n + y^{(k)}v_n + z^{(k)}w_n| \geq e^{-1}.
\]

Take
\[
(x, y, z) \equiv (x_0, y_0, z_0) \pmod{P_3}.
\]

Then
\[
x u_n + y v_n + z w_n
\]
\[
= [(x - x_0 + x^{(k)})u_n + (y - y_0 + y^{(k)})v_n + (z - z_0 + z^{(k)})w_n]
\]
\[
+ [(x_0 - x^{(k)})u_n + (y_0 - y^{(k)})v_n + (z_0 - z^{(k)})w_n]
\]
\[
= A_1 + A_2 \quad \text{(say)}
\]

where $A_1 = (x - x_0 + x^{(k)})u_n + (y - y_0 + y^{(k)})v_n + (z - z_0 + z^{(k)})w_n$

$A_2 = (x_0 - x^{(k)})u_n + (y_0 - y^{(k)})v_n + (z_0 - z^{(k)})w_n$.

But
\[
(x - x_0 + x^{(k)}, y - y_0 + y^{(k)}, z - z_0 + z^{(k)}) \equiv (x_0, y_0, z_0)
\]
\[
\pmod{P_3}.
\]
So \[ |(x - x^*_o + x^*_o(k)) u_n + (y - y^*_o + y^*_o(k)) v_n + (z - z^*_o + z^*_o(k)) w_n| \geq e^{-1}. \]

Also for M large, k is large, so that \[ |A_2| < \frac{1}{e}. \]
Hence
\[ |x u_n + y v_n + z w_n| = |A_1 + A_2| \geq e^{-1}. \]
This proves the result.

Proof of Theorem 20 (Case I): Let \((u_n, v_n, w_n) (-\infty < n < \infty)\) and \((x_o, y_o, z_o)\) be given by lemma 5.8. Suppose \(\theta, \phi, \psi = \bar{\phi}\) are the values taken by \(L_1, L_2', L_3'\) respectively for some
\((x, y, z) = (x_o, y_o, z_o) (mod P_3).\)

Now we assert neither \(\theta = 0\) nor \(\phi = 0 = \psi\). For if possible, suppose \(\theta = 0\). Then from (5.2.2) we get
\[ \frac{1}{\Delta} \max (|\phi \beta_n|, |\psi \gamma_n|) \]
\[ = \frac{|\phi|}{|\Delta|} |\beta_n| . \]
Taking limit as \(n \to \infty\), we get
\[ |x u_n + y v_n + z w_n| \to 0 \]
which contradicts lemma 5.8. Hence \(\theta \neq 0\).

Similarly, \(\phi \neq 0 \neq \psi\).
Since $|\beta_n| + \infty$ as $n + \infty$, $|\beta_n| + 0$ as $n + -\infty$ and $|\beta_n + 1| \geq |\beta_n|$, we can choose an integer $n$ such that

$$|\beta_n|^2 \leq |\theta|^\frac{2}{3} \cdot |\Phi|^\frac{2}{3} \cdot |\Delta|^\frac{4}{3} \leq |\beta_n + 1|^2$$  \hspace{1cm} (5.2.13)

From condition (iii) of lemma 5.6, we get

$$|\alpha_n| \leq |\Delta|^2 \cdot |\beta_n + 1|^{-2} \leq |\Delta|^2 \cdot |\theta|^\frac{2}{3} \cdot |\Phi|^\frac{2}{3} \cdot |\Delta|^\frac{4}{3}$$

$$\leq |\theta|^\frac{2}{3} \cdot |\Phi|^\frac{2}{3} \cdot |\Delta|\frac{2}{3}$$  \hspace{1cm} (5.2.14)

Put

$$l = |\theta| \cdot |\alpha_n|, \ m = |\Phi| \cdot |\beta_n|, \ p = |\theta| \cdot |\Phi| \cdot |\Delta|\frac{2}{3}.$$  

Then from (5.2.2) we get

$$|\Delta| \cdot |x u_n + y v_n + z w_n| = |\theta \alpha_n + \beta \beta_n + \gamma \gamma_n|$$

$$\leq \max (|\theta \alpha_n|, |\Phi \beta_n|, |\gamma \gamma_n|)$$

$$= \max (|\theta \alpha_n|, |\Phi \beta_n|) = \max (l, m) \leq p$$

Hence

$$|x u_n + y v_n + z w_n| \leq p \ |\Delta|^{-1}.$$  

But from lemma 5.8, we get

$$|x u_n + y v_n + z w_n| \geq e^{-1}.$$  

Therefore,

$$p \ |\Delta|^{-1} \geq e^{-1}.$$
or \[ |\phi^2| \geq \frac{|\Delta|^3}{e^3} \]

or \[ \frac{|\Delta|}{e^3} \leq |\phi^2| = |\phi| \cdot |\gamma| (\text{Since } |\phi| = |\gamma|) \]

= \[ L_1(X) L_2'(X) L_3'(X) = L_1(X) (L_2^2(X) + L_3^2(X)) \]

But \( X = (x, y, z) \) is any arbitrary element of \( \mathbb{R}^3 \) congruent to \((x_0, y_0, z_0) \pmod{P_3}\), hence we get the theorem in this case.

Case II: \( M_1(u, v, w) = 0 \) for some non-zero \((u, v, w) \in P_3\),

byt \( M_2(u, v, w) \neq 0 \neq M_3(u, v, w) \) for any non-zero \((u, v, w) \in P_3\).

Let integer \( M \geq 0 \) be given. Then as in Case I, we can find a sequence \( \gamma^C_M \) satisfying the conditions of lemma 5.6.

However in this case, we might end up with an \( N \) such that \( \alpha_N = 0 \). Then lemma 5.7 will be valid, but the proof needs a modification on the lines of lemma 5.7' (on page 125).

With these sequences \( \gamma^C_M \), we proceed as in lemma 5.8. Then either the process of that lemma continues indefinitely in both directions to give a doubly infinite sequence and the proof of Case I applies or we have

**Lemma 5.8':** There exists an infinite set of values \( \alpha_n, \beta_n \).

\[ \gamma_n = \beta_n \text{ (} -\infty < n \leq N \text{)} \text{ of } M_1, M_2, M_3 \text{ corresponding to an infinite sequence } (u_n, v_n, w_n) \in P_3 \text{ with } \gcd(u_n, v_n, w_n) = 1 \]

satisfying the following conditions:
(i) \(|\alpha_n \beta_n^2| \leq e^{-2} |\Delta|^2\).

(ii) \(|\alpha_n| \leq e^{-1} |\alpha_{n - 1}|.

(iii) \(|\alpha_n - 1 \beta_n^2| \leq |\Delta|^2\).

(iv) \(\lim_{n \to \infty} |\beta_n| = 0, \lim_{n \to \infty} |\alpha_n| = \infty\).

(v) \(\alpha_N = 0, \alpha_n \neq 0\) for \(-\infty < n < N\).

(vi) \(|\beta_n| \geq |\beta_{n - 1}|\).

**Lemma 5.7':** For the infinite sequence \((u_n, v_n, w_n)\)

\((-\infty < n < N)\) obtained earlier, there exists \((x_o, y_o, z_o) \in R_3\)
such that for all

\((x, y, z) = (x_o, y_o, z_o) \pmod{P_3},\)

we have

\(|x u_n + y v_n + z w_n| \geq e^{-1} \ (-\infty < n < N)\)

for all \(n \ (-\infty < n < N)\).

**Proof.** Since \(\beta_N \neq 0\), choose \(K = \sigma + i \gamma, \sigma \in K[t], \gamma \in K[t]\)
such that

\(||\beta_n K + \beta_n \bar{K}|| = ||\beta_n K + \gamma_n \bar{K}|| = e^{-1}.\)

Due to condition (ii) of lemma 5.6', we can apply lemma 5.1 with

\(r = 1, \Lambda_n = \alpha_n, \Lambda_n = \beta_n K + \gamma_n \bar{K} \ (-N < n < \infty)\),
and we get \( \xi \in K \{ t \} \) such that
\[
\| \xi \alpha_n + \beta_n \kappa + \gamma_n \overline{\kappa} \| = e^{-1} \quad (- \infty < n < N).
\]
Also \( \alpha_N = 0 \). Thus
\[
\| \xi \alpha_n + \beta_n \kappa + \gamma_n \overline{\kappa} \| = e^{-1} \quad \text{for} \quad - \infty < n \leq N.
\]
Let \((x_0, y_0, z_0) \in R_3\) be a solution of
\[
L_1 (x, y, z) = 2 \xi \Delta, \quad L_2 (x, y, z) = 2 \sigma \Delta, \quad L_3 (x, y, z) = 2 \zeta \Delta
\]
i.e. \( x_0, y_0, z_0 \) satisfies
\[
L_1 (x, y, z) = i \xi \Delta', \quad L_2' (x, y, z) = i \kappa \Delta', \quad L_3' (x, y, z) = i \overline{\kappa} \Delta'.
\]
From (5.2.2) we get
\[
\| x_0 u_n + y_0 v_n + z_0 w_n \| = \| \xi \alpha_n + \beta_n \kappa + \gamma_n \overline{\kappa} \| = e^{-1} \quad (- \infty < n \leq N).
\]
Let \((x, y, z) \equiv (x_0, y_0, z_0) \pmod{P_3}\). Then
\[
(x u_n + y v_n + z w_n) - (x_0 u_n + y_0 v_n + z_0 w_n)
\]
\[
= (x - x_0) u_n + (y - y_0) v_n + (z - z_0) w_n
\]
is an element of \( K[t] \).

Therefore,
\[
\| x u_n + y v_n + z w_n \| = \| x_0 u_n + y_0 v_n + z_0 w_n \| = e^{-1} \quad (- \infty < n \leq N)
\]
Hence
\[ e^{-1} = \| x u_n + y v_n + z w_n \| \leq \| x u_n + y v_n + z w_n \| (- \infty < n \leq N) . \]

Proof of the theorem 20 (Case II): Let \((\alpha_n, \beta_n, \gamma_n) (- \infty < n \leq N)\) be the infinite sequence chosen above and \((x_o, y_o, z_o)\) be given by lemma 5.7. Take
\[(x, y, z) \equiv (x_o, y_o, z_o) \pmod{P_3} .\]

Let \(\theta, \phi, \psi = \overline{\psi}\) be the values taken by \(L_1, L_2', L_3'\) at \((x, y, z)\).

As before we can show \(\theta \neq 0\).

Now suppose \(\phi = 0 = \psi\). Then
\[ |x u_n + y u_n + z w_n| = \left| \frac{i \theta \alpha_N}{\Delta} \right| = 0 \]
which contradicts lemma 5.7'. Therefore \(\phi \neq 0 \neq \psi\).

If we can choose \((u_n, v_n, w_n)\) satisfying (5.2.13), the proof is completed as in Case I. Otherwise we have
\[ |\beta_N|^2 \leq |\theta|^\frac{1}{3} |\phi|^\frac{2}{3} |\Delta|^\frac{1}{3} . \]

From (5.2.2) we get
\[ |x u_n + y v_n + z w_n| = \frac{1}{|\Delta|} \left| \phi \beta_N + \psi \gamma_N \right| \quad \text{(Since } \alpha_N = 0) \]
\[ \leq \frac{|\phi \beta_N|}{|\Delta|} \quad \text{(Since } |\phi \beta_N| = |\psi \gamma_N|) \]
\[ \leq \frac{|\theta|^\frac{1}{3} |\phi|^\frac{1}{3} |\Delta|^\frac{1}{3} |\phi|}{|\Delta|} = \left( \frac{\theta \phi^2}{\Delta} \right)^\frac{1}{3} . \]
But
\[ |x u_N + y v_N + z w_N| \geq e^{-1}. \]

Hence
\[ \left| \frac{\Delta \Phi^2}{\Delta^3} \right| \geq e^{-1} \quad \text{or} \quad |\Phi^2| \geq \frac{|\Delta|}{\epsilon^3}, \]

and the theorem is proved in this case.

Case III: \( M_1 (u, v, w) \neq 0 \) for any non-zero \( (u, v, w) \in P_3 \),

whereas \( M_2 (u, v, w) = 0 = M_3 (u, v, w) \) for some non-zero polynomials \( u, v, w \).

First we prove the following lemma.

Lemma 5.6": Let an integer \( M \geq 0 \) be given. There is a finite set of values \( \alpha_n, \beta_n, \gamma_n = \beta_n (0 \leq n \leq N) \) of \( M_1, M_2, M_3 \) corresponding to polynomials \( (u_n, v_n, w_n) \) with \( \gcd (u_n, v_n, w_n) = 1 \) and satisfying conditions (i), (ii), (iii), (v), (vi) of lemma 5.6 and

\[ (iv'): \beta_0 = 0, \beta_n \neq 0 \text{ for } 0 < n \leq N. \]

Proof: By hypothesis there exists \( (p, q, r) \in P_3, (p, q, r) \neq 0 \)
such that

\[ M_2 (p, q, r) = 0 = M_3 (p, q, r). \]

Consider the solutions of

\[
\begin{aligned}
|M_1 (u, v, w)| &\leq |M_1 (p, q, r)| \\
M_2 (u, v, w) &\neq 0 = M_3 (u, v, w)
\end{aligned}
\]

(5.2.15)
By hypothesis, (5.2.15) has at least one non-zero solution 
\((p, q, r) \in P_3\). Using lemma 1.1, we can see that (5.2.15) has only 
a finite number of solutions \(U = (u, v, w) \in P_3\). Consider 
\[
\min |M_1(u, v, w)|,
\]
where the minimum is taken over non-zero \((u, v, w) \in P_3\) satisfying 
(5.2.15).

As there are only a finite number of solutions of (5.2.15), 
the above minimum is attained, say at \((u_o, v_o, w_o)\). Then 
\[M_1(u_o, v_o, w_o) \neq 0.\]

Then conditions (i), (iv) of lemma 5.6 are satisfied. 
Further if \((u_1, v_1, w_1)\) is any triplet of polynomials satisfying 
conditions (i), (ii), (iii) of lemma 5.6, 
\[|M_2(u_1, v_1, w_1)| \geq 0 = M_2(u_o, v_o, w_o),\]
so that condition (vi) of lemma 5.6 is satisfied.

In fact, we have 
\[|M_2(u_1, v_1, w_1)| > 0.\]

For, if not, by condition (ii) we have 
\[|M_1(u_1, v_1, w_1)| < |M_1(u_o, v_o, w_o)|\]
and this will contradict the choice of \((u_o, v_o, w_o)\). Hence we 
must have \(M_2(u_1, v_1, w_1) \neq 0\).
With this choice of \((u_0, v_0, w_0)\), we proceed as in lemma 5.6 and determine the sequence \(\sigma_M\) satisfying conditions (i), (ii), (iii), (v) and (vi) of lemma 5.6.

Since \(|\beta_n| \geq |\beta_{n-1}|\) and \(\beta_1 \neq 0\), we have the condition (iv') of the lemma.

**Lemma 5.7**: For the sequence obtained in lemma 5.6, there exists \((x_0(M), y_0(M), z_0(M)) \in \mathbb{R}^3\) such that

\[
|x u_n + y v_n + z w_n| \geq e^{-1}
\]  
(5.2.16)

for all \((x, y, z) = (x_0(M), y_0(M), z_0(M)) \pmod{P_3}\) and all triplets \((u_n, v_n, w_n)\) \((n = 0, 1, \ldots, N)\).

The proof of lemma 5.7 is exactly the same as that of lemma 5.7;

**Lemma 5.8**: There exists an infinite sequence \(\sigma\) of triplets \((u_n, v_n, w_n) \in \mathbb{P}_3\) \((0 \leq n < \infty)\) with \(\gcd(u_n, v_n, w_n) = 1\) such that conditions (i), (ii), (iii) and (vi) of lemma 5.6 hold and

\[
\lim_{n \to \infty} |\beta_n| = \infty, \quad \lim_{n \to \infty} |\alpha_n| = 0; \quad \text{and}
\]

\[
\beta_0 = 0, \quad \beta_n \neq 0 \quad \text{for} \quad 0 \leq n < \infty.
\]

Further there is an \((x_0, y_0, z_0) \in \mathbb{R}_3\) such that

\[
|x u_n + y v_n + z w_n| \geq e^{-1} \quad (0 \leq n < \infty)
\]
for all $(x, y, z) \equiv (x_0, y_0, z_0) \pmod{P_3}$.

Proof: Let $(u, v, w) \in P_3$ be such that

$$e^L < |M_1(u, v, w)|$$

for some integer $L$. Then we assert there exist only a finite number of $(u', v', w') \in P_3$ satisfying

(a) $|M_1(u, v, w)| |M_2(u', v', w')|^2 \leq |\Delta|^2$

(b) $|M_1(u', v', w')| \leq e^L$.

The proof of this is exactly as in lemma 5.8.

Now we construct the required sequence $\mathcal{J}$ in the following manner:

Choose $(\bar{u}_0, \bar{v}_0, \bar{w}_0)$ as in lemma 5.6", which occur in infinitely many $\mathcal{J}_M$ (In fact choice of $(u_0, v_0, w_0)$ in lemma 5.6" is independent of choice of $M$ and so we use the same $(u_0, v_0, w_0)$ for all $M$). Now for all $\mathcal{J}_M$, we have

(1) $|M_1(u_0, v_0, w_0)| \cdot |M_2(u_1, v_1, w_1)|^2 \leq |\Delta|^2$

(2) $|M_1(u_1, v_1, w_1)| \leq e^{-1} |M_1(u_0, v_0, w_0)|$.

And by observation made earlier, there are only a finite number of choices for $(u_1, v_1, w_1)$. Therefore, we can pick out $(\bar{u}_1, \bar{v}_1, \bar{w}_1)$, which occur in infinitely many $\mathcal{J}_M$. 
Let \( G_1 \) be the infinite subsequence of those \( \mathcal{J}_M \)'s to which 
\((\overline{u}_1, \overline{v}_1, \overline{w}_1)\) belong. Now we consider only these \( \mathcal{J}_M \)'s.

Now consider the solutions of

\[
\begin{align*}
(i) \quad |M_1(u, v, w)| \cdot |M_2(u, v, w)|^2 &\leq e^{-2} |\Delta|^2 \\
(ii) \quad |M_1(u, v, w)| &\leq e^{-\frac{1}{2}} |M_1(\overline{u}_1, \overline{v}_1, \overline{w}_1)| \\
(iii) \quad |M_1(\overline{u}_1, \overline{v}_1, \overline{w}_1)| \cdot |M_2(u, v, w)|^2 &\leq |\Delta|^2
\end{align*}
\]

Again this has a finite number of solutions \((u_2, v_2, w_2)\).

So we consider \((\overline{u}_2, \overline{v}_2, \overline{w}_2)\), which belongs to infinitely many \( \mathcal{J}_M \).

Let \( G_2 \) be the infinite subsequence of \( G_1 \) to which \((\overline{u}_2, \overline{v}_2, \overline{w}_2)\) belong. Now we consider only these \( \mathcal{J}_M \)'s.

In this way, we get an infinite sequence \((\overline{u}_n, \overline{v}_n, \overline{w}_n)\)
\((0 \leq n < \infty)\) which satisfies conditions (i), (ii), (iii), (vi) of lemma 5.6 and

\[
M_2(\overline{u}_0, \overline{v}_0, \overline{w}_0) = 0, \quad M_2(\overline{u}_n, \overline{v}_n, \overline{w}_n) \neq 0 \text{ for } n \neq 0.
\]

Also by construction

\[
\lim_{n \to \infty} |M_1(\overline{u}_n, \overline{v}_n, \overline{w}_n)| = 0.
\]

As in lemma 5.8, we can see that

\[
\lim_{n \to \infty} |M_2(\overline{u}_n, \overline{v}_n, \overline{w}_n)| = \infty.
\]
In the construction of \((\bar{u}_n, \bar{v}_n, \bar{w}_n)'s\) we have used subsequences \(\mathcal{C}_1, \mathcal{C}_2, \ldots\) which have the same properties as we had in lemma 5.8.

Following the same method as in lemma 5.8, we can get 
\((x_0, y_0, z_0)\) with the required properties.

Proof of theorem in Case III: Let \((u_n, v_n, w_n) (0 \leq n < \infty)\) and 
\((x_0, y_0, z_0)\) be given by lemma 5.8. Suppose \(\theta, \phi, \psi = \bar{\phi}\) are the values assumed by \(L_1, L_2', L_3'\) respectively for some 
\((x, y, z) \equiv (x_0, y_0, z_0) \pmod{P_3}\).

First we assert neither of \(\theta, \phi, \psi\) is zero. For suppose \(\theta = 0\). From (5.2.2) we get 
\[x u_n + y v_n + z w_n = \frac{\left(\phi \beta_n + \psi \gamma_n\right)}{\Delta},\]
Then \(x u_0 + y v_0 + z w_0 = 0\) (Since \(\beta_0 = 0 = \gamma_0\)) and this contradicts lemma 5.8''.

Now suppose \(\phi = 0 = \psi\). Then from (5.2.2) we get 
\[x u_n + y v_n + z w_n = \frac{i \theta \alpha_n}{\Delta},\]
So 
\[|x u_n + y v_n + z w_n| = \left|\frac{i \theta \alpha_n}{\Delta}\right| = \left|\frac{\theta \alpha_n}{\Delta}\right| = 0.\]
And 
\[\lim_{n \to \infty} |x u_n + y v_n + z w_n| = \lim_{n \to \infty} \left|\frac{\theta \alpha_n}{\Delta}\right| = 0.\]
This contradicts lemma 5.8'' again.

Now proof of theorem in this case is exactly the same as in Case I.

Case IV: $M_1(u, v, w) = 0$ for some non-zero polynomials $(u, v, w)$ and $M_2(u, v, w) = 0 = M_3(u, v, w)$ for another set of non-zero polynomials.

First of all, we determine non-zero $(u_o, v_o, w_o) \in P_3$ as in lemma 5.6''. Now proceeding as in lemma 5.6'', if we don't get at any $n$ (for any $M$) such that $\alpha_n = 0$, the rest of the proof of the theorem is the same as in Case III. Otherwise we prove the following lemma.

Lemma 5.6'': There exists a finite set of values $\alpha_n, \beta_n, \gamma_n =$ $\bar{\beta}_n, 0 \leq n \leq N)$ of $M_1, M_2, M_3$ corresponding to polynomials $(u_n, v_n, w_n)$ with $\gcd(u_n, v_n, w_n) = 1$ satisfying conditions (i), (ii), (iii), (vi) of lemma 5.8 and

(iv'') $\beta_0 = 0$, $\beta_n \neq 0$ for $0 < n \leq N$.

(v'') $\alpha_N = 0$, $\alpha_n \neq 0$ for $0 \leq n < N$.

Remark: Since determinant of the forms $M_1, M_2, M_3$ is not zero, $M_1, M_2, M_3$ cannot assume the value zero simultaneously for a non-zero $(u, v, w) \in P_3$ and hence $N \geq 1$, so that $\beta_N = 0$. 
For this finite sequence \( (u_n, v_n, w_n) \) \((0 \leq n \leq N)\) \((N \geq 1)\), proceeding exactly as in lemma 5.7', we get the following lemma:

Lemma 5.7": For the finite sequence \( (u_n, v_n, w_n) \) \((0 \leq n \leq N)\) obtained in lemma 5.6", there exists \((x_0, y_0, z_0) \in R_3\), such that for all \((x, y, z) = (x_0, y_0, z_0) \pmod{P_3}\), we have

\[
|x u_n + y v_n + z w_n| \geq e^{-1} \quad (0 \leq n \leq N).
\]

The proof of the theorem in this case can now be completed as in Case II.

5.3: In this section, we shall prove the following theorem.

Theorem 21: Let \( K \) be a finite field, in which \( x^2 + 1 = 0 \) is not soluble. Let \( L_1, L_2, L_3, L_4 \) be four linear forms in variables \( x, y, z, s \) and with determinant \( \Delta \neq 0 \). Then there exists \((x_0, y_0, z_0, s_0) \in R_4\) such that

\[
|(L_1^2(x) + L_2^2(x))| L_3^2(x) + L_4^2(x)| \geq \frac{|\Delta|}{e^4} \quad (5.3.1)
\]

whenever \( X = (x, y, z, s) \equiv (x_0, y_0, z_0, s_0) \pmod{P_4}\).

Remark 5.6: The result is best possible in the sense that there exist forms \( L_1, L_2, L_3, L_4 \) such that for every \((x_0, y_0, z_0, s_0) \in R_4\), there exists \((x, y, z, s) \equiv (x_0, y_0, z_0, s_0) \pmod{P_4}\), for which

\[
|(L_1^2 + L_2^2)(L_3^2 + L_4^2)| \leq \frac{|\Delta|}{e^4}.
\]
For example, take \( L_1 = x, L_2 = y, L_3 = z, L_4 = s \).

Before proving the actual theorem, we need a few preliminaries.

Suppose the given linear forms are

\[
L_j = a_j x + b_j y + c_j z + d_j s \quad (j = 1, 2, 3, 4).
\]

Write

\[
L_1' = L_1 + i L_2, \quad L_2' = L_1 - i L_2,
\]

\[
L_3' = L_3 + i L_4, \quad L_4' = L_3 - i L_4, \quad (i^2 = -1).
\]

Let \( \Delta' \) be the determinant of \( L_1', L_2', L_3', L_4' \) respectively.

Then

\[
\Delta' = \begin{vmatrix}
 a_1 + i a_2 & b_1 + i b_2 & c_1 + i c_2 & d_1 + i d_2 \\
 a_1 - i a_2 & b_1 - i b_2 & c_1 - i c_2 & d_1 - i d_2 \\
 a_3 + i a_4 & b_3 + i b_4 & c_3 + i c_4 & d_3 + i d_4 \\
 a_3 + i a_4 & b_3 - i b_4 & c_3 - i c_4 & d_3 - i d_4
\end{vmatrix}
\]

\[
= \begin{vmatrix}
 2a_1 & 2b_1 & 2c_1 & 2d_1 \\
 a_1 - i a_2 & b_1 - i b_2 & c_1 - i c_2 & d_1 - i d_2 \\
 2a_3 & 2b_3 & 2c_3 & 2d_3 \\
 a_3 - i a_4 & b_3 - i b_4 & c_3 - i c_4 & d_3 - i d_4
\end{vmatrix}
\]
Thus $\triangle \in K\{t\}$.

Since $x^2 + 1 = 0$ is not soluble in the field $K$, characteristic of the field is not 2. So $|\triangle'| = |-4\triangle| = |\triangle|$.

Let $M_1', M_2', M_3', M_4'$ be forms in variables $u, v, w, r$ such that the coefficients of $M_j'$ are cofactors of the corresponding coefficients of $L_j'$ ($j = 1, 2, 3, 4$) respectively. Then

$$L_1'M_1' + L_2'M_2' + L_3'M_3' + L_4'M_4' = \triangle'(xu + yv + zw + sr)$$

Define

$$M_j = \frac{1}{i} M_j' \quad (j = 1, 2, 3, 4)$$

Then we can see that $M_1 = \overline{M_2}$, $M_3 = \overline{M_4}$. 
Also
\[ \det (M_1, M_2, M_3, M_4) = \det (M_1', M_2', M_3', M_4') = \Delta^3 \] (5.3.2)

and
\[ L_1' M_1 + L_2' M_2 + L_3' M_3 + L_4' M_4 = \frac{\Delta'}{i} (x u + y v + z w + s r) \]
\[ = -i\Delta' (x u + y v + z w + s r). \] (5.3.3)

Now we prove a few lemmas.

**Lemma 5.9:** Let \( \lambda_0, \lambda_1, \ldots, \lambda_n, \ldots \) be a finite or infinite sequence from the field \( \mathbb{K}\{t\} \) such that
\[ |\lambda_{n+1}| \geq e^{\lambda_n}, \quad \lambda_0 \neq 0. \]

Let \( \mu_0, \mu_1, \ldots \) be arbitrary given elements of \( \mathbb{K}\{t\} \). Then there exists \( \xi \in \mathbb{K}\{t\} \) such that
\[ || \lambda_n \xi + \lambda_n \xi \mu_n || = e^{-1} (n = 0, 1, 2, \ldots). \]

**Proof:** First we note the following. Suppose
\[ \lambda_n \xi = \sigma + i\tau. \]

Then
\[ \lambda_n \xi = \sigma - i\tau. \]

And
\[ \lambda_n \xi + \lambda_n \xi = 2\sigma \in \mathbb{K}\{t\}. \]

So that, for \( \lambda_n \xi + \lambda_n \xi \), we have to fix our attention on \( \sigma \).
Without loss of generality, we can assume that the degree of \( \mu_n \) is at most \(-1\). Suppose

\[
\lambda_n = a_{j_n} (n) t^{j_n} + a_{j_n-1} (n) t^{j_n - 1} + \ldots .
\]

\[
+ i \left\{ b_{j_n} (n) t^{j_n} + b_{j_n-1} (n) t^{j_n - 1} + \ldots . \right\}
\]

where \( a \)'s and \( b \)'s belong to \( K \) and at least one of \( a_{j_n} (n) \) and \( b_{j_n} (n) \) is not zero. Let

\[
\lambda_n = r_{-1} (n) t^{-1} + r_{-2} (n) t^{-2} + \ldots .
\]

Take

\[
\zeta = \alpha_{-j_{o-1}} t^{-j_{o-1}} + \alpha_{-j_{o-2}} t^{-j_{o-2}} + \ldots .
\]

\[
+ i \left\{ \beta_{-j_{o-1}} t^{-j_{o-1}} + \beta_{-j_{o-2}} t^{-j_{o-2}} + \ldots . \right\}
\]

where \( \alpha \)'s and \( \beta \)'s belong to \( K \). We shall prove that with suitable choice of \( \alpha \)'s, we have the required result. We choose these by induction.

Coefficient of \( t^{-1} \) in \( \lambda_{j_{o} \overline{\xi}} + \lambda_{o} \overline{\xi} + \mu_{o} \)

\[
= 2 \left\{ a_{j_{o}} (o) \alpha_{-j_{o}-1} - b_{j_{o}} (o) \beta_{-j_{o}-1} \right\} + r_{-1} (o).
\]
Since at least one of \( a_j(0) \), \( b_j(0) \) is not zero, we can choose \( \alpha_{-j-1}, \beta_{-j-1} \in K \) such that coefficient of \( t^{-1} \) in \\
\[ \lambda_0 \xi + \overline{\lambda}_0 \overline{\xi} + \mu_0 \] is not zero. Suppose we have selected already \\
\[ \alpha_{-j+1}, \ldots, \alpha_{-(j_k-1)+1}, \beta_{-(j+1)}, \ldots, \beta_{-(j_k-1)+1}. \]
Since \( j_k \geq j_k - 1 + 1 \), choose \( \alpha_{-(j_k+1)}, \ldots, \alpha_{-j_k} \), \( \beta_{-(j_k+1)}, \ldots, \beta_{j_k} \) arbitrarily and \( \alpha_{-(j+1)}, \beta_{-(j+1)} \)
such that coefficient of \( t^{-1} \) in \\
\[ \lambda_k \xi + \overline{\lambda}_k \overline{\xi} + \mu_k \]
\[ = 2 \left\{ (\alpha_{-(j+1)} a_{j_k}^{(k)} + \ldots + \alpha_{-(j+1)} a_j^{(k)}) \right. \]
\[ - (\beta_{-(j+1)} b_{j_k}^{(k)} + \ldots + \beta_{-(j+1)} b_j^{(k)}) \left. \right\} + r_{-1}(k) \]
is not zero. This is possible, since \( (a_{j_k}^{(k)}, b_{j_k}^{(k)}) \neq (0, 0) \).
Thus the induction is complete and the lemma is proved.

**Lemma 5.10:** Let \( p, q \) be integers such that
\[ e^{2p + 2q} \geq e^{-3} |\Delta|^3. \]
Then there exists a non-zero vector \( U = (u, v, w, r) \in P_4 \) such that
\[ |M_1(U)| = |M_2(U)| \leq e^p, |M_3(U)| = |M_4(U)| \leq e^q \quad (5.3.4) \]
Further (5.3.4) have only a finite number of solutions from \( P_4 \).
Proof: Since $M_1 = \overline{M_2}$ and $M_3 = \overline{M_4}$, let

$$M_1 = N_1 + iN_2, \quad M_2 = N_1 - iN_2,$$
$$M_3 = N_3 + iN_4 \quad \text{and} \quad M_4 = N_3 - iN_4.$$ 

where $N_1, N_2, N_3, N_4$ are linear forms with coefficients from $K\{t\}$. Then by what we have seen earlier, we have

$$|\det (M_1, M_2, M_3, M_4)| = |\det (N_1, N_2, N_3, N_4)| = |\Delta|^3 \neq 0.$$

Since $e^{2p} + 2q \geq e^{-3|\Delta|^3}$, by theorem A of chapter 1, there exists a non-zero vector $U = (u, v, w, r) \in P_4$ satisfying

$$|N_1(U)| \leq e^p, \quad |N_2(U)| \leq e^p, \quad |N_3(U)| \leq e^q, \quad |N_4(U)| \leq e^q. \quad (5.3.5)$$

But, by definition, for any $U \in P_4$,

$$|M_1| = |M_2| = \max \{|N_1|, |N_2|\}$$
and

$$|M_3| = |M_4| = \max \{|N_3|, |N_4|\}$$

So non-zero $U \in P_4$, which satisfies (5.3.5), must also satisfy (5.3.4). and this proves the first part.

To prove that (5.3.4) has only a finite number of solutions, we first observe that any solution of (5.3.4) is also a solution of (5.3.5). But by lemma 1.1, (5.3.5) has only a finite number of solutions $U = (u, v, w, r) \in P_4$. Hence (5.3.4) has only a finite number of solutions.
Remark 5.7: Let $K$ be a field (not necessarily finite) in which $x^2 + 1 = 0$ is not soluble. Let $L_1, L_2, L_3, L_4$ be four linear forms in variables $(x, y, z, s) = X$ with determinant $\Delta \neq 0$. Then lemma 5.10 incidentally proves that there exist a non-zero vector $X \in P_4$ satisfying

$$|L_1^2(X) + L_2^2(X)| \leq e^{-3} |\Delta|.$$

Now we distinguish the following three cases.

(i) $M_1 \neq 0 \neq M_2, M_3 \neq 0 \neq M_4$ for any non-zero $U \in P_4$.

(ii) One and only one of the pairs $(M_1, M_2), (M_3, M_4)$, say $(M_3, M_4)$ assumes the value $(0, 0)$ for some non-zero vector $U \in P_4$.

(iii) $M_1 = 0 = M_2$ for some non-zero $U \in P_4$ and $M_3 = 0 = M_4$ for another non-zero vector $U \in P_4$.

Case I: None of $M_1, M_2, M_3, M_4$ assume the value zero for any non-zero $U \in P_4$.

First we prove the following:

**Lemma 5.11:** Let $M > 0$ be any given integer. Then there exists a finite sequence of values $\alpha_n, \beta_n = \overline{\alpha}_n, \gamma_n, \delta_n = \overline{\gamma}_n \ (0 \leq n \leq N)$ of $M_1, M_2, M_3, M_4$ corresponding to the sequence $\sum M$ of polynomials $(u_n, v_n, w_n, r_n) = 1$. 
such that

\[ |\alpha_n^2 \gamma_n^2| \leq e^{-2} |\Delta|^3. \]

(ii) \[ |\alpha_n| \leq e^{-1} |\alpha_{n-1}| \quad (n \geq 1). \]

(iii) \[ |\gamma_n^2 \alpha_{n-1}^2| \leq |\Delta|^3 \quad (n \geq 1). \]

(iv) \[ |\gamma_0| \leq e^{-M} \quad |\alpha_0| \geq e^M. \]

(v) \[ |\alpha_N| \leq e^{-M}. \]

(vi) \[ |\gamma_n| \geq |\gamma_{n-1}| \quad (n \geq 1). \]

Proof: Let \(|\Delta| = e^d\). Consider the solutions of

\[
\begin{align*}
M_1(u, v, w, r) &= M_2(u, v, w, r) = \max (e^M, e^M + \frac{3d}{2}) \\
M_3(u, v, w, r) &= M_4(u, v, w, r) \leq e^{-M}
\end{align*}
\]

By lemma 5.10, the inequalities (5.3.6) have a non-zero solution \(U = (u, v, w, r) \in P_4\) and the number of such solutions is finite. Consider

\[ \min |M_3(u, v, w, r)| \]

as \((u, v, w, r)\) runs over non-zero solutions from \(P_4\) of inequalities (5.3.6). Since the number of these solutions is finite, this minimum is attained. Since \(M_3 \neq 0\) for any non-zero \(U \in P_4\), this minimum is not zero. Let this minimum be \(e^{-T}\). Then \(T \geq M\).
Consider the solutions of

\[ |M_1(u, v, w, r)| = |M_2(u, v, w, r)| \leq e^{T + \frac{3d}{2}} \]
\[ |M_3(u, v, w, r)| = |M_4(u, v, w, r)| \leq e^{-T - 1} \]

(5.3.7)

By lemma 5.10, the inequalities (5.3.7) have at least one non-zero solution \( U = (u, v, w, r) \in P_4 \). Further the number of such solutions is finite. Hence \( \min |M_3(u, v, w, r)| \) taken over non-zero vectors in \( P_4 \) satisfying (5.3.7) is attained, say at \((u_0, v_0, w_0, r_0)\). Obviously, we must have \( \gcd(u_0, v_0, w_0, r_0) = 1 \).

Let

\[ \alpha_0 = M_1(u_0, v_0, w_0, r_0), \beta_0 = M_2(u_0, v_0, w_0, r_0), \gamma_0 = M_3(u_0, v_0, w_0, r_0) \]
\[ \delta_0 = M_4(u_0, v_0, w_0, r_0). \]

Then \( |\gamma_0| = |\delta_0| \leq e^{-T-1} \leq e^{-M-1} \) (since \( T \geq M \)).

Also for every non-zero \( U = (u, v, w, r) \in P_4 \) satisfying (5.3.7), we must have

\[ |M_1(u, v, w, r)| > e^M. \]

Otherwise, we have

\[ |M_1(u, v, w, r)| \leq e^M. \]
and then this \((u, v, w, r)\) satisfies the system (5.3.6), which contradicts the minimal nature of \(T\). Hence, in particular

\[ |\alpha_o| \geq e^M. \]

Since \((u_o, v_o, w_o, r_o)\) satisfies (5.3.7), \((u_o, v_o, w_o, r_o)\) also satisfies condition (i) of lemma 5.11.

Suppose there exists \((u, v, w, r)\) satisfying

\[ | M_1 (u, v, w, r)| \leq e^{-1} |\alpha_o|. \]

Then, we must have

\[ | M_3 (u, v, w, r)| \geq |\gamma_o|. \]

For, suppose we have

\[ | M_3 (u, v, w, r)| < |\gamma_o|. \]

Then this \((u, v, w, r)\) also satisfies (5.3.7), and this contradicts the minimal nature of \((u_o, v_o, w_o, r_o)\).

Now we shall construct a sequence \(\mathcal{M}^f\) satisfying the conditions of lemma 5.11 by induction. \((u_o, v_o, w_o, r_o)\) satisfies the conditions of the lemma. Suppose \((u_o, v_o, w_o, r_o), \ldots, (u_{n-1}, v_{n-1}, w_{n-1}, r_{n-1})\) have been constructed.
Consider the solutions of
\[
|M_1 (u, v, w, r)| = |M_2 (u, v, w, r)| \leq e^{-1} |\alpha_{n-1}|
\]
(5.3.8)
\[
|M_3 (u, v, w, r)| = |M_4 (u, v, w, r)| \leq |\alpha_{n-1}|^{-1} e^{\frac{2d}{2}}
\]

By lemma 5.10, there exists at least one non-zero solution \((u, v, w, r) \in P_4\) satisfying (5.3.8), and there are only a finite number of such solutions.

Consider
\[
\text{Min} \{M_3 (u, v, w, r)\}
\]
where minimum is being taken over all non-zero vectors \((u, v, w, r) \in P_4\)
satisfying (5.3.8). Since there are only a finite number of such vectors, this minimum is attained, say at \((u_n, v_n, w_n, r_n)\),
obviously, which satisfies conditions (i), (ii), (iii) of lemma 5.11.
It must also satisfy condition (vi), for otherwise it would have been selected at the previous stage. Since
\[
|\alpha_n| \leq e^{-1} |\alpha_{n-1}|
\]
for all \(n\), we must get at an integer \(N'\) such that
\[
|\alpha_n| \leq e^{-M}
\]
for all \(n \geq N'\). We stop at such an \(N\).
Lemma 5.12: There exists $(x_0^{(M)}, y_0^{(M)}, z_0^{(M)}, s_0^{(M)}) \in R_4$ such that

$$|x_n y_n z_n s_n| \geq e^{-1}$$

(5.3.9)

for all $(x, y, z, s) = (x_0^{(M)}, y_0^{(M)}, z_0^{(M)}, s_0^{(M)}) \pmod{P_4}$

for all $n (0 \leq n \leq N)$.

Proof: Consider

$$\Delta_n = \frac{\alpha_n - n}{\Delta}, \mu_n = 0 \quad (n = 0, 1, \ldots, N).$$

Owing to condition (ii) of lemma 5.11,

$$|\Delta_n| \geq e|\Delta_n - 1| \quad (n = 1, \ldots, N).$$

We can, therefore, apply lemma 5.6 and get $\xi \in K^\circ \{t\}$, such that

$$\left| \frac{\alpha_n \xi + \alpha_n \overline{\xi}}{\Delta_n} \right| = \left| \frac{\alpha_n \xi + \beta_n \overline{\xi}}{\Delta_n} \right| = e^{-1} \quad (n = 0, 1, \ldots, N).$$

Let $\xi = \gamma + i \delta$; $\gamma, \delta \in K \{t\}$. Define

$(x_0^{(M)}, y_0^{(M)}, z_0^{(M)}, s_0^{(M)}) \in R_4$ by the equations

$$L_1(x_0^{(M)}, y_0^{(M)}, z_0^{(M)}, s_0^{(M)}) = \gamma, L_2(x_0^{(M)}, y_0^{(M)}, z_0^{(M)}, s_0^{(M)}) = \zeta$$

$$L_3(x_0^{(M)}, y_0^{(M)}, z_0^{(M)}, s_0^{(M)}) = 0 = L_4(x_0^{(M)}, y_0^{(M)}, z_0^{(M)}, s_0^{(M)}).$$
Then \((x_0(M), y_0(M), z_0(M), s_0(M))\) also satisfies

\[
L_1'(x_0(M), y_0(M), z_0(M), s_0(M)) = \xi, \quad L_2'(x_0(M), y_0(M), z_0(M), s_0(M)) = \overline{\xi},
\]

\[
L_3'(x_0(M), y_0(M), z_0(M), s_0(M)) = 0 = L_4'(x_0(M), y_0(M), z_0(M), s_0(M)).
\]

Substituting \(x = x_0(M), y = y_0(M), z = z_0(M), s = s_0(M),\)

\(u = u_n, v = v_n, w = w_n, r = r_n\) in (5.3.3) we get

\[
x_0(M)u_n + y_0(M)v_n + z_0(M)w_n + s_0(M)r_n = \frac{i(\xi \alpha_n^+ \overline{\xi} \alpha_n^{-})}{\Delta'}.
\]

Hence

\[
\|x_0(M)u_n + y_0(M)v_n + z_0(M)w_n + s_0(M)r_n\| = \|\frac{i(\xi \alpha_n^+ \overline{\xi} \alpha_n^{-})}{\Delta'}\|
\]

\[
= \|\frac{\xi \alpha_n^+ \overline{\xi} \alpha_n^{-}}{\Delta'}\| = e^{-1} (n = 0, 1, \ldots, N).
\]

Further if \((x, y, z, s) \notin (x_0(M), y_0(M), z_0(M), s_0(M)) (\text{mod } P_i),\)

we have

\[
(x u_n + y v_n + z w_n + s r_n) - (x_0(M)u_n + y_0(M)v_n + z_0(M)w_n + s_0(M)r_n)
\]

\[
= (x - x_0(M))u_n + (y - y_0(M))v_n + (z - z_0(M))w_n + (s - s_0(M))r_n
\]

and the right hand side is an element of \(K[t]\). Consequently

\[
\|x u_n + y v_n + z w_n + s r_n\| = \|x_0(M)u_n + y_0(M)v_n + z_0(M)w_n + s_0(M)r_n\|
\]

\((n = 0, 1, \ldots, N).\)
Hence

\[ |x u_n + y v_n + z w_n + s r_n| \geq |x u_n + y v_n + z w_n + s r_n| = e^{-1} \]

\[(n = 0, 1, \ldots, N).\]

This proves the lemma.

**Lemma 5.13:** There exists a doubly infinite sequence of polynomials

\((u_n, v_n, w_n, r_n) (-\infty < n < \infty)\) with \(\gcd (u_n, v_n, w_n, r_n) = 1\),

such that conditions (i), (ii), (iii), and (vi) of lemma 5.11 hold and

\[ \lim_{n \to \infty} |\alpha_n| = 0 = \lim_{n \to \infty} (\gamma_n|). \]

Also

\[ \lim_{n \to \infty} |\beta_n| = \infty = \lim_{n \to \infty} |\alpha_n| . \]

Further there is an \((x_0, y_0, z_0, s_0) \subset \mathbb{R}^4\), such that (5.3.9) holds for all \((x, y, z, s) = (x_0, y_0, z_0, s_0) (\mod P_4)\) and for all \(n (-\infty < n < \infty)\).

**Proof:** Let \(L\) be any given integer. First we assert there are only a finite number of \((u_0, v_0, w_0, r_0)\), \((u_0', v_0', w_0', r_0') \subset P_4\) satisfying

(a) \(|M_1(u, v, w, r)|^2 \cdot |M_3(u', v', w', r')|^2 \leq |\Delta|^3\)

(b) \(|M_1(u, v, w, r)|^2 \cdot |M_3(u, v, w, r)|^2 \leq e^{-2} |\Delta|^3\)

(c) \(|M_2(u', v', w', r')| \leq e^{L} < |M_1(u, v, w, r)| .\)
From (a) and (c) we get

$$|M_3(u', v', w', r'|^2 \leq e^{-2L} |\Delta|^3.$$  

Therefore,

$$|M_3(u', v', w', r')| \leq e^{-L + [\frac{3d}{2}]}.$$

But for \((u', v', w', r') \in P_4,$

$$|M_2(u', v', w', r')| = |M_2(u', v', w', r')|$$

and

$$|M_3(u', v', w', r')| = |M_4(u', v', w', r')|.$$

So we have

$$|M_1(u', v', w', r')| = |M_2(u', v', w', r')| \leq e^{L} \text{ (From (c)).}$$

and

$$|M_3(u', v', w', r')| = |M_4(u', v', w', r')| \leq e^{-L + [\frac{3d}{2}]}.$$

By lemma 5.10, there exist only a finite number of \((u', v', w', r') \in P_4, \text{ satisfying above.} \)

Further given non-zero \((u', v', w', r') \in P_4, \text{ we assert there are only a finite number of non-zero } (u, v, w, r) \in P_4. \text{ Since by assumption, } M_3(u', v', w', r') \neq 0, \text{ from (a) we get}\]

$$|M_1(u, v, w, r)|^2 \leq |\Delta|^3 \cdot |M_3(u', v', w', r')|^{-2}$$

Then

$$|M_2(u, v, w, r)| = |M_1(u, v, w, r)| \leq e^{-[\frac{3d}{2}]} |M_3(u', v', w', r')|^{-1}. $$
From (b) we get

\[ |M_3(u, v, w, r)|^2 \leq e^{-2|\Delta|^3} \cdot |M_1(u, v, w, r)|^{-2} \leq e^{-2-2L} |\Delta|^3 \]

Then

\[ |M_3(u, v, w, r)| = |M_4(u, v, w, r)| \leq e^{-L+\frac{3d}{2}}. \]

Now using lemma 1.1 of chapter 1, we see that there are only a finite number of \((u, v, w, r)\in \mathbb{P}_4\) satisfying these.

Now suppose \(M > 0\) is a given integer. Construct a sequence \(c_M\) satisfying the conditions of lemma 5.11. By conditions (iv), (v) and (ii) of lemma 5.6, there exists a unique integer \(N_1\) such that

\[ N \geq N_1 \geq 1 \quad \text{and} \quad \left| \alpha_{N_1} \right| \leq 1 < \left| \alpha_{N_1-1} \right|. \]

Let \(s_M\) be the sequence of \((u_n', v_n', w_n', r_n')\) defined by

\[ (u_n', v_n', w_n', r_n') = (u_n + N_1', v_n + N_1', w_n + N_1', r_n + N_1) \]

for \(-N_1 \leq n \leq N_2 = N - N_1\).

Then conditions (i), (ii), (iii) and (vi) of lemma 5.11 hold for \((u_n', v_n', w_n', r_n')\) and

\[ |\alpha_0'| \leq 1 < |\alpha_{-1}'|, \quad |\alpha_{N_2}'| \leq e^{-M}, \quad |\alpha_{-N_1}'| \geq e^M \] (5.3.10)
We now construct $\mathcal{J}$ by a diagonal process. By the observation already made, there are only a finite number of possibilities for 
$(u_0', v_0', w_0', r_0')$ and $(u_{-1}', v_{-1}', w_{-1}', r_{-1}')$ and so at least one of these must occur infinitely often, as $M$ runs through the set of natural numbers. We fix one such possible pair, say 
$(\overline{u}_0, \overline{v}_0, \overline{w}_0, \overline{r}_0), (\overline{u}_{-1}, \overline{v}_{-1}, \overline{w}_{-1}, \overline{r}_{-1})$. Let $\mathcal{J}_1$ be the subsequence of those $\mathcal{J}_M'$'s to which $(\overline{u}_0, \overline{v}_0, \overline{w}_0, \overline{r}_0)$ and $(\overline{u}_{-1}, \overline{v}_{-1}, \overline{w}_{-1}, \overline{r}_{-1})$ belong. We now consider only these $\mathcal{J}_M'$'s.

For these $M$, conditions (5.3.10) imply that for all $M$
large enough $N_2 \geq 1$, for otherwise $M_1 (u_0', v_0', w_0', r_0') = 0$,
which contradicts the hypothesis.

Denote by $\overline{\alpha}_n, \overline{\beta}_n, \overline{\gamma}_n, \overline{\delta}_n$ the values assumed by
$M_1, M_2, M_3, M_4$ at $(\overline{u}_n, \overline{v}_n, \overline{w}_n, \overline{r}_n)$. Now

$$|\alpha_1'| \leq \varepsilon |\alpha_0| < |\alpha_0| \quad \text{for each } \mathcal{J}_M' \text{'s}.$$  

Again, there are only a finite number of possible choices for
$(u_1, v_1, w_1, r_1) \in P_4$. We pick out $(\overline{u}_1, \overline{v}_1, \overline{w}_1, \overline{r}_1) \in P_4$ with
$\gcd (u_1, v_1, w_1, r_1) = 1$, which occurs infinitely often. Let $\mathcal{J}_2$
be the infinite subsequence of $\mathcal{J}_1$ of those $\mathcal{J}_M'$'s to which
$(\overline{u}_1, \overline{v}_1, \overline{w}_1, \overline{r}_1)$ belong. Now we consider only these $\mathcal{J}_M'$'s. For
these $\mathcal{J}_M'$'s if $M$ is large enough, $N_1 \geq 2$.  


Since

$$| M_1 (\bar{u}_1, \bar{v}_1, \bar{w}_1, \bar{r}_1) | \leq | \bar{\alpha}_1 | < | \alpha_2 ' |,$$

as remarked earlier, there are only a finite number of choices of

$$(u_2', v_2', w_2', r_2').$$

We pick out $(\bar{u}_2', \bar{v}_2', \bar{w}_2', \bar{r}_2')$, which
occurs infinitely often. Let $\bar{G}_3$ be the infinite subsequence of

$\bar{G}_2$ of those $\mathcal{G}_M$'s to which $(\bar{u}_2', \bar{v}_2', \bar{w}_2', \bar{r}_2')$ belong and we
consider only these $\mathcal{G}_M$'s. For these $N_2 \geq 2$ if $M$'s large
enough.

$$| \alpha_2 ' | \leq e^{-1} | \tilde{\alpha}_1 | < | \tilde{\alpha}_1 |$$

and we can pick out $(\bar{u}_2, \bar{v}_2, \bar{w}_2, \bar{r}_2)$ which occurs for infinitely
many $\mathcal{G}_M$'s. Let $\bar{G}_4$ be the infinite subsequence of $\bar{G}_3$ to which

$(\bar{u}_2, \bar{v}_2, \bar{w}_2, \bar{r}_2)$ belong. Continuing this process, we get a doubly
infinite sequence $(\bar{u}_n, \bar{v}_n, \bar{w}_n, \bar{r}_n)$ ($-\infty < n < \infty$), which satisfies
conditions (i), (ii), (iii), (vi) of lemma 5.11.

Obviously, by construction

$$\lim_{n \to \infty} | \tilde{\alpha}_n | = 0, \quad \lim_{n \to -\infty} | \tilde{\alpha}_n | = \infty.$$

Due to condition (i) of this lemma, we must have

$$\lim_{n \to -\infty} | \tilde{\gamma}_n | = 0.$$
Also \( \lim_{n \to \infty} |\gamma_n| = \infty \), for otherwise there exists an integer \( k \) such that

\[
|\gamma_n| \leq e^k
\]

for all \( n \). Then

\[
|M_1(u, v, w, r)| = |M_2(u, v, w, r)| \leq 1
\]

\[
|M_3(u, v, w, r)| = |M_4(u, v, w, r)| \leq e^k
\]

has infinitely many solutions \((u, v, w, r) \in P_4\), which by using lemma 1.1, we can see, is not true. Thus this sequence has all the required properties.

Since there is no danger of confusion, we now suppress the bar and \( \sim \) on \((u_n, v_n, w_n, r_n)\) and \( \alpha_n, \beta_n, \gamma_n, \delta_n \).

During our construction of \((u_i, v_i, w_i, r_i)\)'s, we have used the subsequence \( G_1, G_2, \ldots \) of the sequence of \( \mathcal{M}' \)'s such that

(i) \( G_{i+1} \) is a subsequence of \( G_i \).

(ii) Given any finite set of \((u_j, v_j, w_j, r_j)\)'s, this is a subset of \( G_j \) for \( j \) large.

Let \( G_j^* \) be the first member of \( G_j \). Then the sequence \( G_j^* \) has the following properties:

(1) \( G_j^* = \mathcal{M}_{m_j} \) for some \( j; m_j \to \infty \) as \( j \to \infty \).
(2) Each finite set of \((u_j, v_j, w_j, r_j)\) is a subset of 
\(\mathcal{F}_j^\times\) for all \(j\) large enough.

By lemma 5.12, there exists a sequence
\[
\{(x^{(m_j)}_o, y^{(m_j)}_o, z^{(m_j)}_o, s^{(m_j)}_o) = j = 1, 2, \ldots \}
\]
and we may assume that
\[
| x^{(m_j)}_o | < 1, | y^{(m_j)}_o | < 1, | z^{(m_j)}_o | < 1, | s^{(m_j)}_o | < 1.
\]

By theorem 18, this has a subsequence tending to a limit
\((x_o, y_o, z_o, s_o)\) (say). This choice of \((x_o, y_o, z_o, s_o)\) does what is required. For take \((x, y, z, s) \equiv (x_o, y_o, z_o, s_o) \pmod{P_4} \).

Let \((u_n, v_n, w_n, r_n)\) be any member of this double sequence. For all \(j\) large enough, \((u_n, v_n, w_n, r_n) \in \mathcal{F}_j^\times = \mathcal{F}_k^\times\) (say), and
\[
\left| x u_n + y v_n + z w_n + s r_n \right|
\]
\[
= \left| (x - x_o + x_o(k)) u_n + (y - y_o + y_o(k)) v_n + (z - z_o + z_o(k)) w_n + (s - s_o + s_o(k)) r_n \right|
\]
\[
+ \left| x_o - x_o(k) \right| u_n + \left| y_o - y_o(k) \right| v_n + \left| z_o - z_o(k) \right| w_n + \left| s_o - s_o(k) \right| r_n \right|.
\]

Since
\[
(x - x_o + x_o(k), y - y_o + y_o(k), z - z_o + z_o(k), s - s_o + s_o(k)) \equiv (x_o(k), y_o(k), z_o(k), s_o(k)) \pmod{P_4},
\]

\[
\left( x - x_0 + x_0^{(k)} \right) u_n + \left( y - y_0 + y_0^{(k)} \right) v_n + \left( z - z_0 + z_0^{(k)} \right) w_n + \left( s - s_0 + s_0^{(k)} \right) r_n \geq e^{-1}.
\]

If \( j + \infty \) \( k + \infty \), therefore if \( j \) is large enough,

\[
\left| (x_0 - x_0^{(k)}) u_n + (y_0 - y_0^{(k)}) v_n + (z_0 - z_0^{(k)}) w_n + (s - s_0^{(k)}) r_n \right| < \frac{1}{e}
\]

and

\[
\left| x u_n + y v_n + z w_n + s r_n \right| \geq e^{-1}.
\]

This completes the proof of lemma 5.13.

Proof of theorem 21 in Case I: Let \((x_0, y_0, z_0, s_0) \in \mathbb{R}_4\) be given by lemma 5.13. Take any \((x, y, z, s) \equiv (x_0, y_0, z_0, s_0) \pmod{P_4}\).

Suppose \(a, b = \bar{a}, c, d = \bar{c}\) are the values assumed by \(L_1', L_2', L_3', L_4'\) respectively for \(x, y, z, s\).

First we assert neither of \(a, b, c, d\) is zero. For suppose \(a = 0 = b\). Then from (5.3.3) we get

\[
x u_n + y v_n + z w_n + s r_n = \frac{i(c \gamma_n + d \delta_n)}{\Delta}.
\]

So

\[
\left| x u_n + y v_n + z w_n + s r_n \right| = \left| \frac{i(c \gamma_n + d \delta_n)}{\Delta} \right|
\]

\[
\leq \left| \frac{c \gamma_n}{\Delta} \right| \quad (\text{Since } |c \gamma_n| = |d \delta_n| \text{ and } |\Delta'| = |\Delta|)
\]
Making \( n \to -\infty \) we get

\[
|x_n + y_n + z_n + s r_n| \to 0
\]

and this contradicts that \( x_n + y_n + z_n + s r_n \geq e^{-1} \) for all \( n \). Hence \( a \neq 0 \neq b \). Similarly \( c \neq 0 \neq d \).

Since \( |\gamma_n| \to \infty \) as \( n \to \infty \) and \( |\gamma_n| \to 0 \) as \( n \to -\infty \) and \( |\gamma_n| \geq |\gamma_n| \), we can choose an integer \( n \) such that

\[
|\gamma_n|^2 \leq |\Delta|^\frac{3}{2} \left| \frac{a}{c} \right| \leq |\gamma_n + 1|^2.
\] (5.3.11)

From condition (iii) of lemma 5.13, we get

\[
|\alpha_n|^2 \leq |\Delta|^3 |\gamma_n + 1|^{-2} \leq \left| \frac{c}{a} \right| |\Delta|^\frac{3}{2}.
\] (5.3.12)

Put

\[
l = |ab| \cdot |\alpha_n \beta_n| = |a|^2 |\alpha_n|^2
\]

\[
m = |cd| \cdot |\gamma_n \delta_n| = |c|^2 |\gamma_n|^2
\]

\[
p^2 = |abcd| |\Delta|^3 = |a^2 c^2| |\Delta|^3.
\]

Then from (5.3.11) and (5.3.12) we get

\[
l = |a|^2 |\alpha_n|^2 \leq |ac| \cdot |\Delta|^\frac{3}{2} = p
\]

and

\[
m = |c|^2 |\gamma_n|^2 \leq |ac| \cdot |\Delta|^\frac{3}{2} = p.
\]
From (5.3.3) we get

\[ |\Delta'(x u_n + y v_n + z w_n + s r_n)| = |\Delta(x u_n + y v_n + z w_n + s r_n)| \]
\[ = |a \alpha_n + b \beta_n + c \gamma_n + d \delta_n| \]
\[ \leq \max (|a \alpha_n|, |b \beta_n|, |c \gamma_n|, |d \delta_n|) \]
\[ = \max (|a \alpha_n|, |c \gamma_n|) \]

So

\[ |\Delta(x u_n + y v_n + z w_n + s r_n)|^2 \leq \max (|a \alpha_n|^2, |c \gamma_n|^2) \]
\[ = \max (1, m) \leq p. \]

But from lemma 5.13, we get

\[ |x u_n + y v_n + z w_n + s r_n| \geq e^{-1} \]
\[ \text{i.e.} \quad |(x u_n + y v_n + z w_n + s r_n)|^2 \geq e^{-2} |\Delta|^2 \]

Hence

\[ p \geq e^{-2} |\Delta|^2 \]
\[ \text{or} \quad |abcd|^{\frac{1}{2}} |\Delta|^\frac{3}{2} \geq e^{-2} |\Delta|^2 \]
\[ \text{or} \quad |abcd| \geq \frac{|\Delta|}{e^4}. \]

This proves the theorem in this case.

Case II: \( M_3 = 0 = M_4 \) for some non-zero \( U \in P_4 \), but \( M_1 \neq 0 \neq M_2 \) for any non-zero \( U \in P_4 \).

Lemma 5.11': Let \( M > 0 \) be a given integer. Then there exists a finite sequence of values \( \alpha_n, \beta_n = \overline{\alpha}_n, \gamma_n, \delta_n = \overline{\gamma}_n \) \( (0 \leq n \leq N) \)
of $M_1$, $M_2$, $M_3$, $M_4$ respectively corresponding to polynomials 
$(u_n, v_n, w_n, r_n)$ with $\gcd(u_n, v_n, w_n, r_n) = 1$, satisfying conditions (i), (ii), (iii), (v) and (vi) of lemma 5.11 and

(iv'): $\gamma_0 = 0$, $\gamma_n \neq 0$ for $0 < n \leq N$.

Proof: By hypothesis there exists $(\alpha, \beta, \gamma, \delta) \in P_4$, $(\alpha, \beta, \gamma, \delta) \not= 0$ and with $\gcd(\alpha, \beta, \gamma, \delta) = 1$, such that

$M_3(\alpha, \beta, \gamma, \delta) = 0 = M_4(\alpha, \beta, \gamma, \delta)$.

Consider the solutions of

$|M_1(u, v, w, r)| = |M_2(u, v, w, r)| \leq |M_1(\alpha, \beta, \gamma, \delta)|$

$|M_3(u, v, w, r)| = 0 = |M_4(u, v, w, r)|$

By hypothesis, (5.3.13) has at least one non-zero solution $(\alpha, \beta, \gamma', \delta) \in P_4$. Using lemma 1.1, we can see that (5.3.13) has only a finite number of solutions $U = (u, v, w, r) \in P_4$. Consider

$\Min |M_1(u, v, w, r)|$,

the minimum being taken over all non-zero vectors $(u, v, w, r) \in P_4$ with $\gcd(u, v, w, r) = 1$ satisfying (5.3.13). As there are only a finite number of such vectors, minimum is attained, say at $(u_0, v_0, w_0, r_0)$. Then $M_1(u_0, v_0, w_0, r_0) \not= 0$. 

Now suppose there exists non-zero \((u, v, w, r) \in P_4\) with \(\gcd(u, v, w, r) = 1\), such that

\[
\left| M_1(u, v, w, r) \right| \leq e^{-1} \left| M_1(u_0, v_0, w_0, r_0) \right|.
\]

Then \(M_3(u, v, w, r) \neq 0\). For otherwise, this \((u, v, w, r)\) contradicts the choice of \((u_0, v_0, w_0, r_0)\). In particular if there exists \((u_1, v_1, w_1, r_1)\) satisfying condition (ii) of lemma 5.10, then \(\gamma_1 
eq 0\).

With this choice of \((u_0, v_0, w_0, r_0)\), we proceed as in lemma 5.11, and determine the sequence \(f_M\) satisfying conditions (i), (ii), (iii), (v) and (vi) of lemma 5.11.

Since \(\left| \gamma_n \right| \geq \left| \gamma_{n-1} \right|\) and \(\gamma_1 
eq 0\), we get condition (iv') of lemma 5.11'.

**Lemma 5.12':** For the sequence obtained in lemma 5.11', there exists \((x_0(M), y_0(M), z_0(M), s_0(M)) \in R_4\) such that

\[
\left| x u_n + y v_n + z w_n + s r_n \right| \geq e^{-1}
\]

(5.3.14)

for all \((x, y, z, s) \equiv (x_0(M), y_0(M), z_0(M), s_0(M)) \pmod{P_4}\) and all quadruplets \((u_n, v_n, w_n, r_n)\) \((n = 0, \ldots, N)\).

The proof of lemma 5.12' is the same as that of lemma 5.12.
Lemma 5.13': There exists an infinite sequence of quadruplets 
\[(u_n, v_n, w_n, r_n) \quad (0 \leq n < \infty)\] 
of polynomials with \
gcd \((u_n, v_n, w_n, r_n) = 1\) such that conditions (i), (ii), (iii), 
and (vi) of lemma 5.11 hold and 
\[
\lim_{n \to \infty} |\gamma_n| = \infty, \lim_{n \to \infty} |\alpha_n| = 0 \quad \text{and} \quad \gamma_o = 0, \gamma_n \neq 0 
\text{for } 0 < n < \infty.
\]
Proof: Let \((u, v, w, r)\) be a fixed element of \(P_4\). Let \(L\) be any integer such that 
\[
ed^L < |M_1(u, v, w, r)|.
\]
Then we claim there are only a finite number of non-zero 
\((u', v', w', r')\) \(P_4\) with \(\gcd(u', v', w', r') = 1\) satisfying 
\[
(a) \quad |M_1(u, v, w, r)|^2 |M_3(u', v', w', r')|^2 \leq |\Delta|^3
\]
\[
(b) \quad |M_1(u', v', w', r')| \leq e^L.
\]
The proof of this is exactly the same as in lemma 5.13.

Now choose \((u_o, v_o, w_o, r_o)\) as in lemma 5.11'. We note here that \((u_o, v_o, w_o, r_o)\) as chosen in lemma 5.11' is independent of \(M\) and we can use same \((u_o, v_o, w_o, r_o)\) for all \(\mathcal{C}_M\).

For each \(\mathcal{C}_M\), the corresponding \((u_1, v_1, w_1, r_1)\) satisfies 
\[
|M_3(u_1, v_1, w_1, r_1)|^2 \cdot |M_1(u_o, v_o, w_o, r_o)|^2 \leq |\Delta|^3
\]
and

\[ |M_1(u_1, v_1, w_1, r_1)| \leq e^{-1} |M_1(u_0, v_0, w_0, r_0)| < |M_1(u_0, v_0, w_0, r_0)|. \]

By the above observation, there are only a finitely many choices for \((u_1, v_1, w_1, r_1)\). So we pick out \((\bar{u}_1, \bar{v}_1, \bar{w}_1, \bar{r}_1) \in P_4\) which occurs in infinitely many \(\mathcal{F}_M\). Let \(\mathcal{G}_1\) be the infinite subsequence of those \(\mathcal{F}_M\)'s to which \((\bar{u}_1, \bar{v}_1, \bar{w}_1, \bar{r}_1)\) belong. Now we consider only these \(\mathcal{F}_M\)'s.

Then if \(M\) is large and \(\mathcal{F}_M \in \mathcal{G}_1\), then \(\mathcal{F}_M\) contains more than three terms. Now for each \(\mathcal{F}_M \in \mathcal{G}_1\),

\[ |M_1(u_2, v_2, w_2, r_2)| \leq e^{-1} |M_1(\bar{u}_1, \bar{v}_1, \bar{w}_1, \bar{r}_1)| \]

and

\[ |M_3(u_2, v_2, w_2, r_2)|^2 \cdot |M_1(u_1, v_1, w_1, r_1)|^2 \leq |\Delta|^3 \]

As before, there are only a finite number of choices for \((u_2, v_2, w_2, r_2)\). So pick \((\bar{u}_2, \bar{v}_2, \bar{w}_2, \bar{r}_2) \in P_4\) which occurs in infinitely many \(\mathcal{F}_M\). Let \(\mathcal{G}_2\) be the infinite subsequence of \(\mathcal{G}_1\) of those \(\mathcal{F}_M\)'s to which \((\bar{u}_2, \bar{v}_2, \bar{w}_2, \bar{r}_2)\) belong.

Continuing this process, we have an infinite sequence

\((\bar{u}_n, \bar{v}_n, \bar{w}_n, \bar{r}_n)\) \((0 \leq n < \infty)\) which satisfies conditions (i), (ii), (iii), (vi) of lemma 5.11 and

\[ M_3(\bar{u}_n, \bar{v}_n, \bar{w}_n, \bar{r}_n) = 0, \ M_3(\bar{u}_n, \bar{v}_n, \bar{w}_n, \bar{r}_n) \neq 0 \quad \text{for } n \neq 0. \]
Also by construction

\[
\lim_{n \to \infty} \left| M_1 (\overline{u}_n, \overline{v}_n, \overline{w}_n, \overline{r}_n) \right| = 0
\]

As in lemma 5.13, we can see that

\[
\lim_{n \to \infty} \left| M_3 (\overline{u}_n, \overline{v}_n, \overline{w}_n, \overline{r}_n) \right| = \infty
\]

In the construction of \((\overline{u}_n, \overline{v}_n, \overline{w}_n, \overline{r}_n)\)'s, we have used subsequences \(\mathcal{S}_1, \mathcal{S}_2, \ldots\) which have the same properties, as we had in lemma 5.13.

Following the same method as in lemma 5.13, we can get 
\((x_0, y_0, z_0, s_0)\) with the required properties. In the rest of the proof we suppress the bar.

Proof of theorem in Case II: Let \((u_n, v_n, w_n, r_n) (0 \leq n < \infty)\) and \((x_0, y_0, z_0, s_0)\) be given by lemma 5.13'. Suppose \(a, b = \overline{a}\), \(c, d = \overline{c}\) are the values assumed by \(L_1', L_2', L_3', L_4'\) respectively for some \((x, y, z, s) = (x_0, y_0, z_0, s_0) \mod P_4\).

First we assert that neither of \(a, b, c, d\) is zero. For suppose \(a = 0 = b\). Then from (5.3.3) we get

\[
x u_n + y v_n + z w_n + s r_n = \frac{i(c \gamma + d s_n)}{n} \quad \triangle
\]

So

\[
x u_o + y v_o + z w_o + s r_o = 0 \quad \text{(Since } \gamma_o = 0 = s_o)\]
and this contradicts lemma 5.13'.

Now suppose \( c = 0 = d \). Then from (5.3.3) we get

\[
x u_n + y v_n + z w_n + s r_n = \frac{i(a \alpha_n + b \beta_n)}{\triangle'}
\]

\[
\leq \frac{1}{|\triangle|} \max (|a \alpha_n|, |b \beta_n|) = \left| \frac{a \alpha_n}{|\triangle|} \right|
\]

Making \( n \to \infty \) we get

\[
\lim_{n \to \infty} |x u_n + y v_n + z w_n + s r_n| = 0.
\]

This contradicts lemma 5.13' again.

Now proof of theorem in this case is exactly the same as in Case I.

Case III: \( M_1 = 0 = M_2 \) for some non-zero \( U \in P_4 \) and \( M_3 = 0 = M_4 \) for some other non-zero \( U \in P_4 \).

Then we must have one of the following:

(1) For each \( M \) there exists an \( J_M \) with the properties of lemma 5.11'.

or

(2) There exists a finite sequence \( J \) of quadruplets

\[
(u_n, v_n, w_n, r_n) \in P_4 \ (0 \leq n \leq J)
\]

\[
\gcd (u_n, v_n, w_n, r_n) = 1
\]

satisfying the following
conditions:

(i) \[ |M_1(u_n, v_n, w_n, r_n)|^2 \cdot |M_3(u_n, v_n, w_n, r_n)|^2 \leq e^{-2/|\Delta|^3}. \]

(ii) \[ |M_1(u_n, v_n, w_n, r_n)| \leq |M_1(u_{n-1}, v_{n-1}, w_{n-1}, r_{n-1})| (n \geq 1). \]

(iii) \[ |M_1(u_{n-1}, v_{n-1}, w_{n-1}, r_{n-1})|^2 \cdot |M_3(u_n, v_n, w_n, r_n)|^2 < |\Delta|^3(n \geq 1). \]

(iv') \[ M_3(u_o, v_o, w_o, r_o) = 0 = M_4(u_o, v_o, w_o, r_o), \]

\[ M_3(u_n, v_n, w_n, r_n) \neq 0 \text{ for } 0 < n \leq N. \]

(v') \[ M_1(u_N, v_N, w_N, r_N) = 0, \ M_1(u_n, v_n, w_n, r_n) = 0 \text{ for } 0 < n < N. \]

(vi) \[ |M_3(u_n, v_n, w_n, r_n)| \geq M_3|(u_{n-1}, v_{n-1}, w_{n-1}, r_{n-1})|(n \geq 1). \]

Let \( \alpha_n, \beta_n, \gamma_n, \delta_n \) denote the values of \( M_1, M_2, M_3, M_4 \) respectively at \( (u_n, v_n, w_n, r_n) \). Then we prove the following lemma.

Lemma 5.12"**: For the finite sequence obtained in lemma 5.11", there exists \( (x_o, y_o, z_o, s_o) \in \mathbb{R}_4 \) such that

\[ |x u_n + y v_n + z w_n + s r_n| \geq e^{-1} \]

for all \( (x, y, z, s) = (x_o, y_o, z_o, s_o) \pmod{P_4} \) and all \( n \).

**Proof:** Since \( \alpha_N = 0, \gamma_N = 0 \). So we can choose \( \kappa = \sigma + i \tau; \)
\( \sigma, \tau \in K\{t\} \) such that

\[ ||\gamma_N^\kappa + \overline{\gamma}_N^\kappa||^2 = e^{-1} \quad (5.3.15) \]

Due to condition (ii) of lemma 5.11", we can apply lemma 5.9 with \( \Lambda_n = \alpha_n, \mu_n = \gamma_n^\kappa + \delta_n^\kappa \ (0 \leq n < N) \), and we get
\[ z = \gamma + i \xi \in K \{ t \}; \gamma, \xi \in K \{ t \}, \text{ such that } \]
\[ \| \alpha_n \xi + \bar{\alpha}_n \overline{\xi} + \gamma_n \kappa + \delta_n \overline{\kappa} \| = e^{-1} \quad (0 \leq n < N). \]

Since \( \alpha_N = 0 \), due to (5.3.15) we have
\[ \| \alpha_N \xi + \bar{\alpha}_N \overline{\xi} + \gamma_N \kappa + \delta_N \overline{\kappa} \| = \| \gamma_N \kappa + \overline{\gamma}_N \overline{\kappa} \| = e^{-1} \]
and
\[ \| \alpha_n \xi + \bar{\alpha}_n \overline{\xi} + \gamma_n \kappa + \delta_n \overline{\kappa} \| = e^{-1} \quad (0 \leq n \leq N) \quad (5.3.16) \]

Let \( x_0, y_0, z_0, s_0 \) be the solution of
\[ L_1 (x, y, z, s) = i \triangle', \quad L_2 (x, y, z, s) = \gamma \triangle' \]
\[ L_3 (x, y, z, s) = \overline{\gamma} \triangle', \quad L_4 (x, y, z, s) = \gamma \triangle'. \]

This \( (x_0, y_0, z_0, s_0) \) also satisfies
\[ L_1' (x, y, z, s) = \frac{\xi}{\overline{\xi}} \triangle', \quad L_2' (x, y, z, s) = \overline{\xi} \triangle' \]
\[ L_3' (x, y, z, s) = \frac{\gamma}{\overline{\gamma}} \triangle', \quad L_4' (x, y, z, s) = \gamma \triangle'. \]

From (5.3.3) we get
\[ x_0 u_n + y_0 v_n + z_0 w_n + s_0 r_n = i \left( \frac{\xi \triangle' \alpha_n + \overline{\xi} \triangle' \beta_n + \gamma \kappa + \overline{\gamma} \delta_n}{\triangle} \right) \]
\[ = i \left( \frac{\xi \alpha_n + \overline{\xi} \beta_n + \gamma \kappa + \overline{\gamma} \delta_n}{\triangle} \right). \]

Then from (5.3.16) we get
\[ \| x_0 u_n + y_0 v_n + z_0 w_n + s_0 r_n \| = \| \frac{\xi \alpha_n + \overline{\xi} \beta_n + \gamma \kappa + \overline{\gamma} \delta_n}{\triangle} \| = e^{-1} (0 \leq n \leq N). \]
Take

\[(x, y, z, s) \equiv (x_o, y_o, z_o, s_o) \pmod{P^4}.\]

Then

\[|x u_n + y v_n + z w_n + s r_n| \geq |x u_n + y v_n + z w_n + s r_n| = |x u_o + y v_o + z w_o + s o r_n| = e^{-1} (0 \leq n \leq N).
\]

Proof of theorem in case III: Let \((\alpha_n, \beta_n, \gamma_n, \delta_n) (0 \leq n \leq N)\) be given by lemma 5.1'' and \((x_o, y_o, z_o, s_o)\) be given by lemma 5.12''.

Take

\[(x, y, z, s) = (x_o, y_o, z_o, s_o) \pmod{P^4}.
\]

Let \(a, b = \overline{a}, c, d = \overline{c}\) be the values taken by \(L_{1'}, L_{2'}, L_{3'}, L_{4'}\) respectively at \((x, y, z, s)\). As in case II we can suppose \(a \neq 0 \neq b\). Now suppose \(c = 0 = d\). From (5.3.3) we get

\[x u_n + y v_n + z w_n + s r_n = \frac{i(a \alpha_n + b \beta_n)}{\Delta'}\]

So

\[x u_n + y v_n + z w_n + s r_n = 0.
\]

This contradicts lemma 5.12''. As in (5.3.11), if we can choose \((u_n', v_n', w_n', r_n')\) satisfying

\[|\gamma_{n-1}|^2 \leq |\Delta|^\frac{3}{2} \left|\frac{a}{c}\right| \leq |\gamma_n|^2,\]
we are through. Otherwise we have

\[ |\gamma_N| < |\Delta| \frac{3}{2} \left( \frac{a}{c} \right). \]

But from (5.3.13) and using \( \alpha_N = 0 = \beta_N \) we get

\[ |x u_N + y v_N + z w_N + sr_N| = \left| \frac{c \gamma_N + d \delta_N}{|\Delta|} \right| \leq \left| \frac{c \gamma_N}{\Delta} \right| \]

(Since \( |c \gamma_N| = |d \delta_N| \))

and

\[ |x u_N + y v_N + z w_N + sr_N|^2 \leq \left| \frac{c^2 \gamma_N^2}{|\Delta|^2} \right| \leq \frac{ac}{|\Delta|^{\frac{3}{2}}} \]

But by lemma 5.12" we get

\[ |x u_N + y v_N + z w_N + sr_N| \geq e^{-1} \]

Hence

\[ \frac{|ac|}{|\Delta|^{\frac{3}{2}}} \geq e^{-2} \quad \text{or} \quad a^2 c^2 \geq \frac{|\Delta|}{e^4} \quad \text{or} \quad |abcd| \geq \frac{|\Delta|}{e^4}. \]

This proves the theorem in this case.
BIBLIOGRAPHY


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