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WITH IMPULSE EXCITATION 

DISSERTATION 

Presented in Partial Fulfillment of the Requirements for 
the Degree Doctor of Philosophy in the Graduate 
School of The Ohio State University 

By 


* * * * * * 

The Ohio State University 
1965 

Approved by 

Advisor 
Department of 
Electrical Engineering
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CHAPTER I
THE PROBLEM

It is well known in linear systems theory that an impulse function can be used to determine the transfer function of the system because the Laplace transform of the impulse function is unity. In other words, the inverse transform of the transfer function of a physical system is the impulse response. It follows that the transfer function of a linear system can be determined uniquely by exciting it with an impulse function which can be generated by an impulse generator. It is impossible to generate an ideal impulse by engineering means. However, an impulse can be approximated by a pulse with very short duration and very high magnitude and such a pulse can be produced using various methods (1). Such impulse excitation methods have been used in this work to successfully find the transfer functions of control components such as a two-phase motor (2) and also of a general control system (3).

In general, a nonlinear system which has n degrees of freedom can be characterized by a system of differential equations involving n-dependent-variables (4). Furthermore, one or more of the equations have parameters that are functions of one or more of the dependent variables or of their derivatives with respect to the independent variable,
consequently the superposition theorem used in linear systems is no longer valid since the input and output are no longer directly proportional to each other. It follows that the powerful tool of convolution-integration is no longer applicable. Furthermore, the form of the transient response to a step input is no longer independent of the amplitude of the step, and the response to an arbitrary input cannot be found as the sum of the responses to a series of step or pulse inputs. It is readily seen that a nonlinear control system cannot have a unique transfer function, therefore the problem of how to distinguish and compare such systems, or how to characterize these nonlinear systems as well as to synthesize them into mathematical models arises.

The phase-space method of analysis of a nonlinear problem is one of the most important techniques available in studying nonlinear control system at present. The most valuable feature of the phase-space method is that it tends to provide insight into the physical phenomena which are due to different types of nonlinearities. No other method of analysis seems to convey this insight where nonlinear problems are concerned (5).

The phase-space trajectory is a history of the behavior of the system, and time is used as a parameter along the trajectory curve. The trajectories representing the state-response of the system may often be determined even when the equation of motion cannot be solved. Many nonlinear control
systems can be analyzed in parts in the phase-space or on the phase-plane. Information about the trajectories in all the regions of the space or plane is equivalent to information about the transient performance of the autonomous (the homogeneous solution of a differential equation) nonlinear system. Trajectory segments from each region can be connected by inspection at the boundaries between the regions so as to give trajectories characterizing the nonlinear performance (6).

The phase trajectories for systems with driving functions (or nonautonomous systems) can also be constructed in phase-space or on a phase-plane. But only the simple driving functions such as step functions and ramp functions which permit functional transformations to be performed are possible. Actually only the initial conditions are suitable (5). The approximating of an arbitrary function by a series of step functions has been suggested by R. L. Cosgriff (7) and by a train of impulses with proper weighting by John E. Gibson (8). However, the nature of this problem is quite different from those treated in the works of Cosgriff and Gibson. For each initial condition, there is an autonomous trajectory which takes on exactly one value at every unit of time as traced on the phase-plane except at the singular or critical points. Therefore, the trajectory thus determined by the initial conditions will be unique. Consequently, the state-response of an autonomous system can be evaluated uniquely. The initial conditions of an autonomous system must be given in order to have an autonomous solution.
However, the initial condition as defined in this work is quite apart from that defined by Gibson since the initial condition considered here is determined at $t=a^+$ after excitation by a practical impulse function whose duration is $a$, where Gibson assumes the initial conditions lie on $a$ axis. Furthermore, the practical impulse, which is considered as the driving function of the nonautonomous nonlinear system, has a very high magnitude of $\frac{A}{a}$ and very short duration of $a$. However, even if the duration is short, one still should be able to evaluate the initial conditions at time $t=a^+$. Therefore, this practical impulse serves as a disturbance to the system to establish the initial conditions at the end of the impulse. Consequently, the author suggests that it is advantageous to apply this technique first in time to be followed by Gibson's technique, so that the driving function to a nonautonomous system can be constructed as the superposition of the impulse function followed at $t=a^+$ by the arbitrary input in which one is interested. The arbitrary input therefore will be initiated immediately after the end of the impulse function, that is, at time equal to $a^+$. This procedure will give the proper state response of a nonlinear system subject to an arbitrary driving function. From a physical point of view, this provides the starting point for the phase trajectory which is then subjected to an arbitrary input (represented by a train of impulses with proper weighting). The advantage of this technique is clear, either for
the purpose of comparing the response of a system subjected
to different driving functions or for the comparison of
different systems subjected to the same driving function.

If we have a nonlinear system which has n-degrees of
freedom it can be described by a system of differential
equations involving n dependent variables. In general, the
system is represented by an nth order nonlinear differential
equation which can be expressed as

\[ F( c, \frac{dc}{dt}, \frac{d^2c}{dt^2}, \ldots, \frac{d^n c}{dt^n}) = r(t) \]

where $c(t)$ is the response of the system subject to
the driving function $r(t)$. It is useful to rewrite this as
a system of n equations.

Conventional analysis yields the following expression
for the closed-loop error in response to an arbitrary
function $r(t)$:

\[ \xi(t) = r(t) - c(t) \]

Substituting equation (2) into (1) yields:

\[ F \left[ r(t), \frac{dr}{dt}, \ldots, \frac{d^n r}{dt^n}, \xi(t), \frac{d\xi}{dt}, \ldots, \frac{d^n \xi}{dt^n} \right] - r(t) = 0 \]

By defining a new function $F'(t)$, then (3) becomes

\[ \frac{d^n \xi(t)}{dt^n} + F' \left[ r, \frac{dr}{dt}, \ldots, \frac{d^n r}{dt^n}, \xi, \frac{d\xi}{dt}, \ldots, \frac{d^n \xi}{dt^n} \right] = 0 \]

where

\[ F' \left[ r, \frac{dr}{dt}, \ldots, \frac{d^n r}{dt^n}, \xi, \frac{d\xi}{dt}, \ldots, \frac{d^n \xi}{dt^n} \right] \]
Equation (4) may be expressed as

\[ \begin{align*}
    \dot{x}_1 &= x_2 \\
    \dot{x}_2 &= x_3 \\
    \dot{x}_3 &= x_4 \\
    &\vdots \\
    \dot{x}_n &= -F' \left[ r, \frac{dr}{dt}, \ldots, \frac{d^n r}{dt^n}, \varepsilon, \frac{d\xi}{dt}, \ldots, \frac{d^n \xi}{dt^n} \right]
\end{align*} \]

Where \( x_i \)'s may be referred to as the state variables of the nonlinear system, by letting

\[
\vec{x} = \begin{bmatrix}
    x_1 \\
    x_2 \\
    \vdots \\
    x_n
\end{bmatrix} = \begin{bmatrix}
    \varepsilon \\
    \frac{d\xi}{dt} \\
    \vdots \\
    \frac{d^{n-1}\xi}{dt^{n-1}}
\end{bmatrix} \quad \text{and} \quad \frac{d\vec{x}}{dt} = \begin{bmatrix}
    \dot{x}_1 \\
    \dot{x}_2 \\
    \vdots \\
    \dot{x}_n
\end{bmatrix}
\]

where \( \vec{x} \) is a column vector of \( n \)-dimensional vector.

Therefore, it is possible to construct an \( n \)-dimensional vector space by using the error function \( \varepsilon \), and

\[ \frac{d\xi}{dt}, \frac{d^2\xi}{dt^2}, \ldots, \frac{d^{n-1}\xi}{dt^{n-1}} \]

as independent coordinates.

It is evident that if the initial conditions of the system subject to an approximated impulse function can be determined, then there is no difficulty in finding the state
response of the system to such a driving function. Figure 1 shows the relationships between the excitation, the state response, and the homogeneous nonlinear differential equations which govern the behavior of the physical system. As long as such initial conditions are determined, the state response of the system is uniquely determined since the isoclines are fixed on the phase-space. The property of uniqueness is extremely important because the initial conditions for all systems can be uniquely determined which in turn uniquely determine the state-responses of the systems. Interesting results from the experimental work (Chapter VII) show that there does exist a certain region in the space which gives the saturated state responses of the system. Such regions depend upon the area of the approximated impulse function rather than either the magnitude or duration of the approximated impulse function alone.

If two equivalent impulses (impulses with the same magnitude and duration) are used to excite two unknown systems, and if these two systems respond identically to the impulses (i.e., set up the same initial conditions and then have similar responses), they are identical in this region, and can be represented by the same system model over this region. Obviously, if they react differently, they cannot by represented by the same model.

Further, if two systems have identical initial conditions in phase-space and respond identically, then they are
Fig. 1. The relationships between impulse excitation and state-response of a physical nonlinear system

Fig. 2. State-responses of a third order nonlinear control system
identical and can be represented by the same model, since the impulse serves only to set up the initial conditions.

A group of systems which respond identically and apparently all contain identical system components are identical in their mathematical model are designated as "isomorphic systems." Furthermore, isomorphic systems must be tested for all possible initial conditions.

What is the significant value of determining the state response by impulse function excitation? In answering this question, we note that using impulse excitation we can --

1. Compare two nonlinear systems.
2. Investigate the transient behavior due to different types of nonlinearity.
3. Predict the system’s stability.
4. Predict the existence, amplitude, and frequency of a limit cycle.

Consequently, this investigation focuses on this method of obtaining the initial conditions in phase-space for nonlinear control systems excited by the approximated impulse functions as a useful tool for the analysis and synthesis of system models. If the order of the system is n, then the initial conditions are \( \xi(t), \frac{d\xi}{dt}, \frac{d^2\xi}{dt^2}, \ldots \), \( \frac{d^{n-1}\xi}{dt^{n-1}} \), evaluated at time equal to \( a^+ \), where \( a \) is the duration of the approximated impulse function. The purpose of evaluating and setting such initial conditions is to establish arbitrary initial conditions which in turn will
determine the trajectory in phase-space. In order to solve the nonlinear problem effectively, it is advantageous to study a similar linear transfer function excited by an impulse function and note the response of the system. Insofar as the establishment of the initial conditions is concerned, this can be a basis for analyzing the nonlinear system because the nonlinear system, in general, contains the same linear transfer functions plus nonlinearities. The initial conditions of a linear control system can be uniquely determined. The analysis of the linear system is helpful in nonlinear analysis if the effect of the impulse function on the non-linear elements are clear. This effect is a new impulse function that follows a predicted nonlinear functional transformation and the magnitude of this new impulse function caused by the nonlinear element can be determined. Consequently, it is fruitful to simplify the problem of establishing initial conditions from a nonlinear one to a linear one by merely applying the impulse function, as transformed by the nonlinearities of the system, to the error terminals.

The above result immediately shows that it is possible to classify a great variety of nonlinear control systems into certain general canonical structures as far as the characteristics of their nonlinearities and their relative location in the system are concerned. It will be shown that the initial conditions for a specific system may be obtained quickly by this approach. Fortunately most nonlinear systems belong to these canonical structures and any set of initial conditions can be established using practical
impulse functions for all systems (nonlinear as well as linear systems) unless the signal flow is stopped by nonlinearity, i.e., dead-zone type functional. However, there is the possibility that the technique will work with systems having dead-zones so long as the magnitude of the signal applied before the nonlinearity in the interval immediately succeeding \( t = a^+ \) is greater than the dead-zone constant.

Examples showing the calculation of initial conditions due to impulse excitation for simple systems and several orders of systems are shown in Chapter V and system reduction techniques for a complicated system are also mentioned there.

From another point of view, the impulse excitation to a system can be regarded as a means of inserting a certain amount of energy into the system, and it is interesting to see how this energy travels inside the system. The power spectrum analysis gives the relationships between the magnitude and duration of an impulse function and the energy ratio of the system when an impulse function is used as its input. Therefore, the characteristics of the impulse function can be seen readily.

Experiments have been carried out to show that the technique of exciting the nonlinear system with an approximated impulse function can be applied successfully to the physical system in order to establish the initial condition of the state response of the system and also to determine
how the effect of this excitation differs from that of the conventional sinusoidal excitation. The results of these experiments also show the relationship between the ratio of the impulse magnitude and duration to the state response produced. The practicality in engineering applications of the technique of applying impulse excitation to two arbitrary systems in order to compare them is evident. The state response, which is a function of time, can be transformed into the frequency domain if necessary. On the other hand, the initial conditions can be readily evaluated from the state response resulting from experiment if the system is a second order system. However, for a third order system, the projection of initial conditions on the $\dot{\xi}$ vs $\xi$ plane can also be evaluated from its oscilloscope display by a graphical construction method, but the higher order derivatives are difficult to obtain in this manner. Therefore, the analytical method seems to be more effective in this respect. The experiments show that the impulse function used to excite a system can be varied in both magnitude and duration. However, the state-response might be saturated during certain ranges of this variation and it is also apparent that the region of saturation depends upon the area $A$ of the approximated impulse function rather than its amplitude or duration if the duration of the impulses is much less than all the time constants of the system. Results given in Chapter VII agree fairly well with the analytical derivation in Chapter IV.
So far, approach and results discussed are for the analysis of a nonlinear system. It is hoped that these results can also be applied to the synthesis of nonlinear systems; e.g., to finding the equation of a nonlinear system or the nonlinearities and frequency sensitive functions of the system by assuming that the nonlinear system is an inaccessible "black box". As is true in network synthesis, of course, the answer will not be unique and the order of the system found by using the technique of impulse excitation may not be exactly the same as the actual system, but a high degree of equivalence should exist.
CHAPTER II

IDEAL AND PRACTICAL IMPULSE FUNCTIONS

A. Ideal impulse function

In applied mathematics, the impulse function, which is used as a practical or approximated impulse function in this work, is an important tool. It simplifies the derivation of many results that would otherwise involve complicated arguments. In control engineering, the impulse function determines the transfer function of a linear control system. Another reason for studying the ideal impulse function is that it is convenient to apply to Guillemin's method of evaluation of the Fourier Transform if the given function \( f(t) \) is sufficiently smooth (6). The function \( f(t) \), which may be the response of an arbitrary control system under test, must be determined in the frequency domain in order to find the energy spectrum and its relationships with the practical impulse function.

An ideal unit impulse function can be defined as

\[
\int_{-\infty}^{\infty} \delta(t) \, dt = 1 \quad \delta(t) = 0 \text{ for } t \neq 0
\]

It can be defined as a limit

\[
\delta(t) = \lim_{n \to \infty} f_{n}(t)
\]

14
where $f_n(t)$ defined in the above equation as a sequence of functions satisfying

$$\int_{-\infty}^{\infty} f(t) \, dt = 1 \quad \lim_{n \to \infty} f_n(t) = 0 \text{ for } t \neq 0$$

Furthermore, it can be seen that the ideal impulse function has the property

$$\int_{-\infty}^{\infty} \delta(t) f(t) \, dt = f(0)$$

It is often convenient for the unit impulse function to be the limiting member of some infinite sequence of ordinary functions. Consider the rectangular pulse function

$$f_a(t-t_0) = \begin{cases} \frac{1}{2a} & \text{when } t_0-a < t < t_0+a \\ 0 & \text{otherwise} \end{cases}$$

where $a>0$ for this function

$$\int_{-\infty}^{\infty} f_a(t-t_0) \, dt = 1$$

If we now let $a \to 0$, the width of the pulse approaches zero and the height approaches infinity, whereas the area remains constant at unity. The unit impulse function could therefore be considered to be the limiting member of a rectangular pulse function:

$$\delta(t-t_0) = \lim_{a \to 0} f_a(t-t_0)$$

Although the rectangular pulse function is a simple and convenient way to define the impulse function, it is a discontinuous function as shown in Figure 3.

The Gaussian function is a continuous function and possesses the derivatives for $a>0$.

$$g_a(t-t_0) = \frac{a}{\sqrt{\pi}} e^{-a^2(t-t_0)^2}$$
FIG. 3. RECTANGULAR PULSE FUNCTION

FIG. 4. GAUSSIAN PULSE FUNCTION

FIG. 5. FUNCTION $S_e(t)$
This function is shown in Figure 4. Now, for all values of \( a > 0. \)

\[
\int_{-\infty}^{\infty} g_a(t-t_0) \, dt = 1
\]

Further, the height of \( g_a(t-t_0) \) increases toward infinity as \( a \to \infty \), while the skirts collapse in toward zero. The Gaussian pulse function hence satisfies the requirement of the defining equations (12), (13) for the unit impulse function in the limit as \( a \to \infty \), and we may set

\[
\delta(t-t_0) = \lim_{a \to \infty} g_a(t-t_0)
\]

For the ideal impulse function, the derivative of \( \delta(t) \) is defined as (7)

\[
\delta'(t) = \lim_{\epsilon \to 0} S_{\epsilon}(t)
\]

Where \( S_{\epsilon}(t) = \frac{1}{\epsilon^2} \left[ U(t) - 2U(t-\epsilon) + U(t-2\epsilon) \right] \), and the function \( S_{\epsilon}(t) \) is shown in Figure 5.

B. Practical impulse function

The ideal impulse defined in the last section is non-existent in the physical world. It is impossible to generate an impulse with infinite height and zero duration. The approximated impulse function can be generated by engineering means and it has significant value in application.

The impulse thus generated is called a practical impulse function. The amplitude of the practical impulse has to be large enough relative to the duration in order to approximate an impulse function.
A reasonable approximation sometimes simplifies the problem considerably, and such simplifications are especially useful for nonlinear problems because the nonlinear equations which describe the behavior of the system usually are difficult to solve analytically. Even if they can be solved, the solutions are usually quite involved, such as a series, polynomials, or some other form of the transformed function. Therefore, it is better to solve a complicated problem quickly by the means of reasonable approximations. Fortunately, some complex nonlinear systems may be approximated with reasonable success by simple first or second order equations (3). Because of this, it is profitable to study the properties of the practical impulse function and its application to the nonlinear system problem as a means of simplifying its analysis.

Different forms of practical impulse function can be generated by various possible engineering means. They will be considered as follows.

C. Variable, square impulse function

The variable square impulse function has a height of \( A \) (volts) and duration of \( a \) (seconds). The area under this definition is therefore equal to \( Aa \) volt-seconds. It is clear that this impulse is defined in a different manner than the mathematical impulse function.

Figure 6 shows the variable, square impulse function. However, due to the limitations of the physical device,
FIG. 6. VARIABLE SQUARE IMPULSE FUNCTION

FIG. 7. ACTUALLY GENERATED VARIABLE SQUARE IMPULSE

FIG. 8. FUNCTION $g'(t)$
the impulse actually generated looks more or less like the one shown in Figure 7.

The top of the pulse is not parallel with the horizontal axis (time axis). This happens when it is generated by means of precharged capacitor, or other method. Furthermore, the sides may not be vertical due to the parameters of the circuit which limit its capability of varying the magnitude abruptly. However, since the duration $a$ is so short, it is reasonable to approximate the pulse as having a flat top, instead of taking the average value

$$\frac{1}{2} J(t) \bigg|_{t=0}^{t=a}$$

as the measure of its height. If the variable square impulse function has the form shown in Figure 6 then its derivatives at $t=0$ and $t=a$ are shown in Figure 8.

D. **Modulated impulse function**

Because of their generally lighter weight, ease of amplifier construction, and simplicity of components, a large percentage of the servomechanisms used in instrumentation and computers are a-c carrier systems (9), in preference to d-c systems. The conventional a-c system is characterized by the transmission of its data as the envelope and polarity of a modulated signal. Therefore, the impulse function used has to be modulated by a carrier signal. If $\omega_o$ is the carrier frequency, then the modulated impulse function can be written as

$$A_e \left[ u(t) - u(t-a) \right] \cos \omega_o t.$$

and designated as the function $J_w(t)$. It is shown in Figure 9.
The Fourier Transformation of a practical variable square impulse function may be found by shifting the origin to $\frac{a}{2}$, (for the sake of symmetry of $I(\omega)$)

\begin{equation}
I(\omega) = \frac{1}{\pi} \int g(t) = \frac{A \sin \frac{\alpha \omega}{2}}{\frac{\alpha \omega}{2}}
\end{equation}

and the Fourier Transformation of a modulated impulse function by shifting the origin to $\frac{a}{2}$ gives:

\begin{equation}
I_m(\omega) = \frac{1}{\pi} \int g_m(t) = \frac{A \sin(\omega - \omega_0) \frac{a}{2}}{(\omega - \omega_0)} + \frac{A \sin(\omega + \omega_0) \frac{a}{2}}{(\omega + \omega_0)}
\end{equation}

Both of these functions in the frequency domain are shown in Figure 10 and Figure 11 respectively.

E. Impulse generator

Impulse generators were in use in electrical engineering practice long before modern control theory came into prominence. They have been used to test the insulation of the windings of machines, transformers and other electrical apparatus. Various means have been used to produce the different types of impulses needed for such testing purposes. In the laboratory, impulse voltage waves can be generated by the Marx-circuit (13) in order to test the strength of insulation against surges, for example, due to lightning. Later, the impulse function was used to test the characteristics of control components and systems (2). The following list gives just a few examples of such impulse generator circuits.
FIG. 9. MODULATED IMPULSE FUNCTION — $g_n(t)$

FIG. 10. FREQUENCY SPECTRUM OF THE FUNCTION $g(t)$

FIG. 11. FREQUENCY SPECTRUM OF THE FUNCTION $g_n(t)$
**Marx-circuit.** This type of circuit can imitate lightning waves. In it a bank of capacitor units is charged in parallel with direct current at low voltage and then discharged in series at high voltage through inductance and resistance.

Figure 12 shows such a circuit in its simplest form (14). The transformer charges the capacitors C in parallel through the two-element half wave rectifier tube T and the charging resistors r. The intermediate gaps g between capacitors are broken down in succession by some means, starting at the transformer end, thus connecting the charged capacitors in series and adding their voltages. The combined voltage of all the capacitors in series is applied across gap G, inductance L, and resistance R. The test piece is usually placed across resistance R.

**Electronic impulse generator** (1). The basic concept of surge testing of electrical windings is the application of a precharged capacitor to the terminals of an electrical winding and the subsequent observation or comparison of the voltage transient to determine the electrical properties of the winding. The desire to accurately obtain more test information about the production faults of small d-c or universal armatures has been greatly advanced by the use of electronic comparison surge testing. Small patterns, however, are often obscured in comparison to the absolute value of the
observed waveform. The bridge type surge tester utilizing the method of null detection is used for the evaluation of faults. Such an impulse generator using the electronic techniques has been designated by Weed. (1)

The power units normally used provide for the charging of a single capacitor and its alternate connection first to the standard winding and then to the test winding. The alternate voltage transients are superimposed on the face of a cathode-ray tube. To synchronize the point in time that each transient is started, the actual connection of the capacitor to the winding is made by the firing of a controlled thyratron. The alternate connection between windings is normally accomplished by means of a rotating synchronous
switch. Two traces are actually alternating, but appear simultaneous. These traces with the necessary attenuation are the variation of the total voltage transient resulting from the application of the precharged capacitor. The magnitude of this initial voltage is usually variable from 1000 volts to 10,000 volts. A deviation between traces of 50 volts cannot be observed.

To overcome the difficulty of finding small differences in large absolute signals, null detection may be used. This principle is based on the concept of observing the difference in two large signals rather than observing the large signals themselves. For this application this will require the simultaneous connection of two identical precharged capacitors, one to the test winding and the other to the standard winding. If the controlled simultaneous connection of the two capacitors could be accomplished, then it would be easy to compare the two traces on null basis.

The two simultaneous transients resulting from the capacitor discharged through the test and standard windings may be observed by using the conventional bridge circuit. With the two capacitors connected to ground and the two windings connected in series, all the elements of a standard bridge are present when the thyratrons ionize. The detector is connected between ground and the common
point of the two windings. The simplified circuit is shown in Figure 13 and should be thought of as a transient bridge.

![Simplified circuit of surge unit](image)

**Fig. 13. Simplified circuit of surge unit**

**Variable square impulse generator.** The schematic diagram of this generator is shown in Figure 14. The impulse function generated by this generator can be varied in both magnitude and duration. A set of impulse functions thus generated can be expressed as \( g_{ij}(t) \) where \( i \) is the dummy index for durations and \( j \) is the dummy index for different magnitudes. A battery is used as the source and controlled by two micro switches mounted on the shaft of the constant velocity drive motor, the duration of the impulse is controlled by two cams on the shaft and the magnitude of the impulse is controlled by the potentiometer. The waveform of \( g_{ij}(t) \) thus generated is also shown in Figure 15. One advantage of generating impulse functions by use of micro
FIG. 14. SCHEMATIC DIAGRAM OF A PRACTICAL IMPULSE GENERATOR

FIG. 15. WAVEFORM OF A TYPICAL PRACTICAL IMPULSE.
switches is the achievement vertical sides for the pulse at t=0 and t=a. These sides are much sharper than the one generated by electronic circuits such as shown in Figure 7.

*Modulated impulse generator.* The reason for using a modulated impulse function in testing an a-c carrier system has been discussed in section D of this Chapter.

The basic principle of operation of this generator is the same as that for the variable square impulse generator. However, the source will be a-c instead of d-c in this case. The two micro switches are connected in parallel rather than series as used in the previous case. For this case, one of the switches is normally open and the other has to be normally closed. The schematic diagram of this generator is shown in Figure 16. The wave form of this function \( \mathcal{g}_m(t) \) is also shown in Figure 17. As is true for the previous case, both magnitude and duration can be varied.

This type of impulse generator (as well as the variable square impulse generator) is a repetitive impulse generator because one impulse can be generated when the motor drives the shaft through one revolution. It is apparent that the frequency of the repetitive impulse will depend upon the velocity of the motor as well as the gear ratio of the generator.
Fig. 16. Schematic diagram of a modulated impulse generator

Fig. 17. Waveform of a modulated impulse
CHAPTER III
INITIAL CONDITIONS AND THE SOLUTION OF
AUTONOMOUS NONLINEAR SYSTEMS

A. $n^{th}$ dimension phase-space analysis

A phase-space (state-space) of $n$ dimensions can be used to handle $n$-order systems, but the method will be inconvenient. Even the technique of phase-plane analysis can be extended to the higher order systems. However, the location of trajectories in higher-order space is difficult. It is obvious that some method must be found in order to represent the higher dimension problem on paper. The performance of fourth and higher order systems can only be visualized in terms of the projections of the trajectories, or in terms of the individual variables and their derivatives plotted against time. For the third order system, Ku has developed (13) the method in which he represents state variables $\xi$, $\dot{\xi}$, $\ddot{\xi}$ as three Cartesian Coordinates, and the trajectory could be represented by projections on any two of the three possible planes containing the axes $\xi$, $\dot{\xi}$, or $\xi$, $\ddot{\xi}$, or $\dot{\xi}$, $\ddot{\xi}$. 

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B. Autonomous and nonautonomous systems

The behavior of the second order system can be specified by a curve in the phase-plane. If \( c \) is the output and \( r \) is the input, the differential equation of a general second-order system, whether it is linear or nonlinear, can be expressed as

\[
f(c, \dot{c}, \ddot{c}; t) = r(t)
\]

(20)

Equation (20) can also be written as;

\[
f(c, \dot{c}, \frac{d\dot{c}}{dt}; t) = r(t)
\]

(21)

\[
\frac{dc}{dt} = \dot{c}
\]

\( c \) and \( \dot{c} \) are considered as the dependent variables. It follows that equation (21) is a system of two simultaneous first order differential equations. If the input \( r \) is absent, the system is then designated as an autonomous system (14) and equation (21) becomes;

\[
\frac{d\dot{c}}{dt} = \dot{c} \neq (c, \dot{c})
\]

\[
\frac{dc}{dt} = \dot{c}
\]

(22)

It is apparent that the system of equations does not contain \( t \) explicitly. Dividing the two equations yields;

\[
\frac{d\dot{c}}{dt} = \dot{c} \neq (c, \dot{c})
\]

(23)

Now \( c \) is the independent variable and \( \dot{c} \) is the dependent variable. The behavior of the second order system is then specified by a curve in the phase plane.
On the other hand, if the input $r(t)$ is not absent, then the system is designated to be nonautonomous and $r(t)$ can be called the driving function.

C. The solution of a nonautonomous system

In general, the phase-plane method can be used to analyze a simple driving function such as a step and ramp function. However, for an arbitrary function, the exact solution for a step-type approximation of the input has been given by Cosgriff (7). An input can be approximated by

$$r = a_0 u(t) + a_1 u(t-T) + a_2 u(t-2T) + \ldots$$

where $u(t)$ is a unit step function. The starting point on the phase-plane will be determined by using $a_0$ and the initial conditions of the system response $c(t)$. This point is equivalent to $t = 0$.

Gibson (8) has shown a different approach to the problem. He approximates the actual input to the system by a train of impulses of proper weighting. The repetition rate of such a train of impulses must be high with respect to the response time of the system and the method will, in general, apply only to piecewise linear systems. Each impulse may be considered as an initial condition superimposed upon the final condition of the system at the end of the preceding interval.

The $n^{th}$ order linear ordinary nonhomogeneous differential equation can be expressed as

$$(25) \quad a_n \frac{d^n c}{dt^n} + a_{n-1} \frac{d^{n-1} c}{dt^{n-1}} + \ldots + a_0 c = r(t)$$
where the driving function \( r(t) \) is a Dirac delta function.

\[ r(t) = c \delta(t) \]  

Trimmer has proved (15) that such a driving function has the same effect as an additional term added to the \((n-1)\)st initial condition of the corresponding homogeneous equation

\[ \frac{d^{n-1} c(0^+)}{dt^{n-1}} = \frac{d^{n-1} c(0^-)}{dt^{n-1}} + \frac{c}{a_n} \]  

D. The solution of an autonomous system

The impulse function of large magnitude and short duration of Figure 6 is used as the driving function of a nonlinear system. It serves to establish a set of unique initial conditions for the autonomous system, unless the signal flow is stopped by nonlinearity such as a dead-zone type functional. The initial conditions of the autonomous solution can be established by using a single practical impulse function to excite the system. Therefore, the technique introduced in this work is quite different from Gibson's work where he assumes repetitive impulses which are bounded by the actual magnitude of the arbitrary signal to represent a continuous function. In other words, there is no impulse having magnitude exceeding the maximum value of the signal. For example, one cycle of a sine function is approximated by sixteen impulses which take on different magnitudes but none of them exceeds the maximum value of the sine function. In this work, the impulse function has
a very large magnitude and the duration is very short compared to all the significant time constants of the system.

Again, we can write the linear or piecewise linear differential equation as:

\[
\frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \ldots + a_0 x = r(t)
\]

where \( r(t) \) is defined as:

\[
r(t) = \begin{cases} 
\frac{A}{a} & ; 0 \leq t \leq a \\
0 & ; Otherwise 
\end{cases}
\]

It is interesting to observe that this function may be regarded as a disturbance to the system rather than the driving function since the duration is so short and the function itself is nonrepetitive. On the other hand, the initial conditions can be evaluated at time equal to \( a^+ \). The initial conditions thus evaluated will serve as the initial conditions of the autonomous system on the phase-plane plot.

Furthermore, a nonlinear system containing no memory nonlinearity, has unique initial conditions determined by a practical impulse function which in turn gives the unique state response. In general, a nonlinear system will have a response to a practical impulse, but not necessarily uniquely depending upon the history of the system. The idea can be seen clearly by referring to Figure 1, which gives the procedure of determining the state response of a nonlinear system.

Thus, it may be stated that for each set of initial conditions established by an impulse of different magnitude, a state-space response will take place or may be observed. Moreover, many useful applications can be drawn from this.
Weed (22) has pointed out that the problem of parameter variations (or sensitivity) can be solved on the phase-plane by means of the state-response. In summary, he had the following conclusions:

1. If the initial conditions on the phase-plane are determined, then the state response of a particular control system (linear or nonlinear) are determined uniquely.

2. Different systems in general have different trajectories on the phase-plane even if the initial conditions are the same.

3. If the same impulse function is applied to two identical systems, the state responses, trajectories and the transfer functions obtained from the trajectories are the same.

4. Conversely, if two trajectories on the phase-plane are the same, excited by the same practical impulse function, then the two systems are identical within the bounds of the excitation.
A. Linear control system

A linear control system with negative feedback may be indicated in block diagram form as shown in Figure 18. The function $G(s)$ in general can be written as

$$G(s) = \frac{a_m s^m + a_{m-1} s^{m-1} + \ldots + a_1 s + a_0}{b_n s^n + b_{n-1} s^{n-1} + \ldots + b_1 s + b_0}$$

where $n \geq m$. It can also be expanded into partial fraction form as

$$G(s) = \frac{k_1}{1 + T_1 s} + \frac{k_2}{1 + T_2 s} + \ldots + \frac{k_i}{1 + T_i s}$$

$$+ \ldots + \frac{k_n}{1 + T_n s} = \sum_{i=1}^{n} \frac{k_i}{1 + T_i s}$$

This is the general form of the transfer function in the forward path. Constants, integration, multiple poles and conjugate poles are all special cases of the general form discussed above.

In this chapter, we will deal mainly with the methods of establishing the initial conditions of error-space of the linear system. The mathematical model is very useful in analytical approach. For example, the function $f(t)$ as a
FIG. 18. BLOCK DIAGRAM OF A LINEAR CONTROL SYSTEM

FIG. 19. MATHEMATICAL MODEL OF THE FUNCTION f(t)
consequence of impulse excitation at the error position can be expressed as a mathematical model as shown in Figure 19.

B. **The method of determination of the initial conditions on the error-phase space**

The basic idea of constructing an error-space by using \( \xi, \frac{d\xi}{dt}, \ldots, \frac{d^{n-1}\xi}{dt^{n-1}} \) as independent coordinates has been introduced in Chapter I. The method of determination of the initial conditions by impulse excitation will be presented in this section.

We have two methods of approach to the technique of exciting the actual system by an impulse function. The method used depends upon the nature (time constants) of the system and the length of the impulse duration. These are discussed as follows:

**Impulse excited with the loop closed.** If the duration of the impulse function is long enough that the effect of \( c(t) \) on \( \xi(t) \) cannot be neglected, the transfer function to be used must be of a closed loop type. This is especially true when the modulated impulse function is used to excite a system because it is necessary to allow at least two or three cycles of the carrier signal to go with the variable square impulse function.

Consider the linear system of Figure 18. The initial value of the error function and its derivatives in phase-space can be established by impulse excitation at the error terminals. This can be seen to be
\[ (32) \quad \dot{\varepsilon}(t) = r(t) - c(t) + \mathcal{I}(t) = -c(t) + \mathcal{I}(t) \]

where \( r(t) = 0 \), and \( \mathcal{I}(t) \) represents the impulse function as shown in Figure 6 or Figure 9.

When \( t \) equals \( a^+ \), \( \mathcal{I}(t) \) has just vanished. Therefore, Equation (32) reduces to

\[ (33) \quad \dot{\varepsilon}(t) \bigg|_{t=a^+} = -c(t) \bigg|_{t=a^+} = -\mathcal{L}^{-1} \left[ \frac{I(s) G(s)}{1 + G(s)} \right]_{t=a^+} \]

The derivatives of \( \varepsilon(t) \) can be evaluated if the inverse Laplace Transformation of \( c(s) \) is known.

**Impulse excited with the loop opened.** This method of establishing the initial conditions is based on the following assumptions:

1. \( \varepsilon(t) \) is completely specified with the impulse function, i.e., one shorts out any feedback for the period of the impulse. This can be accomplished by adding a switching device between the feedback path and the error summer as shown in Figure 20. The switch shifts from ground to summer at \( t=a^+ \Delta a \), (where \( \Delta a \) is a very small increment of time). The system then goes back to ordinary feedback operation.

2. The feedback is negligible due to time lags.

3. The feedback is negligible because of the small magnitude of \( c(t) \) during the interval \( t=0 \) to \( t=a^+ \Delta a \).

Assumptions 2 and 3 do occur, but they cannot be regarded as a general case. With respect to establishing initial conditions, assumption 1 is a more favorable one for a generalized approach since no further assumptions are
necessary. If the impulse generator is the type shown in Figure 14 or Figure 16, the switching device can be easily mounted on the same shaft which controls the action of micro-switches.

C. The excitation of a linear control system by an impulse function (closed loop)

The set of initial conditions for a linear system can be established by Equation (33). The function $G(s)$ can be expressed as

$$G(s) = \frac{\sum_{i=1}^{n} \frac{K_i}{1+T_is}}{1+G(s)}$$

form of

$$\sum_{j=1}^{m} \frac{K'_j}{1+T'_js}$$

the open loop case can therefore be applied to the closed loop case as well.

Concerning linear systems, there is little complication in establishing initial conditions with the loop closed as compared to that for the loop opened.
This is not true for a nonlinear system, however, because the over-all behavior of a system which contains several nonlinearities is extremely difficult to analyze. Therefore, many restrictions must be considered when applying an impulse excitation in such a case. This problem will be discussed in the next chapter.

D. The excitation of a linear control system by an impulse function (open loop)

Consider the linear system represented by the model as shown in Figure 20. The switching device will close the feedback path at t=a+. The initial value of the error function and its derivatives in phase-space therefore can be established by an impulse excitation at the error terminals. This can be seen as

\[ \mathcal{E}(t) = r(t) - c(t) + \mathcal{I}(t) = -c(t) + \mathcal{I}(t) \]

where \( r(t) = 0 \), and \( \mathcal{I}(t) \) has just vanished at \( t=a^+ \). Therefore, Equation (34) reduces to

\[ \mathcal{E}(t) \bigg|_{t=a^+} = -c(t) \bigg|_{t=a^+} \]

It is therefore sufficient to evaluate the function \( c(t) \) and its derivatives in order to establish the initial conditions of the error functions.

From Equation (31), we may write:

\[ C(s) = F(s) = I(s) \sum_{i=1}^{n} \frac{k_i}{1+T_i s} \]

Figure 19 gives the same information. However, it is more convenient to work in the time domain. Therefore,
we transform \( C(s) \) into \( c(t) \) by applying the convolution integration.

For \( 0 \leq t \leq a \)

\[
(37) \quad c(t) = \sum_{i=1}^{n} \int_{0}^{t} \frac{A}{T_i} \frac{k}{T_i} e^{-\frac{\tau}{T_i}} d\tau
\]

\[
= \sum_{i=1}^{n} \frac{Ak_i}{a} \left( 1 - e^{-\frac{t}{T_i}} \right)
\]

For \( a < t < \infty \);

\[
(38) \quad c(t) = \sum_{i=1}^{n} \int_{0}^{t} g(\tau) \frac{k}{T_i} e^{-\frac{\tau}{T_i}} d\tau
\]

\[
= \frac{k}{T_1} e^{-\frac{1}{T_1}t} \int_{0}^{a} g(\tau) e^{-\frac{1}{T_1}\tau} d\tau + \frac{k}{T_2} e^{-\frac{1}{T_2}t} \int_{0}^{a} g(\tau) e^{-\frac{1}{T_2}\tau} d\tau + \ldots + \frac{k}{T_n} e^{-\frac{1}{T_n}t} \int_{0}^{a} g(\tau) e^{-\frac{1}{T_n}\tau} d\tau
\]

\[
= \frac{Ak_i}{a} e^{-\frac{1}{T_1}t} \left( e^{-\frac{1}{T_1}a} - 1 \right) + \frac{Ak_i}{a} e^{-\frac{1}{T_2}t} \left( e^{-\frac{1}{T_2}a} - 1 \right) + \ldots + \frac{Ak_i}{a} e^{-\frac{1}{T_n}t} \left( e^{-\frac{1}{T_n}a} - 1 \right) = \sum_{i=1}^{n} \frac{Ak_i}{a} e^{-\frac{1}{T_i}t} \left( e^{-\frac{1}{T_i}a} - 1 \right).
\]

Now, the Equation (35) may be evaluated as

\[
(39) \quad \xi(t) \bigg|_{t=a^+} = -c(t) \bigg|_{t=a^+} = -\sum_{i=1}^{n} \frac{Ak_i}{a}.
\]

\[
= \frac{1}{T_1} a \left( e^{-\frac{1}{T_1}a} - 1 \right) \frac{1}{T_2} a \left( e^{-\frac{1}{T_2}a} - 1 \right) \ldots \frac{1}{T_n} a \left( e^{-\frac{1}{T_n}a} - 1 \right) = \sum_{i=1}^{n} \frac{Ak_i}{a} \left( e^{-\frac{1}{T_i}a} - 1 \right).
\]
The derivatives of $\varepsilon(t)$ may also be evaluated as

\begin{equation}
\frac{d \varepsilon(t)}{dt} = - \frac{d c(t)}{dt} = \sum_{i=1}^{n} \frac{A_k}{a} \left( -1 \right) \left( \frac{1}{T_i} \right) e^{-\frac{1}{T_i} t}.
\end{equation}

\begin{equation}
\frac{d^2 \varepsilon(t)}{dt^2} = - \frac{d^2 c(t)}{dt^2} = \sum_{i=1}^{n} \frac{A_k}{a} \left( -1 \right)^2.
\end{equation}

\begin{equation}
(\frac{1}{T_i})^2 e^{-\frac{1}{T_i} t} (1-e^{-\frac{1}{T_i} t})
\end{equation}

\begin{equation}
\frac{d^m \varepsilon(t)}{dt^m} = - \frac{d^m c(t)}{dt^m} = \sum_{i=1}^{n} \frac{A_k}{a} \left( -1 \right)^m.
\end{equation}

\begin{equation}
(\frac{1}{T_i})^m e^{-\frac{1}{T_i} t} (1-e^{-\frac{1}{T_i} t})
\end{equation}

Thus, it follows that the initial conditions of the $m$th derivative of the error function are:

\begin{equation}
\left. \frac{d^m \varepsilon(t)}{dt^m} \right|_{t=a^+} = \sum_{i=1}^{n} \frac{A_k}{a} \left( -1 \right)^m \left( \frac{1}{T_i} \right)^m e^{-\frac{1}{T_i} a}.
\end{equation}

\begin{equation}
(1-e^{-\frac{1}{T_i} a}) = \sum_{i=1}^{n} \frac{A_k}{a} \left( -1 \right)^m \left( \frac{1}{T_i} \right)^m (e^{-\frac{1}{T_i} a} - 1)
\end{equation}

In conclusion, a set of initial conditions $\varepsilon, \frac{d\varepsilon}{dt}, \ldots,$ $\frac{d^{n-1}\varepsilon}{dt^{n-1}}$ in the error-space can be established by evaluating equations (39) and (43). The dummy index $m$ in equation (43) runs from 1 to $(n-1)$ for an $n$th order linear system.

The following example serves to prove the validity of the technique. Let us use the analog computer to simulate...
a second order control system as shown in Figure 21. The response of the system when it is excited by a variable square impulse function \( \mathcal{I}(t) \) with \( A/a = 14 \ v \) and \( a = 0.14 \ sec \) is shown in Figure 22.

The elements of the analog circuit are chosen to be \( R_1 = 10^6 \) ohms, \( R_2 = 10^5 \) ohms, \( C_1 = 10^{-6} \) farad and \( C_2 = 10^{-6} \) farad. We then have the transfer function:

\[
G(s) = - \frac{0.1s}{(1+0.1s)(1+s)} = - \frac{1/9}{1+0.1s} + \frac{1/9}{1+s}
\]

From Equation (44) and the given information, we can summarize all the constants:

\[
T_1 = 0.1 \ sec, \ T_2 = 1 \ sec, \ K_1 = 1/9, \ K_2 = -1/9, \ A/a = 14 \ v, \ and \ a = 0.14 \ sec.
\]

By substituting these constants into Equation (39),

\[
\mathcal{E}(t) = - \mathcal{C}(t)
\]

\[
\begin{align*}
&= \frac{A K_1}{a} (e^{-a/T_1} - 1) + \frac{A K_2}{a} (e^{-a/T_2} - 1) \\
&= 14 \left[ -\frac{1}{9}(e^{-0.14/0.1} - 1) + \frac{1}{9}(e^{-0.14/1} - 1) \right] \\
&= 0.97533
\end{align*}
\]

Also from Equation (43), we have (for \( m = 1 \)),

\[
\begin{align*}
\frac{d\mathcal{E}(t)}{dt} &= (AK_1/a)(-1)(1/T_1)e^{-a/T_1}(1-e^{-a/T_1}) \\
&\quad + (AK_2/a)(-1)(1/T_2)e^{-a/T_2}(1-e^{-a/T_2}) \\
&= -14 \left[ -(1/9)(1/0.1)(e^{-0.14/0.1} - 1) + \\
&\quad + (1/9)(1/1)(e^{-0.14/1} - 1) \right] \\
&= -11.507
\end{align*}
\]
Fig. 21. Circuit diagram for the computer simulation of a second order control system

\[ G(s) = - \frac{R_2 C_1 s}{(1+R_2 C_2 s)(1+R_1 C_1 s)} \]

Fig. 22. The response of the system excited by a variable square impulse function \( \mathcal{J}(t) \) with \( A/a = 14 \text{ v} \) and \( a = 0.14 \text{ sec} \).
We can also write \( \tan \phi = -11.507 \), or \( \phi = 265.05^\circ \).

For this example, the derivatives of \( m > 1 \) are not required since the system is a second-order system. They can, however, be easily evaluated.

The above result, i.e., \( \xi(t) \bigg|_{a^+} = 0.97533 \) and \( \dot{\xi}(t) \bigg|_{a^+} = -11.507 \) agrees very well with that obtained from measurements made with the oscilloscope, i.e., \( \xi(t) \bigg|_{a^+} = 0.98 \) and \( \dot{\xi}(t) \bigg|_{a^+} = -11.5 \).

E. **Further discussion of initial conditions**

The transfer function \( G(s) \) of second-order or higher-order systems can be expressed by Equation (31) in general. However, it is possible that \( b_0, b_1 \), may be absent, i.e., single or multiple poles at zero may exist. It is also possible that multiple poles are not at the origin. Therefore, it is sometimes desirable to find the output in the time domain in such cases in order to establish initial conditions. We now have

\[
C(s) = F(s) = I(s) \sum_{i=1}^{n-p-q} \frac{K_i}{1 + T_{1i} s} + I(s) \sum_{j=1}^{p} \frac{K_j}{s^j} + I(s) \sum_{k=1}^{q} \frac{K_k}{(1 + T_{3k} s)^k}
\]

In the time domain, \( c(t) \) may be found by applying the Convolution integration.
Integration in the third part of Equation (48) cannot be easily carried out in general form. However, when the positive integer $k$ is specified, there is no difficulty in carrying out this integration. The power of $(t-\tau)$ in the integrand reduces to 1 by performing the integration once. This can be seen as follows:

\begin{align*}
(49) \qquad \int_{0}^{a} (t-\tau)^{k-1} e^{\frac{-1}{T_{3k}} (t-\tau)} d\tau \\
= T_{3k} e^{\frac{-1}{T_{3k}}} [ (t-a)^{k-1} e^{\frac{-1}{T_{3k}} (t-a)} ]
\end{align*}
\[ + T_{3k}(k-1) \int_0^a (t-\tau)^{k-2} e^{-\frac{1}{T_{3k}} \tau} d\tau \]

Finally, we may write:

(50)
\[
c(t) = \sum_{i=1}^{n-p-q} \frac{AK}{a} \left( e^{-\frac{1}{T_{1i}} t} - 1 \right) + \sum_{j=1}^p \frac{AK_{j}}{a_j} \left( t^j - (t-a)^j \right) + \sum_{k=1}^q \frac{AK_{3k}}{a(k-1)! T_{3k}^{k-1}} e^{-\frac{1}{T_{3k}} t} \int_0^a (t-\tau)^{k-2} d\tau \]

Now we can establish the initial conditions as

(51)
\[
\mathcal{E}(t) \bigg|_{t=a^+} = -c(t) \bigg|_{t=a^+} = \sum_{i=1}^{n-p-q} \frac{AK}{a} \left( e^{-\frac{1}{T_{1i}} a^+} - 1 \right) - \sum_{j=1}^p \frac{AK_{j}}{a_j} a^{-j-1} + \sum_{k=1}^q \frac{AK_{3k}}{T_{3k}^{k-1}(k-1)!} a^{k-2} e^{-\frac{1}{T_{3k}} a} \int_0^a (t-\tau)^{k-2} d\tau \]

\[- \sum_{k=1}^q \frac{AK_{3k}}{a(k-2)! T_{3k}^{k-1}} e^{-\frac{1}{T_{3k}} a} \int_0^a (t-\tau)^{k-2} d\tau \]
The second term in Equation (52) will vanish if $m > j$, where $m$ runs from 1 to $n-1$ for an $n^{th}$ order system. The positive integer $j$ runs from 1 to $p$, where $p < n$. Therefore, it is always possible that the second term vanishes after $m = j$. This situation is not true for the third term because there are exponential functions involved.
F. Discussion of the excitation of a linear system by an impulse function with a very short duration

A special case of the open loop approach of the impulse excitation presented in Section D will be discussed in this section. The result can also be applied to a linear system with closed loop approach. We assume that $T_i \gg a$. This is a special case because all the analysis in the last section is not restricted by this assumption.

Let the impulse function of Figure 6 or Figure 9 be applied at the position $b(s)$ of Figure 20. The function $F(s)$ can be found simply by multiplying $I(s)$ with $G(s)$. Thus,

\begin{equation}
F(s) = G(s)I(s) = I(s) \sum_{i=1}^{n} \frac{k_i}{1 + T_i s}
\end{equation}

We can now construct the mathematical model as shown in Figure 19. The function $f(t)$ is formed by summing the individual terms after taking an inverse Laplace transformation (represented by $L^{-1}$). Since the final result is that of the individual terms summed together, it is interesting to observe the effects of the impulse function to a general term, namely, $\frac{k_i}{1 + T_i s}$. All other terms in the partial fraction expansion will be similar.

Consider now

\begin{equation}
F_i(s) = \frac{k_i}{1 + T_i s}I(s) = \frac{Ak_i(1-e^{-as})}{as(1+T_i s)}
\end{equation}

By finding the inverse Laplace transformation, $t-a$

\begin{equation}
f_i(t) = \frac{Ak_i}{a}(1-e^{-\frac{t}{T_i}})u(t) - \frac{Ak_i}{a}(1-e^{-\frac{t-a}{T_i}})u(t-a).
\end{equation}
This may be expressed as:

(56)

\[
f_i(t) = \begin{cases} 
0 & \text{for } t < 0 \\
\frac{A k_i}{a} (1 - e^{-\frac{t}{T_i}}) & \text{for } 0 \leq t \leq a \\
\frac{A k_i}{a} e^{-\frac{t}{T_i}} (1 + \frac{a}{T_i} - 1) & \text{for } t > a 
\end{cases}
\]

The result in Equation (56) may be plotted by using the duration of the impulse function a as a variable parameter as shown in Figure 23.

The responses during the time interval between 0 and a are very close to a linear function since a is very short. On the other hand, the response after a is independent of a by the condition that \( T_i \gg a \). This can be demonstrated by Taylor's series expansion of \( e^{\frac{a}{T_i}} \approx 1 + \frac{a}{T_i} \), and

(57)

\[
f_i(t) = \frac{A k_i}{a} e^{-\frac{t}{T_i}} \left(1 + \frac{a}{T_i} - 1\right) = \frac{A k_i}{T_i} e^{-\frac{t}{T_i}}
\]

for \( t > a \).

The transfer function of a linear system is determined uniquely by the ideal impulse function. However, if a is short enough and the assumption \( T_i \gg a \) holds, the response after a is unaffected by the variable a. Instead, it will depend upon the area of the impulse, namely, A.

Generally speaking, the transfer function of a linear system can be expanded into partial fraction terms as shown in Equation (31). Thus, we can have the conclusion drawn from previous discussions. If a linear control system with
Fig. 23. Plots of $f_1(t)$ by using $a$ as a variable parameter.

Zero response for infinite frequency is subjected to a practical impulse with duration short enough compared to all significant time constants of the control system, the response depends only on the area under the impulse function. Furthermore, the initial conditions established will be dependent upon the area of the practical impulse function only.

G. **Block diagram with the impulse generator for a linear system**

The practical impulse function is added at the error position. It may be regarded as a summer. Figure 24 shows how an impulse generator can actually be connected into a system. It also shows that a typical practical impulse function generator can be achieved by using a delay element (17). It is clear that the practical impulse function is excited at the error terminals while $r(t)$ is absent.
Fig. 24. Block diagram with the impulse generator for a linear system

Fig. 25. The simplified block diagram of an A.C. system
H. Modulated impulse function for carrier systems

The reason for using a carrier technique for control systems has been discussed in section D of Chapter II. The carrier system used in the experiment (as will be seen in Chapter VI) is found to be a typical one in which the output shaft position is controlled with the actuating signal measured by a sychro system. The output of the sychro-control transformer is a sine wave modulated by the information $\mathcal{E}(t)$. The simplified diagram of an a-c system is shown in Figure 25.
CHAPTER V

THE METHOD OF DETERMINING THE INITIAL CONDITIONS OF THE TRAJECTORY OF AN AUTONOMOUS NONLINEAR SYSTEM IN PHASE-SPACE

A. Nonlinearity in a control system

Strictly speaking, there is no ideal linear system in the physical world. Sometimes, we can solve the problem by linearization of the nonlinear elements. However, for a case such as a relay servomechanism, intentional nonlinearity or hysteresis phenomena in the magnetic circuit of a memory device, etc., we must face the problem and solve it either accurately or approximately. The most common nonlinearity, saturation, is found in most useful devices including vacuum tubes, gas tubes, transistors, magnetic amplifiers, and two-phase motors.

The location of the nonlinear elements depends upon the type of system to be analyzed. Some systems have a saturating amplifier in the forward loop of the system in order to amplify the error signal. However, an instrumentation servomechanism may have the nonlinear element in the feedback loop. Because of the nonlinear element in a control system, those techniques employed to analyze linear systems are no longer applicable.
In general, the characteristics of nonlinear elements can be expressed as \( y(t) = f[x(t)] \), where \( y(t) \) is designated as the output time function of the nonlinear element and \( x(t) \) is the input function of time. Many times, this functional relationship is written sectionally, and may or may not be in piecewise linear form.

B. The excitation of a nonlinear control system by an impulse function (closed loop)

The problem of establishing initial conditions for a control system is now extended to the nonlinear system. Utilizing the same assumption stated in Chapter IV, if the practical impulse duration is long enough such that the effect of feedback on the function \( \zeta(t) \) cannot be neglected, we must use the closed loop approach to the problem. The technique of impulse excitation can only be applied to a very limited class of systems because of the appearance of the nonlinearities in the system. This fact can best be recognized by the following illustrative example.

Figure 26 shows a control system having a nonlinear element in the feedback path. Assuming that we know the signal, \( x(t) \) will not saturate from \( t=0 \) to \( t=a+\Delta a \), where \( \Delta a \) is a small measure of time. Then, initial conditions can be established as
(58) $\xi(t) \bigg|_{t=a^+} = -c(t) \bigg|_{t=a^+}$

$$= -L^{-1} \left[ \frac{I(s) mG(s)}{1 + mG(s)} \right]_{t=a^+}$$

where $m$ is the slope of the linear section.

Fig. 26. A nonlinear control system containing a nonlinear element in the feedback path.

The derivatives of $\xi(t)$ at $t=a^+$ can also be evaluated if the inverse Laplace transformation of $C(s)$ is known. It is clear that establishment of the initial conditions is based upon the restriction of the signal $x(t)$. For a complicated system which involves several nonlinearities, it is difficult to write the closed loop transfer function because many restrictions must be made.

C. The excitation of a nonlinear control system by an impulse function (open loop)

The difficulties of exciting a nonlinear control system with the loop closed have been discussed in the preceding section. Because of the difficulties which may occur in solving nonlinear problems, the open loop approach with
reasonable assumptions is introduced in this section. It is interesting to observe that with the technique of the excitation of a practical impulse function to a loop with switching action as shown in Figure 20 the nonlinear problem simplifies a great deal and many useful results follow immediately.

According to the above assumption, as far as the evaluation of the initial conditions is concerned, a closed loop system becomes a cascaded block diagram containing a set of nonlinear characteristics and frequency sensitive functions. Therefore, from the analytical point of view, a single loop system's initial conditions can always be established unless the signal is stopped by a dead-zone nonlinearity. It will be shown later in this chapter that a multi-loop system can always be reduced to a single loop system if the system contains only one nonlinearity. If, however, the nonlinearities are distributed in different loops, the analytical solution is much more complicated. It is, therefore, reasonable to say that the degree of complication of the problem depends upon the number of nonlinearities rather than the order of the system. On the other hand, initial conditions established by experimental methods are always possible unless the signal is stopped by the dead-zone nonlinearity.

The analytical methods presented in this chapter serve to determine how a set of initial conditions in the phase-space can be established. From the conclusion of the
last chapter together with that of this chapter, we can say that all systems (linear or nonlinear) have a unique autonomous response in phase-space depending upon their specific initial conditions.

The technique of impulse excitation as applied to a nonlinear control system in order to establish initial conditions can best be illustrated by considering a few examples. System reduction techniques are useful to obtain this objective. It will be readily observed in the following examples that an impulse function is used as an input to any of a certain group of nonlinearities which results in an impulse output of different amplitude while the duration remains unchanged. The initial conditions can be established accordingly.

D. Establishing initial conditions of a second order nonlinear system (open-loop)

The initial conditions of the trajectory of a second order autonomous nonlinear system on the phase-plane can be established by impulse excitation. Once the initial conditions are determined, we have several available techniques which can be applied in order to find the complete autonomous solution. Those techniques of phase-plane construction worth mentioning include the isocline method, Pell's method and the Delta method (8). The isocline method consists of manipulating the system equation to obtain the loci of constant slope. Given the initial conditions, it is then
possible to sketch the phase-plane trajectory of the system without actually solving the system equation. The Delta method consists in approximating short segments of the phase trajectory with arcs of circles. The Pell's method starts to construct two functions plotted on the plane, then only two triangles are needed to carry out the solution.

A second-order nonlinear system is shown in Figure 27 with a nonlinear element in the forward path. Let us see how the initial conditions are determined in this simple system before studying more complicated nonlinear systems and then hopefully generalizing the problem.

Consider the second order system of Figure 27. The characteristics of the nonlinear element can be expressed as

\[ y(t) = f[x(t)] \]  

Fig. 27. Block diagram of a second order nonlinear system.

The same technique developed in Chapter IV can be applied in nonlinear problems assuming that the function transformation of nonlinear elements is possible. Suppose that the impulse function is applied at the error terminals. \( x(t) \) is then an impulse function with magnitude modified by a factor \( K \). In general, \( y(t) \) is still an impulse function.
with its duration unchanged and its magnitude changed according to Equation (59). However, if the impulse at \( x(t) \) is small, \( y(t) \) may be zero. Under this condition, the establishment of initial conditions becomes impossible. In general, if the impulse function exists at \( y \), then the nonlinear system response will follow and the results of the mathematical models developed in Chapter IV can be used.

Assume that the nonlinear element is followed by a transfer function such as \( \frac{KKM/N}{s(1+T_1s)} \), which has two poles (one at the origin) with positive real residues. This could be the transfer function of a motor and gears, amplifier and motor, hydraulic valve, or an amplidyne with a reasonable approximation of the time constant. If we make the assumption that the control system is de-energized before the impulse function is applied, the impulse will travel through these three blocks. To be more specific, we assume a servo system with a nonlinear transducer with a dead-zone as shown in Figure 28. If an impulse function is applied at \( E(t) \), \( y(t) \) can be found by examining how the impulse travels inside the system. By assuming that the magnitude of the impulse is greater than a constant \( E_1 \), it follows that the equation of the transfer function (shown in Figure 29) can be written as

\[
\frac{y-0}{x-E_1} = K_2 \quad \text{and} \quad y = K_3(x-E_1)
\]
It is evident that the area of the impulse function has been changed from $A$ to $K_s(A-aE_1)$. The new area depends upon four parameters, namely $K_s$, $A$, $a$ and $E_1$. The new impulse function is shown in Figure 29 (b).

The new impulse function can now be considered as the input to $G(s)$. For this special case, it is apparent that a nonlinear problem can be simplified as a linear problem. The mathematical models of the function $f(t)$ developed in the preceding chapter can, therefore, be used. However, the function $I(s)$ should be the new impulse function instead of the old one. In order to illustrate the establishment of the initial conditions for this system, it is preferable to express $G(s)$ as

$$G(s) = \frac{KK_M}{N} \cdot \frac{K_M}{s} = \frac{KK_M}{N} - \frac{KK_{M1}}{N} \cdot \frac{1}{s(1 + T_1s)}$$

Substituting all the known information into Equations (39) and (43) $i=1$, $j=1$. $A=K_s \frac{A-aE_1}{a}$, $a=a$. $K_1=K_1'=\frac{KK_M}{N}$. $T_1=T_1$, $K_j=K_1'=\frac{KK_M}{N}$. 

Fig. 28. Block diagram of a servo with dead zone.
Fig. 29. (a) Characteristics of dead-zone nonlinear element
(b) New impulse function
Since this is a second order system, only $\hat{E}$ and $\frac{dE}{dt}$ are required for phase-plane construction. The location of initial conditions in a particular quadrant of the plane will depend upon the positive or negative sign resulting from the numerical evaluation of Equations (63) and (64).

Consider another nonlinear system which has a nonlinear element with saturation characteristics. This nonlinear element might be a saturating electronic amplifier driving a motor in an instrument servo. The block diagram is shown in Figure 30.
FIG. 30. BLOCK DIAGRAM OF A SERVO WITH SATURATING AMPLIFIER

FIG. 31. BLOCK DIAGRAM OF RELAY SYSTEM WITH A SATURATING FEEDBACK ELEMENT
This system differs from the one just discussed because the impulse function saturates at \( t=0^+ \) if its magnitude is greater than \( K_S E_1 \). Apparently, the area of the impulse at \( y(t) \) is much smaller than \( A \).

Even though the amplitudes of the impulse functions at \( x \) are much greater than \( K_S E_1 \), the set of initial conditions thus established are the same. Therefore, for this special system, a set of different impulse functions establishes the same initial conditions, i.e., a particular point in the phase-plane. Its location depends upon \( K_S E_1 \), \( a \) and the constants of the function \( G(s) \).

The two examples given above serve to illustrate the establishment of the initial conditions for nonlinear control systems, and also illustrate ways in which mathematical models for linear control systems may be used in order to establish initial conditions.

E. Establishing initial conditions of a multi-nonlinearity control system (open-loop)

The problem of evaluating initial conditions for a system containing more than one nonlinearity, will be discussed in this section. Figure 31 shows a control system having a nonlinearity in both the forward and feedback paths as given by George Kovatch (18). It is apparent that the establishment of the initial conditions based on the closed loop approach is complicated. If the impulse duration is short enough and the assumption for an open loop system is
Justified, initial conditions can be established either analytically or experimentally.

Suppose the impulse function of Figure 6 is applied at the $E(t)$ terminal. The signal at $y$ is

$$y(t) = D \quad 0 \leq t \leq a$$

The mathematical model of Figure 19 can now be applied. The function $G(s) = \frac{K}{\prod_{i=1}^{2} (s + r_i)}$ and $F(s)$ can be written as

$$F(s) = I(s) \sum_{i=1}^{2} \frac{K_i}{1 + s T_i}$$

$$f(t) = \sum_{i=1}^{2} f_i(t) = f_1(t) + f_2(t) + f_3(t)

= \sum_{i=1}^{2} \int \left[ \frac{DK_i(1-e^{-as})}{s(1+s T_i)} \right]$$

It is clear that $f(t)$ is the sum of three time functions, each of which has a form similar to Equation (56). It would be formidable to write out the complete solution here; however, the complete form can be easily written if the constants are given. Assume that we have the resultant time function after summing these three individual functions and that it is shown in Figure 32. The next step would be the problem of determining $z(t)$. It can best be solved by graphical techniques as shown in Figure 32.
Fig. 32. Graphical method of nonlinear functional transformation
In general, \( z(t) \) can be analytically expressed sectionally. In this particular example, \( z(t) \) can be expressed in three regions separated by \( t_1 \) and \( t_2 \). If \( t_1 \gg a \), it is sufficient to consider the first region only so far as the evaluation of the initial conditions is concerned.

The last step is to find \( w(t) \). Since the transfer function of \( a-bs \) is a linear one, the convolution integration is applicable.

\[
(68) \quad W(s) = Z(s)(a-bs) = aZ(s) - bsZ(s)
\]

\[
(69) \quad w(t) = \mathcal{L}^{-1} \left[ aZ(s) - bsZ(s) \right] = az(t) - b \frac{dz(t)}{dt}
\]

In the time domain, the first term of Equation (69) is a constant times the original \( z(t) \). The second term, however, is a constant times the derivative of \( z(t) \) since \( s \) represents differentiation. It is clear that the initial condition of \( E(t) \) can be evaluated from Equation (69).

\[
(70) \quad E(t) \bigg|_{t=a^+} = w(t) \bigg|_{t=a^+} = -a z(t) \bigg|_{t=a^+} + b \frac{dz(t)}{dt} \bigg|_{t=a^+}
\]

Therefore, only the information of \( z(t) \) and \( \frac{dz(t)}{dt} \) at \( t=a^+ \) are required in order to establish initial conditions for this particular nonlinear system.

The higher order derivatives can be obtained by successive differentiation of Equation (70). It is true that the graphical method breaks down when evaluating higher order derivatives. The expressions for initial values of \( E, \dot{E}, \ddots, E^{(n-1)} \) can be written as
\[ (71) \quad \frac{d^m \varepsilon(t)}{dt^m} = \frac{d^m w(t)}{dt^m} \quad \text{at } t=a^+ \]

\[ = -a \frac{d^m z(t)}{dt^m} + b \frac{d^{m+1} z(t)}{dt} \quad \text{at } t=a^+ \]

where \( m = 1, 2, \ldots, n-1 \).

In order to establish initial conditions, there is no difficulty in utilizing the analytical approach since in general the constant \( B \) is much greater than \( a \). However, if this were not so, the problem would be even simpler.

F. **Discussions of the dead-zone nonlinearity as a component of a system**

In many control systems, the control engineer prevents the use of the dead-zone nonlinearity as a control component because many ambiguities and undesirable situations may arise. In general, the dead-zone nonlinearity which describes the characteristics of \( x \) and \( y \) can be represented as shown in Figure 33 (a). This nonlinearity can be any piecewise continuous function and can be expressed mathematically as

\[ (72) \quad y = \begin{cases} f(x - \frac{\Delta}{2}), & \text{for } x > \frac{\Delta}{2} \\ 0, & \text{for the interval } -\frac{\Delta}{2} < x < \frac{\Delta}{2} \\ f(x + \frac{\Delta}{2}), & \text{for } x < -\frac{\Delta}{2} \end{cases} \]

Suppose \( x(t) \) is considered to be an impulse function with an amplitude of \( \frac{\Delta}{a} \) where \( \frac{\Delta}{a} \) is smaller than \( \frac{\Delta}{2} \). The output \( y(t) \) will then be identically equal to zero.
Fig. 33. Dead-zone type nonlinearities
However, if the magnitude of $\frac{A}{a^2}$ is greater than the dead-zone constant $\frac{A}{2}$, the output $y(t)$ will still be an impulse function with different amplitudes depending upon the function $f(x - \frac{A}{2})$. Usually such a nonlinearity either transmits or stops an impulse function when it appears immediately after $\xi(t)$. Generally, the dead-zone nonlinear element can appear in both the forward or the feedback path in the control loop or in any location in the system. Therefore, $x(t)$ should be specified as an arbitrary time function. Suppose $x(t)$ is smaller than the dead-zone constant $\frac{A}{2}$ during $a - \Delta a < t < a + \Delta a$ (where $\Delta a$ is a small incremental). The output $y(t)$ will then identically vanish during this interval. Insofar as the establishment of the initial conditions is concerned, it is apparent that no initial conditions can be established. On the other hand, if it is supposed that $x(t)$ is greater than the dead-zone constant $\frac{A}{2}$ during the interval $a - \Delta a < t < a + \Delta a$, then the initial conditions can be evaluated either by analytical or experimental techniques.

The study of the behavior of the dead-zone nonlinearity is important because this is the type of nonlinearity which may cause the technique developed in this work to be invalid. However, with regard to the invalidity of the technique, the appearance of the dead-zone nonlinearity is a necessary but not a sufficient condition. Consequently, it may be concluded that the technique is always applicable
except that the signal flow before the nonlinearity in the system has the instantaneous value less than the dead-zone constant during the time interval, \( a + \Delta a > t > a - \Delta a \).

G. Establishing initial conditions of an \( n \)th order nonlinear control system (open-loop)

Consider that the \( n \)th order linear frequency sensitive function is a part of a nonlinear control system and has the form:

\[
G(s) = \frac{a_0 s^n + a_1 s^{n-1} + \ldots + a_m s + a_0}{b_0 s^n + b_1 s^{n-1} + \ldots + b_m s + b_0}
\]

with \( n \geq m \).

It is necessary to further specify \( G(s) \). In any physical circuit there is at least one zero at infinity, since in any practical circuit, the gain always falls off to zero as the frequency is increased indefinitely. This is the reason for imposing restriction \( n \geq m \) on Equation (73). Truxal (19) suggests three conditions which must be satisfied for \( G(s) \) to be a realizable transfer function.

1. \( |G(j\omega)| \) is the gain function of a physically realizable network if the Paley-Wiener criterion is satisfied. The Paley-Wiener criterion furnishes the rigorous, necessary and sufficient condition for physical realizability in terms of the behavior of \( |G(j\omega)| \) along the \( j\omega \) axis. Specifically, the condition is that

\[
\int_0^\infty \frac{\log |G(j\omega)|}{1 + \omega^2} \, d\omega
\]

have a finite value.
(2) If $G(s)$ is given as the ratio of polynomials with real coefficients, it is realizable if all poles lie in the left-half plane excluding infinity and the $j\omega$-axis.

(3) $g(t)$ is realizable if $g(t) = 0$ for $t < 0$ and $g(t) \to 0$ as $t \to \infty$.

Suppose an arbitrary $n^{th}$ order nonlinear system is restricted to have only one nonlinearity such as shown in Figure 34. The mathematical models can then be constructed as shown in Figure 35. However, the solution to the problem of multi-nonlinearity $n^{th}$ order systems can be obtained by applying the same principle.

H. **Nonlinear system reduction techniques**

In linear system theory, the rules of simplification of block diagrams of a system have been worked out, e.g., Mason's rule or other signal flow diagrams. No attempt is made to repeat these techniques here. Furthermore, the result is fruitful and governed by several concise rules. The situation is quite different in the nonlinear system study. It is, therefore, necessary to carefully consider each individual case. Hopefully, some rules can be developed to simplify the technique for establishing initial conditions for a complicated nonlinear system.

Figure 36 shows a simple forward path of the system composed of one linear transfer function and one nonlinear element. If the impulse of Figure 6 is applied at $x$, $f_1(t)$ can be obtained by applying the convolution integration.
The canonical structures for a system containing one nonlinearity.

(a) Canonical structure I, with $n \geq m$, $q \geq p$.

(b) Canonical structure II, with $n \geq m$, $q \geq p$.

Fig. 34. The canonical structures for a system containing one nonlinearity.
Fig. 35. Mathematical models for a system containing one nonlinearity
On the other hand, \( f_2(t) \) can be found as

\[
(76) \quad f_2(t) = m \int_0^a \frac{A}{a} Ke^{\frac{1}{T}(t-\tau)} d\tau = \frac{KA}{a} \left( e^{\frac{1}{T}a} - 1 \right) e^{-\frac{1}{T}t}
\]

for \( y < B \) \( \frac{f_2(t)}{f_1(t)} = B \) for \( y \geq B \)

Consequently, justification is found in Equations (75) and (76) for stating that the responses will be entirely different by changing the location of the nonlinear element.

Now consider the system that has a nonlinear element in the forward path. The linear transfer function can be replaced as shown in Figure 37 (b). It is obvious that the element in the feedback path cannot be placed ahead of the nonlinear element.
Fig. 37. Nonlinear element as forward path element

On the other hand, if the nonlinear element is located in the feedback path, as shown in Figure 38, $G(s)$ can be removed from the loop and added to the feedback path.

As far as the signal travel due to impulse excitation is concerned, the above operations are allowed.

In summary, the manipulation of block diagrams with regard to this particular problem must follow the rules, as stated below.

1. The locations of linear and nonlinear elements are not interchangeable.

2. The linear transfer function in the feedback path can be removed to the forward path at the right hand side of the nonlinear element (Figure 37).

3. The linear transfer function in the forward path can be moved to the feedback path at the right hand side of
Fig. 38. Nonlinear element as feedback path element

The nonlinear element.

(4) The linear portion can be simplified by usual linear theory.

A single, more fundamental statement may be made to cover these cases, i.e., blocks in the closed loop may be moved around as long as they do not modify the signal entering block N. This covers the situation, but of course, suitable alterations would have to be made to keep the overall transfer function constant, if this were desirable.

The following example serves to illustrate the rules of manipulation of the nonlinear block diagram. Figure 39(a) shows a typical n\textsuperscript{th} order system containing one nonlinear element. It is obvious that (a) can be simplified to (b)
Fig. 39. Simplification the block diagram of a nonlinear system.
by applying rule (4) while (b) to (c) to (d) again requires rule (4). The transformation from (d) to (e) is possible by applying the reverse procedure of rule (2). Finally we have the form shown in (e).

I. **The mathematical model for an \( n \)-th order control system containing multiple nonlinearities**

The rules for simplification of block diagrams of nonlinear systems with vanishing \( r(t) \), and the example worked out by applying these rules in the last section, suggest further investigation of the possibility of constructing a general mathematical model for an \( n \)-th order nonlinear system. For this particular problem, the technique of impulse excitation can be worked out with the aid of signal-flow diagrams and their mathematical models. However, it would be formidable to solve the problem by tracking the signal in the system. It is, therefore, desirable to have a generalized mathematical model.

The location of the nonlinearity in the system is predictable and its location inside the system differs from one system to another. If we know that a component is nonlinear, this particular element is so marked as \( N \). When the system has two or more loops but only one nonlinear component, it is always possible to reduce the block diagram to a single loop by applying the rules developed; the example in the last section proves the validity of this technique. For more complex multiloop systems, it may be best to
Fig. 40. The mathematical model for an \( n \)th order control system containing \( m \) nonlinear elements, \( m = ij + pq \).
preserve the input and output stations. It is advisable to manipulate the linear portion only (rule (4)), but this is not an inviolable rule. In general, it is suggested to generalize the nonlinear systems of \( n \)th order into a single mathematical model. They are shown in Figure 40. Of course, the system of Figure 39 is a special case of the generalized model for \( n \)th order nonlinear systems. Comparison of the mathematical model shown in Figure 39 (e) with that shown in Figure 40 will soon reveal that for the forward paths:

\[
N_{11} = 0; \quad G_{11} = G_1; \quad N_{12} = N; \quad G_{12} = \frac{G_2 G_h}{1 + G_h G_5}; \quad N_{13} = 0. \quad N_{13}, \ G_{13} \ldots \ldots,
\]

\( N_{1j} \), \( G_{1j} \) are replaced by a short circuit, and open all paths from the 2nd path to the \( i \)th path. For the feedback paths:

\[
G'_{11} = 1 + \frac{(1+G)(1+G_h G_5)}{G_4}; \quad N'_1, \ N'_2, \ G'_1, \ldots, \ N'_1, \ G'_q
\]

\( G'_q \) are replaced by a short circuit, and open all paths from the 2nd path to the \( p \)th path.

\[ J. \textbf{Comparison of two nonlinear systems by impulse excitation} \]

The application of the impulse function to two different nonlinear systems will answer the question as to whether or not they have identical characteristics and stability responses. Consider the system shown in Figure 30 representing the case of a saturating electronic amplifier driving a motor in an instrument servo. The equation in the linear region is

\[
(77) \quad \ddot{\varepsilon} + \frac{1}{T_m} \dot{\varepsilon} + \frac{K K K}{N T_m} \varepsilon = 0 \quad -\varepsilon_1 < \varepsilon < +\varepsilon_1
\]
Where $\xi_1$ is the value of $|\xi|$ which causes saturation and $N$ is the gear ratio. For the saturated region

\begin{align}
\ddot{\xi} + \frac{1}{T_M} \dot{\xi} &= + C_1 \quad \xi \leq - \xi_1 \\
\ddot{\xi} + \frac{1}{T_M} \dot{\xi} &= - C_1 \quad \xi \geq + \xi_1
\end{align}

\[
C_1 = \frac{K_K M}{N T_M} \xi_1
\]

We assume further that another system which is exactly the same as the above except that the gains of the nonlinear elements within the linear region of operation are different, i.e., $K_1$ and $K_2$ as shown in Figure 41. Another set of equations can thus be written as follows:

\begin{align}
\ddot{\xi} + \frac{1}{T_M} \dot{\xi} &= + C_2 \quad \xi \leq - \xi_1 \\
\ddot{\xi} + \frac{1}{T_M} \dot{\xi} &= - C_2 \quad \xi \geq + \xi_1
\end{align}

\[
C_2 = \frac{K_K M}{N T_M} \xi_1
\]

Fig. 41. Saturating nonlinearity
On the error phase plane the isoclines are all straight lines. In the linear region the isoclines are radical straight lines emanating from a focus at the origin, while in the saturation region the isoclines are horizontal. The transition from linear to nonlinear operation is indicated by a vertical dividing line at $\mathcal{E} = \epsilon_1$, as shown in Figure 42.

Fig. 42. Phase-plane plot shows the different trajectories of two systems

The state response of the system is the trajectory on the plane if the initial conditions are determined. For these two systems, the trajectories are slightly different since $K_2 \neq K_1$. Likewise we have

\[
C_2 = \frac{K K K}{N T M} \epsilon_1 > C_1 = \frac{K K K}{N T M} \epsilon_1
\]
Furthermore, Figure 30 shows that the saturating values are different, and

\[ K_{s2}\dot{e}_1 > K_{s1}\dot{e}_1 \]

The values \( K_{s2}\dot{e}_1 \) and \( K_{s1}\dot{e}_1 \) are also the magnitudes of the new impulse functions. Thus it follows that the initial values of \( \dot{e} \) and \( \dot{\dot{e}} \) are definitely different since the magnitude of the impulse functions are different. They are located as \( I_1 \) and \( I_2 \) on the phase-plane. The entire response can be different because the gain values in the linear regions of the respective vacuum tubes are different, and this effect can be distinguished clearly by simply exciting the system with an impulse function.

K. Conclusions

Work to this point has shown that the technique of impulse excitation and the investigation of the useful results obtained from impulse excitation can be extended from linear control systems to nonlinear control systems. The objective of this chapter has been to clarify whether it is possible for all systems to be impulse excited or for useful state-responses for control systems to be obtained from impulse data. It is hoped that the basic theoretical studies will lay a foundation for the application of such techniques in linear and nonlinear system studies. Because of the limitation of the closed loop consideration for a complicated nonlinear system with respect to the establishment
of initial conditions, the open loop consideration
(validity based on the assumption) is used in most parts
of this chapter. The technique of the establishment of the
initial conditions has been developed for \( n \)th order non-
linear control systems. The reduction technique for a
complicated nonlinear system has been developed for the
purpose of establishing the initial conditions. The re-
sults of this study can be summarized as follows:

1. All systems (linear and nonlinear) have a unique
autonomous response in phase-space dependent upon their
specific initial conditions.

2. Any set of initial conditions can be established
using practical impulse functions for all systems unless
the signal flow is stopped by a nonlinearity such as a dead-
zone type functional. (This is a necessary but not sufficient
condition).

3. Uniqueness property: A practical impulse function
uniquely establishes a set of initial conditions for a
memoryless system in the phase-space. A set of initial con-
ditions established for a system uniquely determines the
steady-state response of that system if no memory nonlinearity
is contained inside the system. However, the converse may
not be true.

4. If the duration of the practical impulse is much
smaller than all time constants of the system, the initial
conditions established are directly related to the area of
the impulse rather than the magnitude or the duration of the practical impulse function.

5. A controlled practical impulse function (in which both the amplitude and duration can be varied) can be used to vary the initial conditions. However, this is not always true, for example, for a saturating type nonlinearity. The variation of the amplitude of the practical impulse function no longer varies the initial conditions.

6. Comparison of two nonlinear systems: If two nonlinear systems have identical initial conditions in phase-space and respond identically, they are identical and can be represented by the same model within that region.

7. Isomorphic systems: a group of systems which contain corresponding components and have identical mathematical models. Furthermore, isomorphic systems must be tested for all possible initial conditions.

8. An nth order nonlinear control system containing only one nonlinearity can always be reduced to a single loop system.

9. Initial conditions may not be established in some particular regions in the phase-space, such as the origin or the $\mathcal{E}, \dot{\mathcal{E}}, \ldots$, or $\mathcal{E}^{(n-1)}$ axes. Particular regions other than those mentioned may not be established by the practical impulse function depending upon the nature of the systems.
A. General theory

The basic philosophy of identification of a non-linear system is the decision making based on the definition and the properties of both linear and non-linear systems. The differences between the two can be fully used. Of course, a system which is merely a black box can be anything, time varying parameters being one example. As in network synthesis, the uniqueness of the solution can no longer be guaranteed. By following the rules of testing, which are yet to be developed, the system can easily be recognized as either linear or non-linear. Furthermore, the characteristic curve of the nonlinearity can also be recognized.

The basic definitions and properties of linear systems and time varying systems are summarized as follows:

1. Time invariance - a system is time-invariant if its characteristics do not change with time. For example,

\[ y(t) = x(t) + x^2(t) \]  

On the other hand, the system characterized by the input-output relation

\[ y(t) = \cos\omega_0 t x(t) + x^2(t) \]
is not time-invariant, but can be termed "time-varying". Strictly speaking, almost all physical systems are time-varying. As long as the variation in the system characteristics is slow compared to the input variation, one can approximate it as a time invariant system.

If \( T \) is used to denote the transformation or mapping, then for a time invariant system, the relationship between output and input can be expressed as

\[
T \left[ x(t) \right] = y(t)
\]

(84)

The system is time-invariant if and only if

\[
T \left[ kx(t) \right] = kT \left[ x(t) \right] \text{ for all real constants } k \text{ and }
\]

\[
y(t) = ky(t)
\]

(85)

and

\[
T \left[ x(t-t_0) \right] = y(t-t_0) \text{ for all real constants } t_0.
\]

(86)

2. Linearity - although control systems are characterized by differential equations, the concept of linearity can be considered at a more general level.

The term linearity denotes a proportionality relationship between the input and the output. It actually implies more than this. It implies the superposability of inputs and their respective effects, i.e., as given by Zadeh and Desoer (23), the additivity and homogeneity.

A system is homogeneous if and only if for all real constants \( k \), its transformation can always be expressed by

\[
T \left[ kx(t) \right] = kT \left[ x(t) \right]
\]

(87)
A system is additive if and only if for any pair of time functions $x_1(t)$ and $x_2(t)$, its transformation can be expressed by

$$T[x_1(t) + x_2(t)] = T[x_1(t)] + T[x_2(t)] \quad (88)$$

Combining Equation (87) and Equation (88) we obtain

$$T[k_1x_1(t) + k_2x_2(t)] = k_1T[x_1(t)] + k_2T[x_2(t)] \quad (89)$$

Equation (89) appears simple, but it will be shown that it is very important with respect to the identification of a nonlinear system.

This work is restricted to systems with time-invariant characteristics. Therefore, the main interest will be limited to the problem of linearity. Consider a simple example in order to show how the identification of a nonlinear system can be carried out. Figure 43 shows a linear characteristic and Figure 44 gives an arbitrary non-linear functional of the characteristic.

Two practical impulse functions having the same duration but different amplitudes (one is exactly twice as large as the other) are used to excite these two systems. The linear one has a slope of unity while the nonlinear one is an arbitrary continuous function. It is interesting to see that the output shown in Figure 43 preserves the proportionality whereas this is not true for the output of Figure 44. Furthermore, if we assume that the output pulses are of equal magnitude as shown in Figure 44, it can be surmised that there is a dip of the nonlinear functional between...
point 1 and point 2. However, the nonlinear functional shown in Figure 45 will give the same output. It can be easily separated between Figure 44 and Figure 45 simply by applying an additional input between points 2 and 3. The unique solution to the problem can still be obtained.
B. **Block diagram of a general system**

Many practical systems have multiple inputs and outputs, for example, the simplified turbojet engine control and steel rolling mill (8) controls. It is therefore necessary to generalize the system to be recognized. Consider a system with \( n \)-inputs and \( m \)-outputs as shown in Figure 46.

If such a system is linear, we may write

\[
\begin{align*}
    z_1 &= h_{11}x_1 + h_{12}x_2 + \ldots + h_{1n}x_n \\
    z_2 &= h_{21}x_1 + h_{22}x_2 + \ldots + h_{2n}x_n \\
    &\quad \vdots \\
    z_m &= h_{m1}x_1 + h_{m2}x_2 + \ldots + h_{mn}x_n
\end{align*}
\]

Equation (90) can also be expressed in matrix notation as

\[
[z(s)] = [H(s)] [x(s)]
\]
If such a system is non-linear, the above assumptions are no longer true. The output cannot be expressed as linear combinations of the product of inputs and transfer functions. Take a specific $h_{ij}(s)$. It should be replaced by a control path which contains a nonlinear section and a frequency sensitive function such as shown in Figure 47.

For the sake of simplification, $h^*$ is defined as a control path consisting of both frequency sensitive functions and nonlinearities. It is understood that $h^*$ has nothing whatsoever to do with transfer functions or describing functions. The goal is to identify the $h^*_{ij}$.

![Fig. 47. A simple control path $h^*_{ij}$](image)

Usually, it is desired to close the loop around and to include an equalizer or controller in order to stabilize the requirements. By the proper choice of another matrix $[E]$ the satisfactory relationship between $[x]$ and $[z]$ can be found so that stability requirements are met, insofar as the identification problem is concerned. This work will not discuss the details of the design problem.
It is assumed that the input and output terminals are available for a given system, i.e., the impulse function can be applied at any input. (The impulse function cannot be applied at the output terminals unless the system or control path is bilateral). It is evident that

\[ h_{ij}^* = \mathcal{H} \begin{cases} x's = 0, \text{ except } x_i \\ z's = 0, \text{ except } z_j \end{cases} \]

Equation (92) implies that in order to identify the control path \( h_{ij}^* \), it is necessary to apply the test signal at \( x_i \) only and measure the output at \( z_j \) only.

C. Method of identification of a nonlinear control path

In general, the system \( \mathcal{H} \) as shown in Figure 46 is complicated in structure. Therefore, it is desirable to consider a pair of input and output terminals each time. The individual testing will yield the complete solution for the system \( \mathcal{H} \). A control path \( h_{ij}^* \), by itself, can be a complicated combination of nonlinear sections and frequency sensitive functions. It is reasonable to break a complicated path into subpaths as long as the input and output terminals of the sub-paths are distinct. This will make the excitation of the practical impulse function applicable. After the above-mentioned assumptions are made, the problem of identification is narrowed down to consideration of the control paths \( h_{ij}^* \) as shown in Figure 47, (a) and (b). It will be
revealed that the output $z$'s are completely different and can be distinguished by impulse excitation at the input $x$'s.

(a) A control path $h_{ij}^*$ containing a nonlinearity followed by $G(s)$

(b) A control path $h_{ij}^*$ containing $G(s)$ followed by a saturating nonlinearity

Fig. 47

Suppose a practical impulse function is excited at the $x$-terminal of Figure 47 (a). The output, $z$, is a smooth function of time. If the amplitude of the input $x$ varies from zero to $B$, the instantaneous amplitude of the output $z$ will change proportionally according to the variation of the amplitude of the practical impulse function. Apparently, a true impulse function or a practical impulse function of very short duration may not give good results as will be proved by experiments in the next chapter. When the amplitude of the practical impulse function are raised above the
constant $B$, the output $z$ should remain the same because of the saturating characteristics. From the above analysis, it is not difficult to see that if the nonlinear section and $G(s)$ are enclosed inside the black box, it is possible to identify the black box from the information given by the $x's$ and the $z's$. In other words, the signals at $y$ are completely unknown.

It is clear that the technique of identification of a black box is to generate a set of practical impulses with various amplitudes to excite the system at points such as $x$ and then to observe the corresponding variations at $z$.

In section A of this chapter, the idea of identification has been illustrated by considering only one nonlinear characteristic. The basic ideas are just the same, however, because the signals $y$ are still a set of practical impulses. The addition of $G(s)$ has not complicated the problem because it is a linear transfer function and Equation (89) should also be satisfied. Therefore, the information contained at $z$ implies the information contained at $y$ and so the nonlinear characteristics can be easily determined from the $x's$ and $y's$.

![Fig. 48. Nonlinearities](image)

(a) relay (b) dead-zone (c) relay with dead-zone
Suppose that the saturating nonlinear element in Figure 47 (a) is replaced by certain nonlinearities as shown in Figure 48. We can then predict the kind of variation the z's should follow if a set of practical impulses with varying amplitudes is used to excite the point x. For the directional relay, z should apparently remain unchanged unless the practical impulses change polarities. For the dead-zone nonlinear element, as shown in (b), there is no output until the amplitudes of the impulses are greater than $\frac{A}{2}$. From then on, the output should follow the proportionality. For the relay with a dead-zone type nonlinear element, there will be no output until the amplitudes of the impulses have increased and become greater than $\frac{A}{2}$, but all amplitudes greater than $\frac{A}{2}$ should give the same output z.

Now consider the control path shown in Figure 47 (b). This is quite different from the case in (a) because of the interchange of the nonlinearity with G(s). First of all, it is interesting to see how this path can be distinguished from (a). The signal at y is a smooth time function according to the mathematical models developed in Chapter IV. The final form of y(t) is the sum of several individual time functions which depend upon the poles of G(s) and their residues. Assuming an arbitrary time function for y(t), it is interesting to investigate the outputs for various types of nonlinear elements. Consider first the saturating nonlinearity. Certain parts of the output will saturate, as
can be observed by increasing the amplitudes of the impulses. On the contrary, this phenomena can never occur in case (a). Suppose that the nonlinear element is replaced by a directional relay. The output \( z(t) \) in such a case is shown in Figure 49. Such wave forms can never be obtained for case (a). On the other hand, if such wave forms are observed at \( z \), it can easily be surmised that the nonlinearity must be a directional relay. Now, suppose a dead-zone relay is used. The outputs may look like that shown in Figure 50. Again, these wave forms can never appear in case (a) of Figure 47.

**Fig. 49.** The impulse responses of \( h \ast \) containing \( G(s) \) followed by a directional relay.

**Fig. 50.** The impulse responses of \( h \ast \) containing \( G(s) \) followed by a dead-zone relay.
It is interesting at this point to compare Figure 49 with Figure 50. In Figure 49, \( z(t) \) takes on every value along the time axis, but \( z(t) \) identically equals zero in certain sections in Figure 50 because of the appearance of the dead-zone. Therefore, the identification of a non-linearity as a relay or a dead-zone relay can be justified by this phenomena of the \( z(t) \) signal.

D. Discussions concerning the problem of identification

The problem of identification of an arbitrary non-linear system as discussed previously deals with a simple case which includes only a single nonlinearity. Similar to the problem of establishment of initial conditions, the complication of the problem of identification of a nonlinear system depends upon the number of nonlinearities involved, but not upon the order of the system. Suppose there are several nonlinearities and \( G(s) \) functions contained in \( h_{ij} \). The problem of identification is nearly impossible because the known constraints are far less than the demands. Take a simpler case which involves two nonlinearities and one \( G(s) \). The responses \( x(t) \) can be anything, but it is hard to justify for each piece of information of \( x(t) \), exactly which element has contributed to it after the excitation of the impulse function. Therefore, certain information is necessary, for example, the prediction of the rough locations of the nonlinear elements. The identification problem is
then reduced to identifying a simple path containing only one nonlinearity and an $n^{th}$ order $G(s)$.

E. **Synthesis of an unknown nonlinear system with useful blocks**

![Diagram of a nonlinear system]

**Fig. 51.** Block diagram of a nonlinear system - for the purpose of identification as useful blocks.

The problem under investigation in this section is the synthesis of an unknown nonlinear control system using useful blocks. Consider the system as shown in Figure 51. The feedback path of the system can be isolated from the forward path. Therefore, the practical impulse functions can be used to excite both paths. For a complicated path, the path itself can be further divided into subpaths. Thus, in order to illustrate the validity of the technique, only one nonlinearity is considered in the forward path as well as in the feedback path. Making use of the results obtained in Section C of this chapter, it can be said that the relative location of the nonlinear element in the path can be detected. The nonlinearity in either the forward or the feedback path is marked as such. The next step is to recognize exactly what type of nonlinearity is involved. This problem will be
considered in the next section. Of course, the function \( G(s) \) can be obtained as a consequence of the impulse excitation. The answer, however, will not be a unique one, that is, the number of poles and their residues will not be unique. At least, it is possible to conclude that an unknown nonlinear control system can always be synthesized with useful blocks, thus implying, of course, the mathematical model of the nonlinear system. Recall that the same result was arrived at from the establishment of the initial conditions and trajectories in the phase-space approach.

F. **Recognition of nonlinearities and their correct classification**

In general, basic nonlinearities can be classified into four types:

1. Curvature type nonlinearity (a linear section can be considered a special case of this type).
2. Saturating type nonlinearity (a directional relay is a special case of this type).
3. Dead-zone type nonlinearity.
4. Any composition of these nonlinearities.

The problem of recognizing and correctly classifying nonlinearities can best be discussed in three cases.

1. Recognition of a nonlinearity alone:

This is the simplest case and the problem can be easily solved by varying all the possible amplitudes of the impulses.
(from zero to as large as desired). The nonlinear functional can be constructed by mapping (in a one-to-one correspondence) every pair of input and output amplitudes because no memory type nonlinearity is considered. Figures 43, 44 and 45 serve as good examples of this case.

2. Recognition of a nonlinearity preceding G(s):
This type of control path $h^*_i_j$, has the blocks located as shown in Figure 47 (a). By exciting a set of practical impulses at $x$, the curvature nonlinearity gives a set of $z$'s with varying amplitudes. The saturating nonlinearity shows that $z$ will remain unchanged after some amplitude and the dead-zone nonlinearity gives no output for small amplitudes. Of course, the composite nonlinearity will have a combination of such phenomena.

3. Recognition of a nonlinearity following G(s):
The control path $h^*_i_j$ involved in this case has the form shown in Figure 47 (b). It is clear from the analysis given in Section C of this chapter that the curvature nonlinearity gives a set of continuous functions if a set of practical impulses is used to excite point $x$. The saturating nonlinearity will give a set of outputs with certain sections having zero derivatives. The dead-zone nonlinearity will give a set of outputs which have certain sections identically equal to zero. Of course, the composite nonlinearity will show a combination of the phenomena just mentioned.
The theory of recognition of nonlinearities and their correct classification has been developed in this chapter. Experimental verification of these results is presented in the next chapter.
CHAPTER VII
EXPERIMENTAL VERIFICATION

A. General

The theoretical development in this work has been completed. The experimental work reported in this chapter tends to verify the validity of the theory as well as to demonstrate how the method of establishing initial conditions and identification of a nonlinearity may actually be accomplished. The scientific method proposed in this work has to be demonstrated by experimental methods. The results of these experiments lead to new hypothesis. Actually, the experimental work has been conducted concurrently with the theoretical analysis.

There are four experiments reported in this chapter. The systems used include the actual control system as well as analogue computer simulations. The systems include both linear and nonlinear systems. The order of the systems include second and higher orders. Various types of nonlinearities have been used in order to prove the generalized theory. The techniques of impulse excitation include both the open loop and closed loop consideration. Practical impulse generators have been discussed in Chapter II. They are therefore not repeated here. However, the method of
evaluating the results and the discussion of these results are discussed in detail.

B. **Experiment I**

**The system and the experimental setup.** The system used in the first experiment is the General Electric Servo Demonstrator, which contains as components a vacuum tube push-pull amplifier, d-c motor, amplidyne and selsyn control transformers. Nonlinearities appear in the vacuum tube amplifier characteristics and in the amplidyne. There is no attempt to repeat the function of each component of the system here. The photograph of the General Electric Servo Demonstrator is shown in Figure 52. The block diagram of the equipment is shown in Figure 53, and the detailed internal connections of the system and the impulse generator are shown in Figure 54. The photograph of the setup used for generating the practical impulse for this experiment can be seen in Figure 55.

This experiment serves to demonstrate the technique of impulse excitation applied to an actual system with the closed-loop consideration because this is an a-c carrier system and the duration of the practical impulse function used is not short enough that the establishment of the initial conditions must use the model for the closed-loop system approach. Furthermore, the analysis is based on linear approximation; consequently, the system is considered to be a linear one. However, then the
FIGURE 52 PHOTOGRAPH OF THE GENERAL ELECTRIC DEMONSTRATOR AND THE EXPERIMENTAL SET-UP
Fig. 53. Block diagram of experimental testing of the system
Fig. 54. Circuit diagram of the G.E. servo demonstrator and the impulse generator.
FIGURE 55  SET-UP USED FOR GENERATING THE MODULATED IMPULSE

FIGURE 56  SET-UP USED FOR GENERATING THE SINUSOIDAL INPUT
amplitudes of the impulse function are increased, the non-linear phenomena appears. This will be discussed later.

**Bode plot.** If the nonlinearities existent in the system are linearized, the system can be regarded as linear, within a certain degree of approximation. The conventional technique of using a Bode plot can be applied in order to investigate the frequency response of the system. The well-known Bode plot is obtained by applying a sinusoidal driving function to a linear control system in order to find the unique transfer function of the system from the frequency responses. The setup of the low frequency sinusoidal function generator suitable for the band width of this system is shown in Figure 56. Such a generator is connected at the input of the error amplifier. The output rate is obtained from the attached a-c tachometer. This rate may then be integrated to give the output position variations. The Bode plot for this system is shown in Figure 57. The transfer function \( G(s) \) for velocity output (rad/sec) and error (volts) is given by

\[
K_Gv(s) = \frac{102.6875}{(s+0.725)(s+6.25)(s+38.42)}
\]

Apparently, the time constants are 1.38, 0.16 and 0.026 second.

The velocity output (rad/second) can be integrated to give the position output. The transfer function for position output (rads) and error signal (volts) can therefore be expressed as
FIG. 57. BODE DIAGRAM
Impulse excitation and the responses of the system.

The duration of the practical impulses used in this experiment are relatively long, and the impulse excitation to the system has to be a closed loop approach. The responses are, therefore, obtained from a closed loop transfer function.

The closed loop transfer function can be represented by

\[ \frac{K_G(s)}{1+K_G(s)} = \frac{102.6875 s}{s^3 + 45.3955^2 + 135.072s + 174.04s} \]

which can be written as

\[ G'(s) = \frac{K_G(s)}{1+K_G(s)} = \frac{102.6875 s}{(s+42.299)(s^2+3.096s+411.45)} \]

The transformation of \( G'(s) \) into the time domain yields;

\[ g'(t) = \frac{102.6875 e^{-42.299t}}{(1.548-42.299)^2 + 1.31053^2} \]

\[ + \frac{102.6875 e^{-1.548t}}{(42.299-1.548)^2 + 1.31053^2} \cos (1.31053t + \psi) \]

\[ \psi = \tan^{-1} \frac{1.31053}{-1.548} - \tan^{-1} \frac{1.31053}{42.299 - 1.548} \]

\[ g'(t) = -2.61258 e^{-42.299t} + 0.19364776 e^{-1.548t} \cos (1.31053t - 41.83^\circ) \]

The transfer function \( G'(s) \) is for the envelope of the modulated signal. Due to the demodulation process, only the
envelope of the modulated practical impulse is used in order to calculate the output responses.

The transfer function in the time domain for this system can be expressed as

\[ g'(t) = Be^{-a_o t} + De^{-\alpha t} \sin(\beta t + \gamma) \]

where \( B, D, a_o, \alpha, \beta \) and \( \gamma \) are constants.

The responses can be obtained by applying the Convolution Integration.

\[ c(t) = \int_0^a g(\tau) \left[ Be^{-a_o (t-\tau)} + De^{-\alpha (t-\tau)} \sin(b(t-\tau) + \gamma) \right] d\tau \]

\[ = B \int_0^a g(\tau) e^{-a_o (t-\tau)} d\tau + D \int_0^a g(\tau) e^{-\alpha (t-\tau)} \sin(b(t-\tau) + \gamma) d\tau \]

\[ = \frac{AB}{a_o} \left[ \left( e^{a_o} - 1 \right) e^{-a_o t} \right] \]

\[ + \frac{AD}{a} \left[ \frac{e^{\alpha a} (\alpha \cos \beta a + \beta \sin \beta a)}{\alpha^2 + \beta^2} - \frac{\alpha}{\alpha^2 + \beta^2} \right] e^{-\alpha t} \sin(\beta t + \gamma) \]

\[ + \frac{AD}{a} \left[ \frac{e^{\alpha a} (\alpha \sin \beta a - \beta \cos \beta a)}{\alpha^2 + \beta^2} + \frac{\beta}{\alpha^2 + \beta^2} \right] e^{-\alpha t} \cos(\beta t + \gamma) \]

**Establishment of the initial conditions by analytical methods.**

The initial conditions can be easily established by substitution of \( t=a^+ \) into Equation (100) and the higher order \( g(t) \) evaluated at \( t=a^+ \) can also be determined because \( c(t) \) contains only the exponential and sinusoidal functions.
From Equation (98), it is clear that Equation (100) involves all constants except \( \frac{A}{a} \) and \( a \) which describes the envelope of the modulated impulse function. It is clear that different impulse functions will give different initial conditions in the phase-space and they can always be established.

The establishment of the initial conditions by the graphical method. The initial conditions can also be established from the experimental results, that is, by connecting the impulse generator of Figure 55 as an input to the error amplifier of the system shown in Figure 52. The response of the system to the variable magnitude and duration of the impulse functions can be recorded. In this experiment, the amplitude of the impulse function can be varied from 10 volts to 80 volts, and the duration of the impulse function can be varied from 25 milliseconds to 110 milliseconds. The shortest duration of the envelope allows 1.5 to 2 cycles of the carrier signal where the longest duration of the envelope allows about 6 cycles of the carrier signal. For the sake of convenience, the responses are expressed by \( C_{ij} \), where \( i \) is the dummy index for the duration and \( j \) is the dummy index for the amplitude of the impulse function used to excite the system producing the response \( C_{ij} \). The experimental results are shown in Figures 58, 59, 60, 61, 62 and 63. They can be represented by \( C_{1j}, C_{2j}, C_{3j}, C_{4j}, C_{5j}, \) and \( C_{6j} \) respectively.
INPUT: 10V

INPUT: 30V

INPUT: 40V

INPUT: 50V

INPUT: 60V

INPUT: 70V

TIME SCALE 0.1 SEC/CM  OUTPUT: 2V/CM

IMPULSE DURATION: 0.025 SEC

FIGURE 58
INPUT: 5 V

INPUT: 10 V

INPUT: 20 V

INPUT: 30 V

INPUT: 60 V

INPUT: 80 V

TIME SCALE: 0.1 SEC/CM

OUTPUT: 2V/CM

IMPULSE DURATION: 0.05 SEC

FIGURE 59
INPUT :  30V

INPUT :  40V

INPUT :  50V

INPUT :  60V

INPUT :  70V

INPUT :  80V

TIME SCALE 0.1 SEC/CM  OUTPUT: 2V/CM
IMPULSE DURATION : 0.06 SEC

FIGURE 60
TIME SCALE 0.1 SEC/CM
OUTPUT 2V/CM

IMPULSE DURATION : 0.08 SEC

FIGURE 61
INPUT: 10V

INPUT: 40V

INPUT: 60V

INPUT: 30V

INPUT: 50V

INPUT: 80V

TIME SCALE: 0.1 SEC/CM

OUTPUT: 2V/CM

IMPULSE DURATION: 0.10 SEC

FIGURE 62
INPUT: 20V

INPUT: 40V

INPUT: 50V

INPUT: 60V

INPUT: 70V

INPUT: 80V

TIME SCALE 0.1 SEC/CM  OUTPUT: 2V/CM

IMPULSE DURATION: 0.11 SEC

FIGURE 63
A typical response has been enlarged and is shown in Figure 6h.

The responses of the system have the behavior of decay damping which can also be predicted from analytical considerations because the closed loop transfer function shows complex conjugate poles.

It has been shown that the initial conditions can always be established analytically for a linear system with either a closed loop or an open loop approach and so is also true for nonlinear systems with an open loop approximation. As far as the nonlinear system is concerned, the initial conditions can always be established from experimental data, unless the signal travel is stopped by dead-zone nonlinearity. Unfortunately, only the initial conditions for \( \xi \) and \( \frac{d\xi}{dt} \) can be obtained from the graphical method.

The initial condition of the column vector \( \begin{bmatrix} \xi & \frac{d\xi}{dt} & \frac{d^2\xi}{dt^2} & \ldots & \frac{d^{n-1}\xi}{dt^{n-1}} \end{bmatrix} \) can be evaluated analytically by taking the successive derivatives if the system equation exists. However, graphical measurement of them would cause many difficulties even if the system is only third order. In this respect, the analytical method seems to be superior to the graphical technique for higher order systems. However, it is always possible to measure \( \frac{\xi}{t=a^+} \) and \( \frac{d\xi}{dt} \bigg|_{t=a^+} \) from experimental results. For higher order systems, this implies a pair of initial conditions, on the \( \xi \) vs \( \frac{d\xi}{dt} \) plane,
FIG. 64. A TYPICAL RESPONSE TO IMPULSE EXCITATION
which in turn is the projection of the actual initial condition on \( \xi \) vs \( \frac{d\xi}{dt} \) plane.

It is of interest to investigate the method of measuring this pair of initial conditions graphically.

Designate the maximum value of the output function \( c(t) \) as \( C_{\text{max}} \) and then draw a line parallel to the time axis. The tangent at \( t = a^+ \) can be drawn and intersects the time axis and the line parallel to time axis just constructed. \( T \) is then defined as the time interval between these two intersecting points. It follows that \( \tan \phi = \frac{C_{\text{max}}}{T} \)

Therefore, the pair of initial conditions yield,

\[
\begin{align*}
\xi \bigg|_{t = a^+} &= -c(t) \bigg|_{t = a^+} \\
\frac{d\xi}{dt} \bigg|_{t = a^+} &= -\tan \phi = -\frac{C_{\text{max}}}{T}
\end{align*}
\]

Conventionally, \( c(t) \) is used to designate the output of the system. If the system has unity negative feedback, the expressions in Equation (101) and (102) are true. Therefore, \( c(t) \) really represents the signal in the feedback path before the summer.

Figure 65 shows a modulated response. The envelope contains the information because the envelope of the transfer function as well as the envelope of the impulse function have been used. Therefore, the initial conditions should be established from the envelope of the response. Figure 65 shows how the initial conditions are established according
FIG. 65. THE GRAPHICAL METHOD OF EVALUATING THE INITIAL CONDITIONS ON $\phi$ VS $\dot{\phi}$ PLANE.
to Equations (101) and (102). The pair of initial conditions show, of course, the projection of the actual initial condition in three dimensional space on the $\dot{\xi}$ vs $\xi$ phase-plane.

Discussion of the responses if the durations are very short. The time constants of the open loop transfer function are 1.38 second, 0.16 second and 0.026 second respectively, while the durations of the impulses used to excite the system range from 0.025 second to 0.110 second. Generally speaking, the durations of the practical impulses are less than the time constants of the system, but it is improper to say that they are much less than all the time constants of the system. Even though this assumption is not strong enough, it is possible to show some correlation with respect to the discussion presented in Section F of Chapter IV.

Comparison of the responses can be accomplished in various ways. For this system, the areas of the responses are difficult to measure. Therefore, three peak values of the responses are measured. The responses intersects the time coordinate several times and the time measured between two distinct time values is designated as the rise-time. Furthermore, the rise-times are nearly equal, no matter which practical impulse function is applied. Therefore, the first three distinct peak values defined should represent the behavior of the response of the system. The results of measurements are summarized in Table 1 which shows 36 practical impulse functions $(\mathcal{G}_{ij})$ and 36 responses $(\mathcal{C}_{ij})$. 
Table I. Data of the Experiment I

<table>
<thead>
<tr>
<th>Impulse Function ((\mathcal{G}_{ij}))</th>
<th>System Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>A/a Dur.</td>
<td>1st (v)</td>
</tr>
<tr>
<td>A/m (m/s)sec</td>
<td></td>
</tr>
<tr>
<td>(C_{11})</td>
<td>10 25</td>
</tr>
<tr>
<td>(C_{12})</td>
<td>30 25</td>
</tr>
<tr>
<td>(C_{13})</td>
<td>40 25</td>
</tr>
<tr>
<td>(C_{14})</td>
<td>50 25</td>
</tr>
<tr>
<td>(C_{15})</td>
<td>60 25</td>
</tr>
<tr>
<td>(C_{16})</td>
<td>70 25</td>
</tr>
<tr>
<td>(C_{21})</td>
<td>5 50</td>
</tr>
<tr>
<td>(C_{22})</td>
<td>10 50</td>
</tr>
<tr>
<td>(C_{23})</td>
<td>20 50</td>
</tr>
<tr>
<td>(C_{24})</td>
<td>30 50</td>
</tr>
<tr>
<td>(C_{25})</td>
<td>60 50</td>
</tr>
<tr>
<td>(C_{26})</td>
<td>80 50</td>
</tr>
<tr>
<td>(C_{31})</td>
<td>30 60</td>
</tr>
<tr>
<td>(C_{32})</td>
<td>40 60</td>
</tr>
<tr>
<td>(C_{33})</td>
<td>50 60</td>
</tr>
<tr>
<td>(C_{34})</td>
<td>60 60</td>
</tr>
<tr>
<td>(C_{35})</td>
<td>70 60</td>
</tr>
<tr>
<td>(C_{36})</td>
<td>80 60</td>
</tr>
<tr>
<td>(C_{41})</td>
<td>10 80</td>
</tr>
<tr>
<td>(C_{42})</td>
<td>20 80</td>
</tr>
<tr>
<td>(C_{43})</td>
<td>30 80</td>
</tr>
</tbody>
</table>
Table I (continued)

<table>
<thead>
<tr>
<th>Impulse Function</th>
<th>System Output</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A/a</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>C44</td>
<td>40</td>
</tr>
<tr>
<td>C45</td>
<td>60</td>
</tr>
<tr>
<td>C46</td>
<td>80</td>
</tr>
<tr>
<td>C51</td>
<td>10</td>
</tr>
<tr>
<td>C52</td>
<td>30</td>
</tr>
<tr>
<td>C53</td>
<td>40</td>
</tr>
<tr>
<td>C54</td>
<td>50</td>
</tr>
<tr>
<td>C55</td>
<td>60</td>
</tr>
<tr>
<td>C56</td>
<td>80</td>
</tr>
<tr>
<td>C61</td>
<td>20</td>
</tr>
<tr>
<td>C62</td>
<td>40</td>
</tr>
<tr>
<td>C63</td>
<td>50</td>
</tr>
<tr>
<td>C64</td>
<td>60</td>
</tr>
<tr>
<td>C65</td>
<td>70</td>
</tr>
<tr>
<td>C66</td>
<td>80</td>
</tr>
</tbody>
</table>

Abbreviations:

A/a : Magnitude of the impulse function.
A : Area of the impulse function.
Dur. : Duration of the impulse function.
v : volt
ms : milli-second
v-sec : volt-second
r.t. : rise time
The first three peak values of $C_{ij}$ as well as the first three rise-times are shown.

Figures 66, 67, 68 show the plots of the first three peak values of the responses versus the area of the practical impulses. They are plotted from data given in Table I. The results of these curves tend to show the theoretical development in Section F of the Chapter IV which concludes that the responses will depend upon area $A$ of the impulses rather than the amplitudes or the durations of these impulses. If an arbitrary value of area $A$ (volt-second) is chosen for different durations of the impulses (the amplitude of the impulses will be different, of course, because the areas remain the same) the peak values are nearly equal. If the assumption is made that durations are much less than all the time constants of the system, then these curves naturally tend to be even closer. Another interesting result can be observed from Figures 66, 67 and 68. These curves tend to saturate, which implies, of course, that these curves are bounded with increasing amplitudes of the impulses (since the durations are not increased without bound, increase of the area means increase of amplitude). It can best be explained that such phenomena is due to the existence of the saturating nonlinearities in the system.

Prediction of the region of the initial conditions established by the practical impulse function. Physically, an $n$-dimension vector space is difficult to visualize.
Fig. 66. The plot of the 1st peak value of output vs the area of the impulse.
Fig 67: The plot of the 2nd peak value of output vs the area of the impulse
FIG. 68. THE PLOT OF THE 3RD PEAK VALUE OF OUTPUT VS THE AREA OF THE IMPULSE
The investigation of the projection may, however, be important. To an arbitrary control system, it is always possible to establish the initial conditions experimentally in the $\dot{e}$ vs $e$ plane except when the signal travel is stopped by the dead-zone nonlinearity. It will be shown in this chapter that the problem of the prediction of the location of the initial conditions depends upon the type of nonlinearities which exist inside the system. For this system, from an analytical analysis point of view, Equation (100) shows that the set of initial conditions will depend upon $A$ and $a$ only because all the remaining parameters are merely constants. A set of initial conditions on the $\dot{e}$ vs $e$ plane can be obtained from Figures 58 through Figure 63 by graphical technique just introduced. The results of such evaluation are shown in Table II and located on the $\dot{e}$ vs $e$ plane in Figure 69. It can be very well predicted that the set of initial conditions will be located in the third quadrant because the output responses are all positive when evaluated at $t=a^+$, likewise, so are the derivatives, as shown in Figures 58 through 63.

It is interesting to note that for those practical impulse functions (approximately for areas greater than 3.5) the saturating responses, as shown in Figures 66, 67 and 68, their initial conditions are located in a particular area on the $\dot{e}$ vs $e$ plane as shown in Figure 69. Because the system is a third order system, the actual initial conditions
FIG. 69. THE INITIAL CONDITIONS WHICH CHARACTERIZE THE SYSTEM PROJECTED ON THE $\xi$ VS. $\dot{\xi}$ PLANE
should therefore be located inside a particular volume in the space. However, there do exist a few initial conditions which be outside the particular region. Generally speaking, such a region is not predictable until some of the initial conditions are found. The location of such a region varies from system to system. It is significant to point out that some systems may not have such a region at all, e.g., the system containing an ideal relay element.
Table II  Initial Conditions of $E$ and $iE$ of Experiment I

<table>
<thead>
<tr>
<th>$i$</th>
<th>Duration</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>25ms</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$c(t)$ at $t=a^+$</td>
<td>0.150</td>
<td>0.17</td>
<td>0.185</td>
<td>0.390</td>
<td>0.400</td>
<td>0.380</td>
</tr>
<tr>
<td></td>
<td>$\tan \phi$</td>
<td>0.087</td>
<td>0.160</td>
<td>0.216</td>
<td>0.781</td>
<td>0.249</td>
<td>0.701</td>
</tr>
<tr>
<td>2</td>
<td>$c(t)$ at $t=a^+$</td>
<td>0.100</td>
<td>0.100</td>
<td>0.180</td>
<td>0.400</td>
<td>0.700</td>
<td>0.540</td>
</tr>
<tr>
<td></td>
<td>$\tan \phi$</td>
<td>0.118</td>
<td>0.149</td>
<td>0.165</td>
<td>0.176</td>
<td>0.384</td>
<td>0.364</td>
</tr>
<tr>
<td>3</td>
<td>$c(t)$ at $t=a^+$</td>
<td>0.400</td>
<td>0.560</td>
<td>0.600</td>
<td>0.640</td>
<td>0.600</td>
<td>0.500</td>
</tr>
<tr>
<td></td>
<td>$\tan \phi$</td>
<td>0.323</td>
<td>0.425</td>
<td>0.296</td>
<td>0.394</td>
<td>0.404</td>
<td>0.414</td>
</tr>
<tr>
<td>4</td>
<td>$c(t)$ at $t=a^+$</td>
<td>0.360</td>
<td>0.380</td>
<td>0.390</td>
<td>0.540</td>
<td>0.560</td>
<td>0.780</td>
</tr>
<tr>
<td></td>
<td>$\tan \phi$</td>
<td>0.306</td>
<td>0.286</td>
<td>0.249</td>
<td>0.414</td>
<td>0.384</td>
<td>0.487</td>
</tr>
<tr>
<td>5</td>
<td>$c(t)$ at $t=a^+$</td>
<td>0.380</td>
<td>0.620</td>
<td>0.640</td>
<td>0.640</td>
<td>0.620</td>
<td>0.420</td>
</tr>
<tr>
<td></td>
<td>$\tan \phi$</td>
<td>0.249</td>
<td>0.287</td>
<td>0.306</td>
<td>0.354</td>
<td>0.374</td>
<td>0.325</td>
</tr>
<tr>
<td>6</td>
<td>$c(t)$ at $t=a^+$</td>
<td>0.600</td>
<td>0.580</td>
<td>0.600</td>
<td>0.700</td>
<td>0.540</td>
<td>0.850</td>
</tr>
<tr>
<td></td>
<td>$\tan \phi$</td>
<td>0.268</td>
<td>0.306</td>
<td>0.354</td>
<td>0.384</td>
<td>0.287</td>
<td>0.364</td>
</tr>
</tbody>
</table>

Remark: $j$: dummy index for the amplitudes of $q_{ij}$, $j=1,2,\ldots,6$. $i$: index for different durations of the practical impulse functions, $i=1,2,\ldots,6$ means Figures 58, 59,\ldots,63 respectively. $i=1, 25$ ms $i=3, 60$ ms. $i=5, 100$ ms. $i=2, 50$ ms. $i=4, 80$ ms. $i=6, 110$ ms.
The concept of the equivalent nonlinearity of a system.

The problem of identification of a nonlinearity can be performed if the open loop excitation technique is used. It will be shown in later experiments that identification of a nonlinear system can be achieved. However, the system used in this experiment has been linearized and is excited with the loop closed. Therefore, a closed loop transfer function $KG(s)$ has been used throughout the analysis. Even though $1 + KG(s)$ the system is considered to be a linear one, it behaves differently when the amplitude of the impulses increases. In other words, the proportionality property is no longer preserved. It may be useful to define the equivalent nonlinearity of a system which serves at least to indicate the threshold amplitude of the impulse function which violates the proportionality property. The system will be represented, of course, in an equivalent opened loop form as shown in Figure 70.

![Figure 70. Equivalent nonlinearity of a system.](image)

The identification of $N_{eq}$ (equivalent nonlinearity) of a system excited with the loop closed can be accomplished by taking the measurements from system responses and comparing
their amplitudes on a per unit basis against the actual volts of the amplitudes of the impulse functions. Theoretically speaking, the measurement can be taken at any instant of the responses. Figures 71, 72 and 73 show the data taken from three peak values and they agree very well. The duration of the impulse functions is also used as an independent variable in plotting the curves. The results obtained from the responses by actually applying the impulses to the system show that when the durations of the impulses are short, the data are quite apart from the actual curve. For longer durations, however, (from 80 ms. to 110 ms), the curves obtained are limited to a very narrow band, as indicated in the figures. It may be concluded that the durations of impulses should be longer for the identification purpose because the responses of the system with the loop opened will die out more quickly for very short durations of the practical impulses than that of the longer durations. Consequently, comparison of the instantaneous value of the responses become difficult and inaccurate. This is contrary to the necessary requirement in the problem of establishing initial conditions.
FIG. 71: FUNCTIONAL RELATIONSHIP OF THE NONLINEAR ELEMENT OF THE SYSTEM (1) FROM 1\textsuperscript{st} PEAK MEASUREMENT.
FIG. 72 (2) FROM 2ND PEAK MEASUREMENT.
FIG. 73 (3) FROM 3rd PEAK MEASUREMENT.
C. Experiment II

The system and the experimental setup. The technique of impulse excitation with the closed loop approach has been demonstrated in Experiment I in which linearization of the system has been assumed. For most nonlinear systems, it is advantageous to excite a system with the open loop approach. Experiments II through V will show how this can be done.

Figure 74 shows a 4th order nonlinear system containing a nonlinearity $N$. The nonlinearity $N$ is an ideal relay in this particular experiment and is simulated by a STRUTHERS DUNN perfect relay (Figure 75) and the frequency sensitive function is simulated by a REAC electronic analogue computer (Figure 76). Figure 77 shows the ideal relay characteristics and Figure 78 shows the schematic diagram of the nonlinearity. The mechanism of the ideal relay consists of coils and contacts. The coil will energize with signals having amplitudes of 1 volt or more. The contactor will hold as long as the coil is energized. Of course, for a practical impulse input, the output $y$ will also be a practical impulse function with a constant amplitude of 12 volts. Such an output is then utilized as the input to the frequency sensitive function, which is simulated by the analogue computer. The circuit of the analogue computer simulation for a 4th order transfer function is shown in Figure 79.
Fig. 74. Block diagram of the system for Experiment II.

Fig. 75. Photograph of the ideal relay and nonlinearity simulation setup.
Fig. 76. Photograph of the REAC electronic analogue computer used to simulate the transfer functions in Experiments II, III, IV and V.
Fig. 77 Ideal relay characteristic

Fig. 78. Schematic diagram of the ideal relay nonlinearity
Fig. 79. A circuit for the simulation of the transfer function of G(s)

\[
G(s) = \frac{-s^2 (1+0.45)}{(1+0.25)(1+0.5s)(1+s)(1+1.25)}
\]
The analytical method for establishment of initial conditions. The frequency sensitive function has four poles and it may be written as

\[ G(s) = \frac{-s^2(1+0.4s)}{(1+0.2s)(1+0.5s)(1+s)(1+1.2s)} \]

\[ = \frac{-10(s^3 + 2.5s^2)}{s^4 + 8.833s^3 + 23.653s^2 + 24.156s + 8.33} \]

It may also be expressed in partial fraction form as

\[ G(s) = \sum_{i=1}^{4} \frac{K_i}{1+\frac{s}{T_i}} = -0.833 \frac{1}{1+0.2s} - 0.954 \frac{1}{1+0.5s} + 7.48 \frac{1}{1+s} - 2.835 \frac{1}{1+1.2s} \]

where \( K_1 = -0.833, K_2 = -0.954, K_3 = 7.48, K_4 = -2.835 \) and \( T_1 = 0.2, T_2 = 0.5, T_3 = 1, T_4 = 1.2 \) seconds.

The amplitude of the practical impulse function is constant and is equal to 12 volts. However, the durations can be varied as long as they are much less than the time constants \( T_1, T_2, T_3, \) and \( T_4 \). Letting \( a = 0.1 \) second the initial conditions can be established by substituting the constants into Equations (51) and (52) which yields

\[ E(t) = -c(t) \]

\[ = \sum_{i=1}^{4} \frac{AK_i}{a} (e^{-\frac{t}{T_i} - 1}) \]

\[ = 12 \left( 0.833(e^{-0.2} - 1) + 0.954(e^{-0.5} - 1) \right) \]
The initial conditions on the $\dot{\xi}$ vs $\xi$ phase plane subject to an impulse duration $a=0.1$ second are $(0.3216, -20.33604)$.

The experimental method of establishment of initial conditions. The responses of the system subjected to a set of practical impulses are recorded by a graphical recording instrument. The variation of the amplitude of the impulses at $x$ has no effect on the output $y$ as can be seen from Figure 78. Therefore, the initial conditions vary according to the changes in the durations which are different from Experiment I. Figures 80 to 92 show a set of responses $c(t)$ with different duration $a$.

Measuring the $\dot{\xi}$ and $\ddot{\xi}$ at $t=a^+$ from the experimental results yields $\dot{\xi} = \tan \theta = \tan -87.2^\circ = -20.45$ and $\ddot{\xi} = -0.320$.
This agrees with the analytical approach in the last section.

For the purpose of establishing initial conditions in this experiment, it is suggested that the duration of the impulses be no longer than 0.10 second, since the time constants of the system are 0.20, 0.50, 1, and 1.2 seconds. Durations "a" greater than 0.10 have been shown in Figures 87 through 92, for identification of the nonlinearity.
(a) \( \frac{A}{a} = 12v, a=0.04 \text{ second.} \)

(b) The output of the system (Experiment II)
Vertical scale: 0.4 v/cm
Time scale: 0.4 sec/cm

(c) The output of the system (Experiment IV)
Vertical scale: 4v/cm
Time scale: 0.4 sec/cm

Fig. 80. The practical impulse function and the outputs of systems.
(a) \( \frac{A}{a} = 12v, a = 0.05 \text{ second.} \)

(b) The output of the system (Experiment II)
Vertical scale: 0.4 v/cm
Time scale: 0.4 sec/cm

(c) The output of the system (Experiment IV)
Vertical scale: 4 v/cm
Time scale: 0.4 sec/cm

Fig. 81. The practical impulse function and the outputs of systems.
(a) $\frac{A}{a} = 12v$, $a = 0.06$ second.

(b) The output of the system (Experiment II)
Vertical scale: 0.4 v/cm
Time scale: 0.4 sec/cm

(c) The output of the system (Experiment IV)
Vertical scale: 4 v/cm
Time scale: 0.4 sec/cm

Fig. 82. The practical impulse function and the outputs of systems.
(a) $\frac{\Delta}{a} = 12v$, $a = 0.08$ second.

(b) The output of the system (Experiment II)
Vertical scale: 0.4 v/cm
Time scale: 0.4 sec/cm

(c) The output of the system (Experiment IV)
Vertical scale: 4 v/cm
Time scale: 0.4 sec/cm

Fig. 83. The practical impulse function and the outputs of systems.
(a) \( \frac{A}{a} = 12v, a = 0.09 \text{ second.} \)

(b) The output of the system (Experiment II)
Vertical scale: 0.4 v/cm
Time scale: 0.4 sec/cm

(c) The output of the system (Experiment IV)
Vertical scale: 4 v/cm
Time scale: 0.4 sec/cm

Fig. 84. The practical impulse function and the outputs of systems.
(a) \[ A_a = 12v, \ a = 0.10 \text{ second.} \]

(b) The output of the system (Experiment II)
Vertical scale: 0.4 v/cm
Time scale: 0.4 sec/cm

(c) The output of the system (Experiment IV)
Vertical scale: 4 v/cm
Time scale: 0.4 sec/cm

Fig. 85. The practical impulse function and the outputs of systems.
(a) \( \frac{A}{a} = 12v, \ a = 0.11 \text{ second.} \)

(b) The output of the system (Experiment II)
Vertical scale: 0.4 v/cm
Time scale 0.4 sec/cm

(c) The output of the system (Experiment IV)
Vertical scale: 4 v/cm
Time scale: 0.4 sec/cm

Fig. 86. The practical impulse function and the outputs of systems.
Fig. 87. The practical impulse function and the outputs of systems.
Fig. 88. The practical impulse function and the outputs of systems.
Fig. 89. The practical impulse function and the outputs of systems.
(a) \( \frac{\Delta v}{a} = 12v, \ a = 0.16 \text{ second}. \)

(b) The output of the system (Experiment II)  
Vertical scale: 0.4 v/cm  
Time scale: 0.4 sec/cm

(c) The output of the system (Experiment IV)  
Vertical scale: 4 v/cm  
Time scale: 0.4 sec/cm

Fig. 90. The practical impulse function and the outputs of systems.
(a) $\Delta a = 12v$, $a = 0.20$ second.

(b) The output of the system (Experiment II)
Vertical scale: $0.4 \ v/cm$
Time scale: $0.4 \ sec/cm$

(c) The output of the system (Experiment IV)
Vertical scale: $4 \ v/cm$
Time scale: $0.4 \ sec/cm$

Fig. 91. The practical impulse function and the outputs of systems.
(a) $\frac{\Delta}{a} = 12v, \; a = 0.25 \text{ second.}$

(b) The output of the system (Experiment II)
Vertical scale: 0.4 v/cm
Time scale: 0.4 sec/cm

(c) The output of the system (Experiment IV)
Vertical scale: 4 v/cm
Time scale: 0.4 sec/cm

Fig. 92. The practical impulse function and the outputs of systems.
Prediction of the region of location of initial conditions. The set of initial conditions in this experiment is located in a bounded region in the third quadrant of the \( \dot{\mathcal{E}} - \mathcal{E} \) plane as shown in Figure 93. This phenomenon can be seen from either analytical or experimental results. Equations (105) and (106) show that the values of \( A, a, K_1, T_1 \) are bounded, as well as the ratio \( \frac{A}{a} \) (equal to 12v).

Fig. 93. The region of the initial conditions established.

The value of the factor \( (e^{-\frac{a}{T_1}} - 1) \) is always less than unity. Therefore, the initial conditions thus evaluated are always bounded. Figures 80 through 92 show that the above statement is true and shed more insight as to the sign of \( \mathcal{E} \) and \( \dot{\mathcal{E}} \) evaluated at \( t = a^+ \). The initial conditions should be located in the third quadrant as shown in Figure 93. Figure 94 shows the response of the system subjected to an impulse with a
Fig. 94. The response of the system (Experiment II) with $A = 12v$, $a = 0.64$ second.
Output: Vertical scale: 0.4 v/cm
Time scale: 0.4 sec/cm
duration of 0.64 second which tends to reach the averaged
time constant (0.725 second) of the system. The initial
conditions established in this case are different from the
previous ones, because the sign of $\dot{C}$ evaluated at $t=a^+$ will
be positive, which implies, of course, that the region will
extend to the second quadrant as well. However, such long
durations of the impulse function are not recommended.

Discussion of responses if durations are very short.
In order to demonstrate the discussion of a special case
as developed in section F of Chapter IV from experimental
results, the plot of amplitude versus area of the impulses
must be found. In this experiment, the amplitude $A_a$ is kept
constant because of the ideal relay element. Then the
variation of "a" always results in unequal areas. Conse­
quently, we cannot proceed with such a discussion.

Identification of the perfect relay nonlinearity.
The perfect relay nonlinearity can be identified by applying
a set of impulse functions at $x$ and observing the responses
of the system. Observe that the instantaneous amplitudes
of the responses change when the duration of the impulses
change. Therefore, it is not wise to vary the amplitude and
the duration of impulses at the same time. If the duration
of the impulses is kept constant (0.10 second is a typical
value), no matter what values of amplitude are chosen for the
impulses, the responses are always as shown in Figure 85 (b).
Such a nonlinearity can therefore be successfully identified
and this has been proved by actual recordings.
D. **Experiment III**

**The system and the experimental setup.** The system used in this experiment is a second order system with a saturating nonlinearity. The block diagram of the system is shown in Figure 95 (a). It is to be remembered that the application of the impulse function to the system uses the idea of the open-loop system. The nonlinearity N which has a saturating characteristic is simulated by a clamping circuit using a rectifier with a forward resistance of 3 ohms as shown in Figure 95(b). If the signal has an amplitude less than 14 volts, the output X(s) gives the same practical impulse function. However, when the input has an amplitude greater than 14 volts, the rectifier will conduct and the output X(s) will always give a practical impulse function with an amplitude of 14 volts.

The frequency sensitive function G(s) is simulated on the REAC analogue computer. The circuit diagram for this computer was shown in Figure 21.

**Impulse excitation and responses of the system.** A set of impulse functions is fed into the error terminals and the responses c(t) are recorded. These responses are shown in Figures 96-1 through 96-7. Both the duration and the amplitude are varied. The responses of the system are expressed by \( C_{i,j} \), where \( i \) is the dummy index for the duration and \( j \) is the dummy index for the amplitude. Both \( i \) and \( j \) range from 1 to 7. The \( A \) having amplitudes greater than 14 volts give
Fig. 95(a) Block diagram of the nonlinear system of Experiment III

Fig. 95(b) Diagram of the nonlinearity N in Fig.
FIG. 96.1. IMPULSE RESPONSES OF THE SYSTEM SHOWN IN FIG. 95(a).

IMPULSE DURATION (a) : 20 ms
TIME SCALE : 0.2 SEC/CM  OUTPUT (c_{ij}) : 0.1 V/CM
A/a: 10V

C_{25}

A/a: 12V

C_{26}

A/a: 14V

C_{27}

IMPULSE DURATION (a): 40 ms

TIME SCALE: 0.2 SEC/CM
OUTPUT (v_{ij}): 0.2V/CM

FIG. 96 -2.
A/a : 10V

A/a : 12V

A/a : 14V

IMPULSE DURATION (a) : 60 ms

TIME SCALE : 0.2 SEC/CM

OUTPUT (Cij) : 0.2 V/CM

FIG. 96 - 3.
IMPULSE DURATION (a) : 80 ms

TIME SCALE: 0.2 SEC/CM OUTPUT (C_y) : 0.2 V/CM

FIG. 96-4.
A/a : 10V

C55

A/a : 12V

C56

A/a : 14V

C57

IMPULSE DURATION (a) : 100 ms

TIME SCALE: 0.2SEC/CM  OUTPUT (C) : 0.2V/CM

FIG. 96-5.
IMPULSE DURATION (a) : 120 ms

TIME SCALE: 0.2 SEC/CM     OUTPUT (Gij): 0.2V/CM

FIG. 96 - 6.
Impulse duration (a): 140 ms
Time scale: 0.2 sec/cm  Output (Cij): 0.2 V/cm

Fig. 96-7.
the same response as that for \( \frac{A}{a} \) equal to 14 volts. In other words, they are the same as \( C_i \), where \( i = 1, 2, \ldots, 7 \).

**The establishment of the initial conditions.** It is clear that the initial conditions established will depend upon both the amplitude and the duration of the impulse functions and they may be evaluated either analytically or experimentally. The response of the system with \( \frac{A}{a} = 14 \) volts and \( a = 0.14 \) second has been chosen as an example illustrated in section D of Chapter IV (pp. 44-46) to prove that the analytical and experimental establishment of the initial conditions agree with each other. No attempt is made to repeat them here.

**Prediction of the region of all possible locations of the initial conditions.** The frequency sensitive function used in this experiment may be written as

\[
G(s) = \frac{2}{1} \frac{K_1}{1 + sT_1} = \frac{1}{2} \frac{1}{1 + 0.1s} + \frac{1}{1 + s}
\]

where \( K_2 = -K_1 = \frac{1}{9} \)

Writing Equation (45) as

\[
(45) \quad \underline{E(t)} = \frac{AK_1}{a} \left[ (e^{-\frac{a}{T_1}} - 1) - (e^{-\frac{a}{T_2}} - 1) \right]
\]

\[
(46) \quad \frac{d\underline{E(t)}}{dt} = -A \frac{K_1}{a} \left[ \frac{1}{T_1} (e^{-\frac{a}{T_1}} - 1) - \frac{1}{T_2} (e^{-\frac{a}{T_2}} - 1) \right]
\]
Because of the fact of equal residues in absolute value in this special case, the above equation leads to an interesting result. If $C_1$, $C_2$ are two distinct constants, Equation (45) (46) can be expressed as

\begin{align}
\dot{\xi}(t) |_{t=a^+} &= C_1A \\
\dot{\xi}(t) |_{t=a^+} &= C_2A
\end{align}

It is clear that the pair of initial conditions established will be a straight line on the $\dot{\xi}$ versus $\xi$ plane if the duration $a$ is kept constant. For different durations, there will be a set of straight lines passing through the origin as shown in Figure 97.

![Diagram](image)

**Fig. 97.** The possible location of the initial conditions.

The time constants in this system are 0.1 and 1 second. Since one of these is very short, the major parts of initial conditions established from Figure 96 are not acceptable. Only the first two data $C_{1j}$ and $C_{2j}$ are allowed. Data $C_{3j}$ to $C_{7j}$
show that the durations are too long and nearly equal to the time constant of 0.1 second. However, the discussion of the responses' dependency on the area of the impulses in section F of Chapter IV is not desirable.

The identification of the saturating type nonlinearity. Even though data $C_{3j}$ to $C_{7j}$ cannot be used to establish the initial conditions of the system, they show an even better effect with respect to the problem of identification of a saturating type nonlinearity. Figure 98 shows the characteristics of the nonlinearity identified by different durations of the practical impulses from measurements of the relative amplitudes of the responses.

E. Experiment IV

The system and the experimental setup. The block diagram of the nonlinear system used for this experiment is the same as that of experiment II except that the nonlinear element has a dead-zone constant of 12 volts and $G(s)$ is a second order function with time constants of 0.5 second and 1.2 seconds. Reference should be made to Figure 74 for the block diagram and Figure 78 for the schematic diagram of the nonlinearity simulation. In Figure 74, the function $G(s)$ should be $\frac{s^2}{(1+0.5s)(1+1.2s)}$ for this experiment. In Figure 78, everything remains unchanged except the relay which should be a dead-zone type relay instead of an ideal one. The dead-zone relay used in this experiment has a
Fig. 98. Identified saturating nonlinearity.
dead-zone constant of 12 v d-c, which means that the contactor will not act unless the d-c voltage applied at the coil terminals is greater than or equal to 12 volts. Photographs of such a dead-zone type relay and the simulation setup are shown in Figure 99. The nonlinear characteristic thus obtained is also shown in Figure 100. The circuit for simulating the G(s) function which may be programmed on the REAC analogue computer is shown in Figure 101.

The technique of impulse excitation again uses the open-loop approach and the responses of the system are different because of the dead-zone element that has been introduced in the system. This experiment together with the next one serves to demonstrate the effect of the dead-zone nonlinearity as far as establishment of the initial conditions is concerned.

**Impulse excitation and the responses of the system.**

A set of practical impulses is injected at the input terminals to the nonlinear element. If the amplitude of these impulses is less than 12 volts, the contactor will not close and therefore, no output is observed. However, when the amplitude of the impulses is greater than 12 volts, the output of the systems gives the same responses regardless of the input impulses. Therefore, the variations in the output signals depend upon the duration, but certainly not on the amplitudes in the saturating section. Figures
Fig. 99 Photograph of the simulation of the dead-zone relay nonlinearity.

Fig. 100 The characteristics of the dead-zone relay.
Fig. 101. A circuit for \( G(s) = \frac{-s^2}{(1+0.5s)(1+1.25)} \) which may be programmed on the REAC electronic analogue computer.
The establishment of the initial conditions. The initial conditions can be established either analytically or experimentally. Suppose, for example, a typical impulse with \( A_a = 12 \) volts and \( a = 0.1 \) second is chosen. The initial conditions may be established analytically as follows.

Place \( G(s) \) into the following form.

\[
G(s) = \frac{2}{1} \frac{K}{1 + sT_1} = \frac{2.8563}{1 + 0.5s} - \frac{1.1896}{1 + 1.2s}
\]

with \( K_1 = 2.8563 \), \( K_2 = -1.1896 \) and \( T_1 = 0.5 \) second, \( T_2 = 1.2 \) seconds.

The initial conditions may be obtained by substituting the known constants into Equations (51) and (52).

\[
\begin{align*}
\xi(t) & = -c(t) \\
\Bigg|_{t=a^+} & \quad \Bigg|_{t=10^{-1}} \\
& = \sum_{i=1}^{2} \frac{AK_i}{a}(e^{-\frac{1}{T_i}a} - 1) \\
& = 12 \left[ 2.8563 \left( e^{-0.1} - 1 \right) - 1.1896 \left( e^{-0.1} - 1 \right) \right] \\
& = -5.140212
\end{align*}
\]

\[
\begin{align*}
\frac{d\xi(t)}{dt} & = -\frac{dc(t)}{dt} \\
\Bigg|_{t=a^+} & \quad \Bigg|_{t=a^+} \\
& = \sum_{i=1}^{2} \frac{AK_i}{a} (-1) \frac{1}{T_i} \frac{1}{T_i} (e^{-\frac{1}{T_i}a} - 1) \\
& = -12 \left[ 2.8563 \frac{-0.1}{0.5} \left( e^{0.5} - 1 \right) - 1.1896 \frac{-0.1}{1.2} \left( e^{1.2} - 1 \right) \right] \\
& = 11.42736
\end{align*}
\]
Now measurement of the initial conditions in Figure 87 (c) yields \( \dot{\xi} \bigg|_{t=a^+} = -5.60 \), \( \ddot{\xi} \bigg|_{t=a^+} = 12.03 \) which shows that both are in very close agreement.

**Prediction of the region of all possible initial conditions.** For the case in which the initial conditions established by the impulse excitation are plotted versus the amplitudes \( \dot{A}_a \), it is interesting to observe the plane to be divided into two distinct regions as shown in Figure 102.

![Fig. 102. The regions of the established initial conditions of Experiment IV.](image-url)

In the \( \dot{\xi} \) versus \( \xi \) plane, it is difficult to visualize the division of the distinct regions because of small amplitudes and \( \dot{A}_a \) might have larger initial conditions, especially the derivatives. Furthermore, when the order of the system becomes high enough, this is even more difficult to predict. We may conclude that for a dead-zone nonlinearity, this
division can always be expressed in a configuration as shown in Figure 102.

**Discussion of a special case if the durations are very short.** All the durations less than 0.1 second from Figures 80 through 92 can be considered as a << $T_1$, since the duration 0.1 second is only $\frac{1}{2}$ and $\frac{1}{12}$ of the $T_1$ and $T_2$ respectively. Suppose 0.02 second is used. This shows that the duration is only $\frac{1}{22}$ and $\frac{1}{60}$ of the time constants $T_1$ and $T_2$ respectively. Therefore, for those values of $a$ less than 0.1 second, the durations are good enough to be used to establish the initial conditions. The discussion of section F, Chapter IV, cannot be verified for the same reason as is discussed in Experiment II.

**The identification of a dead-zone relay nonlinearity.** A dead-zone relay nonlinearity can be easily identified by injecting a set of practical impulses at the error terminals. Whenever the amplitude of the impulses is less than 12 volts, no responses can be observed. When the amplitudes are equal to or greater than 12 volts, and if the duration $a$ is kept constant, no change of the responses can be observed. This phenomenon has been observed in this experiment.
F. Experiment V

The system and the experimental setup. The system used in this experiment is exactly the same as that used in Experiment II except for the characteristics of the nonlinearity. The frequency sensitive function has order four and can be programmed in the REAC electronic analogue computer. The circuit diagram has been shown in Figure 79, (the standard symbols may be referred to in reference 18) which requires four summing integrators and one summing amplifier. The photograph of the programming board for the function $G(s)$ is shown in Figure 103. The characteristics of the nonlinearity used in this experiment are shown in Figure 104, and the schematic diagram of the simulation of such a nonlinearity is also shown in Figure 105. This combined nonlinearity is accomplished by using a dead-zone relay with the dead-zone constant 12 volts d-c. The slope of the straight line after the dead-zone constant is unity.

Impulse excitation and the responses of the system. The responses of the system excited by a set of impulses under an open-loop consideration may be observed from the output recordings. It is clear that if the amplitude of the impulses is less than 12 volts, no responses will be observed. However, if the amplitude of the impulses is equal to or greater than 12 volts, the responses will not saturate as was observed in Experiment II. Therefore, the problem of establishing the initial conditions in this case is more
Fig. 103 REAC analogue computer board for simulation of $G(s)$. 

\[
\frac{-s^2 (1+0.4s)}{(1+0.2s)(1+0.5s)(1+s)(1+1.2s)}
\]
Fig. The combined nonlinearity characteristics controlled micro switches

Fig. 105 Schematic diagram of the nonlinear element with dead-zone characteristic
complicated than was that for Experiment II. Figure 106 shows the responses of the system resulting from excitation with practical impulses with amplitudes of 14 volts and 17 volts and durations of 0.04, 0.08, 0.12, 0.16, 0.25, and 0.36 second. The set of responses with an amplitude of 12 volts is exactly the same as shown in Figure 80 (b) through Figure 92 (b). They are not repeated here.

Prediction of all possible regions of the initial conditions which can be established by the practical impulse excitation. If the duration of the impulses is assumed to be much shorter than all time constants, the regions for the established initial conditions are essentially the same as shown in Figure 93. Assume that the nonlinearity used in this system has a saturating characteristic instead of the linear one. It is then interesting to compare the region if $A_s$ is used as an independent variable, where $A_s$ is the amplitude just before the function $G(s)$ block. Figure 107 shows their differences.

For the dead-zone relay (Figure 100), the established initial conditions occupy the real lines and cannot be elsewhere, whereas those established for the combined nonlinearity (Figure 104) occupy the planes above these real lines because the impulses at the output of the nonlinearity are theoretically boundless. This shows that for different types of nonlinearities, it is always separable as far as the regions of the initial conditions are concerned.
Fig. 106 The practical impulse functions and the outputs of the system (Experiment V)
(e) $\frac{A}{a} = 14v, a = 0.12$ second

(f) The output of the system
Vertical scale: 0.4 v/cm  Time scale: 0.4 sec/cm

(g) $\frac{A}{a} = 14v, a = 0.16$ second

(h) The output of the system
Vertical scale: 0.4 v/cm  Time scale: 0.4 sec/cm

Fig. 106 The practical impulse functions and the outputs of the system (Experiment V)
(i) $\frac{A}{a} = 14v$, $a = 0.25$ second

(j) The output of the system  
Vertical scale: $0.4 \text{ v/cm}$  Time scale: $0.4 \text{ sec/cm}$

(k) $\frac{A}{a} = 14v$, $a = 0.36$ second

(l) The output of the system  
Vertical scale: $0.4 \text{ v/cm}$  Time scale: $0.4 \text{ sec/cm}$

Fig. 106 The practical impulse functions and the outputs of the system (Experiment V)
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Discussion of the special case when the durations are very short. The discussion presented in section F of Chapter IV may well be illustrated in this experiment. During the dead-zone period, there is nothing to be discussed because no output can be observed. However, for an amplitude greater than the dead-zone constant, interesting results may be observed. Consider Figure 106 (m) and Figure 82(a). Both impulses give the same area, 0.72 volt-second. (0.04 x 18 and 0.06 x 12 respectively). Now, observe the output responses of Figure 106 (n) and Figure 82 (b). Both outputs give the same responses by comparing many instantaneous values, especially true for the time after duration a. For the other data, they agree quite well in conjunction with the theoretical development.
CHAPTER VIII
CONCLUSION

The work begins with the analysis of various practical impulse functions and their generators. The controllable practical impulse functions and modulated practical impulse functions have actually been generated and these function generators have been used to obtain the experimental results. Control of the practical impulse function is important insofar as the investigation of the problem is concerned. The variation of the amplitude of a practical impulse function is important for both the problem of establishing the initial conditions as well as the identifying of a nonlinear system. On the other hand, the variation of the duration of the impulse function generally gives different initial conditions regardless of the type of nonlinearities encountered in the system. The variation of the duration does not help to accomplish the identification, but when the duration is too short, it does effect the accuracy of the identification.

The relationships between the initial conditions and the state-response have been introduced in Chapter III. The technique of establishing the initial conditions and the mathematical models for the systems have been presented in
Chapters IV and V for linear and nonlinear systems respectively. Insofar as the linear systems are concerned, initial conditions can always be established regardless of the open-loop or closed-loop approach. On the other hand, the nonlinear system can advantageously be excited with the open-loop approach. This is especially true when several nonlinearities are involved in a system. Without making any assumptions in order to solve a nonlinear problem, it is difficult to establish the initial conditions, even if the system has a simple appearance with respect to its block diagram. Therefore, the open-loop approach has been used in solving most nonlinear systems.

The conclusions at the end of Chapter V serve to summarize the problem of establishing the initial conditions for both linear and nonlinear systems. It is not intended to repeat them here, but they can certainly serve as conclusions to the first few chapters. The situation in which the initial conditions could not be established are also discussed in Chapter V. This occurs, of course, only when the system is nonlinear.

The mathematical models for a nonlinear control system developed in this work consists of the nonlinearities and frequency sensitive function sections. This approach is widely used and is practical because the system is composed of components which may or may not be represented by linear functions.
One outstanding feature of this work is the use of a single impulse function instead of a periodical function to excite the system. The impulse functions generated, however, are periodic with a very long period. This implies, of course, that the second impulse arrives after the responses of the previous one has nearly died out. The periodic sinusoidal signal has been used in the Bode plot for the linear system and in the describing function for nonlinear system analysis. For example, the describing function cannot be used to solve the identification problem because the frequency and the amplitude are independent variables. It is also not admissible in the phase-space or phase-plane analysis because the input is a periodic function and the phase-space analysis admits only the initial conditions.

Various systems have been chosen in the experiments to verify the theories developed in previous chapters. The initial conditions can always be established from experimental data unless the signal travel is stopped by the dead-zone nonlinearities. If the open loop excitation is assumed, the $\dot{\epsilon}$ and $\ddot{\epsilon}$ evaluated at $t=a^+$ can always be found no matter how complicated the system. Unfortunately, the higher order systems are difficult to visualize graphically. However, the initial conditions projected on the $\epsilon$ vs $\dot{\epsilon}$ plane can always be studied. The experimental technique utilizing the impulse function is very important, especially in studying
the characteristics of a system which is changing because of the changes occurring in the system's environment.

Finally, the technique of analysis of control systems presented in this work tries to lay a foundation for the identification technique for system-characteristic adaptive systems which are a particular class of adaptive systems whose adjustments are based on the measurement of the system characteristics. Practical impulse response gives a good approximation in the system characteristics, which in turn determines the "Figure of Merit" of the system.
BIBLIOGRAPHY


